The Foundations of Analysis

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March 22, 2013

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1 Introduction

Here we introduce the real numbers. On the one hand, this is a modern introduction based on morphisms between objects in an algebraic category. On the other hand, it is an ancient introduction with 24 of the theorems dating back to about 300 B.C. It is hoped that the complete and elementary nature of this work will show that it is practical to introduce the real numbers from a categorical perspective to students who may never study abstract algebra, but will use real numbers on a regular basis in their future course work and professional careers.

In this endeavor we have been inspired by Landau's little book *Grundlagen der Analysis*[1] from which we have appropriated the title of the present work.

2 Magnitude Spaces

Definition 2.1 *A* ***magnitude space*** *is a nonempty set M together with a binary operation on M, which we usually denote by $+$, such that for any $a, b, c \in M$*

(i)(associativity) $a + (b + c) = (a + b) + c$,

(ii)(commutativity) $a + b = b + a$, and

(iii)(trichotomy) exactly one of the following is true: $a = b + d$ for some $d \in M$, or $a = b$, or $b = a + d$ for some $d \in M$.

Definition 2.2 (The whole is greater than the part.) *If a and b are elements of a magnitude space M and $b = a + d$ for some $d \in M$, we say a is*

less than *b and write $a < b$ or equivalently we say b is* ***greater than*** *a and write $b > a$.*

Remark 2.1 *Since a and b are smaller than $a + b$ and hence not equal to $a + b$, a magnitude space does not have an additive identity or zero element.*

The concept of magnitude spaces has a long history. That the concept was recognized as a formal abstraction before 300 B.C. is evidenced by the following two quotes. The first is Proposition 16 from Book V of Euclid's *Elements.*

> **Given**–Four proportionate magnitudes a, b, c, d with a to b the same as c to d.
>
> **To be Shown**–The alternates will also be a proportion, a to c the same as b to d.

The second, dating from a generation or two before Euclid, is a comment by Aristotle (Posterior Analytics I, 5, 20).

> Alternation used to be demonstrated separately of numbers, lines, solids, and time intervals, though it could have been proved of them all by a single demonstration. Because there was no single name to denote that in which numbers, lines, solids, and time intervals are identical, and because they differed specifically from one another, this property was proved of each of them separately. Today, however, the proof is commensurately universal, for they do not possess this attribute as numbers, lines, solids, and time intervals, but as manifesting this generic character which they are postulated as possessing universally.

We do not have an explicit list of the properties which the various kinds of magnitudes are postulated as possessing universally from this time period. All we have are a number of "Common Notions" such as "the whole is greater than the part" and "if equals are subtracted from equals, the remainders will be equal." The only complete work on proportions surviving from this period is Book V of Euclid's *Elements.* Here we find rigorous reasoning without an explicit foundation. We do not know if there was an explicit foundation generally known at the time. What we do know is that, starting with the definition of magnitude spaces above, we can establish a foundation by which the propositions in Book V of Euclid can be proved in accordance with present day standards of rigor.[2] And from the propositions in Book V of Euclid there follows the theory establishing the real numbers in the modern categorical sense.

3 Functions

Here we review function terminology and a few basic theorems.

Definition 3.1 *To indicate that φ is a function from a set S into a set S', we write $\varphi : S \to S'$. We also refer to functions as mappings and if $\varphi a = b$ we say that φ **maps** a into b.*

Definition 3.2 *Two functions φ and ψ from a set S into a set S' are **equal** if for all $a \in S$, $\varphi a = \psi a$. And in this case we write $\varphi = \psi$.*

Definition 3.3 *A function $\varphi : S \to S'$ is **one-to-one** if for all $a, b \in S$, $a \neq b$ implies $\varphi a \neq \varphi b$.*

Definition 3.4 *A function $\varphi : S \to S'$ is **onto** if for each $a' \in S'$ there is some $a \in S$ such that $\varphi a = a'$. And in this case we say that φ maps S **onto** S'.*

Definition 3.5 *If $\varphi_1 : S_1 \to S_2$ and $\chi : S_2 \to S_3$ are functions, then we define the **composition** $\chi \circ \varphi : S_1 \to S_3$ by $(\chi \circ \varphi)\, a = \chi\,(\varphi a)$ for all $a \in S_1$. In most cases, we omit the $\circ$ symbol between the functions and define the composition $\chi\varphi : S_1 \to S_3$ by $(\chi\varphi)\, a = \chi\,(\varphi a)$*

Definition 3.6 *The **identity function** $i_S : S \to S$ maps every element of a set S to itself. I.e. for all $a \in S$, $i_S a = a$.*

Theorem 3.1 *Composition of functions is associative*

Proof Assume S, S', S'' and $\varphi : S \to S'$, $\chi : S' \to S''$, $\psi : S'' \to S'''$ are functions. If $a \in S$ then

$$\begin{aligned}
(\psi\,(\chi\varphi))\, a &= \psi\,((\chi\varphi)\, a) \ \ \text{(Definition 3.5)} \\
&= \psi\,(\chi\,(\varphi a)) \ \ \text{(Definition 3.5)} \\
&= (\psi\chi)\,(\varphi a) \ \ \text{(Definition 3.5)} \\
&= ((\psi\chi)\,\varphi)\, a \ \text{(Definition 3.5)}
\end{aligned}$$

and therefore $\psi\,(\chi\varphi) = (\psi\chi)\,\varphi$ by Definition 3.2. ■

Theorem 3.2 *The identity function $i_S : S \to S$ is one-to-one and onto.*

Proof Suppose $a, b \in S$ and $a \neq b$. Now $i_S a = a$ and $i_S b = b$ by Definition 3.6 and hence $i_S a \neq i_S b$. Therefore i_S is one-to one according to Definition 3.3.

And for any $a \in S$ there is some $c \in S$ (namely $c = a$) such that $i_S c = a$ by Definition 3.6. Therefore i_S is onto according to Definition 3.4. ■

Theorem 3.3 *If $\varphi : S \to S'$, $\psi : S' \to S$, and $\psi\varphi = i_S$, then ψ is onto.*

Proof For any $a \in S$

$$\begin{aligned} a &= i_S a \text{ (Definition 3.6)} \\ &= (\psi\varphi)\, a \text{ (Definition 3.2)} \\ &= \psi\,(\varphi a)\,. \text{ (Definition 3.5)} \end{aligned}$$

Thus there is an element in S' (namely φa) which ψ maps into a. Therefore ψ is onto according to Definition 3.4. ■

Remark 3.1 *In the preceding theorem, we could also conclude that φ is one-to-one.*

4 Basic Equalities and Inequalities

Definition 4.1 *A binary relation $<$ on a set S is* ***trichotomous*** *if for all $a, b \in S$, exactly one of the following is true: $b < a$, or $a = b$, or $a < b$.*

Definition 4.2 *A binary relation $<$ on a set S is* ***transitive*** *if for all $a, b, c \in S$, $a < b$ and $b < c$ implies $a < c$.*

Definition 4.3 *A binary relation is a* ***strict linear order*** *if it is trichotomous and transitive.*

Definition 4.4 *Binary relations $<$ and $>$ on a set S are* ***inverses*** *(to each other) if for all $a, b \in S$, $a < b$ if and only if $b > a$.*

Theorem 4.1 *If S and S' are sets with trichotomous relations $<$ (and inverse relations $>$) and $\varphi : S \to S'$ is a function such that for all $a, b \in S$, $a < b$ implies $\varphi a < \varphi b$, then for all $a, b \in S$, φa has to φb the same relation ($<$, $=$, or $>$) as a has to b, and φ is one-to-one.*

Proof For $a, b \in S$ the three mutually exclusive cases

$$a < b, \text{ or } a = b, \text{ or } a > b$$

imply, by assumption, the three mutually exclusive cases

$$\varphi a < \varphi b, \text{ or } \varphi a = \varphi b, \text{ or } \varphi a > \varphi b$$

respectively. The three converse implications follow from trichotomy. For instance, assume $\varphi a < \varphi b$. If $a = b$, then $\varphi a = \varphi b$ which contradicts the assumption. If $a > b$, then $\varphi a > \varphi b$ which also contradicts the assumption. Since $a = b$ and $a > b$ are incompatible with our assumption, $a < b$ by trichotomy. Thus $\varphi a < \varphi b$ implies $a < b$.

I say φ is one-to-one. For if $a, b \in S$ and $a \neq b$, then $a < b$ or $a > b$ by trichotomy and hence $\varphi a < \varphi b$ or $\varphi b < \varphi a$ and hence $\varphi a \neq \varphi b$ by trichotomy. We have now shown that if $a \neq b$ then $\varphi a \neq \varphi b$. Therefore φ is one-to-one according to Definition 3.3. ■

In the remainder of this section lower case variables a, b, c, and d are elements of a magnitude space M.

Theorem 4.2 *The $<$ relation in a magnitude space is trichotomous.*

Proof From Definition 2.1, exactly one of the following is true: $a = b + d$ for some $d \in M$, $a = b$, or $b = a + d$ for some $d \in M$. From Definition 2.2, exactly one of the following is true: $b < a$, or $a = b$, or $a < b$. Therefore the $<$ relation in a magnitude space is trichotomous according to Definition 4.1. ■

Theorem 4.3 (Translation Invariance) *If $b < c$, then $a + b < a + c$.*

Proof If $b < c$, then $c = b + d$ for some d by Definition 2.2. Hence

$$a + c = a + (b + d) = (a + b) + d$$

by Definition 2.1 (associativity) and therefore $a + b < a + c$ according to Definition 2.2. ■

Theorem 4.4 (If equals are added to equals or unequals ...) *$a+b$ has to $a + c$ the same relation ($<$, $=$, or $>$) as b has to c.*

Proof Fix a and define a function $\varphi : M \to M$ by $\varphi b = a + b$. If $b < c$, then $\varphi b < \varphi c$ by the preceding theorem. Therefore $\varphi b = a + b$ has to $\varphi c = a + c$ the same relation ($<$, $=$, or $>$) as b has to c by Theorem 4.1. $\blacksquare$

Remark 4.1 *If $a < b$, then, from Definition 2.2, there is some d such that $b = a + d$. In fact there is only one such element. For if $a + d = a + d'$, then $d = d'$ by the preceding theorem.*

Definition 4.5 *If $a < b$, we define $b - a$ to be the unique element d such that $b = a + d$.*

Theorem 4.5 *If $a < b$, then $b - a < b$.*

Proof Definitions 2.2 and 4.5 $\blacksquare$

Remark 4.2 *If $a < b$, then $(b - a) + a = a + (b - a) = b$ since $(b - a)$ is, by definition, the unique element which when added to a yields b.*

Theorem 4.6 (Transitivity of $<$) *If $a < b$ and $b < c$, then $a < c$ and $c - a = (c - b) + (b - a)$.*

Proof If $a < b$ and $b < c$, then $c = (c - b) + b = (c - b) + (b - a) + a$ by Definition 4.5. Hence $a < c$ by Definition 2.2 and $c - a = (c - b) + (b - a)$ according to Definition 4.5. $\blacksquare$

Theorem 4.7 *The $<$ relation in a magnitude space is a strict linear order.*

Proof The $<$ relation in a magnitude space is trichotomous by Theorem 4.2. And it is transitive by the preceding theorem and Definition 4.2. Therefore the $<$ relation in a magnitude space is a strict linear order according to Definition 4.3. $\blacksquare$

Theorem 4.8 (If equals are subtracted from equals or unequals ...) *If $b > a$ and $c > a$, then $b - a$ has to $c - a$ the same relation ($<$, $=$, or $>$) as b has to c.*

Proof $a + (b - a)$ has to $a + (c - a)$ the same relation ($<$, $=$, or $>$) as $(b - a)$ has to $(c - a)$ by Theorem 4.4. But $a + (b - a) = b$ and $a + (c - a) = c$ by Definition 4.5. Therefore $b - a$ has to $c - a$ the same relation ($<$, $=$, or $>$) as b has to c. $\blacksquare$

Theorem 4.9 *If $a > b$, then a has to $b + c$ the same relation ($<$, $=$, or $>$) as $a - b$ has to c.*

Proof $b + (a - b)$ has to $b + c$ the same relation ($<$, $=$, or $>$) as $a - b$ has to c by Theorem 4.4. And $a = b + (a - b)$ by Definition 4.5. Therefore a has to $b + c$ the same relation ($<$, $=$, or $>$) as $a - b$ has to c. ■

5 Magnitude Space Embeddings

In this section M, M', and M'' are magnitude spaces.

Definition 5.1 *A mapping $\varphi : M \to M'$ is a **homomorphism** if $\varphi(a + b) = \varphi a + \varphi b$ for all $a, b \in M$. A homomorphism which is one-to-one is an **embedding**.*

Definition 5.2 *An embedding $\varphi : M \to M'$ which is onto as a mapping is an **isomorphism**. If there is an isomorphism from M onto M' we say M is **isomorphic** to M'.*

Definition 5.3 *A homomorphism $\varphi : M \to M$ of a magnitude space into itself is an **endomorphism** and an endomorphism which is one-to-one and onto as a map is an **automorphism**.*

Definition 5.4 *If $\varphi : M \to M'$ and $\chi : M \to M'$ are two functions, their **sum** is the function $(\varphi + \chi) : M \to M'$ defined by $(\varphi + \chi)\, a = \varphi a + \chi a$ for all $a \in M$.*

The next two theorems show that homomorphisms between magnitude spaces are always embeddings.

Theorem 5.1 *If $\varphi : M \to M'$ is a homomorphism and $a < b$, then $\varphi a < \varphi b$ and $\varphi b - \varphi a = \varphi(b - a)$.*

Proof If φ is a homomorphism and $a < b$, then

$$\begin{aligned} \varphi b &= \varphi(a + (b - a)) \text{ (Definition 4.5)} \\ &= \varphi a + \varphi(b - a). \text{ (Definition 5.1)} \end{aligned}$$

Hence $\varphi a < \varphi b$ according to Definition 2.2 and $\varphi b - \varphi a = \varphi(b - a)$ according to Definition 4.5. ■

Theorem 5.2 *If $\varphi : M \to M'$ is a homomorphism, then φa has to φb the same relation ($<$, $=$, or $>$) as a has to b and φ is an embedding.*

Proof From the preceding theorem, $a < b$ implies $\varphi a < \varphi b$. Hence φa has to φb the same relation ($<$, $=$, or $>$) as a has to b and φ is one-to-one by Theorem 4.1. And since φ is a homomorphism and is one-to-one, φ is an embedding according to Definition 5.1. $\blacksquare$

Theorem 5.3 *The identity function i_M in a magnitude space M is an automorphism.*

Proof For any $a, b \in M$, $i_M (a + b) = a + b = i_M a + i_M b$ by Definition 3.6. Hence i_M is a homomorphism according to Definition 5.1. And i_M is one-to-one and onto by Theorem 3.2. Therefore i_M is an automorphism according to Definitions 5.1 and 5.3. $\blacksquare$

Theorem 5.4 *The sum of two embeddings is an embedding.*

Proof If $\varphi : M \to M'$ and $\chi : M \to M'$ are embeddings and $a, b \in M$, then

$$\begin{aligned}(\varphi + \chi)(a + b) &= \varphi (a + b) + \chi (a + b) \ \ \text{(Definition 5.4)}\\ &= (\varphi a + \varphi b) + (\chi a + \chi b) \ \ \text{(Definition 5.1)}\\ &= (\varphi a + \chi a) + (\varphi b + \chi b) \ \ \text{(commutativity and associativity of +)}\\ &= (\varphi + \chi) a + (\varphi + \chi) b. \ \ \text{(Definition 5.4)}\end{aligned}$$

Therefore $\varphi + \chi$ is an embedding by Definition 5.1 and Theorem 5.2. $\blacksquare$

Theorem 5.5 *The composition of two embeddings is an embedding.*

Proof If $\varphi : M \to M'$ and $\chi : M' \to M''$ are embeddings and $a, b \in M$, then

$$\begin{aligned}(\chi\varphi)(a + b) &= \chi (\varphi (a + b)) \ \ \text{(Definition 3.5)}\\ &= \chi (\varphi a + \varphi b) \ \ \text{(Definition 5.1)}\\ &= \chi (\varphi a) + \chi (\varphi b) \ \ \text{(Definition 5.1)}\\ &= (\chi\varphi) a + (\chi\varphi) b. \ \ \text{(Definition 3.5)}\end{aligned}$$

Therefore $\varphi\chi$ is an embedding by Definition 5.1 and Theorem 5.2. $\blacksquare$

6 Classification of Magnitude Spaces

Definition 6.1 *Let $<$ be a strict linear order with inverse $>$. By $a \leq b$ we shall mean $a < b$ or $a = b$. By $a \geq b$ we shall mean $a > b$ or $a = b$.*

Definition 6.2 *Let S be a set with a strict linear order $<$ and let A be a nonempty subset of S. We say $b \in S$ is a **lower bound** of A if $b \leq a$ for every $a \in A$. We say that b is a **smallest** or **least** element of A if b is a lower bound of A and $b \in A$. We say $b \in S$ is an **upper bound** of A if $a \leq b$ for every $a \in A$. We say that b is a **largest** or **greatest** element of A if b is an upper bound of A and $b \in A$.*

Definition 6.3 *A magnitude space is **discrete** if it has a smallest element; otherwise it is **nondiscrete**.*

Definition 6.4 *A magnitude space M is **well ordered** if every nonempty subset of M has a smallest element.*

Definition 6.5 *A magnitude space is **complete** if every nonempty subset with an upper bound has a least upper bound or, in other words, the set of upper bounds has a least element.*

Definition 6.6 *A magnitude space is **continuous** if it is complete and nondiscrete.*

Definition 6.7 (Hölder) *A magnitude space M is **Archimedean** if for any element $a \in M$ and any nonempty subset A with an upper bound, there is some element $\zeta \in M$ such that $\zeta \in A$ and $\zeta + a \notin A$.*

Remark 6.1 *The property of being Archimedean is usually defined in terms of integral multiples.[3]*

Theorem 6.1 *If $a \leq b$ and $b < c$, then $a < c$. And if $a < b$ and $b \leq c$, then $a < c$.*

Proof Assume $a \leq b$ and $b < c$. Then $a < b$ or $a = b$ by Definition 6.1. If $a < b$ then since also $b < c$, $a < c$ by Theorem 4.6. If $a = b$ then since also $b < c$, $a < c$. Thus in both cases $a < c$. The second part of the theorem follows by similar reasoning. ■

Theorem 6.2 *If an element is greater than an upper bound of a set, then it is an upper bound of the set but not an element of the set.*

Proof Let S be a linearly ordered set, A a nonempty subset of S, a an upper bound of A, and $a < b$. I say b is an upper bound of A and $b \notin A$.

If $c \in A$, then $c \leq a$ by Definition 6.2. And $a < b$ by assumption. Thus $c \leq b$ by Theorem 6.1 and Definition 6.1. We have now shown that if $c \in A$, then $c \leq b$. Therefore b is an upper bound of A.

Now suppose $b \in A$. Then $b \leq a$ by Definition 6.2. But this contradicts the assumption $a < b$ by Definition 6.1 and Theorem 4.2. Therefore $b \notin A$. ■

Theorem 6.3 *If an element is less than a lower bound of a set, then it is a lower bound of the set but not an element of the set.*

Proof Similar to proof of previous theorem. ■

Theorem 6.4 *If M is a discrete magnitude space with smallest element a, and $b \in M$, then there is no $c \in M$ such that $b < c < b + a$.*

Proof Suppose there is such a c. Then $(b + a) - b = ((b + a) - c) + (c - b)$ by Theorem 4.6. But $(b + a) - b = a$ by Definition 4.5. Hence $c - b < a$ by Definition 2.2. But a is the smallest element of M by assumption and so we have a contradiction. ■

Theorem 6.5 *A well ordered magnitude space is discrete and complete.*

Proof Let M be a well ordered magnitude space. Since M is a subset of M, M has a smallest element by Definition 6.4. Therefore M is discrete according to Definition 6.3.

Now let A be any nonempty subset of M with an upper bound. If B is the set of upper bounds of A, then B is a nonempty subset of M and hence B has a least element by Definition 6.4. Therefore M is complete according to Definition 6.5. ■

Theorem 6.6 (Hölder) *A complete magnitude space is Archimedean.*

Proof Let M be a complete magnitude space, $a \in M$, and A a nonempty subset of M with an upper bound. Since M is complete, A has a least upper bound ζ' by Definition 6.5.

Case 1: $\zeta' \leq a$. Since Let ζ be any element of A. Then $a < \zeta + a$ by Definition 2.2 and so $\zeta' < \zeta + a$ by Theorem 6.1. But then $\zeta + a$ is greater than an upper bound of A and hence $\zeta + a \notin A$ by Theorem 6.2.

Case 2: $a < \zeta'$. Since $\zeta' - a < \zeta'$ by Theorem 4.5, $\zeta' - a$ is less than the least upper bound of A and hence is not an upper bound of A by Theorem 6.3. Thus there is some $\zeta \in A$ such that $\zeta' - a < \zeta$ by Definition 6.2. And $\zeta' < \zeta + a$ by Theorem 4.9 and so $\zeta + a$ is greater than an upper bound of A and hence $\zeta + a \notin A$ by Theorem 6.2.

We have shown, in both cases, that there is some $\zeta \in M$ such that $\zeta \in A$ and $\zeta + a \notin A$. Therefore M is Archimedean according to Definition 6.7. ■

Theorem 6.7 *A discrete Archimedean magnitude space is well ordered.*

Proof Assume M is a discrete Archimedean magnitude space with smallest element a and A is a nonempty subset of M. I say A has a smallest element.

Let B be the set of all lower bounds of A. If $b \in A$, then $a \leq b$ by Definition 6.2. Therefore $a \in B$ and so B is nonempty. Now let c be any fixed element of A. If $b \in B$, then $b \leq c$. Thus c is an upper bound of B according to Definition 6.2 and so B is a nonempty subset of M with an upper bound.

But M is Archimedean and hence there is some element $\zeta \in M$ such that $\zeta \in B$ and $\zeta + a \notin B$ by Definition 6.7. Or, in other words, ζ is a lower bound of A and $\zeta + a$ is not a lower bound of A. And because $\zeta + a$ is not a lower bound of A, there is some $b \in A$ such that $b < \zeta + a$ by Definitions 6.2 and 6.1 and trichotomy. And since ζ is a lower bound of A, $\zeta \leq b$ by Definition 6.2. But $\zeta < b$ and $b < \zeta + a$ is impossible by Theorem 6.4. Thus $\zeta = b$ by Definition 6.1. Therefore $\zeta \in A$ and ζ is a lower bound of A and hence ζ is the smallest element of A according to Definition 6.2. ■

7 Well Ordered Magnitude Spaces

We next prove a form of mathematical induction for well ordered magnitude spaces.

Theorem 7.1 *If M is a well ordered magnitude space with smallest element a, and A is subset of M containing a such that $c \in A$ implies $c+a \in A$, then $A = M$.*

Proof Suppose $A \neq M$. Then the set B consisting of those elements in M which are not in A is nonempty and hence has a smallest element by Definition 6.4. Let b be the smallest element of B. Then b is a lower bound of B and $b \in B$ by Definition 6.2. Now a is the smallest element of M by assumption and hence $a \leq b$ by Definition 6.2. And b is not equal to a since $a \in A$ by assumption and $b \in B$, a set having no element in A. And from $a \leq b$ and $a \neq b$ follows $a < b$ by Definition 6.1. But then $b - a < b$ by Theorem 4.5. Thus $b - a$ is less than a lower bound of B and hence $b - a \notin B$ by Theorem 6.3. Hence $b - a \in A$. But, by assumption, $b - a \in A$ implies $(b - a) + a \in A$. And $(b - a) + a = b$ by Definition 4.5. Thus b is an element of A and of B which is impossible. ■

In the following theorem an embedding of a well ordered magnitude space into an arbitrary magnitude space is constructed inductively. The general approach is that of Dedekind.[4]

Theorem 7.2 *If M is a well ordered magnitude space with smallest element a, M' is an arbitrary magnitude space, and $a' \in M'$, then there exists a unique function $\varphi : M \to M'$ such that*

(i) $\varphi a = a'$ and

(ii) $\varphi b = \varphi (b - a) + a'$ for all $b > a$.

Proof First, there can be at most one function satisfying the two conditions above. For if there are two distinct functions φ and ψ each satisfying the two conditions, then there must be a smallest $b \in M$ for which $\varphi b \neq \psi b$ by Definition 6.4. Now $\varphi a = a' = \psi a$ since each of φ and ψ are assumed to have property (i) above. Thus $a \neq b$ since $\varphi b \neq \psi b$. And a is the smallest element of M by assumption and hence $a \leq b$ by Definition 6.2. And from $a \neq b$ and $a \leq b$ follows $a < b$ by Definition 6.1. Hence $b - a < b$ by Theorem 4.5. And since b is the smallest element of M such that $\varphi b \neq \psi b$, $\varphi (b - a) = \psi (b - a)$. Therefore

$$\begin{aligned}
\varphi b &= \varphi (b - a) + a' \text{ (property (ii) above)} \\
&= \psi (b - a) + a' \text{ (Theorem 4.4)} \\
&= \psi b \text{ (property (ii) above)}
\end{aligned}$$

which is a contradiction.

It remains to show that there exists a function φ with the specified properties. To this end, for each $b \in M$ let M_b be the set of those elements in M which are less than or equal to b. I say that for each $b \in M$ there is a unique function $\varphi_b : M_b \to M'$ such that

(i) $\varphi_b a = a'$ and

(ii) $\varphi_b c = \varphi_b (c - a) + a'$ for $a < c \leq b$.

That there cannot be two distinct functions with these properties for a given $b \in M$ can be shown by the same argument as given in the beginning of the proof. To prove the existence of one such function for each $b \in M$, let A be the set of all elements b in M for which there is a unique function $\varphi_b : M_b \to M'$ as described above. In the case of $b = a$, $M_a = \{a\}$ and the function $\varphi_a : M_a \to M'$ defined by $\varphi_a a = a'$ has the required properties. Thus $a \in A$. Now suppose $b \in A$. We can then define a function $\varphi_{b+a} : M_{b+a} \to M'$ in terms of the unique function $\varphi_b : M_b \to M'$ according to

$$\varphi_{b+a} c = \begin{cases} \varphi_b c & \text{if} \quad c \leq b \\ \varphi_b b + a' & \text{if} \quad c = b + a \end{cases} .$$

And since $b \in A$, (i) $\varphi_{b+a} a = a'$ and (ii) $\varphi_{b+a} c = \varphi_{b+a} (c - a) + a'$ for all $c \in M_{b+a}$. We have now shown that if $b \in A$, then $b + a \in A$. Hence $A = M$ by the preceding theorem and for each $b \in M$ there exists a unique function φ_b satisfying the two conditions above.

Now define the function $\varphi : M \to S$ according to $\varphi b = \varphi_b b$. We then have

$$\varphi a = \varphi_a a = a'$$

and for any $b > a$

$$\varphi b = \varphi_b b = \varphi_b (b - a) + a' = \varphi_{b-a} (b - a) + a' = \varphi (b - a) + a'.$$

■

Theorem 7.3 *If M is a well ordered magnitude space, M' is any magnitude space, a is the smallest element in M, and a' is any element in M', then there exists a unique embedding $\varphi : M \to M'$ such that $\varphi a = a'$.*

Proof From the preceding theorem there is a unique function $\varphi : M \to M'$ such that (i) $\varphi a = a'$ and (ii) $\varphi b = \varphi (b - a) + a'$ for any $b > a$. Note that from the second property it follows that

$$\begin{aligned}\varphi(b+a) &= \varphi((b+a)-a)+a' \text{ (definition of } \varphi) \\ &= \varphi b + a'. \text{ (Definition 4.5)}\end{aligned}$$

for any $b \in M$ since $b+a > a$.

I say that φ is an embedding. To see this, fix $c \in M$ and define A to be the set of all elements d in M such that $\varphi(c+d) = \varphi c + \varphi d$. Now $a \in A$ since $\varphi(c+a) = \varphi c + a' = \varphi c + \varphi a$ by note above. And if $d \in A$, then

$$\begin{aligned}\varphi(c+(d+a)) &= \varphi((c+d)+a) \text{ (associativity)} \\ &= \varphi(c+d)+a' \text{ (note above)} \\ &= \varphi c + \varphi d + a' \text{ } (d \in A) \\ &= \varphi c + \varphi(d+a) \text{ (note above)}\end{aligned}$$

and so $d+a \in A$. Therefore, $A = M$ by Theorem 7.1. And since our choice of $c \in M$ was arbitrary, $\varphi(c+d) = \varphi c + \varphi d$ for all $c, d \in M$. Thus φ is an embedding by Definition 5.1 and Theorem 5.2.

Now suppose ψ is some other embedding of M into M' which maps a into a'. Then $\psi a = a'$and, if $b > a$ then

$$\begin{aligned}\psi b &= \psi((b-a)+a) \text{ (Definition 4.5)} \\ &= \psi(b-a)+\psi a \text{ (Definition 5.1)} \\ &= \psi(b-a)+a'. \text{ (assumed property of } \varphi)\end{aligned}$$

But by the preceding theorem, there is only one such mapping and hence $\psi = \varphi$. ■

Theorem 7.4 *Any two well ordered magnitude spaces are isomorphic.*

Proof Assume that M and M' are well ordered magnitude spaces with smallest elements a and a' respectively. There is an embedding $\varphi : M \to M'$ such that $\varphi a = a'$ and an embedding $\varphi' : M' \to M$ such that $\varphi' a' = a$ from the preceding theorem. The composition $\varphi\varphi' : M' \to M'$ is an embedding by Theorem 5.5 and the identity mapping $i_{M'} : M' \to M'$ is also an embedding by Theorem 5.3. But

$$(\varphi\varphi')\, a' = \varphi(\varphi' a') = \varphi a = a' = i_{M'} a'$$

and so $\varphi\varphi' = i_{M'}$ by the preceding theorem. Therefore φ is onto by Theorem 3.3 and hence is an isomorphism according to Definition 5.2. ■

8 Natural Numbers and Integral Multiples

Definition 8.1 *Pick any well ordered magnitude space and denote it by* $\mathbb{N}$ *and denote the smallest element of* $\mathbb{N}$ *by* 1. *Since all well ordered magnitude spaces are isomorphic (Theorem 7.4) and we will use only the algebraic properties that the well ordered magnitude spaces have in common, it does not matter which well ordered magnitude space is chosen to play the role of the "number system"* $\mathbb{N}$. *We call* $\mathbb{N}$ *the* ***natural numbers****.*

Definition 8.2 *If* $n \in \mathbb{N}$ *and* a *is an element of a magnitude space* M, *then the* ***integral multiple*** na *is given by* $na = \varphi_{1,a}n$ *where* $\varphi_{1,a}$ *is the unique embedding of* $\mathbb{N}$ *into* M *which maps* 1 *into* a *(Theorem 7.3).*

Theorem 8.1 *For each* $a \in M$, $(n+1)\,a = na + a$.

Proof Let $\varphi_{1,a}$ be the unique embedding of $\mathbb{N}$ into M which maps 1 into a. Then

$$\begin{aligned} (n+1)\,a &= \varphi_{1,a}\,(n+1) \;\; \text{(Definition 8.2)} \\ &= \varphi_{1,a}n + \varphi_{1,a}1 \;\text{(Definition 5.1)} \\ &= na + a. \;\text{(Definition 8.2)} \end{aligned}$$

■

Remark 8.1 *For each* a *in a magnitude space* M

$$\begin{aligned} 1a &= \varphi_{1,a}1 = a \\ (1+1)\,a &= \varphi_{1,a}\,(1+1) = \varphi_{1,a}1 + a = a + a \\ (1+1+1)\,a &= \varphi_{1,a}\,(1+1+1) = \varphi_{1,a}\,(1+1) + a = a + a + a \\ &\vdots \end{aligned}$$

Theorem 8.2 *If* $\chi : M \to M'$ *is an embedding, then* $\chi\,(na) = n\,(\chi a)$ *for every* $a \in M$ *and* $n \in \mathbb{N}$.

Proof Let a be fixed. There exists a unique embedding $\varphi_{1,a}$ of $\mathbb{N}$ into M which maps 1 into a and a unique embedding $\varphi_{1,\chi a}$ of $\mathbb{N}$ into M' which maps

1 into χa by Theorem 7.3. Now $\chi\varphi_{1,a} : \mathbb{N} \to M'$ is an embedding by Theorem 5.5 and

$$\begin{aligned} (\chi\varphi_{1,a})\,1 &= \chi\left(\varphi_{1,a}1\right) \text{ (Definition 3.5)} \\ &= \chi\left(a\right). \text{ (definition of } \varphi_{1,a}) \end{aligned}$$

Therefore each of $\varphi_{1,\chi a}$ and $\chi\varphi_{1,a}$ are embeddings of $\mathbb{N}$ into M' which map 1 (the smallest element in $\mathbb{N}$) into the same element $\chi a \in M'$ and hence $\varphi_{1,\chi a} = \chi\varphi_{1,a}$ by Theorem 7.3. Therefore

$$\begin{aligned} n\left(\chi a\right) &= \varphi_{1,\chi a}n \text{ (Definition 8.2)} \\ &= \left(\chi\varphi_{1,a}\right)n \text{ (Definition 3.2)} \\ &= \chi\left(\varphi_{1,a}n\right) \text{ (Definition 3.5)} \\ &= \chi\left(na\right). \text{ (Definition 8.2)} \end{aligned}$$

■

Theorem 8.3 *If $\varphi : M \to M'$ is an embedding and $a, b \in M$, then for each pair $m, n \in \mathbb{N}$, $m\left(\varphi a\right)$ has to $n\left(\varphi b\right)$ the same relation ($<$, $=$, or $>$) as ma has to nb.*

Proof First, $m\left(\varphi a\right) = \varphi\left(ma\right)$ and $n\left(\varphi b\right) = \varphi\left(nb\right)$ by the preceding theorem. Second, $\varphi\left(ma\right)$ has to $\varphi\left(nb\right)$ the same relation ($<$, $=$, or $>$) as ma has to nb by Theorem 5.2. Therefore $m\left(\varphi a\right)$ has to $n\left(\varphi b\right)$ the same relation ($<$, $=$, or $>$) as ma has to nb. ■

9 Embeddings and Ratios

The preceding theorem is the connecting point between modern algebraic definitions of number systems and the classical theory of ratios. The classical theory is based on the following two definitions.

Definition 9.1 (Euclid V, Definition 4) *Two elements a, b in a magnitude space are said to* ***have a ratio*** *if there are natural numbers m, n such that $ma > b$ and $nb > a$.*

Definition 9.2 (Euclid V, Definitions 5, 6, and 7) *Let M and M' be magnitude spaces. We say that a pair $a, b \in M$ has the* ***same ratio*** *as* *(or are **proportional** to) a pair $a', b' \in M'$ if ma has to nb the same relation ($<$, $=$, or $>$) as ma' has to nb' for every $m, n \in \mathbb{N}$. And in this case we write $a : b = a' : b'$. And if for some $m, n \in \mathbb{N}$, $ma > nb$ and $ma' \leq nb'$, then we say a has to b a **greater ratio** than a' has to b' and we write $a : b > a' : b'$ or $a' : b' < a : b$.*

We are particularly interested in magnitude spaces in which every pair of elements have a ratio. The next two theorems establish that a magnitude space has this property if and only if it is an Archimedean magnitude space.

Theorem 9.1 *If every pair of elements of a magnitude space have a ratio, then the magnitude space is Archimedean.*

Proof Assume M is a magnitude space in which every pair of elements have a ratio. Let $a \in M$ and let A be a nonempty subset of M which has an upper bound. I say there is some $\zeta \in M$ such that $\zeta \in A$ and $\zeta + a \notin A$.

Let B be the set of all natural numbers n such that na is an upper bound of A. If b is any upper bound of A, then there is some natural number n such that $na > b$ because a and b have a ratio by assumption and Definition 9.1. Thus B is nonempty. And the natural numbers are a well ordered magnitude space by Definition 8.1. Hence B has a smallest element by Definition 6.4. Let n be the smallest element of B.

Case 1: $n = 1$. Pick any $\zeta \in A$. Then $na < \zeta + a$.

Case 2: $n > 1$. Since n is the smallest natural number such that na is an upper bound of A and $n - 1 < n$ by Theorem 4.5, it follows that $(n-1)\,a$ is not an upper bound of A. Therefore there must be some $\zeta \in A$ such that $(n-1)\,a < \zeta$ by Definition 6.2. Thus $(n-1)\,a + a < \zeta + a$ by Theorem 4.3 and

$$\begin{aligned}(n-1)\,a + a &= ((n-1)+1)\,a \text{ (Theorem 8.1)}\\ &= na. \text{ (Definition 4.5)}\end{aligned}$$

Therefore $na < \zeta + a$.

In both cases, $\zeta \in A$ and $\zeta + a$ is greater than an upper bound of A and hence $\zeta + a \notin A$ by Theorem 6.2. Therefore M is an Archimedean magnitude space according to Definition 6.7. ■

Theorem 9.2 *If a and b are elements of an Archimedean magnitude space, then there is some natural number n such that $na > b$.*

Proof Suppose $na \leq b$ for all natural numbers n. Let A be the set of all integral multiples of a. Then b is an upper bound of A by Definition 6.2. Thus there is some element $\zeta \in M$ such that $\zeta \in A$ and $\zeta + a \notin A$ by Definition 6.7. But if $\zeta \in A$, then $\zeta = na$ for some natural number n and $\zeta + a = na + a$. And $na + a = (n+1)\,a$ by Theorem 8.1. Hence $\zeta + a$ is an integral multiple of a and hence $\zeta + a \in A$ which is a contradiction. ∎

Theorem 9.3 *Any two elements of an Archimedean magnitude space have a ratio.*

Proof Previous theorem and Definition 9.1. ∎

Theorem 9.4 *If M and M' are Archimedean magnitude spaces and $\varphi : M \to M'$ is an embedding, then $\varphi a : \varphi b = a : b$ for every $a, b \in M$.*

Proof Any two elements of M have a ratio from the previous theorem. And likewise any two elements of M' have a ratio. Fix $a, b \in M$. For any two natural numbers m and n, $m\,(\varphi a)$ has to $n\,(\varphi b)$ the same relation ($<$, $=$, or $>$) as ma has to nb by Theorem 8.3. Therefore $\varphi a : \varphi b = a : b$ according to Definition 9.2. ∎

10 Classical Theory of Ratios

Henceforth we shall consider only Archimedean magnitude spaces.

In this section variables a, b, c, d, e, f are all elements of a magnitude space M, a', b', c', d', e', f' are all elements of a magnitude space M', and so on. Variables j, k, m, n are natural numbers.

The propositions in this section appear in the exact order as the propositions in Book V of The Elements. The proofs are likewise similar with the following exceptions.

1. Proofs for Propositions 1, 2, and 3 are based on Definition 8.2 and Theorem 7.3.

2. The proofs of Propositions 10 and 18 follow those of Robert Simson.[5]

Proposition 1 *$n(a + b) = na + nb$.*

Proof Let a and b be fixed. There exists a unique embeddings $\varphi_{1,a}$, $\varphi_{1,b}$, and $\varphi_{1,a+b}$ of $\mathbb{N}$ into M which map 1 into a, b, and $a+b$ respectively by Theorem 7.3. Note that $\varphi_{1,a}+\varphi_{1,b}$ is an embedding of $\mathbb{N}$ into M by Theorem 5.4 and

$$\begin{aligned}(\varphi_{1,a}+\varphi_{1,b})\,1 &= \varphi_{1,a}1+\varphi_{1,b}1 \text{ (Definition 5.4)}\\ &= a+b. \text{ (definitions of } \varphi_{1,a} \text{ and } \varphi_{1,b})\end{aligned}$$

Thus $\varphi_{1,a+b}$ and $\varphi_{1,a}+\varphi_{1,b}$ are each embeddings of $\mathbb{N}$ into M which map 1 into $a+b$ and hence $\varphi_{1,a+b}=\varphi_{1,a}+\varphi_{1,b}$ by Theorem 7.3. Therefore

$$\begin{aligned}n(a+b) &= \varphi_{1,a+b}n \text{ (Definition 8.2)}\\ &= (\varphi_{1,a}+\varphi_{1,b})\,n \text{ (Definition 3.2)}\\ &= \varphi_{1,a}n+\varphi_{1,b}n \text{ (Definition 5.4)}\\ &= na+nb. \text{ (Definition 8.2)}\end{aligned}$$

■

Proposition 2 $(m+n)\,a = ma+na.$

Proof Let a be fixed. There exists a unique embedding $\varphi_{1,a}$ of $\mathbb{N}$ into M which maps 1 into a by Theorem 7.3 and

$$\begin{aligned}(m+n)\,a &= \varphi_{1,a}(m+n) \text{ (Definition 8.2)}\\ &= \varphi_{1,a}m+\varphi_{1,a}n \text{ (Definition 5.1)}\\ &= ma+na. \text{ (Definition 8.2)}\end{aligned}$$

■

Proposition 3 $(mn)\,a = m\,(na).$

Proof Let a be fixed. There exists a unique embedding $\varphi_{1,a}$ of $\mathbb{N}$ into M which maps 1 into a by Theorem 7.3 and

$$\begin{aligned}(mn)\,a &= \varphi_{1,a}(mn) \text{ (Definition 8.2)}\\ &= m\,(\varphi_{1,a}n) \text{ (Theorem 8.2)}\\ &= m\,(na)\,. \text{ (Definition 8.2)}\end{aligned}$$

■

Proposition 4 *If $a : b = a' : b'$, then $ja : kb = ja' : kb'$.*

Proof If $a : b = a' : b'$, then for any two numbers m and n

$$\begin{aligned} m\,(ja) > n\,(kb) &\implies (mj)\,a > (nk)\,b \text{ (Proposition 3)} \\ &\implies (mj)\,a' > (nk)\,b' \text{ (Definition 9.2)} \\ &\implies m\,(ja') > n\,(kb')\,. \text{ (Proposition 3).} \end{aligned}$$

And the same argument applies with $>$ replaced by $=$ or $<$. Therefore $ja : kb = ja' : kb'$ according to Definition 9.2. ■

Proposition 5 *If $a > b$, then $na > nb$ and $na - nb = n\,(a - b)$. And, more generally, na has to nb the same relation ($<$, $=$, or $>$) as a has to b.*

Proof Fix n and let $\varphi : M \to M$ be the function defined by $\varphi a = na$. Then $\varphi\,(a + b) = \varphi a + \varphi b$ for any a and b by Proposition 1 and hence φ is a homomorphism according to Definition 5.1. Hence Theorems 5.1 and 5.2 apply. ■

Proposition 6 *If $m > n$, then $ma > na$ and $ma - na = (m - n)a$. And, more generally, ma has to na the same relation ($<$, $=$, or $>$) as m has to n.*

Proof Fix a and let $\varphi : \mathbb{N} \to M$ be the function defined by $\varphi n = na$. Then $\varphi\,(m + n) = \varphi m + \varphi n$ for any m and n by Proposition 2 and hence φ is a homomorphism according to Definition 5.1. Hence Theorems 5.1 and 5.2 apply. ■

Proposition 7 *If $a = b$, then $a : c = b : c$ and $c : a = c : b$.*

Proof If $a = b$, then $ma = mb$ by Proposition 5. Thus for any m and n, ma has the same relation to nc ($<$, $=$, or $>$) as mb has to nc. Therefore $a : c = b : c$ according to Definition 9.2. A similar argument shows that $c : a = c : b$. ■

Proposition 8 *If $a > b$, then $a : c > b : c$ and $c : b > c : a$.*

Proof Assume $a > b$. Since magnitude spaces are assumed to be Archimedean, there is some m such that $m\,(a - b) > c$ by Theorem 9.2. And from Proposition 5, $ma > mb$ and $m\,(a - b) = ma - mb$. Therefore $ma - mb > c$. And $ma > mb + c$ by Theorem 4.9.

Likewise, the set A of all natural numbers k such that $kc > mb$ is nonempty by Theorem 9.2. And since the natural numbers are well ordered by Definition 8.1, A has a smallest element by Definition 6.4. Let n be the smallest element of A.

I say $ma > nc$. Suppose, on the contrary, that $nc \geq ma$. Then, since it was shown that $ma > mb + c$, it follows that $nc > mb + c$ by Theorem 6.1 and

$$\begin{aligned}
nc > mb + c &\implies nc > mb + 1c \text{ (Definition 8.2)}\\
&\implies nc - 1c > mb \text{ (Theorem 4.9)}\\
&\implies (n-1)\,c > mb \text{ (Proposition 5)}\\
&\implies (n-1) \in A. \text{ (definition of } A\text{)}
\end{aligned}$$

But $(n-1) < n$ by Theorem 4.5 and since n is the smallest element of A, n is a lower bound of A by Definition 6.2, and hence $(n-1) \notin A$ by Theorem 6.3 which is a contradiction. Therefore, indeed, $ma > nc$.

And n was chosen so that $nc > mb$. Therefore, $ma > nc$ and $mb \leq nc$ by Definition 6.1 and hence $a : c > b : c$ according to Definition 9.2.

Likewise, $nc > mb$ and $nc \leq ma$ by Definition 6.1 and hence $c : b > c : a$ according to Definition 9.2. ■

Proposition 9 *If $a : c = b : c$, then $a = b$. And if $c : a = c : b$ then $a = b$.*

Proof Assume $a : c = b : c$. If $a > b$, then $a : c > b : c$ by the preceding theorem. But this is not consistent with the assumption. And for the same reason $b > a$ is not consistent with the assumption. Therefore $a = b$. The second part of the theorem is proved in a similar manner. ■

Proposition 10 *If $a : c > b : c$, then $a > b$. And if $c : a > c : b$, then $b > a$.*

Proof Assume $a : c > b : c$. Then there are numbers m and n such that $ma > nc$ and $mb \leq nc$ by Definition 9.2. And from $ma > nc$ and $nc \geq mb$ follows $ma > mb$ by Theorem 6.1. Therefore, $a > b$ by Proposition 5. The second part of the theorem is proved in the same way. ■

Proposition 11 *If $a : b = a' : b'$ and $a'' : b'' = a' : b'$, then $a : b = a'' : b''$.*

Proof If $a : b = a' : b'$ and $a'' : b'' = a' : b'$, and m and n are any two numbers

$$\begin{aligned} ma > nb &\implies ma' > nb' \text{ (Definition 9.2)} \\ &\implies ma'' > nb''. \text{ (Definition 9.2)} \end{aligned}$$

And the same argument applies with $>$ replaced by $=$ or $<$. Therefore $a : b = a'' : b''$ by Definition 9.2. ■

Proposition 12 *If $a : b = c : d$ then $a : b = (a + c) : (b + d)$.*

Proof Assume $a : b = c : d$. If $ma > nb$, then $mc > nd$ by Definition 9.2 and hence

$$\begin{aligned} m(a + c) &= ma + mc \text{ (Proposition 1)} \\ &> nb + mc \text{ (Theorem 4.4)} \\ &> nb + nd \text{ (Theorem 4.4)} \\ &= n(b + d). \text{ (Proposition 1)} \end{aligned}$$

Therefore $ma > nb \implies m(a + c) > n(b + d)$. And the same argument applies with $>$ replaced by $=$ or $<$. Therefore $a : b = (a + c) : (b + d)$ by Definition 9.2. ■

Proposition 13 *If $a : b = a' : b'$ and $a' : b' > a'' : b''$, then $a : b > a'' : b''$. And if $a : b = a' : b'$ and $a'' : b'' > a' : b'$, then $a'' : b'' > a : b$.*

Proof Assume $a : b = a' : b'$ and $a' : b' > a'' : b''$. Then there are numbers m and n such that $ma' > nb'$ and $ma'' \leq nb''$ by Definition 9.2. And because $ma' > nb'$, also $ma > nb$ by Definition 9.2. Therefore $ma > nb$ and $ma'' \leq nb''$ and hence $a : b > a'' : b''$ by Definition 9.2. The second part of the theorem is proved in a similar manner. ■

Proposition 14 *If $a : b = c : d$, then a has the same relation to c ($<$, $=$, or $>$) as b has to d.*

Proof Assume $a : b = c : d$. Then

$$\begin{aligned} a > c &\implies a : b > c : b \text{ (Proposition 8)} \\ &\implies c : d > c : b \text{ (Proposition 13)} \\ &\implies b > d \text{ (Proposition 10)} \end{aligned}$$

and by the same argument $a < c \implies b < d$. Likewise

$$\begin{aligned} a = c &\implies a : b = c : b \text{ (Proposition 7)} \\ &\implies c : d = c : b \text{ (Proposition 11)} \\ &\implies b = d. \text{ (Proposition 9)} \end{aligned}$$

■

Proposition 15 *$a : b = ka : kb$.*

Proof The theorem is true for $k = 1$ since $1a = a$ and $1b = b$ by Definition 8.2. Now assume the theorem is true for k. Then $a : b = ka : kb$ and hence $(ka + a) : (kb + b) = a : b$ by Proposition 12. But $ka + a = (k + 1)\, a$ and $kb + b = (k + 1)\, b$ Theorem 8.1. Therefore$(k + 1)\, a : (k + 1)\, b = a : b$ and the theorem is true for $k + 1$. Therefore the theorem is true for all k by Theorem 7.1. ■

Proposition 16 *If $a : b = c : d$, then $a : c = b : d$.*

Proof Assume $a : b = c : d$ and let m, n be any two natural numbers. Then

$$ma : mb = a : b \text{ and } nc : nd = c : d.$$

by the preceding theorem and hence

$$ma : mb = nc : nd$$

by Proposition 11. Therefore ma has the same relation to nc ($<$, $=$, or $>$) as mb has to nd by Proposition 14 and hence $a : c = b : d$ by Definition 9.2. ■

Remark 10.1 *An alternate formulation of the following theorem is that $a : b = a' : b'$ implies $(a - b) : b = (a' - b') : b'$*

Proposition 17 *If $(a + b) : b = (a' + b') : b'$, then $a : b = a' : b'$.*

Proof Assume $(a + b) : b = (a' + b') : b'$. For any m and n

$$\begin{aligned} ma > nb &\implies ma + mb > nb + mb \text{ (Theorem 4.4)} \\ &\implies m\,(a + b) > (n + m)\, b \text{ (Propositions 1 and 2)} \\ &\implies m\,(a' + b') > (n + m)\, b' \text{ (Definition 9.2)} \\ &\implies ma' + mb' > nb' + mb' \text{ (Propositions 1 and 2)} \\ &\implies ma' > nb'. \text{ (Theorem 4.4)} \end{aligned}$$

And the same argument applies with $>$ replaced by $=$ or $<$. Therefore $a : b = a' : b'$ according to Definition 9.2. ■

Proposition 18 *If $a : b = a' : b'$, then $(a + b) : b = (a' + b') : b'$.*

Proof Assume $a : b = a' : b'$ and consider the integral multiples $m\,(a + b)$ and nb.

Now $a+b > b$ by Definition 2.2 and hence $m\,(a + b) > mb$ by Proposition 5.

If $m = n$ then $m\,(a + b) > nb$. And if $m > n$, then $mb > nb$ by Proposition 6 and from $m\,(a + b) > mb$ and $mb > nb$ follows $m\,(a + b) > nb$ by Theorem 4.6. In summary, if $m \geq n$, then $m\,(a + b) > nb$ and also by the same argument $m\,(a' + b') > nb'$.

Now suppose $m < n$. In this case, $mb < nb$ by Proposition 6. Thus

$$\begin{aligned} m\,(a + b) < nb &\Longrightarrow ma + mb < nb \text{ (Proposition 1)} \\ &\Longrightarrow ma < nb - mb \text{ (Theorem 4.9)} \\ &\Longrightarrow ma < (n - m)\,b \text{ (Proposition 6)} \\ &\Longrightarrow ma' < (n - m)\,b' \text{ (Definition 9.2)} \\ &\Longrightarrow ma' < nb' - mb' \text{ (Proposition 6)} \\ &\Longrightarrow ma' + mb' < nb' \text{ (Theorem 4.9)} \\ &\Longrightarrow m\,(a' + b') < nb'. \text{ (Proposition 1)} \end{aligned}$$

And the same argument as above applies with $<$ replaced with $=$ or $>$.

We have now shown that for all m and n, $m\,(a + b)$ has to nb the same relation ($<$, $=$, or $>$) as $m\,(a' + b')$ has to nb' and hence $(a + b) : b = (a' + b') : b'$ according to Definition 9.2. ■

Proposition 19 *If $(a + b) : (c + d) = a : c$, then $b : d = a : c$.*

Proof

$$\begin{aligned} (a + b) : (c + d) = a : c &\Longrightarrow (a + b) : a = (c + d) : c \text{ (Proposition 16)} \\ &\Longrightarrow b : a = d : c \text{ (Proposition 17)} \\ &\Longrightarrow b : d = a : c. \text{ (Proposition 16)} \end{aligned}$$

■

Proposition 20 *If $a : b = a' : b'$ and $b : c = b' : c'$, then a has to c the same relation ($<$, $=$, or $>$) as a' has to c'.*

Proof Assume $a : b = a' : b'$ and $b : c = b' : c'$. Then $c : b = c' : b'$ by Definition 9.2 and hence

$$\begin{aligned} a > c &\implies a : b > c : b \text{ (Proposition 8)} \\ &\implies a' : b' > c : b \text{ (Proposition 13)} \\ &\implies a' : b' > c' : b' \text{ (Proposition 13)} \\ &\implies a' > c'. \text{ (Proposition 10)} \end{aligned}$$

And similar arguments apply with $>$ replaced by $=$ or $<$. ■

Proposition 21 *If $a : b = b' : c'$ and $b : c = a' : b'$, then a has to c the same relation ($<$, $=$, or $>$) as a' has to c'.*

Proof Assume $a : b = b' : c'$ and $b : c = a' : b'$. Then $c : b = b' : a'$ by Definition 9.2 and hence

$$\begin{aligned} a > c &\implies a : b > c : b \text{ (Proposition 8)} \\ &\implies b' : c' > c : b \text{ (Proposition 13)} \\ &\implies b' : c' > b' : a' \text{ (Proposition 13)} \\ &\implies a' > c'. \text{ (Proposition 10)} \end{aligned}$$

And similar arguments apply with $>$ replaced by $=$ or $<$. ■

Proposition 22 *If $a : b = a' : b'$ and $b : c = b' : c'$, then $a : c = a' : c'$.*

Proof Assume $a : b = a' : b'$ and $b : c = b' : c'$ and let m and n be any two natural numbers. Then

$$(ma) : (mb) = (ma') : (mb') \text{ and } (mb) : (nc) = (mb') : (nc')$$

by Proposition 4. And ma has to nc the same relation ($<$, $=$, or $>$) as ma' has to nc' by Proposition 20. Therefore $a : c = a' : c'$ according to Definition 9.2. ■

Proposition 23 *If $a : b = b' : c'$ and $b : c = a' : b'$, then $a : c = a' : c'$.*

Proof Assume $a : b = b' : c'$ and $b : c = a' : b'$ and let m and n be any two natural numbers. Then

$$(ma) : (nb) = (mb') : (nc') \text{ and } (mb) : (nc) = (ma') : (nb') .$$

by Proposition 4. And ma has to nc the same relation ($<$, $=$, or $>$)as ma' has to nc' by Proposition 21. Therefore $a : c = a' : c'$ according to Definition 9.2. ■

Proposition 24 *If $a : b = c : d$ and $e : b = f : d$, then $(a + e) : b = (c + f) : d$.*

Proof Assume $a : b = c : d$ and $e : b = f : d$ or, equivalently,

$$a : b = c : d \text{ and } b : e = d : f.$$

Then $a : e = c : f$ by Proposition 22 and $(a + e) : e = (c + f) : f$ by Proposition 18. And $e : b = f : d$ by assumption. Therefore $(a + e) : b = (c + f) : d$ by Proposition 22. ■

11 Embeddings and the Fourth Proportional

In this section a, b, c are elements of a magnitude space M and a', b', c' are elements of a magnitude space M'.

Definition 11.1 *If $a : b = a' : b'$, then we say that b' is a **fourth proportional** to a, b, and a'.*

Theorem 11.1 *If there is a fourth proportional to a, b, and a', then it is unique.*

Proof Suppose b' and c' are each fourth proportionals to a, b, and a'. Then $a : b = a' : b'$ and $a : b = a' : c'$ by Definition 11.1. Hence $a' : b' = a' : c'$ by Proposition 11 and $b' = c'$ by Proposition 9. ■

Theorem 11.2 *If $\varphi : M \to M'$ is an embedding which maps a into a', then φb is the fourth proportional to a, b, a' for each b.*

Proof Theorem 9.4 and Definition 11.1. ■

Theorem 11.3 *If a and a' are fixed and there is, for each b, a fourth proportional to a, b, a', then there is an embedding $\varphi : M \to M'$ which maps a into a'.*

Proof Let a and a' be fixed and suppose that there is, for each b, a fourth proportional to a, b, a'. Then for each b there is exactly one fourth proportional to a, b, a' by Theorem 11.1. Thus we can define a function $\varphi : M \to M'$ such that φb is the fourth proportional to a, b, and a'.

I say φ is an embedding. For any two elements b and c of M, we have by assumption

$$a : b = a' : \varphi b, \quad a : c = a' : \varphi c, \quad \text{and } a : b + c = a' : \varphi\,(b + c)$$

or equivalently by Definition 9.2

$$\varphi b : a' = b : a, \quad \varphi c : a' = c : a, \quad \text{and } \varphi\,(b + c) : a' = b + c : a.$$

Thus

$$\varphi b + \varphi c : a' = b + c : a$$

by Proposition 24,

$$\varphi\,(b + c) : a' = \varphi b + \varphi c : a'$$

by Proposition 11, and

$$\varphi\,(b + c) = \varphi b + \varphi c$$

by Proposition 9. We have now shown that φ is homomorphism according to Definition 5.1 and hence φ is an embedding by Theorem 5.2. And φa is the fourth proportional to a, a, and a', or in other words $a : a = a' : \varphi a$. And since $1a = 1a$, it follows that $1a' = 1\,(\varphi a)$ by Definition 9.2. Therefore $a' = \varphi a$ by Definition 8.2. ■

In fact the embedding constructed in the preceding theorem is the unique embedding of M into M' which maps a into a'. This and a bit more is established in the next theorem.

Theorem 11.4 *If φ and χ are embeddings from M into M', then φa has to χa the same relation ($<$, $=$, or $>$) as φb has to χb. (In particular if $\varphi a = \chi a$ for one a, then $\varphi b = \chi b$ for all b and $\varphi = \chi$.)*

Proof Assume φ and χ are embeddings from M into M'. Then

$$\varphi a : \varphi b = a : b \text{ and } \chi a : \chi b = a : b$$

by Theorem 9.4 and hence

$$\varphi a : \varphi b = \chi a : \chi b$$

by Proposition 11. Therefore φa has to χa the same relation ($<$, $=$, or $>$) as φb has to χb by Proposition 14. ■

Theorem 11.5 *If φ and χ are endomorphisms of M, then $\varphi\chi = \chi\varphi$.*

Proof Assume φ and χ are endomorphisms of M. Then for any $a \in M$,

$$\varphi(\chi a) : \varphi a = \chi a : a \text{ and } \varphi a : a = \chi(\varphi a) : \chi a$$

by Theorem 9.4,

$$\varphi(\chi a) : a = \chi(\varphi a) : a$$

by Proposition 23, and

$$\varphi(\chi a) = \chi(\varphi a)$$

by Proposition 9. Therefore $(\varphi\chi)\, a = (\chi\varphi)\, a$ by Definition 3.5 and $\varphi\chi = \chi\varphi$ by Definition 3.2. ■

12 Continuous Magnitude Spaces

In this section we come to the central theorem: If a is an element of an Archimedean magnitude space M and a' is an element of a continuous magnitude space M', then there is a unique embedding of M into M' which maps a into a'. Before tackling the main theorem we need to establish some results concerning ratios which, although very basic, have not been required prior to this section.

Theorem 12.1 *If $a : b > a' : b'$ and $ma \leq nb$, then $ma' < nb'$.*

Proof $ma : mb = a : b$ by Proposition 15 and $a : b > a' : b'$ by assumption. Hence $ma : mb > a' : b'$ by Proposition 13. Likewise, $a' : b' = ma' : mb'$

by Proposition 15 and hence $ma : mb > ma' : mb'$ by Proposition 13. Thus there exist j and k such that

$$j\,(ma) > k\,(mb) \text{ and } j\,(ma') \leq k\,(mb')$$

by Definition 9.2. And by assumption, $ma \leq nb$ and so $j\,(ma) \leq j\,(nb)$ by Proposition 5. And also, from above, $j\,(ma) > k\,(mb)$ and hence $k\,(mb) < j\,(nb)$ by Theorem 6.1 and hence $km < jn$ by Propositions 3 and 6. Hence $k\,(mb') < j\,(nb')$ by Propositions 6 and 3. And also, from above, $j\,(ma') \leq k\,(mb')$ and hence $j\,(ma') < j\,(nb')$ by Theorem 6.1 and hence $ma' < nb'$ by Proposition 5; the very thing to be shown. ■

Theorem 12.2 *If $a : b > a' : b'$ and $a' : b' > a'' : b''$, then $a : b > a'' : b''$.*

Proof Assume $a : b > a' : b'$ and $a' : b' > a'' : b''$. From the first ratio inequality there are natural numbers m and n such that $ma > nb$ and $ma' \leq nb'$ by Definition 9.2. And also $ma'' < nb''$ by the preceding theorem. Hence $ma > nb$ and $ma'' < nb''$ and therefore $a : b > a'' : b''$ according to Definition 9.2. ■

Theorem 12.3 *If there are natural numbers j, k such that $ja > kb$ and $ja' = kb'$, then there are natural numbers m, n such that $ma > nb$ and $ma' < nb'$.*

Proof Assume $ja > kb$ and $ja' = kb'$. Since magnitude spaces are assumed to be Archimedean, there is some natural number p such that $p\,(ja - kb) > 1a$ by Theorem 9.2. Therefore

$$\begin{aligned}
p\,(ja - kb) > a &\implies p\,(ja) - p\,(kb) > 1a \text{ (Proposition 5)} \\
&\implies (pj)\,a - (pk)\,b > 1a \text{ (Proposition 3)} \\
&\implies (pj)\,a > (pk)\,b + 1a \text{ (Theorem 4.9)} \\
&\implies (pj)\,a - 1a > (pk)\,b \text{ (Theorem 4.9)} \\
&\implies (pj - 1)\,a > (pk)\,b. \text{ (Theorem 8.1)}
\end{aligned}$$

And since $ja' = kb'$,

$$\begin{aligned}
(pj - 1)\,a' &< (pj)\,a' \text{ (Theorem 4.5 and Proposition 6)} \\
&= p\,(ja') \text{ (Proposition 3)} \\
&= p\,(kb') \text{ } (ja' = kb') \\
&= (pk)\,b' \text{ ((Proposition 3)}
\end{aligned}$$

and hence $(pj-1)\,a' < (pk)\,b'$. We now have two natural numbers $m = pj-1$ and $n = pk$ such that $ma > nb$ and $ma' < nb'$. $\blacksquare$

Theorem 12.4 *If $a : b \neq a' : b'$, then $a : b > a' : b'$ or $a' : b' > a : b$.*

Proof If $a : b \neq a' : b'$, then there are two natural numbers j, k for which at least one of following six cases is true:

$$\begin{aligned}
&1)\ ja < kb \text{ and } ja' = kb'\\
&2)\ ja < kb \text{ and } ja' > kb'\\
&3)\ ja = kb \text{ and } ja' < kb'\\
&4)\ ja = kb \text{ and } ja' > kb'\\
&5)\ ja > kb \text{ and } ja' < kb'\\
&6)\ ja > kb \text{ and } ja' = kb'
\end{aligned}$$

In each of the cases 2, 4, 5, and 6, $a : b > a' : b'$ or $a' : b' > a : b$ according to Definition 9.2. In case 1, there are two natural numbers m, n such that $na < mb$ and $na' > mb'$ by the preceding theorem and hence $a' : b' > a : b$. In a similar fashion we can show that in case 3 $a : b > a' : b'$. $\blacksquare$

Theorem 12.5 *If M is nondiscrete, then for any $a \in M$ and $n \in \mathbb{N}$ there is some $b \in M$ such that $nb < a$.*

Proof Assume M is nondiscrete. Then the theorem is true for $n = 1$ since there is some element $b < a$ by Definition 6.3 and $1b = b$ by Definition 8.2. Now suppose the theorem is true for n and $nb < a$. There is some $c \in M$ such that $c < b$ by Definition 6.3. Let d be the smaller of $(b-c)$ and c so that $d \leq b-c$ and $d \leq c$. There is some $e \in M$ such that $e < d$ by Definition 6.3 and hence $e < b-c$ and $e < c$ by Theorem 6.1. Hence

$$\begin{aligned}
e + e &< (b-c) + e \ (e < b-c \text{ and Theorem 4.4})\\
&< (b-c) + c \ (e < c \text{ and Theorem 4.4})\\
&= b. \ (\text{Definition 4.5})
\end{aligned}$$

And hence $e+e<b$ by Theorem 4.6. Now $1 \leq n$ by Definition 8.1 and hence $n+1 \leq n+n$ by Theorem 4.4. Therefore

$$\begin{aligned}
(n+1)\,e &\leq (n+n)\,e \text{ (Proposition 6)} \\
&= ne+ne \text{ (Proposition 2)} \\
&= n\,(e+e) \text{ (Proposition 1)} \\
&< nb \text{ (Proposition 5)} \\
&< a \text{ (assumption)}
\end{aligned}$$

and so $(n+1)\,e<a$ by Theorem 6.1 and hence the theorem is true for $n+1$. Therefore the theorem is true for all n by Theorem 7.1. ■

Theorem 12.6 *If M and M' are magnitude spaces, M is nondiscrete, $a,b \in M$, $a',b' \in M'$, then if $a:b>a':b'$ there exists some $c \in M$ such that $a:b>c:b>a':b'$, and if $a:b<a':b'$ there exists some $c \in M$ such that $a:b<c:b<a':b'$.*

Proof If $a:b>a':b'$, then there are $m,n \in \mathbb{N}$ such that

$$ma>nb \text{ and } ma' \leq nb'$$

by Definition 9.2. From the preceding theorem, there is some $d \in M$ such that

$$ma-nb>md.$$

And

$$\begin{aligned}
ma-nb>md &\implies ma>md+nb \text{ (Theorem 4.9)} \\
&\implies ma-md>nb \text{ (Theorem 4.9)} \\
&\implies m\,(a-d)>nb. \text{ (Proposition 5)}
\end{aligned}$$

Letting $c=a-d$ we have

$$mc>nb \text{ and } ma' \leq nb'$$

and hence $c:b>a':b'$. And since $a>c$, from Proposition 8, $a:b>c:b$. The second part of the theorem is proved in a similar manner. ■

With these preliminaries out of the way, we are ready for the main theorem in this section.

Theorem 12.7 *If M is a magnitude space, M' is a continuous magnitude space, $a \in M$, and $a' \in M'$, then for each $b \in M$ there is a fourth proportional $b' \in M'$ to a, b, and a'.*

Proof Assume M' is continuous and hence complete and nondiscrete by Definition 6.6. Let $a, b \in M$ and $a' \in M'$ be given. Let

$$A = \{c' \in M' \mid c' : a' < b : a\} \text{ and } B = \{c' \in M' \mid c' : a' > b : a\}.$$

We first show that each of these sets is nonempty. Since M is Archimedean, there is some natural number m such that $mb > 1a$ by Theorem 9.2. And since M' is nondiscrete, there is some $c' \in M'$ such that $mc' < 1a'$ from Theorem 12.5. Then $mb > 1a$ and $mc' < 1a'$; hence $b : a > c' : a'$ by Definition 9.2 and $c' \in A$. Likewise, there is some natural number n such that $1b < na$ by Theorem 9.2. Let $c' = na' + a'$. Then $1c' > na'$ and $1b < na$; hence $c' : a' > b : a$ by Definition 9.2 and $c' \in B$.

If $c' \in A$ and $d' \in B$, then $c' : a' < b : a$ and $b : a < d' : a'$. Thus $c' : a' < d' : a'$ from Theorem 12.2 and hence $c' < d'$ from Proposition 10. Thus every element of B is an upper bound of A and every element of A is a lower bound of B according to Definition 6.2.

Since M' is complete, A has a least upper bound b' by Definition 6.5. I say b' is a fourth proportional to a, b, and a'. That is, $a : b = a' : b'$ or equivalently $b' : a' = b : a$. Suppose that b' is not a fourth proportional to a, b, and a'. Then $b' : a' < b : a$ or $b' : a' > b : a$ from Theorem 12.4, or what is the same $b' \in A$ or $b' \in B$.

If $b' \in A$ (i.e. $b' : a' < b : a$), then there is a $c' \in M'$ such that $b' : a' < c' : a' < b : a$ from Theorem 12.6 and hence $c' \in A$ and $b' < c'$. But b' is an upper bound for A which is a contradiction. Therefore $b' \notin A$.

If $b' \in B$ (i.e. $b' : a' > b : a$), then from Theorem 12.6 there is a $c' \in M'$ such that $b' : a' > c' : a' > b : a$ and hence $c' \in B$ and $b' > c'$. But since $c' \in B$, c' is an upper bound of A and it is smaller than the least upper bound of A which is a contradiction. Therefore $b' \notin B$.

We have now shown that b' is not an element of A or B which contradicts b' not being the fourth proportional to a, b, and a'. Therefore, indeed, b' is the fourth proportional to a, b, and a'. ■

Theorem 12.8 *If M is a magnitude space, M' is a continuous magnitude space, $a \in M$, and $a' \in M'$, then there exists a unique embedding $\varphi : M \to M'$ such that $\varphi a = a'$.*

Proof Preceding theorem, Theorem 11.3, and Theorem 11.4. ■

Theorem 12.9 *If M and M' are continuous magnitude spaces, $a \in M$ and $a' \in M'$, then there exists a unique isomorphism $\varphi : M \to M'$ such that $\varphi a = a'$.*

Proof Since M' is continuous, there exists a unique embedding $\varphi : M \to M'$ such that $\varphi a = a'$ by the preceding theorem. And since M is continuous, there is also a unique embedding $\chi : M' \to M$ such that $\chi a' = a$ by the preceding theorem. But then $\varphi \circ \chi$ is an embedding of M' into M' by Theorem 5.5. And $\varphi \circ \chi$ maps a' into a'. But the identity map $i_{M'}$ on M' is an embedding by Theorem 5.3 and also $i_{M'}$ maps a' into a'. Hence $\varphi \circ \chi = i_{M'}$ by Theorem 11.4. Hence φ is onto by Theorem 3.3 and therefore is an isomorphism according to Definition 5.2. ■

Theorem 12.10 *Any two continuous magnitude spaces are isomorphic.*

Proof Let M and M' be any two continuous magnitude spaces. Pick any $a \in M$ and $a' \in M'$. There is an embedding $\varphi : M \to M'$ such that $\varphi a = a'$ by Theorem 12.8. The embedding φ is an isomorphism by the preceding theorem and therefore M is isomorphic to M' according to Definition 5.2. ■

13 Real Numbers

Let us now review the definition of the natural numbers and define the positive real numbers.

1. Well ordered magnitude spaces are complete (Theorem 6.5) and hence Archimedean (Theorem 6.6).

2. If M is a well ordered magnitude space, M' is any magnitude space (not necessarily Archimedean), a is the smallest element in M, and a' is any element in M', then there exists a unique embedding $\varphi : M \to M'$ such that $\varphi a = a'$ (Theorem 7.3).

3. Any two well ordered magnitude spaces are isomorphic (Theorem 7.4).

Because a well ordered magnitude space can be embedded into any magnitude space, we say that well ordered magnitude spaces are **minimal** magnitude spaces. And for the same reason we also say that the well ordered magnitude spaces are minimal Archimedean magnitude spaces. We defined the natural numbers as an arbitrary representative of the well ordered magnitude spaces.

1. Continuous magnitude spaces are complete (Definition 6.6) and hence Archimedean (Theorem 6.6).

2. If M is an Archimedean magnitude space, M' is a continuous magnitude space, $a \in M$, and $a' \in M'$, then there exists a unique embedding $\varphi : M \to M'$ such that $\varphi a = a'$ (Theorem 12.8).

3. Any two continuous magnitude spaces are isomorphic (Theorem 12.10).

Because any Archimedean magnitude space can be embedded into a continuous magnitude space, we say that continuous magnitudes are **maximal** Archimedean magnitude spaces. We now define the positive real numbers as an arbitrary representative of the continuous magnitude spaces.

Definition 13.1 *Pick any continuous magnitude space and denote it by $\mathbb{R}_+$ and pick any element of $\mathbb{R}_+$ and denote it by 1. Since all continuous magnitude spaces are isomorphic (Theorem 12.10) and we will use only the algebraic properties that the continuous magnitude spaces have in common, it does not matter which continuous magnitude space is chosen to play the role of the "number system" $\mathbb{R}_+$. We call $\mathbb{R}_+$ the* ***positive real numbers****.*

When we defined the natural numbers, we immediately defined an integral multiple na where n is a natural number and a is an element of a magnitude space. The remainder of this work examines similar constructions of multiples where the multiplier is not necessarily a natural number. This will lead us to real multiples of real numbers which is the familiar binary product operator in $\mathbb{R}_+$.

14 Magnitude Embedding Spaces

Definitions of multiples and products are based on embeddings of one magnitude space into another. It is useful at the onset to consider the general case.

Definition 14.1 *$H(M, M')$ is the set of all embeddings from M into M'.*

It may happen that $H(M, M')$ is the empty set. For example, if M is nondiscrete and M' is discrete, there are no embeddings of M into M'. If $H(M, M')$ is not empty and $\varphi, \chi \in H(M, M')$, then we have already shown (Theorem 5.4) that $\varphi + \chi \in H(M, M')$.

Theorem 14.1 *If $H(M, M')$ is nonempty, then $H(M, M')$ is a magnitude space.*

Proof Let $\varphi, \chi, \psi \in H(M, M')$.

(i)For any $a \in M$

$$\begin{aligned}(\varphi + (\chi + \psi))\, a &= \varphi a + (\chi + \psi)\, a \text{ (Definition 5.4)}\\ &= \varphi a + (\chi a + \psi a) \text{ (Definition 5.4)}\\ &= (\varphi a + \chi a) + \psi a \text{ (Definition 2.1)}\\ &= (\varphi + \chi)\, a + \psi a \text{ (Definition 5.4)}\\ &= ((\varphi + \chi) + \psi)\, a \text{ (Definition 5.4)}\end{aligned}$$

and hence $\varphi + (\chi + \psi) = (\varphi + \chi) + \psi$ according to Definition 3.2.

(ii)For any $a \in M$

$$\begin{aligned}(\varphi + \chi)\, a &= \varphi a + \chi a \text{ (Definition 5.4)}\\ &= \chi a + \varphi a \text{ (Definition 2.1)}\\ &= (\chi + \varphi)\, a \text{ (Definition 5.4)}\end{aligned}$$

and hence $\varphi + \chi = \chi + \varphi$ according to Definition 3.2.

(iii)Fix $a \in M$. Since $\varphi a, \chi a \in M'$, exactly one of the following is true:

$$\begin{aligned} &\varphi a = \chi a, \\ &\varphi a > \chi a, \text{ or} \\ &\chi a > \varphi a \end{aligned}$$

by Theorem 4.2. In the first case, Theorem 11.4 shows that $\varphi b = \chi b$ for all $b \in M$ and hence $\varphi = \chi$ according to Definition 3.2. In the second case, $\varphi b > \chi b$ for all $b \in M$ by Theorem 11.4. We can then define a function $d : M \to M'$ by

$$\delta b = \varphi b - \chi b.$$

For any $b, c \in M$

$$\begin{aligned} \varphi(b+c) &= \varphi b + \varphi c \text{ (Definition 5.1)} \\ &= ((\varphi b - \chi b) + \chi b) + ((\varphi c - \chi c) + \chi c) \text{ (Definition 4.5)} \\ &= ((\varphi b - \chi b) + (\varphi c - \chi c)) + (\chi b + \chi c) \text{ (associativity and commutativity of } + \text{)} \\ &= (\delta b + \delta c) + (\chi b + \chi c) \text{ (definition of } \delta\text{)} \\ &= (\delta b + \delta c) + \chi(b+c) \text{ (Definition 5.1)} \end{aligned}$$

and hence $\varphi(b+c) - \chi(b+c) = (\delta b + \delta c)$ by Definition 4.5. But $\delta(b+c) = \varphi(b+c) - \chi(b+c)$ by the definition of δ. Hence $\delta(b+c) = \delta b + \delta c$ and δ is an element of $H(M, M')$ by Definition 5.1 and Theorem 5.2. And, of course,

$$\begin{aligned} \varphi(b) &= \chi b + (\varphi b - \chi b) \text{ (Definition 4.5)} \\ &= \chi b + \delta b \text{ (definition of } \delta\text{)} \\ &= (\chi + \delta) b \text{ (Definition 5.4)} \end{aligned}$$

for all $b \in M$ and hence $\varphi = \chi + \delta$ according to Definition 3.2. A similar argument applies in the third case where $\chi a > \varphi a$. So, in summary, if $\varphi, \chi \in H(M, M')$, then exactly one of the following is true:

$$\begin{aligned} &\varphi = \chi, \\ &\varphi = \chi + \delta \text{ for some } \delta \in H(M, M'), \text{ or} \\ &\chi = \varphi + \delta \text{ for some } \delta \in H(M, M'). \end{aligned}$$

We have now shown that $H(M, M')$, with the usual addition of embeddings, is a magnitude space according to Definition 2.1. ■

15 Magnitude Endomorphism Spaces

A special case of magnitude space embeddings are endomorphisms; that is embeddings of a magnitude space into itself.

Definition 15.1 *If M is a magnitude space, then we denote the set of endomorphisms of M by $E(M)$.*

Of course $E(M) = H(M, M)$. Now in general $H(M, M')$ can be empty. But $E(M)$ always has elements. In fact for each natural number n the mapping $\varphi : M \to M$ defined by $\varphi a = na$ is an embedding of M into M and hence is an element of $E(M)$.

If $\varphi, \chi \in E(M)$ then $\varphi + \chi$ and $\varphi \circ \chi$ are each elements of $E(M)$ by Theorems 5.4 and 5.5. We have already shown that $E(M) = H(M, M)$ is a magnitude space with the $+$ operator on elements of $E(M)$ (Theorem 14.1). Also we know that the composition operator $\circ$ is commutative (Theorem 11.5) and associative (Theorem 3.1).

Theorem 15.1 *If $\varphi, \chi, \psi \in E(M)$ then $\varphi \circ (\chi + \psi) = \varphi \circ \chi + \varphi \circ \psi$ and $(\varphi + \chi) \circ \psi = \varphi \circ \psi + \chi \circ \psi$.*

Proof If $a \in M$ then

$$\begin{aligned}
(\varphi \circ (\chi + \psi))\, a &= \varphi\left((\chi + \psi)\, a\right) \text{ (Definition 3.5)}\\
&= \varphi\left(\chi a + \psi a\right) \text{ (Definition 5.4)}\\
&= \varphi\left(\chi a\right) + \varphi\left(\psi a\right) \text{ (Definition 5.1)}\\
&= (\varphi \circ \chi)\, a + (\varphi \circ \psi)\, a \text{ (Definition 3.5)}\\
&= (\varphi \circ \chi + \varphi \circ \psi)\, a \text{ (Definition 5.4)}
\end{aligned}$$

and hence $\varphi \circ (\chi + \psi) = \varphi \circ \chi + \varphi \circ \psi$ by Definition 3.2. The second part of the theorem can be proved in a similar manner. ■

Theorem 15.2 *If $\varphi, \chi, \psi \in E(M)$ then $\varphi \circ \chi$ has to $\varphi \circ \psi$ the same relation ($<$, $=$, or $>$) as χ has to ψ and $\chi \circ \varphi$ has to $\psi \circ \varphi$ the same relation ($<$, $=$, or $>$) as χ has to ψ.*

Proof Fix $\varphi \in E(M)$ and let the mapping $\Psi : E(M) \to E(M)$ be defined by $\Psi\chi = \varphi \circ \chi$. If $\chi, \psi \in E(M)$, then

$$\begin{aligned}\Psi(\chi + \psi) &= \varphi \circ (\chi + \psi) \ \text{(definition of } \Psi\text{)} \\ &= \varphi \circ \chi + \varphi \circ \psi \ \text{(Theorem 15.1)} \\ &= \Psi\chi + \Psi\psi \ \text{(definition of } \Psi\text{)}\end{aligned}$$

and hence Ψ is a homomorphism by Definition 5.1. Therefore $\varphi \circ \chi = \Psi\chi$ has to $\varphi \circ \psi = \Psi\psi$ the same relation ($<$, $=$, or $>$) as χ has to ψ (and Ψ is an embedding) by Theorem 5.2. The second part of the theorem can be proved in a similar manner. ■

Theorem 15.3 *If $\varphi \in E(M)$, then $\varphi \circ i_M = i_M \circ \varphi = \varphi$.*

Proof If $a \in M$, then

$$\begin{aligned}(\varphi \circ i_M)\, a &= \varphi(i_M a) \ \text{(Definition 3.5)} \\ &= \varphi a \ \text{(Definition 3.6)}\end{aligned}$$

and therefore $\varphi \circ i_M = \varphi$ by Definition 3.2. And $\varphi \circ i_M = i_M \circ \varphi$ since endomorphisms commute by Theorem 11.5. ■

At this point we would like to point out that the magnitude space $E(M)$ for an arbitrary (Archimedean) magnitude space has exactly those properties that we associate with addition and multiplication of positive numbers.

1. $E(M)$ with the addition operator is a magnitude space (Theorem 14.1).
   (a) $\varphi + (\chi + \psi) = (\varphi + \chi) + \psi$
   (b) $\varphi + \chi = \chi + \varphi$
   (c) Exactly one of the following is true: $\varphi = \chi$, or $\varphi = \chi + \delta$ for some $\delta \in E(M)$, or $\chi = \varphi + \delta$ for some $\delta \in E(M)$.

2. The composition operator is associative and commutative (Theorems 3.1 and 11.5).
   (a) $\psi \circ (\chi \circ \varphi) = (\psi \circ \chi) \circ \varphi$
   (b) $\varphi \circ \chi = \chi \circ \varphi$

3. Composition distributes over addition (Theorem 15.1).

   (a) $\varphi \circ (\chi + \psi) = \varphi \circ \chi + \varphi \circ \psi$

   (b) $(\varphi + \chi) \circ \psi = \varphi \circ \psi + \chi \circ \psi$

4. There is an identity element for the composition operator (Theorem 15.3).

   (a) $\varphi \circ i_M = \varphi$

   (b) $i_M \circ \varphi = \varphi$

5. Composition on left or right preserves order relations in the magnitude space $E(M)$ (Theorem 15.2)

   (a) $\varphi \circ \chi$ has to $\varphi \circ \psi$ the same relation ($<$, $=$, or $>$) as χ has to ψ

   (b) $\chi \circ \varphi$ has to $\psi \circ \varphi$ the same relation ($<$, $=$, or $>$) as χ has to ψ

16 Generalized Multiple

In Section 14 we showed $H(M, M')$ is itself a magnitude space. The question naturally arises of how $H(M, M')$ might be related to the magnitude spaces M and M'. In this section we consider the special case in which M is a magnitude space with some distinguished element 1, and M' is any magnitude space such that for each element $a' \in M'$ there is an embedding of M into M' which maps 1 into a'. We already have two examples of this special case:

1. M is a well ordered magnitude space and 1 is the smallest element in M; and M' is an arbitrary (Archimedean) magnitude space (Theorem 7.3).

2. M is an arbitrary (Archimedean) magnitude space and 1 is any element in M; and M' is a continuous magnitude space (Theorem 12.8).

In this section variables a, b are elements of M and a', b' are elements of M'.

Definition 16.1 *Let the mapping $\Psi : M' \to H(M, M')$ be defined such that for each $a' \in M'$, $\Psi a'$ is the unique element of $H(M, M')$ which maps 1 into a'. For any $a \in M$, $a' \in M'$ we define aa' to be $(\Psi a')\, a$. Note that $aa' \in M'$.*

Theorem 16.1 *The map $\Psi : M' \to H(M, M')$ is an isomorphism.*

Proof Note that $H(M, M')$ is a magnitude space by Theorem 14.1. Let a' and b' be any two elements of M'. By assumption there exist elements $\Psi a', \Psi b', \Psi(a' + b') \in H(M, M')$ which map 1 into $a', b', a' + b'$ respectively. Now $\Psi a' + \Psi b' \in H(M, M')$ by Theorem 5.4 and

$$\begin{aligned}(\Psi a' + \Psi b') 1 &= (\Psi a') 1 + (\Psi b') 1 \text{ (Definition 5.4)} \\ &= a' + b'. \text{ (definition of } \Psi a' \text{ and } \Psi b')\end{aligned}$$

Thus $\Psi(a' + b')$ and $\Psi a' + \Psi b'$ are each embeddings which map 1 into $a' + b'$ and hence $\Psi(a' + b') = \Psi a' + \Psi b'$ by Theorem 11.4. Therefore Ψ is an embedding by Definition 5.1 and Theorem 5.2.

Now if $\chi \in H(M, M')$, then χ maps 1 into $\chi 1$. And also $\Psi(\chi 1) \in H(M, M')$ maps 1 into $\chi 1$. Therefore $\chi = \Psi(\chi 1)$ by Theorem 11.4. Hence Ψ is onto and therefore Ψ is an isomorphism from M' onto $H(M, M')$ according to Definition 5.2. ■

Theorem 16.2 $1a' = a'$

Proof $1a' = (\Psi a') 1 = a'$ by Definition 16.1. ■

The following two theorems were proved separately for integral multiples in Propositions 1 and 2.

Theorem 16.3 $a(a' + b') = aa' + ab'$

Proof

$$\begin{aligned}a(a' + b') &= (\Psi(a' + b')) a \text{ (Definition 16.1)} \\ &= (\Psi a' + \Psi b') a \text{ (Definition 5.1)} \\ &= (\Psi a') a + (\Psi b') a \text{ (Definition 5.4)} \\ &= aa' + ab' \text{ (Definition 16.1)}\end{aligned}$$

■

Theorem 16.4 $(a + b) a' = aa' + ba'$

Proof

$$\begin{aligned}(a+b)\,a' &= (\Psi a')\,(a+b) \ \text{(Definition 16.1)}\\ &= (\Psi a')\,a + (\Psi a')\,b \ \text{(Definition 5.1)}\\ &= aa' + ba' \ \text{(Definition 16.1)}\end{aligned}$$

■

The following two theorems were proved separately for integral multiples in Propositions 5 and 6.

Theorem 16.5 *If $a' > b'$, then $aa' > ab'$ and $aa' - ab' = a\,(a' - b')$. And, more generally, aa' has to ab' the same relation ($<$, $=$, or $>$) as a' has to b'.*

Proof Fix a and let $\varphi : M' \to M'$ be the function defined by $\varphi a' = aa'$. Then $\varphi\,(a' + b') = \varphi a' + \varphi b'$ for any a' and b' by Theorem 16.3 and hence φ is a homomorphism according to Definition 5.1. Hence Theorems 5.1 and 5.2 apply. ■

Theorem 16.6 *If $a > b$, then $aa' > ba'$ and $aa' - ba' = (a - b)a'$. And, more generally, aa' has to ba' the same relation ($<$, $=$, or $>$) as a has to a.*

Proof Fix a' and let $\varphi : M \to M$ be the function defined by $\varphi a = aa'$. Then $\varphi\,(a + b) = \varphi a + \varphi b$ for any a and b by Theorem 16.4 and hence φ is a homomorphism according to Definition 5.1. Hence Theorems 5.1 and 5.2 apply. ■

17 Generalized Product Operator

In this section we assume that M is a magnitude space with some distinguished element 1 such that for each element $a \in M$ there is an embedding $\Psi a \in E\,(M) = H\,(M, M)$ which maps 1 into a. In other words, we assume the same thing as in the previous section but, in addition, $M = M'$. In this case we will write a product $ab = (\Psi b)\,a$ as $a \cdot b$ to emphasize that we have here a binary operator. Here are two examples of this special case.

1. M is a well ordered magnitude space and 1 is the smallest element in M (for instance, $M = \mathbb{N}$).

2. M is a continuous magnitude space and 1 is an arbitrary element in M (for instance, $M = \mathbb{R}_+$).

From the preceding section $\Psi : M \to E(M)$ is an isomorphism and, since Ψ is an isomorphism,

$$\Psi(a+b) = \Psi a + \Psi b$$

where the addition operator on the right hand side represents the addition of the endomorphisms Ψa and Ψb and the result is another endomorphism.

Theorem 17.1 $\Psi 1 = i_M$

Proof $\Psi 1$ is an embedding of M into M which maps 1 into 1. And the identity function i_M is also an embedding of M into M which maps 1 into 1. Therefore $\Psi 1 = i_M$ by Theorem 11.4. ■

Theorem 17.2 $\Psi(a \cdot b) = \Psi a \circ \Psi b$

Proof First $(\Psi(a \cdot b))\,1 = a \cdot b$ from the definition of Ψ. Second

$$\begin{aligned}
(\Psi a \circ \Psi b)\,1 &= (\Psi b \circ \Psi a)\,1 \text{ (Theorem 11.5)} \\
&= \Psi b\,((\Psi a)\,1) \text{ (Definition 3.5)} \\
&= (\Psi b)\,a \text{ (definition of } \Psi) \\
&= a \cdot b \text{ (Definition 16.1)}
\end{aligned}$$

and hence also $(\Psi a \circ \Psi b)\,1 = a \cdot b$. Therefore $\Psi(a \cdot b) = \Psi a \circ \Psi b$ by Theorem 11.4. ■

Remark 17.1 *By means of the preceding theorem, we can show that the product binary operator $\cdot$ on M has analogous properties as the binary operator $\circ$ on $E(M)$ as summarized at the end of Section 15.*

Theorem 17.3 $a \cdot b = b \cdot a$

Proof

$$\begin{aligned}
\Psi(a \cdot b) &= \Psi a \circ \Psi b \text{ (preceding theorem)} \\
&= \Psi b \circ \Psi a \text{ (Theorem 11.5)} \\
&= \Psi(b \cdot a) \text{ (preceding theorem)}
\end{aligned}$$

Thus $\Psi(a \cdot b) = \Psi(b \cdot a)$ and therefore $a \cdot b = b \cdot a$ by Theorem 5.2. ■

Theorem 17.4 $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Proof

$$\begin{aligned}\Psi((a \cdot b) \cdot c) &= \Psi(a \cdot b) \circ \Psi c \text{ (Theorem 17.2)} \\ &= (\Psi a \circ \Psi b) \circ \Psi c \text{ (Theorem 17.2)} \\ &= \Psi a \circ (\Psi b \circ \Psi c) \ \text{ (Theorem 3.1)} \\ &= \Psi a \circ \Psi(b \cdot c) \ \text{ (Theorem 17.2)} \\ &= \Psi(a \cdot (b \cdot c)) \ \text{ (Theorem 17.2)}\end{aligned}$$

Thus $\Psi((a \cdot b) \cdot c) = \Psi(a \cdot (b \cdot c))$ and therefore $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ by Theorem 5.2. ■

18 Symmetric Magnitude Spaces

From above, we have the definitions of binary product operators for the natural numbers and the positive real numbers and we have shown the properties which these two binary operators have in common. There are, of course, differences between the product operators for the natural and positive real numbers. In particular, given any $x, y \in \mathbb{R}_+$ there is a $q \in \mathbb{R}_+$ (called the quotient of y and x) such that $y = q \cdot x$. Once again it is useful to develop this additional property of the real numbers from a general perspective; for there are also noncontinuous magnitude spaces in which every pair of elements has a quotient.

Definition 18.1 *A magnitude space M is **symmetric** if there is, for every pair $a, b \in M$, an endomorphism of M which maps a into b.*

Remark 18.1 *Alternatively, a magnitude space M is **symmetric** if, for every three elements $a, b, c \in M$, there is a fourth proportional (i.e. an element $d \in M$ such that $a : b = c : d$).*

Proof Theorem 12.8. ■

Theorem 18.1 *If M is a symmetric magnitude, then for each $\varphi \in E(M)$, there is a corresponding $\chi \in E(M)$ such that $\chi \circ \varphi = \varphi \circ \chi = i_M$ and each of φ and χ are onto (and thus automorphisms).*

Proof Let $\varphi \in E(M)$ and pick any element $a \in M$. Since M is a symmetric magnitude space, there is another embedding $\chi \in E(M)$ such that χ maps φa into a. Then $\chi \circ \varphi$ maps a into a. And also i_M maps a into a by Definition 3.6. Hence $\chi \circ \varphi = i_M$ by Theorem 11.4. But also $\chi \circ \varphi = \varphi \circ \chi$ by Theorem 11.5 and so $\varphi \circ \chi = i_M$. Therefore φ and χ are each onto by Theorem 3.3 and so φ and χ are automorphisms according to Definition 5.3. ■

In the remainder of this section M is a symmetric magnitude space with one element designated by 1. Then for any $a \in M$, there is a unique embedding of M into M which maps 1 into a. From the preceding section, we can define an isomorphism $\Psi : M \to E(M)$ such that for $a \in M$, Ψa is the unique element in $E(M)$ which maps 1 into a. And, as before, we can define a binary operator on M according to $a \cdot b = (\Psi b)\, a$. A property of this binary product which differs from the product of two natural numbers is given in the following theorem.

Theorem 18.2 *If $a, b \in M$, there exists a unique $d \in M$ such that $b = d \cdot a$.*

Proof The embedding Ψa from M into M is an automorphism by the preceding theorem and hence is one-to-one and onto by Definition 5.3. Hence there is a unique $d \in M$ such that $(\Psi a)\, d = b$ by Definitions 3.3 and 3.4. Therefore there is a unique $d \in M$ such that $d \cdot a = b$ by Definition 16.1. ■

Definition 18.2 *For $a, b \in M$, we denote by b/a the unique element d of M such that $b = d \cdot a$.*

Theorem 18.3 *b has to a the same relation ($<$, $=$, or $>$) as b/a has to 1.*

Proof $(b/a) \cdot a$ has to $1 \cdot a$ the same relation ($<$, $=$, or $>$) as b/a has to 1 by Theorem 16.6. But $b = (b/a) \cdot a$ by Definition 18.2 and $1 \cdot a = a$ by Theorem 16.2. Therefore b has to a the same relation ($<$, $=$, or $>$) as b/a has to 1. ■

19 Power Functions

We have mentioned above that it is worthwhile to view the formation of products in a general way. As a concrete example, let us consider the definition of x^y where x and y are positive real numbers and $x > 1$. In this section $\cdot$ is the product operator in $\mathbb{R}_+$.

Definition 19.1 *Let $\mathbb{R}_{>1}$ be the elements of $\mathbb{R}_+$ which are greater than 1.*

Theorem 19.1 *If $x, y \in \mathbb{R}_{>1}$, then $x \cdot y \in \mathbb{R}_{>1}$.*

Proof If $x, y \in \mathbb{R}_{>1}$, then $x > 1$ and $y > 1$ by Definition 19.1. And $x \cdot y > 1 \cdot y$ by Theorem 16.6 and $1 \cdot y = y$ by Theorem 16.2. Thus $x \cdot y > y$ and $y > 1$. Therefore $x \cdot y > 1$ by Theorem 4.6 and $x \cdot y \in \mathbb{R}_{>1}$ by Definition 19.1. ■

Theorem 19.2 $\mathbb{R}_{>1}$ *with the multiplicative operator $\cdot$ is a magnitude space.*

Proof The multiplicative operator $\cdot$ is a binary operator on $\mathbb{R}_{>1}$ by the preceding theorem. And $\cdot$ is associative and commutative by Theorems 17.4 and 17.3.

It remains to show that the binary operator $\cdot$ on $\mathbb{R}_{>1}$ is trichotomous. Let $x, y \in \mathbb{R}_{>1}$. Note that $y = (y/x) \cdot x = x \cdot (y/x)$ and $x = (x/y) \cdot y = y \cdot (x/y)$ by Definition 18.2 and Theorem 17.3. From trichotomy in $\mathbb{R}_+$, exactly one of the following is true: $y < x$, or $y = x$, or $y > x$.

Now in each case, there is a unique $d \in \mathbb{R}_+$, namely $d = x/y$, such that $x = y \cdot d$ by Theorem 18.2. But $d = x/y \in \mathbb{R}_{>1}$ only if $y < x$. Similarly, in each case, there is a unique $d \in \mathbb{R}_+$, namely $d = y/x$ such that $y = x \cdot d$. But $d = y/x \in \mathbb{R}_{>1}$ only if $y > x$. Thus there are three mutually exclusive cases: $x = y \cdot d$ for some $d \in \mathbb{R}_{>1}$, or $x = y$, or $y = x \cdot d$ for some $d \in \mathbb{R}_{>1}$. Thus $\mathbb{R}_{>1}$ with the $\cdot$ binary operator has the trichotomy property in Definition 2.1. ■

Theorem 19.3 $\mathbb{R}_{>1}$ *with the multiplicative operator $\cdot$ is a continuous magnitude space.*

Proof We begin with an elementary observation. If A is a nonempty set with some element a greater than 1 and b is an upper bound of A, then $b > 1$. For if b is an upper bound of A, then $a \leq b$ by Definition 6.2. And from $1 < a$ and $a \leq b$ follows $1 < b$ by Theorem 6.1.

Definition 2.2 defines $<$ and $>$ in a magnitude space in terms of the binary operator of the magnitude space. In the proof of the preceding theorem, it may be observed that for $x, y \in \mathbb{R}_{>1}$, x has to y the same relation ($<$, $=$, or $>$) in the order defined by $+$ binary operator as x has to y in the order defined defined by the $\cdot$ operator.

Now let A be a nonempty subset of $\mathbb{R}_{>1}$ with a nonempty set B of upper bounds with respect to the order defined by the $\cdot$ operator. From the preliminary observation, B coincides with the set of upper bounds of A with respect to the order defined by the $+$ operator. But $\mathbb{R}_+$ is continuous by Definition 13.1 and hence B has a smallest element with respect to the order defined by the + operator by Definition 6.6. Therefore B has a smallest element with respect to the order defined by the $\cdot$ operator. Therefore $\mathbb{R}_{>1}$ is continuous by Definition 6.6. ■

Up to the preceding theorem, we have consistently used the symbol + for the binary operator in a magnitude space. To be precise, if we have two magnitude spaces M and M', the binary operators are in general not the same. If $a, b \in M$ and $a', b' \in M'$ then the + sign in the expression $a + b$ is understood to be the binary operator in M and the + sign in the expression $a' + b'$ is understood to be the binary operator in M'. Admittedly it would be more precise to denote the binary operator in M' by $+'$ and to write $a' +' b'$ but we have left it up to reader to make this distinction. In particular we defined a map $\varphi : M \to M'$ to be a homomorphism if for all $a, b \in M$, $\varphi(a + b) = (\varphi a) + (\varphi b)$ and the first + sign refers to the binary operator in M while the second + sign refers to the binary operator in M'. And if we had a different symbol for the binary operator in M', say $\times$, we would of course say that $\varphi : M \to M'$ is an homomorphism if for all $a, b \in M$, $\varphi(a + b) = (\varphi a) \times (\varphi b)$. In the present case, we denote the binary operator in the magnitude space $\mathbb{R}_{>1}$ by $\cdot$ and so by a homomorphism $\mathbb{R}_+$ into $\mathbb{R}_{>1}$ we mean a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_{>1}$ such that for all $x, y \in \mathbb{R}_+$, $\varphi(x + y) = (\varphi x) \cdot (\varphi y)$.

Having shown that $\mathbb{R}_{>1}$ is a continuous magnitude space, we know that for each $y \in \mathbb{R}_{>1}$ there exists a unique embedding of $\mathbb{R}_+$ into $\mathbb{R}_{>1}$ which maps 1 into y by Theorem 12.8. The assumptions of Section 16 are satisfied with $M = \mathbb{R}_+$ and $M' = \mathbb{R}_{>1}$. And, in accordance with Section 16, for each $x \in \mathbb{R}_{>1}$ we define Ψx to be the unique embedding of $\mathbb{R}_+$ into $\mathbb{R}_{>1}$ which maps 1 into x. For $y \in \mathbb{R}_+$ and $x \in \mathbb{R}_{>1}$ we then have, as before, a product $yx = (\Psi x)\, y \in \mathbb{R}_{>1}$. To avoid confusing this definition of yx with multiplication in $\mathbb{R}_+$ we make the following definition.

Definition 19.2 *If $x \in \mathbb{R}_{>1}$ then we denote by Ψx the unique embedding of $\mathbb{R}_+ \to \mathbb{R}_{>1}$ which maps 1 into x. And for $y \in \mathbb{R}_+$, we denote $(\Psi x)\, y$ by x^y.*

Theorem 19.4 $(x_1 \cdot x_2)^y = x_1^y \cdot x_2^y$

Proof Theorem 16.3. ■

Theorem 19.5 $x^{y_1 + y_2} = x^{y_1} \cdot x^{y_2}$

Proof Theorem 16.4. ■

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