AN ELEMENTARY TREATISE

ON

LAPLACE'S FUNCTIONS, LAMÉ'S FUNCTIONS,
AND BESSEL'S FUNCTIONS.

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BY

I. TODHUNTER, M.A. F.R.S.
HONORARY FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE.

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PREFACE.

THE present volume is devoted mainly to an investigation of the properties of the remarkable expressions which were first introduced to the notice of mathematicians by Legendre, and are now known as Laplace's Coefficients and Functions. Some account of these expressions is given in various works, but their importance in modern researches suggests the advantage of a more complete and systematic development of them than has hitherto appeared in England. The work now published will it is hoped be found sufficiently elementary for those who are commencing the subject, and at the same time adequate in extent to the wants of the advanced student.

The book is composed of four parts. The first part consists of twelve Chapters, in which the expressions are considered as functions of only a single variable; in this form they were first introduced by Legendre, and it is convenient to denote them, thus restricted, by his name. The second part consists of eight Chapters, in which the expressions are considered as functions of two variables; this is the form in which they present themselves in the writings of Laplace. The third part consists of nine Chapters which treat of Lamé's Functions; these may be regarded as an extension of Laplace's Functions. The fourth part consists of seven

Chapters which treat of Bessel's Functions; these are not connected with the main subject of the book, but as they are becoming very prominent in the applications of mathematics to physics it may be convenient to find an exposition of them here.

The demonstrations which are adopted have been carefully chosen so as to bring under the attention of students some of the most instructive processes of modern analysis. Thus the work may be regarded both as an account of the Functions to which it is specially devoted, and also as a continuation of the two volumes already published on the Differential and Integral Calculus respectively; the three together form a connected treatise on the higher department of pure mathematics.

In conducting the work through the press, I have had the valuable assistance of the Rev. J. Sephton, M.A., Head Master of the Liverpool Institute, formerly Fellow of St John's College, Cambridge.

I. TODHUNTER.

ST JOHN'S COLLEGE,
November, 1875.

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CHAPTER I.

INTRODUCTION.

1. THE mathematical expressions to which the present volume is mainly devoted were first introduced by Legendre in some researches relating to the Figure of the Earth, and were much cultivated by himself and by Laplace in their investigations of this important problem of Physical Astronomy. In the *History of the Mathematical Theories of Attraction and of the Figure of the Earth* will be found an account of the origin and early progress of the branch of analysis which we are now about to expound.

2. Suppose that the expression $(1 - 2\alpha x + \alpha^2)^{-\frac{1}{2}}$ is expanded in a series of ascending powers of α; the coefficient of α^n will be a function of x which we shall denote by $P_n(x)$, and shall call *Legendre's Coefficient of the* nth *order.* The term *Laplace's Coefficient* is generally used when for x we substitute the value $\cos\theta \cos\theta_1 + \sin\theta \sin\theta_1 \cos(\phi - \phi_1)$, where we regard θ and ϕ as variables, and θ_1 and ϕ_1 as constants; so that Laplace's Coefficient is a function of two independent variables. But the term Laplace's Coefficient is sometimes employed even for what we propose to call Legendre's Coefficient.

3. Other names have also been suggested for the celebrated expressions which we are about to discuss: thus the Germans call them *Kugelfunctionen,* and in France the corresponding name *fonctions sphériques* has been used; Sir William Thomson and Professor Tait call them *spherical harmonics.* The name *Laplace's Functions* appears to have been first introduced by the late Dr Whewell, and has been generally adopted in England. In analogy with this, other

functions which we shall hereafter notice are associated with the names of eminent mathematicians, as *Lamé's Functions*, and *Bessel's Functions*.

The relation between Laplace's *Coefficients* and Laplace's *Functions* will be explained hereafter.

4. The researches of Legendre and Laplace were originally published in the volumes of the Paris Academy of Sciences; those of Legendre are reproduced with extended generality in his *Exercices de Calcul Intégral*, and those of Laplace are reproduced in his *Mécanique Céleste*. In more recent times other mathematicians have in various memoirs contributed improvements and extensions; and moreover the following separate works on the subject have appeared:

Recherches sur les Fonctions de Legendre par N. C. Schmit. ...Bruxelles, 1858. This consists of 80 octavo pages, besides the Title and Preface; on pages 72...75 is a list of memoirs on the subject.

Die Theorie der Kugelfunktionen. Von Dr. Georg Sidler. Bern, 1861. This consists of 71 quarto pages, and forms a good elementary treatise on the subject; it contains several references to the original memoirs.

Handbuch der Kugelfunctionen von Dr. E. Heine,... Berlin, 1861. This consists of 382 large octavo pages, besides the Title and Preface; it is a very elaborate work with abundant references to the original memoirs, and should be studied by those who wish to devote special attention to this branch of analysis. It discusses very fully the results which follow from the substitution of imaginary values for the variables in the expressions; but this development is somewhat abstruse, and belongs rather to the pure analyst, than to the cultivator of mathematical physics, for whom the subject in its simpler form is specially valuable.

5. Although I do not profess to have made that close investigation into the history of this subject, beyond its earlier stages, which I have prosecuted with respect to some other parts of mathematical science, yet I have incidentally paid some attention to it. One important memoir has been overlooked by the three writers mentioned in Art. 4; it is that by Rodrigues to which I drew attention in the *History*

of the Mathematical Theories of Attraction, Arts. 1176...1193. Three expansions which I shall give in Arts. 19, 21, and 23 are ascribed by Heine in his pages 8 and 15 to Dirichlet; they had however previously appeared in Murphy's Treatise on *Electricity*, Cambridge, 1833, in the more general forms from which I have deduced them.

6. As we have said in Art. 2, if $(1-2\alpha x+\alpha^2)^{-\frac{1}{2}}$ be expanded in a series of ascending powers of α, the coefficient of α^n is a function of x which is called *Legendre's Coefficient of the* n^th^ *order*: we may call it briefly *Legendre's* n^th^ *Coefficient*. We shall denote it by $P_n(x)$, but for the sake of simplicity we shall often omit the x, and thus use merely P_n. French writers very commonly use X_n for the same thing. We proceed to develope P_n explicitly.

7. We have $(1-2\alpha x+\alpha^2)^{-\frac{1}{2}}=\{1-\alpha(2x-\alpha)\}^{-\frac{1}{2}}$; expand by the Binomial Theorem; thus we obtain

$$1+\frac{1}{2}\alpha(2x-\alpha)+\frac{1\,.\,3}{2\,.\,4}\alpha^2(2x-\alpha)^2+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\alpha^3(2x-\alpha)^3+\ldots$$
$$+\frac{1\,.\,3\ldots(2n-1)}{2\,.\,4\ldots 2n}\alpha^n(2x-\alpha)^n+\ldots.$$

Suppose the various powers of $2x-\alpha$ to be expanded; and then pick out of each term the part which involves α^n, beginning with the last term which is here expressed. Thus we obtain

$$P_n=\frac{1\,.\,3\,.\,5\ldots(2n-1)}{\lfloor n}x^n-\frac{1\,.\,3\,.\,5\ldots(2n-3)}{2\lfloor n-2}x^{n-2}$$
$$+\frac{1\,.\,3\,.\,5\ldots(2n-5)}{2\,.\,4\lfloor n-4}x^{n-4}-\ldots.$$

If n be even, the last term is $(-1)^{\frac{n}{2}}\dfrac{1\,.\,3\,.\,5\ldots(n-1)}{2\,.\,4\ldots n}$, and if n be odd it is $(-1)^{\frac{n-1}{2}}\dfrac{3\,.\,5\ldots n}{2\,.\,4\ldots(n-1)}x$.

Thus $P_n(x)$ is a rational integral function of x of the degree n, and it involves only even powers of x, or only odd powers of x, according as n is even or odd.

We see that $P_n(-x)=(-1)^nP_n(x)$.

8. We may also put P_n in the form

$$\frac{1.3.5\ldots(2n-1)}{\lfloor n} \left\{x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} - \ldots\right\}.$$

9. If we remove, by cancelling, the odd integers which occur in the denominators of the numerical factors we obtain the following results, in which we take first examples of Legendre's Coefficients of *even* orders, and next examples of those of *odd* orders:

$$P_0 = 1,$$

$$P_2 = \frac{3}{2} x^2 - \frac{1}{2},$$

$$P_4 = \frac{5.7}{2.4} x^4 - \frac{3.5}{2.4} 2x^2 + \frac{1.3}{2.4},$$

$$P_6 = \frac{7.9.11}{2.4.6} x^6 - \frac{5.7.9}{2.4.6} 3x^4 + \frac{3.5.7}{2.4.6} 3x^2 - \frac{1.3.5}{2.4.6};$$

and generally

$$P_{2n} = \Sigma (-1)^{n-s} \frac{(2s+1)(2s+3)\ldots(2s+2n-1)}{2.4\ldots 2n} \chi(n, s) x^{2s};$$

where Σ denotes a summation with respect to s from 0 to n, both inclusive; and $\chi(n, s)$ stands for $\frac{n(n-1)\ldots(n-s+1)}{\lfloor s}$.

$$P_1 = x,$$

$$P_3 = \frac{5}{2} x^3 - \frac{3}{2} x,$$

$$P_5 = \frac{7.9}{2.4} x^5 - \frac{5.7}{2.4} 2x^3 + \frac{3.5}{2.4} x,$$

$$P_7 = \frac{9.11.13}{2.4.6} x^7 - \frac{7.9.11}{2.4.6} 3x^5 + \frac{5.7.9}{2.4.6} 3x^3 - \frac{3.5.7}{2.4.6} x;$$

and generally

$$P_{2n+1} = \Sigma (-1)^{n-s} \frac{(2s+3)(2s+5)\ldots(2s+2n+1)}{2.4\ldots2n} \chi(n,s)\, x^{2s+1},$$

where Σ and $\chi(n,s)$ have the same meaning as before.

It will be observed that $\chi(n,s)$ is an integer, being in fact equal to the number of the combinations of n things taken s at a time.

10. The numerical factors which occur in the preceding Article admit in some cases of further reduction; and they can be put in such forms that the denominators of the fractions consist entirely of powers of 2. Thus for example

$$P_6 = \frac{231}{16} x^6 - \frac{315}{16} x^4 + \frac{105}{16} x^2 - \frac{5}{16}.$$

It is easily seen that this must be the case. For in the first expansion which we have given in Art. 7, we obtain as the general term $\frac{1.3\ldots(2m-1)}{2.4\ldots2m} a^m (2x-a)^m$, that is $\frac{\lfloor 2m}{2^m \lfloor m\, 2^m \lfloor m} a^m (2x-a)^m$, that is $\frac{\lfloor 2m}{\lfloor m \lfloor m} \left(\frac{a}{2}\right)^m \left(x - \frac{a}{2}\right)^m$. Now $\frac{\lfloor 2m}{\lfloor m \lfloor m}$ is an integer, and hence the coefficient of a^n will not have any number in the denominator except 2^n. We may go a step further; for $\frac{\lfloor 2m}{\lfloor m \lfloor m} = 2\frac{\lfloor 2m-1}{\lfloor m \lfloor m-1}$, and is therefore necessarily an *even* integer; and thus the numerical factors of $P_n(x)$ will not involve in the denominators any power of 2 higher than 2^{n-1}.

11. The expression for $P_n(x)$ may be put in a very compact form first given by Rodrigues, namely

$$P_n = \frac{1}{2^n \lfloor n} \frac{d^n (x^2-1)^n}{dx^n}.$$

For let $(x^2-1)^n$ be expanded by the Binomial Theorem, and let the result be differentiated n times with respect to x, then it will be found that the term which involves x^{n-2s}

$= \text{the product of } \dfrac{(-1)^s x^{n-2s}}{2^n |\underline{n}} \dfrac{n(n-1)\ldots(n-s+1)}{|\underline{s}}$ into

$(2n-2s)(2n-2s-1)\ldots(n-2s+1)$

$$= \frac{(-1)^s x^{n-2s}(2n-2s)(2n-2s-1)\ldots\ldots(n-2s+1)}{2^n |\underline{n-s}\, |\underline{s}}.$$

Again, in the formula obtained for $P_n(x)$ in Art. 7 we see that the term which involves x^{n-2s}

$$= \frac{(-1)^s x^{n-2s} 1 . 3 . 5 \ldots (2n-2s-1)}{2 . 4 \ldots 2s\, |\underline{n-2s}} = \frac{(-1)^s x^{n-2s} |\underline{2n-2s}}{2^n |\underline{n-s}\, |\underline{s}\, |\underline{n-2s}}$$

$$= \frac{(-1)^s x^{n-2s}(2n-2s)(2n-2s-1)\ldots(n-2s+1)}{2^n |\underline{n-s}\, |\underline{s}}.$$

This agrees with the former result, and thus the identity of the two forms of expression for $P_n(x)$ is established.

12. Another mode of investigating the expression of the preceding Article for $P_n(x)$ may be noticed.

Assume $\sqrt{(1-2ax+a^2)} = 1 - ay$;

therefore $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(1-2ax+a^2)}}$.

Hence we require the coefficient of a^n in the expansion of $\dfrac{dy}{dx}$ in a series proceeding according to ascending powers of a.

Now $1 - 2ax + a^2 = (1-ay)^2 = 1 - 2ay + a^2y^2$;

therefore $y = x + a\dfrac{y^2-1}{2}$.

The general term of the expansion of y in powers of a may now be obtained by the aid of Lagrange's Theorem: see *Differential Calculus*, page 117. It is $\dfrac{a^n}{2^n |\underline{n}} \dfrac{d^{n-1}(x^2-1)^n}{dx^{n-1}}$;

and therefore the general term in the expansion of $\dfrac{dy}{dx}$ is

$$\frac{a^n}{2^n |\underline{n}} \frac{d^n(x^2-1)^n}{dx^n}.$$

CHAPTER II.

OTHER FORMS OF LEGENDRE'S COEFFICIENTS.

13. In the preceding Chapter we have given the most important expressions for Legendre's Coefficients; in the present Chapter we shall investigate some other forms which are frequently useful.

In applications of the theory x is very often equal to the cosine of an angle; we shall denote it by $\cos\theta$, and shall proceed to develope $P_n(\cos\theta)$ in cosines of multiples of θ.

14. We have, putting ι for $\sqrt{-1}$,

$$(1-2\alpha\cos\theta+\alpha^2)^{-\frac{1}{2}} = \{1-\alpha(e^{\iota\theta}+e^{-\iota\theta})+\alpha^2\}^{-\frac{1}{2}}$$
$$= (1-\alpha e^{\iota\theta})^{-\frac{1}{2}}(1-\alpha e^{-\iota\theta})^{-\frac{1}{2}}.$$

Expand each factor by the Binomial Theorem; thus we obtain

$$\left\{1+\frac{1}{2}\alpha e^{\iota\theta}+\frac{1\,.\,3}{2\,.\,4}\alpha^2e^{2\iota\theta}+\ldots\right\}\left\{1+\frac{1}{2}\alpha e^{-\iota\theta}+\frac{1\,.\,3}{2\,.\,4}\alpha^2e^{-2\iota\theta}+\ldots\right\}.$$

Multiply the two series together, and pick out the term which involves α^n; it will be found that the coefficient of this term is

$$\frac{1\,.\,3\ldots(2n-1)}{2\,.\,4\ldots2n}e^{n\iota\theta}+\frac{1}{2}\cdot\frac{1\,.\,3\ldots(2n-3)}{2\,.\,4\ldots(2n-2)}e^{(n-2)\iota\theta}$$
$$+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{1\,.\,3\ldots(2n-5)}{2\,.\,4\ldots(2n-4)}e^{(n-4)\iota\theta}+\ldots+\frac{1\,.\,3\ldots(2n-1)}{2\,.\,4\ldots2n}e^{-n\iota\theta}.$$

Now put for each exponential its value derived from the formula $e^{r\theta} = \cos r\theta + \iota \sin r\theta$; then the imaginary part disappears, and we have the following result:

$$P_n(\cos\theta) = \frac{1.3\ldots(2n-1)}{2^n \underline{|n}} \left\{\cos n\theta + \frac{1.n}{1.(2n-1)}\cos(n-2)\theta + \frac{1.3.n(n-1)}{1.2(2n-1)(2n-3)}\cos(n-4)\theta + \ldots\right\}.$$

The series within the brackets is to continue until it terminates of itself by the occurrence of zero as a factor; so that there are $n+1$ terms in the series, and the last of them is $\cos(n-2n)\theta$, which is equal to $\cos n\theta$.

15. We may state the result with respect to the series within the brackets of the preceding Article in another form, thus: if n be odd continue the series to $\frac{n+1}{2}$ terms, and double every term; if n be even continue the series to $\frac{n}{2}+1$ terms, and double every term except the last.

16. The formula of Art. 14 leads to the important result that $P_n(\cos\theta)$ *has its greatest value when* $\theta = 0$. The value in this case may be found most simply by recurring to the definition; $P_n(1)$ is the coefficient of α^n in the expansion of $(1-2\alpha+\alpha^2)^{-\frac{1}{2}}$, that is in the expansion of $(1-\alpha)^{-1}$; and so the value is unity.

17. In Art. 14 we put $\cos\theta$ for the general symbol x, so that we assumed x to be not greater than unity; but a formula analogous to that obtained in Art. 14 will hold when x is greater than unity.

For assume $2x = \xi + \xi^{-1}$, and $2\sqrt{(x^2-1)} = \xi - \xi^{-1}$,

so that $\xi = x + \sqrt{(x^2-1)}$, and $\xi^{-1} = x - \sqrt{(x^2-1)}$.

Then $$P_n(x) = \frac{1.3\ldots(2n-1)}{2^n \underline{|n}} \left\{\xi^n + \frac{1.n}{1.(2n-1)}\xi^{n-2} + \frac{1.3.n(n-1)}{1.2.(2n-1)(2n-3)}\xi^{n-4} + \ldots\right\}.$$

The series within the brackets is to continue until it terminates of itself by the occurrence of zero as a factor; so that there are $n+1$ terms in the series, and the last of them is ξ^{-n}.

To demonstrate this formula we observe that the right-hand member when developed will become a rational integral function of x, and the left-hand member is always such by Art. 7. Moreover, we know by Art. 14 that the two members are identically equivalent when x has any value less than unity. Hence they are always identically equivalent.

18. By Art. 11 we have

$$P_n(x)=\frac{1}{2^n\lfloor n}\frac{d^n(x^2-1)^n}{dx^n}=\frac{1}{2^n\lfloor n}\frac{d^n}{dx^n}(x-1)^n(x+1)^n$$

$$=\frac{1}{2^n\lfloor n}\frac{d^n}{dx^n}(x-1)^n(2+x-1)^n.$$

Let $(2+x-1)^n$ be expanded in ascending powers of $x-1$; thus

$$P_n(x)=\frac{1}{2^n\lfloor n}\frac{d^n}{dx^n}\left\{2^n(x-1)^n+n2^{n-1}(x-1)^{n+1}\right.$$

$$\left.+\frac{n(n-1)}{1.2}2^{n-2}(x-1)^{n+2}+\dots\right\}$$

$$=1+\frac{(n+1)n}{1^2}\frac{x-1}{2}+\frac{(n+2)(n+1)n(n-1)}{1^2.2^2}\frac{(x-1)^2}{2^2}$$

$$+\frac{(n+3)(n+2)(n+1)n(n-1)(n-2)}{1^2.2^2.3^2}\frac{(x-1)^3}{2^3}+\dots$$

19. For a particular case of the preceding Article put $x=\cos\theta$, then $x-1=-2\sin^2\frac{\theta}{2}$; thus

$$P_n(\cos\theta)=1-\frac{(n+1)n}{1^2}\sin^2\frac{\theta}{2}$$

$$+\frac{(n+2)(n+1)n(n-1)}{1^2.2^2}\sin^4\frac{\theta}{2}-\dots$$

This may also be obtained in the following way:

$$(1-2a\cos\theta+a^2)^{-\frac{1}{2}}=\left\{1-2a\left(1-2\sin^2\frac{\theta}{2}\right)+a^2\right\}^{-\frac{1}{2}}$$

$$=\left\{(1-a)^2+4a\sin^2\frac{\theta}{2}\right\}^{-\frac{1}{2}}.$$

Expand by the Binomial Theorem; thus we obtain for the general term

$$(-1)^m\frac{1.3...(2m-1)}{\lfloor m}\frac{\left(2a\sin^2\frac{\theta}{2}\right)^m}{(1-a)^{2m+1}},\text{ that is }\frac{(-1)^m\lfloor 2m}{\lfloor m\lfloor m}\frac{\left(a\sin^2\frac{\theta}{2}\right)^m}{(1-a)^{2m+1}}.$$

Expand $(1-a)^{-2m-1}$ in ascending powers of a, and pick out the term which involves a^n; in this way we obtain finally as before

$$P_n(\cos\theta)=1-\frac{(n+1)n}{1^2}\sin^2\frac{\theta}{2}$$

$$+\frac{(n+2)(n+1)n(n-1)}{1^2.2^2}\sin^4\frac{\theta}{2}-\ldots$$

20. Again, we have

$$P_n(x)=\frac{1}{2^n\lfloor n}\frac{d^n(x^2-1)^n}{dx^n}=(-1)^n\frac{1}{2^n\lfloor n}\frac{d^n}{dx^n}(x+1)^n(1-x)^n$$

$$=(-1)^n\frac{1}{2^n\lfloor n}\frac{d^n}{dx^n}(x+1)^n(2-x-1)^n.$$

Let $(2-x-1)^n$ be expanded in ascending powers of $(x+1)^n$; thus

$$P_n(x)=\frac{(-1)^n}{2_n\lfloor n}\frac{d^n}{dx^n}\left\{2^n(x+1)^n-n\,2^{n-1}(x+1)^{n+1}\right.$$

$$\left.+\frac{n(n-1)}{1.2}2^{n-2}(x+1)^{n+2}-\ldots\right\}$$

$$=(-1)^n\left\{1-\frac{(n+1)n}{1^2}\frac{x+1}{2}+\frac{(n+2)(n+1)n(n-1)}{1^2.2^2}\frac{(x+1)^2}{2^2}\right.$$

$$\left.-\frac{(n+3)(n+2)(n+1)n(n-1)(n-2)}{1^2.2^2.3^2}\frac{(x+1)^3}{2^3}+\ldots\right\}.$$

21. For a particular case of the preceding Article put $x = \cos\theta$, then $x + 1 = 2\cos^2\frac{\theta}{2}$; thus

$$P_n(\cos\theta) = (-1)^n \left\{1 - \frac{(n+1)n}{1^2}\cos^2\frac{\theta}{2} + \frac{(n+2)(n+1)n(n-1)}{1^2 . 2^2}\cos^4\frac{\theta}{2} - \dots\right\}.$$

This may also be obtained by putting $(1 - 2\alpha\cos\theta + \alpha^2)^{-\frac{1}{2}}$ in the form $\left\{(1+\alpha)^2 - 4\alpha\cos^2\frac{\theta}{2}\right\}^{-\frac{1}{2}}$ and proceeding as in Art. 19; or it may be deduced from the result of that Article by changing θ into $\pi - \theta$, and α into $-\alpha$.

22. By the theorem of Leibnitz, given in the *Differential Calculus*, Art. 80, we have

$$\frac{d^n}{dx^n}(x+1)^n(x-1)^n = (x+1)^n\frac{d^n(x-1)^n}{dx^n} + \frac{n}{1}\frac{d(x+1)^n}{dx}\frac{d^{n-1}(x-1)^n}{dx^{n-1}} + \frac{n(n-1)}{1.2}\frac{d^2(x+1)^n}{dx^2}\frac{d^{n-2}(x-1)^n}{dx^{n-2}} + \dots$$

Hence $P_n = \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x+1)^n(x-1)^n$

$$= \frac{1}{2^n}\left[(x+1)^n + \left(\frac{n}{1}\right)^2(x+1)^{n-1}(x-1) + \left\{\frac{n(n-1)}{1.2}\right\}^2(x+1)^{n-2}(x-1)^2 + \dots\right].$$

23. For a particular case of the preceding Article put $x = \cos\theta$; then $x + 1 = 2\cos^2\frac{\theta}{2}$, and $x - 1 = -2\sin^2\frac{\theta}{2}$; thus

$$P_n(\cos\theta) = \cos^{2n}\frac{\theta}{2}\left[1 - \left\{\frac{n}{1}\tan\frac{\theta}{2}\right\}^2 + \left\{\frac{n(n-1)}{1.2}\tan^2\frac{\theta}{2}\right\}^2 - \dots\right].$$

24. We have $(1-2\alpha x+\alpha^2)^{-\frac{1}{2}}=\{(1-\alpha x)^2+\alpha^2(1-x^2)\}^{-\frac{1}{2}}$.

Expand by the Binomial Theorem; thus we obtain for the general term

$$(-1)^m \frac{1.3...(2m-1)}{2^m \underline{|m}} \frac{\{\alpha^2(1-x^2)\}^m}{(1-\alpha x)^{2m+1}},$$

that is

$$\frac{(-1)^m \underline{|2m}}{2^{2m}\underline{|m}\,\underline{|m}} \frac{\alpha^{2m}(1-x^2)^m}{(1-\alpha x)^{2m+1}}.$$

Expand $(1-\alpha x)^{-2m-1}$ in ascending powers of αx, and pick out the term which involves α^n; we find after reduction that this is $\frac{(-1)^m \underline{|n}\,\alpha^n(1-x^2)^m x^{n-2m}}{2^{2m}\underline{|m}\,\underline{|m}\,\underline{|n-2m}}$. Hence, putting t^2 for $\frac{1-x^2}{x^2}$, we obtain finally

$$P_n(x)=x^n\left\{1-\frac{n(n-1)}{2^2}t^2+\frac{n(n-1)(n-2)(n-3)}{2^2.4^2}t^4-...\right\}.$$

25. For a particular case of the preceding Article put $x=\cos\theta$, then $\frac{1-x^2}{x^2}=\tan^2\theta$; thus

$$P_n(\cos\theta)=\cos^n\theta\left\{1-\frac{n(n-1)}{2^2}\tan^2\theta\right.$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{2^2.4^2}\tan^4\theta-......\right\}.$$

26. In all these expansions we may if we please suppose α to be so small as to ensure the convergence of the series. We know, by Art. 16, that $P_n(\cos\theta)$ cannot exceed unity; and thus the series of which the general term is $P_n(\cos\theta)\,\alpha^n$ is convergent if α is less than unity.

CHAPTER III.

PROPERTIES OF LEGENDRE'S COEFFICIENTS.

27. WE have from Art. 7,

$$1 = P_0,$$

$$x = P_1,$$

$$x^2 = \frac{2}{3}P_2 + \frac{1}{3},$$

$$x^3 = \frac{2}{5}P_3 + \frac{3}{5}x = \frac{2}{5}P_3 + \frac{3}{5}P_1,$$

$$x^4 = \frac{8}{35}P_4 + \frac{6}{7}x^2 - \frac{3}{35} = \frac{8}{35}P_4 + \frac{6}{7}\left(\frac{2}{3}P_2 + \frac{1}{3}\right) - \frac{3}{35}.$$

Proceeding in this way we see that any positive integral power of x may be expressed in terms of Legendre's Coefficients. The expression for x^n will be of the form

$$a_n P_n + a_{n-2} P_{n-2} + a_{n-4} P_{n-4} + \ldots,$$

where a_n, a_{n-2}, a_{n-4}, ... are certain numerical coefficients. The expression terminates with $a_0 P_0$, or with $a_1 P_1$, according as n is even or odd. The practical determination of the values of a_n, a_{n-2}, ... is facilitated by some propositions in the Integral Calculus to which we now proceed.

28. To shew that $\int_{-1}^{1} P_m P_n dx = 0$, if m and n are unequal, and that $\int_{-1}^{1} P_n P_n dx = \frac{2}{2n+1}$.

Consider the integral $\int \frac{dx}{\sqrt{a - bx}\sqrt{a' - b'x}}$, that is

$$\int \frac{dx}{\sqrt{\{aa' - (ab' + a'b)x + bb'x^2\}}}.$$

We shall find by the *Integral Calculus*, Art. 14, that

$$\int \frac{dx}{\sqrt{a-bx}\,\sqrt{a'-b'x}} = \frac{2}{\sqrt{bb'}} \log \left\{ \sqrt{b\,(a'-b'x)} - \sqrt{b'\,(a-bx)} \right\}.$$

Thus

$$\int \frac{dx}{\sqrt{1-2\alpha x+\alpha^2}\,\sqrt{1-2\beta x+\beta^2}}$$

$$= \frac{1}{\sqrt{\alpha\beta}} \log \left\{ \sqrt{2\alpha\,(1-2\beta x+\beta^2)} - \sqrt{2\beta(1-2\alpha x+\alpha^2)} \right\}.$$

Then, by taking the integral between the limits -1 and 1, we have

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-2\alpha x+\alpha^2}\,\sqrt{1-2\beta x+\beta^2}} = \frac{1}{\sqrt{\alpha\beta}} \log \frac{1+\sqrt{\alpha\beta}}{1-\sqrt{\alpha\beta}}$$

$$= 2\left\{1 + \frac{\alpha\beta}{3} + \frac{\alpha^2\beta^2}{5} + \frac{\alpha^3\beta^3}{7} + \ldots\right\}.$$

Now the expression under the integral sign in the left-hand member of this equation is, by Art. 6, equal to

$$(1+\alpha P_1+\alpha^2 P_2+\ldots+\alpha^n P_n+\ldots)\,(1+\beta P_1+\beta^2 P_2+\ldots+\beta^n P_n+\ldots).$$

Hence, by equating the coefficients of like terms, we see that

$$\int_{-1}^{1} P_m P_n\, dx = 0,$$

if m and n are unequal; and that

$$\int_{-1}^{1} P_n P_n\, dx = \frac{2}{2n+1}.$$

29. We have shewn in Art. 27 that

$$x^n = a_n P_n + a_{n-2} P_{n-2} + a_{n-4} P_{n-4} + \ldots.$$

Let a_m denote any one of the numerical factors; multiply by P_m and integrate between the limits -1 and 1: thus, by the aid of Art. 28, we have

$$\int_{-1}^{1} P_m x^n\, dx = \frac{2a_m}{2m+1},$$

therefore $$a_m = \frac{2m+1}{2}\int_{-1}^{1} P_m x^n\,dx.$$

Thus the numerical factors can be expressed as definite integrals.

30. It follows from Arts. 28 and 29 that if m and n are positive integers, and m greater than n, then

$$\int_{-1}^{1} P_m x^n\,dx = 0.$$

This is one of the most important properties of Legendre's Coefficients. It will be convenient to change the notation and express the result thus: if m and n are positive integers, and m less than n, then

$$\int_{-1}^{1} P_n x^m\,dx = 0.$$

31. The result of the preceding Article may also be obtained in another way.

Let y be any function of x. By integration by parts we have

$$\int P_n y\,dx = \xi_1 y - \int \xi_1 \frac{dy}{dx}\,dx,$$

where ξ_1 stands for $\int P_n dx$.

By Art. 11, we have $\xi_1 = \dfrac{1}{2^n \underline{|n}} \dfrac{d^{n-1}}{dx^{n-1}} (x+1)^n (x-1)^n$; and this vanishes both when $x = -1$ and when $x = 1$. Thus

$$\int_{-1}^{1} P_n y\,dx = -\int_{-1}^{1} \xi_1 \frac{dy}{dx}\,dx.$$

In the same way we find that

$$\int_{-1}^{1} \xi_1 \frac{dy}{dx}\,dx = -\int_{-1}^{1} \xi_2 \frac{d^2y}{dx^2}\,dx,$$

where ξ_2 stands for $\int \xi_1 dx$, that is for $\dfrac{1}{2^n \underline{|n}} \dfrac{d^{n-2}}{dx^{n-2}} (x+1)^n (x-1)^n$.

Proceeding in this way, we have finally

$$\int_{-1}^{1} P_n y\,dx = (-1)^n \int_{-1}^{1} \xi_n \frac{d^n y}{dx^n}\,dx,$$

where
$$\xi_n = \frac{1}{2^n \lfloor n} (x+1)^n (x-1)^n.$$

Hence if y be a rational integral function of x of a lower dimension than the n^{th}, we have

$$\int_{-1}^{1} P_n y\,dx = 0.$$

32. We shall now shew that no other rational integral function of x of the n^{th} degree except the product of a constant into $P_n(x)$ has the important property noticed in Art. 30; that is if $\phi(x)$ be a rational integral function of x of the n^{th} degree, such that $\int_{-1}^{1} \phi(x)\,x^m dx = 0$, when m is any positive integer less than n, then $\phi(x)$ must be of the form $CP_n(x)$, where C is some constant.

Let $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$,... denote a series of functions of x formed in succession according to these laws;

$$\phi_1(x) = \int_{-1}^{x} \phi(x)\,dx,$$

$$\phi_2(x) = \int_{-1}^{x} \phi_1(x)\,dx,$$

$$\phi_3(x) = \int_{-1}^{x} \phi_2(x)\,dx,$$

and so on.

By integration by parts, we have

$$\int_{-1}^{x} \phi(x)\,x^m dx = \phi_1(x)\,x^m - m\phi_2(x)\,x^{m-1} + m(m-1)\,\phi_3(x)\,x^{m-2} - \ldots + (-1)^m \lfloor m\,\phi_{m+1}(x).$$

Now if m have any positive integral value between 0 and $n-1$, both inclusive, $\int_{-1}^{x} \phi(x)\,x^m dx$ is by supposition zero when $x=1$. Put for m in succession the values 0, 1,...$n-1$ in the preceding equation; thus we see that $\phi_1(x)$, $\phi_2(x)$,...$\phi_n(x)$ all vanish when $x=1$: that is, $\phi_n(x)$ and its successive differential coefficients down to the $(n-1)^{th}$ all vanish when $x=1$. Moreover by the laws of formation $\phi_n(x)$ and its successive differential coefficients down to the $(n-1)^{th}$ all vanish when $x=-1$. And $\phi_n(x)$ is of the degree $2n$ in terms of x. Hence, by the *Theory of Equations*, Art. 75, it follows that $\phi_n(x)$ is of the form $A(x+1)^n(x-1)^n$, where A is a constant. Therefore

$$\phi(x) = A\frac{d^n}{dx^n}(x+1)^n(x-1)^n.$$

Thus, by Art. 11, it follows that $\phi(x) = CP_n(x)$, where C is some constant.

33. If m is a positive integer less than n, and $n-m$ is an even number, then

$$\int_0^1 x^m P_n dx = 0.$$

For by Art. 7, we have $(-x)^m P_n(-x) = (-1)^{m+n} x^m P_n(x)$

$= x^m P_n(x)$ when $n-m$ is even.

Therefore in this case

$$\int_0^1 x^m P_n dx = \frac{1}{2}\int_{-1}^1 x^m P_n dx = 0.$$

34. We shall now determine the value of $\int_0^1 x^k P_n dx$, where k is any positive number, whole or fractional.

We know that

$$P_n = ax^n + \beta x^{n-2} + \gamma x^{n-4} + \dots,$$

where $\alpha, \beta, \gamma, \dots$ are certain numerical factors.

Hence

$$\int_0^1 x^k P_n dx = \frac{\alpha}{k+n+1} + \frac{\beta}{k+n-1} + \frac{\gamma}{k+n-3} + \dots$$

I. Suppose n even. Then the number of the fractions in the expression just given will be $\frac{n}{2}+1$. If we bring the fractions to a common denominator, we obtain for the result

$$\frac{K}{(k+n+1)(k+n-1)(k+n-3)\dots(k+1)},$$

where K is some rational integral function of k of the degree $\frac{n}{2}$. Now we know by Art. 33 that K will vanish when k has any of the following values, $n-2, n-4, \dots 2, 0$: hence K must be of the form $\lambda k(k-2)(k-4)\dots(k-n+2)$, where λ is independent of k, since K is of the degree $\frac{n}{2}$. Moreover by the way in which K was obtained, since λ is the coefficient of the highest power of k, we must have

$$\lambda = \alpha+\beta+\gamma+\dots;$$

that is, $\lambda = P_n(1) = 1$, by Art. 16.

Therefore when n is *even*

$$\int_0^1 x^k P_n dx = \frac{k(k-2)(k-4)\dots(k-n+2)}{(k+n+1)(k+n-1)\dots(k+1)}.$$

It will be seen that the investigation and the result will also hold in this case when k is negative, provided that it be numerically less than unity.

II. Suppose n odd.

By proceeding as in the former case we find that the sum of the fractions is

$$\frac{K}{(k+n+1)(k+n-1)(k+n-3)\dots(k+2)},$$

where K is some rational integral function of k of the degree $\frac{n-1}{2}$. Then K must be of the form

$$\lambda(k-1)(k-3)(k-5)\dots(k-n+2),$$

and as before we find that $\lambda = 1$.

Therefore when n is *odd*

$$\int_0^1 x^k P_n dx = \frac{(k-1)(k-3)(k-5)\ldots(k-n+2)}{(k+n+1)(k+n-1)\ldots(k+2)}.$$

It will be seen that the investigation and the result will also hold in this case when k is negative, provided that it be numerically less than 2.

Hence $\int_{-1}^1 x^k P_n dx$ can be immediately found; supposing that if k be a fraction the denominator is an odd number when the fraction is in its lowest terms, so that the expression may be *real* throughout the range of integration. For if $x^k P_n$ changes sign with x the definite integral is zero, and if $x^k P_n$ does not change sign with x the value of the definite integral is *twice* the value corresponding to the limits 0 and 1.

35. For a particular case of the preceding Article let k be a positive integer not less than n, and let $k-n$ be *even*.

First suppose k even, and therefore n even. Take the result in I; multiply both numerator and denominator by $1.3\ldots(k-1)$, and also by $2.4\ldots(k-n)$: thus we obtain

$$\frac{\lfloor k}{2.4\ldots(k-n)\,1.3.5\ldots(k+n+1)}.$$

Next suppose k odd, and therefore n odd. Take the result in II; multiply both numerator and denominator by $1,3\ldots k$, and also by $2.4\ldots(k-n)$: thus we again obtain

$$\frac{\lfloor k}{2.4\ldots(k-n)\,1.3.5\ldots(k+n+1)}.$$

As an example we have

$$\int_0^1 x^n P_n dx = \frac{\lfloor n}{1.3.5\ldots(2n+1)} = \frac{2^n \lfloor n \lfloor n}{\lfloor 2n+1}.$$

36. We can now definitely express x^n in terms of Legendre's coefficients, n being a positive integer.

By Art. 29 we have

$$x^n = a_n P_n + a_{n-2} P_{n-2} + a_{n-4} P_{n-4} + \ldots,$$

where any numerical factor a_m is determined by the equation

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} x^n P_m dx.$$

Therefore, as in this case $n-m$ is even, we have, by the method of Art. 33,

$$a_m = (2m+1) \int_0^1 x^n P_m dx;$$

and therefore, by Art. 35,

$$a_m = \frac{(2m+1) \lfloor n}{2.4 \ldots (n-m)\, 1.3.5 \ldots (n+m+1)}.$$

Hence, finally,

$$x^n = \frac{\lfloor n}{1.3.5 \ldots (2n+1)} \left\{ (2n+1) P_n + (2n-3) \frac{2n+1}{2} P_{n-2} \right.$$
$$\left. + (2n-7) \frac{(2n+1)(2n-1)}{2.4} P_{n-4} + \ldots \right\}.$$

37. As an example we will express the function $\frac{1}{y-x}$ by the aid of Legendre's coefficients, under two conditions which will appear in the course of the process.

The first condition is that y be greater than x; then we have

$$\frac{1}{y-x} = \frac{1}{y} + \frac{x}{y^2} + \frac{x^2}{y^3} + \ldots,$$

where the infinite series is convergent.

Now express each power of x in a series of Legendre's coefficients by Art. 36, and then collect all the terms which involve the same coefficient. Thus $P_n(x)$ will arise from $\frac{x^n}{y^{n+1}}, \frac{x^{n+2}}{y^{n+3}}, \frac{x^{n+4}}{y^{n+5}}, \ldots$; and for the multiplier of it, from $\dfrac{x^n}{y^{n+1}}$ we get $\dfrac{(2n+1)\,\underline{|n}}{1.3.5\ldots(2n+1)}\cdot\dfrac{1}{y^{n+1}}$,

from $\dfrac{x^{n+2}}{y^{n+3}}$ we get $\dfrac{(2n+1)\,\underline{|n+2}}{2.1.3\ldots(2n+3)}\cdot\dfrac{1}{y^{n+3}}$,

from $\dfrac{x^{n+4}}{y^{n+5}}$ we get $\dfrac{(2n+1)\,\underline{|n+4}}{2.4.1.3\ldots(2n+5)}\cdot\dfrac{1}{y^{n+5}}$,

and so on.

Thus let $(2n+1)\,Q_n(y) =$

$$\frac{\underline{|n}}{1.3\ldots(2n-1)}\left\{y^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}y^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}y^{-n-5}+\ldots\right\};$$

then
$$\frac{1}{y-x}=\Sigma\,(2n+1)\,Q_n(y)\,P_n(x),$$

where Σ denotes summation with respect to n from 0 to infinity.

As the second condition we require that y should be greater than unity, in order that the series denoted by $Q_n(y)$ may be convergent. See *Algebra*, Art. 775.

38. To express $\dfrac{dP_n}{dx}$ in terms of Legendre's coefficients.

The powers of x which $\dfrac{dP_n}{dx}$ involves are the following, x^{n-1}, x^{n-3}, x^{n-5}, ... ; we may therefore assume that

$$\frac{dP_n}{dx}=a_{n-1}P_{n-1}+a_{n-3}P_{n-3}+a_{n-5}P_{n-5}+\ldots,$$

where a_{n-1}, a_{n-3}, a_{n-5}, ... are numerical factors to be determined. Let a_m denote any one of them; then, by Art. 28,

$$a_m=\frac{2m+1}{2}\int_{-1}^{1}P_m\frac{dP_n}{dx}\,dx\,\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now, by Art. 30, we see that for all the values of m with which we are here concerned

$$0 = \int_{-1}^{1} P_n \frac{dP_m}{dx} dx \text{ ....(2).}$$

Multiply (2) by $\frac{2m+1}{2}$ and add to (1); thus

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} \left(P_m \frac{dP_n}{dx} + P_n \frac{dP_m}{dx}\right) dx.$$

But $$\int \left(P_m \frac{dP_n}{dx} + P_n \frac{dP_m}{dx}\right) dx = P_m P_n ;$$

and as $n - m$ is here an odd number we have $P_m P_n = 1$ when $x = 1$, and $= -1$ when $x = -1$.

Therefore $$a_m = 2m + 1.$$

Thus $\frac{dP_n}{dx} = (2n-1) P_{n-1} + (2n-5) P_{n-3} + (2n-9) P_{n-5} + \dots$;

the last term is $3P_1$ if n is even, and P_0 if n is odd.

39. In Art. 14 we have expressed $P_n (\cos \theta)$ in terms of cosines of multiples of θ. Now if $f(\theta)$ denote any function of θ we can expand $f(\theta)$ in a series of the form

$$a_1 \sin \theta + a_2 \sin 2\theta + a_3 \sin 3\theta + \dots,$$

where $a_1, a_2, a_3 \dots$ are numerical factors: see *Integral Calculus*, Chapter XIII. The expansion will hold for values of θ between 0 and π, excluding however these limiting values unless $f(\theta)$ vanishes when $\theta = 0$ and when $\theta = \pi$. All the numerical factors are determined by the general formula

$$a_m = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin m\theta \, d\theta.$$

We shall now apply this process to the case in which $f(\theta) = P_n (\cos \theta)$.

We shall first shew that a_m is zero if m is less than $n + 1$.

We know that $\sin m\theta = M \times \sin\theta$, where M denotes a rational integral function of $\cos\theta$, of the degree $m-1$: see *Plane Trigonometry*, Art. 288. Thus

$$\int_0^\pi P_n(\cos\theta)\sin m\theta\,d\theta = \int_0^\pi P_n(\cos\theta)\,M\sin\theta\,d\theta$$

$$= \int_{-1}^1 P_n(x)\,M dx,$$

where M is now supposed to be expressed as a function of x, by putting x for $\cos\theta$.

Hence by Art. 30 it follows that a_m is zero if m is less than $n+1$.

We shall next shew that a_m is zero if $m-n$ is equal to any even number.

For M being expressed as a function of x as before, the product $P_n(x)M$ will involve only *odd* powers of x, and therefore the integral of it between the limits -1 and 1 will vanish.

Thus we have to find a_m only for the cases in which m has the following values, $n+1$, $n+3$, $n+5$,...

Now, by Art. 15, we may put $P_n(\cos\theta)$ in the form

$$2b_n\cos n\theta + 2b_{n-2}\cos(n-2)\theta + 2b_{n-4}\cos(n-4)\theta + \ldots,$$

observing that if n is odd the last term will be $2b_1\cos\theta$, and if n is even the last term will be b_0.

Hence $$P_n(\cos\theta)\sin m\theta = b_n\{\sin(m+n)\theta + \sin(m-n)\theta\}$$
$$+ b_{n-2}\{\sin(m+n-2)\theta + \sin(m-n+2)\theta\}$$
$$+ b_{n-4}\{\sin(m+n-4)\theta + \sin(m-n+4)\theta\} + \ldots.$$

Integrate between the limits 0 and π for θ; thus since $m-n$ is odd we obtain

$$2b_n\left(\frac{1}{m+n} + \frac{1}{m-n}\right) + 2b_{n-2}\left(\frac{1}{m+n-2} + \frac{1}{m-n+2}\right)$$
$$+ 2b_{n-4}\left(\frac{1}{m+n-4} + \frac{1}{m-n+4}\right) + \ldots,$$

the last term being $2b_1\left(\frac{1}{m+1}+\frac{1}{m-1}\right)$ if n is odd, and $\frac{2b_0}{m}$ if n is even.

Let $m=n+2k+1$; then the expression becomes

$$2b_n\left(\frac{1}{2n+2k+1}+\frac{1}{2k+1}\right)+2b_{n-2}\left(\frac{1}{2n+2k-1}+\frac{1}{2k+3}\right)$$
$$+2b_{n-4}\left(\frac{1}{2n+2k-3}+\frac{1}{2k+5}\right)+\ldots.$$

Bring all these fractions to a common denominator; thus we obtain

$$\frac{K}{(2k+1)(2k+3)\ldots(2k+2n+1)},$$

where K denotes a rational integral function of k of the degree n. Now K must vanish when k has any of the values $-1, -2, \ldots -n$; for in all these cases $\sin m\theta$ becomes numerically equal to $\sin \mu\theta$, where μ has some positive integral value which is less than $n+1$, and therefore, by what has been already shewn, $\int_0^\pi P_n(\cos\theta)\sin m\theta\,d\theta$ vanishes. Hence K must be of the form $\lambda(k+1)(k+2)\ldots(k+n)$, where λ is independent of k.

Also from the way in which K was obtained we see that, according as n is odd or even,

$$\lambda=2^{n+1}(2b_n+2b_{n-2}+\ldots+2b_1),$$

or $$\lambda=2^{n+1}(2b_n+2b_{n-2}+\ldots+2b_2+b_0);$$

so that in both cases $\lambda=2^{n+1}P_n(1)=2^{n+1}$.

Hence $\int_0^\pi P_n(\cos\theta)\sin(n+2k+1)\theta\,d\theta$

$$=\frac{2^{n+1}(k+1)(k+2)\ldots(k+n)}{(2k+1)(2k+3)\ldots(2k+2n+1)};$$

and a_{n+2k+1} is equal to the product of this into $\frac{2}{\pi}$.

Thus finally $P_n(\cos\theta)$

$$=\frac{4}{\pi}\cdot\frac{2.4\ldots 2n}{1.3\ldots(2n+1)}\left\{\sin(n+1)\theta+\frac{1.(n+1)}{1.(2n+3)}\sin(n+3)\theta\right.$$

$$+\frac{1.3.(n+1)(n+2)}{1.2.(2n+3)(2n+5)}\sin(n+5)\theta$$

$$\left.+\frac{1.3.5(n+1)(n+2)(n+3)}{1.2.3(2n+3)(2n+5)(2n+7)}\sin(n+7)\theta+\ldots\right\}.$$

The value of $\int_0^\pi P_n(\cos\theta)\sin(n+2k+1)\theta\,d\theta$ can be put in the form

$$\frac{\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\ldots\left(1+\frac{n}{k}\right)}{\left(k+\frac{1}{2}\right)\left(1+\frac{3}{2k}\right)\left(1+\frac{5}{2k}\right)\ldots\left(1+\frac{2n+1}{2k}\right)};$$

thus we see that it is less than $\frac{1}{k+\frac{1}{2}}$, and is therefore indefinitely small when k is indefinitely large.

40. In the general formula of the preceding Article for $P_n(\cos\theta)$ put $n=0$; thus

$$1=\frac{4}{\pi}\left\{\sin\theta+\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta+\ldots\right\}.$$

Again, in the same formula put $n=1$; thus

$$\cos\theta=\frac{2}{\pi}\left\{\frac{4}{3}\sin 2\theta+\frac{8}{15}\sin 4\theta+\frac{12}{35}\sin 6\theta+\ldots\right\}.$$

These results are well known: see *Integral Calculus*, Arts. 311 and 312.

41. We shall now shew that the roots of the equation $P_n(x)=0$ are all real and unequal, and comprised between the limits -1 and $+1$.

I. Suppose *n even.* By Art. 30 we have $\int_{-1}^{1} P_n dx = 0$. Hence P_n must change sign once at least between $x = -1$ and $x = 1$.

Let a denote a value of x at which a change of sign takes place. Then since $P_n(-x) = P_n(x)$ it follows that $P_n(x) = (x^2 - a^2) Y_{n-2}$, where Y_{n-2} is a rational integral function of x of the degree $n - 2$.

Again, by Art. 30, we have $\int_{-1}^{1} (x^2 - a^2) P_n dx = 0$; therefore $\int_{-1}^{1} (x^2 - a^2)^2 Y_{n-2} dx = 0$. Hence Y_{n-2} must change sign once at least between $x = -1$ and $x = 1$. Then, as before, we see that $Y_{n-2} = (x^2 - b^2) Z_{n-4}$, where Z_{n-4} is a rational integral function of x of the degree $n - 4$.

Proceeding in this way we obtain finally

$$P_n = A(x^2 - a^2)(x^2 - b^2)(x^2 - c^2) \ldots,$$

where the number of the factors $x^2 - a^2$, $x^2 - b^2$, $x^2 - c^2, \ldots$ is $\frac{n}{2}$, and A is some numerical coefficient, since P_n is of the degree n.

Thus we see that the equation $P_n(x) = 0$ has n roots lying between -1 and $+1$.

We have still to shew that the factors of P_n are all different. If possible suppose that two of them are alike, so that $P_n = (x^2 - a^2)^2 Z_{n-4}$. By Art. 30 we have $\int_{-1}^{1} P_n Z_{n-4} dx = 0$, so that $\int_{-1}^{1} \frac{P_n^2}{(x^2 - a^2)^2} dx = 0$; but this is obviously impossible. Hence the factors of P_n must be all different.

II. Suppose *n odd.* In this case $P_n(0) = 0$. By Art. 30 we have $\int_{-1}^{1} x P_n dx = 0$; and since $P_n(-x) = -P_n(x)$ it follows that $x P_n(x)$ must change sign once at least between $x = -1$

and $x=1$. Let a denote a value of x at which a change of sign takes place. Then since $\frac{P_n(x)}{x}$ involves only even powers of x, it follows that $P_n(x)=x(x^2-a^2)Y_{n-3}$, where Y_{n-3} is a rational integral function of x of the degree $n-3$.

Again, by Art. 30, we have $\int_{-1}^{1} x(x^2-a^2)P_n dx=0$; therefore $\int_{-1}^{1} x^2(x^2-a^2)^2 Y_{n-3}dx=0$. Hence Y_{n-3} must change sign once at least between $x=-1$ and $x=1$. Then, as before, we see that $Y_{n-3}=(x^2-b^2)Z_{n-5}$, where Z_{n-5} is a rational integral function of x of the degree $n-5$.

Proceeding in this way we obtain finally

$$P_n=Ax(x^2-a^2)(x^2-b^2)(x^2-c^2)\ldots,$$

where the number of the factors x^2-a^2, x^2-b^2, $x^2-c^2, \ldots$ is $\frac{n-1}{2}$, and A is some numerical coefficient, since P_n is of the degree n.

Thus we see that the equation $P_n(x)=0$ has n roots lying between -1 and $+1$.

In the same manner as in I we may shew that the factors of P_n are all different.

42. Since the roots of the equation $P_n(x)=0$ are all comprised between -1 and $+1$, it is obvious that $P_n(x)$ can never vanish when x is numerically greater than unity. This can also be readily inferred from some of the expressions previously given for $P_n(x)$.

Thus in Art. 17 if ξ be expressed in terms of x, and reductions effected, we obtain only powers and products of x and x^2-1 with positive numerical factors; so that the whole is necessarily positive when x is positive and greater than unity. And as $P_n(-x)=(-1)^n P_n(x)$ it follows that $P_n(x)$ will not vanish when x is negative and numerically greater than unity.

The same conclusion may also be deduced from Art. 24.

43. Take the equation $\Sigma \alpha^n P_n = (1 - 2\alpha x + \alpha^2)^{-\frac{1}{2}}$, where Σ denotes a summation with respect to n from 0 to ∞; put $\frac{p}{\sqrt{(1+k^2x^2)}}$ for α, and suppose p numerically less than unity, so as to ensure a convergent series. Thus

$$\Sigma \frac{p^n P_n}{(1+k^2x^2)^{\frac{n}{2}}} = \left\{1 - \frac{2px}{\sqrt{(1+k^2x^2)}} + \frac{p^2}{1+k^2x^2}\right\}^{-\frac{1}{2}}.$$

Assume $\frac{x}{\sqrt{(1+k^2x^2)}} = y$; then $x^2 = \frac{y^2}{1-k^2y^2}$,

and
$$1 - k^2y^2 = \frac{1}{1+k^2x^2}.$$

Hence
$$\Sigma \frac{p^n P_n}{(1+k^2x^2)^{\frac{n}{2}}} = \{1 - 2py + p^2(1-k^2y^2)\}^{-\frac{1}{2}}$$
$$= \frac{k}{\{1+k^2+p^2k^2 - (1+pk^2y)^2\}^{\frac{1}{2}}},$$

and
$$dy = \frac{dx}{(1+k^2x^2)^{\frac{3}{2}}},$$

therefore
$$\Sigma \frac{p^n P_n\, dx}{(1+k^2x^2)^{\frac{n+3}{2}}} = \frac{k\, dy}{\{1+k^2+p^2k^2 - (1+pk^2y)^2\}^{\frac{1}{2}}}.$$

By integration we have

$$\Sigma p^n \int \frac{P_n\, dx}{(1+k^2x^2)^{\frac{n+3}{2}}} = \frac{1}{pk} \sin^{-1} \frac{1+pk^2y}{\sqrt{(1+k^2+p^2k^2)}} = \frac{\phi}{pk} \text{ say.}$$

Take the integral with respect to x between the limits -1 and 1; the corresponding limits with respect to y are $-\frac{1}{\sqrt{(1+k^2)}}$, and $\frac{1}{\sqrt{(1+k^2)}}$.

In order to simplify the expression on the right-hand side of the equation let $\tan A = k$, and $\tan B = \frac{pk}{\sqrt{(1+k^2)}}$. Therefore

$$\cos A = \frac{1}{\sqrt{(1+k^2)}}, \text{ and } \cos B = \frac{\sqrt{(1+k^2)}}{\sqrt{(1+k^2+p^2k^2)}};$$

therefore

$$\frac{1 \pm \dfrac{pk^2}{\sqrt{(1+k^2)}}}{\sqrt{(1+k^2+p^2k^2)}} = \cos A \cos B\,(1 \pm \tan A \tan B) = \cos(A \mp B).$$

Thus the value of ϕ at the upper limit is $\frac{\pi}{2} - A + B$, and at the lower limit $\frac{\pi}{2} - A - B$. Hence

$$\Sigma p^n \int_{-1}^{1} \frac{P_n\,dx}{(1+k^2x^2)^{\frac{n+2}{2}}} = \frac{1}{pk}\left\{\frac{\pi}{2} - A + B - \left(\frac{\pi}{2} - A - B\right)\right\}$$

$$= \frac{2B}{pk} = \frac{2}{pk}\tan^{-1}\frac{pk}{\sqrt{(1+k^2)}}.$$

Expand $\tan^{-1}\dfrac{pk}{\sqrt{(1+k^2)}}$ in powers of $\dfrac{pk}{\sqrt{(1+k^2)}}$; thus

$$\Sigma p^n \int_{-1}^{1} \frac{P_n\,dx}{(1+k^2x^2)^{\frac{n+2}{2}}} = \Sigma\,(-1)^n \frac{2p^{2n}k^{2n}}{(2n+1)\,(1+k^2)^{\frac{2n+1}{2}}},$$

where both summations extend from $n=0$ to $n=\infty$.

Hence equating the coefficients of the powers of p we see that $\displaystyle\int_{-1}^{1} \frac{P_n\,dx}{(1+k^2x^2)^{\frac{n+2}{2}}}$ is zero if n be odd, and is equal to $\dfrac{2\,(-1)^{\frac{n}{2}}\,k^n}{(n+1)\,(1+k^2)^{\frac{n+1}{2}}}$ if n be even.

CHAPTER IV.

THE COEFFICIENTS EXPRESSED BY DEFINITE INTEGRALS.

44. LET a and b denote real quantities of which a is positive and greater than b; then will

$$\int_0^\pi \frac{d\phi}{a + b\cos\phi} = \frac{\pi}{\sqrt{(a^2 - b^2)}} \quad \text{..............(1).}$$

For we may assume $\frac{b}{a} = \frac{2c}{1+c^2}$, where c is less than unity;

thus $$\int_0^\pi \frac{d\phi}{a + b\cos\phi} = \frac{1}{a}\int_0^\pi \frac{d\phi}{1 + \frac{2c}{1+c^2}\cos\phi}$$

$$= \frac{1+c^2}{a}\int_0^\pi \frac{d\phi}{1 + c^2 + 2c\cos\phi} = \frac{1+c^2}{a}\cdot\frac{\pi}{1-c^2},$$

by *Integral Calculus*, Art. 296.

And $$\frac{\pi}{\sqrt{(a^2 - b^2)}} = \frac{\pi}{a\sqrt{\left(1 - \frac{b^2}{a^2}\right)}} = \frac{\pi(1+c^2)}{a(1-c^2)}.$$

Thus (1) is established.

Now in (1) put $a = 1 - \alpha x$, and $b = \alpha\sqrt{(x^2 - 1)}$. We may suppose x positive and greater than unity, and α negative, so that a and b are both real and a is positive; moreover $a^2 - b^2 = 1 - 2\alpha x + \alpha^2$, which is positive.

Therefore from (1) we get

$$\frac{1}{\pi}\int_0^\pi \frac{d\phi}{1 - ax + a\sqrt{(x^2-1)}\cos\phi} = \frac{1}{(1-2ax+a^2)^{\frac{1}{2}}}.$$

Hence expanding both sides in ascending powers of a, and equating the coefficients of a^n, we have, by the definition of Art. 6,

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \{x - \sqrt{(x^2-1)}\cos\phi\}^n d\phi \ldots\ldots\ldots(2).$$

Thus $P_n(x)$ is expressed as a definite integral. This formula is due to Laplace, *Mécanique Céleste*, Livre XI, Chapitre II.

45. In obtaining equation (2) of the preceding Article we found it convenient to suppose x positive and greater than unity; but it is obvious from the nature of the result that it is true for all values of x. For if $\{x - \sqrt{(x^2-1)}\cos\phi\}^n$ is expanded, and the terms integrated between the limits 0 and π, then all the terms which involve odd powers of $\sqrt{(x^2-1)}$ will vanish. Hence we obtain finally a rational integral function of x, and as this is identical with $P_n(x)$ when x is positive and greater than unity, it must be identical with $P_n(x)$ for all values of x.

46. The definite integral in Art. 44 can easily be made to reproduce some of our former expansions.

For example $\frac{1}{\pi}\int_0^\pi \{x - \sqrt{(x^2-1)}\cos\phi\}^n d\phi$

$$= \frac{1}{\pi}\int_0^\pi \left\{x^n - nx^{n-1}\sqrt{(x^2-1)}\cos\phi + \frac{n(n-1)}{\underline{|2}}x^{n-2}(x^2-1)\cos^2\phi - \ldots\right\} d\phi.$$

As we have said in Art. 45 the *odd* powers of $\sqrt{(x^2-1)}$ will disappear from this expression, so that it reduces to

$$\frac{1}{\pi}\int_0^\pi \left\{x^n + \frac{n(n-1)}{\underline{|2}}x^{n-2}(x^2-1)\cos^2\phi + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}}x^{n-4}(x^2-1)^2\cos^4\phi + \ldots\right\} d\phi.$$

Thus by the *Integral Calculus*, Art. 35, we obtain

$$P_n = x^n + \frac{n(n-1)}{2^2} x^{n-2}(x^2-1) + \frac{n(n-1)(n-2)(n-3)}{2^2 . 4^2} x^{n-4}(x^2-1)^2 + \dots.$$

This coincides with Art. 24.

47. It is obvious from the preceding Article that we may also take

$$P_n = \frac{1}{\pi}\int_0^\pi \{x + \sqrt{(x^2-1)}\cos\phi\}^n \, d\phi,$$

for this is really identical with equation (2) of Art. 44 when the expansion and integration are effected.

48. We will now give another example of the use of the definite integral. We have $x + \sqrt{(x^2-1)}\cos\phi$

$$= \left\{\sqrt{\frac{x+1}{2}} + \sqrt{\frac{x-1}{2}} e^{\phi\iota}\right\}\left\{\sqrt{\frac{x+1}{2}} + \sqrt{\frac{x-1}{2}} e^{-\phi\iota}\right\},$$

where ι is put for $\sqrt{-1}$. Thus if $\tau = \sqrt{\dfrac{x-1}{x+1}}$ we have

$$\{x + \sqrt{(x^2-1)}\cos\phi\}^n = \left(\frac{x+1}{2}\right)^n (1 + \tau e^{\phi\iota})^n (1 + \tau e^{-\phi\iota})^n.$$

By expanding and multiplying out we can arrange the product $(1 + \tau e^{\phi\iota})^n (1 + \tau e^{-\phi\iota})^n$ in the form

$$a_0 + a_1 \cos\phi + a_2 \cos 2\phi + \dots,$$

and thus when we integrate with respect to ϕ from 0 to π every term vanishes except the first; therefore

$$P_n = \left(\frac{x+1}{2}\right)^n a_0;$$

and $a_0 = 1 + n^2\tau^2 + \left\{\dfrac{n(n-1)}{1 . 2}\right\}^2 \tau^4 + \dots.$

This coincides with Art. 22.

49. We will now shew that the definite integral obtained in Art. 44 may be transformed when x is positive and greater than unity so as to give the formula

$$P_n = \frac{1}{\pi}\int_0^\pi \frac{d\psi}{\{x+\sqrt{(x^2-1)}\cos\psi\}^{n+1}}.$$

For assume a new variable ψ connected with ϕ by the relation

$$\cos\phi = \frac{x\cos\psi + \sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}\cos\psi},$$

which leads to

$$\sin\phi = \frac{\sin\psi}{x+\sqrt{(x^2-1)}\cos\psi},$$

$$x-\sqrt{(x^2-1)}\cos\phi = \frac{1}{x+\sqrt{(x^2-1)}\cos\psi},$$

$$d\phi = \frac{d\psi}{x+\sqrt{(x^2-1)}\cos\psi},$$

Since x is supposed greater than unity $x+\sqrt{(x^2-1)}\cos\psi$ can never vanish, and it is always positive, as x is supposed positive: thus as ψ continually increases from 0 to π we have ϕ also continually increasing from 0 to π. Hence

$$\int_0^\pi \{x-\sqrt{(x^2-1)}\cos\phi\}^n\, d\phi = \int_0^\pi \frac{d\psi}{\{x+\sqrt{(x^2-1)}\cos\psi\}^{n+1}}.$$

50. Suppose $x=\cos\theta$; then by equation (2) of Art. 44 we have

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^\pi (\cos\theta - \iota\sin\theta\cos\phi)^n\, d\phi;$$

this expression for $P_n(\cos\theta)$ involves the imaginary symbol ι.

Dirichlet however has expressed $P_n(\cos\theta)$ by means of definite integrals in which the imaginary symbol does not occur; and we now proceed to his investigation.

We have $\quad \dfrac{1}{\sqrt{(1-2\alpha\cos\theta+\alpha^2)}} = \Sigma\alpha^n P_n(\cos\theta),$

where Σ denotes a summation from $n=0$ to $n=\infty$.

Let $\alpha = \cos\phi + \iota \sin\phi$; then $\Sigma\alpha^n P_n(\cos\theta)$ takes the form $H + \iota K$, where

$$H = \Sigma \cos n\phi P_n(\cos\theta), \quad K = \Sigma \sin n\phi P_n(\cos\theta) \ldots\ldots(1).$$

We must now separate $\dfrac{1}{\sqrt{(1 - 2e^{\iota\phi}\cos\theta + e^{2\iota\phi})}}$ into its real and imaginary parts. We have $1 - 2e^{\iota\phi}\cos\theta + e^{2\iota\phi}$

$$= e^{\iota\phi}(e^{\iota\phi} + e^{-\iota\phi}) - 2e^{\iota\phi}\cos\theta = 2e^{\iota\phi}(\cos\phi - \cos\theta).$$

We suppose both θ and ϕ to lie between 0 and π.

If θ is greater than ϕ then $\sqrt{(\cos\phi - \cos\theta)}$ is real; thus

$$\frac{1}{\sqrt{(1 - 2e^{\iota\phi}\cos\theta + e^{2\iota\phi})}} = \frac{e^{-\frac{\iota\phi}{2}}}{\sqrt{2(\cos\phi - \cos\theta)}} = \frac{\cos\frac{\phi}{2} - \iota\sin\frac{\phi}{2}}{\sqrt{2(\cos\phi - \cos\theta)}}.$$

If θ is less than ϕ then $\sqrt{(\cos\theta - \cos\phi)}$ is real; and if we multiply the numerator and the denominator of the fraction already obtained by ι, that is by $e^{\frac{\iota\pi}{2}}$, we obtain

$$\frac{1}{\sqrt{(1 - 2e^{\iota\phi}\cos\theta + e^{2\iota\phi})}} = \frac{e^{\frac{\iota(\pi-\phi)}{2}}}{\sqrt{2(\cos\theta - \cos\phi)}} = \frac{\sin\frac{\phi}{2} + \iota\cos\frac{\phi}{2}}{\sqrt{2(\cos\theta - \cos\phi)}}.$$

Hence we deduce that

$$H = \frac{\cos\frac{\phi}{2}}{\sqrt{2(\cos\phi - \cos\theta)}} \text{ when } \theta \text{ is greater than } \phi,$$

$$\text{and } = \frac{\sin\frac{\phi}{2}}{\sqrt{2(\cos\theta - \cos\phi)}} \text{ when } \theta \text{ is less than } \phi;$$

$$K = -\frac{\sin\frac{\phi}{2}}{\sqrt{2(\cos\phi - \cos\theta)}} \text{ when } \theta \text{ is greater than } \phi,$$

$$\text{and } = \frac{\cos\frac{\phi}{2}}{\sqrt{2(\cos\theta - \cos\phi)}} \text{ when } \theta \text{ is less than } \phi.$$

Now from the equation $H = \Sigma \cos n\phi P_n(\cos\theta)$ we obtain

$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\pi H \cos n\phi \, d\phi,$$

for every positive integral value of n, except when n is zero, and then we have

$$P_0 = \frac{1}{\pi}\int_0^\pi H \, d\phi.$$

Again, from the equation $K = \Sigma \sin n\phi P_n(\cos\theta)$ we obtain in like manner

$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\pi K \sin n\phi \, d\phi$$

for every positive integral value of n, excluding zero.

Hence with the values which have been already obtained for H and K we have

$$P_n(\cos\theta) =$$

$$\frac{2}{\pi}\int_0^\theta \frac{\cos n\phi \cos\frac{\phi}{2}}{\sqrt{2(\cos\phi - \cos\theta)}} d\phi + \frac{2}{\pi}\int_\theta^\pi \frac{\cos n\phi \sin\frac{\phi}{2}}{\sqrt{2(\cos\theta - \cos\phi)}} d\phi \ldots (2);$$

this holds for every positive integral value of n, except when $n = 0$, and then only half the expression on the right-hand side must be taken:

and $P_n(\cos\theta) =$

$$-\frac{2}{\pi}\int_0^\theta \frac{\sin n\phi \sin\frac{\phi}{2}}{\sqrt{2(\cos\phi - \cos\theta)}} d\phi + \frac{2}{\pi}\int_\theta^\pi \frac{\sin n\phi \cos\frac{\phi}{2}}{\sqrt{2(\cos\theta - \cos\phi)}} d\phi \ldots (3);$$

this holds for every positive integral value of n, excluding zero.

The formulæ (2) and (3) are Dirichlet's expressions for $P_n(\cos\theta)$ by means of definite integrals.

51. Multiply the first of equations (1) of the preceding Article by $\sin\frac{\phi}{2}$, and the second by $\cos\frac{\phi}{2}$, and add, using the values obtained for H and K: thus we get

$$\Sigma \sin \frac{2n+1}{2} \phi P_n (\cos \theta) = 0 \text{ when } \theta \text{ is greater than } \phi,$$

$$\text{and} = \frac{1}{\sqrt{2 (\cos \theta - \cos \phi)}} \text{ when } \theta \text{ is less than } \phi.$$

Again, multiply the first of equations (1) by $\cos \frac{\phi}{2}$, and the second by $- \sin \frac{\phi}{2}$, and add, using the values obtained for H and K: thus we get

$$\Sigma \cos \frac{2n+1}{2} \phi P_n (\cos \theta)$$

$$= 0 \text{ when } \theta \text{ is less than } \phi,$$

$$\text{and} = \frac{1}{\sqrt{2 (\cos \phi - \cos \theta)}} \text{ when } \theta \text{ is greater than } \phi.$$

52. From equations (2) and (3) of Art. 50 we have by addition and subtraction respectively:

$$P_n (\cos \theta) =$$

$$\frac{1}{\pi} \int_0^\theta \frac{\cos \frac{2n+1}{2} \phi}{\sqrt{2 (\cos \phi - \cos \theta)}} d\phi + \frac{1}{\pi} \int_\theta^\pi \frac{\sin \frac{2n+1}{2} \phi}{\sqrt{2 (\cos \theta - \cos \phi)}} d\phi,$$

$$0 = \int_0^\theta \frac{\cos \frac{2n-1}{2} \phi}{\sqrt{2 (\cos \phi - \cos \theta)}} d\phi - \int_\theta^\pi \frac{\sin \frac{2n-1}{2} \phi}{\sqrt{2 (\cos \theta - \cos \phi)}} d\phi;$$

these hold for all positive integral values of n, including zero in the first formula, but excluding it in the second.

53. The investigation of Art. 50 is not quite satisfactory owing to the substitution of an imaginary symbol for α; hence it is advisable to verify the equations (2) and (3) of that Article. We begin with equation (2).

Let the first integral which occurs in (2) be denoted by A_n and the second by B_n; we shall shew that $\Sigma \alpha^n (A_n + B_n)$ is equal to $(1 - 2\alpha \cos \theta + \alpha^2)^{-\frac{1}{2}}$, which amounts to shewing that $A_n + B_n = P_n (\cos \theta)$.

In the first place A_n is finite; for

$$A_n = \frac{2}{\pi}\int_0^\theta \frac{\cos n\phi \cos\frac{\phi}{2}}{\sqrt{2}\,(\cos\phi - \cos\theta)}\,d\phi = \frac{1}{\pi}\int_0^\theta \frac{\cos n\phi \cos\frac{\phi}{2}}{\sqrt{\left(\sin^2\frac{\theta}{2} - \sin^2\frac{\phi}{2}\right)}}\,d\phi.$$

Now as $\cos\frac{\phi}{2}$ retains the same sign within the range of the integration we know by the *Integral Calculus*, Art. 40, that

$$A_n = \frac{\gamma}{\pi}\int_0^\theta \frac{\cos\frac{\phi}{2}}{\sqrt{\left(\sin^2\frac{\theta}{2} - \sin^2\frac{\phi}{2}\right)}}\,d\phi,$$

where γ is some value assumed by $\cos n\phi$ within the range of integration. Hence the value of A_n is less than

$$\frac{1}{\pi}\int_0^\theta \frac{\cos\frac{\phi}{2}}{\sqrt{\left(\sin^2\frac{\theta}{2} - \sin^2\frac{\phi}{2}\right)}}\,d\phi,$$

that is, less than unity; so that A_n is finite.

Since A_n is less than unity the series of which $\alpha^n A_n$ is the general term is convergent if α is numerically less than unity. This series, putting for A_n its value, is

$$\frac{1}{\pi}\int_0^\theta \frac{\cos\frac{\phi}{2}}{\sqrt{\left(\sin^2\frac{\theta}{2} - \sin^2\frac{\phi}{2}\right)}}\left\{\frac{1}{2} + \alpha\cos\phi + \alpha^2\cos 2\phi + \alpha^3\cos 3\phi \ldots\right\}d\phi.$$

Now the sum of the infinite series between the brackets is known by *Plane Trigonometry*, Art. 333, to be

$$\frac{1}{2}\,\frac{1-\alpha^2}{1-2\alpha\cos\phi+\alpha^2}.$$

Thus $$\Sigma\,\alpha^n A_n = \frac{1-\alpha^2}{2\pi}\int_0^\theta \frac{\cos\frac{\phi}{2}}{\sqrt{\left(\sin^2\frac{\theta}{2} - \sin^2\frac{\phi}{2}\right)}}\cdot\frac{d\phi}{1-2\alpha\cos\phi+\alpha^2}.$$

Assume $\sin\frac{\phi}{2} = \sin\frac{\theta}{2}\sin\psi$; then

$$\frac{\cos\frac{\phi}{2}\,d\phi}{\sqrt{\left(\sin^2\frac{\theta}{2} - \sin^2\frac{\phi}{2}\right)}} = 2d\psi,$$

and $$1 - 2\alpha\cos\phi + \alpha^2 = (1-\alpha)^2 + 4\alpha\sin^2\frac{\theta}{2}\sin^2\psi$$

$$= (1-\alpha)^2\cos^2\psi + \left\{(1-\alpha)^2 + 4\alpha\sin^2\frac{\theta}{2}\right\}\sin^2\psi$$

$$= (1-\alpha)^2\cos^2\psi + (1 - 2\alpha\cos\theta + \alpha^2)\sin^2\psi.$$

Hence $$\Sigma\,\alpha^n A_n = \frac{1-\alpha^2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\psi}{(1-\alpha)^2\cos^2\psi + (1-2\alpha\cos\theta+\alpha^2)\sin^2\psi}$$

$$= \frac{1-\alpha^2}{\pi}\cdot\frac{1}{\sqrt{(1-\alpha)^2(1-2\alpha\cos\theta+\alpha^2)}}\cdot\frac{\pi}{2} = \frac{1+\alpha}{2\sqrt{(1-2\alpha\cos\theta+\alpha^2)}}.$$

Next consider $\Sigma\,\alpha^n B_n$. We have

$$B_n = \frac{2}{\pi}\int_\theta^\pi \frac{\cos n\phi\sin\frac{\phi}{2}}{\sqrt{2}\,(\cos\theta - \cos\phi)}\,d\phi;$$

by changing ϕ into $\pi - \phi'$ we obtain

$$B_n = (-1)^n\frac{2}{\pi}\int_0^{\pi-\theta} \frac{\cos n\phi'\cos\frac{\phi'}{2}}{\sqrt{2}\,(\cos\phi' - \cos(\pi-\theta))}\,d\phi'.$$

Hence $(-1)^n B_n$ is the same function of $\pi - \theta$ as A_n is of θ; and thus $\Sigma\alpha^n B_n$ can be obtained from $\Sigma\alpha^n A_n$ by changing θ into $\pi - \theta$ and α into $-\alpha$. Hence

$$\Sigma\alpha^n B_n = \frac{1-\alpha}{2\sqrt{(1-2\alpha\cos\theta+\alpha^2)}}.$$

Therefore $$\Sigma\alpha^n(A_n + B_n) = \frac{1}{\sqrt{(1-2\alpha\cos\theta+\alpha^2)}},$$

which was to be shewn.

We shall next verify the equation (3) of Art. 50.

Let the first integral which occurs in (3) be denoted by C_n, and the second by E_n; then as the equation is asserted to hold for all positive integral values of n except zero, and that $a^0 P_0 = 1$, we must shew that

$$\Sigma a^n (C_n + E_n) = (1 - 2a\cos\theta + a^2)^{-\frac{1}{2}} - 1;$$

the summation extending from $n = 1$ to $n = \infty$.

We can shew as before that the series of which the general term is $a^n C_n$ is convergent when a is less than unity. This series, putting for C_n its value, is

$$-\frac{1}{\pi}\int_0^\theta \frac{\sin\frac{\phi}{2}}{\sqrt{\left(\sin^2\frac{\theta}{2} - \sin^2\frac{\phi}{2}\right)}} \{a\sin\phi + a^2\sin 2\phi + a^3\sin 3\phi + \ldots\} d\phi.$$

Now the sum of the infinite series between the brackets is known by *Plane Trigonometry*, Art. 333, to be

$$\frac{a\sin\phi}{1 - 2a\cos\phi + a^2}.$$

Thus $$\Sigma a^n C_n = -\frac{a}{\pi}\int_0^\theta \frac{\sin\frac{\phi}{2}}{\sqrt{\left(\sin^2\frac{\theta}{2} - \sin^2\frac{\phi}{2}\right)}} \cdot \frac{\sin\phi\, d\phi}{1 - 2a\cos\phi + a^2}.$$

Assume $\sin\frac{\phi}{2} = \sin\frac{\theta}{2}\sin\psi$; thus

$$\Sigma a^n C_n = -\frac{4a}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2\frac{\phi}{2}\, d\psi}{1 - 2a\cos\phi + a^2}.$$

But $4a\sin^2\frac{\phi}{2} = 1 - 2a\cos\phi + a^2 - (1-a)^2$; so that

$$\Sigma a^n C_n = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \left\{-1 + \frac{(1-a)^2}{1 - 2a\cos\phi + a^2}\right\} d\psi,$$

and thus, by the aid of what has already been given, we have

$$\Sigma a^n C_n = -\frac{1}{2} + \frac{1-a}{2\sqrt{(1-2a\cos\theta + a^2)}}.$$

We may deduce the value of $\Sigma a^n E_n$ from that of $\Sigma a^n C_n$ in the same way as we deduced the value of $\Sigma a^n B_n$ from that of $\Sigma a^n A_n$, namely by changing θ into $\pi - \theta$ and a into $-a$. Thus

$$\Sigma a^n E_n = -\frac{1}{2} + \frac{1+a}{2\sqrt{(1-2a\cos\theta + a^2)}}.$$

Therefore $\Sigma a^n (C_n + E_n) = -1 + \dfrac{1}{\sqrt{(1-2a\cos\theta + a^2)}}$,

which was to be shewn.

CHAPTER V.

DIFFERENTIAL EQUATION WHICH IS SATISFIED BY LEGENDRE'S COEFFICIENTS.

54. LET $$V = \frac{1}{\sqrt{(1-2ax+a^2)}},$$

then $$\frac{dV}{dx} = \frac{a}{(1-2ax+a^2)^{\frac{3}{2}}} = aV^3,$$

$$\frac{dV}{da} = \frac{x-a}{(1-2ax+a^2)^{\frac{3}{2}}} = (x-a)\,V^3;$$

hence $$\frac{d^2V}{dx^2} = 3aV^2\frac{dV}{dx} = 3a^2V^5,$$

$$\frac{d^2V}{da^2} = -V^3 + 3(x-a)\,V^2\frac{dV}{da} = -V^3 + 3(x-a)^2V^5.$$

Therefore

$$(1-x^2)\frac{d^2V}{dx^2} + a^2\frac{d^2V}{da^2} = V^3\{3a^2(1-x^2)\,V^2 - a^2 + 3a^2(x-a)^2V^2\};$$

and $$3a^2(1-x^2) + 3a^2(x-a)^2 = 3a^2(1-2ax+a^2) = \frac{3a^2}{V^2}:$$

thus $$(1-x^2)\frac{d^2V}{dx^2} + a^2\frac{d^2V}{da^2} = 2a^2V^3.$$

Also $$2x\frac{dV}{dx} - 2a\frac{dV}{da} = 2a^2V^3.$$

Therefore, by subtraction,

$$(1-x^2)\frac{d^2V}{dx^2}-2x\frac{dV}{dx}+a^2\frac{d^2V}{da^2}+2a\frac{dV}{da}=0;$$

this may also be written thus:

$$\frac{d}{dx}\left\{(1-x^2)\frac{dV}{dx}\right\}+\frac{d}{da}\left\{a^2\frac{dV}{da}\right\}=0 \text{............} (1).$$

By definition we have $V=\Sigma a^n P_n$; substitute the value of V in (1), and equate to zero the coefficient of a^n: thus

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}+n(n+1)P_n=0 \text{............} (2).$$

This shews that Legendre's n^{th} Coefficient must satisfy the differential equation (2), which may also be written thus:

$$(1-x^2)\frac{d^2P_n}{dx^2}-2x\frac{dP_n}{dx}+n(n+1)P_n=0 \text{.........} (3).$$

55. We have shewn in Art. 41 that the roots of the equation $P_n(x)=0$ are all real and unequal, and comprised between the values -1 and $+1$. Part of this proposition may be deduced immediately from the formula

$$P_n(x)=\frac{1}{2^n \lfloor n}\frac{d^n(x^2-1)^n}{dx^n}.$$

For the roots of the equation $(x^2-1)^n=0$ are all real; namely, n of them equal to -1, and n of them equal to $+1$: hence, by the *Theory of Equations*, Art. 105, the roots of the equation $P_n(x)=0$ are all real, and comprised between the values -1 and $+1$.

Thus to complete the proposition we have only to shew that the roots of the equation $P_n(x)=0$ are all *unequal*; and this will follow from (3) of Art. 54. For we know by the *Theory of Equations*, Art. 79, that if the equation $P_n(x)=0$ has two roots equal to a, then $P_n(x)$ and $\frac{dP_n(x)}{dx}$ both vanish when $x=a$; hence from (3) it follows that $\frac{d^2P_n(x)}{dx^2}$ will also vanish when $x=a$. And proceeding in this way, and using the results obtained by successive differentiation of (3), we should find that all the differential coefficients of $P_n(x)$ down to $\frac{d^nP_n(x)}{dx^n}$ vanish when $x=a$. But this is impossible; for we know by Art. 8 that $\frac{d^nP_n(x)}{dx^n}=1.3.5\ldots(2n-1)$; and so it does not vanish.

56. The following relation holds between three successive Coefficients of Legendre:

$$(n+1)P_{n+1}-(2n+1)xP_n+nP_{n-1}=0.$$

For it appears from the process of Art. 54 that

$$(1-2\alpha x+\alpha^2)\frac{dV}{d\alpha}+(\alpha-x)V=0.$$

Put for V its value $\Sigma\alpha^nP_n$, and then equate to zero the coefficient of α^n; thus we obtain

$$(n+1)P_{n+1}-2nxP_n+(n-1)P_{n-1}+P_{n-1}-xP_n=0,$$

that is, $\qquad (n+1)P_{n+1}-(2n+1)xP_n+nP_{n-1}=0\ldots\ldots(4).$

57. From equation (4) by changing n into $n-1$ we obtain

$$nP_n-(2n-1)xP_{n-1}+(n-1)P_{n-2}=0,$$

and then we may again change n into $n-1$, and so on.

From the equations thus obtained we see that $P_n, P_{n-1}\ldots$ constitute a series of terms which possess the same essential properties as *Sturm's Functions*; see *Theory of Equations*, Chapter XIV. These properties are that no two consecutive terms of the series can simultaneously vanish, and that when one term vanishes the preceding and succeeding terms have contrary signs. Moreover when $x=1$ all the terms are positive, and when $x=-1$ the signs are determined by $P_r(-1)=(-1)^r$, so that they are alternatively positive and negative. Hence by the application of Sturm's method we obtain another demonstration of the whole theorem of Art. 41.

Also we see that between two consecutive roots of the equation $P_n(x) = 0$ there is one, and only one, root of the equation $P_{n-1}(x) = 0$. For let h and k denote two consecutive roots of the equation $P_n(x) = 0$, and suppose h the less. Then if there were no root of the equation $P_{n-1}(x) = 0$ between h and k the number of permanences of sign exhibited by the series when x is a little greater than k would be the same as the number when x is a little less than h: but this is impossible, for the former number exceeds the latter by 2. Hence there must be *one* root of the equation $P_{n-1}(x) = 0$ between h and k. And there cannot be more than one; for otherwise the whole number of roots of the equation $P_{n-1}(x) = 0$ would be greater than $n - 1$; which is impossible.

58. From equation (3) of Art. 44 we have

$$xP_{n-1} - P_{n-2}$$

$$= \frac{1}{\pi}\int_0^\pi \{x - \sqrt{(x^2-1)}\cos\phi\}^{n-2}\{x^2 - x\sqrt{(x^2-1)}\cos\phi - 1\}\,d\phi$$

$$= \frac{x^2-1}{\pi}\int_0^\pi \{x - \sqrt{(x^2-1)}\cos\phi\}^{n-2}\left\{1 - \frac{x\cos\phi}{\sqrt{(x^2-1)}}\right\}d\phi$$

$$= \frac{x^2-1}{n-1}\frac{d}{dx}P_{n-1};$$

thus $$(n-1)(xP_{n-1} - P_{n-2}) = (x^2-1)\frac{dP_{n-1}}{dx} \quad \text{.........(5)}.$$

Again from Art. 49 we have

$$xP_{n-1} - P_n = \frac{1}{\pi}\int_0^\pi \frac{x\{x + \sqrt{(x^2-1)}\cos\phi\} - 1}{\{x + \sqrt{(x^2-1)}\cos\phi\}^{n+1}}\,d\phi$$

$$= \frac{x^2-1}{\pi}\int_0^\pi \frac{1 + \frac{x}{\sqrt{(x^2-1)}}\cos\phi}{\{x + \sqrt{(x^2-1)}\cos\phi\}^{n+1}}\,d\phi$$

$$= -\frac{x^2-1}{n}\frac{d}{dx}P_{n-1};$$

thus $$n(xP_{n-1} - P_n) = -(x^2-1)\frac{dP_{n-1}}{dx} \quad \text{..........(6)}.$$

The formula in Art. 49 by the aid of which (6) has been obtained was demonstrated only for the case in which x is positive and greater than unity; but as (6) expresses an identity between certain rational integral functions of x, it is manifest that since it holds when x is positive and greater than unity it holds for all values of x.

By adding (5) and (6) we obtain

$$-nP_n + (2n-1)\,xP_{n-1} - (n-1)\,P_{n-2} = 0\,;$$

this agrees substantially with (4).

59. Other relations resembling those of the preceding Article may be obtained. Thus, take the fundamental equation

$$\frac{1}{\sqrt{(1-2ax+a^2)}} = P_0 + P_1 a + P_2 a^2 + P_3 a^3 + \ldots\,;$$

differentiate with respect to x, and then divide by a; we obtain

$$\frac{1}{(1-2ax+a^2)^{\frac{3}{2}}} = \frac{dP_1}{dx} + a\frac{dP_2}{dx} + a^2\frac{dP_3}{dx} + \ldots\ldots\ldots \quad (7).$$

Also from the fundamental equation, by differentiating with respect to a, we get

$$\frac{x-a}{(1-2ax+a^2)^{\frac{3}{2}}} = P_1 + 2P_2 a + 3P_3 a^2 + \ldots\ldots\ldots\ldots \quad (8).$$

From (7) and (8) we get

$$(x-a)\left\{\frac{dP_1}{dx} + a\frac{dP_2}{dx} + a^2\frac{dP_3}{dx} + \ldots\right\} = P_1 + 2P_2 a + 3P_3 a^2 + \ldots$$

Hence, by equating the coefficients of a^{n-1}, we get

$$x\frac{dP_n}{dx} - \frac{dP_{n-1}}{dx} = nP_n \ldots\ldots\ldots\ldots\ldots\ldots \quad (9).$$

Again from (7) we have

$$\frac{1}{\sqrt{(1-2ax+a^2)}} = (1-2ax+a^2)\left\{\frac{dP_1}{dx} + a\frac{dP_2}{dx} + a^2\frac{dP_3}{dx} + \ldots\right\}.$$

Substitute for the left-hand member its value from the fundamental equation, and then equate the coefficients of α^n; thus

$$P_n = \frac{dP_{n+1}}{dx} - 2x\frac{dP_n}{dx} + \frac{dP_{n-1}}{dx} \text{ .... (10).}$$

From (9) and (10) we have

$$P_n = \frac{dP_{n+1}}{dx} - \frac{dP_{n-1}}{dx} - 2nP_n,$$

so that
$$\frac{dP_{n+1}}{dx} - \frac{dP_{n-1}}{dx} = (2n+1)P_n \text{ ....(11).}$$

60. In equation (11) change n successively into $n-2$, $n-4, \dots$ and add the results; thus we have a new demonstration of the result obtained in Art. 38.

61. By integrating (11) we obtain

$$(2n+1)\int_{-1}^{x} P_n dx = P_{n+1} - P_{n-1} \text{ ....(12),}$$

for the right-hand member vanishes when $x = -1$, so that no constant term is required.

Similarly
$$(2n+1)\int_{x}^{1} P_n dx = P_{n-1} - P_{n+1} \text{ .... (13).}$$

62. The differential equation (2) of Art. 54 serves as the foundation of an instructive demonstration of part of the theorem of Art. 28.

For by virtue of the differential equation we have

$$-n(n+1)\int P_m P_n dx = \int P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx;$$

integrate the right-hand member by parts, and take -1 and $+1$ as the limits of the integration: thus we obtain

$$n(n+1)\int_{-1}^{1} P_m P_n dx = \int_{-1}^{1}(1-x^2)\frac{dP_m}{dx}\frac{dP_n}{dx}dx.$$

In precisely the same way we may shew that

$$m(m+1)\int_{-1}^{1} P_m P_n dx = \int_{-1}^{1} (1-x^2)\frac{dP_m}{dx}\frac{dP_n}{dx}dx.$$

Therefore $m(m+1)\int_{-1}^{1} P_m P_n dx = n(n+1)\int_{-1}^{1} P_m P_n dx.$

Hence if m and n are different we must have

$$\int_{-1}^{1} P_m P_n dx = 0.$$

If we consider the *indefinite integral* we obtain by the method of this Article

$$\{m(m+1) - n(n+1)\}\int P_m(x) P_n(x) dx$$

$$= \left\{P_m(x)\frac{dP_n(x)}{dx} - P_n(x)\frac{dP_m(x)}{dx}\right\}(1-x^2):$$

this may be immediately verified by differentiation.

From this formula we can find the value of $\int P_m(x) P_n(x) dx$ between any assigned limits; for example

$$\{m(m+1) - n(n+1)\}\int_0^1 P_m(x) P_n(x) dx$$

$$= \text{the value when } x = 0 \text{ of } \left\{P_n(x)\frac{dP_m(x)}{dx} - P_m(x)\frac{dP_n(x)}{dx}\right\}.$$

By Art. 7 the right-hand member vanishes if m and n are both odd, or both even. Put $2m$ for m and $2n-1$ for n; thus $\{2m(2m+1) - (2n-1)2n\}\int_0^1 P_{2m}(x) P_{2n-1}(x) dx$

$$= \text{the value when } x = 0 \text{ of } -\left\{P_{2m}(x)\frac{dP_{2n-1}(x)}{dx}\right\}$$

$$= (-1)^{m+n}\frac{1.3.5\ldots(2m-1)}{2.4\ldots 2m}\cdot\frac{1.3.5\ldots(2n-1)}{2.4\ldots(2n-2)}.$$

As an example we may shew from this formula that

$$\int_0^1 P_{2m}(x) P_{2m-1}(x) dx = \int_0 P_{2m}(x) P_{2m+1}(x) dx.$$

63. The differential equation (2) of Art. 54 will be modified in various ways by the transformation of the independent variable: we will notice some of these.

I. Put $x = \cos\theta$; then (2) becomes

$$\frac{d}{d\theta}\left(\sin\theta\frac{dP_n}{d\theta}\right) + n(n+1)\sin\theta P_n = 0,$$

or
$$\frac{d^2P_n}{d\theta^2} + \cot\theta\frac{dP_n}{d\theta} + n(n+1)P_n = 0.$$

II. Let $x^2 + \rho^2 = 1$; then (2) becomes

$$\sqrt{(\rho^2-1)}\frac{d}{d\rho}\left\{\rho\sqrt{(\rho^2-1)}\frac{dP_n}{d\rho}\right\} - n(n+1)\rho P_n = 0,$$

or
$$\rho.(\rho^2-1)\frac{d^2P_n}{d\rho^2} + (2\rho^2-1)\frac{dP_n}{d\rho} - n(n+1)\rho P_n = 0.$$

III. Let $2x = \xi + \xi^{-1}$; then (2) becomes

$$-\frac{2\xi^2}{\xi^2-1}\frac{d}{d\xi}\left\{\frac{\xi^2-1}{2}\frac{dP_n}{d\xi}\right\} + n(n+1)P_n = 0,$$

or
$$\xi^2(\xi^2-1)\frac{d^2P_n}{d\xi^2} + 2\xi^3\frac{dP_n}{d\xi} - n(n+1)(\xi^2-1)P_n = 0.$$

64. The differential equation may be employed to deduce various expansions of P_n; we will take one example and thus verify the expression for $P_n(\cos\theta)$ in a series of sines of multiples of θ which was obtained in Art. 39.

Assume then that

$$P_n(\cos\theta) = a_1\sin\theta + a_2\sin 2\theta + a_3\sin 3\theta + \ldots;$$

and put this value in the differential equation I of Art. 63, which may be expressed thus:

$$\sin\theta\left\{\frac{d^2P_n}{d\theta^2} + n(n+1)P_n\right\} + \cos\theta\frac{dP_n}{d\theta} = 0,$$

The term $a_m \sin m\theta$ gives rise to

$$a_m \left[\sin\theta \sin m\theta \left\{n(n+1) - m^2\right\} + m\cos\theta\cos m\theta\right],$$

that is to

$$\frac{a_m}{2}\left\{\cos(m-1)\theta - \cos(m+1)\theta\right\}\left\{n(n+1) - m^2\right\}$$
$$+ \frac{ma_m}{2}\left\{\cos(m-1)\theta + \cos(m+1)\theta\right\}.$$

The sum of all such expressions is zero by virtue of the differential equation; hence multiplying by 2, and rearranging, the following sum is zero:

$$a_1 n(n+1)$$
$$+ \;a_2\left\{n(n+1) - 2^2 + 2\right\}\cos\theta$$
$$+ \left[a_3\left\{n(n+1) - 3^2 + 3\right\} - a_1\left\{n(n+1) - 1^2 - 1\right\}\right]\cos 2\theta$$
$$+ \ldots\ldots$$
$$+ \left[a_m\left\{n(n+1) - m^2 + m\right\} - a_{m-2}\left\{n(n+1) - (m-2)^2 - (m-2)\right\}\right]\cos m\theta$$
$$+ \ldots\ldots.$$

As this must vanish for all values of θ, we find in succession $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, ... $a_n = 0$. Then when $m = n + 1$, we see that the coefficient of $\cos(n+1)\theta$ vanishes, whatever finite value a_{n+1} may have. Also a_{n+2}, a_{n+4}, a_{n+6}, ... $= 0$. And a_{n+1}, a_{n+3}, are connected by the law

$$a_m = \frac{(m-n-2)(m+n-1)}{(m-n-1)(m+n)}\, a_{m-2}.$$

Thus we obtain $P_n(\cos\theta)$

$$= a_{n+1}\left\{\sin(n+1)\theta + \frac{1\,.\,(n+1)}{1\,.\,(2n+3)}\sin(n+3)\theta\right.$$
$$\left. + \frac{1\,.\,3\,.\,(n+1)(n+2)}{1\,.\,2\,.\,(2n+3)(2n+5)}\sin(n+5)\theta + \ldots\right\}.$$

This agrees with Art. 39 as to the terms between the brackets, but leaves the value of a_{n+1} as yet undetermined. The differential equation will not enable us to determine a_{n+1}; for that equation will not be changed in form if instead of P_n we substitute the product of P_n into any constant factor. We may use the formula

$$a_{n+1} = \frac{2}{\pi}\int_0^\pi P_n(\cos\theta)\sin(n+1)\theta d\theta;$$

and since $a_{n-1} = 0$, we have

$$0 = \frac{2}{\pi}\int_0^\pi P_n(\cos\theta)\sin(n-1)\theta d\theta;$$

therefore, by subtraction,

$$a_{n+1} = \frac{4}{\pi}\int_0^\pi P_n(\cos\theta)\cos n\theta\sin\theta d\theta.$$

Now $2\cos n\theta = 2^n\cos^n\theta +$ terms involving lower powers of $\cos\theta$; hence, by Art. 30,

$$a_{n+1} = \frac{2}{\pi}\int_{-1}^1 P_n(x)\, 2^n x^n dx$$

$$= \frac{2}{\pi}\int_{-1}^1 (1-x^2)^n dx, \text{ by Art. 32,}$$

$$= \frac{2}{\pi}\int_0^\pi \sin^{2n+1}\psi d\psi = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin^{2n+1}\psi d\psi$$

$$= \frac{4}{\pi}\cdot\frac{2n(2n-2)\dots 2}{(2n+1)(2n-1)\dots 3}.$$

This agrees with Art. 39.

CHAPTER VI.

THE COEFFICIENTS OF THE SECOND KIND.

65. We have seen in Art. 54 that $P_n(x)$ satisfies a certain differential equation of the second order: according to the known theory of differential equations we infer that there must also be another solution, and this we proceed to investigate.

66. Take the differential equation

$$(1-x^2)\frac{d^2z}{dx^2}-2x\frac{dz}{dx}+n(n+1)z=0,$$

and find a solution in the form of a series proceeding according to ascending powers of x.

Assume $\qquad z=x^m+a_2x^{m+2}+a_4x^{m+4}+\ldots,$

substitute in the differential equation, and equate to zero the coefficient of x^{m+2r}. Thus we find that

$$a_{2r+2}(m+2r+2)(m+2r+1)$$
$$-a_{2r}\left\{(m+2r)(m+2r-1)+2(m+2r)-n(n+1)\right\}=0,$$

therefore $\qquad a_{2r+2}=\dfrac{(2r+m+n+1)(2r+m-n)}{(2r+m+2)(2r+m+1)}a_{2r}.$

This holds for every positive integral value of r.

But in the differential equation there will still remain the term $m(m-1)x^{m-2}$, and to make this vanish we must have either $m=0$ or $m=1$.

Take $m=0$; then the series becomes

$$1-\frac{n(n+1)}{\lfloor 2}x^2+\frac{(n-2)n(n+1)(n+3)}{\lfloor 4}x^4-\dots.$$

Take $m=1$; then the series becomes

$$x-\frac{(n-1)(n+2)}{\lfloor 3}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{\lfloor 5}x^5-\dots.$$

Now if n be even the first series consists of a finite number of terms, and the second of an infinite number; if n be odd the first series consists of an infinite number of terms, and the second of a finite number.

The series are of the kind called *hypergeometrical.* The general form of such series is

$$1+\frac{\alpha.\beta}{1.\gamma}t+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}t^2 \\ +\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}t^3+\dots,$$

and this is conveniently denoted by $F(\alpha, \beta, \gamma, t)$.

Thus the first and second series are denoted respectively by

$$F\left(-\frac{n}{2}, \frac{n+1}{2}, \frac{1}{2}, x^2\right), \text{ and } xF\left(-\frac{n-1}{2}, \frac{n+2}{2}, \frac{3}{2}, x^2\right).$$

In both series α, β, γ are such that $\alpha+\beta-\gamma=0$.

The series which is infinite is convergent if x is less than unity, but divergent if x is greater than unity or equal to unity: see *Algebra*, Art. 775.

67. We infer that of the two series obtained in the preceding Article that which is finite $= CP_n(x)$, where C is some constant. The other series furnishes, at least when x is less than unity, a second solution of the differential equation.

68. As another example we may proceed to find a solution of the differential equation of Art. 66, in a series proceeding according to descending powers of x.

Assume $\quad z = x^m + a_2 x^{m-2} + a_4 x^{m-4} + \dots,$

substitute in the differential equation and equate to zero the coefficient of x^{m-2r-2}. Thus we find that

$$a_{2r}(m-2r)(m-2r-1) -$$
$$a_{2r+2}\left\{(m-2r-2)(m-2r-3) + 2(m-2r-2) - n(n+1)\right\} = 0.$$

This holds for every positive integral value of r.

But in the differential equation there will still remain the term

$$x^m\left\{n(n+1) - m(m+1)\right\},$$

and to make this vanish we must have

$$n(n+1) - m(m+1) = 0,$$

so that either $m = n$ or $m = -n-1$.

Take $m = n$; then the series becomes

$$x^n - \frac{n(n-1)}{2.(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \dots,$$

so that it is finite, and of the form $CP_n(x)$, where C is a constant.

Take $m = -n-1$; then the series becomes

$$\frac{1}{x^{n+1}} + \frac{(n+1)(n+2)}{2.(2n+3)} \cdot \frac{1}{x^{n+3}}$$
$$+ \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} \cdot \frac{1}{x^{n+5}} + \dots;$$

and in the notation of Art. 66 this will be denoted by

$$\frac{1}{x^{n+1}}F\left(\frac{n+1}{2}, \frac{n+2}{2}, \frac{2n+3}{2}, x^{-2}\right):$$

this is an infinite series, convergent if x is greater than 1, but divergent in other cases.

If $Q_n(x)$ have the meaning assigned in Art. 37 this infinite series $= CQ_n(x)$, where C is a constant.

69. We know from Art. 63 that by assuming

$$\xi = x + \sqrt{(x^2-1)}$$

the differential equation of Art. 66 may be transformed into

$$\xi^2(1-\xi^2)\frac{d^2z}{d\xi^2} - 2\xi^3\frac{dz}{d\xi} - n(n+1)(1-\xi^2)z = 0.$$

Assume $\quad z = \xi^m + a_2\xi^{m-2} + a_4\xi^{m-4} + \ldots;$

then, by the same method as before, we shall find that

$$a_{2r+2} = \frac{(n+1+2r-m)(n+m-2r)}{(n+2+2r-m)(n+m-2r-1)}a_{2r},$$

and moreover that $m(m+1) - n(n+1) = 0$.

Thus either $m = n$ or $m = -n-1$; and we obtain two series which, expressed in the usual notation, are

$$\xi^n F\left(\frac{1}{2},\ -n,\ -\frac{2n-1}{2},\ \xi^{-2}\right),$$

and
$$\xi^{-n-1} F\left(\frac{1}{2},\ n+1,\ \frac{2n+3}{2},\ \xi^{-2}\right).$$

The former series will be found to be the product of a constant into $P_n(x)$, by comparing it with the formula given in Art. 17. Hence we infer that the product of the latter series into some constant will be equal to the $Q_n(x)$ of Art. 68; or, which is the same thing, that

$$Q_n(x) = \lambda\xi^{-n-1} F\left(\frac{1}{2},\ n+1,\ \frac{2n+3}{2},\ \xi^{-2}\right);$$

where λ is some constant.

To determine this constant we observe that according to Art. 37 we have $x^{n+1}Q_n(x) = \dfrac{\lfloor n}{1.3.5\ldots(2n+1)}$ when x is infinite. But when x is infinite

$$\lambda x^{n+1} \xi^{-n-1} F\left(\frac{1}{2},\ n+1,\ \frac{2n+3}{2},\ \xi^{-2}\right) = \frac{\lambda}{2^{n+1}};$$

therefore $$\lambda = \frac{2^{n+1} \lfloor n}{1.3.5\ldots(2n+1)}.$$

70. Hence besides the solution of the differential equation of Art. 66, which is furnished by $P_n(x)$, we have always another solution when x is either less than unity or greater than unity: namely in the former case the solution found in Art. 66; and in the latter case that found in Art. 68 or Art. 69. The second solution is presented in the form of an infinite series.

71. We may however express the second solution in a finite form. Take the differential equation

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0.$$

We know that $P_n(x)$ is a solution, so that

$$(1-x^2)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n+1)P_n = 0.$$

Let ζ denote the other solution, so that

$$(1-x^2)\frac{d^2\zeta}{dx^2} - 2x\frac{d\zeta}{dx} + n(n+1)\zeta = 0.$$

Multiply the former equation by ζ, and the latter by P_n, and subtract: thus

$$(1-x^2)\left\{P_n\frac{d^2\zeta}{dx^2} - \zeta\frac{d^2P_n}{dx^2}\right\} = 2x\left\{P_n\frac{d\zeta}{dx} - \zeta\frac{dP_n}{dx}\right\},$$

that is $$(1-x^2)\frac{d}{dx}\left\{P_n\frac{d\zeta}{dx} - \zeta\frac{dP_n}{dx}\right\} = 2x\left\{P_n\frac{d\zeta}{dx} - \zeta\frac{dP_n}{dx}\right\}.$$

Hence by integration we obtain

$$\log\left\{P_n\frac{d\zeta}{dx} - \zeta\frac{dP_n}{dx}\right\} = \text{constant} - \log(x^2-1),$$

or $$= \text{constant} - \log(1-x^2),$$

according as x is greater than unity or less than unity.

Hence, in both cases, C being a constant, we have

$$P_n \frac{d\zeta}{dx} - \zeta \frac{dP_n}{dx} = \frac{C}{x^2 - 1};$$

therefore $$\frac{d}{dx}\left(\frac{\zeta}{P_n}\right) = \frac{C}{(P_n)^2 (x^2 - 1)};$$

therefore $$\zeta = CP_n \int \frac{dx}{(P_n)^2 (x^2 - 1)}.$$

Thus we have the second solution expressed in a finite form; and by properly determining the constant C, and keeping to the former meaning of $Q_n(x)$, we shall have

$$Q_n(x) = CP_n \int \frac{dx}{(P_n)^2 (x^2 - 1)}.$$

72. The integration denoted in the formula of the preceding Article may be effected.

Let $\alpha, \beta, \gamma, \ldots$ denote the roots of the equation $P_n(x) = 0$, which we know are all real and unequal. Then by the theory of the decomposition of rational fractions explained in the *Integral Calculus*, Chapter II, we have

$$\frac{1}{(P_n)^2 (x^2 - 1)} = \frac{h}{x-1} + \frac{k}{x+1} + \Sigma \frac{A}{(x-\alpha)^2} + \Sigma \frac{A'}{x-\alpha},$$

where h, k, A, A' are constants; and Σ denotes a summation to be made by considering all the roots $\alpha, \beta, \gamma, \ldots$, which will give rise to other constants like A and A'.

We proceed to determine these constants.

We have $h = \dfrac{1}{(P_n)^2 (x+1)}$, when $x = 1$, so that $h = \dfrac{1}{2}$,

and $$k = \frac{1}{(P_n)^2 (x-1)}, \text{ when } x = -1, \text{ so that } k = -\frac{1}{2}.$$

Also $$A = \frac{(x-\alpha)^2}{(P_n)^2(x^2-1)}, \text{ when } x=\alpha,$$

and $$A' = \frac{d}{dx}\left\{\frac{(x-\alpha)^2}{(P_n)^2(x^2-1)}\right\}, \text{ when } x=\alpha.$$

We shall now shew that $A'=0$.

Let $P_n = (x-\alpha) R$, so that

$$A' = \frac{d}{dx}\left\{\frac{1}{R^2(x^2-1)}\right\}, \text{ when } x=\alpha,$$

$$= -2\,\frac{(x^2-1)\dfrac{dR}{dx}+Rx}{R^3(x^2-1)^2}, \text{ when } x=\alpha.$$

Substitute $(x-\alpha) R$ for P_n in the equation (3) of Art. 54; thus

$$(1-x^2)\left\{(x-\alpha)\frac{d^2R}{dx^2}+2\frac{dR}{dx}\right\} - 2x\left\{(x-\alpha)\frac{dR}{dx}+R\right\}$$
$$+n(n+1)(x-\alpha)R=0,$$

so that when $x=\alpha$ we have $(1-x^2)\dfrac{dR}{dx}-Rx=0$; therefore $A'=0$. Hence we have

$$Q_n(x) = CP_n(x)\int\left\{\frac{1}{2(x-1)}-\frac{1}{2(x+1)}+\Sigma\frac{A}{(x-\alpha)^2}\right\}dx.$$

Therefore if x is greater than unity we may write

$$Q_n(x) = -CP_n(x)\left\{\frac{1}{2}\log\frac{x+1}{x-1}+\Sigma\frac{A}{x-\alpha}+C_1\right\}\ \ldots.(1),$$

and if x is less than unity

$$Q_n(x) = -CP_n(x)\left\{\frac{1}{2}\log\frac{1+x}{1-x}+\Sigma\frac{A}{x-\alpha}+C_1\right\}\ \ldots.(2),$$

where C_1 denotes a constant.

We do not mean to assert that C and C_1 must have the same values when x is less than unity as when x is greater than unity; but only that C and C_1 do not change in (1) so long as x is greater than unity, and do not change in (2) so long as x is less than unity.

73. Let us suppose for example that x is greater than unity; then the right-hand member of (1) is an expression with *two* arbitrary constants, which satisfies the differential equation of Art. 66; hence it is the complete solution of that equation, and by giving suitable values to the constants will coincide with any special solution which may have been obtained. Take for example the series at the end of Art. 68. This vanishes when x is infinite. But the part between the brackets in (1) reduces to C_1 when x is infinite; hence the whole expression will not vanish unless $C_1 = 0$. Take $C_1 = 0$; then by properly determining C this expression (1) must coincide with the series at the end of Art. 68.

74. Suppose for a particular case that $n = 1$. Take $C_1 = 0$; and put

$$\log \frac{x+1}{x-1} = \log \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = 2\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \ldots\right).$$

Also in this case $\alpha = 0$, and $A = -1$.

Thus we obtain from (1)

$$Q_n(x) = -Cx\left\{\frac{1}{3x^3} + \frac{1}{5x^5} + \ldots\right\};$$

and this agrees with the result at the end of Art. 68.

75. In like manner if x is less than unity the formula (2) of Art. 72, by giving suitable values to the two arbitrary constants, will coincide with any special solution. For instance, take $n = 1$; then we get

$$-Cx\left\{\frac{1}{2}\log\frac{1+x}{1-x} - \frac{1}{x} + C_1\right\}.$$

This will coincide with the first series of Art. 66, if we put $C_1 = 0$ and expand $\log \frac{1+x}{1-x}$ in ascending powers of x.

76. We have seen that if (1) of Art. 72 is to coincide with the result of Art. 68 we must have $C_1 = 0$: it will be convenient to determine the connection between C and other constants which present themselves in our process.

Let h be a constant, and suppose that we put

$$Q_n = h\left\{\frac{1}{x^{n+1}} + \frac{(n+1)(n+2)}{2.(2n+3)} \cdot \frac{1}{x^{n+3}} + \dots\right\},$$

so that Q_n reduces to hx^{-n-1} when x is very great.

We know that $P_n = kx^n +$ terms in $x^{n-2}, x^{n-4}, \dots$;

where $$k = \frac{1.3\dots(2n-1)}{\lfloor n}.$$

By Art. 71 we have $P_n \frac{dQ_n}{dx} - Q_n \frac{dP_n}{dx} = \frac{C}{x^2-1}$,

so that when x is very great

$$-\frac{hk(2n+1)}{x^2} = \frac{C}{x^2-1},$$

and therefore $$C = -hk(2n+1).$$

For instance, if we put $C = -1$, so as to give to (1) of Art. 72 its simplest form, we have $hk(2n+1) = 1$; so that $h = \frac{1}{(2n+1)k}$. This value of h makes the Q_n of the present Article exactly coincident with the Q_n of Art. 37.

77. Taking then for simplicity $C_1 = 0$ and $C = -1$ in (1) of Art. 72, we have, when x is greater than unity,

$$Q_n(x) = P_n(x)\left\{\frac{1}{2}\log\frac{x+1}{x-1} + \Sigma\frac{A}{x-a}\right\};$$

this agrees with Art. 37, and we shall use this as the value of $Q_n(x)$ when x is greater than unity.

When x is less than unity we shall take

$$Q_n(x) = P_n(x)\left\{\frac{1}{2}\log\frac{1+x}{1-x} + \Sigma\frac{A}{x-a}\right\}.$$

78. We have then by the preceding Article, for the case in which x is greater than unity,

$$Q_n(x) = \frac{1}{2}P_n(x)\log\frac{x+1}{x-1} - R,$$

where R denotes a certain rational integral function of x of the degree $n-1$. We shall now express R in terms of Legendre's Coefficients.

Substitute this value of $Q_n(x)$ in the differential equation of Art. 66, which we know it satisfies; thus we obtain

$$\frac{1}{2}\log\frac{x+1}{x-1}\left\{(1-x^2)\frac{d^2P_n}{dx^2} - 2x\frac{dP_n}{dx} + n(n+1)P_n\right\}$$

$$+2\frac{dP_n}{dx} - \left\{(1-x^2)\frac{d^2R}{dx^2} - 2x\frac{dR}{dx} + n(n+1)R\right\} = 0.$$

By Art. 54 this reduces to

$$(1-x^2)\frac{d^2R}{dx^2} - 2x\frac{dR}{dx} + n(n+1)R = 2\frac{dP_n}{dx};$$

and therefore, by Art. 38,

$$(1-x^2)\frac{d^2R}{dx^2} - 2x\frac{dR}{dx} + n(n+1)R$$

$$= 2\left\{(2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + \dots\right\}\dots(3).$$

Assume now $R = a_1P_{n-1} + a_2P_{n-3} + a_3P_{n-5} + \dots$; where $a_1, a_2, a_3, \dots$ are constants to be determined.

When P_{n-r} is put for R in the left-hand side of (3) it reduces to $\left\{n(n+1)-(n-r)(n-r+1)\right\}P_{n-r}$, that is to $r(2n+1-r)P_{n-r}$. Hence by comparison with the right-hand side of (3) we see that if r be even a_r vanishes, and that if r be odd $a_r=\dfrac{2(2n-2r+1)}{r(2n+1-r)}$. Thus finally

$$R=\frac{2n-1}{1.n}P_{n-1}+\frac{2n-5}{3(n-1)}P_{n-3}+\frac{2n-9}{5(n-2)}P_{n-5}+\ldots.(4).$$

The series in (4) ends with the term involving P_1 if n be even, and with the term involving P_0 if n be odd.

79. In obtaining (4) we began by supposing x greater than unity; but it is obvious from the form of the result that it is universally true; for the rational integral function $-P_n(x)\Sigma\dfrac{A}{x-a}$, being equal to the rational integral function which forms the right-hand member of (4) when x is greater than unity, must always be equal to it.

In future we shall cease to distinguish between the forms (1) and (2); that is, we shall use (1) and leave to the student the task of examining if necessary how far the investigations apply also to (2).

80. We may shew in another way that

$$Q_n(x)=\frac{1}{2}P_n(x)\log\frac{x+1}{x-1}-R,$$

where R denotes a rational integral function of the degree $n-1$. For by Art. 37 we have

$$\frac{1}{x-y}=\Sigma(2n+1)\,Q_n(x)\,P_n(y);$$

therefore by Art. 28,

$$Q_n(x)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(y)\,dy}{x-y};$$

this may be written

$$Q_n(x)=-\frac{1}{2}\int_{-1}^{1}\frac{P_n(x)-P_n(y)}{x-y}dy+\frac{1}{2}P_n(x)\int_{-1}^{1}\frac{dy}{x-y}.$$

The expression $\frac{P_n(x) - P_n(y)}{x-y}$ is obviously a rational integral function of x and y of the degree $n-1$, and after integration with respect to y between the limits will be a rational integral function of x of the degree $n-1$. Also $\int_{-1}^{1} \frac{dy}{x-y} = \log \frac{x+1}{x-1}$. Thus the required result is obtained.

81. It is found convenient to use the symbol D to stand for $\frac{d}{dx}$, for abbreviation; thus $\frac{d^n v}{dx^n}$ is often denoted by $D^n v$. In like manner the symbol I may be used for integration; so that $\int v\,dx$ may be denoted by Iv; and if $\int v\,dx$ is to be integrated again we may denote the operation by $I^2 v$: and generally if the operation of integration is to be performed n times in succession we may denote this by $I^n v$.

These abbreviations will enable us to present some results in a compact form.

In the next five Articles we shall use C to denote a constant without assuming that the *same* constant is always to be understood: we shall also use C with various suffixes for constants under the same liberty of interpretation.

82. We know that $P_n(x) = C\frac{d^n(x^2-1)^n}{dx^n}$, which we may write thus,

$$P_n(x) = CD^n(x^2-1)^n \quad\text{.................}(5).$$

Now we saw in Art. 68 that a series for $Q_n(x)$ can be derived from one for $P_n(x)$ by changing n into $-n-1$; and thus we are led to conjecture that an equation of the following form will hold:

$$Q_n(x) = CD^{-n-1}(x^2-1)^{-n-1}.$$

But according to an interpretation of symbols suggested by the fact that integration is the reverse of differentiation, we may presume that D^{-n-1} is equivalent to I^{n+1}; so that we should have

$$Q_n(x) = CI^{n+1}\frac{1}{(x^2-1)^{n+1}} \quad\text{..............}(6),$$

or, which is the same thing,

$$D^{n+1}Q_n(x) = \frac{C}{(x^2-1)^{n+1}} \quad \text{..................(7).}$$

We have then to establish (6), or its equivalent (7), to which we have been led by analogy.

83. Take the expression for Q_n given at the beginning of Art. 78, namely

$$Q_n = \frac{1}{2} P_n \log\frac{x+1}{x-1} - R,$$

and differentiate $n+1$ times.

The $(n+1)^{\text{th}}$ differential coefficient of R is zero. Apply the theorem of Leibnitz with respect to the first term in Q_n. The $(n+1)^{\text{th}}$ differential coefficient of P_n is zero. The first differential coefficient of $\log\frac{x+1}{x-1}$ is $-\frac{2}{x^2-1}$; and every succeeding differential coefficient will introduce another power of x^2-1 into the denominator. Thus the $(n+1)^{\text{th}}$ differential coefficient of Q_n, when all the terms are brought to a common denominator, will be of the form $\frac{T}{(x^2-1)^{n+1}}$. Moreover T must be a constant. For if the highest power of x in T were x^m, then when x is very large $\frac{d^{n+1}Q_n}{dx^{n+1}}$ would be of the same order as $x^{-2n-2+m}$; whereas we know from Art. 68 that it must be of the same order as x^{-2n-2}. Hence T is constant, and thus (7) is established.

Or we might verify (7) by differentiating $n+1$ times the expression found for Q_n in Art. 68.

84. We shall now obtain the result of the preceding Article in another way.

Take the differential equation

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0 \quad \text{.........(8).}$$

Differentiate; then after reduction we obtain

$$(1-x^2)\frac{d^3z}{dx^3} - 4x\frac{d^2z}{dx^2} + (n-1)(n+2)\frac{dz}{dx} = 0.$$

Differentiate again; then after reduction we obtain

$$(1-x^2)\frac{d^4z}{dx^4}-6x\frac{d^3z}{dx^3}+(n-2)(n+3)\frac{d^2z}{dx^2}=0.$$

Proceeding in this way we find after m differentiations

$$(1-x^2)\frac{d^{m+2}z}{dx^{m+2}}-2(m+1)x\frac{d^{m+1}z}{dx^{m+1}}+(n-m)(n+m+1)\frac{d^mz}{dx^m}=0 \qquad \text{.......................(9)}.$$

Now the general solution of (8) is

$$z=C_1P_n(x)+C_2Q_n(x),$$

and hence we see that the general value of $\frac{d^mz}{dx^m}$ in (9) is

$$C_1\frac{d^mP_n(x)}{dx^m}+C_2\frac{d^mQ_n(x)}{dx^m}.$$

Let $m=n$; then (9) becomes

$$(1-x^2)\frac{d^{n+2}z}{dx^{n+2}}-2(n+1)x\frac{d^{n+1}z}{dx^{n+1}}=0.$$

This can be obviously solved; put u for $\frac{d^{n+1}z}{dx^{n+1}}$: thus

$$(1-x^2)\frac{du}{dx}=2(n+1)xu;$$

therefore $$\frac{1}{u}\frac{du}{dx}=-\frac{2(n+1)x}{x^2-1};$$

therefore $$\log u=-\log(x^2-1)^{n+1}+\text{a constant};$$

therefore $$u=\frac{C}{(x^2-1)^{n+1}};$$

thus $$\frac{d^nu}{dx^n}=C\int\frac{dx}{(x^2-1)^{n+1}}.$$

Hence it follows that by giving suitable values to C_1 and C_2 we must have

$$C_1 \frac{d^n P_n}{dx^n} + C_2 \frac{d^n Q_n}{dx^n} = C \int \frac{dx}{(x^2-1)^{n+1}}.$$

But $\frac{d^n P_n}{dx^n}$ is a constant; and thus

$$C_2 \frac{d^{n+1} Q_n}{dx^{n+1}} = \frac{C}{(x^2-1)^{n+1}};$$

this agrees with the result of Art. 83.

85. We may observe that equation (9) may be put in the form

$$(n-m)(n+m+1)(1-x^2)^m \frac{d^m z}{dx^m} + \frac{d}{dx}\left\{(1-x^2)^{m+1} \frac{d^{m+1} z}{dx^{m+1}}\right\} = 0;$$

this will be satisfied when for z we put $P_n(x)$. This equation with respect to $P_n(x)$ has been called *Ivory's Equation;* it was given by Ivory in the *Philosophical Transactions* for 1812, page 50.

86. Again, suppose a quantity ζ to be determined by the differential equation

$$(1-x^2)\frac{d^2\zeta}{dx^2} + 2(m-1)x\frac{d\zeta}{dx} + (n-m+1)(n+m)\zeta = 0 \ldots (10).$$

If we differentiate this r times in succession, we obtain

$$(1-x^2)\frac{d^{r+2}\zeta}{dx^{r+2}} + 2(m-r-1)x\frac{d^{r+1}\zeta}{dx^{r+1}}$$
$$+ (n-m+r+1)(n+m-r)\frac{d^r\zeta}{dx^r} = 0.$$

Thus if $r=m$ we have

$$(1-x^2)\frac{d^{m+2}\zeta}{dx^{m+2}} - 2x\frac{d^{m+1}\zeta}{dx^{m+1}} + (n+1)n\frac{d^m\zeta}{dx^m} = 0 \ldots\ldots (11);$$

which is of the same form as (8).

Now if $m = n$ equation (10) becomes

$$(1 - x^2)\frac{d^2\zeta}{dx^2} + 2(n-1)x\frac{d\zeta}{dx} + 2n\zeta = 0;$$

one solution of this is $\zeta = C(x^2 - 1)^n$, as may be immediately verified. Then, by the process of Art. 71, we can find the other solution; and thus the general solution will be found to be

$$\zeta = C_1(x^2 - 1)^n \int \frac{dx}{(x^2 - 1)^{n+1}},$$

where a second arbitrary constant may be supposed to be involved in the integral. Or if we prefer to denote this constant explicitly, we may take for the general solution

$$\zeta = C_1(x^2 - 1)^n \int \frac{dx}{(x^2 - 1)^{n+1}} + C_2(x^2 - 1)^n.$$

Hence the solution of (11) if $m = n$ is obtained by taking this value of ζ and differentiating n times. But we know that the solution of (11) is of the form $C_3 P_n(x) + C_4 Q_n(x)$. Hence by proper adjustments of the constants we must have

$$C_3 P_n(x) + C_4 Q_n(x) = C_1 \frac{d^n}{dx^n}\left\{(x^2 - 1)^n \int \frac{dx}{(x^2 - 1)^{n+1}}\right\} + C_2 \frac{d^n}{dx^n}(x^2 - 1)^n.$$

As we know that $Q_n(x)$ does not contain any positive power of x, at least when x is greater than unity, we infer that

$$Q_n(x) = C\frac{d^n}{dx^n}\left\{(x^2 - 1)^n \int \frac{dx}{(x^2 - 1)^{n+1}}\right\} \text{............(12).}$$

This gives another form for $Q_n(x)$. By comparing it with that furnished by equation (6), we infer that for some value of the constant C we must have

$$C\Gamma^{n+1}\frac{1}{(x^2 - 1)^{n+1}} = \frac{d^n}{dx^n}\left\{(x^2 - 1)^n \int \frac{dx}{(x^2 - 1)^{n+1}}\right\}.$$

The constant C may be determined by supposing x indefinitely great; for then the equation becomes

$$CI^{n+1}\frac{1}{x^{2n+2}}=\frac{d^n}{dx^n}\left\{x^{2n}\int\frac{dx}{x^{2n+2}}\right\}$$

$$=-\frac{1}{2n+1}\frac{d^n}{dx^n}\left(\frac{1}{x}\right);$$

this gives $$C=\lfloor 2n.$$

87. Since the general solution of (11) is

$$\frac{d^m\zeta}{dx^m}=C_3P_n(x)+C_4Q_n(x),$$

it follows that the general solution of (10) is

$$\zeta=C_3I^mP_n(x)+C_4I^mQ_n(x),$$

and we may use for $Q_n(x)$ either of the forms (6) and (12).

CHAPTER VII.

APPROXIMATE VALUES OF COEFFICIENTS OF HIGH ORDERS.

88. SUPPOSE x positive and greater than unity. We have by Art. 17,

$$P_n(x) = k\xi^n \left\{1 + \frac{1 \cdot n}{1 \cdot (2n-1)} \xi^{-2} + \frac{1 \cdot 3 \cdot n(n-1)}{1 \cdot 2 \cdot (2n-1)(2n-3)} \xi^{-4} + \dots\right\},$$

where k stands for $\dfrac{1 \cdot 3 \dots (2n-1)}{2^n \lfloor n}$.

When n is indefinitely increased the series between the brackets becomes ultimately

$$1 + \frac{1}{2}\xi^{-2} + \frac{1 \cdot 3}{2 \cdot 4}\xi^{-4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\xi^{-6} + \dots,$$

that is $(1 - \xi^{-2})^{-\frac{1}{2}}$.

Thus $P_n(x) = k\xi^n \{(1 - \xi^{-2})^{-\frac{1}{2}} + \epsilon\}$,

where ϵ denotes a quantity which diminishes indefinitely as n increases indefinitely.

Now $k = \dfrac{\lfloor 2n}{2^{2n} \lfloor n \lfloor n}$; and by applying the formula given in the *Integral Calculus*, Art. 282, we see that when n is very great we have approximately $k = \dfrac{1}{\sqrt{n\pi}}$.

Thus finally when x is positive and greater than unity, and n very large, we have approximately

$$P_n(x) = \frac{1}{\sqrt{n\pi}} \frac{\xi^n}{\sqrt{(1-\xi^{-2})}}.$$

We suppose x positive and greater than unity in order that ξ may be greater than unity, and so the series between the brackets convergent when n is very large.

The case in which x is negative and numerically greater than unity may be made to depend on that in which x is positive by the relation $P_n(-x) = (-1)^n P_n(x)$.

89. Now suppose x numerically less than unity. Put $\cos\theta$ for x. In Art. 39 we have shewn that

$$P_n(\cos\theta) = \frac{4}{\pi k(2n+1)} \left\{ \sin(n+1)\theta + \frac{1.(n+1)}{1.(2n+3)} \sin(n+3)\theta + \frac{1.3.(n+1)(n+2)}{1.2(2n+3)(2n+5)} \sin(n+5)\theta + \dots \right\},$$

where k has the same value as in Art. 88.

If we suppose n to increase indefinitely the series between the brackets takes ultimately the form

$$\sin(n+1)\theta + \frac{1}{2}\sin(n+3)\theta + \frac{1.3}{2.4}\sin(n+5)\theta + \dots,$$

that is
$$\sin n\theta \left\{ \cos\theta + \frac{1}{2}\cos 3\theta + \frac{1.3}{2.4}\cos 5\theta + \dots \right\} + \cos n\theta \left\{ \sin\theta + \frac{1}{2}\sin 3\theta + \frac{1.3}{2.4}\sin 5\theta + \dots \right\}.$$

We have then to find equivalents for the two infinite series just indicated.

Let t be a quantity less than unity;

put
$$t\cos\theta + \frac{1}{2}t^3\cos 3\theta + \frac{1.3}{2.4}t^5\cos 5\theta + \dots = C,$$

and
$$t\sin\theta + \frac{1}{2}t^3\sin 3\theta + \frac{1.3}{2.4}t^5\sin 5\theta + \dots = S.$$

Thus both C and S denote convergent series.

Then $\quad C+\iota S = te^{\iota\theta} + \frac{1}{2} t^3 e^{3\iota\theta} + \frac{1.3}{2.4} t^5 e^{5\iota\theta} + \ldots$

$$= \frac{te^{\iota\theta}}{\sqrt{(1-t^2 e^{2\iota\theta})}} = \frac{te^{\iota\theta}}{\sqrt{(1-t^2\cos 2\theta - t^2 \iota \sin 2\theta)}}.$$

Assume $1-t^2\cos 2\theta = \rho\cos\phi$, and $t^2 \sin 2\theta = \rho \sin\phi$;

so that $\rho^2 = 1 - 2t^2\cos 2\theta + t^4$, and $\tan\phi = \dfrac{t^2 \sin 2\theta}{1-t^2\cos 2\theta}$.

Then $C+\iota S =$

$$\frac{te^{\iota\theta}}{\sqrt{\rho}\, e^{-\frac{\iota\phi}{2}}} = \frac{t}{\sqrt{\rho}} e^{\iota\left(\theta+\frac{\phi}{2}\right)} = \frac{t}{\sqrt{\rho}}\left\{\cos\left(\theta+\frac{\phi}{2}\right) + \iota \sin\left(\theta+\frac{\phi}{2}\right)\right\}:$$

so that $\quad C = \dfrac{t}{\sqrt{\rho}}\cos\left(\theta+\dfrac{\phi}{2}\right)$, and $S = \dfrac{t}{\sqrt{\rho}}\sin\left(\theta+\dfrac{\phi}{2}\right)$.

These results may be admitted to hold so long as t is less than unity. Assume them to hold even when t is equal to unity. We have then

$$\rho^2 = 2(1-\cos 2\theta), \text{ so that } \sqrt{\rho} = \sqrt{2\sin\theta};$$

$$\tan\phi = \frac{\sin 2\theta}{1-\cos 2\theta} = \frac{\cos\theta}{\sin\theta} = \tan\left(\frac{\pi}{2}-\theta\right), \text{ so that } \phi = \frac{\pi}{2}-\theta.$$

Hence when n is very great we have approximately

$$P_n(\cos\theta) = \frac{2}{\pi k n}\,\frac{\sin n\theta\cos\left(\theta+\frac{\phi}{2}\right) + \cos n\theta \sin\left(\theta+\frac{\phi}{2}\right)}{\sqrt{2\sin\theta}}$$

$$= \frac{2}{\pi k n}\,\frac{\sin\left(n\theta+\theta+\frac{\phi}{2}\right)}{\sqrt{2\sin\theta}} = \frac{2}{\pi k n}\,\frac{\sin\left(n\theta+\frac{\theta}{2}+\frac{\pi}{4}\right)}{\sqrt{2\sin\theta}};$$

and as $k = \dfrac{1}{\sqrt{n\pi}}$ approximately we have finally as an approximation when n is very great

$$P_n(\cos\theta) = \frac{\sqrt{2}}{\sqrt{n\pi\sin\theta}}\cos\left(n\theta+\frac{\theta}{2}-\frac{\pi}{4}\right) \ldots\ldots\ldots (1).$$

90. The result obtained in the preceding Article is due to Laplace; it cannot be accepted with great confidence: it does not lead in any obvious way to the value unity when $\theta=0$, which we know ought to hold for all values of n.

Laplace himself gave two investigations, both in the *Mécanique Céleste,* one in Livre XI. § 3, and the other in the *Supplément au* 5[e] *Volume;* they differ from that of Art. 89, but do not seem more satisfactory. We will reproduce the latter of them.

By Art. 63 we know that

$$\frac{d^2 P_n}{d\theta^2}+\cot\theta\frac{dP_n}{d\theta}+n(n+1)P_n=0.$$

Assume that

$$P_n=u\cos a\theta+u'\sin a\theta, \qquad\qquad (2),$$

where u and u' are functions of θ to be determined, and $a=\sqrt{n(n+1)}$. Substitute in the differential equation, and equate to zero the coefficients of $\sin a\theta$ and $\cos a\theta$. Thus

$$\left.\begin{aligned} 2\frac{du}{d\theta}+u\cot\theta &= \frac{1}{a}\left(\frac{d^2u'}{d\theta^2}+\frac{du'}{d\theta}\cot\theta\right) \\ 2\frac{du'}{d\theta}+u'\cot\theta &= -\frac{1}{a}\left(\frac{d^2u}{d\theta^2}+\frac{du}{d\theta}\cot\theta\right) \end{aligned}\right\} \qquad (3).$$

If we neglect the terms divided by a, which is large since n is supposed large, these equations become

$$2\frac{du}{d\theta}+u\cot\theta=0, \qquad 2\frac{du'}{d\theta}+u'\cot\theta=0;$$

and hence we obtain

$$u=\frac{H}{\sqrt{\sin\theta}}, \quad u'=\frac{H'}{\sqrt{\sin\theta}},$$

where H and H' are arbitrary constants.

These may be regarded as first approximations to the true values of u and u'; we may then assume

$$u=\frac{H}{\sqrt{\sin\theta}}+\frac{X}{a}, \qquad u'=\frac{H'}{\sqrt{\sin\theta}}+\frac{X'}{a},$$

and substitute these values in the differential equations (3) and proceed to find, at least approximately, X and X'.

But we shall confine ourselves to the first approximation, so that we have from (2)

$$P_n = \frac{1}{\sqrt{\sin\theta}}(H\cos a\theta + H'\sin a\theta)$$

$$= \frac{C}{\sqrt{\sin\theta}}\cos(a\theta+\gamma),$$

where C and γ denote certain constants.

And as $a = \sqrt{n(n+1)}$ we have approximately $a = n + \frac{1}{2}$,

so that $$P_n = \frac{C}{\sqrt{\sin\theta}}\cos\left(n\theta + \frac{\theta}{2} + \gamma\right).$$

To determine the constant γ we observe that if n be odd $P_n = 0$ when $\theta = \frac{\pi}{2}$; this leads to $\gamma = -\frac{\pi}{4}$, so that

$$P_n = \frac{C}{\sqrt{\sin\theta}}\cos\left(n\theta + \frac{\theta}{2} - \frac{\pi}{4}\right).$$

To determine the constant C we observe that if n be even and denoted by $2m$ we have by Art. 7, when $\theta = \frac{\pi}{2}$,

$$P_n = \frac{\lfloor 2m}{2^{2m}\lfloor m \lfloor m}(-1)^m;$$

therefore $$C = \frac{\lfloor 2m}{2^{2m}\lfloor m \lfloor m};$$

and by approximating as in Art. 88, we have

$$C = \frac{1}{\sqrt{m\pi}} = \frac{\sqrt{2}}{\sqrt{n\pi}}.$$

Thus our result agrees with (1).

91. Laplace's other investigation of (1) starts with the expression of $P_n(x)$ by means of a definite integral given in Art. 44; we shall not reproduce this. It is however easy to shew that when n is very large $P_n(x)$ is very small if x is numerically less than unity.

For we have $P_n(x) = \frac{1}{\pi}\int_0^\pi \{x - \iota\sqrt{(1-x^2)}\cos\phi\}^n d\phi$.

Assume $x = \rho\cos\psi$, and $\sqrt{(1-x^2)}\cos\phi = \rho\sin\psi$;

thus $$P_n(x) = \frac{1}{\pi}\int_0^\pi \rho^n\{\cos n\psi + \iota\sin n\psi\}\, d\phi.$$

The imaginary part vanishes and we get

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \rho^n\cos n\psi\, d\phi.$$

Now when n is very large the value of this expression is very small on two accounts; ρ^n is very small except when $x = 1$; and $\cos n\psi$ fluctuates very rapidly in sign.

92. Another investigation of the value of $P_n(x)$ when n is very large is given by M. Ossian Bonnet in Liouville's *Journal de Mathématiques*, Vol. XVII. pages 270...277.

We have $\frac{d^2P_n}{d\theta^2} + \cot\theta\frac{dP_n}{d\theta} + n(n+1)P_n = 0$.

Assume $P_n = u(\sin\theta)^{-\frac{1}{2}}$; thus we obtain

$$\frac{d^2u}{d\theta^2} + \left(n+\frac{1}{2}\right)^2 u = -\frac{u}{4\sin^2\theta};$$

putting m for $n+\frac{1}{2}$ we have $\frac{d^2u}{d\theta^2} + m^2u = -\frac{u}{4\sin^2\theta}$ ...... (4).

Multiply by $\sin m\theta$ and integrate; thus

$$\sin m\theta\frac{du}{d\theta} - mu\cos m\theta = C_1 - \frac{1}{4}\int_a^\theta \frac{u\sin m\theta}{\sin^2\theta}\,d\theta \text{...... (5)},$$

where C_1 is an arbitrary constant, and a a fixed quantity which may however be as small as we please.

In precisely the same manner, by multiplying (4) by $\cos m\theta$ and integrating, we obtain

$$\cos m\theta \frac{du}{d\theta} + mu \sin m\theta = C_2 - \frac{1}{4}\int_a^\theta \frac{u \cos m\theta}{\sin^2\theta} d\theta \text{ ...... (6).}$$

Eliminate $\frac{du}{d\theta}$ between (5) and (6); thus

$$mu = C_2 \sin m\theta - C_1 \cos m\theta$$
$$-\frac{1}{4}\sin m\theta \int_a^\theta \frac{u \cos m\theta}{\sin^2\theta} d\theta + \frac{1}{4}\cos m\theta \int_a^\theta \frac{u \sin m\theta}{\sin^2\theta} d\theta.$$

This may be expressed more concisely; for let u' denote the same function of θ' that u denotes of θ: then

$$-\sin m\theta \int_a^\theta \frac{u \cos m\theta}{\sin^2\theta} d\theta + \cos m\theta \int_a^\theta \frac{u \sin m\theta}{\sin^2\theta} d\theta$$
$$= -\sin m\theta \int_a^\theta \frac{u' \cos m\theta'}{\sin^2\theta'} d\theta' + \cos m\theta \int_a^\theta \frac{u' \sin m\theta'}{\sin^2\theta'} d\theta'$$
$$= \int_a^\theta \frac{u' \sin m(\theta' - \theta)}{\sin^2\theta'} d\theta'.$$

Thus expressing the constants C_1 and C_2 in terms of two new constants b and β, we have

$$u = \frac{b\cos(m\theta + \beta)}{m} + \frac{1}{4m}\int_a^\theta \frac{u' \sin m(\theta' - \theta)}{\sin^2\theta'} d\theta' \text{......(7).}$$

Denote this for abbreviation thus:

$$u = \frac{b\cos(m\theta + \beta)}{m} + \frac{1}{4m}\psi(\theta);$$

then, by substituting the corresponding value of u' in (7), we get

$$u = \frac{b\cos(m\theta + \beta)}{m} + \frac{b}{4m^2}\int_a^\theta \frac{\cos(m\theta' + \beta) \sin m(\theta' - \theta)}{\sin^2\theta'} d\theta'$$
$$+ \frac{1}{16m^2}\int_a^\theta \frac{\psi(\theta') \sin m(\theta' - \theta)}{\sin^2\theta'} d\theta'.$$

The last term on the right-hand side involves u', for u' occurs in $\psi(\theta')$. The process of substitution may then be performed again if we please; and so on.

Finally it will be necessary to determine the values of b and β: we observe that they are constant with respect to θ, but M. Bonnet assumes that they are constant with respect to n, and this appears to me a serious fault in the rest of his process; in fact, quantities are retained which are of the same order as those which are neglected.

93. We will briefly advert to the value of $Q_n(x)$ when n is very large, supposing x positive and greater than unity.

We have by Art. 69,

$$Q_n(x) = \lambda \xi^{n-1} F\left(\frac{1}{2},\ n+1,\ \frac{2n+3}{2},\ \xi^2\right):$$

this becomes approximately when n is very great

$$Q_n(x) = \frac{\lambda \xi^{n-1}}{\sqrt{(1-\xi^2)}};$$

and $\lambda = \dfrac{2}{(2n+1)k}$, where k is the same as in Art. 88: thus approximately

$$\lambda = \sqrt{\frac{\pi}{n}}.$$

CHAPTER VIII.

ASSOCIATED FUNCTIONS.

94. THERE are certain functions analogous to $P_n(x)$ which present themselves naturally in the course of our investigations, and we now propose to consider them. They may be called *Associated Functions of the First Kind.*

95. We have seen in Art. 47 that

$$P_n(x) = \int_0^\pi \{x + \sqrt{(x^2-1)} \cos\phi\}^n d\phi \quad \text{............} \quad (1).$$

Now we may expand $\{x + \sqrt{(x^2-1)} \cos\phi\}^n$ in a series proceeding according to powers of $\cos\phi$, and then the powers of $\cos\phi$ may be transformed into cosines of multiples of ϕ; thus finally $\{x + \sqrt{(x^2-1)}\cos\phi\}^n$ may be arranged in the form

$$a_0 + a_1 \cos\phi + a_2 \cos 2\phi + \ldots + a_n \cos n\phi,$$

where $a_0, a_1, a_2, \ldots a_n$ are functions of x, but do not contain ϕ.

Hence it follows from (1) that

$$P_n(x) = \int_0^\pi a_0 d\phi = a_0\pi;$$

therefore $$a_0 = \frac{1}{\pi} P_n(x) = \frac{1}{\pi 2^n \lfloor n} \frac{d^n(x^2-1)^n}{dx^n}.$$

We shall now determine the value of a_m, where a_m denotes any one of the series $a_1, a_2, \ldots a_n$.

96. We have

$$x+\sqrt{(x^2-1)}\cos\phi = x+\sqrt{(x^2-1)}\frac{e^{\iota\phi}+e^{-\iota\phi}}{2}$$

$$=\frac{2xe^{\iota\phi}+\sqrt{(x^2-1)}(e^{2\iota\phi}+1)}{2e^{\iota\phi}}=\frac{2x\sqrt{(x^2-1)}\,e^{\iota\phi}+(x^2-1)(e^{2\iota\phi}+1)}{2\sqrt{(x^2-1)}\,e^{\iota\phi}}$$

$$=\frac{(x+z)^2-1}{2z},$$

where z is put for $\sqrt{(x^2-1)}\,e^{\iota\phi}$.

Thus $2^n\{x+\sqrt{(x^2-1)}\cos\phi\}^n=\left\{\frac{(x+z)^2-1}{z}\right\}^n$.

Now we may expand $\{(x+z)^2-1\}^n$ in powers of z, by Taylor's Theorem; and thus if u stand for $(x^2-1)^n$ we get

$2^n\{x+\sqrt{(x^2-1)}\cos\phi\}^n$

$$=\frac{1}{z^n}\left\{u+z\frac{du}{dx}+\frac{z^2}{\underline{|2}}\frac{d^2u}{dx^2}+\ldots+\frac{z^{2n}}{\underline{|2n}}\frac{d^{2n}u}{dx^{2n}}\right\}.$$

The series ends with the last term which is here expressed, because u is of the degree $2n$ in x.

Re-arranging the terms we obtain

$2^n\{x+\sqrt{(x^2-1)}\cos\phi\}^n$

$$=\frac{1}{\underline{|n}}\frac{d^nu}{dx^n}+\frac{z}{\underline{|n+1}}\frac{d^{n+1}u}{dx^{n+1}}+\frac{z^2}{\underline{|n+2}}\frac{d^{n+2}u}{dx^{n+2}}+\ldots+\frac{z^n}{\underline{|2n}}\frac{d^{2n}u}{dx^{2n}}$$

$$+\frac{z^{-1}}{\underline{|n-1}}\frac{d^{n-1}u}{dx^{n-1}}+\frac{z^{-2}}{\underline{|n-2}}\frac{d^{n-2}u}{dx^{n-2}}+\ldots+z^{-n}u.$$

Now put $e^{\iota\phi}\sqrt{(x^2-1)}$ for z; then the series resolves itself into two parts, a real and an imaginary part. From Art. 95 we know that the result is entirely real, so that the imaginary part must disappear. This imaginary part consists of n terms, of which the m^{th} is

$$\iota\left\{\frac{(x^2-1)^{\frac{m}{2}}}{\underline{|n+m}}\frac{d^{n+m}u}{dx^{n+m}}-\frac{(x^2-1)^{-\frac{m}{2}}}{\underline{|n-m}}\frac{d^{n-m}u}{dx^{n-m}}\right\}\sin m\phi.$$

Hence we see that these terms must separately vanish; so that we obtain the formula

$$\frac{(x^2-1)^{\frac{m}{2}}}{\lfloor n+m} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}} = \frac{(x^2-1)^{-\frac{m}{2}}}{\lfloor n-m} \frac{d^{n-m}(x^2-1)^n}{dx^{n-m}} \dots\text{(2)};$$

this holds for positive integral values of m from 1 to n inclusive.

Hence finally we have

$$2^n \{x+\sqrt{(x^2-1)}\cos\phi\}^n$$

$$= \frac{1}{\lfloor n} \frac{d^n(x^2-1)^n}{dx^n} + 2\Sigma \frac{(x^2-1)^{\frac{m}{2}}}{\lfloor n+m} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}} \cos m\phi \dots\text{(3)};$$

where Σ denotes a summation with respect to m from 1 to n inclusive. Moreover by (2) we may if we please change m to $-m$ in (3).

97. Now the functions which we propose to consider are the coefficients of the cosines in (3).

We see that the coefficient of $\cos m\phi$ is

$$2\frac{(x^2-1)^{\frac{m}{2}}}{\lfloor n+m} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}}.$$

It will be found that

$$\frac{d^{n+m}(x^2-1)^n}{dx^{n+m}} = \frac{\lfloor 2n}{\lfloor n-m} \Big\{ x^{n-m} - \frac{(n-m)(n-m-1)}{2.(2n-1)} x^{n-m-2}$$

$$+ \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2.4.(2n-1)(2n-3)} x^{n-m-4} - \dots \Big\}.$$

We shall denote the series between the brackets by $\varpi(m,n)$; so that

$$\varpi(m,n) = \frac{\lfloor n-m}{\lfloor 2n} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}} = \frac{\lfloor n-m}{\lfloor 2n} 2^n \lfloor n \frac{d^m P_n(x)}{dx^m}:$$

therefore $$\varpi(m,n) = \frac{\lfloor n-m}{1.3.5\dots(2n-1)} \frac{d^m P_n(x)}{dx^m} \dots\dots\dots\text{(4)}.$$

Thus we may express (3) in the form

$$2^n \{x + \sqrt{(x^2-1)} \cos\phi\}^n$$
$$= \frac{1}{\lfloor n}\frac{d^n (x^2-1)^n}{dx^n} + 2\Sigma \frac{\lfloor 2n}{\lfloor n+m \lfloor n-m}(x^2-1)^{\frac{m}{2}} \varpi(m,n) \cos m\phi.$$

We may if we please replace the first term

$$\frac{1}{\lfloor n}\frac{d^n (x^2-1)^n}{dx^n} \text{ by } \frac{\lfloor 2n}{\lfloor n \lfloor n}\varpi(0,n);$$

so that $\frac{2^n}{\lfloor 2n}\left\{x + \sqrt{(x^2-1)} \cos\phi\right\}^n$

$$= \frac{\varpi(0,n)}{\lfloor n \lfloor n} + 2\Sigma \frac{(x^2-1)^{\frac{m}{2}}}{\lfloor n+m \lfloor n-m}\varpi(m,n) \cos m\phi.$$

In cases where it is convenient to express the variable we might use $\varpi(m,n,x)$ instead of the shorter $\varpi(m,n)$.

98. It will be seen that we arrived indirectly at equation (2) of Art. 96; but it may be established in a direct manner. The result may be put in this slightly generalised form:

$$\frac{(x+a)^m (x+b)^m}{\lfloor n+m}\frac{d^{n+m}(x+a)^n (x+b)^n}{dx^{n+m}} = \frac{1}{\lfloor n-m}\frac{d^{n-m}(x+a)^n(x+b)^n}{dx^{n-m}}.$$

To demonstrate this, develop the two members by the aid of the theorem of Leibnitz; use D for $\frac{d}{dx}$, for abbreviation.

Then in the development of $D^{n+m}(x+a)^n(x+b)^n$, the first term which does not vanish is $\frac{\lfloor n+m}{\lfloor n \lfloor m} D^n(x+a)^n D^m (x+b)^n$,

that is $\frac{\lfloor n+m}{\lfloor n \lfloor m}\lfloor n \frac{\lfloor n}{\lfloor n-m}(x+b)^{n-m}$; and in like manner the r^{th} term of the development, counting this as the first term, will be found to be

$$\frac{\lfloor n+m}{\lfloor n-r+1 \lfloor m+r-1} D^{n-r+1}(x+a)^n D^{m+r-1}(x+b)^n,$$

that is

$$\frac{\underline{|n+m}}{\underline{|n-r+1}\;\underline{|m+r-1}}\,\frac{\underline{|n}}{\underline{|r-1}}(x+a)^{r-1}\,\frac{\underline{|n}}{\underline{|n-m-r+1}}(x+b)^{n-m-r+1}:$$

we will denote this by A.

Similarly we find that the r^{th} term in the development of $D^{n-m}(x+a)^n(x+b)^n$ is

$$\frac{\underline{|n-m}}{\underline{|n-m-r+1}\;\underline{|r-1}}D^{n-m-r+1}(x+a)^n D^{r-1}(x+b)^n,$$

that is

$$\frac{\underline{|n-m}}{\underline{|n-m-r+1}\;\underline{|r-1}}\,\frac{\underline{|n}}{\underline{|m+r-1}}(x+a)^{m+r-1}\,\frac{\underline{|n}}{\underline{|n-r+1}}(x+b)^{n-r+1};$$

we will denote this by B.

Then we see that

$$\frac{(x+a)^m(x+b)^m}{\underline{|n+m}}A=\frac{1}{\underline{|n-m}}B;$$

and this establishes the required result.

99. The functions which we denote by $(x^2-1)^{\frac{m}{2}}\varpi(m,n)$ are called *Associated Functions of the First Kind:* Heine denotes them by $P_m^{\;n}(x)$.

100. We have seen that the differential equation (9) of Art. 84 is satisfied when $P_n(x)$ is put for z. Hence from equation (4) of Art. 97 it follows that

$$(1-x^2)\frac{d^2\varpi(m,n)}{dx^2}-2(m+1)x\frac{d\varpi(m,n)}{dx}$$
$$+(n-m)(n+m+1)\varpi(m,n)=0\;\ldots\ldots\;(5).$$

Now the expression which we have denoted by $(x^2-1)^{\frac{m}{2}}\varpi(m,n)$ is equivalent to $(x^2-1)^{-\frac{m}{2}}\varpi(-m,n)$, as we see by Art. 96. Hence we have

$$\varpi(m,n)=(x^2-1)^{-m}\varpi(-m,n),$$

and substituting in (5) we find that

$$(1-x^2)\frac{d^2\varpi(-m,n)}{dx^2}+2(m-1)x\frac{d\varpi(-m,n)}{dx}$$
$$+(n+m)(n-m+1)\varpi(-m,n)=0 \;\ldots\ldots\; (6).$$

It will be seen that (6) differs from (5) only as to the sign of m.

We have deduced (6) from (5) without assuming anything relating to $\varpi(m,n)$ except that it satisfies (5). If then we get the general solutions of (5) and (6) we may equate the latter to the product of $(x^2-1)^m$ into the former.

Now we know from Art. 84 that the general solution of (5) is

$$\varpi(m,n)=C_1D^mP_n(x)+C_2D^mQ_n(x)\,;$$

and we know from Art. 87 that the general solution of (6) may be expressed thus:

$$\varpi(-m,n)=C_3D^{-m}P_n(x)+C_4D^{-m}Q_n(x).$$

Hence by proper adjustment of the constants we shall have

$$(x^2-1)^m\{C_1D^mP_n(x)+C_2D^mQ_n(x)\}$$
$$=C_3D^{-m}P_n(x)+C_4D^{-m}Q_n(x).$$

By considering the integral and the fractional functions of x which occur in this relation, we see that it must break up into the two

$$(x^2-1)^m\,C_1D^mP_n(x)=C_3D^{-m}P_n(x)\;\ldots\ldots\ldots(7),$$

and

$$(x^2-1)^m\,C_2D^mQ_n(x)=C_4D^{-m}Q_n(x)\;\ldots\ldots\ldots(8):$$

these hold for positive integral values of m not exceeding n.

101. Equation (7) coincides with a result already obtained in Arts. 96 and 98.

Equation (8) takes various forms, according to the expression we use for $Q_n(x)$: see equations (6) and (12) of

Chapter VI. Thus we have the following results, in which C denotes some constant:

$$(x^2-1)^m I^{n-m+1}\frac{1}{(x^2-1)^{n+1}} = CI^{n+m+1}\frac{1}{(x^2-1)^{n+1}},$$

$$(x^2-1)^m I^{n-m+1}\frac{1}{(x^2-1)^{n+1}} = CD^{n-m}\left\{(x^2-1)^n\int\frac{dx}{(x^2-1)^{n+1}}\right\},$$

$$(x^2-1)^m D^{n+m}\left\{(x^2-1)^n\int\frac{dx}{(x^2-1)^{n+1}}\right\} = CI^{n+m+1}\frac{1}{(x^2-1)^{n+1}},$$

$$(x^2-1)^m D^{n+m}\left\{(x^2-1)^n\int\frac{dx}{(x^2-1)^{n+1}}\right\}$$
$$= CD^{n-m}\left\{(x^2-1)^n\int\frac{dx}{(x^2-1)^{n+1}}\right\}.$$

The constant C may be determined by special examination in each case, as in Art. 86.

We shall find in the first and fourth cases

$$C = \frac{\lfloor n+m}{\lfloor n-m};$$

in the second case $\quad C = \dfrac{1}{\lfloor 2n}\dfrac{\lfloor n+m}{\lfloor n-m};$

and in the third case $\quad C = \lfloor 2n \dfrac{\lfloor n+m}{\lfloor n-m}.$

Of the first and second cases one will follow from the other by the aid of the result obtained in Art. 86, if we integrate that result m times; in like manner of the third and fourth cases one will follow from the other.

102. We see by Art. 100 that $\varpi(m, n)$ satisfies the differential equation (5), namely

$$(1-x^2)\frac{d^2\varpi(m,n)}{dx^2} - 2(m+1)x\frac{d\varpi(m,n)}{dx}$$
$$+ (n-m)(n+m+1)\varpi(m,n) = 0.$$

Put $y=(x^2-1)^{\frac{m}{2}}\varpi(m, n)$, so that $\varpi(m, n)=y(x^2-1)^{-\frac{m}{2}}$; substitute in (5), and we thus obtain

$$(1-x^2)^2\frac{d^2y}{dx^2}-2x(1-x^2)\frac{dy}{dx}$$
$$+\{n(n+1)-m^2-n(n+1)x^2\}y=0. \quad \text{.........} \ (9).$$

Conversely we may deduce (5) from (9) by putting $y=(x^2-1)^{\frac{m}{2}}\varpi(m, n)$. As the general solution of (5) is known we know that of (9), namely

$$y=(x^2-1)^{\frac{m}{2}}\{C_1D^mP_n(x)+C_2D^mQ_n(x)\}.$$

By Art. 100 this is equivalent to

$$y=(x^2-1)^{-\frac{m}{2}}\{C_3D^{-m}P_n(x)+C_4D^{-m}Q_n(x)\}.$$

103. Put ϖ for $\varpi(m, n)$ for abbreviation; thus we have from (5)

$$(1-x^2)\frac{d^2\varpi}{dx^2}-2(m+1)x\frac{d\varpi}{dx}+(n-m)(n+m+1)\varpi=0.$$

We shall transform this by a substitution of which we have already made use; namely $2x=\xi+\xi^{-1}$,

so that $$2\sqrt{(x^2-1)}=\xi-\xi^{-1}.$$

Now $$\frac{d\varpi}{dx}=\frac{d\varpi}{d\xi}\frac{d\xi}{dx}=\frac{d\varpi}{d\xi}\left(1+\frac{x}{\sqrt{(x^2-1)}}\right)=\frac{d\varpi}{d\xi}\left(1+\frac{\xi+\xi^{-1}}{\xi-\xi^{-1}}\right)$$
$$=2\frac{d\varpi}{d\xi}\frac{\xi^2}{\xi^2-1};$$

$$\frac{d^2\varpi}{dx^2}=2\frac{d}{d\xi}\left(\frac{d\varpi}{d\xi}\frac{\xi^2}{\xi^2-1}\right)\frac{d\xi}{dx}=4\left(\frac{\xi^2}{\xi^2-1}\right)^2\frac{d^2\varpi}{d\xi^2}-\frac{8\xi^3}{(\xi^2-1)^3}\frac{d\varpi}{d\xi}:$$

therefore $$(1-x^2)\frac{d^2\varpi}{dx^2}=-\frac{1}{4}\left(\frac{\xi^2-1}{\xi}\right)^2\frac{d^2\varpi}{dx^2}=-\xi^2\frac{d^2\varpi}{d\xi^2}+\frac{2\xi}{\xi^2-1}\frac{d\varpi}{d\xi},$$

and $$2(m+1)x\frac{d\varpi}{dx}=\frac{2(m+1)(\xi^2+1)\xi}{\xi^2-1}\frac{d\varpi}{d\xi}.$$

Hence by substitution and reduction we finally obtain

$$\xi^2(\xi^2-1)\frac{d^2\varpi}{d\xi^2}+2\xi\{m+(m+1)\xi^2\}\frac{d\varpi}{d\xi}$$
$$-(n-m)(n+m+1)(\xi^2-1)\varpi=0. \quad\ldots\ldots\ldots (10).$$

From this differential equation we shall obtain a series for ϖ proceeding according to descending powers of ξ.

Assume $\quad \varpi=a_0\xi^s+a_2\xi^{s-2}+a_4\xi^{s-4}+\ldots,$

substitute in (10) and equate to zero the coefficient of ξ^{s-2r}; thus

$$a_{2r+2}(s-2r-2)(s-2r-3)-a_{2r}(s-2r)(s-2r-1)$$
$$+2m\,a_{2r}(s-2r)+2(m+1)\,a_{2r+2}(s-2r-2)$$
$$-(n-m)(n+m+1)(a_{2r+2}-a_{2r})=0. \quad\ldots\ldots\ldots (11).$$

Moreover in order that the coefficient of ξ^{s+2} may vanish, we must have $s(s-1)+2(m+1)s-(n-m)(n+m+1)=0$, that is, $s(s+2m+1)-(n-m)(n+m+1)=0$; so that $s=n-m$ is a solution.

From (11) we have by reduction

$$a_{2r+2}\left\{(s-2r-2)(s-2r+2m-1)-(n-m)(n+m+1)\right\}$$
$$=a_{2r}\left\{(s-2r)(s-2r-2m-1)-(n-m)(n+m+1)\right\}.$$

Substitute $n-m$ for s, and we obtain finally

$$a_{2r+2}=\frac{(2r+2m+1)(n-m-r)}{(r+1)(2n-2r-1)}a_{2r}.$$

Thus we get

$$\varpi=a_0\left\{\xi^{n-m}+\frac{(n-m)(2m+1)}{1\,.\,(2n-1)}\xi^{n-m-2}\right.$$
$$\left.+\frac{(n-m)(n-m-1)(2m+1)(2m+3)}{1\,.\,2\,.\,(2n-1)(2n-3)}\xi^{n-m-4}+\ldots\right\};$$

the series between the brackets is to be continued until it terminates of itself.

The value of a_0 may be found by comparing the first term of this expansion with the first term of the expansion of ϖ in powers of x, which is given in Art. 97, and supposing x indefinitely great: thus we get $a_0 = 2^{-n+m}$.

If we put $\cos\theta$ for x we have $\xi = e^{\iota\theta}$; then the imaginary part must disappear from the expression for ϖ, and we obtain

$$\varpi = 2^{-n+m}\left\{\cos(n-m)\,\theta + \frac{(n-m)(2m+1)}{1\,.\,(2n-1)}\cos(n-m-2)\,\theta\right.$$

$$\left. + \frac{(n-m)(n-m-1)(2m+1)(2m+3)}{1\,.\,2\,.\,(2n-1)(2n-3)}\cos(n-m-4)\,\theta + \ldots\right\};$$

the series between the brackets is to be continued until it terminates of itself.

104. The last formula shews that if x is not greater than unity then ϖ is greatest when x is equal to unity. This value of ϖ may be found most readily in the following manner.

By (4) we have

$$\varpi(m, n) = \frac{\underline{|n-m}}{1\,.\,3\,.\,5\ldots(2n-1)}\,\frac{d^m P_n(x)}{dx^m};$$

and, by Art. 18, when $x = 1$ we have

$$\frac{d^m P_n(x)}{dx^m} = \frac{(n+m)(n+m-1)\ldots(n-m+1)}{\underline{|m}\,\underline{|m}}\,.\,\frac{\underline{|m}}{2^m}$$

$$= \frac{\underline{|n+m}}{2^m\,\underline{|m}\,\underline{|n-m}};$$

so that when $x = 1$ we have

$$\varpi(m, n) = \frac{\underline{|n+m}}{2^m\,\underline{|m}\,1\,.\,3\,.\,5\ldots(2n-1)}.$$

105. If in the process of Art. 103 we change the sign of m, we shall obtain an expansion for $\varpi(-m, n)$; and thus we deduce another formula for $\varpi(m, n)$ by aid of the relation $\varpi(m, n) = (x^2-1)^{-m}\varpi(-m, n)$ given in Art. 100.

106. We know from Art. 97 that $\varpi(m, n, \cos\theta)$

$$= \cos^{n-m}\theta - \frac{(n-m)(n-m-1)}{2.(2n-1)}\cos^{n-m-2}\theta + \frac{(n-m)\dots(n-m-3)}{2.4.(2n-1)(2n-3)}\cos^{n-m-4}\theta - \dots.$$

It is obvious that by virtue of the relation

$$\sin^2\theta + \cos^2\theta = 1,$$

this series may be put in the form

$$b_0\cos^{n-m}\theta + b_1\cos^{n-m-2}\theta\sin^2\theta + b_2\cos^{n-m-4}\theta\sin^4\theta + \dots$$

It will be found that we shall thus obtain $\varpi(m, n, \cos\theta)$.

$$= \varpi(m, n, 1)\left\{\cos^{n-m}\theta - \frac{(n-m)(n-m-1)}{4.(m+1)}\cos^{n-m-2}\theta\sin^2\theta + \frac{(n-m)\dots(n-m-3)}{4^2.(m+1)(m+2)}\cos^{n-m-4}\theta\sin^4\theta - \dots\right\} \dots\dots(12).$$

To establish this, let us suppose that the original series is denoted by

$$\cos^{n-m}\theta + a_1\cos^{n-m-2}\theta + a_2\cos^{n-m-4}\theta + \dots;$$

divide by $\cos^{n-m}\theta$, and put t for $\tan^2\theta$: then we must have

$$1 + a_1(1+t) + a_2(1+t)^2 + a_3(1+t)^3 + \dots = b_0 + b_1t + b_2t^2 + b_3t^3 + \dots,$$

and from this identity we are to find $b_0, b_1, b_2, b_3, \dots$

Equate the terms independent of t; thus we have

$$b_0 = 1 + a_1 + a_2 + a_3 + \dots,$$

that is $$b_0 = \varpi(m, n, 1).$$

Equate the coefficients of t^r; thus we have

$$b_r = a_r + (r+1)\,a_{r+1} + \frac{(r+2)(r+1)}{\lfloor 2}\,a_{r+2} + \frac{(r+3)(r+2)(r+1)}{\lfloor 3}\,a_{r+3} + \dots$$

$$= a_r \left\{1 + (r+1)\frac{a_{r+1}}{a_r} + \frac{(r+2)(r+1)}{\lfloor 2}\frac{a_{r+2}}{a_r} + \frac{(r+3)(r+2)(r+1)}{\lfloor 3}\frac{a_{r+3}}{a_r} + \dots\right\}$$

$$= a_r \left\{1 - \frac{(n-m-2r)(n-m-2r-1)}{2\,.\,(2n-2r-1)} + \frac{(n-m-2r)\dots(n-m-2r-3)}{2\,.\,4\,.\,(2n-2r-1)(2n-2r-3)} - \dots\right\}$$

$$= a_r \varpi(m+r,\ n-r,\ 1).$$

Similarly $b_{r+1} = a_{r+1}\varpi(m+r+1,\ n-r-1,\ 1)$.

Therefore $\dfrac{b_{r+1}}{b_r} = \dfrac{a_{r+1}}{a_r}\,.\,\dfrac{\varpi(m+r+1,\ n-r-1,\ 1)}{\varpi(m+r,\ n-r,\ 1)}$

$$= -\frac{(n-m-2r)(n-m-2r-1)}{2(2n-2r-1)}\,.\,\frac{\varpi(m+r+1,\ n-r-1,\ 1)}{\varpi(m+r,\ n-r,\ 1)};$$

by Art. 104 we find that this reduces to

$$\frac{b_{r+1}}{b_r} = -\frac{(n-m-2r)(n-m-2r-1)}{4(m+r+1)},$$

and by this law we obtain the series given in (12).

107. According to Arts. 97 and 99 the associated functions of the *first kind* are defined to be the product of a certain constant into $(x^2-1)^{\frac{m}{2}}\dfrac{d^mP_n(x)}{dx^m}$. In like manner the associated functions of the *second kind* are defined to be the product of a certain constant into $(x^2-1)^{\frac{m}{2}}\dfrac{d^mQ_n(x)}{dx^m}$.

Now

$$\frac{d^m Q_n(x)}{dx^m} = \lambda \left\{ x^{-n-m-1} + \frac{(n+m+1)(n+m+2)}{2.(2n+3)} x^{-n-m-3} + \ldots \right\},$$

where $\lambda = \frac{(-1)^m \lfloor n+m}{1.3.5\ldots(2n+1)}$. See Art. 37.

Hence we may conveniently take for the associated functions of the second kind the expression

$$\frac{1}{\lambda}(x^2-1)^{\frac{m}{2}} \frac{d^m Q_n(x)}{dx^m}.$$

108. The associated functions of the second kind may be put in various forms by the use of the various expressions which have been found for $Q_n(x)$.

For example, we have by Art. 37

$$\frac{1}{y-x} = \Sigma(2n+1) P_n(x) Q_n(y).$$

Differentiate m times with respect to y; thus

$$(-1)^m \frac{\lfloor m}{(y-x)^{m+1}} = \Sigma(2n+1) P_n(x) \frac{d^m Q_n(y)}{dy^m};$$

and therefore by Art. 28

$$\frac{d^m Q_n(y)}{dy^m} = \frac{(-1)^m \lfloor m}{2} \int_{-1}^{1} \frac{P_n(x)\,dx}{(y-x)^{m+1}}.$$

Hence, changing the notation, we have

$$\frac{d^m Q_n(x)}{dx^m} = \frac{(-1)^m \lfloor m}{2} \int_{-1}^{1} \frac{P_n(t)\,dt}{(x-t)^{m+1}}.$$

109. We shall not find it necessary to discuss the associated functions of the second kind beyond one more formula, which we will now give. Put

$$y \text{ for } C_1 (x^2-1)^{\frac{m}{2}} \frac{d^m P_n(x)}{dx^m} \text{ and } z \text{ for } C_2 (x^2-1)^{\frac{m}{2}} \frac{d^m Q_n(x)}{dx^m},$$

where C_1 and C_2 are constants; then we know that y and z both satisfy equation (9) of Art. 102, so that

$$(1-x^2)^2\frac{d^2y}{dx^2}-2x(1-x^2)\frac{dy}{dx}+\left\{n(n+1)-m^2-n(n+1)x^2\right\}y=0,$$

$$(1-x^2)^2\frac{d^2z}{dx^2}-2x(1-x^2)\frac{dz}{dx}+\left\{n(n+1)-m^2-n(n+1)x^2\right\}z=0.$$

Multiply the former by z and the latter by y, and subtract; thus

$$(x^2-1)\frac{d}{dx}\left\{y\frac{dz}{dx}-z\frac{dy}{dx}\right\}=-2x\left(y\frac{dz}{dx}-z\frac{dy}{dx}\right).$$

Hence by integration

$$y\frac{dz}{dx}-z\frac{dy}{dx}=\frac{C}{x^2-1}\ \ldots\ldots\ldots\ldots\ldots\ (13),$$

where C is a constant.

Then by integrating again,

$$\frac{z}{y}=-C\int_x^\infty\frac{dx}{(x^2-1)y^2}.$$

No additional constant is now required, because each side vanishes when x is infinite.

Now let C_1 and C_2 have such values that y and z represent exactly the associated functions of the first and second kind respectively. Then when x is very great we have ultimately $y=x^n$ and $z=x^{-n-1}$: see Arts. 99 and 107.

Hence by (13) we have $C=-(2n+1)$, and thus finally

$$\frac{z}{y}=(2n+1)\int_x^\infty\frac{dx}{(x^2-1)y^2}.$$

CHAPTER IX.

CONTINUED FRACTIONS.

110. It is shewn in the *Algebra*, Art. 801, that the quotient obtained by dividing a certain hypergeometrical series by another, namely, $\frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)}$, can be developed into a continued fraction.

For a special case we may suppose $\beta = 0$; and then $F(\alpha, \beta, \gamma, x)$ becomes unity, so that we obtain a continued fraction equal to $F(\alpha, 1, \gamma+1, x)$, that is equal to the series

$$1 + \frac{\alpha}{\gamma+1}x + \frac{\alpha(\alpha+1)}{(\gamma+1)(\gamma+2)}x^2 + \dots$$

As an example, suppose $\alpha = \frac{1}{2}$, and $\gamma = \frac{1}{2}$; and put $\frac{1}{y^2}$ for x: then we have a continued fraction for

$$1 + \frac{1}{3}y^{-2} + \frac{1}{5}y^{-4} + \dots,$$

that is for $\frac{y}{2}\log\frac{1+\frac{1}{y}}{1-\frac{1}{y}}$, that is for $\frac{y}{2}\log\frac{y+1}{y-1}$.

Hence, dividing by y, we obtain a continued fraction for $\frac{1}{2}\log\frac{y+1}{y-1}$; and the form of it is

$$\cfrac{y^{-1}}{1 - \cfrac{a_1 y^{-2}}{1 - \cfrac{a_2 y^{-2}}{1 - \cfrac{a_3 y^{-2}}{1 - \dots}}}}$$

that is,

$$\cfrac{1}{y-\cfrac{a_1}{y-\cfrac{a_2}{y-\cfrac{a_3}{y-\dots}}}}$$

Moreover $a_1=\frac{1}{3}$, $a_2=\frac{2.2}{3.5}$, $a_3=\frac{3.3}{5.7}$, $a_4=\frac{4.4}{7.9}$, ...

All this can be easily verified from the Article in the *Algebra* already cited.

111. But we now propose to find a continued fraction for $\frac{1}{2}\log\frac{x+1}{x-1}$ without the use of the general theory, merely by the aid of Legendre's Coefficients; and this process we give, not for the sake of the result which may be obtained in the way already noticed, but for the exemplification of the use of Legendre's Coefficients.

112. Consider the continued fraction

$$\cfrac{1}{x-\cfrac{a_1}{x-\cfrac{a_2}{x-\dots}}}$$

Let U_n denote the numerator and E_n the denominator of the n^{th} convergent to this continued fraction. Then

$$\left.\begin{array}{lll} U_1=1, & U_2=x, & U_3=x^2-a_2, \dots \\ E_1=x, & E_2=x^2-a_1, & E_3=x^3-(a_1+a_2)\,x, \dots \end{array}\right\}\ (1).$$

And we have in the usual way

$$\left.\begin{array}{l} U_n=x\,U_{n-1}-a_{n-1}\,U_{n-2}, \dots\dots\dots\dots\dots\dots \\ E_n=x\,E_{n-1}-a_{n-1}\,E_{n-2}. \dots\dots\dots\dots\dots\dots \end{array}\right\}\ (2).$$

Thus U_n is of the degree $n-1$ with respect to x, and E_n is of the degree n with respect to x.

From (2) we obtain

$$U_n E_{n-1} - E_n U_{n-1} = a_{n-1}\,(U_{n-1}E_{n-2} - U_{n-2}E_{n-1});$$

and from repeated applications of this formula we find with the aid of (1) that

$$U_n E_{n-1} - E_n U_{n-1} = a_{n-1} a_{n-2} a_{n-3} \dots a_1 \dots\dots\dots (3).$$

From (3) we obtain $\dfrac{U_{n+1}}{E_{n+1}} - \dfrac{U_n}{E_n} = \dfrac{a_1 a_2 \dots a_n}{E_n E_{n+1}}$,

$$\frac{U_{n+2}}{E_{n+2}} - \frac{U_{n+1}}{E_{n+1}} = \frac{a_1 a_2 \dots a_{n+1}}{E_{n+1} E_{n+2}}.$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Proceeding thus, adding the results, and denoting by λ the limit of $\dfrac{U_r}{E_r}$ when r is infinite, on the assumption that there is such a limit we get

$$\lambda - \frac{U_n}{E_n} = a_1 a_2 \dots a_n \left\{ \frac{1}{E_n E_{n+1}} + \frac{a_{n+1}}{E_{n+1} E_{n+2}} + \dots \right\}.$$

Thus we see that $\lambda E_n - U_n$ is such that if it be expanded in descending powers of x there will be no term with an exponent, algebraically greater than $-(n+1)$.

113. We can now arrive at some results respecting the forms of U_n and E_n. It will be found that

U_n is of the form $x^{n-1} + b_2 x^{n-3} + b_4 x^{n-5} + \dots$,

and E_n is of the form $x^n + c_2 x^{n-2} + c_4 x^{n-4} + \dots$;

that is, U_n contains only x^{n-1} and powers of x in which the exponent is $n-1$ diminished by some even number, while E_n contains only x^n and powers of x in which the exponent is n diminished by some even number. These laws follow immediately from (1) and (2).

114. We must now distinguish two cases.

I. Suppose *n even;* then E_n is of the form

$$x^n + c_2 x^{n-2} + c_4 x^{n-4} + \dots\dots + c_n.$$

II. Suppose *n odd;* then E_n is of the form

$$x(x^{n-1} + c_2 x^{n-3} + \dots\dots + c_{n-1}).$$

In both cases the product λE_n is to be free from the terms

$$x^{-1},\ x^{-2},\ \ldots\ldots\ x^{-n}.$$

Moreover we propose to take

$$\lambda = \frac{1}{2}\log\frac{x+1}{x-1} = x^{-1} + \frac{x^{-3}}{3} + \frac{x^{-5}}{5} + \ldots.$$

115. In case I. we find that no even power of x will occur in the product λE_n; and in order that $x^{-1}, x^{-3}, \ldots x^{-n+1}$ may disappear, we must have the following equations satisfied:

$$\frac{c_n}{1} + \frac{c_{n-2}}{3} + \ldots\ldots + \frac{1}{n+1} = 0,$$

$$\frac{c_n}{3} + \frac{c_{n-2}}{5} + \ldots\ldots + \frac{1}{n+3} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{c_n}{n-1} + \frac{c_{n-2}}{n+1} + \ldots\ldots + \frac{1}{2n-1} = 0.$$

Thus we have $\frac{n}{2}$ equations to determine the $\frac{n}{2}$ quantities

$$c_n,\ c_{n-2}, \ldots c_2.$$

Instead of solving these equations directly, we may proceed indirectly.

It is obvious that these $\frac{n}{2}$ equations amount to the following:

$$\int_{-1}^{1} E_n\,dx = 0,\ \int_{-1}^{1} E_n x^2 dx = 0, \ldots \int_{-1}^{1} E_n x^{n-2}\,dx = 0;$$

and since E_n involves only even powers of x, we know that

$$\int_{-1}^{1} E_n x dx = 0,\ \int_{-1}^{1} E_n x^3 dx = 0, \ldots \int_{-1}^{1} E_n x^{n-1}\,dx = 0.$$

Hence it follows, by Art. 32, that E_n must be of the form $k\,\frac{d^n(x^2-1)^n}{dx^n}$, where k is some constant.

116. In case II, by proceeding in the same way as for case I, we shall again arrive at the result that E_n is of the form $k\dfrac{d^n(x^2-1)^n}{dx^n}$.

117. Since we know that the first term of E_n is x^n, it follows that $k=\dfrac{\lfloor n}{\lfloor 2n}$. Thus

$$E_n=\frac{2^n\lfloor n\lfloor n}{\lfloor 2n}P_n(x).$$

118. We have next to find U_n.

Since $\lambda-\dfrac{U_n}{E_n}$ involves only $x^{-n-1}, x^{-n-2},\ldots$, it follows that U_n is equal to the integral part of the product λE_n, that is, to the integral part of

$$\frac{1}{2}\log\frac{x+1}{x-1}\cdot\frac{2^n\lfloor n\lfloor n}{\lfloor 2n}P_n(x).$$

But by Art. 78 we have

$$\frac{1}{2}P_n(x)\log\frac{x+1}{x-1}=R+Q_n(x),$$

where R is integral, and $Q_n(x)$ is fractional.

Hence it follows that $U_n=\dfrac{2^n\lfloor n\lfloor n}{\lfloor 2n}R$, where R has the value found in Art. 78.

119. Thus if $\frac{1}{2}\log\dfrac{x+1}{x-1}$ can be developed into a continued fraction of the form given at the beginning of Art. 112, we have determined the n^{th} convergent. It remains to shew that $\frac{1}{2}\log\dfrac{x+1}{x-1}$ really can be developed in this form; and also to find $a_1, a_2, a_3, \ldots$.

We know that $\dfrac{1}{2}\log\dfrac{x+1}{x-1}=\dfrac{R}{P_n(x)}+\dfrac{Q_n(x)}{P_n(x)}$.

Now suppose, as we do throughout this process, that x is greater than unity; then $Q_n(x)$ vanishes when n is indefinitely great. Hence $\frac{1}{2}\log\frac{x+1}{x-1}$ = the limit of $\frac{R}{P_n(x)}$ when n is indefinitely great.

We know by Art. 56 that

$$nP_n(x) - (2n-1)\,xP_{n-1}(x) + (n-1)\,P_{n-2}(x) = 0;$$

let Y_n stand for $\frac{2^n \lfloor n \lfloor n}{\lfloor 2n} P_n(x)$, that is for

$$\frac{\lfloor n}{1.3.5\ldots(2n-1)} P_n(x);$$

thus $$Y_n(x) - xY_{n-1}(x) + \frac{(n-1)^2}{(2n-3)(2n-1)} Y_{n-2}(x) = 0,$$

so that $$Y_n(x) = xY_{n-1}(x) - a_{n-1}\,Y_{n-2}(x);\ \ldots\ldots\ldots (4),$$

where $$a_{n-1} = \frac{(n-1)^2}{(2n-3)(2n-1)}.$$

Multiply both sides of (4) by $\frac{1}{2}\log\frac{x+1}{x-1}$; then each term gives rise to an integral and a fractional part, and denoting by $Z_n(x)$ the integral part of $\frac{1}{2}Y_n\log\frac{x+1}{x-1}$, we get

$$Z_n(x) = xZ_{n-1}(x) - a_{n-1}Z_{n-2}(x)\ \ldots\ldots\ldots (5).$$

From (4) and (5) we see that $\frac{Z_n}{Y_n}$ can be put in a continued fraction of the required form, extending as far as the component $\frac{a_{n-1}}{x}$. And $\frac{Z_n}{Y_n}$ is equal to $\frac{R}{P_n(x)}$.

Also $a_1 = \frac{1.1}{1.3}$, $a_2 = \frac{2.2}{3.5}$, and generally

$$a_m = \frac{m^2}{(2m-1)(2m+1)}.$$

CHAPTER X.

APPROXIMATE QUADRATURE.

120. SUPPOSE that we require the value of a certain integral between definite limits, say $\int_{-1}^{1} f(x)\,dx$; if the indefinite integral is known, we can at once by taking the values at the limits determine the definite integral. But if the indefinite integral is not known, we are in general compelled to use processes of approximation, and such processes may also be advantageous in some cases where the indefinite integral is known, but is of a very complex form. One of the most obvious applications of the result is to find the area of a figure bounded by a given curve, certain fixed ordinates, and the axis of abscissæ; and thus it is frequently described as the approximate determination of the areas of curves, or in old language as the approximate quadrature of curves.

121. The matter is discussed in the *Integral Calculus*, Chapter VII, and various rules concerning it are there given; these rules all imply that we draw *equidistant* ordinates between the two fixed ordinates. The method of Gauss, which we are about to explain, implies also that intermediate ordinates are drawn, but *not at equal distances*, and in fact proposes to determine the law of succession of these ordinates in such a manner as to ensure the most advantageous result.

122. Let $f(x)$ denote any function of x, which is supposed to remain continuous between the limits -1 and $+1$ for x. Now a function of x can always be found, which is rational and integral and of the degree $n-1$, and which is equal in value to $f(x)$ when x has any one of n specified values.

For let $a_1, a_2, \ldots a_n$ denote these specified values; put

$$\psi(x) = (x - a_1)(x - a_2)\ldots\ldots(x - a_n),$$

$$\phi(x) = \psi(x)\left\{\frac{f(a_1)}{(x - a_1)\psi'(a_1)} + \frac{f(a_2)}{(x - a_2)\psi'(a_2)} + \ldots\ldots + \frac{f(a_n)}{(x - a_n)\psi'(a_n)}\right\}:$$

then $\phi(x)$ is such a function as is required.

For $\phi(x)$ is obviously rational and integral and of the degree $n - 1$. Also the value of $\frac{\psi(x)}{x - a_r}$ when $x = a_r$ is $\psi'(a_r)$; and thus the value of $\phi(x)$ when $x = a_r$ is $f(a_r)$. Moreover there is only one such function. For if there could be another denote it by $\chi(x)$. Then $\phi(x)$ and $\chi(x)$ are equal when x has any of the values $a_1, a_2, \ldots a_n$; thus $\phi(x) - \chi(x)$ vanishes for n different values of x, which is impossible, since $\phi(x) - \chi(x)$ is of the degree $n - 1$ at the highest.

123. We may suppose that the n values $a_1, a_2, \ldots a_n$ all fall between -1 and $+1$; thus, using geometrical language, the curves $y = \phi(x)$ and $y = f(x)$ have n points in common, corresponding to abscissæ between -1 and $+1$: and $\int_{-1}^{1} \phi(x)\,dx$ may be taken as an approximate value of $\int_{-1}^{1} f(x)\,dx$, subject of course to some examination of the amount of the error thus introduced.

124. Let $\frac{1}{\psi'(a_r)}\int_{-1}^{1}\frac{\psi(x)}{x - a_r}\,dx$ be denoted by A_r; then

$$\int_{-1}^{1} \phi(x)\,dx = A_1 f(a_1) + A_2 f(a_2) + \ldots\ldots + A_n f(a_n) \ \ldots\ldots \ (1).$$

Now here it will be observed that A_r is quite independent of the form of the function $f(x)$; so that when $A_1, A_2, \ldots\ldots A_n$ have once been calculated, we can use them in (1) whatever $f(x)$ may be.

125. The older methods of approximate quadrature used, as we have said, *equidistant* ordinates. According to this method we should have

$$a_1 = -1,\ a_n = 1,\ a_r = -1 + \frac{2(r-1)}{n-1} = \frac{2r-n-1}{n-1},$$

so that $$a_{n-r+1} = -1 + \frac{2(n-r)}{n-1} = \frac{n-2r+1}{n-1} = -a_r.$$

Thus $\psi(x) = (x-a_1)(x+a_1)(x-a_2)(x+a_2)\ldots\ldots;$
so that if n be even $\psi(x)$ involves only factors of the form $x^2 - a^2$, but if n be odd one factor is x.

Hence $$\psi(-x) = (-1)^n \psi(x);$$
and therefore $$-\psi'(-x) = (-1)^n \psi'(x),$$
so that $$\psi'(-x) = (-1)^{n+1} \psi'(x).$$

Now
$$\begin{aligned} A_{n-r+1} &= \frac{1}{\psi'(a_{n-r+1})}\int_{-1}^{1} \frac{\psi(x)\,dx}{x - a_{n-r+1}} \\ &= \frac{1}{\psi'(-a_r)}\int_{-1}^{1} \frac{\psi(x)\,dx}{x + a_r} \\ &= \frac{1}{(-1)^{n+1}\psi'(a_r)}\int_{-1}^{1} \frac{\psi(-x)\,dx}{a_r - x} \\ &= \frac{1}{(-1)^{n+1}\psi'(a_r)}\int_{-1}^{1} \frac{(-1)^n\psi(x)\,dx}{a_r - x} \\ &= \frac{1}{\psi'(a_r)}\int_{-1}^{1} \frac{\psi(x)\,dx}{x - a_r} = A_r. \end{aligned}$$

Thus the quantities $A_1, A_2, \ldots\ldots A_n$ are such that those which are equidistant from the first and the last are equal.

126. The *error* which arises from taking the approximate quadrature instead of the real quadrature is

$$\int_{-1}^{1} f(x)\,dx - \Sigma A_r f(a_r);$$

here and throughout the Chapter Σ denotes a summation with respect to r from $r=1$ to $r=n$, both inclusive.

Now if $f(x)$ be a rational integral function of x of a degree not exceeding $n-1$ this will vanish, for then $f(x)$ is identical with $\phi(x)$, and there is no *error* at all. This holds then for the ordinary process of approximate quadrature, since it holds whatever may be the law by which $a_1, a_2, \ldots\ldots a_n$ are determined.

Gauss proposed to take $a_1, a_2, \ldots\ldots a_n$ in such a manner that the error should also vanish when $f(x)$ is any rational integral function of x of a degree not exceeding $2n-1$. To this we now proceed.

Suppose then that $f(x)$ is of the degree $2n-1$. Since $f(x) - \phi(x)$ vanishes when x has any of the values $a_1, a_2, \ldots a_n$, it follows that $f(x) - \phi(x)$ is divisible by $\psi(x)$. Assume then that $\frac{f(x) - \phi(x)}{\psi(x)} = c_0 + c_1 x + c_2 x^2 + \ldots\ldots + c_{n-1} x^{n-1}$,

so that $f(x) = \phi(x) + \psi(x) \{c_0 + c_1 x + c_2 x^2 + \ldots\ldots + c_{n-1} x^{n-1}\}$.

By ascribing suitable values to $c_0, c_1, \ldots c_n$, we may obtain every possible form of $f(x)$ of the degree $2n-1$, under the condition that $f(x) - \phi(x)$ vanishes for the n specified values of x.

In order then that $\int_{-1}^{1} f(x)\,dx - \int_{-1}^{1} \phi(x)\,dx$ may vanish for every possible form of $f(x)$ of the assigned degree, we must have $\int_{-1}^{1} x^r \psi(x)\,dx = 0$ for all positive integral values of r between 0 and $n-1$ inclusive. Hence it follows by Art. 32 that $\psi(x)$ must be of the form $CP_n(x)$, where C is a constant; and therefore the roots of $\psi(x) = 0$ must be those of $P_n(x) = 0$. This determines the law of succession of the quantities $a_1, a_2, \ldots a_n$.

Since the coefficient of x^n in $\psi(x)$ is supposed to be unity we must have

$$C = \frac{\lfloor n}{1 \,.\, 3 \,.\, 5 \ldots (2n-1)}.$$

127. Since by Art. 7 we have $a_n = -a_1$, and $a_{n-r+1} = -a_r$, it follows by Art. 125 that $A_{n-r+1} = A_r$. When n is odd the middle term of the set $a_1, a_2, \ldots a_n$ is zero.

128. Thus we see that if $f(x)$ be rational and integral and of the degree $2n-1$ at the highest then $\int_{-1}^{1} f(x)\,dx$ is

exactly equal to $\int_{-1}^{1} \phi(x)\,dx$, when $a_1, a_2, \ldots a_n$ are the roots of $P_n(x)=0$; or, to use geometrical language, the area of the figure bounded by a portion of the curve $y=f(x)$, two fixed ordinates, and the axis of abscissæ, can be determined *exactly* when besides the two fixed ordinates we know n intermediate ordinates at suitably selected intervals.

We proceed to consider the amount of error which the method of Gauss involves when $f(x)$ is no longer restricted to be of the degree $2n-1$ at the highest.

129. Suppose that $f(x)$ can be expanded in a convergent series so that

$$f(x)=b_0+b_1x+b_2x^2+\ldots+b_nx^n+\ldots\ldots\ldots \quad (2).$$

The whole *error* is $\int_{-1}^{1} f(x)\,dx-\Sigma A_r f(a_r)$. Put for $f(x)$ and for $f(a_r)$ their expansions from (2); then the *error* will consist of a series of terms of which the type is

$$b_m\left\{\int_{-1}^{1} x^m\,dx-\Sigma A_r a_r^m\right\};$$

we will denote this by $b_m E_m$.

Now we know from Art. 126 that E_m vanishes if m be not greater than $2n-1$, so that the whole *error* reduces to

$$b_{2n}E_{2n}+b_{2n+1}E_{2n+1}+b_{2n+2}E_{2n+2}+\ldots\ldots$$

130. We have first to observe that all the terms with *odd* suffixes will disappear from the preceding series; that is, $2p+1$ being any odd number, we shall have

$$\int_{-1}^{1} x^{2p+1}\,dx-\Sigma A_r a_r^{2p+1}=0.$$

For $\int_{-1}^{1} x^{2p+1}\,dx$ is obviously zero; and $\Sigma A_r a_r^{2p+1}$ is zero by reason of the facts mentioned in Art. 127.

131. Consider then E_{2p}, that is $\int_{-1}^{1} x^{2p}\,dx-\Sigma A_r a_r^{2p}$, that is $\frac{2}{2p+1}-\Sigma A_r a_r^{2p}$; it is obvious that this is equal to the

coefficient of z^{-2p-1} in the development of $\log\frac{z+1}{z-1}-z\Sigma\frac{A_r}{z^2-a_r^2}$ in descending powers of z.

Now $$A_r=\frac{1}{P'_n(a_r)}\int_{-1}^{1}\frac{P_n(x)\,dx}{x-a_r};$$

let $$\chi(z)=\int_{-1}^{1}\frac{P_n(x)-P_n(z)}{x-z}\,dx \qquad (3),$$

then $$\chi(a_r)=\int_{-1}^{1}\frac{P_n(x)}{x-a_r}\,dx, \text{ for } P_n(a_r)=0;$$

thus $$A_r=\frac{\chi(a_r)}{P'_n(a_r)} \qquad (4).$$

But $\chi(z)$ is a rational integral function of z of the degree $n-1$, and therefore by Art. 122 we have

$$\chi(z)=P_n(z)\,\Sigma\,\frac{\chi(a_r)}{P'_n(a_r)(z-a_r)} \qquad (5).$$

Thus from (4) and (5) we get

$$\chi(z)=P_n(z)\,\Sigma\,\frac{A_r}{z-a_r}.$$

But by Art. 127 we can also write this

$$\chi(z)=P_n(z)\,\Sigma\,\frac{A_r}{z+a_r},$$

and therefore, by addition,

$$\chi(z)=zP_n(z)\,\Sigma\,\frac{A_r}{z^2-a_r^2}.$$

Hence $z\Sigma\frac{A_r}{z^2-a_r^2}=\frac{\chi(z)}{P_n(z)}$, and therefore E_{2p} is equal to the coefficient of z^{-2p-1} in the development of $\log\frac{z+1}{z-1}-\frac{\chi(z)}{P_n(z)}$ in descending powers of z.

But by (3) we have

$$\chi(z)=-P_n(z)\int_{-1}^{1}\frac{dx}{x-z}+\int_{-1}^{1}\frac{P_n(x)}{x-z}\,dx$$
$$=P_n(z)\log\frac{z+1}{z-1}-2Q_n(z), \text{ by Art. 80.}$$

Hence finally E_{2p} is equal to the coefficient of z^{-2p-1} in the development of $\frac{2Q_n(z)}{P_n(z)}$ in descending powers of z.

Let $\mu = \frac{2}{2n+1}\left\{\frac{1 . 2 . 3 \ldots n}{1 . 3 . 5 \ldots (2n-1)}\right\}^2$; then we have

$$\frac{2Q_n(z)}{P_n(z)} = \mu \frac{z^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} z^{-n-3} + \ldots}{z^n - \frac{n(n-1)}{2(2n-1)} z^{n-2} + \ldots}.$$

If this be developed in descending powers of z we obtain

$$\mu z^{-2n-1} + \frac{\mu}{2}\left\{\frac{(n+1)(n+2)}{2n+3} + \frac{n(n-1)}{2n-1}\right\} z^{-2n-3} + \ldots;$$

thus we have $$E_{2n} = \mu,$$

$$E_{2n+2} = \frac{\mu}{2}\left\{\frac{(n+1)(n+2)}{2n+3} + \frac{n(n-1)}{2n-1}\right\}.$$

132. We may investigate somewhat more closely the extent of the error to which the new method of approximation is exposed.

By Arts. 72 and 77 we have

$$\frac{2Q_n(z)}{P_n(z)} = \log\frac{z+1}{z-1} + 2\Sigma \frac{C_r}{z-a},$$

where $$C_r = \frac{(z-a_r)^2}{\{P_n(z)\}^2 (z^2-1)} \text{ when } z = a_r,$$

so that $$C_r = \frac{1}{\{P'_n(a_r)\}^2 (a_r^2 - 1)}.$$

Thus

$$\frac{2Q_n(z)}{P_n(z)} = \log\frac{z+1}{z-1} + 2\Sigma \frac{1}{\{P'_n(a_r)\}^2 (a_r^2-1)(z-a_r)}.$$

But since $a_{n-r+1} = -a_r$ we may write this thus:

$$\frac{2Q_n(z)}{P_n(z)} = \log\frac{z+1}{z-1} + 2z\Sigma \frac{1}{\{P'_n(a_r)\}^2 (a_r^2-1)(z^2-a_r^2)}.$$

Let this be developed in descending powers of z, then we find that the coefficient of z^{-2p-1}, that is E_{2p}, is

$$\frac{2}{2p+1} - 2\Sigma \frac{1}{1-a_r^2} \left\{ \frac{a_r^p}{P'_n(a_r)} \right\}^2.$$

By comparing this with the value of E_{2p} at the beginning of Art. 131, we see that $A_r = \frac{1}{1-a_r^2} \left\{ \frac{1}{P'_n(a_r)} \right\}^2$. This furnishes a new expression for A_r, and shews that it is necessarily positive.

Let S_{2p} stand for $\Sigma A_r a_r^{2p}$, so that $E_{2p} = \frac{2}{2p+1} - S_{2p}$. Let β denote the numerically greatest of the quantities $a_1, a_2, \dots a_n$; then since $A_1, A_2, \dots A_n$ are positive it is obvious that S_{2p+2} is less than $\beta^2 S_{2p}$. But we know that E_{2n-2} is zero, so that $S_{2n-2} = \frac{2}{2n-1}$; hence it follows that $S_{2n+2q-2}$ is less than $\frac{2\beta^{2q}}{2n-1}$, and therefore $E_{2n+2q-2}$ cannot differ from $\frac{2}{2n+2q-1}$ by so much as $\frac{2\beta^{2q}}{2n-1}$. We may observe that each of the quantities $A_1, A_2, \dots A_n$ is less than 2. For since E_{2p} is zero when p is zero, we have

$$A_1 + A_2 + \dots + A_n = 2.$$

Moreover when n is even each of the quantities is less than unity, since any two equidistant from the first and the last are equal.

133. Let us now make some comparison between Gauss's method and the old method of *equidistant* ordinates. We suppose that n ordinates are used besides the extreme ordinates. Suppose as before that $f(x)$ can be put in the form (2). Then according to the old method the *error* may be denoted by $b_n E_n + b_{n+1} E_{n+1} + b_{n+2} E_{n+2} + \dots$, and by the principles of Art. 130 this reduces to $b_n E_n + b_{n+2} E_{n+2} + b_{n+4} E_{n+4} + \dots$ if n be even, and to $b_{n+1} E_{n+1} + b_{n+3} E_{n+3} + b_{n+5} E_{n+5} + \dots$ if n be *odd*.

According to Gauss's method the *error* may be denoted by $b_{2n}E_{2n} + b_{2n+2}E_{2n+2} + b_{2n+4}E_{2n+4} + \dots$. But it must be remembered that such a symbol as E_m does not denote the same thing in the two methods; for this reason, and because $b_n, b_{n+1}, \dots$ are not known until $f(x)$ is specially assigned, we cannot make any close arithmetical comparison between the two methods.

If the expansion of $f(x)$ is extremely convergent, so that the quantities $b_n, b_{n+1}, b_{n+2}, \dots$ form a rapidly diminishing series, we may draw two general inferences.

I. In the application of the old method if n be an odd number, then n ordinates are as advantageous as $n + 1$.

II. The new method by using n ordinates is about as advantageous as the old method would be by the use of $2n$ ordinates.

134. There is another mode of investigating the results of Art. 131 which may be noticed. We propose in fact, using the notation of Art. 122, to find the value of

$$\int_{-1}^{1} \left\{f(x) - \phi(x)\right\} dx.$$

Now since $f(x) - \phi(x)$ vanishes when $\psi(x)$ vanishes, we will assume that $f(x) - \phi(x)$ is divisible by $\psi(x)$; this would certainly be true if the expansion of $f(x)$ consisted of a *finite* number of terms, and on the supposition that the expansion of $f(x)$ is highly convergent, we may admit that $f(x)$ may be treated practically as if there were only a finite number of terms.

Let then $$f(x) - \phi(x) = \psi(x)\,\chi(x),$$

so that $\chi(x)$ may be considered to be equal to the expansion of $\dfrac{f(x) - \phi(x)}{\psi(x)}$ in ascending powers of x.

Now $\phi(x)$ is of one dimension lower than $\psi(x)$, and so the expansion of $\dfrac{\phi(x)}{\psi(x)}$ will consist only of negative powers of x; hence these negative powers will cancel those arising from the expansion of $\dfrac{f(x)}{\psi(x)}$, and leave $\chi(x)$ equal to the aggregate of the terms in the expansion of $\dfrac{f(x)}{\psi(x)}$ which involve *positive* powers of x.

Suppose then $\dfrac{1}{\psi(x)} = \dfrac{\beta_0}{x^n} + \dfrac{\beta_2}{x^{n+2}} + \dfrac{\beta_4}{x^{n+4}} + \ldots;$ ....(6),

for it is obvious that the other powers of x will not occur in the expansion, since $\psi(x)$ involves only x^n and other terms in which the exponent differs from n by an *even* number. Since

$$\psi(x) = x^n - \frac{n(n-1)}{2.(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} - \ldots,$$

the values of $\beta_0, \beta_2, \beta_4, \ldots$ are found in succession from the equations

$$1 = \beta_0,$$

$$0 = \beta_2 - \frac{n(n-1)}{2.(2n-1)} \beta_0,$$

$$0 = \beta_4 - \frac{n(n-1)}{2.(2n-1)} \beta_2 + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} \beta_0,$$

....

Now the *error* $= \displaystyle\int_{-1}^{1} \left\{f(x) - \phi(x)\right\} dx = \int_{-1}^{1} \psi(x)\, \chi(x)\, dx;$

and $\chi(x) =$ that part of $\dfrac{f(x)}{\psi(x)}$ which involves *positive* powers of x

$$= b_n\beta_0 + b_{n+1}\beta_0 x + b_{n+2}(\beta_0 x^2 + \beta_2) + b_{n+3}(\beta_0 x^3 + \beta_2 x)$$
$$+ b_{n+4}(\beta_0 x^4 + \beta_2 x^2 + \beta_4) + \ldots,$$

by (2) and (6): and thus the *error* becomes

$$b_n B_0 + b_{n+1} B_1 + b_{n+2} B_2 + \ldots,$$

where B_m stands for $\displaystyle\int_{-1}^{1} (\beta_0 x^m + \beta_2 x^{m-2} + \beta_4 x^{m-4} + \ldots)\, \psi(x)\, dx.$

Now $\int_{-1}^{1} x^m \psi(x)\,dx$ vanishes if m is less than n; and thus the *error* reduces to $b_{2n}B_n + b_{2n+1}B_{n+1} + b_{2n+2}B_{n+2} + \ldots.$

Also $\psi(x) = CP_n(x)$, where C stands for $\dfrac{\lfloor n}{1.3.5\ldots(2n-1)}$; and thus the *error*

$$= Cb_{2n}\int_{-1}^{1} \beta_0 x^n P_n(x)\,dx + Cb_{2n+1}\int_{-1}^{1} \beta_0 x^{n+1} P_n(x)\,dx$$

$$+ Cb_{2n+2}\int_{-1}^{1} (\beta_0 x^{n+2} + \beta_2 x^n)\,P_n(x)\,dx + \ldots.$$

The integrations may be effected by Art. 35, and thus giving to μ the same value as in Art. 131, we find that the *error*

$$= \mu b_{2n}\beta_0 + \mu b_{2n+2}\left[\beta_0 \frac{(n+1)(n+2)}{2.(2n+3)} + \beta_2\right]$$

$$+ \mu b_{2n+4}\left[\beta_0 \frac{(n+1)\ldots(n+4)}{2.4.(2n+3)(2n+5)} + \beta_2 \frac{(n+1)(n+2)}{2.(2n+3)} + \beta_4\right] + \ldots$$

135. We have supposed throughout that the limits of the integration are -1 and $+1$; but by an easy transformation we can adapt the process to the case of any other limits. Suppose, for example, that we put $x = 2\xi - 1$; then $\xi = 0$ when $x = -1$, and $\xi = 1$ when $x = 1$, so that

$$\int_{-1}^{1} f(x)\,dx = 2\int_{0}^{1} f(2\xi - 1)\,d\xi.$$

Let $f(2\xi - 1)$ be called $\phi(\xi)$; then

$$\int_{0}^{1} \phi(\xi)\,d\xi = \frac{1}{2}\int_{-1}^{1} f(x)\,dx,$$

or

$$\int_{0}^{1} \phi(\xi)\,d\xi = \frac{1}{2}\int_{-1}^{1} \phi\left(\frac{1+x}{2}\right)dx \;\ldots\ldots\ldots\ldots\ldots(7),$$

and thus we shall have approximately by Gauss's method

$$\int_0^1 \phi(\xi)\, d\xi = \frac{1}{2} \Sigma A_r \phi\left(\frac{1+a_r}{2}\right) \qquad \text{............(8).}$$

Let $\frac{A_r}{2} = C_r$, and $\frac{1+a_r}{2} = \gamma_r$: then approximately

$$\int_0^1 \phi(\xi)\, d\xi = \Sigma C_r \phi(\gamma_r) \qquad \text{................ (9).}$$

Gauss has calculated the quantities of which C_r and γ_r are the types, for all values of n from 1 to 7 inclusive; we will give his results in an abridged form at the end of the Chapter.

It will be observed that $\gamma_1, \gamma_2, \ldots \gamma_n$ are the roots of the equation $\frac{d^n (x^2-1)^n}{dx^n} = 0$, when for x we put $2\xi - 1$; so that they are the roots of $\frac{d^n \xi^n (\xi - 1)^n}{d\xi^n} = 0$; that is they are the roots of

$$\xi^n - \frac{n^2}{1 \,.\, 2n} \xi^{n-1} + \frac{n^2 (n-1)^2}{1 \,.\, 2 \,.\, 2n (2n-1)} \xi^{n-2} + \ldots = 0.$$

The roots of $P_n(x) = 0$ can be obtained from the values which we shall give for $\gamma_1, \gamma_2, \ldots \gamma_n$, by the relation $a_r = 2\gamma_r - 1$.

Again, to estimate the *error* produced by using (9), suppose that

$$\phi\left(\frac{1}{2} + \frac{x}{2}\right) = L_0 + L_1 \frac{x}{2} + L_2 \left(\frac{x}{2}\right)^2 + \ldots,$$

then as this is the expansion of $f(x)$ the former notation and the present are connected by the relations

$$b_{2n} = \frac{L_{2n}}{2^{2n}}, \quad b_{2n+2} = \frac{L_{2n+2}}{2^{2n+2}}, \ldots.$$

Moreover from (7) and (8) we see that the expression for the *error* will be *half* that formerly obtained; so that it will be

$$\frac{\mu}{2} b_{2n} + \frac{\mu}{4} \left\{ \frac{(n+1)(n+2)}{2n+3} + \frac{n(n-1)}{2n-1} \right\} b_{2n+2} + \ldots,$$

that is

$$\frac{\mu}{2^{2n+1}} L_{2n} + \frac{\mu}{2^{2n+4}} \left\{ \frac{(n+1)(n+2)}{2n+3} + \frac{n(n-1)}{2n-1} \right\} L_{2n+2} + \ldots.$$

136. We will now give the numerical values required in the formula (9) for the values of n from 1 to 7 inclusive.

$n = 1$

$\gamma_1 = \cdot 5$

$C_1 = 1$

$n = 2$

$\gamma_1 = \cdot 2113248654$

$\gamma_2 = \cdot 7886751346$

$C_1 = C_2 = \cdot 5$

$n = 3$

$\gamma_1 = \cdot 1127016654$

$\gamma_2 = \cdot 5$

$\gamma_3 = \cdot 8872983346$

$C_1 = C_3 = \frac{5}{18}$

$C_2 = \frac{4}{9}$

$n = 4$

$\gamma_1 = \cdot 0694318442$

$\gamma_2 = \cdot 3300094782$

$\gamma_3 = \cdot 6699905218$

$\gamma_4 = \cdot 9305681558$

$C_1 = C_4 = \cdot 1739274226$

$C_2 = C_3 = \cdot 3260725774$

$n = 5$

$\gamma_1 = \cdot 0469100770$

$\gamma_2 = \cdot 2307653449$

$\gamma_3 = \cdot 5$

$\gamma_4 = \cdot 7692346551$

$\gamma_5 = \cdot 9530899230$

$C_1 = C_5 = \cdot 1184634425$

$C_2 = C_4 = \cdot 2393143352$

$C_3 = \cdot 2844444444$

$n = 6$

$\gamma_1 = \cdot 0337652429$

$\gamma_2 = \cdot 1693953068$

$\gamma_3 = \cdot 3806904070$

$\gamma_4 = \cdot 6193095930$

$\gamma_5 = \cdot 8306046932$

$\gamma_6 = \cdot 9662347571$

$C_1 = C_6 = \cdot 0856622462$

$C_2 = C_5 = \cdot 1803807865$

$C_3 = C_4 = \cdot 2339569673$

$$n = 7$$

$\gamma_1 = \cdot 0254460438286202$

$\gamma_2 = \cdot 1292344072003028$

$\gamma_3 = \cdot 2970774243113015$

$\gamma_4 = \cdot 5$

$\gamma_5 = \cdot 7029225756886985$

$\gamma_6 = \cdot 8707655927996972$

$\gamma_7 = \cdot 9745539561713798$

$C_1 = C_7 = \cdot 0647424830844348$

$C_2 = C_6 = \cdot 1398526957446384$

$C_3 = C_5 = \cdot 1909150252525595$

$C_4 = \cdot 2089795918367347$

CHAPTER XI.

EXPANSION OF FUNCTIONS IN TERMS OF LEGENDRE'S COEFFICIENTS.

137. We have seen in Art. 27 that any positive integral power of x can be expressed in terms of Legendre's Coefficients; and hence also any rational integral function of x can be so expressed. We have next to determine whether *any function* whatever of x can be so expressed; this matter however is somewhat difficult, and we shall treat it very briefly here, as it will come before us again when we consider the more general Coefficients which we call Laplace's, and which include Legendre's as a particular case.

138. Let $f(x)$ denote any function of x; if possible suppose that

$$f(x) = a_0 + a_1 P_1(x) + a_2 P_2(x) + \dots\dots\dots\dots (1),$$

where $a_0, a_1, a_2, \dots$ are constants at present undetermined.

Let n be any positive integer; multiply both sides of (1) by $P_n(x)$, and integrate between the limits -1 and $+1$; thus by Art. 28

$$\int_{-1}^{1} P_n(x) f(x)\, dx = \frac{2a_n}{2n+1},$$

therefore $$a_n = \frac{2n+1}{2} \int_{-1}^{1} P_n(x) f(x)\, dx \dots\dots\dots\dots (2).$$

Thus if $f(x)$ can be expressed in the form (1) the constants $a_1, a_2, \dots$ must have the values assigned by (2).

The formula (2) implies that $f(x)$ remains finite between the limits -1 and $+1$ of x: this condition then must be understood in all which follows.

139. Since the constants in (1) are thus determined it follows implicitly that there can be *only one form* for the expression of a function in terms of Legendre's Coefficients; this may be shewn more explicitly in the following manner.

If possible suppose that

$$f(x) = a_0 + a_1 P_1(x) + a_2 P_2(x) + \ldots,$$

and also

$$= b_0 + b_1 P_1(x) + b_2 P_2(x) + \ldots.$$

By subtraction,

$$0 = a_0 - b_0 + (a_1 - b_1) P_1(x) + (a_2 - b_2) P_2(x) + \ldots.$$

Let n be any positive integer; multiply by $P_n(x)$ and integrate between the limits -1 and $+1$: thus by Art. 28

$$0 = \frac{2(a_n - b_n)}{2n+1}.$$

Therefore $a_n = b_n$: and thus the two expressions coincide.

140. We have shewn that if $f(x)$ *can* be expressed in terms of Legendre's Coefficients the expression takes a single definite form; but we have still to shew that such a mode of expression is always possible. This we shall do, at least partially and indirectly, by finding the value of

$$\Sigma \frac{2n+1}{2} P_n(x) \int_{-1}^{1} P_n(x) f(x)\, dx,$$

where Σ denotes summation with respect to n from zero to infinity. We shall require an auxiliary proposition that will now be given.

141. If $\phi(x)$ be such that it is always finite and that $\int_p^q x^n \phi(x)\, dx$ vanishes, where p and q are fixed, and n takes successively every positive integral value, then $\phi(x)$ must be always zero between the limits p and q.

For if $\phi(x)$ be not always zero between these limits it must change sign once or oftener. Suppose $\phi(x)$ to change its sign m times, and let $x_1, x_2, \ldots x_m$ denote the values of x at which the changes take place. Let

$$\psi(x) = (x - x_1)(x - x_2) \ldots (x - x_m);$$

then by multiplying out we have

$$\psi(x) = x^m + A_1 x^{m-1} + A_2 x^{m-2} + \ldots + A_m,$$

where $A_1, A_2, \ldots A_m$ are constants.

Now we have by supposition

$$\int_p^q x^n \phi(x)\, dx = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

In (3) put for n in succession $m, m-1, \ldots 0$; and add the results multiplied respectively by $1, A_1, \ldots A_m$. Thus we get

$$\int_p^q \psi(x)\, \phi(x)\, dx = 0.$$

But this is manifestly absurd, for $\psi(x)$ and $\phi(x)$ change sign together, so that $\psi(x)\, \phi(x)$ does not change sign.

The condition that $\phi(x)$ is to remain finite is introduced because we can have no confidence in the results of integration when the function to be integrated becomes infinite.

142. We now proceed to find the value of

$$\Sigma \frac{2n+1}{2} P_n(x) \int_{-1}^1 P_n(x) f(x)\, dx.$$

We *assume* that it is finite, and denote it by $F(x)$; so that

$$F(x) = \Sigma \frac{2n+1}{2} P_n(x) \int_{-1}^1 P_n(x) f(x)\, dx.$$

Multiply by $P_n(x)$ and integrate between the limits -1 and $+1$; thus

$$\int_{-1}^1 P_n(x)\, F(x)\, dx = \int_{-1}^1 P_n(x) f(x)\, dx;$$

therefore $$\int_{-1}^1 P_n(x) \{F(x) - f(x)\}\, dx = 0 \ldots\ldots\ldots\ldots(4).$$

Now we know that x^n can be expressed in a series of Legendre's Coefficients; let then

$$x^n = c_n P_n(x) + c_{n-1} P_{n-1}(x) + c_{n-2} P_{n-2}(x) + \ldots.$$

Multiply (4) by c_n, then change n in succession into $n-1$, $n-2$, ..., and add; thus

$$\int_{-1}^{1} x^n \{F(x) - f(x)\}\, dx = 0.$$

This then holds for every positive integral value of n; and hence by Art. 141 we have $F(x) - f(x) = 0$; therefore $F(x) = f(x)$.

This process is given in Liouville's *Journal de Mathématiques*, Vol. II.

143. Thus we see that if the series denoted by

$$\Sigma \frac{2n+1}{2} P_n(x) \int_{-1}^{1} P_n(x) f(x)\, dx$$

is *really finite*, it is equivalent to $f(x)$; the difficulty is to shew generally that the series is finite, and as we have said we shall return to the subject.

144. As an example suppose it required to express x in a series of Legendre's Coefficients, where k is a positive fraction, proper or improper, which reduced to its lowest terms has an *even* number for numerator.

Then $\int_{-1}^{1} x^k P_n(x)\, dx = 0$, when n is odd,

and $\qquad\qquad = 2\int_{0}^{1} x^k P_n(x)\, dx$, when n is even.

Thus, by Art. 34,

$$x^k = \frac{1}{k+1} + \frac{5k}{(k+1)(k+3)} P_2(x) + \frac{9k(k-2)}{(k+1)(k+3)(k+5)} P_4(x) + \dots$$
$$+ \frac{(4m+1)k(k-2)\dots(k-2m+2)}{(k+1)(k+3)\dots(k+2m+1)} P_{2m}(x) + \dots.$$

It will be seen that after a certain term the numerical factors are alternately positive and negative; and it may be shewn that they are ultimately indefinitely small: hence the series is certainly finite if x is numerically less than unity.

To shew that the numerical factor is ultimately indefinitely small we observe that it bears a finite ratio to

$$\frac{4m+1}{k+2m+1} \cdot \frac{2 . 4 . 6 \dots (2m-2)}{3 . 5 . 7 \dots (2m-1)},$$

that is to
$$\frac{4m+1}{k+2m+1} \frac{\{2^{m-1} \lfloor m-1\}^2}{\lfloor 2m-1},$$

and the ordinary mode of approximation will shew that this vanishes when m is infinite. *Integral Calculus*, Art. 282.

145. As another example we will express $\frac{1}{\sqrt{(1-x^2)}}$ in a series of Legendre's Coefficients.

In Art. 14 suppose n *even*, say $=2r$; then the term independent of θ will be found to be $\left\{\frac{1.3.5 \dots (2r-1)}{2.4 \dots 2r}\right\}^2$; and thus $\frac{1}{\pi}\int_0^\pi P_{2r}(\cos\theta)\,d\theta$ is equal to this expression.

Now
$$\int_{-1}^1 \frac{P_n(x)\,dx}{\sqrt{(1-x^2)}} = \int_0^\pi P_n(\cos\theta)\,d\theta;$$

this is zero if n be odd, and if n be even it has the value just found. Thus by (1) and (2) we have

$$\frac{1}{\sqrt{(1-x^2)}} = \frac{\pi}{2}\left\{1 + 5\left(\frac{1}{2}\right)^2 P_2(x) + 9\left(\frac{1 . 3}{2 . 4}\right)^2 P_4(x) + 13\left(\frac{1 . 3 . 5}{2 . 4 . 6}\right)^2 P_6(x) + \dots\right\} \dots\dots\dots (5).$$

If we put $x=0$ we deduce

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1 . 3}{2 . 4}\right)^3 - 13\left(\frac{1 . 3 . 5}{2 . 4 . 6}\right)^3 + \dots.$$

146. Again, we will express $\frac{x}{\sqrt{(1-x^2)}}$ in a series of Legendre's Coefficients.

In Art. 14 suppose n *odd*, say $=2r+1$; then the term which involves $\cos\theta$ will be found to be

$$2\left\{\frac{1.3.5\ldots(2r-1)}{2.4\ldots 2r}\right\}^2\frac{2r+1}{2r+2}\cos\theta;$$

and thus

$$\frac{1}{\pi}\int_0^\pi P_{2r+1}(\cos\theta)\cos\theta\,d\theta=\left\{\frac{1.3.5\ldots(2r-1)}{2.4\ldots 2r}\right\}^2\frac{2r+1}{2r+2}.$$

Now $$\int_{-1}^1\frac{P_n(x)\,x\,dx}{\sqrt{(1-x^2)}}=\int_0^\pi P_n(\cos\theta)\cos\theta\,d\theta;$$

this is zero if n be even, and if n be odd it has the value just found. Thus by (1) and (2) we have

$$\frac{x}{\sqrt{(1-x^2)}}=\frac{\pi}{2}\left\{3\frac{1}{2}P_1(x)+7\left(\frac{1}{2}\right)^2\frac{3}{4}P_3(x)+11\left(\frac{1.3}{2.4}\right)^2\frac{5}{6}P_5(x)+\ldots\right\}$$
$$\ldots\ldots\ldots\ldots(6).$$

147. Integrate (5), making use of Art. 61; thus

$$\frac{2}{\pi}\sin^{-1}x=3\left(\frac{1}{2}\right)^2P_1(x)+7\left(\frac{1}{2.4}\right)^2P_3(x)$$
$$+11\left(\frac{1.3}{2.4.6}\right)^2P_5(x)+15\left(\frac{1.3.5}{2.4.6.8}\right)^2P_7(x)+\ldots.$$

Integrate (6), making use of Art. 61; thus

$$\frac{2}{\pi}\sqrt{(1-x^2)}=\frac{1}{2}-5\left(\frac{1}{2}\right)^2\cdot\frac{1}{4}P_2(x)-9\left(\frac{1}{2.4}\right)^2\frac{3}{6}P_4(x)$$
$$-13\left(\frac{1.3}{2.4.6}\right)^2\frac{5}{8}P_6(x)+\ldots.$$

148. Multiply the left-hand member of (5) by $\dfrac{1}{\sqrt{(1-2ax+a^2)}}$, and the right-hand member by the equivalent series $1+P_1a+P_2a^2+\ldots$; then integrate between the limits -1 and $+1$: thus we get

$$\int_{-1}^{1} \frac{dx}{\sqrt{(1-x^2)}\,\sqrt{(1-2\alpha x+\alpha^2)}},$$

that is
$$\int_0^{\pi} \frac{d\theta}{\sqrt{(1-2\alpha\cos\theta+\alpha^2)}}$$

$$=\pi\left\{1+\left(\frac{1}{2}\right)^2\alpha^2+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2\alpha^4+\left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2\alpha\ +\ldots\right\}.$$

In a similar manner we obtain from (6)

$$\int_{-1}^{1} \frac{x\,dx}{\sqrt{(1-x^2)}\,\sqrt{(1-2xx+\alpha^2)}},$$

that is
$$\int_0^{\pi} \frac{\cos\theta\,d\theta}{\sqrt{(1-2\alpha\cos\theta+\alpha^2)}}$$

$$=\pi\left\{\frac{1}{2}\,\alpha+\left(\frac{1}{2}\right)^2\frac{3}{4}\,\alpha^3+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2\frac{5}{6}\,\alpha^5+\ldots\right\}.$$

The examples of Arts. 145...148 are taken from Crelle's *Journal für...Mathematik*, Vol. 56.

CHAPTER XII.

MISCELLANEOUS PROPOSITIONS.

149. IN Art. 96 we have shewn that

$$\{x+\sqrt{(x^2-1)}\cos\phi\}^n = a_0 + a_1\cos\phi + a_2\cos 2\phi + \ldots + a_n\cos n\phi,$$

where $$a_m = \frac{2}{2^n}\frac{(x^2-1)^{\frac{m}{2}}}{\lfloor n+m}\frac{d^{n+m}(x^2-1)^n}{dx^{n+m}},$$

or as we have expressed it in Art. 97,

$$a_m = \frac{2}{2^n}\frac{\lfloor 2n}{\lfloor n+m\,\lfloor n-m}(x^2-1)^{\frac{m}{2}}\varpi(m,n);$$

but when $m=0$ we take only half of these expressions.

Now let x be positive and greater than unity, and suppose that we expand $\dfrac{1}{\{x+\sqrt{(x^2-1)}\cos\phi\}^{n+1}}$ in the form

$$b_0 + b_1\cos\phi + b_2\cos 2\phi + \ldots + b_n\cos n\phi + \ldots,$$

where b_0, b_1, b_2,... are functions of x which do not involve ϕ; then it is found that so long as m is not greater than n the fraction $\dfrac{a_m}{b_m}$ is *independent of* x: indeed as a_m is zero when m is greater than n, we may say simply that $\dfrac{a_m}{b_m}$ is always independent of x.

This has already appeared in the case in which m is zero; for we have in fact shewn in Art. 49 that $\dfrac{a_0}{b_0}=1$. We shall now investigate the general proposition.

150. We know that if m is greater than zero,

$$a_m = \frac{2}{\pi}\int_0^\pi \{x + \sqrt{(x^2-1)}\cos\phi\}^n \cos m\phi\, d\phi\,;$$

we shall denote the definite integral by J_n, so that

$$a_m = \frac{2}{\pi} J_n.$$

Also we know that if m is greater than zero,

$$b_m = \frac{2}{\pi}\int_0^\pi \frac{\cos m\phi\, d\phi}{\{x + \sqrt{(x^2-1)}\cos\phi\}^{n+1}}\,;$$

we shall denote the definite integral by J_{-n-1}, so that

$$b_m = \frac{2}{\pi} J_{-n-1}.$$

We shall now transform the definite integral J_n.

It is shewn in the *Differential Calculus*, Art. 369, that

$$\frac{\sin m\phi}{m} = \lambda \frac{d^{m-1}(1-t^2)^{m-\frac{1}{2}}}{dt^{m-1}},$$

where $t = \cos\phi$ and $\lambda = \dfrac{(-1)^{m-1}}{1.3.5\ldots(2m-1)}$.

Hence $$\cos m\phi\, d\phi = \lambda \frac{d^m(1-t^2)^{m-\frac{1}{2}}}{dt^m}\, dt.$$

Substitute in J_n; thus it becomes

$$\lambda \int_{+1}^{-1} \{x + t\sqrt{(x^2-1)}\}^n \frac{d^m(1-t^2)^{m-\frac{1}{2}}}{dt^m}\, dt.$$

Integrate by parts, and then again by parts, and so on until the operation has been performed m times; then since $\dfrac{d^r(1-t^2)^{m-\frac{1}{2}}}{dt^r}$ vanishes at the limits so long as r is less than m, which is the case in our process, we obtain

$$J_n = \frac{\lambda(-1)^m \,\underline{|n}\, (x^2-1)^{\frac{m}{2}}}{\underline{|n-m}} \int_{+1}^{-1} \{x + t\sqrt{(x^2-1)}\}^{n-m} (1-t^2)^{m-\frac{1}{2}}\, dt.$$

Then restoring $\cos\phi$ for t we obtain

$$\frac{(-1)^{m+1}\lfloor n-m}{\lambda\lfloor n}(x^2-1)^{-\frac{m}{2}}J_n$$

$$=\int_0^\pi \{x+\sqrt{(x^2-1)}\cos\phi\}^{n-m}\sin^{2m}\phi\,d\phi \quad\text{......(1).}$$

If we apply a similar transformation to J_{-n-1} we obtain

$$-\frac{\lfloor n}{\lambda\lfloor n+m}(x^2-1)^{-\frac{m}{2}}J_{-n-1}$$

$$=\int_0^\pi \frac{\sin^{2m}\phi\,d\phi}{\{x+\sqrt{(x^2-1)}\cos\phi\}^{n+m+1}} \quad\text{.............(2).}$$

We shall now shew that the definite integrals on the right-hand sides of (1) and (2) are equal; this gives in fact the demonstration of the statement of Art. 149.

First change ϕ into $\pi-\phi$ in the definite integral in (1); then it becomes $\int_0^\pi \{x-\sqrt{(x^2-1)}\cos\phi\}^{n-m}\sin^{2m}\phi\,d\phi$.

Now use the same transformation as in Art. 49, namely,

$$\cos\phi=\frac{x\cos\psi+\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}\cos\psi},$$

which leads to
$$\sin\phi=\frac{\sin\psi}{x+\sqrt{(x^2-1)}\cos\psi},$$

$$x-\sqrt{(x^2-1)}\cos\phi=\frac{1}{x+\sqrt{(x^2-1)}\cos\psi},$$

$$d\phi=\frac{d\psi}{x+\sqrt{(x^2-1)}\cos\psi};$$

thus
$$\int_0^\pi \{x-\sqrt{(x^2-1)}\cos\phi\}^{n-m}\sin^{2m}\phi\,d\phi$$

$$=\int_0^\pi \frac{\sin^{2m}\psi\,d\psi}{\{x+\sqrt{(x^2-1)}\cos\psi\}^{n+m+1}}.$$

Hence from (1) and (2) we have

$$\frac{(-1)^{m+1}\lfloor n-m}{\lfloor n} J_n = -\frac{\lfloor n}{\lfloor n+m} J_{-n-1},$$

so that

$$\frac{J_n}{J_{-n-1}} = \frac{\lfloor n \lfloor n}{\lfloor n+m \lfloor n-m} (-1)^m.$$

We have thus two forms for the associated functions of the first kind analogous to the two forms for P_n in Arts. 47 and 49; namely

$$(x^2-1)^{\frac{m}{2}} \varpi(m, n)$$
$$= \frac{2^n}{\pi} \frac{\lfloor n+m \lfloor n-m}{\lfloor 2n} \int_0^{\pi} \{x + \sqrt{(x^2-1)} \cos\phi\}^n \cos m\phi \, d\phi,$$

and also when x is positive and greater than unity

$$(x^2-1)^{\frac{m}{2}} \varpi(m, n)$$
$$= \frac{(-1)^m \lfloor n}{\pi 1.3.5 \ldots (2n-1)} \int_0^{\pi} \frac{\cos m\phi \, d\phi}{\{x + \sqrt{(x^2-1)} \cos\phi\}^{n+1}}.$$

151. The process given in Art. 96 for the expansion of $\{x + \sqrt{(x^2-1)} \cos\phi\}^n$ may be generalised.

For if $\psi(x)$ denote any function of x we have

$$x + \psi(x) \cos\phi = \frac{\{x + e^{\iota\phi}\psi(x)\}^2 + \{\psi(x)\}^2 - x^2}{2e^{\iota\phi}\psi(x)},$$

and hence the expansion of $\{x + \psi(x) \cos\phi\}^n$ may be found thus: expand $\left\{\frac{(x+z)^2 + a^2}{2z}\right\}^n$ in powers of z, at the end put $e^{\iota\phi}\psi(x)$ for z, and $\{\psi(x)\}^2 - x^2$ for a^2. We shall thus obtain for the general term

$$\frac{1}{2^n}\left\{\frac{[\psi(x)]^m}{\lfloor n+m} D^{n+m}(x^2+a^2)^n + \frac{[\psi(x)]^{-m}}{\lfloor n-m} D^{n-m}(x^2+a^2)^n\right\} \cos m\phi,$$

where D stands for $\frac{d}{dx}$; and at the same time we obtain the theorem

$$\frac{[\psi(x)]^m}{\lfloor n+m} D^{n+m}(x^2+a^2)^n = \frac{[\psi(x)]^{-m}}{\lfloor n-m} D^{n-m}(x^2+a^2)^n :$$

in these formulæ the value of a^2 is to be substituted after the differentiations are performed.

152. We may exhibit $P_n(x)$ as a determinant. For put

$$A_n + A_{n-1}x + A_{n-2}x^2 + \ldots + A_0 x^n = V \ldots\ldots\ldots\ldots (3),$$

where $A_n, A_{n-1}, \ldots A_0$ are constants; let these constants be determined by the conditions that $\int_{-1}^{1} Vx^m dx = 0$, when m is any positive integer not greater than n.

Put $a_r = \frac{1}{2}\int_{-1}^{1} x^r dx$, so that $a_r = \frac{1}{r+1}$ if r is even, and $= 0$ if r is odd. Then by putting for m in succession the values $0, 1, 2, \ldots n-1$, we obtain the following n equations:

$$\left.\begin{array}{l} A_n a_0 \;\; + A_{n-1}a_1 + \ldots + A_0 a_n \;\;\;\; = 0 \\ A_n a_1 \;\; + A_{n-1}a_2 + \ldots + A_0 a_{n+1} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ A_n a_{n-1} + A_{n-1}a_n + \ldots + A_0 a_{2n-1} = 0 \end{array}\right\} \ldots\ldots (4).$$

We may consider that (3) and (4) form $n+1$ equations for expressing $A_0, A_1, \ldots A_n$ in terms of $V, x, a_0, a_1, \ldots a_{2n-1}$; thus we get by the *Theory of Equations*, Art. 388,

$$A_0 \times M = V \times N,$$

where M stands for the determinant

$$\left|\begin{array}{llll} a_0, & a_1, \; a_2, & \ldots, & a_n \\ a_1, & a_2, \; a_3, & \ldots & a_{n+1} \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ a_{n-1}, & a_n, \; a_{n+1}, & \ldots & a_{2n-1} \\ 1, & x, \;\; x^2, & \ldots & x^n \end{array}\right|$$

and N stands for the determinant obtained from M by omitting the extreme right-hand column and the lowest row.

Now we know by Art. 32 that $P_n(x)$ is the product of a certain constant into V, and as the coefficient of x^n in $P_n(x)$ is $\frac{1.3.5\ldots(2n-1)}{\lfloor n}$, we have

$$P_n(x) = \frac{1.3.5\ldots(2n-1)}{\lfloor n}\,\frac{V}{A_0} = \frac{1.3.5\ldots(2n-1)}{\lfloor n}\,\frac{M}{N}.$$

Thus $P_n(x)$ is expressed as the product of the determinant M into the constant factor $\frac{1.3.5\ldots(2n-1)}{\lfloor n\, N}$.

153. The value found for $P_n(x)$ verifies immediately the property that $\int_{-1}^{1} P_n(x)\,x^m dx = 0$, when m is any positive integer less than n.

For since $V = \frac{A_0 M}{N}$ the value of $\frac{1}{2}\int_{-1}^{1} Vx^m dx$ will be found to be $\frac{A_0\mu}{N}$, where μ is obtained from M by changing the last row of M into

$$a_m,\ a_{m+1},\ a_{m+2},\ \ldots\ a_{m+n}.$$

But thus μ has two rows identical, and therefore vanishes by the *Theory of Equations*, Art. 371.

154. Since a_r is zero when r is odd, it will be found that we can separate the equations (4) into two groups, one involving $A_0, A_2, A_4, \ldots$, and the other involving $A_1, A_3, A_5, \ldots$ The number of equations in the latter group will be the same as the number of the quantities $A_1, A_3, A_5, \ldots$; and as the right-hand member of each equation is zero we obtain $A_1 = 0$, $A_3 = 0$, $A_5 = 0, \ldots$. The former group of equations in conjunction with (3) will serve to find A_0; we shall obtain a result which we may express thus:

$$A_0 \times M_1 = V \times N_1,$$

where M_1 and N_1 are determinants.

If $n=2r$ we have for M_1 the form

$$\begin{vmatrix} a_0, & a_2, & a_4, & \dots a_{2r} \\ a_2, & a_4, & a_6, & \dots a_{2r+2} \\ \dots & \dots & \dots & \dots \\ a_{2r-2}, & a_{2r}, & a_{2r+2}, & \dots a_{4r-2} \\ 1, & x^2, & x^4, & \dots x^{2r} \end{vmatrix}$$

If $n=2r+1$ we have for M_1 the form

$$\begin{vmatrix} a_2, & a_4, & a_6, & \dots a_{2r+2} \\ a_4, & a_6, & a_8, & \dots a_{2r+4} \\ \dots & \dots & \dots & \dots \\ a_{2r}, & a_{2r+2}, & a_{2r+4}, & \dots a_{4r} \\ x, & x^3, & x^5, & \dots x^{2r+1} \end{vmatrix}$$

In each case N_1 is formed from M_1 by omitting the extreme right-hand column and the lowest row.

As before we have

$$P_n(x)=\frac{1.3.5\dots(2n-1)}{\lfloor n}\frac{V}{A_0},$$

so that $$P_n(x)=\frac{1.3.5\dots(2n-1)}{\lfloor n}\frac{M_1}{N_1}.$$

Articles 152...154 are taken from the *Comptes Rendus* of the French *Institut,* Vol. XLVII.

155. In Art. 102 we saw that if $y=\varpi(m,n)(x^2-1)^{\frac{m}{2}}$, then

$$(1-x^2)^2\frac{d^2y}{dx^2}-2x(1-x^2)\frac{dy}{dx}+\{n(n+1)-m^2-n(n+1)x^2\}y=0;$$

we will denote y by $\phi(m,n)$, and proceed to some properties of this function.

156. Let K_m stand for $\int_{-1}^{1} \phi(m, n)\, \phi(m, \nu)\, dx$, then K_m will vanish if n and ν are different.

For from the differential equation of Art. 155 we obtain

$$\int_{-1}^{1} \phi(m, \nu) \frac{d}{dx} \left\{(1-x^2) \frac{d\phi(m, n)}{dx}\right\} dx$$

$$= m^2 \int_{-1}^{1} \frac{\phi(m, n)\, \phi(m, \nu)}{1-x^2}\, dx - n(n+1) \int_{-1}^{1} \phi(m, n)\, \phi(m, \nu)\, dx.$$

By two integrations by parts the left-hand member becomes

$$\int_{-1}^{1} \phi(m, n) \frac{d}{dx} \left\{(1-x^2) \frac{d\phi(m, \nu)}{dx}\right\} dx;$$

and this by the differential equation

$$= m^2 \int_{-1}^{1} \frac{\phi(m, n)\, \phi(m, \nu)}{1-x^2}\, dx - \nu(\nu+1) \int_{-1}^{1} \phi(m, n)\, \phi(m, \nu)\, dx.$$

Therefore

$$\{\nu(\nu+1) - n(n+1)\} \int_{-1}^{1} \phi(m, n)\, \phi(m, \nu)\, dx = 0;$$

and therefore if n and ν are different $K_m = 0$.

157. We shall next find the value of K_m when $\nu = n$.

By Art. 97 we see that

$$\phi(m, n) = \frac{\lfloor n-m}{\lfloor 2n} (x^2-1)^{\frac{m}{2}} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}};$$

by Art. 96 we are allowed to change m into $-m$ in the expression here given without altering its value, so that we have also

$$\phi(m, n) = \frac{\lfloor n+m}{\lfloor 2n} (x^2-1)^{-\frac{m}{2}} \frac{d^{n-m}(x^2-1)^n}{dx^{n-m}}.$$

Hence we have

$$\int_{-1}^{1} \{\phi(m, n)\}^2 dx = \frac{\lfloor n+m \lfloor n-m}{\lfloor 2n \lfloor 2n} \int_{-1}^{1} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}} \frac{d^{n-m}(x^2-1)^n}{dx^{n-m}}\, dx.$$

Integrate by parts; thus

$$\int_{-1}^{1} \frac{d^{n+m}(x^2-1)^n}{dx^{n+m}} \frac{d^{n-m}(x^2-1)^n}{dx^{n-m}} dx$$
$$= -\int_{-1}^{1} \frac{d^{n+m-1}(x^2-1)^n}{dx^{n+m-1}} \frac{d^{n-m+1}(x^2-1)^n}{dx^{n-m+1}} dx.$$

Integrate by parts again; and so on until we arrive at

$$(-1)^m \int_{-1}^{1} \frac{d^n(x^2-1)^n}{dx^n} \frac{d^n(x^2-1)^n}{dx^n} dx;$$

this $= (-1)^m \{\lfloor n \, 2^n\}^2 \int_{-1}^{1} \{P_n(x)\}^2 dx = \dfrac{2(-1)^m}{2n+1} \{\lfloor n \, 2^n\}^2.$

Thus finally when $\nu = n$ we have

$$K_m = \frac{2(-1)^m}{2n+1} \frac{\lfloor n+m \lfloor n-m}{\{1 \cdot 3 \cdot 5 \ldots (2n-1)\}^2}.$$

158. It will be convenient to state the results of the last two Articles in another notation by the aid of equation (4) of Art. 97. We have then

$$\int_{-1}^{1} \frac{d^m P_n(x)}{dx^m} \frac{d^m P_\nu(x)}{dx^m} (1-x^2)^m dx = 0,$$

if n and ν are different; and

$$\int_{-1}^{1} \left\{\frac{d^m P_n(x)}{dx^m}\right\}^2 (1-x^2)^m dx = \frac{2 \lfloor n+m}{(2n+1) \lfloor n-m}.$$

159. We shall now establish the following relation:

$$(2n-1)\, \phi(m, n) = nx\phi(m, n-1) + (x^2-1) \frac{d}{dx} \phi(m, n-1).$$

By using the formula quoted at the beginning of Art. 157 and reducing, and putting D for $\dfrac{d}{dx}$, the proposed relation takes the form

$$\frac{n-m}{2n} D^{n+m}(x^2-1)^n$$
$$= nxD^{n+m-1}(x^2-1)^{n-1} + mxD^{n+m-1}(x^2-1)^{n-1} + (x^2-1) D^{n+m}(x^2-1)^{n-1};$$

and $D^{n+m}(x^2-1)^n = D^{n+m-1}\dfrac{d}{dx}(x^2-1)^n = 2nD^{n+m-1}x(x^2-1)^{n-1}$;

so that the relation becomes

$$(n-m)D^{n+m-1}x(x^2-1)^{n-1}$$
$$= (n+m)xD^{n+m-1}(x^2-1)^{n-1} + (x^2-1)D^{n+m}(x^2-1)^{n-1} \ldots\ldots (5).$$

We shall establish (5) by induction. Assume that it is true, and differentiate both sides; thus

$$(n-m)D^{n+m}x(x^2-1)^{n-1} = (n+m)xD^{n+m}(x^2-1)^{n-1}$$
$$+ (x^2-1)D^{n+m+1}(x^2-1)^{n-1} + (n+m)D^{n+m-1}(x^2-1)^{n-1}$$
$$+ 2xD^{n+m}(x^2-1)^{n-1} \ldots\ldots (6).$$

But by the theorem of Leibnitz,

$$D^{n+m}x(x^2-1)^{n-1} = xD^{n+m}(x^2-1)^{n-1} + (m+n)D^{n+m-1}(x^2-1)^{n-1};$$

and thus (6) may be written

$$(n-m)D^{n+m}x(x^2-1)^{n-1} = (n+m)xD^{n+m}(x^2-1)^{n-1}$$
$$+ (x^2-1)D^{n+m+1}(x^2-1)^{n-1} + D^{n+m}x(x^2-1)^{n-1}$$
$$+ xD^{n+m}(x^2-1)^{n-1}:$$

this is what we should get from (5) by changing m into $m+1$; so that if (5) be true for any value of m it is true when m is changed into $m+1$.

But (5) is true when $m=0$; for then it becomes

$$nD^{n-1}x(x^2-1)^{n-1} = nxD^{n-1}(x^2-1)^{n-1} + (x^2-1)D^n(x^2-1)^{n-1},$$

that is $\quad n(n-1)D^{n-2}(x^2-1)^{n-1} = (x^2-1)D^n(x^2-1)^{n-1}$;

and this is a particular case of equation (2) of Art. 96, namely, what we obtain by putting 1 for m, and changing n to $n-1$.

160. The results of Arts. 156 and 157 enable us to extend to the function $\phi(m, n)$ some propositions which hold with respect to $P_n(x)$; this will be seen in the next three Articles.

161. Suppose that a function $f(x)$ can be expressed in the form

$$f(x) = a_0\phi(m, m) + a_1\phi(m, m+1) + a_2\phi(m, m+2) + \ldots,$$

where a_0, a_1, a_2,... are numerical factors to be determined. Then these numerical factors may be determined by the general formula

$$a_r\int_{-1}^{1}\{\phi(m, m+r)\}^2 dx = \int_{-1}^{1} f(x)\,\phi(m, m+r)\,dx.$$

Moreover there is only one such mode of expressing $f(x)$. See Arts. 138 and 139.

162. Again, suppose we have the series

$$b_0\phi(0, n) + b_1\phi(1, n) + b_2\phi(2, n) + \ldots + b_n\phi(n, n);$$

then, if this series vanish for every value of x, the numerical factors b_0, b_1,... b_n must all be zero.

For suppose that $x = 1$; then $\phi(1, n)$, $\phi(2, n)$... all vanish; and therefore $b_0\phi(0, n) = 0$; therefore $b_0 = 0$.

Then we have $b_1\phi(1, n) + b_2\phi(2, n) \ldots + b_n\phi(n, n)$ always zero; divide by $\sqrt{(x^2 - 1)}$, and then put $x = 1$; thus we find that $b_1 = 0$; and so on.

This process assumes that $\dfrac{\phi(m, n)}{(x^2-1)^{\frac{m}{2}}}$ does not vanish when $x = 1$, that is, that $\varpi(m, n)$ does not vanish when $x = 1$; and this we know to be the case from Art. 103.

163. Suppose that a function $f(x)$ can be expressed in the form

$$f(x) = b_0\phi(0, n) + b_1\phi(1, n) + b_2\phi(2, n) \ldots + b_n\phi(n, n);$$

then the numerical factors b_0, b_1, b_2, ... may be determined in succession, thus:

$$b_0 = \frac{f(x)}{\phi(0, n)}, \quad b_1 = \frac{f(x) - b_0\phi(0, n)}{\phi(1, n)\sqrt{(x^2 - 1)}}, \ldots$$

where in the expressions on the right-hand side we must put 1 for x. There will be only one such mode of expressing $f(x)$.

164. In various investigations of mixed mathematics we obtain with more or less rigour modes of expressing a given function analogous to those of Arts. 138, 161, and 163. It is usually shewn in a satisfactory manner that if such a mode of expression is possible it can be effected in one definite manner; but it is rarely decisively shewn that such a mode of expression is certainly possible. We will give one example.

Suppose that a homogeneous sphere is heated in such a manner that the temperature is the same at all points equally distant from the centre; and let the sphere be placed in a medium of which the temperature is constant; then it is shewn in various treatises on the mathematical theory of heat that in order to determine the temperature at any time t of the points of the sphere which are at the distance x from the centre, we must find a quantity u which satisfies the following conditions: the equation

$$\frac{du}{dt} = c^2 \frac{d^2u}{dx^2} \qquad (7)$$

must hold, whatever t may be, for all values of x comprised between 0 and the radius of the sphere, which we will denote by l; and the equation

$$\frac{du}{dx} + hu = 0 \qquad (8)$$

must hold when $x = l$, whatever t may be. Here c and h are certain constants. Then the temperature at the time t of the points of the sphere which are at the distance x from the centre will be $\frac{u}{x}$.

Now we will assume that there is some expression for u in terms of x and t which does satisfy these conditions; that is, we assume that the problem has a solution. We will also assume that as u is a function of t it may be expanded in a series proceeding according to ascending powers of e^{-t}; this assumption may be in some degree justified by Burmann's Theorem; see *Differential Calculus*, Chapter IX.

We assume then that u can be expressed in a series of the form

$$u = A_1 e^{-a_1^2 t} + A_2 e^{-a_2^2 t} + \ldots \ldots\ldots\ldots\ldots\ldots (9),$$

where $A_1, A_2, \ldots$ are functions of x; and $a_1^2, a_2^2, \ldots$ are constants: these are now to be determined.

Substitute from (9) in (7); then we obtain an equation which must be true for all values of t, and which leads therefore to the set of equations

$$-a_1^2 A_1 = c^2 \frac{d^2 A_1}{dx^2}, \quad -a_2^2 A_2 = c^2 \frac{d^2 A_2}{dx^2}, \ldots$$

Thus we get $A_1 = B_1 \sin\left(\frac{a_1 x}{c} + C_1\right)$, $A_2 = B_2 \sin\left(\frac{a_2 x}{c} + C_2\right), \ldots$

where $B_1, B_2, \ldots C_1, C_2, \ldots$ are constants which remain to be determined.

In the present problem we must have $C_1, C_2, \ldots$ zero, in order that the temperature at the centre of the sphere may be finite. Therefore

$$A_1 = B_1 \sin \frac{a_1 x}{c}, \quad A_2 = B_2 \sin \frac{a_2 x}{c}, \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Substitute from (9) in (8); then we obtain an equation which must be true for all values of t: by the aid of (10) this leads to a set of equations of the form

$$a \cos \frac{al}{c} + hc \sin \frac{al}{c} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

where a stands for any of the quantities $a_1, a_2, \ldots$

Put $a = c\rho$, then (11) becomes

$$\rho \cos \rho l + h \sin \rho l = 0 \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

Thus we obtain $u = \Sigma B \sin \rho x e^{-c^2 \rho^2 t} \ldots\ldots\ldots\ldots\ldots\ldots (13),$

where Σ denotes a summation which is to be effected by giving to ρ the values which satisfy (12), and to B the values which correspond to those of ρ. The connexion between B and ρ must now be investigated.

The value of u in terms of x, when $t=0$, may be supposed to be given arbitrarily; denote it by $\phi(x)$: then we must have

$$\phi(x) = \Sigma B \sin \rho x \quad \text{.......................} \quad (14).$$

Let ρ_1 and ρ_2 denote two of the values of ρ; and B_1 and B_2 the corresponding values of B. Multiply both sides of (14) by $\sin \rho_1 x$, and integrate from $x=0$ to $x=l$. Then, since

$$\int \sin \rho_1 x \sin \rho_2 x \, dx = \frac{\sin(\rho_1 - \rho_2)x}{2(\rho_1 - \rho_2)} - \frac{\sin(\rho_1 + \rho_2)x}{2(\rho_1 + \rho_2)}$$

$$= \frac{-\rho_1 \sin \rho_2 x \cos \rho_1 x + \rho_2 \sin \rho_1 x \cos \rho_2 x}{\rho_1^2 - \rho_2^2},$$

we find by the aid of (12) that $\int_0^l \sin \rho_1 x \sin \rho_2 x \, dx = 0.$

And
$$\int_0^l \sin^2 \rho_1 x \, dx = \frac{l}{2} - \frac{\sin \rho_1 l \cos \rho_1 l}{2\rho_1}.$$

Thus we get
$$B_1 = \frac{2\rho_1 \int_0^l \phi(x) \sin \rho_1 x \, dx}{\rho_1 l - \sin \rho_1 l \cos \rho_1 l}.$$

Similarly B_2, B_3... may be determined.

Substitute in (13); thus we get

$$u = \Sigma \frac{2\rho \sin \rho x e^{-c^2\rho^2 t} \int_0^l \phi(x) \sin \rho x \, dx}{\rho l - \sin \rho l \cos \rho l}.$$

Thus the value of u is determined. We obtain indirectly the following theorem: if $\phi(x)$ denotes any function of x, which satisfies (8) when $x=l$, but is otherwise arbitrary, then

$$\phi(x) = \Sigma \frac{2\rho \sin \rho x \int_0^l \phi(x) \sin \rho x \, dx}{\rho l - \sin \rho l \cos \rho l}.$$

This result was first obtained by Fourier: see his *Théorie Analytique de la Chaleur*, page 350; and Poisson's *Théorie Mathématique de la Chaleur*, pages 171 and 294.

CHAPTER XIII.

LAPLACE'S COEFFICIENTS.

165. We have defined Legendre's n^{th} Coefficient as the Coefficient of a^n in the development of $(1-2ax+a^2)^{-\frac{1}{2}}$ in a series of ascending powers of a; thus this Coefficient is a function of x, and we denote it by $P_n(x)$.

Let $\cos\gamma$ be put for x; then the Coefficient becomes a function of $\cos\gamma$ which we denote by $P_n(\cos\gamma)$.

Suppose two points on the surface of a sphere, and let their positions be determined in the usual manner by two elements which we may call *latitude* and *longitude*; let $\frac{\pi}{2}-\theta$ be the latitude, and ϕ the longitude of one point; let $\frac{\pi}{2}-\theta'$ be the latitude, and ϕ' the longitude of the other point; let γ be the arc which joins the two points: then by Spherical Trigonometry

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi').$$

Suppose this value of $\cos\gamma$ substituted in Legendre's n^{th} Coefficient; then it becomes what we call *Laplace's n^{th} Coefficient*: we denote it by Y_n, and we proceed to discuss the form and the properties of this Coefficient.

It will be observed that Y_n is thus a function of four quantities, namely θ, θ', ϕ, and ϕ'; we shall in general regard θ and ϕ as variable, and θ' and ϕ' as constant, but it will be found that no difficulty will arise if we have in some cases to regard θ' and ϕ' also as variables.

The geometrical language about the sphere which we have introduced is not necessary, for we might have stated the connexion between γ and the new variables merely as an arbitrary choice of notation. But with the aid of the spherical triangle, which is formed by connecting the two points and each of them with the pole, a distinctness and reality are given to the subject which will be found very advantageous.

166. Throughout the following investigations we shall use μ for $\cos\theta$, whenever it may be convenient; this gives $d\mu = -\sin\theta d\theta$. Similarly we shall use μ' for $\cos\theta'$; this gives $d\mu' = -\sin\theta' d\theta'$.

Thus we have

$$\cos\gamma = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')$$

$$= \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}(\cos\phi\cos\phi' + \sin\phi\sin\phi').$$

We shall sometimes use ψ for $\phi - \phi'$.

167. We shall first establish a certain differential equation.

Let $$U = \frac{1}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}};$$

then $$\frac{dU}{dx} = \frac{x'-x}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{3}{2}}},$$

$$\frac{d^2U}{dx^2} = -\frac{1}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{3}{2}}}$$
$$+ \frac{3(x-x')^2}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{5}{2}}}.$$

Similar expressions hold for $\frac{d^2U}{dy^2}$ and $\frac{d^2U}{dz^2}$; and thus by addition we have

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now assume

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta;$$

then by *Differential Calculus*, Art. 207, equation (1) transforms to

$$\frac{d^2U}{dr^2}+\frac{1}{r^2}\frac{d^2U}{d\theta^2}+\frac{2}{r}\frac{dU}{dr}+\frac{\cot\theta}{r^2}\frac{dU}{d\theta}+\frac{1}{r^2\sin^2\theta}\frac{d^2U}{d\phi^2}=0;$$

this may also be written

$$r\frac{d^2(Ur)}{dr^2}+\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dU}{d\mu}\right\}+\frac{1}{1-\mu^2}\frac{d^2U}{d\phi^2}=0\ldots(2).$$

This differential equation was first given by Laplace, and may be called *Laplace's differential equation*.

Let us also assume

$$x'=r'\sin\theta'\cos\phi',\quad y'=r'\sin\theta'\sin\phi',\quad z'=r'\cos\theta';$$

then
$$U=\frac{1}{(r^2-2\lambda rr'+r'^2)^{\frac{1}{2}}},$$

where λ stands for $\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$.

Suppose r' greater than r; we may put U in the form

$$\frac{1}{r'}\left\{1-2\lambda\frac{r}{r'}+\frac{r^2}{r'^2}\right\}^{-\frac{1}{2}},$$

and by expanding we obtain for U the convergent series

$$U=\frac{1}{r'}+Y_1\frac{r}{r'^2}+Y_2\frac{r^2}{r'^3}+Y_3\frac{r^3}{r'^4}+\ldots\ldots\ldots\ldots\ (3).$$

Substitute this value of U in (2), and equate the coefficient of r^n to zero; thus

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dY_n}{d\mu}\right\}+\frac{1}{1-\mu^2}\frac{d^2Y_n}{d\phi^2}+n(n+1)Y_n=0\ldots(4).$$

If we suppose r greater than r', we have instead of (3)

$$U=\frac{1}{r}+Y_1\frac{r'}{r^2}+Y_2\frac{r'^2}{r^3}+Y_3\frac{r'^3}{r^4}+\ldots;$$

and by equating to zero the coefficient of r^{-n-1} we again obtain (4).

168. Confining our attention for the present to μ and ϕ as the variables, we see from the equation

$$\cos\gamma = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}(\cos\phi\cos\phi' + \sin\phi\sin\phi')$$

that $\cos\gamma$ is an expression of the first degree with respect to these three terms, μ, $\sqrt{1-\mu^2}\cos\phi$, and $\sqrt{1-\mu^2}\sin\phi$. Hence as $P_n(\cos\gamma)$ is of the n^{th} degree in $\cos\gamma$, it follows that Y_n will be of the n^{th} degree in the three terms μ, $\sqrt{1-\mu^2}\cos\phi$, and $\sqrt{1-\mu^2}\sin\phi$; that is, the aggregate of the exponents of these three terms in any element of Y_n will not exceed n.

Also, since $\cos\gamma = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')$, we see that the powers of $\cos\gamma$ may be developed in powers of $\cos(\phi-\phi')$; and then these powers may be transformed by Plane Trigonometry into cosines of multiples of $\phi-\phi'$. In this way we see that Y_n may be arranged in a series of cosines of multiples of $\phi-\phi'$. As such a term as $\cos m(\phi-\phi')$ can arise only from the powers m, $m+2$, $m+4$, ... of $\cos(\phi-\phi')$, it follows that $(1-\mu^2)^{\frac{m}{2}}$ must be a factor of the element which involves $\cos m(\phi-\phi')$; and the other factor will be of the form

$$A_0\mu^{n-m} + A_1\mu^{n-m-2} + A_2\mu^{n-m-4} + \dots,$$

where A_0, A_1, A_2, ... are independent of μ. We will denote this by B_m. Thus Y_n is of the form

$$B_0 + B_1\sqrt{1-\mu^2}\cos\psi + \dots + B_m(1-\mu^2)^{\frac{m}{2}}\cos m\psi + \dots$$
$$+ B_n(1-\mu^2)^{\frac{n}{2}}\cos n\psi.$$

Substitute this value in (4), observing that $\dfrac{d^2Y_n}{d\phi^2} = \dfrac{d^2Y_n}{d\psi^2}$; and equate to zero the coefficient of $\cos m\psi$; thus

$$(1-\mu^2)\frac{d^2}{d\mu^2}\{B_m(1-\mu^2)^{\frac{m}{2}}\} - 2\mu\frac{d}{d\mu}\{B_m(1-\mu^2)^{\frac{m}{2}}\} - \frac{B_m m^2(1-\mu^2)^{\frac{m}{2}}}{1-\mu^2}$$
$$+ n(n+1)B_m(1-\mu^2)^{\frac{m}{2}} = 0;$$

when this is developed it becomes

$$(1-\mu^2)^{\frac{m}{2}+1}\frac{d^2B_m}{d\mu^2}-2(m+1)\mu(1-\mu^2)^{\frac{m}{2}}\frac{dB_m}{d\mu}$$
$$+B_m\left\{(1-\mu^2)\frac{d^2(1-\mu^2)^{\frac{m}{2}}}{d\mu^2}-2\mu\frac{d(1-\mu^2)^{\frac{m}{2}}}{d\mu}-m^2(1-\mu^2)^{\frac{m}{2}-1}\right.$$
$$\left.+n(n+1)(1-\mu^2)^{\frac{m}{2}}\right\}=0;$$

that is, $$(1-\mu^2)^{\frac{m}{2}+1}\frac{d^2B_m}{d\mu^2}-2(m+1)\mu(1-\mu^2)^{\frac{m}{2}}\frac{dB_m}{d\mu}$$
$$+B_m\{n(n+1)-m^2-m\}(1-\mu^2)^{\frac{m}{2}}=0.$$

This may be written

$$\frac{d}{d\mu}\left\{(1-\mu^2)^{m+1}\frac{dB_m}{d\mu}\right\}+(n-m)(n+m+1)(1-\mu^2)^m B_m=0.$$

Substitute for B_m the series which it represents in this equation, and equate the coefficient of $(1-\mu^2)^m\mu^{n-m-2s}$ to zero; thus, using p for $n-m-2s$, we have

$$-p(p-1)A_s-2(m+1)pA_s+(n-m)(n+m+1)A_s$$
$$+(p+2)(p+1)A_{s-1}=0.$$

Thus by reduction we get

$$A_s=-\frac{(n-m-2s+2)(n-m-2s+1)}{2s(2n-2s+1)}A_{s-1}.$$

Hence we find that

$$B_m=A_0\left\{\mu^{n-m}-\frac{(n-m)(n-m-1)}{2.(2n-1)}\mu^{n-m-2}\right.$$
$$\left.+\frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2.4.(2n-1)(2n-3)}\mu^{n-m-4}-\dots\right\}.$$

The expression within the brackets may be denoted by $\varpi(m, n, \mu)$; thus the term in Y_n which involves $\cos m\psi$ is $A_0(1-\mu^2)^{\frac{m}{2}}\varpi(m, n, \mu)\cos m\psi$, where A_0 is independent of μ and ψ.

But this term must be the same function of μ' that it is of μ, because μ and μ' occur symmetrically in Y_n; so that we see A_0 must contain $(1-\mu'^2)^{\frac{m}{2}}\varpi(m,n,\mu')$ as a factor. Hence finally the term in Y_n which involves $\cos m\psi$ is

$$C(1-\mu^2)^{\frac{m}{2}}(1-\mu'^2)^{\frac{m}{2}}\varpi(m,n,\mu)\,\varpi(m,n,\mu')\cos m\psi,$$

where C is some numerical factor independent of μ, μ', and ψ. The value of C must now be found.

I. Suppose $n-m$ even. Then in $\varpi(m,n,\mu)$ there is a term independent of μ, and therefore a term independent of μ' in $\varpi(m,n,\mu')$; so that if we put $\mu=0$ and $\mu'=0$, the above term becomes $C\{\varpi(m,n,0)\}^2\cos m\psi$, that is

$$C\left\{\frac{(n-m)(n-m-1)\dots 1}{2.4\dots(n-m)(2n-1)(2n-3)\dots(n+m+1)}\right\}^2\cos m\psi,$$

that is

$$C\left\{\frac{1.3.5\dots(n-m-1)\,1.3.5\dots(n+m-1)}{1.3.5\dots(2n-1)}\right\}^2\cos m\psi.$$

But when μ and μ' vanish the function to be expanded becomes $(1-2\alpha\cos\psi+\alpha^2)^{-\frac{1}{2}}$, and we have to pick out the term which involves $\cos m\psi$ in the coefficient of α^n. It will be found by Art. 14 that this has the factor

$$2\,\frac{1.3.5\dots(n-m-1)\,1.3.5\dots(n+m-1)}{2.4\dots(n-m)\,2.4\dots(n+m)},$$

that is $$2\,\frac{\{1.3.5\dots(n-m-1)\,1.3.5\dots(n+m-1)\}^2}{\lfloor n-m\,\lfloor n+m};$$

but only half of this is to be taken when $m=0$.

Thus we get $C=2\,\dfrac{\{1.3.5\dots(2n-1)\}^2}{\lfloor n-m\,\lfloor n+m}$,

but only half of this is to be taken when $m=0$.

II. Suppose $n-m$ odd. Then in $\varpi(m,n,\mu)$ the lowest power of μ is the first, and the lowest power of μ' in $\varpi(m,n,\mu')$ is the first. Hence we find that a part of the term in Y_n which involves $\cos m\psi$ has the factor

$$C\mu\mu'\left\{\frac{(n-m)(n-m-1)\dots 2}{2.4\dots(n-m-1)(2n-1)(2n-3)\dots(n+m+2)}\right\}^2,$$

that is $$C\mu\mu'\left\{\frac{3.5\dots(n-m)\,1.3.5\dots(n+m)}{1.3.5\dots(2n-1)}\right\}^2.$$

Also, if we neglect powers of μ and μ' above the first, we have

$$\{1-2\alpha(\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\psi)+\alpha^2\}^{-\frac{1}{2}}$$
$$=(1-2\alpha\cos\psi+\alpha^2)^{-\frac{1}{2}}+\alpha\mu\mu'(1-2\alpha\cos\psi+\alpha^2)^{-\frac{3}{2}};$$

the second term on the right-hand side

$$=\alpha\mu\mu'(1-\alpha e^{\iota\psi})^{-\frac{3}{2}}(1-\alpha e^{-\iota\psi})^{-\frac{3}{2}}$$
$$=\alpha\mu\mu'\left\{1+\frac{3}{2}\alpha e^{\iota\psi}+\frac{3.5}{2.4}\alpha^2e^{2\iota\psi}+\dots\right\}\left\{1+\frac{3}{2}\alpha e^{-\iota\psi}+\frac{3.5}{2.4}\alpha^2e^{-2\iota\psi}+\dots\right\}.$$

We want from this the coefficient of $\alpha^n\mu\mu'\cos m\psi$; it will be found to be

$$2\frac{3.5\dots(n-m)\,3.5\dots(n+m)}{2.4\dots(n-m-1)\,2.4\dots(n+m-1)},$$

that is $$2\frac{\{3.5\dots(n-m)\,3.5\dots(n+m)\}^2}{\underline{|n-m}\;\underline{|n+m}};$$

but only half of this is to be taken when $m=0$.

Hence we get as before $C=2\dfrac{\{1.3.5\dots(2n-1)\}^2}{\underline{|n-m}\;\underline{|n+m}}$;

but only half of this is to be taken when $m=0$.

Thus finally we have

$$Y_n=\lambda\Sigma\frac{(1-\mu^2)^{\frac{m}{2}}(1-\mu'^2)^{\frac{m}{2}}}{\underline{|n-m}\;\underline{|n+m}}\varpi(m,n,\mu)\,\varpi(m,n,\mu')\cos m\psi,$$

where Σ denotes a summation with respect to m from 0 to n both inclusive; and $\lambda=2\{1.3.5\dots(2n-1)\}^2$, except when $m=0$, and then we must take only half this value.

Or we may write separately the term which corresponds to $m=0$, and thus we have

$$Y_n = \frac{\{1.3.5\ldots(2n-1)\}^2}{\underline{|n}\ \underline{|n}}\,\varpi(0, n, \mu)\,\varpi(0, n, \mu')$$

$$+\lambda\Sigma\,\frac{(1-\mu^2)^{\frac{m}{2}}(1-\mu'^2)^{\frac{m}{2}}}{\underline{|n-m}\ \underline{|n+m}}\,\varpi(m, n, \mu)\,\varpi(m, n, \mu')\cos m\psi,$$

where Σ now denotes a summation with respect to m from 1 to n both inclusive. It will be observed that the symbol ϖ has the same meaning here as in Art. 97.

169. For examples we may give explicitly the values of the first three of Laplace's Coefficients.

$$Y_1 = \mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\psi,$$

$$Y_2 = \frac{9}{4}\left(\mu^2-\frac{1}{3}\right)\left(\mu'^2-\frac{1}{3}\right) + 3(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\mu\mu'\cos\psi$$
$$+\frac{3}{4}(1-\mu^2)(1-\mu'^2)\cos 2\psi,$$

$$Y_3 = \frac{25}{4}\left(\mu^3-\frac{3}{5}\mu\right)\left(\mu'^3-\frac{3}{5}\mu'\right)$$
$$+\frac{75}{8}(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\left(\mu^2-\frac{1}{5}\right)\left(\mu'^2-\frac{1}{5}\right)\cos\psi$$
$$+\frac{15}{4}(1-\mu^2)(1-\mu'^2)\mu\mu'\cos 2\psi + \frac{5}{8}(1-\mu^2)^{\frac{3}{2}}(1-\mu'^2)^{\frac{3}{2}}\cos 3\psi.$$

170. From the value of Y_n given at the end of Art. 168 we have immediately

$$\int_0^{2\pi} Y_n\,d\phi = 2\pi\left\{\frac{1.3.5\ldots(2n-1)}{\underline{|n}}\right\}^2\varpi(0, n, \mu)\,\varpi(0, n, \mu').$$

This result was obtained by Legendre in a very laborious manner in his earliest researches on the subject; see *History of the Theories of Attraction*...Art. 787.

By Art. 97 the result may also be written

$$\int_0^{2\pi} Y_n\,d\phi = 2\pi P_n(\cos\theta)\,P_n(\cos\theta').$$

CHAPTER XIV.

LAPLACE'S COEFFICIENTS. ADDITIONAL INVESTIGATIONS.

171. In the preceding Chapter we have given all that is absolutely necessary with respect to the form of Laplace's Coefficients; in the present Chapter we shall shew how the results may be obtained by other modes of investigation, and shall express some of the formulæ in a slightly different manner. The preceding Chapter was almost independent of the processes already exhibited in this work; in the present Chapter, however, we shall make more use of those processes.

172. The determination of the value of C in Art. 168 is troublesome from the fact that two cases have to be considered, namely, that in which $n-m$ is even and that in which $n-m$ is odd. Perhaps the following investigation, which depends on an examination of the highest power of μ instead of the lowest, may be simpler.

Suppose $\mu' = \mu$; then

$$C(1-\mu^2)^{\frac{m}{2}}(1-\mu'^2)^{\frac{m}{2}}\varpi(m, n, \mu)\,\varpi(m, n, \mu')\cos m\psi$$

becomes $\qquad C(1-\mu^2)^m\{\varpi(m, n, \mu)\}^2\cos m\psi.$

The highest power of μ in this expression is μ^{2n}, and its coefficient is $C(-1)^m\cos m\psi$.

Also when $\mu' = \mu$ the function which is to be expanded becomes

$$\{1-2\alpha[\mu^2+(1-\mu^2)\cos\psi]+\alpha^2\}^{-\frac{1}{2}},$$

that is $\qquad \{1-2\alpha[\cos\psi+\mu^2(1-\cos\psi)]+\alpha^2\}^{-\frac{1}{2}}.$

When this is expanded in powers of α the coefficient of α^n will involve $\dfrac{1.3.5\ldots(2n-1)}{\lfloor n}\mu^{2n}(1-\cos\psi)^n$; and we must pick out from this the coefficient of $\cos m\psi$, when $(1-\cos\psi)^n$ is put in the form of cosines of multiples of ψ.

But $(1-\cos\psi)^n = 2^n\sin^{2n}\dfrac{\psi}{2} = 2^n\left\{\dfrac{e^{\frac{\iota\psi}{2}} - e^{-\frac{\iota\psi}{2}}}{2\iota}\right\}^{2n}$

$$= \frac{1}{2^n}\Sigma\frac{(-1)^m \lfloor 2n}{\lfloor n-m \, \lfloor n+m} 2\cos m\psi,$$

where Σ denotes a summation with respect to m from 0 to n both inclusive; except that we must take only half of the value when $m=0$.

Thus $\quad C = 2\dfrac{1.3.5\ldots(2n-1)}{\lfloor n} \times \dfrac{\lfloor 2n}{2^n \lfloor n-m \, \lfloor n+m}$

$$= 2\frac{\{1.3.5\ldots(2n-1)\}^2}{\lfloor n-m \, \lfloor n+m},$$

but only half this value must be taken when $m=0$.

This agrees with Art. 168.

173. In Art. 97 we have seen that

$$\varpi(m, n, \mu) = \frac{\lfloor n-m}{1.3\ldots(2n-1)}\frac{d^m P_n(\mu)}{d\mu^m},$$

also $$P_n(\mu) = \frac{1}{2^n \lfloor n}\frac{d^n(\mu^2-1)^n}{d\mu^n}.$$

Thus

$$\{1.3.5\ldots(2n-1)\}^2\frac{(1-\mu^2)^{\frac{m}{2}}(1-\mu'^2)^{\frac{m}{2}}}{\lfloor n+m \, \lfloor n-m}\varpi(m, n, \mu)\,\varpi(m, n, \mu')$$

$$= \frac{1}{2^{2n}\lfloor n \lfloor n}\frac{\lfloor n-m}{\lfloor n+m}(1-\mu^2)^{\frac{m}{2}}(1-\mu'^2)^{\frac{m}{2}}\frac{d^{n+m}(1-\mu^2)^n}{d\mu^{n+m}}\frac{d^{n+m}(1-\mu'^2)^n}{d\mu'^{n+m}}$$

$$= (1-\mu^2)^{\frac{m}{2}}(1-\mu'^2)^{\frac{m}{2}}\frac{\lfloor n-m}{\lfloor n+m}\frac{d^{2m}M}{d\mu^m d\mu'^m},$$

where $$M = \frac{1}{\{2^n \lfloor n\}^2}\frac{d^{2n}}{d\mu^n d\mu'^n}(1-\mu^2)^n(1-\mu'^2)^n.$$

Thus from Art. 168 we have

$$Y_n = M + 2\Sigma (1-\mu^2)^{\frac{m}{2}} (1-\mu'^2)^{\frac{m}{2}} \frac{\lfloor n-m}{\lfloor n+m} \frac{d^{2m} M}{d\mu^m d\mu'^m} \cos m\psi,$$

where Σ denotes a summation with respect to m from 1 to n, both inclusive.

174. It will be observed that in Arts. 168 and 173 there is nothing to restrict the values of μ and μ' to be unity or less than unity, though it may be often convenient to suppose that $\mu = \cos\theta$ and $\mu' = \cos\theta'$. If we make these suppositions we may write the result of Art. 173 explicitly thus:

$$Y_n = M + \frac{2 \sin\theta \sin\theta' \cos\psi}{n(n+1)} \frac{d^2 M}{d\mu d\mu'}$$

$$+ \frac{2 \sin^2\theta \sin^2\theta' \cos 2\psi}{(n-1)n(n+1)(n+2)} \frac{d^4 M}{d\mu^2 d\mu'^2}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{2 \sin^n\theta \sin^n\theta' \cos n\psi}{\lfloor 2n} \frac{d^{2n} M}{d\mu^n d\mu'^n}.$$

175. We will now give another mode of obtaining the expression for Laplace's Coefficients.

We begin by shewing, as in the beginning of Art. 168, that Y_n must be of the form $\Sigma u_m \cos m\psi$, where Σ denotes summation with respect to m from 0 to n inclusive, and u_m is some function of μ and μ' which is to be determined.

Substitute this expression for Y_n in the differential equation (4) of Art. 167, observing that $\frac{d^2 Y_n}{d\phi^2} = \frac{d^2 Y_n}{d\psi^2}$; then equating to zero the coefficient of $\cos m\psi$, we get

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du_m}{d\mu}\right\} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\} u_m = 0.$$

This differential equation coincides with (9) of Art. 102, and its solution is of the form

$$u_m = (\mu^2 - 1)^{\frac{m}{2}} \{C_1 D^m P_n(\mu) + C_2 D^m Q_n(\mu)\},$$

where D stands for $\frac{d}{d\mu}$, and C_1 and C_2 are constants with respect to μ, though they may involve μ'.

But in the present case we must have $C_2 = 0$, because u_m is necessarily finite when $\mu = 1$, whereas $D^m Q_n(\mu)$ is then infinite, as we know from the form of $Q_n(\mu)$; see Art. 37.

Hence $$u_m = C_1 (\mu^2 - 1)^{\frac{m}{2}} D^m P_n(\mu).$$

But as u_m involves μ and μ' symmetrically, we see in the same manner that

$$u_m = C_0 (\mu'^2 - 1)^{\frac{m}{2}} D^m P_n(\mu'),$$

where D now stands for $\frac{d}{d\mu'}$, and C_0 is constant with respect to μ'. Hence it follows that

$$u_m = b_m (\mu^2 - 1)^{\frac{m}{2}} (\mu'^2 - 1)^{\frac{m}{2}} \frac{d^m P_n(\mu)}{d\mu^m} \frac{d^m P_n(\mu')}{d\mu'^m},$$

where b_m is a constant independent both of μ and μ'.

And $$Y_n = \Sigma u_m \cos m\psi,$$

where u_m has the value just expressed, and Σ denotes a summation with respect to m from 0 to n, both inclusive.

By the use of the notation of Art. 97 we may also express the result thus:

$$Y_n = \Sigma h_m (\mu^2 - 1)^{\frac{m}{2}} (\mu'^2 - 1)^{\frac{m}{2}} \varpi(m, n, \mu)\, \varpi(m, n, \mu') \cos m\psi,$$

where h_m is also a constant, and is connected with b_m by the relation

$$\left\{\frac{\lfloor n-m}{1 \cdot 3 \cdot 5 \ldots (2n-1)}\right\}^2 h_m = b_m.$$

It remains to determine the value of the constant h_m in the last expression for Y_n. This may be done precisely as the value of C was found in Art. 172, for the h_m of the present Article is equal to the C of Art. 172 multiplied into $(-1)^m$.

Thus we find

$$h_m = 2\frac{\{1.3.5\ldots(2n-1)\}^2}{\lfloor n-m \, \lfloor n+m}(-1)^m;$$

and hence $\quad b_m = 2\frac{\lfloor n-m}{\lfloor n+m}(-1)^m;$

but only half these values must be taken when $m=0$.

176. There is still another method of obtaining the expression for Y_n which deserves notice; this does not use *Laplace's differential equation* to which we have had recourse in the investigations already given.

177. If A, B, and C are real quantities, and A positive, and also $A^2-B^2-C^2$ positive, then

$$\int_0^{2\pi}\frac{dt}{A+B\cos t+C\sin t}=\frac{2\pi}{\sqrt{(A^2-B^2-C^2)}}.$$

For assume $B=\rho\cos\gamma$, and $C=\rho\sin\gamma$; thus

$$\int_0^{2\pi}\frac{dt}{A+B\cos t+C\sin t}=\int_0^{2\pi}\frac{dt}{A+\rho\cos(t-\gamma)}=\int_{-\gamma}^{2\pi-\gamma}\frac{d\tau}{A+\rho\cos\tau}.$$

Now the last integral is independent of γ, for its differential coefficient with respect to γ is zero, by the *Integral Calculus*, Chapter IX.: thus the value of the integral is the same as if γ were zero.

Therefore the expression

$$=\int_0^{2\pi}\frac{d\tau}{A+\rho\cos\tau}=\frac{2\pi}{\sqrt{(A^2-\rho^2)}}\text{, by Art. 44,}$$

$$=\frac{2\pi}{\sqrt{(A^2-B^2-C^2)}}.$$

178. Now $P_n(z)$ is the coefficient of a^n in the expansion of $(1-2az+a^2)^{-\frac{1}{2}}$, and we obtain Y_n when for z we put

$$\mu\mu' - \sqrt{\mu^2-1}\,\sqrt{\mu'^2-1}\cos(\phi-\phi').$$

Thus we get $1-2az+a^2$

$$= (\mu - a\mu')^2 - \{\sqrt{(\mu^2-1)}\cos\phi - a\sqrt{(\mu'^2-1)}\cos\phi'\}^2$$
$$- \{\sqrt{(\mu^2-1)}\sin\phi - a\sqrt{(\mu'^2-1)}\sin\phi'\}^2,$$

say $= A^2 - B^2 - C^2$.

Suppose μ positive and greater than μ', so that $\mu - a\mu'$ is positive when a is small enough; then, by Art. 177,

$$\frac{2\pi}{\sqrt{(1-2az+a^2)}}$$
$$= \int_0^{2\pi} \frac{dt}{\mu + \cos(\phi-t)\sqrt{(\mu^2-1)} - a\{\mu' + \cos(\phi'-t)\sqrt{(\mu'^2-1)}\}}.$$

Expand the expression under the integral sign in a series of ascending powers of a; thus we get

$$Y_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\{\mu' + \cos(\phi'-t)\sqrt{(\mu'^2-1)}\}^n}{\{\mu + \cos(\phi-t)\sqrt{(\mu^2-1)}\}^{n+1}}\,dt \;\ldots\ldots(1).$$

Now we know by Art. 149 that

$$\{\mu' + \cos(\phi'-t)\sqrt{(\mu'^2-1)}\}^n$$
$$= a_0 + a_1\cos(\phi'-t) + a_2\cos 2(\phi'-t) + \ldots + a_n\cos n(\phi'-t),$$

and that

$$\frac{1}{\{\mu+\cos(\phi-t)\sqrt{(\mu^2-1)}\}^{n+1}} = b_0 + b_1\cos(\phi-t) + \ldots + b_n\cos n(\phi-t) + \ldots;$$

hence $Y_n = a_0b_0 + \frac{1}{2}a_1b_1\cos(\phi-\phi') + \frac{1}{2}a_2b_2\cos 2(\phi-\phi') + \ldots$

$$+ \frac{1}{2}a_nb_n\cos n(\phi-\phi').$$

Moreover, by Arts. 149 and 150,

$$a_m = \frac{2}{2^n}\frac{\lfloor 2n\,(\mu'^2-1)^{\frac{m}{2}}}{\lfloor n-m\,\lfloor n+m}\,\varpi(m, n, \mu'),$$

$$b_m = \frac{\lfloor n-m \lfloor n+m}{\lfloor n \lfloor n}(-1)^m \frac{2}{2^n}\frac{\lfloor 2n\,(\mu^2-1)^{\frac{m}{2}}}{\lfloor n-m \lfloor n+m}\varpi(m, n, \mu)$$

$$= \frac{2(-1)^m \lfloor 2n}{2^n \lfloor n \lfloor n}(\mu^2-1)^{\frac{m}{2}}\varpi(m, n, \mu);$$

so that, except when $m=0$,

$$a_m b_m = \frac{4\{1.3...(2n-1)\}^2}{\lfloor n-m \lfloor n+m}(1-\mu^2)^{\frac{m}{2}}(1-\mu'^2)^{\frac{m}{2}}\varpi(m, n, \mu)\varpi(m, n, \mu'),$$

and $$a_0 b_0 = \frac{\{1.3...(2n-1)\}^2}{\lfloor n \lfloor n}\varpi(0, n, \mu)\varpi(0, n, \mu').$$

Strictly speaking the result is obtained on the supposition that μ^2-1 and μ'^2-1 are positive; but it is obvious from the form of the result that it holds universally.

179. It will be seen that the definite integral obtained for Y_n in (1) includes both the definite integrals given as expressions for Legendre's n^{th} Coefficient in Art. 49.

For if we put $\mu=1$, we get

$$P_n(\mu') = \frac{1}{2\pi}\int_0^{2\pi}\{\mu' + \cos(\phi'-t)\sqrt{(\mu'^2-1)}\}^n dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\{\mu' + \cos\tau\sqrt{(\mu'^2-1)}\}^n d\tau$$

$$= \frac{1}{\pi}\int_0^{\pi}\{\mu' + \cos\tau\sqrt{(\mu'^2-1)}\}^n d\tau.$$

And if we put $\mu'=1$, we get

$$P_n(\mu) = \frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{\{\mu + \cos(\phi-t)\sqrt{(\mu^2-1)}\}^{n+1}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{d\tau}{\{\mu + \cos\tau\sqrt{(\mu^2-1)}\}^{n+1}}$$

$$= \frac{1}{\pi}\int_0^{\pi}\frac{d\tau}{\{\mu + \cos\tau\sqrt{(\mu^2-1)}\}^{n+1}}.$$

180. The process of Art. 178 involves the equality of two definite integrals which may also be established in another way.

We know that

$$P_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \{z - \cos y \sqrt{(z^2-1)}\}^n \, dy \text{ ......(2)};$$

let $z = xx_1 - \sqrt{(x^2-1)}\sqrt{(x_1^2-1)}\cos(\phi - \phi_1)$; then in Art. 178, we obtain another form for $P_n(z)$, namely

$$P_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\{x_1 + \cos(\phi_1 - y)\sqrt{(x_1^2-1)}\}^n}{\{x + \cos(\phi - y)\sqrt{(x^2-1)}\}^{n+1}} dy \text{......(3)}.$$

We propose then to establish in a direct manner the equality of the right-hand members of (2) and (3).

Put $y - \phi = \chi$; thus the right-hand member of (3) becomes

$$\frac{1}{2\pi}\int_{-\phi}^{2\pi-\phi} \frac{\{x_1 + \cos(\chi + \phi - \phi_1)\sqrt{(x_1^2-1)}\}^n}{\{x + \cos\chi\sqrt{(x^2-1)}\}^{n+1}} d\chi.$$

If we vary ϕ in the limits of this definite integral it does not affect the result; and so the definite integral

$$= \int_0^{2\pi} \frac{\{x_1 + \cos(\chi + \phi - \phi_1)\sqrt{(x_1^2-1)}\}^n}{\{x + \cos\chi\sqrt{(x^2-1)}\}^{n+1}} d\chi.$$

Put β for $\phi_1 - \phi$; and thus we get

$$\int_0^{2\pi} \frac{\{x_1 + \cos(\chi - \beta)\sqrt{(x_1^2-1)}\}^n}{\{x + \cos\chi\sqrt{(x^2-1)}\}^{n+1}} d\chi.$$

Separate this definite integral into two parts, one between the limits 0 and π, and the other between the limits π and 2π; and in the second part change χ into $2\pi - \chi$: thus we get

$$\int_0^{\pi} \frac{\{x_1 + \cos(\chi-\beta)\sqrt{(x_1^2-1)}\}^n + \{x_1 + \cos(\chi+\beta)\sqrt{(x_1^2-1)}\}^n}{\{x + \cos\chi\sqrt{(x^2-1)}\}^{n+1}} d\chi.$$

Now transform this by a process like that of Art. 49; assume

$$\cos\chi = \frac{x\cos\psi - \sqrt{(x^2-1)}}{x - \cos\psi\sqrt{(x^2-1)}};$$

this leads to

$$\sin\chi = \frac{\sin\psi}{x - \cos\psi\sqrt{(x^2-1)}},$$

$$x + \cos\chi\sqrt{(x^2-1)} = \frac{1}{x - \cos\psi\sqrt{(x^2-1)}},$$

$$d\chi = \frac{d\psi}{x - \cos\psi\sqrt{(x^2-1)}}.$$

Thus the definite integral becomes

$$\int_0^\pi (A - B\cos\psi - C\sin\psi)^n d\psi + \int_0^\pi (A - B\cos\psi + C\sin\psi)^n d\psi,$$

where $A = xx_1 - \cos\beta\sqrt{(x^2-1)}\sqrt{(x_1^2-1)},$

$B = x_1\sqrt{(x^2-1)} - x\sqrt{(x_1^2-1)}\cos\beta,$

$C = \sqrt{(x_1^2-1)}\sin\beta.$

Hence we see that $A^2 - B^2 - C^2 = 1$, so that

$$B^2 + C^2 = A^2 - 1 = z^2 - 1;$$

and therefore we may assume

$$B = \sqrt{(z^2-1)}\cos\alpha, \text{ and } C = \sqrt{(z^2-1)}\sin\alpha.$$

The definite integral thus

$$= \int_0^\pi \{z - \sqrt{(z^2-1)}\cos(\psi-\alpha)\}^n d\psi + \int_0^\pi \{z - \sqrt{(z^2-1)}\cos(\psi+\alpha)\}^n d\psi$$

$$= \int_0^{2\pi} \{z - \sqrt{(z^2-1)}\cos(\psi-\alpha)\}^n d\psi$$

$$= \int_0^{2\pi} \{z - \sqrt{(z^2-1)}\cos\psi\}^n d\psi.$$

Thus the definite integral is reduced to the form in (2); and this is what was to be done.

181. In the expressions which have been given for Laplace's Coefficients we have made much use of the function introduced in Art. 97 and denoted by the symbol ϖ. Hence the various forms which are obtained for this function in Arts. 103...106 become of practical interest; and two others to which we now proceed may deserve notice.

182. Suppose $n-m$ *even*. Then it is obvious from the formula at the beginning of Art. 106 that $\varpi(m, n, \cos\theta)$ might be expressed in a series of powers of $\sin\theta$; this series might be deduced from that formula, but an independent investigation will be simpler.

Let $y=(x^2-1)^{\frac{m}{2}}\varpi(m, n, x)$; then y satisfies the differential equation (9) of Art 102. Put $x=\cos\theta$; then this differential equation becomes

$$\frac{d^2y}{d\theta^2}+\cot\theta\frac{dy}{d\theta}+\left\{n(n+1)-\frac{m^2}{\sin^2\theta}\right\}y=0\text{.......(4).}$$

We know then that this equation has a solution of the form

$$y=c_0\sin^n\theta+c_1\sin^{n-2}\theta+c_2\sin^{n-4}\theta+\ldots$$

Substitute this value of y in (4) and let

$$n+m=2\rho,\quad n-m=2\sigma:$$

we shall obtain after reduction

$$c_r=-\frac{(\rho-r+1)(\sigma-r+1)}{r\left(\rho+\sigma-\frac{2r-1}{2}\right)}c_{r-1}\text{............(5).}$$

By direct comparison of the value of y with that of $\varpi(m, n, \cos\theta)$ at the beginning of Art. 106 we see that

$$c_0=(-1)^{\frac{m}{2}}(-1)^{\frac{n-m}{2}}=(-1)^{\frac{n}{2}};$$

therefore $$y=(-1)^{\frac{n}{2}}\left\{\sin^n\theta-\frac{\rho.\sigma}{1.\left(\rho+\sigma-\frac{1}{2}\right)}\sin^{n-2}\theta\right.$$

$$\left.+\frac{\rho(\rho-1)\sigma(\sigma-1)}{\underline{|2}\left(\rho+\sigma-\frac{1}{2}\right)\left(\rho+\sigma-\frac{3}{2}\right)}\sin^{n-4}\theta-\ldots\right\}.$$

It will be seen that y is symmetrical in terms of ρ and σ; this might have been anticipated because y is unchanged in value when the sign of m is changed: see Art. 100. Divide the expression for y by $(-1)^{\frac{m}{2}} \sin^m \theta$; thus we get

$$\varpi(m, n, \cos\theta) = (-1)^\sigma \left\{\sin^{2\sigma}\theta - \frac{\rho \cdot \sigma}{1 \cdot \left(\rho+\sigma-\frac{1}{2}\right)} \sin^{2\sigma-2}\theta \right.$$

$$\left. + \frac{\rho(\rho-1)\sigma(\sigma-1)}{\underline{|2}\left(\rho+\sigma-\frac{1}{2}\right)\left(\rho+\sigma-\frac{3}{2}\right)} \sin^{2\sigma-4}\theta - \dots\right\}.$$

183. Suppose $n-m$ *odd.* Then we see that y will take the form

$$\cos\theta\,\{c_0 \sin^{n-1}\theta + c_1 \sin^{n-3}\theta + c_2 \sin^{n-5}\theta + \dots\}.$$

The differential equation (4) may be expressed thus:

$$\frac{d}{d\theta}\left(\frac{dy}{d\theta}\sin\theta\right) + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\} y \sin\theta = 0.$$

Substitute the value of y; then it will be found that the term which involves c_r is

$$c_r\{(n-2r-1)^2(\sin\theta)^{n-2r-2} - (n-2r)(n-2r+1)(\sin\theta)^{n-2r}\}\cos\theta$$

$$+ c_r\left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}(\sin\theta)^{n-2r}\cos\theta.$$

Hence we see that

$$c_r\{n(n+1)-(n-2r)(n-2r+1)\} + c_{r-1}\{(n-2r+1)^2 - m^2\} = 0.$$

Put $n+m = 2\rho+1$, and $n-m = 2\sigma+1$; then

$$c_r = -\frac{(\rho-r+1)(\sigma-r+1)}{r\left(\rho+\sigma+1-\frac{2r-1}{2}\right)} c_{r-1}.$$

Also by direct comparison we get $c_0 = (-1)^{\frac{n-1}{2}}$.

Hence finally we shall have

$$\varpi(m, n, \cos\theta) = (-1)^{\sigma} \cos\theta \left\{ \sin^{2\sigma}\theta - \frac{\rho \cdot \sigma}{1 \cdot \left(\rho + \sigma + \frac{1}{2}\right)} \sin^{2\sigma - 2}\theta \right.$$

$$\left. + \frac{\rho(\rho-1)\sigma(\sigma-1)}{\underline{|2} \left(\rho + \sigma + \frac{1}{2}\right)\left(\rho + \sigma - \frac{1}{2}\right)} \sin^{2\sigma - 4}\theta - \ldots \right\}.$$

CHAPTER XV.

LAPLACE'S FUNCTIONS.

184. We have already used the differential equation which Laplace's Coefficients satisfy; see equation (4) of Art. 167. We proceed to some further consideration of this equation.

185. We shall first shew how it may be deduced from the more simple equation of Legendre's Coefficients. We known by Art. 54 that $P_n(z)$ satisfies the differential equation

$$(1-z^2)\frac{d^2P_n(z)}{dz^2}-2z\frac{dP_n(z)}{dz}+n(n+1)P_n(z)=0.$$

Assume

$$z=a\cos\theta+b\sin\theta\cos\phi+c\sin\theta\sin\phi,$$

where a, b, and c are constants.

Then

$$\frac{dz}{d\theta}=-a\sin\theta+b\cos\theta\cos\phi+c\cos\theta\sin\phi,$$

$$\frac{d^2z}{d\theta^2}=-z,$$

$$\frac{dz}{d\phi}=(-b\sin\phi+c\cos\phi)\sin\theta,$$

$$\frac{d^2z}{d\phi^2}=-(b\cos\phi+c\sin\phi)\sin\theta.$$

Hence we find that

$$\frac{d^2P_n}{d\theta^2}+\cot\theta\frac{dP_n}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2P_n}{d\phi^2}=A\frac{d^2P_n}{dz^2}+B\frac{dP_n}{dz},$$

where

$$A=\left(\frac{dz}{d\theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{dz}{d\phi}\right)^2$$

$$=(-a\sin\theta+b\cos\theta\cos\phi+c\cos\theta\sin\phi)^2$$

$$+(-b\sin\phi+c\cos\phi)^2,$$

and

$$B=\frac{d^2z}{d\theta^2}+\cot\theta\frac{dz}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2z}{d\phi^2}.$$

Thus we see that $A+z^2=a^2+b^2+c^2$,

so that $$A=a^2+b^2+c^2-z^2;$$

and $B=-2z$. Hence if $a^2+b^2+c^2=1$, we have

$$(1-z^2)\frac{d^2P_n}{dz^2}-2z\frac{dP_n}{dz}+n(n+1)P_n$$

$$=\frac{d^2P_n}{d\theta^2}+\cot\theta\frac{dP_n}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2P_n}{d\phi^2}+n(n+1)P_n;$$

and therefore the last expression is zero.

186. Any function which satisfies the partial differential equation (4) of Art. 167 may be called a Laplace's *Function* of the n^{th} order. The variables it will be observed are θ and ϕ, and $\mu=\cos\theta$. Thus Laplace's *Coefficients* are particular cases of Laplace's *Functions;* for the *Coefficients* all satisfy the equation (4) of Art. 167. We shall continue to use Y_n to denote Laplace's *Coefficient* of the n^{th} order, and shall use other symbols as X_n and Z_n to denote a Laplace's *Function* of the n^{th} order.

187. Let m and n be different positive integers. Let X_m be a Laplace's Function of the order m, and Z_n a Laplace's

Function of the order n; then under certain conditions which will appear in the course of the investigation we shall have

$$\int_{-1}^{1}\int_{0}^{2\pi} X_m Z_n \, d\mu \, d\phi = 0.$$

For by the differential equation of Laplace's Functions we have

$$m(m+1) X_m = -\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dX_m}{d\mu}\right\} - \frac{1}{1-\mu^2}\frac{d^2X_m}{d\phi^2};$$

and therefore $\displaystyle\int_{-1}^{1}\int_{0}^{2\pi} X_m Z_n \, d\mu \, d\phi$

$$= -\frac{1}{m(m+1)}\int_{-1}^{1}\int_{0}^{2\pi}\left[\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dX_m}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2X_m}{d\phi^2}\right] Z_n \, d\mu \, d\phi.$$

By integrating by parts twice we find that

$$\int\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dX_m}{d\mu}\right\} Z_n \, d\mu = (1-\mu^2)\frac{dX_m}{d\mu} Z_n - (1-\mu^2)\frac{dZ_n}{d\mu} X_m$$
$$+\int\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dZ_n}{d\mu}\right\} X_m \, d\mu;$$

therefore

$$\int_{-1}^{1}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dX_m}{d\mu}\right\} Z_n \, d\mu = \int_{-1}^{1}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dZ_n}{d\mu}\right\} X_m \, d\mu.$$

Again, by integrating by parts twice we have

$$\int\frac{d^2X_m}{d\phi^2} Z_n \, d\phi = \frac{dX_m}{d\phi} Z_n - \frac{dZ_n}{d\phi} X_m + \int\frac{d^2Z_n}{d\phi^2} X_m \, d\phi;$$

therefore $\displaystyle\int_{0}^{2\pi}\frac{d^2X_m}{d\phi^2} Z_n \, d\phi = \int_{0}^{2\pi}\frac{d^2Z_n}{d\phi^2} X_m \, d\phi,$

assuming that X_m and $\dfrac{dX_m}{d\phi}$ *have the same values respectively when* $\phi = 0$ *and when* $\phi = 2\pi$, *and making a similar assumption with respect to* Z_n and $\dfrac{dZ_n}{d\phi}$.

Hence $\int_{-1}^{1}\int_{0}^{2\pi} X_m Z_n \, d\mu \, d\phi$

$$= -\frac{1}{m(m+1)}\int_{-1}^{1}\int_{0}^{2\pi}\left[\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dZ_n}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2Z_n}{d\phi^2}\right] X_m \, d\mu \, d\phi$$

$$= \frac{n(n+1)}{m(m+1)}\int_{-1}^{1}\int_{0}^{2\pi} X_m Z_n \, d\mu \, d\phi,$$

by the differential equation of Laplace's Functions.

Hence since m and n are supposed different

$$\int_{-1}^{1}\int_{0}^{2\pi} X_m Z_n \, d\mu \, d\phi = 0.$$

188. In addition to the conditions which are expressly stated in the preceding Article, we have of course one which is always implied in applications of the Integral Calculus, namely that the functions which occur are to remain *finite* throughout the range of the integration; these functions here are X_m and Z_n and their first and second differential coefficients with respect to μ and ϕ.

189. In future whenever we speak of Laplace's Functions we shall always suppose them to be limited by the conditions stated in Arts. 187 and 188.

190. The differential equation of Laplace's Functions has been integrated in a symbolical form by Mr Hargreave; and after him by Professor Donkin and Professor Boole; see Boole's *Differential Equations*, Chapter XVII. The result though very interesting theoretically has not hitherto been used in practical applications.

191. Take the general expression for Y_n which is given in Art. 168; consider it as a function of θ and ϕ, putting $\phi - \phi'$ for ψ. This expression then may be said to consist of $2n+1$ terms, namely one corresponding to $m=0$, and two corresponding to every other value of m not greater than n: the two are of the form

$$K_m(1-\mu^2)^{\frac{m}{2}}\varpi(m, n, \mu)\cos m\phi, \text{ and } L_m(1-\mu^2)^{\frac{m}{2}}\varpi(m, n, \mu)\sin m\phi,$$

where K_m and L_m are independent of μ and ϕ.

Each of the $2n+1$ terms will separately satisfy the differential equation of Laplace's Functions; for the whole expression satisfies that equation, and thus the terms which involve $\sin m\phi$ and $\cos m\phi$ must separately vanish.

192. We shall now shew that any Laplace's Function which is a *rational integral function* of $\cos\theta$, $\sin\theta\cos\phi$, and $\sin\theta\sin\phi$, consisting of a finite number of terms, is of the form

$$A_0\varpi(0, n, \cos\theta) + \Sigma\{A_m C_m + B_m S_m\},$$

where C_m stands for $\sin^m\theta\,\varpi(m, n, \cos\theta)\cos m\phi$, and S_m stands for $\sin^m\theta\,\varpi(m, n, \cos\theta)\sin m\phi$, and A_m and B_m denote arbitrary constants; also Σ denotes a summation with respect to m from 1 to n, both inclusive. It will be seen that the conditions which we here impose on our Laplace's Function *include* those of Art. 189, but are more restrictive still.

To demonstrate this we observe that any rational integral function of $\cos\theta$, $\sin\theta\cos\phi$, and $\sin\theta\sin\phi$, may be put in the form $\Sigma(u_m\cos m\phi + v_m\sin m\phi)$, where u_m and v_m are functions of θ only, and Σ denotes summation with respect to m. Substitute in the differential equation of Laplace's Functions; then it will be found that u_m and v_m must both be values of ζ which satisfy the differential equation

$$\frac{d^2\zeta}{d\theta^2} + \cot\theta\frac{d\zeta}{d\theta} + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}\zeta = 0.$$

Put x for $\cos\theta$; then this differential equation coincides with equation (9) of Art. 102, and therefore the solution is

$$\zeta = (x^2-1)^{\frac{m}{2}}\{H_1D^mP_n(x) + H_2D^mQ_n(x)\},$$

where H_1 and H_2 are arbitrary constants.

But since ζ is in this case to be *rational and integral and of a finite number of terms*, we must have $H_2 = 0$.

Thus $\zeta = (x^2-1)^{\frac{m}{2}}H_1D^mP_n(x)$; and this vanishes if m is greater than n. And as $\varpi(m, n, x)$ is equal to the product of a constant into $D^mP_n(x)$ we have finally

$$\zeta = (x^2-1)^{\frac{m}{2}}K\varpi(m, n, x),$$

where K is a constant. This establishes the proposition.

193. The expression given at the beginning of the preceding Article denotes Laplace's Function of the n^{th} order under the restrictive conditions there enunciated. We may give various forms to this expression by means of the various developments which have been obtained for $\varpi(m, n, x)$ or for $(x^2-1)^{\frac{m}{2}}\varpi(m, n, x)$.

For example, let y_σ denote the series which is between the brackets in the value of y of Art. 182; and let z_σ denote the series which can be obtained from y_σ by changing $\rho+\sigma$ into $\rho+\sigma+1$ in denominators; then it will be found that the Laplace's Function

$$=\Sigma y_\sigma\{b_\sigma\cos(\rho-\sigma)\phi+c_\sigma\sin(\rho-\sigma)\phi\}$$
$$+\Sigma z_\sigma\cos\theta\{\beta_\sigma\cos(\rho-\sigma)\phi+\gamma_\sigma\sin(\rho-\sigma)\phi\}.$$

Here b_σ, c_σ, β_σ, γ_σ are arbitrary constants, and Σ denotes summation with respect to σ. In the first part of the expression ρ is to be determined by the equation $\rho+\sigma=n$; and the summation is to be from 0 to the greatest integer in $\frac{n}{2}$, both inclusive. In the second part of the expression ρ is to be determined by the equation $\rho+\sigma=n-1$; and the summation is to be from 0 to the greatest integer in $\frac{n-1}{2}$, both inclusive.

194. We shall now find the value of $\int_{-1}^{1}\int_0^{2\pi}X_nZ_n\,d\mu\,d\phi$, where X_n and Z_n are two Laplace's Functions of the order n limited by the respective conditions of Art. 192. We may take

$$X_n=\Sigma\sin^m\theta\,\varpi(m, n, \cos\theta)\,(A_m\cos m\phi+B_m\sin m\phi),$$
$$Z_n=\Sigma\sin^m\theta\,\varpi(m, n, \cos\theta)\,(G_m\cos m\phi+H_m\sin m\phi),$$

where A_m, B_m, G_m, and H_m denote constants; and Σ denotes summation with respect to m from 0 to n, both inclusive.

Multiply, and integrate with respect to ϕ from 0 to 2π; thus

$$\int_0^{2\pi}X_nZ_n\,d\phi=\pi\Sigma\sin^{2m}\theta\,\{\varpi(m, n, \cos\theta)\}^2(A_mG_m+B_mH_m),$$

except when $m=0$, and then for π we must put 2π.

The next step then is to find the value of

$$\int_{-1}^{1} \sin^{2m}\theta \, \{\varpi(m, n, \cos\theta)\}^2 \, d\mu,$$

that is of $\int_{-1}^{1} (1 - x^2)^m \{\varpi(m, n, x)\}^2 \, dx.$

By Art. 97 the expression to be evaluated is

$$\left\{\frac{\lfloor n-m}{\lfloor 2n}\right\}^2 \int_{-1}^{1} (1-x^2)^m \{D^{n+m}(x^2-1)^n\}^2 dx,$$

and this by equation (2) of Art. 96

$$= (-1)^m \frac{\lfloor n-m \, \lfloor n+m}{\lfloor 2n \, \lfloor 2n} \int_{-1}^{1} D^{n+m}(x^2-1)^n D^{n-m}(x^2-1)^n \, dx.$$

By successive integration by parts we have

$$\int_{-1}^{1} D^{n+m}(x^2-1)^n D^{n-m}(x^2-1)^n dx = (-1)^{n-m} \int_{-1}^{1} (x^2-1)^n D^{2n}(x^2-1)^n dx$$

$$= \lfloor 2n \, (-1)^{n-m} \int_{-1}^{1} (x^2-1)^n dx = \lfloor 2n \, (-1)^{-m} \int_{-1}^{1} (1-x^2)^n dx$$

$$= (-1)^{-m} \lfloor 2n \, \frac{2n(2n-2)\dots 2}{(2n+1)(2n-1)\dots 3} \, 2.$$

Hence we obtain

$$\int_{-1}^{1} \sin^{2m}\theta \{\varpi(m, n, \cos\theta)\}^2 d\mu = \frac{2 \lfloor n-m \, \lfloor n+m}{\lfloor 2n} \, \frac{2n.(2n-2)\dots 2}{(2n+1)(2n-1)\dots 3},$$

and thus finally $\int_{-1}^{1}\int_{0}^{2\pi} X_n Z_n \, d\mu \, d\phi$

$$= 2\pi \frac{2n(2n-2)\dots 2}{\lfloor 2n \, (2n+1)\dots 3} \Sigma \lfloor n-m \, \lfloor n+m \, (A_m G_m + B_m H_m)$$

$$= \frac{2\pi}{(2n+1)\{1.3.5\dots(2n-1)\}^2} \Sigma \lfloor n-m \, \lfloor n+m \, (A_m G_m + B_m H_m),$$

but for the case of $m = 0$ we must double the term.

Thus we may express the result in the following manner:

$$\int_{-1}^{1}\int_{0}^{2\pi} X_n Z_n \, d\mu \, d\phi = \text{the product of } \frac{2\pi}{(2n+1)\{1.3.5\ldots(2n-1)\}^2}$$

$$\text{into } \left\{2\,\lfloor n\,\lfloor n\, A_0 G_0 + \Sigma\,\lfloor n-m\,\lfloor n+m\,(A_m G_m + B_m H_m)\right\},$$

where Σ now denotes a summation with respect to m from 1 to n, both inclusive.

195. As a particular case of the preceding Article suppose the *function* X_n to be the *Coefficient* Y_n. By Art. 168

$$A_m = \frac{2.\{1.3.5\ldots(2n-1)\}^2}{\lfloor n-m\,\lfloor n+m} \sin^m\theta'\,\varpi\,(m, n, \cos\theta')\cos m\phi',$$

and B_m may be obtained from this by changing $\cos m\phi'$ into $\sin m\phi'$: but when $m = 0$ we must take half these values. Hence we have $\int_{-1}^{1}\int_{0}^{2\pi} Y_n Z_n \, d\mu \, d\phi$

$$= \frac{4\pi}{2n+1}\,\Sigma \sin^m\theta'\,\varpi\,(m, n, \cos\theta')\,(G_m \cos m\phi' + H_m \sin m\phi')$$

$$= \frac{4\pi}{2n+1} Z'_n,$$

where Z'_n is what Z_n becomes when for θ and ϕ we put θ' and ϕ' respectively.

This is a very important result.

196. Hence, for example, we have

$$\int_{-1}^{1}\int_{0}^{2\pi} Y_n^2 \, d\mu \, d\phi = \frac{4\pi}{2n+1},$$

because $Y_n' = 1$.

CHAPTER XVI.

EXPANSION OF FUNCTIONS.

197. IN the course of Laplace's researches on Attractions and the Figure of the Earth he obtained incidentally the remarkable result that any function of the spherical coordinates μ and ϕ might be expressed in a series of Laplace's Functions. The demonstration however was not very satisfactory and other investigations have been given since.

198. We shall first shew that a function can be expressed in only one way in terms of Laplace's Functions. Let $F(\mu, \phi)$ denote a given function, and if possible suppose that

$$F(\mu, \phi) = X_1 + X_2 + X_3 + \ldots\ldots,$$

and also $$= Z_1 + Z_2 + Z_3 + \ldots\ldots;$$

where X_m and Z_m denote Laplace's Functions of the order m. Then by subtraction

$$0 = X_1 - Z_1 + X_2 - Z_2 + X_3 - Z_3 + \ldots.$$

Multiply by Y_n, and perform the double integration with respect to μ and ϕ. Then, by Art. 187,

$$0 = \int_{-1}^{1}\int_{0}^{2\pi} Y_n (X_n - Z_n)\, d\mu d\phi;$$

therefore, by Art. 195,

$$0 = X_n' - Z_n';$$

where X_n' denotes the value of X_n when we put θ' for θ and ϕ' for ϕ; and a similar meaning belongs to Z_n'.

Thus since $X_n' = Z_n'$ whatever θ' and ϕ' may be, it is obvious that X_n is identical with Z_n.

199. In the simple case where a given function is a *rational integral function* of $\cos\theta$, $\sin\theta\cos\phi$, and $\sin\theta\sin\phi$, there is no difficulty in shewing that the function can be expressed in a series of Laplace's Functions.

Any constant quantity may be considered as a Laplace's Function of the order zero; since it will satisfy the differential equation of Laplace's Functions when we put $n=0$.

Next take any rational integral function of $\cos\theta$, $\sin\theta\cos\phi$, and $\sin\theta\sin\phi$ of the *first* degree. This must be of the form

$$A_1\cos\theta + A_2\sin\theta\cos\phi + A_3\sin\theta\sin\phi + A_4,$$

where A_1, A_2, A_3, and A_4 are constants.

Here A_4 is a Laplace's Function of the order zero as we have just seen; and $A_1\cos\theta$, $A_2\sin\theta\cos\phi$, $A_3\sin\theta\sin\phi$ are all Laplace's Functions of the first order, as we may infer from the known form of Y_1, or as we may verify by actual substitution in the differential equation of Laplace's Functions.

Next take a rational integral function of the *second* degree. This must be of the form

$$B_1\cos^2\theta + B_2\sin^2\theta\cos^2\phi + B_3\sin^2\theta\sin^2\phi$$
$$+ B_4\cos\theta\sin\theta\cos\phi + B_5\cos\theta\sin\theta\sin\phi + B_6\sin^2\theta\cos\phi\sin\phi,$$

omitting terms of the first order, for these as we have already seen can be exhibited as Laplace's Functions.

We may express these six terms thus

$$C_1\left(\cos^2\theta - \frac{1}{3}\right) + C_2\sin^2\theta\cos 2\phi + C_3$$
$$+ \frac{1}{2}B_6\sin^2\theta\sin 2\phi + B_4\cos\theta\sin\theta\cos\phi + B_5\cos\theta\sin\theta\sin\phi,$$

where C_1, C_2, C_3 are all constant, as well as B_4, B_5, B_6.

Here C_3 will be a Laplace's Function of the order zero, and the other terms will be Laplace's Functions of the second order, as may be seen in the manner already indicated.

But without giving any more examples let us proceed to the general investigation.

A rational integral function of $\cos\theta$, $\sin\theta\cos\phi$, and $\sin\theta\sin\phi$ will be an assemblage of terms of the form $(\cos\theta)^p(\sin\theta\cos\phi)^q(\sin\theta\sin\phi)^r$ multiplied into constants.

Now $\cos^q\phi\sin^r\phi$ can be expressed as a series of cosines of multiples of ϕ, or of sines of multiples of ϕ, according as r is even or odd. Thus $(\sin\theta\cos\phi)^q(\sin\theta\sin\phi)^r$ may be expressed as the product of $(\sin\theta)^{q+r}$ into a series of sines of multiples of ϕ or cosines of multiples of ϕ. When this is done for all the terms in the given rational integral function, we shall find that a term $\cos k\phi$ or $\sin k\phi$ is multiplied by a power of $\sin\theta$, of which the index is k, or k increased by some *even* number.

Hence if f denote any rational integral function, we can express it thus

$$f = F_0 + F_1\sin\theta\cos\phi + F_2\sin^2\theta\cos 2\phi + F_3\sin^3\theta\cos 3\phi + \ldots$$
$$+ G_1\sin\theta\sin\phi + G_2\sin^2\theta\sin 2\phi + G_3\sin^3\theta\sin 3\phi + \ldots,$$

where $F_0, F_1, F_2, \ldots G_1, G_2 \ldots$ denote rational integral functions of $\cos\theta$.

Now any one of these, say F_m, may be divided into two parts, one an even function of $\cos\theta$, and the other an odd function of $\cos\theta$. Let $F_m = u_m + v_m$, where u_m denotes the even function, and v_m the odd function.

Suppose then

$$u_m = a_0\cos^{2\lambda}\theta + a_2\cos^{2\lambda-2}\theta + a_4\cos^{2\lambda-4}\theta + \ldots,$$

where $a_0, a_2, a_4, \ldots$ are constants.

By Art. 97 we see that

$$u_m - a_0\varpi(m, m+2\lambda, \cos\theta) = k\cos^{2\lambda-2}\theta + \ldots,$$

that is $u_m - a_0\varpi(m, m+2\lambda, \cos\theta)$ is of two dimensions lower than u_m as to powers of $\cos\theta$.

Proceeding in this way we see that we can express u_m thus:

$$u_m = b_0\varpi(m, m, \cos\theta) + b_2\varpi(m, m+2, \cos\theta)$$
$$+ b_4\varpi(m, m+4, \cos\theta) + \ldots,$$

where $b_0, b_2, b_4, \ldots$ are constants.

Similarly we may shew that

$$v_m = b_1 \varpi(m, m+1, \cos\theta) + b_3 \varpi(m, m+3, \cos\theta) + \ldots,$$

where $b_1, b_3, \ldots$ are constants.

Thus $F_m = b_0 \varpi(m, m, \cos\theta) + b_1 \varpi(m, m+1, \cos\theta)$
$$+ b_2 \varpi(m, m+2, \cos\theta) + \ldots.$$

In like manner G_m may be expressed.

Then by Art. 191 we see that f takes the form of a set of Laplace's Functions; the highest *order* being determined by the greatest value of n which occurs in the expressions of which the type is $\varpi(m, n, \cos\theta)$.

200. But we wish to shew that *any* function of θ and ϕ can be expressed in a series of Laplace's Functions; that is, we no longer restrict ourselves to the case of a rational integral function of $\cos\theta$, $\sin\theta\cos\phi$, and $\sin\theta\sin\phi$. We shall give a process which is in substance frequently repeated in the writings of Poisson: see for instance his *Théorie Mathématique de la Chaleur.*

We have by definition

$$\frac{1}{\sqrt{(1-2\alpha x+\alpha^2)}} = 1 + P_1(x)\,\alpha + P_2(x)\,\alpha^2 + \ldots\ldots(1).$$

Differentiate with respect to α; thus

$$\frac{x-\alpha}{(1-2\alpha x+\alpha^2)^{\frac{3}{2}}} = P_1(x) + 2P_2(x)\,\alpha + 3P_3(x)\,\alpha^2 + \ldots(2).$$

Multiply (2) by 2α, and add to (1); thus

$$\frac{1-\alpha^2}{(1-2\alpha x+\alpha^2)^{\frac{3}{2}}} = 1 + 3P_1(x)\,\alpha + 5P_2(x)\,\alpha^2 + \ldots$$
$$+ (2n+1)\,P_n(x)\,\alpha^n + \ldots\ldots(3).$$

Now substitute for x the value

$$\mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\cos(\phi-\phi');$$

and integrate both sides between the limits -1 and 1 for μ and 0 and 2π for ϕ. For brevity we shall retain the symbol x on the left-hand side; but shall change $P_n(x)$ to Y_n on the right-hand side. Thus

$$\int_{-1}^{1}\int_{0}^{2\pi}\frac{1-\alpha^2}{(1-2\alpha x+\alpha^2)^{\frac{3}{2}}}\,d\mu\,d\phi$$

$$=\int_{-1}^{1}\int_{0}^{2\pi}\{1+3Y_1\alpha+5Y_2\alpha^2+\ldots+(2n+1)\,Y_n\alpha^n+\ldots\}\,d\mu\,d\phi.$$

Now by the property of Laplace's Coefficients given in Art. 187 all the terms on the right-hand side disappear except the first, and thus we get

$$\int_{-1}^{1}\int_{0}^{2\pi}\frac{1-\alpha^2}{(1-2\alpha x+\alpha^2)^{\frac{3}{2}}}\,d\mu\,d\phi=4\pi.$$

201. Thus we see that the value of the preceding definite integral is independent of α: this very remarkable result may be confirmed by another method.

We know, by Art. 165, that x may be considered to represent the cosine of the arc drawn on the surface of a sphere from a certain fixed point of which the coordinates are θ' and ϕ' to a certain variable point of which the coordinates are θ and ϕ. Denote the former point by P', and the latter by P. Let γ denote the arc PP', and χ the angle between $P'P$ and a fixed arc through P'. Then we may in fact transform the double integral by expressing it in terms of the new variables γ and χ. The element of spherical surface $d\mu\,d\phi$ will be equivalent to $\sin\gamma\,d\gamma\,d\chi$, that is to $-d\cos\gamma\,d\chi$, that is to $-dx\,d\chi$. Thus we get

$$\int_{-1}^{1}\int_{0}^{2\pi}\frac{1-\alpha^2}{(1-2\alpha x+\alpha^2)^{\frac{3}{2}}}\,d\mu\,d\phi=\int_{-1}^{1}\int_{0}^{2\pi}\frac{1-\alpha^2}{(1-2\alpha x+\alpha^2)^{\frac{3}{2}}}\,dx\,d\chi$$

$$=2\pi\int_{-1}^{1}\frac{1-\alpha^2}{(1-2\alpha x+\alpha^2)^{\frac{3}{2}}}\,dx.$$

Now $$\int\frac{dx}{(1-2\alpha x+\alpha^2)^{\frac{3}{2}}}=\frac{1}{\alpha(1-2\alpha x+\alpha^2)^{\frac{1}{2}}},$$

therefore

$$\int_{-1}^{1} \frac{dx}{(1-2ax+a^2)^{\frac{3}{2}}} = \frac{1}{a}\left\{\frac{1}{1-a} - \frac{1}{1+a}\right\} = \frac{2}{1-a^2}.$$

Thus as before

$$\int_{-1}^{1}\int_{0}^{2\pi} \frac{1-a^2}{(1-2ax+a^2)^{\frac{3}{2}}}\, d\mu d\phi = 4\pi.$$

202. Put ζ for $\dfrac{1-a^2}{(1-2ax+a^2)^{\frac{3}{2}}}$, where

$$x = \mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\cos(\phi-\phi').$$

Then we have shewn that $\int_{-1}^{1}\int_{0}^{2\pi} \zeta d\mu d\phi = 4\pi$. This result is true however near a may be to unity. But if the difference between unity and a is infinitesimal, it is obvious that ζ is also infinitesimal except when the denominator of it is very small: this can happen only when x is indefinitely near to unity, that is when $\theta-\theta'$ and $\phi-\phi'$ are both infinitesimal.

If we consider ζ to represent an ordinate which corresponds to the two variables μ and ϕ, then $\int_{-1}^{1}\int_{0}^{2\pi} \zeta d\mu d\phi$ will represent a certain volume; and we see that when $1-a$ is infinitesimal, the elements of this volume are insensible except close to the point at which $\theta=\theta'$ and $\phi=\phi'$. At this point the ordinate becomes very great. The volume however is always finite, namely 4π.

203. Let $F(\theta, \phi)$ denote any function of θ and ϕ which is always finite between the limits of μ and ϕ with which we are concerned. By Art. 200 we have

$$\int_{-1}^{1}\int_{0}^{2\pi} \frac{1-a^2}{(1-2ax+a^2)^{\frac{3}{2}}}\, F(\theta,\phi)\, d\mu d\phi$$

$$= \int_{-1}^{1}\int_{0}^{2\pi} F(\theta,\phi)\{1+3Y_1 a+5Y_2 a^2+\ldots+(2n+1)Y_n a^n+\ldots\}\, d\mu d\phi.$$

Denote the left-hand member by X, then we may express the result thus, $X = U_0 + \alpha U_1 + \alpha^2 U_2 + \ldots$, where

$$U_n = (2n+1)\int_{-1}^{1}\int_{0}^{2\pi} Y_n F(\theta, \phi)\, d\mu d\phi.$$

This relation being always true when α is any proper fraction, we may assume that it holds even up to the limit when α is unity. The limit of the right-hand member is obtained by putting unity for α. We must investigate the limit of the left-hand member.

Let ζ have the same meaning as in Art. 202. Since ζ ultimately vanishes, except when $\mu - \mu'$ and $\phi - \phi'$ are infinitesimal, we may change the limits of the integral $\int_{-1}^{1}\int_{0}^{2\pi} F(\theta, \phi)\, \zeta d\mu d\phi$ to any others which include the values $\mu = \mu'$ and $\phi = \phi'$. Thus the limits may be $\mu' - \beta$ and $\mu' + \beta$ for μ, where β is infinitesimal, and $\phi' - \gamma$ and $\phi' + \gamma$ for ϕ, where γ is infinitesimal.

Hence we reduce the integral to

$$\int_{\mu'-\beta}^{\mu'+\beta}\int_{\phi'-\gamma}^{\phi'+\gamma} F(\theta, \phi)\, \zeta d\mu d\phi.$$

Next we observe, that since ζ is always positive, we have

$$\int_{\mu'-\beta}^{\mu'+\beta}\int_{\phi'-\gamma}^{\phi'+\gamma} F(\theta, \phi)\, \zeta d\mu d\phi = f\int_{\mu'-\beta}^{\mu'+\beta}\int_{\phi'-\gamma}^{\phi'+\gamma} \zeta d\mu d\phi,$$

where f is some value which $F(\theta, \phi)$ takes between the limits of the integrations: see *Integral Calculus*, Art. 40. And since these limits are ultimately indefinitely close to μ' and ϕ' respectively, we have ultimately $f = F(\theta', \phi')$. Also $\iint \zeta d\mu d\phi$ between the limits $= 4\pi$. Thus finally

$$4\pi F(\theta', \phi') =$$

$$\int_{-1}^{1}\int_{0}^{2\pi} \{1 + 3Y_1 + 5Y_2 + \ldots + (2n+1) Y_n + \ldots\} F(\theta, \phi) d\mu d\phi.$$

This shews that $F(\theta', \phi')$ can be expressed in a series of Laplace's Functions; for Y_n is a Laplace's Function of μ' and ϕ' of the order n, and when it is integrated with respect to μ and ϕ it is still such. It is often convenient to express the result thus

$$4\pi F(\theta', \phi') = U_0 + U_1 + U_2 + \ldots,$$

where $$U_n = (2n+1) \int_{-1}^{1} \int_{0}^{2\pi} Y_n F(\theta, \phi)\, d\mu d\phi.$$

204. By interchanging the symbols θ and θ', and also ϕ and ϕ', we get

$$4\pi F(\theta, \phi) =$$
$$\int_{-1}^{1} \int_{0}^{2\pi} \{1 + 3Y_1 + 5Y_2 + \ldots + (2n+1) Y_n + \ldots\} F(\theta', \phi')\, d\mu', d\phi';$$

it is unnecessary to make any change in the general symbol Y_n, for that involves θ and θ' symmetrically, and also ϕ and ϕ' symmetrically.

Thus $F(\theta, \phi)$ is here exhibited in the form of a series of Laplace's Functions; the Function of the n^{th} order being

$$\frac{2n+1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} Y_n F(\theta', \phi')\, d\mu' d\phi'.$$

205. In Art. 203 suppose that $F(\theta', \phi')$ is itself a *Laplace's Function* of the n^{th} order; then by Art. 187 all the terms in the series disappear except one, and we have

$$4\pi F(\theta', \phi') = (2n+1) \int_{-1}^{1} \int_{0}^{2\pi} Y_n F(\theta, \phi)\, d\mu d\phi;$$

this agrees with the last result of Art. 195.

206. Let the definite integral

$$\frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \frac{(1-\alpha^2) F(\theta, \phi)}{(1 - 2\alpha x + \alpha^2)^{\frac{3}{2}}}\, d\mu d\phi$$

be denoted by Q for brevity; then we have shewn in Art. 203 that the value of Q when α is unity is $F(\theta', \phi')$: Poisson himself puts some of the reasoning by which this is obtained in a more formal manner, but not I think more decisively. The result holds so long as θ' lies between 0 and π, and ϕ' between 0 and 2π; but at these limits exceptions occur which we proceed to notice.

207. Suppose $\phi'=0$. There are now *two* values of ϕ which in conjunction with $\theta=\theta'$ make the denominator of ζ vanish, namely $\phi=0$ and $\phi=2\pi$.

We have

$$\int_{-1}^{1}\int_{0}^{2\pi} F(\theta,\phi)\,\zeta d\mu d\phi$$

$$=\int_{-1}^{1}\int_{0}^{\pi} F(\theta,\phi)\,\zeta d\mu d\phi+\int_{-1}^{1}\int_{\pi}^{2\pi} F(\theta,\phi)\,\zeta d\mu d\phi,$$

and we will consider separately the two expressions on the right-hand side.

Take $\int_{-1}^{1}\int_{0}^{\pi} F(\theta,\phi)\,\zeta d\mu d\phi$. Since ζ vanishes throughout the range of integration, except when ϕ and $\theta-\theta'$ are very small, we may reduce this to $\int_{\mu'-\beta}^{\mu'+\beta}\int_{0}^{\gamma} F(\theta,\phi)\,\zeta d\mu d\phi$, where β and γ are infinitesimal. In the next place we may take this to be ultimately equal to $F(\theta',0)\int_{\mu'-\beta}^{\mu'+\beta}\int_{0}^{\gamma} \zeta d\mu d\phi$. Then without causing any sensible difference we may change this to $F(\theta',0)\int_{-1}^{1}\int_{0}^{\pi} \zeta d\mu d\phi$; and this is equal to $2\pi F(\theta',0)$; for if we return to the process of Art. 201, and suppose $\phi'=0$, we shall obtain half the result there given, now that the limits of ϕ are 0 and π instead of 0 and 2π. Thus finally

$$\int_{-1}^{1}\int_{0}^{\pi} F(\theta,\phi)\,\zeta d\mu d\phi=2\pi F(\theta',0).$$

In the same manner it may be shewn that

$$\int_{-1}^{1}\int_{\pi}^{2\pi} F(\theta,\phi)\,\zeta d\mu d\phi=2\pi F(\theta',2\pi).$$

Hence, when $\phi'=0$, we have

$$Q=\tfrac{1}{2}\{F(\theta',0)+F(\theta',2\pi)\}.$$

208. Suppose $\phi'=2\pi$. Then, adopting the same method as in the preceding Article, we shall arrive at the same result. Thus the value of Q, when $\phi'=0$ or $\phi'=2\pi$, is the half-sum of the values of $F(\theta',\phi')$ for these values of ϕ'.

209. Suppose $\theta' = 0$. Then the denominator of ζ vanishes when $\theta = 0$, whatever ϕ may be. Here $1 - 2ax + a^2$ reduces to $1 - 2a\cos\theta + a^2$, and ζ vanishes in the limit except when θ vanishes. Thus

$$\int_{-1}^{1}\int_{0}^{2\pi} F(\theta, \phi)\, \frac{1 - a^2}{(1 - 2a\cos\theta + a^2)^{\frac{3}{2}}}\, d\mu\, d\phi \text{ reduces to}$$

$$\int_{0}^{2\pi} F(0, \phi) \left\{\int_{-1}^{1} \frac{(1 - a^2)\, d\mu}{(1 - 2a\cos\theta + a^2)^{\frac{3}{2}}}\right\} d\phi,$$

and $\displaystyle\int_{-1}^{1} \frac{(1 - a^2)\, d\mu}{(1 - 2a\cos\theta + a^2)^{\frac{3}{2}}} = 2$, as is shewn in Art. 201.

Thus finally $$Q = \frac{1}{2\pi}\int_{0}^{2\pi} F(0, \phi)\, d\phi.$$

Thus, when $\theta' = 0$, we may say that Q is the *mean* of the values of $F(0, \phi)$.

210. Suppose $\theta' = \pi$. Then adopting the same method as in the preceding Article, we shall find that

$$Q = \frac{1}{2\pi}\int_{0}^{2\pi} F(\pi, \phi)\, d\phi;$$

so we may say that Q is the *mean* of the values of $F(\pi, \phi)$.

211. There is still one more remark to make respecting the value of Q. The process which we have given does not require that the function $F(\theta, \phi)$ should have the same *form* throughout the range of integration; the result will remain unaffected, unless the change of form occurs at the value $\theta = \theta'$ or at the value $\phi = \phi'$. Suppose, for instance, that for the values of θ less than θ' we have $F(\theta, \phi)$ equal to $\xi(\theta, \phi)$, and that for the values of θ greater than θ' we have $F(\theta, \phi)$ equal to $\chi(\theta, \phi)$; then it will easily be found on examination that

$$Q = \tfrac{1}{2}\{\xi(\theta', \phi') + \chi(\theta', \phi')\}.$$

A similar remark holds if a change of form in $F(\theta, \phi)$ occurs when $\phi = \phi'$.

212. It will be observed that the general term of the series in Art. 204 has the factor $2n+1$, and thus there may be room to suspect that the terms ultimately become very great. It may however be shewn that the terms do in general become indefinitely small when n is indefinitely great.

For consider $\int_{-1}^{1}\int_{0}^{2\pi} Y_n F(\theta', \phi')\, d\mu'\, d\phi'$;

by reason of the differential equation which Laplace's Coefficients satisfy, given in Art. 167, this definite integral is equal to the product of $-\dfrac{1}{n(n+1)}$ into

$$\int_{-1}^{1}\int_{0}^{2\pi}\left[\frac{d}{d\mu'}\left\{(1-\mu'^2)\frac{dY_n}{d\mu'}\right\}+\frac{1}{1-\mu'^2}\frac{d^2Y_n}{d\phi'^2}\right]F(\theta', \phi')\, d\mu'\, d\phi'.$$

By a double integration by parts, as in Art. 187, this may be transformed so as to become equal to the product of $-\dfrac{1}{n(n+1)}$ into

$$\int_{-1}^{1}\int_{0}^{2\pi}\left[\frac{d}{d\mu'}\left\{(1-\mu'^2)\frac{dF(\theta',\phi')}{d\mu'}\right\}+\frac{1}{1-\mu'^2}\frac{d^2F(\theta',\phi')}{d\phi'^2}\right]Y_n\, d\mu'\, d\phi',$$

assuming that $F(\theta', \phi')$ has the same value when $\phi' = 2\pi$ as when $\phi' = 0$; and *assuming* the same thing with respect to $\dfrac{dF(\theta', \phi')}{d\phi'}$.

Now the greatest value of Y_n is unity; hence, if $F(\theta',\phi')$ and its first and second differential coefficients with respect to θ' and ϕ' are always finite, and if moreover $\dfrac{d^2F(\theta', \phi')}{d\phi'^2}$ vanishes when $\mu' = -1$ or $=1$, then the definite integral in the last expression is finite, whatever n may be. If then we denote by k a value which it never surpasses, the term is numerically less than $\dfrac{k}{n(n+1)}$. Hence the general term in Art. 204 is numerically less than $\dfrac{(2n+1)k}{4\pi n(n+1)}$; and is therefore indefinitely small when n is indefinitely great.

213. It will be observed that the preceding investigation does not shew that the series obtained in Art. 204 is *convergent*, but only that the terms are ultimately indefinitely small.

In Art. 203 we assumed with Poisson as obvious a proposition which may be stated thus: the limit of $\Sigma(2n+1)a^n u_n$ is equal to $\Sigma(2n+1)u_n$ when the latter is a convergent series. For a formal demonstration we may refer to Abel's *Œuvres Complètes*, Vol. I. pages 69 and 70.

214. The proposition that a given function of θ and ϕ may be expressed in a series of Laplace's Functions is one of the utmost importance in the higher parts of mathematical physics. The demonstration of Poisson, though very instructive, cannot be considered perfectly conclusive, and we shall give two other investigations in the subsequent Chapters; we will here briefly notice a third, which was published by M. Ossian Bonnet in Liouville's *Journal de Mathématiques*. To this Professor Heine, on his page 266, refers without any remark, and M. Resal, on page 169 of his *Traité élémentaire de Mécanique Céleste*, pronounces it *à l'abri de toute objection*.

M. Bonnet alludes to Poisson's demonstration, and says it assumes that the given function and its differential coefficients with respect to θ and ϕ are continuous, whereas these conditions may not be fulfilled in very simple cases. M. Bonnet considers that the only entirely rigorous demonstration hitherto given is one by Lejeune Dirichlet; he proposes his own as more direct than this. M. Bonnet's process is very laborious, and it seems to me unsound, as resting on the unsatisfactory investigation of the value of Legendre's Function for a very high order, to which I have alluded in Art. 92.

CHAPTER XVII.

OTHER INVESTIGATIONS OF THE EXPANSION OF FUNCTIONS.

215. THE following investigation is due to M. Darboux, and is given in Bertrand's *Calcul Intégral*, pages 544...546.

It is required to find the sum of the first n terms of the series of which the r^{th} term is

$$\frac{2r+1}{4\pi}\int_0^\pi\int_0^{2\pi} Y_n F(\theta', \phi') \sin\theta' d\theta' d\phi' ;$$

and in fact to shew that when n increases indefinitely the limit of the sum is $F(\theta, \phi)$.

The variables θ' and ϕ' may be regarded as polar coordinates determining the position of a point on the surface of a sphere of radius unity. Change the coordinates, and take the point (θ, ϕ) as the new pole; let θ_1 and ϕ_1 be the new coordinates which determine the position of (θ', ϕ'): then

$$\cos\theta_1 = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

Also the element of surface $\sin\theta' \, d\theta' \, d\phi'$ may be replaced by $\sin\theta_1 \, d\theta_1 \, d\phi_1$. Hence the above r^{th} term becomes

$$\frac{2r+1}{4\pi}\int_0^\pi\int_0^{2\pi} P_n(\cos\theta_1) F(\theta_1, \phi_1) \sin\theta_1 \, d\theta_1 \, d\phi_1,$$

where $F(\theta_1, \phi_1)$ denotes what $F(\theta', \phi')$ becomes when the coordinates are changed, and $P_n(\cos\theta_1)$ is Legendre's n^{th} Coefficient, being equivalent to Laplace's n^{th} Coefficient Y_n.

Integrate with respect to ϕ_1, and put

$$\int_0^{2\pi} F(\theta_1, \phi_1)\, d\phi_1 = 2\pi f(\cos\theta_1);$$

so that $f(\cos\theta_1)$ may be considered as the *mean* value of $F(\theta_1, \phi_1)$ round a small circle distant θ_1 from the pole.

Thus $\int_0^{\pi}\int_0^{2\pi} P_n(\cos\theta_1)\, F(\theta_1, \phi_1)\sin\theta_1\, d\theta_1\, d\phi_1$

$$= 2\pi\int_0^{\pi} P_n(\cos\theta_1) f(\cos\theta_1)\sin\theta_1\, d\theta_1.$$

Put $\cos\theta_1$ for x; then the right-hand member becomes $2\pi\int_{-1}^{1} P_n(x) f(x)\, dx$: thus the proposed series reduces to

$$\frac{1}{2}\int_{-1}^{1} f(x)\,\{P_0(x) + 3P_1(x) + 5P_2(x) + \ldots + (2n+1)\,P_n(x)\}\, dx.$$

By means of equation (11) of Art. 59, this

$$= \frac{1}{2}\int_{-1}^{1} f(x)\left\{\frac{dP_n(x)}{dx} + \frac{dP_{n+1}(x)}{dx}\right\} dx.$$

Now by integration by parts, we have

$$\frac{1}{2}\int f(x)\left\{\frac{dP_n(x)}{dx} + \frac{dP_{n+1}(x)}{dx}\right\} dx = \frac{1}{2} f(x)\,\{P_n(x) + P_{n+1}(x)\}$$

$$- \frac{1}{2}\int f'(x)\,\{P_n(x) + P_{n+1}(x)\}\, dx.$$

At the limit -1 we have

$$P_n(x) + P_{n+1}(x) = (-1)^n + (-1)^{n+1} = 0;$$

at the limit 1 we have $P_n(x) + P_{n+1}(x) = 1 + 1 = 2$. Thus

$$\frac{1}{2}\int_{-1}^{1} f(x)\left\{\frac{dP_n(x)}{dx} + \frac{dP_{n+1}(x)}{dx}\right\} dx$$

$$= f(1) - \frac{1}{2}\int_{-1}^{1} f'(x)\,\{P_n(x) + P_{n+1}(x)\}\, dx.$$

When n is very large we know that $P_n(x)$ and $P_{n+1}(x)$ are insensible, except when x is indefinitely close to -1 or 1; thus the integral $\int_{-1}^{1} f'(x)\{P_n(x)+P_{n+1}(x)\}\,dx$ may be considered to vanish ultimately: at least this will be the case if $f'(x)$ is always finite.

And $f(1)$ is the value of $\frac{1}{2\pi}\int_0^{2\pi} F(\theta_1, \phi_1)\,d\phi_1$ when $\cos\theta_1=1$, that is when $\theta_1=0$; so that $f(1)$ is the mean value of $F(\theta_1, \phi_1)$ round an infinitesimal circle close to the pole, that is in fact the value of $F(\theta_1, \phi_1)$ at the pole, that is $F(\theta, \phi)$.

Thus the required result is established.

216. In the process of M. Darboux, suppose that we integrate between β and 1, where β is very near to unity; we get the same value as if we integrate between -1 and 1. For $\frac{d}{dx}\{P_n(x)+P_{n+1}(x)\}$ is very large when x is close to unity, but is insensible in other cases. Thus

$$\int_{-1}^{1} f(x)\frac{d}{dx}\{P_n(x)+P_{n+1}(x)\}\,dx=\int_{\beta}^{1} f(x)\frac{d}{dx}\{P_n(x)+P_{n+1}(x)\}\,dx$$

$$=f(\xi)\int_{\beta}^{1}\frac{d}{dx}\{P_n(x)+P_{n+1}(x)\}\,dx,$$

where ξ is between 1 and β,

$$=f(\xi)\{P_n(1)+P_{n+1}(1)\}=2f(\xi)=2f(1) \text{ ultimately.}$$

217. Although the process of M. Darboux is simple in appearance, it may be doubted whether it ought to be accepted as satisfactory. We cannot regard $P_n(x)+P_{n+1}(x)$ as finite when x is unity and as vanishing when x differs insensibly from unity, without treating $\frac{d}{dx}\{P_n(x)+P_{n+1}(x)\}$ as infinite when x is unity; and we cannot depend on the results of integration when the expression to be integrated becomes infinite within the range of integration. The process of M. Darboux has the advantage of leading very naturally to the special results of Arts. 207...211.

218. We ought not to overlook the fact that Poisson's treatment *may* be put in a form which involves the same kind of difficulty as we have pointed out in that of M. Darboux.

In Art. 203 we have a result which may be written thus:

$$X = U_0 + aU_1 + a^2 U_2 + a^3 U_3 + \dots,$$

where U_n stands for $(2n+1)\int_0^{2\pi}\int_{-1}^{1} F(\theta, \phi)\, Y_n\, d\phi\, d\mu$,

and X stands for $\int_0^{2\pi}\int_{-1}^{1} \frac{(1-a^2)\, F(\theta, \phi)\, d\phi\, d\mu}{(1-2ax+a^2)^{\frac{3}{2}}}$.

Then we find the limiting value when $a=1$, and thus obtain

$$F(\theta', \phi') = U_0 + U_1 + U_2 + U_3 + \dots$$

Now there is nothing that compels us to modify the form of the right-hand member of the last result, and express it thus:

$$\int_0^{2\pi}\int_{-1}^{1} \{Y_0 + 3Y_1 + 5Y_2 + \dots + (2n+1)Y_n + \dots\}\, F(\theta, \phi)\, d\phi\, d\mu.$$

If the quantity under the integral sign were always finite, this modification would present no difficulty; but the fact is that the expression

$$Y_0 + 3Y_1 + 5Y_2 + \dots + (2n+1)\, Y_n + \dots$$

is of a very peculiar kind; it is always zero except when $\theta = \theta'$ and $\phi = \phi'$, and then it is infinite. Hence the proposed modification cannot be effected without risk of error, and as there is no necessity for it in Poisson's method, we shall do well to avoid it.

219. The main parts of Poisson's process have been called *Poisson's Theorem*, and presented in the following form.

Let ∇ be used as an abbreviation for $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$; let

$r = \sqrt{(x^2+y^2+z^2)}$, and $r' = \sqrt{(x'^2+y'^2+z'^2)}$, and $a = \frac{r}{r'}$.

Let $x = r\cos\theta,\quad y = r\sin\theta\cos\phi,\quad z = r\sin\theta\sin\phi,$

$$x' = r'\cos\theta',\quad y' = r'\sin\theta'\cos\phi',\quad z' = r'\sin\theta'\sin\phi',$$

$$p = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

Let $V = \int_0^{2\pi}\int_0^{\pi}\frac{(1-\alpha^2)\,F(\theta', \phi')\,d\phi'\sin\theta'\,d\theta'}{(1-2\alpha p+\alpha^2)^{\frac{3}{2}}}$;

and suppose α less than unity.

Then V satisfies the equation $\nabla V = 0$, and reduces to $4\pi F(\theta, \phi)$ when $\alpha = 1$.

To establish the first part of this statement, put

$$\sigma = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = \frac{1}{r'\sqrt{(1-2\alpha p+\alpha^2)}}.$$

We know by Art. 167 that σ satisfies the equation $\nabla\sigma = 0$.

And $\sigma = \frac{1}{r'}\{1 + \alpha P_1 + \alpha^2 P_2 + \alpha^3 P_3 + \ldots\}$,

where P_m is put for shortness instead of $P_m(p)$.

Since then σ satisfies $\nabla\sigma = 0$, whatever α may be, it follows that $\alpha^m P_m$ will satisfy the same condition.

Now $\alpha\frac{d\sigma}{d\alpha} = \frac{1}{r'}\{\alpha P_1 + 2\alpha^2 P_2 + 3\alpha^3 P_3 + \ldots\}$;

hence $\alpha\frac{d\sigma}{d\alpha}$ satisfies the condition; therefore $\sigma + 2\alpha\frac{d\sigma}{d\alpha}$ also satisfies it, that is $\frac{1-\alpha^2}{r'(1-2\alpha p+\alpha^2)^{\frac{3}{2}}}$. Hence $\frac{1-\alpha^2}{(1-2\alpha p+\alpha^2)^{\frac{3}{2}}}$ will satisfy the condition; and therefore V will, that is

$$\nabla V = 0.$$

This establishes the first part of the statement; the second part is established in Art. 203.

See *Cours de Physique Mathématique* by E. Mathieu, pages 175...177.

220. Suppose in the general theorem of Art. 203 that the given function does not involve ϕ'; we may write the result thus,

$$F(\theta) = U_0 + U_1 + U_2 + \dots U_n + \dots,$$

where $$U_n = \frac{2n+1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} Y_n F(\theta') \, d\mu' \, d\phi'.$$

But by Art. 170 we have

$$\int_0^{2\pi} Y_n \, d\phi' = 2\pi P_n (\cos\theta) \, P_n (\cos\theta'),$$

so that $$U_n = \frac{2n+1}{2} P_n (\cos\theta) \int_{-1}^{1} P_n (\cos\theta') \, F(\theta') \, d\mu'.$$

Thus if we suppose $F(\theta) = f(\cos\theta)$, and change the notation by putting x for $\cos\theta$, and x' for $\cos\theta'$, we get

$$f(x) = \Sigma \frac{2n+1}{2} P_n (x) \int_{-1}^{1} P_n (x') f(x') \, dx'.$$

This is the theorem already imperfectly treated in Chapter XI.; it is here established, for the case in which x is less than unity; that is to say, the truth of it is made to rest on the same assumptions as the investigation of Art. 203.

221. The method of Dirichlet, as we saw in Art. 214, is commended by Bonnet; it is also emphatically praised by Heine: see page 266 of his work. Sidler too holds the same opinion: see page 56 of his work. Accordingly, swayed by the judgment of these eminent mathematicians, we shall reproduce it. But as similar principles have been employed to establish the truth of the well-known developments of functions in sines and cosines of multiple angles, we shall treat this simpler question in Chapter XVIII., and then proceed in Chapter XIX. to the investigation with which we are more immediately concerned.

CHAPTER XVIII.

EXPANSION OF A FUNCTION IN SINES AND COSINES OF MULTIPLE ANGLES.

222. We have already treated this subject in Chapter XIII. of the *Integral Calculus*, where we have reproduced investigations due to Lagrange and Poisson respectively.

Let $f(x)$ denote any function of x; then one of the theorems thus obtained may be stated in the following form:

$$f(x) = \frac{1}{2} u_0 + u_1 + u_2 + u_3 + \dots,$$

where
$$u_n = \frac{2}{\pi} \cos nx \int_0^\pi f(t) \cos nt\, dt.$$

The process we are about to give treats the problem in a reverse order; instead of *obtaining* this development we shall *verify* it by seeking the value of the sum of the infinite series $\frac{1}{2} u_0 + u_1 + u_2 + \dots$. The process is taken substantially from Schlömilch's *Compendium der Höheren Analysis*.

223. Let $\phi(t)$ be a function of t which is continuous between the limits a and b of t; we propose to find the limit when n is indefinitely increased of $\int_a^b \phi(t) \sin nt\, dt$.

We have

$$\int \phi(t) \sin nt\, dt = -\frac{\phi(t) \cos nt}{n} + \frac{1}{n} \int \phi'(t) \cos nt\, dt;$$

therefore
$$\int_a^b \phi(t) \sin nt\, dt = \frac{1}{n} \{\phi(a) \cos na - \phi(b) \cos nb\} + \frac{1}{n} \int_a^b \phi'(t) \cos nt\, dt.$$

Now let us assume that $\phi'(t)$ retains the same sign from $t=a$ to $t=b$, so that $\phi(t)$ continually increases or continually diminishes from $t=a$ to $t=b$; then by the *Integral Calculus*, Art. 40, we have

$$\int_a^b \phi'(t)\cos nt\,dt = \cos n\tau \int_a^b \phi'(t)\,dt = \cos n\tau\,\{\phi(b)-\phi(a)\},$$

where τ is some value of t lying between a and b. Thus

$$\int_a^b \phi(t)\sin nt\,dt = \frac{1}{n}\{\phi(a)\cos na - \phi(b)\cos nb\}$$
$$+\frac{\cos n\tau}{n}\{\phi(b)-\phi(a)\}.$$

Hence when n increases indefinitely we have

$$\int_a^b \phi(t)\sin nt\,dt = 0.$$

224. If $\phi(t)$ does not increase or decrease continually through the whole interval from a to b, we may subdivide this interval into smaller intervals, throughout each of which this condition holds. For example, suppose a, c, e, b in ascending order of magnitude, and suppose that $\phi(t)$ continually increases as t increases from a to c, then continually decreases as t increases from c to e, and then again continually increases as t increases from e to b. By Art. 223 the integral $\int\phi(t)\sin nt\,dt$ taken through each of these intervals vanishes, and therefore as before $\int_a^b \phi(t)\sin nt\,dt = 0$. This assumes, however, that the number of these subordinate intervals is *finite*; if it be infinite we have as a result an infinite number of infinitesimals, which is not necessarily zero. For example, we must not put $\phi(t) = \sin nt$.

225. We have supposed that $\phi(t)$ is a *continuous* function of t; this involves two conditions, namely, that $\phi(t)$ is always *finite*, and that $\phi(t)$ varies infinitesimally when t varies infinitesimally. The latter condition, however, is unnecessary; that is, $\phi(t)$ may change its form any *finite* number of times within the range. Suppose for instance that c is intermediate

between a and b, and that $\phi(t)$ passes from one finite value to a different finite value when t passes through the value c. Then divide the interval from a to b into two intervals, one from a to c, and the other from c to b. By Art. 223 the integral $\int \phi(t) \sin ntdt$ vanishes through each of these intervals, and therefore as before $\int_a^b \phi(t) \sin ntdt = 0$.

226. Now let $\phi(t) = \frac{f(x+t)-f(x)}{\sin t}$. Suppose that $a=0$, and that b is less than π. Assume that $f(x+t)$ is finite for all values of t from 0 to b. Then by Arts. 223...225 zero is the value when n is infinite of $\int_0^b \frac{f(x+t)-f(x)}{\sin t} \sin ntdt$.

227. It may appear that our process requires that $\phi(t)$ should be finite when $t=0$; and by evaluating $\phi(t)$, when $t=0$, we see that this is secured if $f'(x)$ is finite. But it is not necessary to impose this condition, because although the denominator of $\phi(t)$ vanishes when $t=0$, yet $\sin nt$ also vanishes; and thus we escape the presence of an infinite element in the definite integral.

228. It follows from Art. 226 that when n is infinite the limit of $\int_0^b \frac{\sin nt}{\sin t} f(x+t)\, dt =$ the limit of $f(x) \int_0^b \frac{\sin nt}{\sin t} dt$.

We proceed to find the limit of $\int_0^b \frac{\sin nt}{\sin t} dt$.

We have $\int_0^b \frac{\sin nt}{\sin t} dt = \int_0^{\frac{1}{2}\pi} \frac{\sin nt}{\sin t} dt + \int_{\frac{1}{2}\pi}^b \frac{\sin nt}{\sin t} dt$.

Now the second integral on the right-hand side vanishes by Art. 223, for $\frac{1}{\sin t}$ is always finite within the range of the integration. Thus we have only to find the value of the first integral on the right-hand side.

Hitherto we have spoken of n becoming *infinite*, but it is sufficient for our purpose to consider n as having a special kind of infinite value, namely, an infinite odd positive integral value. Suppose that $n=2m+1$. Then we have

$$\frac{\sin nt}{\sin t} = 1 + 2\{\cos 2t + \cos 4t + \ldots + \cos 2mt\};$$

therefore
$$\int_0^{\frac{1}{2}\pi} \frac{\sin nt}{\sin t}\, dt = \frac{1}{2}\pi.$$

Thus $\frac{1}{2}\pi$ is the limit required. Hence finally if b is between 0 and π the limit when n is an infinite odd positive integer of $\int_0^b \frac{\sin nt}{\sin t} f(x+t)\, dt$ is $\frac{1}{2}\pi f(x)$.

229. It will be found on examination that if c be any constant, positive or negative, we may put $f(x+ct)$ instead of $f(x+t)$; and thus we see that the limit when n is infinite of $\int_0^b \frac{\sin nt}{\sin t} f(x+ct)\, dt$ is $\frac{1}{2}\pi f(x)$.

230. The result of Art. 228 holds so long as b is less than π, but not when $b=\pi$; for then the function denoted by $\phi(t)$ in Art. 226 becomes infinite when $t=b$. We will consider this case.

$$\int_0^{\pi} \frac{\sin nt}{\sin t} f(x+t)\, dt$$
$$= \int_0^{\frac{1}{2}\pi} \frac{\sin nt}{\sin t} f(x+t)\, dt + \int_{\frac{1}{2}\pi}^{\pi} \frac{\sin nt}{\sin t} f(x+t)\, dt.$$

Put in the second integral on the right-hand side $t=\pi-t'$; then remembering that n is an odd integer, we have

$$\int_{\frac{1}{2}\pi}^{\pi} \frac{\sin nt}{\sin t} f(x+t)\, dt = \int_0^{\frac{1}{2}\pi} \frac{\sin nt'}{\sin t'} f(x+\pi-t')\, dt';$$

and in the definite integral we may change t' to t.

Thus $\int_0^{\pi} \frac{\sin nt}{\sin t} f(x+t)\, dt$

$$= \int_0^{\frac{1}{2}\pi} \frac{\sin nt}{\sin t} f(x+t)\, dt + \int_0^{\frac{1}{2}\pi} \frac{\sin nt}{\sin t} f(x+\pi-t)\, dt;$$

and by Art. 229 the limit of the right-hand member when n is infinite is $\frac{1}{2}\pi f(x) + \frac{1}{2}\pi f(x+\pi)$.

231. We now proceed to find the value of the following expression:

$$\int_0^{\pi} \left\{\frac{1}{2} + \cos(t+x) + \cos 2(t+x) + \cos 3(t+x) + \dots\right\} f(t)\, dt.$$

Suppose that the series within the brackets instead of being infinite extends only to the term $\cos m(t+x)$ inclusive: then the expression, by *Plane Trigonometry*, Art. 304,

$$= \int_0^{\pi} \frac{\sin \frac{2m+1}{2}(t+x)}{2 \sin \frac{1}{2}(t+x)} f(t)\, dt,$$

and we have to find the limit to which this tends when m increases indefinitely. Put $\frac{1}{2}(t+x) = t'$, and $2m+1 = n$; then the integral becomes $\int_{\frac{1}{2}x}^{\frac{1}{2}(\pi+x)} \frac{\sin nt'}{\sin t'} f(2t'-x)\, dt'$; and this

$$= \int_0^{\frac{1}{2}(\pi+x)} \frac{\sin nt'}{\sin t'} f(2t'-x)\, dt' - \int_0^{\frac{1}{2}x} \frac{\sin nt'}{\sin t'} f(2t'-x)\, dt'.$$

If $x=0$ the second integral on the right-hand side vanishes, and the first is equal to $\frac{\pi}{2} f(0)$ by Art. 229.

If x is between 0 and π the two integrals are equal by Art. 229; and thus the result is zero.

If $x = \pi$ the expression reduces to $\int_{\frac{1}{2}\pi}^{\pi} \frac{\sin nt'}{\sin t'} f(2t'-\pi)\, dt'$;

put $t' = \pi - t$, and this becomes $\int_0^{\frac{1}{2}\pi} \frac{\sin nt}{\sin t} f(\pi - 2t)\, dt$, which is equal to $\frac{\pi}{2} f(\pi)$.

232. Again consider in like manner the following expression:

$$\int_0^\pi \left\{\frac{1}{2} + \cos(t-x) + \cos 2(t-x) + \cos 3(t-x) + \ldots\right\} f(t)\, dt.$$

This reduces in the manner already shewn to

$$\int_{-\frac{1}{2}x}^{\frac{1}{2}(\pi - x)} \frac{\sin nt'}{\sin t'} f(2t' + x)\, dt',$$

where n is to be made infinite; and this

$$= \int_0^{\frac{1}{2}(\pi - x)} \frac{\sin nt'}{\sin t'} f(2t' + x)\, dt' + \int_{-\frac{1}{2}x}^{0} \frac{\sin nt'}{\sin t'} f(2t' + x)\, dt',$$

or changing the form of the second term it

$$= \int_0^{\frac{1}{2}(\pi - x)} \frac{\sin nt}{\sin t} f(2t + x)\, dt + \int_0^{\frac{1}{2}x} \frac{\sin nt}{\sin t} f(x - 2t)\, dt.$$

If $x = 0$ the second integral vanishes, and the first is equal to $\frac{\pi}{2} f(0)$ by Art. 229.

If x is between 0 and π each integral is equal to $\frac{\pi}{2} f(x)$ by Art. 229; and thus the result is $\pi f(x)$.

If $x = \pi$ the first integral vanishes, and the second is equal to $\frac{\pi}{2} f(\pi)$.

233. From the results obtained in Arts. 231 and 232, we deduce by addition and subtraction the two following, in which Σ denotes a summation with respect to positive integral values of i from one to infinity:

$\frac{1}{\pi}\int_0^\pi f(t)\,dt + \frac{2}{\pi}\Sigma\cos ix\int_0^\pi \cos it\, f(t)\,dt$ is equal to $f(x)$ for all values of x between 0 and π, both inclusive;

$\frac{2}{\pi}\Sigma\sin ix\int_0^\pi \sin it\, f(t)\,dt$ is equal to $f(x)$ for all values of x between 0 and π, both exclusive.

234. The formulæ just established coincide with what we obtain when we put $l=\pi$ in equations (3) and (4) of Art. 309 of the *Integral Calculus*. We may establish these equations (3) and (4) in the same way as we have just established the more simple cases; or we may deduce these equations (3) and (4) by putting $\frac{\tau\pi}{l}$ for t, and $\frac{\xi\pi}{l}$ for x, in the more simple cases.

235. We have in the preceding investigations expressly stated that the function denoted by $f(t+x)$ is not to become infinite within the range of integration; this condition may however be to some extent relaxed, as we shall now shew.

Put S for $\dfrac{\sin\frac{2m+1}{2}t}{\sin\frac{1}{2}t}$; then we have shewn in Art. 231 that when m is made infinite $\int_0^\pi Sf(t)\,dt = \pi f(0)$. We add now that this formula will hold even if $f(t)$ become infinite within the range of the integration, provided that $\int f(t)\,dt$ remains infinitesimal when taken between limits which are indefinitely close but include the value of t which makes $f(t)$ infinite.

Let τ be the value of t which makes $f(t)$ infinite, and let e and η be infinitesimals. Divide the interval from 0 to π into three, the first from 0 to $\tau-e$, the second from $\tau-e$ to $\tau+\eta$, and the third from $\tau+\eta$ to π. Then the value of $\int Sf(t)\,dt$ for the second interval vanishes by our supposition; we shall shew that the value for the first interval is $\pi f(0)$, and that the value for the third interval is zero.

Let $\chi(t)$ denote a function which coincides with $f(t)$ when t is between 0 and $\tau - e$, and is zero when t is between $\tau - e$ and π.

Then, by Art. 231, we have $\int_0^\pi S\chi(t)\,dt = \pi\chi(0)$, that is

$$\int_0^{\tau-e} Sf(t)\,dt = \pi f(0).$$

Again, let $\chi(t)$ now denote a function which is zero when t is between 0 and $\tau+\eta$, and coincides with $f(t)$ when t is between $\tau+\eta$ and π.

Then, by Art. 231, we have $\int_0^\pi S\chi(t)\,dt = \pi\chi(0) = 0$,

that is
$$\int_{\tau+\eta}^\pi Sf(t)\,dt = 0.$$

236. The result obtained in Art. 223 on which the subsequent investigations mainly depend may also be established in another manner.

Suppose that $\beta = \alpha + \dfrac{2\pi}{n}$, so that $\int_\alpha^\beta \sin nt\,dt = 0$.

Let c be the least value of $\phi(t)$ between the limits α and β, and assume $\phi(t) = c + u$. Then

$$\int_\alpha^\beta \phi(t)\sin nt\,dt = \int_\alpha^\beta (c+u)\sin nt\,dt = \int_\alpha^\beta u\sin nt\,dt.$$

Let p be the greatest value of u between the limits $t=\alpha$ and $t=\beta$, then $\int_\alpha^\beta u\sin nt\,dt$ cannot be so great as $\int_\alpha^\beta p\,dt$, that is as $p(\beta-\alpha)$.

In this way we can shew by dividing the interval $b-a$ into smaller portions, that when $b-a$ is a multiple of $\dfrac{2\pi}{n}$ the value of $\int_a^b \phi(t)\sin nt\,dt$ cannot be so great as $p(b-a)$, where p is the extreme difference that can exist between the greatest and the least values of $\phi(t)$ comprised between one subordinate pair of limits, as α and β.

But when n is made indefinitely great, the difference between α and β becomes indefinitely small; and hence $\phi(t)$ cannot experience an appreciable change in the interval between α and β; so that p ultimately vanishes.

The process though not extremely rigid throws some light on the theorem; it shews that what is essential in $\phi(t)$ is that there should be only an infinitesimal change corresponding to an infinitesimal change in t. Hence if n should occur in $\phi(t)$ the theorem may cease to be applicable; this happens in the case already noticed in Art. 224, in which $\phi(t) = \sin nt$.

As in Art. 225 we may extend our conclusion to the case in which the form of $\phi(t)$ changes any finite number of times within the range of integration.

CHAPTER XIX.

DIRICHLET'S INVESTIGATION.

237. Let $F(\theta', \phi')$ denote any function of θ' and ϕ' which remains finite throughout the range of integration; and let

$$U_n = \frac{2n+1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} F(\theta', \phi')\, Y_n\, d\phi'\, d\mu';$$

then it is required to find the value of the infinite series

$$U_0 + U_1 + U_2 + \ldots + U_n + \ldots$$

238. We begin with a particular case, from which we shall be able to deduce the general result required. We suppose that θ which occurs in Y_n is zero. Then Y_n becomes a function of θ' only, and we have with the notation of Art. 13,

$$Y_n = P_n(\cos\theta').$$

Then we may put

$$U_n = \frac{2n+1}{4\pi}\int_{-1}^{1}\left\{\int_0^{2\pi} F(\theta', \phi')\, d\phi'\right\} P_n(\cos\theta')\, d\mu'.$$

Here $\frac{1}{2\pi}\int_0^{2\pi} F(\theta', \phi')\, d\phi'$ will be a function of θ' only, and for shortness we will denote it by $f(\theta')$; so that $f(\theta')$ may be described as the *mean value* of $F(\theta', \phi')$ taken round a small circle at the distance θ' from the pole.

Thus $$U_n = \frac{2n+1}{2}\int_{-1}^{1} f(\theta')\, P_n(\cos\theta')\, d\mu'.$$

To avoid accents we shall use t instead of θ', so that

$$U_n = \frac{2n+1}{2}\int_0^{\pi} f(t)\, P_n(\cos t)\sin t\, dt.$$

239. We shall now seek the value of the sum of the first n terms of the infinite series; that is, the sum of

$$U_0 + U_1 + U_2 + \ldots + U_n,$$

and this we shall separate into two parts.

Let $$T_1 = \frac{1}{2}\int_0^\pi f(t)\{P_0 + P_1 + P_2 + \ldots + P_n\} \sin t dt,$$

and $$T_2 = \int_0^\pi f(t)\{P_1 + 2P_2 + 3P_3 + \ldots + nP_n\} \sin t dt;$$

where P_r is now put for shortness instead of $P_r(\cos t)$: then our proposed series is equal to $T_1 + T_2$.

240. Consider first T_1. By Art. 50 we have

$$P_r(\cos t) = \frac{2}{\pi}\int_0^t \frac{\cos\frac{1}{2}z \cos rz dz}{\sqrt{(2\cos z - 2\cos t)}} + \frac{2}{\pi}\int_t^\pi \frac{\sin\frac{1}{2}z \cos rz dz}{\sqrt{(2\cos t - 2\cos z)}},$$

but only half of the expression on the right-hand side is to be taken when $r = 0$.

Hence we find that $2\pi T_1 =$

$$\int_0^\pi \left[\int_0^t \frac{S\cos\frac{1}{2}z dz}{\sqrt{(2\cos z - 2\cos t)}} + \int_t^\pi \frac{S\sin\frac{1}{2}z dz}{\sqrt{(2\cos t - 2\cos z)}}\right] f(t) \sin t\, dt,$$

where S stands for $1 + 2\cos z + 2\cos 2z + \ldots + 2\cos nz$.

By *Plane Trigonometry*, Art. 304, we know that

$$S = \frac{\sin\frac{2n+1}{2}z}{\sin\frac{1}{2}z};$$

and so this value may be substituted for S.

241. We shall now change the order of the two integrations involved in the expression for $2\pi T_1$.

Let a be any constant, and u any function of x and y; then from simple geometrical considerations, or from the theory explained in the *Integral Calculus*, Chapter XI., we have

$$\int_0^a \left[\int_0^x u\,dy\right] dx = \int_0^a \left[\int_y^a u\,dx\right] dy.$$

By applying this formula to the present case we obtain

$$\int_0^\pi \left[\int_0^t \frac{S\cos\frac{1}{2}z\,dz}{\sqrt{(2\cos z - 2\cos t)}}\right] f(t)\sin t\,dt$$

$$= \int_0^\pi \left[\int_z^\pi \frac{S\cos\frac{1}{2}z f(t)\sin t\,dt}{\sqrt{(2\cos z - 2\cos t)}}\right] dz$$

$$= \int_0^\pi \left[\int_z^\pi \frac{f(t)\sin t\,dt}{\sqrt{(2\cos z - 2\cos t)}}\right] S\cos\frac{1}{2}z\,dz;$$

$$\int_0^\pi \left[\int_t^\pi \frac{S\sin\frac{1}{2}z\,dz}{\sqrt{(2\cos t - 2\cos z)}}\right] f(t)\sin t\,dt$$

$$= \int_0^\pi \left[\int_0^z \frac{S\sin\frac{1}{2}z f(t)\sin t\,dt}{\sqrt{(2\cos t - 2\cos z)}}\right] dz$$

$$= \int_0^\pi \left[\int_0^z \frac{f(t)\sin t\,dt}{\sqrt{(2\cos t - 2\cos z)}}\right] S\sin\frac{1}{2}z\,dz.$$

Thus
$$2\pi T_1 = \int_0^\pi \left[\cos\frac{1}{2}z\int_z^\pi \frac{f(t)\sin t\,dt}{\sqrt{(2\cos z - 2\cos t)}} + \sin\frac{1}{2}z\int_0^z \frac{f(t)\sin t\,dt}{\sqrt{(2\cos t - 2\cos z)}}\right] S\,dz.$$

242. The expression here enclosed within brackets is a function of z only, and we will denote it by $\chi(z)$ for shortness, so that $T_1 = \frac{1}{2\pi}\int_0^\pi \chi(z)\,S\,dz.$

Now we have shewn in Art. 231 that the limit of $\frac{1}{2\pi}\int_0^\pi \chi(z)\, S dz$ when n is indefinitely increased is $\frac{1}{2}\chi(0)$; and without using the preceding Chapter the same result will follow from any method of expanding a function in a series of cosines of multiple angles; for such a method gives

$$\chi(z) = \frac{1}{2} b_0 + b_1 \cos z + b_2 \cos 2z + b_3 \cos 3z + \ldots,$$

where $$b_m = \frac{2}{\pi}\int_0^\pi \chi(t) \cos mt\, dt,$$

and so when $z = 0$ we have

$$\chi(0) = \frac{1}{2} b_0 + b_1 + b_2 + b_3 + \ldots$$

Thus ultimately $T_1 = \frac{1}{2}\chi(0)$, that is

$$T_1 = \frac{1}{2}\int_0^\pi f(t) \cos \frac{1}{2} t\, dt.$$

243. The result just obtained depends on the assumption that $\chi(z)$ is finite throughout the range of the integration. It is easily shewn that this condition is satisfied by examining separately the two terms in $\chi(z)$.

For we assume that $f(t)$ is finite through the range of the integration with respect to t; therefore by the *Integral Calculus*, Art. 40,

$$\int_z^\pi \frac{f(t) \sin t\, dt}{\sqrt{(2\cos z - 2\cos t)}} = f(\tau)\int_z^\pi \frac{\sin t\, dt}{\sqrt{(2\cos z - 2\cos t)}},$$

where τ is some value of t between z and π.

And $\int_z^\pi \frac{\sin t\, dt}{\sqrt{(2\cos z - 2\cos t)}} = \sqrt{(2\cos z - 2\cos \pi)}$, which is finite.

In the same manner we may shew that the other term in $\chi(z)$ is finite.

244. We now consider the series which we denoted by T_2 in Art. 239. We have by Art. 50

$$P_r(\cos t) = -\frac{2}{\pi}\int_0^t \frac{\sin\frac{1}{2}z\sin nz\,dz}{\sqrt{(2\cos z - 2\cos t)}} + \frac{2}{\pi}\int_t^\pi \frac{\cos\frac{1}{2}z\sin nz\,dz}{\sqrt{(2\cos t - 2\cos z)}}.$$

Hence we find that $\pi T_2 =$

$$\int_0^\pi \left[\int_0^t \frac{-S'\sin\frac{1}{2}z\,dz}{\sqrt{(2\cos z - 2\cos t)}} + \int_t^\pi \frac{S'\cos\frac{1}{2}z\,dz}{\sqrt{(2\cos t - 2\cos z)}}\right] f(t)\sin t\,dt,$$

where S' stands for $2(\sin z + 2\sin 2z + 3\sin 3z + \ldots + n\sin nz)$; we see that $S' = -\frac{dS}{dz}$.

245. Next we change the order of the two integrations involved in the expression for πT_2. Proceeding as in Arts. 241 and 242 we arrive at the result

$$T_2 = \frac{1}{\pi}\int_0^\pi \xi(z)\,S'dz,$$

where $\xi(z)$ stands for

$$-\sin\frac{1}{2}z\int_z^\pi \frac{f(t)\sin t\,dt}{\sqrt{(2\cos z - 2\cos t)}} + \cos\frac{1}{2}z\int_0^z \frac{f(t)\sin t\,dt}{\sqrt{(2\cos t - 2\cos z)}}.$$

246. The function $\xi(z)$ is finite throughout the range of integration, as we see by the method of Art. 243. It will be necessary however for our purpose to shew something more, namely that the function is *continuous*, so that it experiences only an infinitesimal change when z does. To shew this we examine separately the two terms of which $\xi(z)$ consists; take for example the second term, and it will be seen that the first term may be treated in a similar way.

We have then in fact to shew that

$$\int_0^{z+\zeta} \frac{f(t)\sin t\,dt}{\sqrt{2\cos t - 2\cos(z+\zeta)}} - \int_0^z \frac{f(t)\sin t\,dt}{\sqrt{2\cos t - 2\cos z}} \text{ vanishes with } \zeta.$$

This expression is equal to $\int_z^{z+\zeta} \frac{f(t)\sin t\,dt}{\sqrt{2\cos t - 2\cos(z+\zeta)}}$

$$-\int_0^z f(t)\left\{\frac{\sin t}{\sqrt{2\cos t - 2\cos z}} - \frac{\sin t}{\sqrt{2\cos t - 2\cos(z+\zeta)}}\right\}dt;$$

and we will take these two integrals separately.

Let g denote the numerically greatest value of $f(t)$ between the values z and $z+\zeta$ of the variable; then the former integral is numerically less than

$$g\int_z^{z+\zeta} \frac{\sin t\,dt}{\sqrt{2\cos t - 2\cos(z+\zeta)}}.$$

But $\int \frac{\sin t\,dt}{\sqrt{2\cos t - 2\cos(z+\zeta)}} = -\sqrt{2\cos t - 2\cos(z+\zeta)}$;

thus the former integral is less than $g\sqrt{2\cos z - 2\cos(z+\zeta)}$, and therefore vanishes with ζ.

Next we treat the latter integral. Let g now denote the numerically greatest value of $f(t)$ between the values 0 and z of the variable; then the integral is numerically less than

$$g\int_0^z\left\{\frac{\sin t}{\sqrt{2\cos t - 2\cos z}} - \frac{\sin t}{\sqrt{2\cos t - 2\cos(z+\zeta)}}\right\}dt,$$

that is less than

$$g\{\sqrt{2-2\cos z} - \sqrt{2-2\cos(z+\zeta)} + \sqrt{2\cos z - 2\cos(z+\zeta)}\},$$

which vanishes with ζ.

247. We shall require immediately the values of $\xi(0)$, $\xi(\pi)$, and $\xi'(0)$; they may be conveniently determined now.

It is obvious that $\xi(0)$ and $\xi(\pi)$ are both zero.

We proceed then to investigate the value of $\xi'(0)$.

For shortness, put $\xi(z) = -r\sin\frac{1}{2}z + s\cos\frac{1}{2}z$,

so that $r = \int_z^\pi \frac{f(t)\sin t\,dt}{\sqrt{(2\cos z - 2\cos t)}}$ and $s = \int_0^z \frac{f(t)\sin t\,dt}{\sqrt{(2\cos t - 2\cos z)}}$.

Thus $\xi'(z) = -\frac{r}{2}\cos\frac{1}{2}z - \frac{s}{2}\sin\frac{1}{2}z - \sin\frac{1}{2}z\frac{dr}{dz} + \cos\frac{1}{2}z\frac{ds}{dz}$;

and therefore $$\xi'(0) = -\frac{r}{2} + \frac{ds}{dz};$$

where on the right-hand side we are to put 0 for z. This assumes that $\frac{dr}{dz}$ is not infinite when $z=0$, an assumption which will be justified immediately.

Now the value of $\frac{ds}{dz}$ when z is zero is the limit when z is zero of the expression $\frac{1}{z}\int_0^z \frac{f(t)\sin t\,dt}{\sqrt{(2\cos t - 2\cos z)}}$. We know that this expression $= \frac{f(\tau)}{z}\int_0^z \frac{\sin t\,dt}{\sqrt{(2\cos t - 2\cos z)}}$, where τ is some value of t between 0 and z. And

$$\int_0^z \frac{\sin t\,dt}{\sqrt{(2\cos t - 2\cos z)}} = 2\sin\frac{1}{2}z,$$

so that the expression $= \frac{2f(\tau)\sin\frac{1}{2}z}{z}$, and the limit of it when $z=0$ is $f(0)$.

In a similar manner we can shew that $\frac{dr}{dz}$ is finite when $z=0$; and we shall not require its precise value.

Thus finally $\xi'(0) = f(0) - \frac{1}{2}\int_0^\pi f(t)\cos\frac{1}{2}t\,dt$.

248. Now return to the value of T_2. We have

$$T_2 = \frac{1}{\pi}\int_0^\pi \xi(z)\,S'\,dz = -\frac{1}{\pi}\int_0^\pi \xi(z)\frac{dS}{dz}dz.$$

Integrate by parts; thus $T_2 = \frac{1}{\pi}\int_0^\pi \xi'(z)\,S\,dz$.

Therefore when n is indefinitely increased, the limit of T_2 is $\xi'(0)$, the value of which was found in Art. 247.

249. Hence $T_1 + T_2 = f(0)$.

Thus the limit of $T_1 + T_2$ is $\frac{1}{2\pi}\int_0^{2\pi} F(0, \phi')\, d\phi'$.

This will coincide with $F(0, \phi')$ when $F(0, \phi')$ is independent of ϕ'. In other cases it will be what we may call the *mean value* of $F(0, \phi')$.

250. Thus we have established the required result in the particular case contemplated in Art. 238, namely that in which θ is zero.

We may state in words what has been shewn.

Suppose a spherical surface, let $F(\theta', \phi')$ denote the *density* at any point, or rather at any element of surface, say at S. Then the integral in U_n involves the product of the element of the surface, into the density, into a certain function Y_n of the arc which joins the element of surface to a fixed point in the sphere. In the case in which $\theta = 0$ let us call that fixed point A; then we see that ΣU_n is equal to the mean density round the fixed point.

Now if θ be not $= 0$, let us call the fixed point C. Then Y_n becomes the same function of the arc CS as it was in the former case of the arc AS. Hence the value of ΣU_n will now be the mean density at C; that is it will be $F(\theta, \phi)$. Thus the problem proposed in Art. 237 is solved.

CHAPTER XX.

MISCELLANEOUS THEOREMS.

251. To shew that $\dfrac{2^m \lfloor m \lfloor n-m}{\lfloor n+m} \dfrac{d^m P_n(x)}{dx^m} \dfrac{d^m P_n(x_1)}{dx_1^m}$

$$= D^m P_n(\xi) + \frac{(1-x^2)(1-x_1^2)}{2.(2m+2)} D^{m+2} P_n(\xi)$$
$$+ \frac{(1-x^2)^2(1-x_1^2)^2}{2.4.(2m+2)(2m+4)} D^{m+4} P_n(\xi) + \ldots,$$

where ξ stands for xx_1, and D for $\dfrac{d}{d\xi}$.

To prove this we observe that Laplace's n^{th} Coefficient is $P_n(z)$, where $z = xx_1 + \sqrt{1-x^2}\sqrt{1-x_1^2}\cos\psi$. Put t for $\sqrt{1-x^2}\sqrt{1-x_1^2}\cos\psi$, then $P_n(z)$ becomes a function of $\xi + t$, say $F(\xi + t)$; and this by Taylor's Theorem is equal to

$$F(\xi) + \frac{dF(\xi)}{d\xi} t + \frac{1}{\lfloor 2} \frac{d^2 F(\xi)}{d\xi^2} t^2 + \ldots$$

Pick out the coefficient of $\cos m\psi$ from this, and equate it to the $(-1)^m b_m (1-x^2)^{\frac{m}{2}} (1-x_1^2)^{\frac{m}{2}} \dfrac{d^m P_n(x)}{dx^m} \dfrac{d^m P_n(x_1)}{dx_1^m}$ of Art. 175, that is by the same Article to

$$\frac{2 \lfloor n-m}{\lfloor n+m} (1-x^2)^{\frac{m}{2}} (1-x_1^2)^{\frac{m}{2}} \frac{d^m P_n(x)}{dx^m} \frac{d^m P_n(x_1)}{dx_1^m}.$$

Now the first term in the series above given for $F(\xi + t)$ which involves $\cos m\psi$ is $\dfrac{t^m}{\lfloor m} D^m F(\xi)$, and this will give for the coefficient of $\cos m\psi$ the expression

$$\frac{1}{\underline{|m}}(1-x^2)^{\frac{m}{2}}(1-x_1^2)^{\frac{m}{2}}\frac{1}{2^{m-1}}D^m F(\xi).$$

The next term which involves $\cos m\psi$ is $\frac{t^{m+2}}{\underline{|m+2}} D^{m+2}F(\xi)$, and this will give for coefficient the expression

$$\frac{1}{\underline{|m+2}}(1-x^2)^{\frac{m+2}{2}}(1-x_1^2)^{\frac{m+2}{2}}\frac{m+2}{2^{m+1}}D^{m+2}F(\xi).$$

Next we get

$$\frac{1}{\underline{|m+4}}(1-x^2)^{\frac{m+4}{2}}(1-x_1^2)^{\frac{m+4}{2}}\frac{(m+4)(m+3)}{2^{m+3}\underline{|2}}D^{m+4}F(\xi).$$

And so on. Thus we obtain the required result.

252. In the formula of the preceding Article put $x_1=0$; then we get an expression for $\frac{d^m P_n(x)}{dx^m}$ arranged in powers of $1-x^2$. There will be two cases.

I. Suppose $n-m$ even. Then $\frac{d^m P_n(x_1)}{dx_1^m}$ contains a term which does not vanish when $x_1=0$; and a similar remark holds with respect to $D^m P_n(\xi)$, $D^{m+2}P_n(\xi)$....

Thus we get

$$\frac{2^m \underline{|m}\,\underline{|n-m}}{\underline{|n+m}}\frac{d^m P_n(x)}{dx^m}=1+\frac{p(q+1)}{1\,.\,2^2\,.\,(m+1)}(x^2-1)$$
$$+\frac{p(p-2)(q+1)(q+3)}{\underline{|2}\,.\,2^4\,.\,(m+1)(m+2)}(x^2-1)^2$$
$$+\frac{p(p-2)(p-4)(q+1)(q+3)(q+5)}{\underline{|3}\,.\,2^6\,.\,(m+1)(m+2)(m+3)}(x^2-1)^3$$
$$+\ldots$$

where p stands for $n-m$ and q for $n+m$.

II. Suppose $n-m$ odd. After the operations denoted in the preceding Article have been performed divide by x_1, and then put $x_1=0$. Thus we get

$$\frac{2^m \lfloor m \lfloor n-m}{\lfloor n+m} \frac{d^m P_n(x)}{dx^m} = xS, \text{ where}$$

$$S = 1 + \frac{(p-1)(q+2)}{1 \cdot 2^2 \cdot (m+1)} (x^2-1)$$
$$+ \frac{(p-1)(p-3)(q+2)(q+4)}{\lfloor 2 \cdot 2^4 \cdot (m+1)(m+2)} (x^2-1)^2$$
$$+ \frac{(p-1)(p-3)(p-5)(q+2)(q+4)(q+6)}{\lfloor 3 \cdot 2^6 \cdot (m+1)(m+2)(m+3)} (x^2-1)^3$$
$$+ \ldots$$

253. The theorem for the expansion of a function in terms of Legendre's Coefficients may be enunciated thus

$$\phi(x) = \Sigma \frac{2n+1}{2} P_n(x) \int_{-1}^{1} P_n(x') \phi(x') \, dx',$$

where Σ denotes summation with respect to n from 0 to ∞.

In Art. 220 we have deduced this as a particular case of the expansion of a function of two variables in terms of Laplace's Functions. We will now give another investigation.

Let ξ stand for xx'. We know by Art. 251 that

$$P_n(x) P_n(x') = P_n(\xi) + \frac{(1-x^2)(1-x'^2)}{2^2} \frac{d^2 P_n(\xi)}{d\xi^2}$$
$$+ \frac{(1-x^2)^2 (1-x'^2)^2}{2^2 \cdot 4^2} \frac{d^4 P_n(\xi)}{d\xi^4} + \ldots$$

Now we know by Art. 200 that

$$\Sigma (2n+1) P_n(\xi) = \text{the limit when } \alpha = 1 \text{ of } \frac{1-\alpha^2}{(1-2\alpha\xi+\alpha^2)^{\frac{3}{2}}};$$

hence $\Sigma (2n+1) \dfrac{(1-x^2)(1-x'^2)}{2^2} \dfrac{d^2 P_n(\xi)}{d\xi^2}$ = the limit when $\alpha = 1$ of $\dfrac{(1-x^2)(1-x'^2)}{2^2} \dfrac{d^2}{d\xi^2} \dfrac{1-\alpha^2}{(1-2\alpha\xi+\alpha^2)^{\frac{3}{2}}}$ = the limit when $\alpha = 1$ of $\dfrac{(1-x^2)(1-x'^2)}{2^2} \dfrac{3 \cdot 5\alpha^2 (1-\alpha^2)}{(1-2\alpha\xi+\alpha^2)^{\frac{7}{2}}}$.

In like manner $\Sigma\,(2n+1)\,\frac{(1-x^2)^2\,(1-x'^2)^2}{2^2\,.\,4^2}\,\frac{d^4P_n(\xi)}{d\xi^4}=$ the limit when $\alpha=1$ of $\frac{(1-x^2)^2\,(1-x'^2)^2}{2^2\,.\,4^2}\,\frac{3\,.\,5\,.\,7\,.\,9\alpha^4\,(1-\alpha^2)}{(1-2\alpha\xi+\alpha^2)^{\frac{11}{2}}}$.

In this way we can transform $\Sigma\,(2n+1)\,P_n(x)\,P_n(x')$, and putting $\alpha=1$ in the limit we see that the expression will vanish provided the following series is *convergent:*

$$1+\frac{3\,.\,5}{2^2}\,\tau+\frac{3\,.\,5\,.\,7\,.\,9}{2^2\,.\,4^2}\,\tau^2+\frac{3\,.\,5\,.\,7\,.\,9\,.\,11\,.\,13}{2^2\,.\,4^2\,.\,6^2}\,\tau^3+\ldots$$

where τ stands for $\frac{(1-x^2)\,(1-x'^2)}{(1-2\xi+1)^2}$, that is for $\frac{(1-x^2)\,(1-x'^2)}{4\,(1-\xi)^2}$.

The application of the usual rule shews that the series is convergent so long as $\frac{(1-x^2)\,(1-x'^2)}{(1-\xi)^2}$ is numerically less than unity. This will be the case provided x and x' are unequal and both less than unity.

Hence we see that $\Sigma\,\frac{2n+1}{2}\,P_n(x)\,P_n(x')$ is indefinitely small for every value within the range of integration, except when $x'=x$; what the value is then we shall not require to know.

Therefore $\Sigma\,\frac{2n+1}{2}\int_{-1}^{1}P_n(x)\,P_n(x')\,\phi(x')\,dx'$

$$=\Sigma\,\frac{2n+1}{2}\int_{\beta}^{\gamma}P_n(x)\,P_n(x')\,\phi(x')\,dx';$$

where the limits β and γ may be indefinitely close provided the value x is comprised between them.

Next we transform the last expression into

$$\phi(x)\,\Sigma\,\frac{2n+1}{2}\int_{\beta}^{\gamma}P_n(x)\,P_n(x')\,dx';$$

and then again since $\Sigma\,\frac{2n+1}{2}\,P_n(x)\,P_n(x')$ vanishes except when $x' = x$, we may transform this to

$$\phi(x) \Sigma \frac{2n+1}{2} \int_{-1}^{1} P_n(x) P_n(x') \, dx',$$

that is to $$\phi(x) \Sigma \frac{2n+1}{2} P_n(x) \int_{-1}^{1} P_n(x') \, dx'.$$

But $\int_{-1}^{1} P_n(x') \, dx' = 0$ except when $n = 0$, and then it $= 2$. Thus finally we obtain $\phi(x)$.

The preceding investigation seems to throw some light on the nature of the result. It has the advantage of being quite independent of the theorem that a function of two variables can be expressed in a series of Laplace's Functions.

254. Let $U = \dfrac{1}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}}$;

put $r^2 = x^2 + y^2 + z^2$, $r'^2 = x'^2 + y'^2 + z'^2$, $\cos\theta = \dfrac{xx' + yy' + zz'}{rr'}$.

Then $U = \dfrac{1}{r\left\{1 - \dfrac{2r'}{r}\cos\theta + \dfrac{r'^2}{r^2}\right\}^{\frac{1}{2}}}$; so that if U be expanded in powers of $\dfrac{r'}{r}$ we have $P_n(\cos\theta)$ for the coefficient of $\left(\dfrac{r'}{r}\right)^n$, and therefore $P_n(\cos\theta)$ may be considered to be a function of $\dfrac{xx' + yy' + zz'}{rr'}$.

Now we see that this function has the following properties:

It is symmetrical with respect to the two sets of variables; that is if x and x' be interchanged it is not altered, and similarly for y and y', and for z and z'. Since $\cos\theta$ is raised to the power n in $P_n(\cos\theta)$ it follows that the function when expressed in terms of x, y, z and x', y', z' will have $(rr')^n$ in the denominator. Hence if we make this the common denominator, the numerator will involve each of the variables to the n^{th} power, and it will be homogeneous with respect to each set of variables. Thus if one term of the numerator be $Ax'^{\alpha}y'^{\beta}z'^{\gamma}$, where A does not involve x' or y' or z', we shall have $\alpha+\beta+\gamma=n$.

We might take the original form of U and develop it in powers of x', y', z' by the usual theorem for developing a function of three independent variables. Thus we shall get for the type of the terms in the development

$$\frac{(-x')^{\alpha}(-y')^{\beta}(-z')^{\gamma}}{\lfloor\alpha\,\lfloor\beta\,\lfloor\gamma}\frac{d^{\alpha+\beta+\gamma}V}{dx^{\alpha}dy^{\beta}dz^{\gamma}},$$

where V stands for $\frac{1}{\sqrt{(x^2+y^2+z^2)}}$. All the terms of the degree n will be found by taking α, β, γ of various positive integral values subject to the condition $\alpha+\beta+\gamma=n$.

Suppose $\alpha+\beta+\gamma=n$; then the type of the terms just expressed takes the form $\frac{(-x')^{\alpha}(-y')^{\beta}(-z')^{\gamma}N}{r^{2n+1}}$, where N is a homogeneous function of x, y, z of the degree n.

Thus we infer that

$$\frac{r'^n}{r^{n+1}}P_n(\cos\theta)=\Sigma\frac{(-x')^{\alpha}(-y')^{\beta}(-z')^{\gamma}}{\lfloor\alpha\,\lfloor\beta\,\lfloor\gamma}\frac{d^{\alpha+\beta+\gamma}V}{dx^{\alpha}dy^{\beta}dz^{\gamma}},$$

when for $\cos\theta$ we put $\frac{xx'+yy'+zz'}{rr'}$; the Σ denotes a summation for all values of α, β, γ consistent with the condition $\alpha+\beta+\gamma=n$.

We may confirm this by supposing that r' is very small compared with r; and then our result is in fact obtained by equating terms of the same order of small quantities. The result is of such a nature that it is then true for all relative values of r and r'.

255. Suppose we have to develop in terms of Laplace's Functions a function of which we do not know the analytical form, but only various numerical values. For instance, we might require an expression in terms of Laplace's Functions

for the mean temperature at any point of the surface of the globe; we may imagine this expression to be some function of the latitude and longitude of the point, and may seek to determine the developed form of the function from the numerical values given by observation at various places. We shall devote the remainder of the Chapter to this subject.

256. Let $F(\theta, \phi)$ denote the function, and suppose that

$$F(\theta, \phi) = Z_0 + Z_1 + Z_2 + \dots + Z_n,$$

where Z_k denotes a Laplace's Function of the order k.

We suppose that the development of $F(\theta, \phi)$ converges with sufficient rapidity to enable us to stop with the term Z_n. In Z_k there are $2k+1$ constants; and thus in the development of $F(\theta, \phi)$ there are altogether $(n+1)^2$ constants; we must shew how these can be determined.

By Art. 192 we have

$$Z_k = \Sigma \sin^m \theta \, D^m P_k \, (A_{k,m} \cos m\phi + B_{k,m} \sin m\phi),$$

where Σ denotes summation with respect to m from 0 to k inclusive, D stands for $\frac{d}{dx}$, and P_k for $P_k(x)$; also $x = \cos\theta$. Moreover $A_{k,m}$ and $B_{k,m}$ are constants. Then $F(\theta, \phi)$ is to be obtained by summing the values of Z_k from $k=0$ to $k=n$ inclusive.

We may also put $F(\theta, \phi)$ in the form

$$F(\theta, \phi) = \Sigma_m (C_m \cos m\phi + S_m \sin m\phi) \dots\dots\dots (1),$$

where Σ_m denotes summation with respect to m from 0 to n, both inclusive; also

$$\left.\begin{aligned} C_m &= \sin^m \theta \, \Sigma_k \, A_{k,m} D^m P_k \\ S_m &= \sin^m \theta \, \Sigma_k \, B_{k,m} D^m P_k \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (2),$$

where Σ_k denotes summation with respect to k from m to n, both inclusive.

257. We first determine from (1) the values of the quantities of which C_m and S_m are the types.

Let $\alpha = \frac{2\pi}{2n+1}$; suppose that in $F(\theta, \phi)$ we put for ϕ in succession the values $0, \alpha, 2\alpha, \ldots\ldots, 2n\alpha$; and that the corresponding values of $F(\theta, \phi)$ are known. Then we have for all values of k from 0 to $2n$, both inclusive,

$$F(\theta, k\alpha) = C_0 + C_1 \cos k\alpha + C_2 \cos 2k\alpha + \ldots\ldots + C_n \cos nk\alpha$$
$$+ S_1 \sin k\alpha + S_2 \sin 2k\alpha + \ldots\ldots + S_n \sin nk\alpha.$$

Multiply this equation first by $\cos ks\alpha$, and next by $\sin ks\alpha$; and sum for all values of k from 0 to $2n$, both inclusive. Then apply the following Trigonometrical formulæ, which are easily established, and which we have used in the *Integral Calculus*, Chapter XIII:

$\Sigma \cos ks\alpha \cos ks'\alpha = 2n+1$ when s and s' are both zero,

$= \frac{1}{2}(2n+1)$ when s and s' are equal but not zero,

$= 0$ when s and s' are unequal.

$\Sigma \cos ks\alpha \sin ks'\alpha = 0.$

$\Sigma \sin ks\alpha \sin ks'\alpha = 0$ when s and s' are both zero,

$= \frac{1}{2}(2n+1)$ when s and s' are equal but not zero,

$= 0$ when s and s' are unequal.

Hence we obtain

$$\left.\begin{aligned} C_s &= \frac{2}{2n+1} \Sigma F(\theta, k\alpha) \cos ks\alpha \\ S_s &= \frac{2}{2n+1} \Sigma F(\theta, k\alpha) \sin ks\alpha \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(3),$$

where Σ denotes summation with respect to k from 0 to $2n$, both inclusive; but for C_0 we must take only half the value which the formula would give.

258. Now from (2) we have

$$\left.\begin{aligned}
C_n &= \sin^n\theta\, A_{n,n} D^n P_n \\
C_{n-1} &= \sin^{n-1}\theta \{A_{n-1,n-1} D^{n-1} P_{n-1} + A_{n,n-1} D^{n-1} P_n\} \\
C_{n-2} &= \sin^{n-2}\theta \{A_{n-2,n-2} D^{n-2} P_{n-2} + A_{n-1,n-2} D^{n-2} P_{n-1} + A_{n,n-2} D^{n-2} P_n\} \\
&\ldots\ldots\ldots \\
C_0 &= A_{0,0} P_0 + A_{1,0} P_1 + \ldots\ldots + A_{n-1,0} P_{n-1} + A_{n,0} P_n
\end{aligned}\right\} \ldots\ldots\ldots\ldots(4).$$

A similar set of equations holds in which S with suffixes occurs instead of C with suffixes, and B with suffixes instead of A with suffixes.

Now it will be seen that the first of equations (4) involves only one constant to be determined, namely $A_{n,n}$; thus it will be sufficient to know one value of the quantity denoted by C_n, that is the value of C_n for *one* value of the polar distance θ. The second of equations (4) involves two constants, namely $A_{n-1,n-1}$ and $A_{n,n-1}$; thus in order to determine them we must know the value of C_{n-1} for *two* values of the polar distance θ. In like manner C_{n-2} must be known for *three* values of the polar distance θ; and so on.

259. But suppose that the values of the quantities denoted by C with suffixes are known for *more* values of the polar distance θ than we have seen to be necessary; for example, suppose that C_{n-2} is known for *four* values of the polar distance θ: then we have more equations than are necessary to determine the constants denoted by A with suffixes. Two ways have been proposed for treating such a case.

We may use the method of least squares, or any other method which the theory of probability supplies, as advantageous for obtaining the best results from a system of linear equations which exceeds in number the number of unknown quantities to be found. This method is that suggested by Gauss in order to express the elements of the earth's magnetism as functions of the latitude and longitude.

Or, when a suitable number of values is given, we may treat the equations in another way which is simple and convenient, though it does not possess any recommendation from the theory of probability. If the equations though more numerous than is absolutely necessary are all consistent with each other the results obtained will be exact. If the equations though not absolutely consistent are very nearly so, we may assume that our results will be reasonably satisfactory. To this method we now proceed.

260. Suppose that we have a number of values of x given, and that to each value corresponds a certain coefficient ξ; and suppose that the values of x and the coefficients are so adjusted that the following relation holds for all positive integral values of s from 0 to $2n$ inclusive:

$$\Sigma\, \xi x^s = \int_{-1}^{1} x^s\, dx \quad \text{.....................(5)},$$

where the summation indicated on the left-hand side is to extend over all the given values of x.

It follows from (5) that if $f(x)$ denote any rational integral function of x, of which the degree is not higher than $2n$, then

$$\Sigma\, \xi f(x) = \int_{-1}^{1} f(x)\, dx.$$

Now apply this equation to the formulæ obtained in Art. 28; then so long as $k+\kappa$ is not greater than $2n$,

$$\left.\begin{aligned} \Sigma\, \xi P_k P_\kappa &= 0 \text{ when } k \text{ and } \kappa \text{ are unequal,} \\ &= \frac{2}{2k+1} \text{ when } \kappa = k \end{aligned}\right\} \text{......(6)}.$$

In like manner by aid of the formulæ obtained in Art. 158, we have

$$\left.\begin{aligned} \Sigma\, \xi (1-x^2)^s D^s P_k D^s P_\kappa &= 0 \text{ when } k \text{ and } \kappa \text{ are unequal,} \\ &= \frac{2\,\lfloor k+s}{(2k+1)\,\lfloor k-s} \text{ when } \kappa = k \end{aligned}\right\} \text{...(7)}.$$

The summation indicated on the left-hand side in (6) and (7) is to extend over the same given values of x as that in (5).

261. The relation (5) amounts to a system of $2n+1$ linear equations to be satisfied by the coefficients of which ξ is the type. We take then $2n+1$ values of x as arbitrarily given, and the summations in (5), (6), (7) will refer to these $2n+1$ given values. It will be remembered that we have $x = \cos\theta$, so that when x is given the polar distance θ is given.

Suppose now that for all these $2n+1$ polar distances we have the values of C_s and S_s determined by (3). Take from (4) the expression for C_s, multiply it by $\xi \sin^s\theta D^s P_k$ and form the sum for the $2n+1$ polar distances. Thus

$$\Sigma\xi\, C_s \sin^s\theta\, D^s P_k = \Sigma_\lambda \{A_{\lambda,s}\Sigma\xi\,(1-x^2)^s D^s P_\lambda D^s P_k\},$$

where Σ denotes summation with respect to the $2n+1$ polar distances, and Σ_λ denotes summation with respect to λ from $\lambda = s$ to $\lambda = n$, both inclusive.

By means of (7) all the terms on the right-hand side vanish except when $\lambda = k$; and thus we obtain

$$\Sigma\xi\, C_s \sin^s\theta\, D^s P_k = \frac{2}{2k+1}\frac{\lfloor k+s}{\lfloor k-s} A_{k,s} \quad \text{..........(8).}$$

This determines $A_{k,s}$.

Similarly $$\Sigma\xi\, S_s \sin^s\theta\, D^s P_k = \frac{2}{2k+1}\frac{\lfloor k+s}{\lfloor k-s} B_{k,s} \quad \text{.......(9).}$$

This determines $B_{k,s}$.

262. We proceed to express in a convenient form the coefficients of which ξ is the type.

Let $x_0, x_1, \dots x_{2n}$ denote the given values of x, so that for positive integral values of s from 0 to $2n$ inclusive

$$\xi_0 x_0^s + \xi_1 x_1^s + \xi_2 x_2^s + \dots\dots + \xi_{2n} x_{2n}^s = \int_{-1}^{1} x^s\, dx \quad \text{...(10).}$$

Put $\psi(x) = (x - x_0)(x - x_1) \ldots\ldots (x - x_{2n})$.

When we divide $\psi(x)$ by one of its factors, for example by the first factor, we obtain an expression which is equivalent to a rational function of x of the degree $2n$.

Let $$\psi_0(x) = \frac{\psi(x)}{x - x_0} = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_{2n} x^{2n};$$

then we know that this expression will vanish for all the given values of x except x_0.

Multiply equations (10) in order by $a_0, a_1, \ldots a_{2n}$, and add; thus

$$\xi_0 \psi_0(x_0) + \xi_1 \psi_0(x_1) + \ldots\ldots + \xi_{2n} \psi_0(x_{2n}) = \int_{-1}^{1} \psi_0(x)\, dx,$$

so that $$\xi_0 \psi_0(x_0) = \int_{-1}^{1} \psi_0(x)\, dx.$$

This we may write thus

$$\xi_0 (x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_{2n})$$
$$= \int_{-1}^{1} (x - x_1)(x - x_2) \ldots (x - x_{2n})\, dx,$$

or $$\xi_0 \left[\frac{d}{dx} \psi(x)\right]_0 = \int_{-1}^{1} \frac{\psi(x)}{x - x_0}\, dx,$$

where $\left[\frac{d}{dx} \psi(x)\right]_0$ indicates that $\psi(x)$ is to be differentiated and then x_0 put for x.

Thus ξ_0 is determined; and similarly we may determine $\xi_1, \xi_2, \ldots\ldots \xi_{2n}$.

263. We will now change our suppositions. Instead of $2n + 1$ *given* values of x we will suppose there are $n + 1$ values *to be determined* as well as the $n + 1$ corresponding values of ξ. We may then assume $2n + 2$ conditions, and these shall be that the following relation holds for all positive integral values of s from 0 to $2n + 1$ inclusive,

$$\xi_0 x_0^s + \xi_1 x_1^s + \ldots\ldots + \xi_n x_n^s = \int_{-1}^{1} x^s\, dx \ldots\ldots\ldots (11).$$

Then the equations (6) and (7) will hold so long as $k+\kappa$ is not greater than $2n+1$.

We proceed to eliminate from (11) the quantities ξ_0, ξ_1, ... ξ_n.

Put $$\chi = \frac{\xi_0}{x-x_0} + \frac{\xi_1}{x-x_1} + \ldots\ldots + \frac{\xi_n}{x-x_n}.$$

When we develop the fractions in ascending powers of x, we find that the general term of χ is

$$\frac{1}{x^{s+1}}\{\xi_0 x_0^s + \xi_1 x_1^s + \ldots\ldots + \xi_n x_n^s\}.$$

Hence by (11) we have

$$\chi = \Sigma \frac{1}{x^{s+1}} \int_{-1}^{1} x^s\,dx + \frac{R}{x^{2n+3}} \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where Σ denotes summation with respect to s from 0 to $2n+1$ both inclusive, and R is an infinite series of the form

$$b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \ldots$$

Now χ is a fraction of the form

$$\frac{B_0 x^n + B_1 x^{n-1} + \ldots\ldots + B_{n-1}x + B_n}{x^{n+1} + A_1 x^n + \ldots\ldots + A_n x + A_{n+1}} \ldots\ldots\ldots\ldots(13)$$

where the denominator $= (x-x_0)(x-x_1)\ldots\ldots(x-x_n)$.

Let us denote the denominator by $\varpi(x)$, so that the quantities $x_0, x_1, \ldots\ldots x_n$ are the roots of the equation

$$\varpi(x) = 0.$$

From (12) and (13) we have

$$B_0 x^n + B_1 x^{n-1} + \ldots\ldots + B_{n-1}x + B_n$$

$=$ the product of $(x^{n+1} + A_1 x^n + \ldots\ldots + A_{n+1})$

into $(H_0 x^{-1} + H_1 x^{-2} + \ldots\ldots + H_{2n+1} x^{-2n-2} + Rx^{-2n-3})$,

where $H_k = \int_{-1}^{1} x^k dx = \frac{2}{k+1}$ or 0, according as k is even or odd.

Equate the coefficients of the powers of x; thus

$$H_{n+1} + A_1 H_n + \ldots\ldots + A_{n+1} H_0 = 0,$$
$$H_{n+2} + A_1 H_{n+1} + \ldots\ldots + A_{n+1} H_1 = 0,$$
$$\ldots\ldots\ldots\ldots$$
$$H_{2n+1} + A_1 H_{2n} + \ldots\ldots + A_{n+1} H_n = 0;$$
$$B_0 = H_0,$$
$$B_1 = H_1 + A_1 H_0,$$
$$\ldots\ldots\ldots\ldots$$
$$B_n = H_n + A_1 H_{n-1} + \ldots + A_n H_0.$$

The former group consists of $n+1$ equations between the quantities $A_1, A_2, \ldots A_{n+1}$, which will suffice to determine them. If we restore for H_k its value $\int_{-1}^{1} x^k dx$, we find that these $n+1$ equations are all cases of the following, obtained by giving to s positive integral values from 0 to n, both inclusive:

$$\int_{-1}^{1} x^s (x^{n+1} + A_1 x^n + \ldots + A_{n+1})\, dx = 0,$$

that is

$$\int_{-1}^{1} x^s \varpi(x)\, dx = 0.$$

Hence it follows, by Art. 32, that $\varpi(x) = CP_{n+1}(x)$, where C is a constant. Thus the values $x_0, x_1, \ldots x_n$ in (11) are the roots of the equation $P_{n+1}(x) = 0$.

Then, as in Art. 262, we find that

$$\xi_0 \left[\frac{d}{dx} P_{n+1}(x)\right]_0 = \int_{-1}^{1} \frac{P_{n+1}(x)}{x - x_0}\, dx,$$

where $\left[\frac{d}{dx} P_{n+1}(x)\right]_0$ indicates that $P_{n+1}(x)$ is to be differentiated, and then x_0 put for x. Similarly we can find $\xi_1, \xi_2, \ldots \xi_n$. Hence the coefficients $\xi_0, \xi_1, \ldots \xi_n$ are identical with the quantities, the type of which is A_r obtained in Gauss's process of integration; see Art. 131.

264. As a particular case, let us suppose that the function denoted by F does not involve ϕ, so that it reduces to $F(\theta)$. Then, by Art. 257,

$$C_s = \frac{2F(\theta)}{2n+1} \Sigma \cos ks\alpha, \qquad S_s = \frac{2F(\theta)}{2n+1} \Sigma \sin ks\alpha,$$

except that when $s=0$ we take only half the value given by the first formula.

Now when $s=0$ we have $\Sigma \cos ks\alpha = 2n+1$, and in other cases $\Sigma \cos ks\alpha = 0$; also $\Sigma \sin ks\alpha = 0$.

Thus S_s always vanishes; and C_s always vanishes except when $s=0$, and then we have $C_0 = F(\theta)$: or putting $f(x)$ instead of $F(\theta)$ we have $C_0 = f(x)$.

Hence by (8) we have $A_{k,0} = \frac{2k+1}{2} \Sigma \xi f(x) P_k$.

The constants denoted by A with suffixes vanish by (8) except when the second suffix is zero; the constants denoted by B with suffixes always vanish by (9). Thus the value of Z_k of Art. 256 reduces to $\frac{2k+1}{2} P_k \Sigma \xi f(x) P_k$.

Hence we obtain for the development of the function $f(x)$,

$$f(x) = p_0 P_0 + p_1 P_1 + \ldots + p_n P_n,$$

where $p_k =$

$$\frac{2k+1}{2} \{f(x_0)\xi_0 P_k(x_0) + f(x_1)\xi_1 P_k(x_1) + \ldots + f(x_n)\xi_n P_k(x_n)\} \ldots (14).$$

But we know by Art. 138 that the exact development of $f(x)$ is

$$f(x) = q_0 P_0 + q_1 P_1 + \ldots + q_n P_n + \ldots\ldots\ldots\ldots (15),$$

where $$q_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k \, dx.$$

If we make use of this formula in (14) we find that

$$p_k = \frac{2k+1}{2} \Sigma_s q_s \{\Sigma \xi P_s P_k\} \ldots\ldots\ldots\ldots\ldots\ldots (16),$$

where Σ denotes summation with respect to all the $n+1$ values of x, and Σ_s denotes summation with respect to s from 0 to ∞. By virtue of equations (6), which with the present notation hold so long as $k+s$ is not greater than $2n+1$, the right-hand side of (16) may be reduced. The term which corresponds to $s=k$ becomes simply q_k; all the other terms vanish so long as s is not greater than $2n+1-k$: thus we obtain

$$p_k = q_k + q_{2n+2-k}E_{2n+2-k} + q_{2n+3-k}E_{2n+3-k} + \ldots\ldots\ldots (17),$$

where $E_m = \frac{2k+1}{2}\Sigma\xi P_m P_k$, the summation extending to all the $n+1$ values of x.

For instance,

$$p_0 = q_0 + q_{2n+2}E_{2n+2} + q_{2n+3}E_{2n+3} + q_{2n+4}E_{2n+4} + \ldots,$$

$$p_1 = q_1 + q_{2n+1}E_{2n+1} + q_{2n+2}E_{2n+2} + q_{2n+3}E_{2n+3} + \ldots,$$

where it must be observed that the symbols E with suffixes have different meanings in the two lines; in the first line $E_m = \frac{1}{2}\Sigma\xi P_m$, and in the second line $E_m = \frac{3}{2}\Sigma\xi P_m P_1$.

From (15) we have $\int_{-1}^{1} f(x)\,dx = 2q_0$

$$= 2p_0 - 2\{q_{2n+2}E_{2n+2} + q_{2n+3}E_{2n+3} + \ldots\}, \text{ by (17).}$$

Hence by (14) we obtain

$$\int_{-1}^{1} f(x)\,dx = \xi_0 f(x_0) + \xi_1 f(x_1) + \ldots + \xi_n f(x_n)$$
$$- 2\{q_{2n+2}E_{2n+2} + q_{2n+3}E_{2n+3} + \ldots\}.$$

In this expression for $\int_{-1}^{1} f(x)\,dx$ the first part is identical with the $\Sigma A_r f(a_r)$ of Art. 126, so that the second part gives us a new expression for the *error* which arises in taking the approximate quadrature for the real quadrature.

CHAPTER XXI.

SPECIAL CURVILINEAR COORDINATES.

265. In some investigations of mixed mathematics, certain coordinates introduced by Lamé have been found very useful: these we shall now explain. Lamé's own investigations on the subject were first given by him in various memoirs, and afterwards reproduced in two works entitled *Leçons sur les fonctions inverses des transcendantes et les surfaces isothermes*, 1857; and *Leçons sur les coordonnées curvilignes et leurs diverses applications*, 1859. These coordinates are also explained in the *Cours de Physique Mathématique* of M. Mathieu, 1873.

266. Consider the following three equations where x, y, z denote variable coordinates:

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - b^2} + \frac{z^2}{\lambda^2 - c^2} = 1,$$

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} + \frac{z^2}{\mu^2 - c^2} = 1,$$

$$\frac{x^2}{\nu^2} + \frac{y^2}{\nu^2 - b^2} + \frac{z^2}{\nu^2 - c^2} = 1.$$

Suppose b^2 less than c^2, λ^2 greater than c^2, μ^2 between b^2 and c^2, and ν^2 less than b^2: then the first equation represents an ellipsoid, the second an hyperboloid of one sheet, and the third an hyperboloid of two sheets.

We shall sometimes denote these surfaces by S_1, S_2, S_3 respectively.

267. Suppose the preceding three equations to exist simultaneously; then x, y, z will be the coordinates of a point or points at which the surfaces intersect. The values of x^2, y^2, z^2 which satisfy these equations simultaneously are easily found to be

$$x^2 = \frac{\lambda^2\mu^2\nu^2}{b^2c^2}, \quad y^2 = \frac{(\lambda^2 - b^2)(\mu^2 - b^2)(\nu^2 - b^2)}{b^2(b^2 - c^2)},$$

$$z^2 = \frac{(\lambda^2 - c^2)(\mu^2 - c^2)(\nu^2 - c^2)}{c^2(c^2 - b^2)}.$$

These values may be immediately obtained from the general formulæ given in the *Theory of Equations*, Art. 291.

Or we may proceed thus. The three equations of Art. 266 may be considered as expressing the fact that

$$1 - \frac{x^2}{\rho} - \frac{y^2}{\rho - b^2} - \frac{z^2}{\rho - c^2}$$

vanishes when $\rho = \lambda^2$ or μ^2 or ν^2. Hence we have

$$1 - \frac{x^2}{\rho} - \frac{y^2}{\rho - b^2} - \frac{z^2}{\rho - c^2} = \frac{(\rho - \lambda^2)(\rho - \mu^2)(\rho - \nu^2)}{\rho(\rho - b^2)(\rho - c^2)};$$

for no constant factor is required since each side becomes unity when ρ is infinite. Then if we decompose the right member into partial fractions, in the manner explained in the *Integral Calculus*, Chapter II., we obtain

$$x^2 = \frac{\lambda^2\mu^2\nu^2}{b^2c^2}, \quad y^2 = \frac{(\lambda^2 - b^2)(\mu^2 - b^2)(\nu^2 - b^2)}{b^2(b^2 - c^2)},$$

$$z^2 = \frac{(\lambda^2 - c^2)(\mu^2 - c^2)(\nu^2 - c^2)}{c^2(c^2 - b^2)}.$$

Since by extracting the square roots of the last equations we obtain three double signs, we see that the surfaces of Art. 266 have *eight* points of intersection.

268. *Through any point in space one such system of surfaces as that of Art. 266 can be drawn, and only one,* b *and* c *being fixed quantities.*

For let (x, y, z) denote the point; and let it be required to find t from the equation

$$\frac{x^2}{t}+\frac{y^2}{t-b^2}+\frac{z^2}{t-c^2}=1.$$

This may be considered as a cubic equation in t, and by observing the changes of sign in the left-hand member as t varies, we see that there is one root of the equation between 0 and b^2, one root between b^2 and c^2, and one greater than c^2. We suppose here that none of the three quantities x, y, z is zero.

269. *The three surfaces of Art.* 266 *are mutually at right angles at the points of intersection.*

Denote the first equation by $u=0$, and the second by $v=0$; then the condition that the surfaces may intersect at right angles is

$$\frac{du}{dx}\frac{dv}{dx}+\frac{du}{dy}\frac{dv}{dy}+\frac{du}{dz}\frac{dv}{dz}=0,$$

that is $$\frac{x^2}{\lambda^2\mu^2}+\frac{y^2}{(\lambda^2-b^2)(\mu^2-b^2)}+\frac{z^2}{(\lambda^2-c^2)(\mu^2-c^2)}=0.$$

Now this condition is fulfilled at the points of intersection as we see by subtracting the second equation of Art. 266 from the first.

Similarly the other two surfaces intersect at right angles.

270. By adding the values of x^2, y^2, and z^2 in Art. 267, we obtain

$$x^2+y^2+z^2=\lambda^2+\mu^2+\nu^2-b^2-c^2.$$

271. By extracting the square roots of the expressions in Art. 267, we obtain

$$x=\frac{\lambda\mu\nu}{bc},\ y=\frac{\sqrt{(\lambda^2-b^2)(\mu^2-b^2)(b^2-\nu^2)}}{b\sqrt{(c^2-b^2)}},\ z=\frac{\sqrt{(\lambda^2-c^2)(c^2-\mu^2)(c^2-\nu^2)}}{c\sqrt{(c^2-b^2)}}.$$

Some convention as to signs is necessary in order to ensure that the last formulæ shall have due generality; and the following is found sufficient by Lamé. Out of the *nine* quantities λ, μ, ν, $\sqrt{(\lambda^2 - b^2)}$, $\sqrt{(\mu^2 - b^2)}$, $\sqrt{(b^2 - \nu^2)}$, $\sqrt{(\lambda^2 - c^2)}$, $\sqrt{(c^2 - \mu^2)}$, $\sqrt{(c^2 - \nu^2)}$, *three* are taken to be susceptible of either sign, namely ν, $\sqrt{(\mu^2 - b^2)}$ and $\sqrt{(\lambda^2 - c^2)}$; the rest are considered always positive. Thus the expressions for x, y, and z have each one factor which may be either positive or negative.

272. The quantities λ, μ, ν are called *elliptic coordinates.* When they are given we may suppose the surfaces of Art. 266 to be constructed, and their common points determined. Or we may find x, y, and z from the formulæ of Art. 271.

It will be observed that if we merely know λ, μ, and ν, the point in space is not completely determined; for there are *eight* points corresponding to assigned values of λ, μ, and ν. If however we attend to the sign of ν, according to the convention of Art. 271, the number of points is reduced to *four.*

273. Suppose in the first equation of Art. 266 that λ varies; we thus obtain a series of ellipsoids, all *confocal,* that is all having the same points for the foci of their principal sections. We may suppose λ to commence with a value infinitesimally greater than c, and then one of the axes of the ellipsoid is infinitesimal, namely that which is in length equal to $2\sqrt{(\lambda^2 - c^2)}$. Then λ may be supposed to increase indefinitely.

Similarly in the second equation of Art. 266, if μ varies we obtain a series of confocal hyperboloids of one sheet. The limits between which μ may vary are from a value infinitesimally greater than b to a value infinitesimally less than c. At the former limit the real axis which is in length equal to $2\sqrt{(\mu^2 - b^2)}$ vanishes, and at the latter limit the conjugate axis which is in length equal to $2\sqrt{(c^2 - \mu^2)}$ vanishes.

Finally, in the third equation of Art. 266, if ν varies we obtain a series of confocal hyperboloids of two sheets. The limits between which ν may vary are from an infinitesimal value to a value infinitesimally less than b. At the former limit the real axis which is in length equal to 2ν vanishes, and at the latter limit the conjugate axis which is in length equal to $2\sqrt{(b^2 - \nu^2)}$ vanishes.

274. Take the logarithms of the formulæ in Art. 267, and differentiate. Thus

$$dx = \frac{xd\lambda}{\lambda} + \frac{xd\mu}{\mu} + \frac{xd\nu}{\nu},$$

$$dy = \frac{y\lambda d\lambda}{\lambda^2 - b^2} + \frac{y\mu d\mu}{\mu^2 - b^2} + \frac{y\nu d\nu}{\nu^2 - b^2},$$

$$dz = \frac{z\lambda d\lambda}{\lambda^2 - c^2} + \frac{z\mu d\mu}{\mu^2 - c^2} + \frac{z\nu d\nu}{\nu^2 - c^2}.$$

Square and add; then by the aid of the equations of Art. 266, we obtain

$$\begin{aligned} dx^2 + dy^2 + dz^2 = &\left\{\frac{x^2}{\lambda^4} + \frac{y^2}{(\lambda^2 - b^2)^2} + \frac{z^2}{(\lambda^2 - c^2)^2}\right\} \lambda^2 d\lambda^2 \\ &+ \left\{\frac{x^2}{\mu^4} + \frac{y^2}{(\mu^2 - b^2)^2} + \frac{z^2}{(\mu^2 - c^2)^2}\right\} \mu^2 d\mu^2 \\ &+ \left\{\frac{x^2}{\nu^4} + \frac{y^2}{(\nu^2 - b^2)^2} + \frac{z^2}{(\nu^2 - c^2)^2}\right\} \nu^2 d\nu^2. \end{aligned}$$

But by the formulæ of Art. 267 we shall obtain

$$\frac{x^2}{\lambda^4} + \frac{y^2}{(\lambda^2 - b^2)^2} + \frac{z^2}{(\lambda^2 - c^2)^2} = \frac{(\lambda^2 - \mu^2)(\lambda^2 - \nu^2)}{\lambda^2 (\lambda^2 - b^2)(\lambda^2 - c^2)};$$

and then by symmetry

$$\frac{x^2}{\mu^4} + \frac{y^2}{(\mu^2 - b^2)^2} + \frac{z^2}{(\mu^2 - c^2)^2} = \frac{(\mu^2 - \nu^2)(\mu^2 - \lambda^2)}{\mu^2 (\mu^2 - b^2)(\mu^2 - c^2)};$$

and $$\frac{x^2}{\nu^4} + \frac{y^2}{(\nu^2 - b^2)^2} + \frac{z^2}{(\nu^2 - c^2)^2} = \frac{(\nu^2 - \lambda^2)(\nu^2 - \mu^2)}{\nu^2 (\nu^2 - b^2)(\nu^2 - c^2)}.$$

Hence, putting ds^2 for $dx^2 + dy^2 + dz^2$, we have

$$\begin{aligned} ds^2 = &\frac{d\lambda^2 (\lambda^2 - \mu^2)(\lambda^2 - \nu^2)}{(\lambda^2 - b^2)(\lambda^2 - c^2)} + \frac{d\mu^2 (\mu^2 - \nu^2)(\mu^2 - \lambda^2)}{(\mu^2 - b^2)(\mu^2 - c^2)} \\ &+ \frac{d\nu^2 (\nu^2 - \lambda^2)(\nu^2 - \mu^2)}{(\nu^2 - b^2)(\nu^2 - c^2)}. \end{aligned}$$

Here ds denotes the distance between the point (x, y, z) and the point $(x+dx,\ y+dy,\ z+dz)$.

275. Suppose we put $d\lambda=0$ in the result of the preceding Article; then the two points both lie on the surface S_1, and the formula becomes

$$ds^2=\frac{d\mu^2\,(\mu^2-\nu^2)\,(\mu^2-\lambda^2)}{(\mu^2-b^2)\,(\mu^2-c^2)}+\frac{d\nu^2\,(\nu^2-\lambda^2)\,(\nu^2-\mu^2)}{(\nu^2-b^2)\,(\nu^2-c^2)}.$$

276. Suppose we put $d\mu=0$ and $d\nu=0$ in the result of Art. 274; then the two points both lie on the surface S_2 and also on the surface S_3, and the formula becomes

$$ds^2=\frac{d\lambda^2\,(\lambda^2-\mu^2)\,(\lambda^2-\nu^2)}{(\lambda^2-b^2)\,(\lambda^2-c^2)}.$$

This is therefore the value of the square of the length of the infinitesimal straight line drawn normally to S_1, to meet the adjacent surface of the same family as S_1, in which the parameter has the value $\lambda+d\lambda$.

A similar expression holds for the infinitesimal distance between S_2 and the adjacent surface of the same family, and also for the infinitesimal distance between S_3 and the adjacent surface of the same family.

We shall now give some examples of the use of the formulæ which have been obtained.

277. Let $d\sigma$ denote an element of the surface of a solid, p the perpendicular from a fixed origin on the element; then $\frac{1}{3}p\,d\sigma$ represents the volume of an infinitesimal cone having its vertex at the origin, and having $d\sigma$ for base. Thus the volume of the whole solid $=\frac{1}{3}\int p\,d\sigma$, the integral being taken between appropriate limits.

We will apply this to the ellipsoid given by the first equation of Art. 266, taking the origin of coordinates as the vertex of the cones.

We have by the usual formulæ of Solid Geometry

$$\frac{1}{p^2} = \left(\frac{x}{\lambda^2}\right)^2 + \left(\frac{y}{\lambda^2 - b^2}\right)^2 + \left(\frac{z}{\lambda^2 - c^2}\right)^2;$$

transforming this by the aid of the expressions in Art. 267 we obtain

$$p = \frac{\lambda \sqrt{(\lambda^2 - b^2)(\lambda^2 - c^2)}}{\sqrt{(\lambda^2 - \mu^2)(\lambda^2 - \nu^2)}}.$$

Also by Art. 276 we have

$$d\sigma = \frac{d\mu \sqrt{(\mu^2 - \nu^2)(\mu^2 - \lambda^2)}}{\sqrt{(\mu^2 - b^2)(\mu^2 - c^2)}} \times \frac{d\nu \sqrt{(\nu^2 - \lambda^2)(\nu^2 - \mu^2)}}{\sqrt{(\nu^2 - b^2)(\nu^2 - c^2)}}$$

$$= \frac{d\mu \, d\nu (\mu^2 - \nu^2) \sqrt{(\lambda^2 - \mu^2)(\lambda^2 - \nu^2)}}{\sqrt{(b^2 - \nu^2)(c^2 - \nu^2)(c^2 - \mu^2)(\mu^2 - b^2)}}.$$

If we integrate between the limits 0 and b for ν, and b and c for μ, we obtain one-eighth of the volume of the ellipsoid whose semi-axes are λ, $\sqrt{(\lambda^2 - b^2)}$ and $\sqrt{(\lambda^2 - c^2)}$. Thus

$$\frac{1}{3} \lambda \sqrt{(\lambda^2 - b^2)(\lambda^2 - c^2)} \int_b^c \int_0^b \frac{(\mu^2 - \nu^2) \, d\mu \, d\nu}{\sqrt{(b^2 - \nu^2)(c^2 - \nu^2)(c^2 - \mu^2)(\mu^2 - b^2)}}$$

$$= \frac{1}{8} \cdot \frac{4\pi}{3} \cdot \lambda \sqrt{(\lambda^2 - b^2)(\lambda^2 - c^2)};$$

and therefore $$\int_b^c \int_0^b \frac{(\mu^2 - \nu^2) \, d\mu \, d\nu}{\sqrt{(b^2 - \nu^2)(c^2 - \nu^2)(c^2 - \mu^2)(\mu^2 - b^2)}} = \frac{\pi}{2}.$$

278. Let ω denote any element of area on the plane (x, y), and let z be the corresponding ordinate of a solid; then the volume of the solid is found by taking the integral $\int z \, d\omega$ between proper limits. If $d\sigma$ denote an element of the surface, and γ the angle between the normal to $d\sigma$ and the axis of z, we may put $\cos \gamma \, d\sigma$ for $d\omega$. Thus the volume $= \int z \cos \gamma \, d\sigma$.

We will apply this to the ellipsoid given by the first equation of Art. 266. We have by the usual formulæ of Solid Geometry

$$\cos\gamma = \frac{z}{\lambda^2 - c^2}\left\{\left(\frac{x}{\lambda^2}\right)^2 + \left(\frac{y}{\lambda^2 - b^2}\right)^2 + \left(\frac{z}{\lambda^2 - c^2}\right)^2\right\}^{-\frac{1}{2}};$$

transposing this by the aid of the expressions in Art. 267 we obtain

$$\cos\gamma = \frac{\lambda\sqrt{(\lambda^2 - b^2)(c^2 - \mu^2)(c^2 - \nu^2)}}{c\sqrt{(c^2 - b^2)(\lambda^2 - \mu^2)(\lambda^2 - \nu^2)}}.$$

Hence proceeding as in Art. 277 we obtain finally

$$\int_b^c \int_0^b \frac{(\mu^2 - \nu^2)\sqrt{(c^2 - \mu^2)(c^2 - \nu^2)}\, d\mu d\nu}{\sqrt{(b^2 - \nu^2)(\mu^2 - b^2)}} = \frac{\pi}{6} c^2 (c^2 - b^2).$$

279. If we take the three expressions furnished by Art. 276 we find that an element of volume of a solid may be denoted by $H d\lambda d\mu d\nu$ where

$$H = \frac{(\lambda^2 - \mu^2)(\lambda^2 - \nu^2)(\mu^2 - \nu^2)}{\sqrt{(\lambda^2 - b^2)(\lambda^2 - c^2)(\mu^2 - b^2)(c^2 - \mu^2)(b^2 - \nu^2)(c^2 - \nu^2)}}.$$

Apply this expression to the ellipsoid given by the first equation of Art. 266; then proceeding as in Art. 277 we obtain

$$\int_0^\lambda \int_b^c \int_0^b H d\lambda d\mu d\nu = \frac{\pi}{6} \lambda\sqrt{(\lambda^2 - b^2)(\lambda^2 - c^2)}.$$

280. There is another system deserving of notice in which the ellipsoid is replaced by a sphere and the two hyperboloids by cones. Here we have

$$x^2 + y^2 + z^2 = r^2,$$

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} + \frac{z^2}{\mu^2 - c^2} = 0,$$

$$\frac{x^2}{\nu^2} + \frac{y^2}{\nu^2 - b^2} + \frac{z^2}{\nu^2 - c^2} = 0.$$

These equations give

$$x^2 = \frac{r^2\mu^2\nu^2}{b^2c^2}, \quad y^2 = \frac{r^2(\mu^2 - b^2)(\nu^2 - b^2)}{b^2(b^2 - c^2)}, \quad z^2 = \frac{r^2(\mu^2 - c^2)(\nu^2 - c^2)}{c^2(c^2 - b^2)}.$$

It is easy to shew, as in Art. 269, that the surfaces represented by the three equations intersect at right angles.

281. We may apply the formulæ of Art. 280 to obtain an expression for the surface of a sphere of radius r.

If we proceed as in Art. 277 we shall find that the area of an infinitesimal element of the surface is

$$\frac{(\mu^2 - \nu^2) r^2 d\mu d\nu}{\sqrt{(b^2 - \nu^2)(c^2 - \nu^2)(c^2 - \mu^2)(\mu^2 - b^2)}};$$

and if this be integrated between the limits 0 and b for ν, and b and c for μ, we obtain one-eighth part of the surface of the sphere, that is $\frac{\pi}{2} r^2$. Hence

$$\int_b^c \int_0^b \frac{(\mu^2 - \nu^2) d\mu d\nu}{\sqrt{(b^2 - \nu^2)(c^2 - \nu^2)(c^2 - \mu^2)(\mu^2 - b^2)}} = \frac{\pi}{2}.$$

This agrees with Art. 277.

CHAPTER XXII.

GENERAL CURVILINEAR COORDINATES.

282. In the preceding Chapter we have given an account of a special system of curvilinear coordinates; we shall now treat the subject more generally.

283. Let there be three surfaces represented by the equations

$$\left.\begin{aligned} f_1(x, y, z) &= \rho_1, \\ f_2(x, y, z) &= \rho_2, \\ f_3(x, y, z) &= \rho_3. \end{aligned}\right\} \quad \text{.........................} \quad (1).$$

Here x, y, z are variable coordinates and ρ_1, ρ_2, ρ_3 are parameters which are constant for any surface; but by varying a parameter we obtain a family of corresponding surfaces. For shortness we may denote the surface of the first family for which the parameter has the value ρ_1 by the words *the surface* ρ_1; and similarly *the surface* ρ_2 will denote the surface of the second family, for which the parameter has the value ρ_2; and a like meaning will apply to the words *the surface* ρ_3.

284. To given values of x, y, z in (1) will correspond definite values of ρ_1, ρ_2, ρ_3; that is, for every point of space the parameters of the three surfaces can be determined. Conversely, if ρ_1, ρ_2, ρ_3 are given the values of x, y, z may be theoretically found; that is, the points (x, y, z) may be considered to be known when the three parameters are given.

285. Put $$\left(\frac{d\rho_1}{dx}\right)^2 + \left(\frac{d\rho_1}{dy}\right)^2 + \left(\frac{d\rho_1}{dz}\right)^2 = h_1^2,$$

$$\left(\frac{d\rho_2}{dx}\right)^2 + \left(\frac{d\rho_2}{dy}\right)^2 + \left(\frac{d\rho_2}{dz}\right)^2 = h_2^2,$$

$$\left(\frac{d\rho_3}{dx}\right)^2 + \left(\frac{d\rho_3}{dy}\right)^2 + \left(\frac{d\rho_3}{dz}\right)^2 = h_3^2.$$

Let a_1, b_1, c_1 denote the cosines of the angles which the normal at (x, y, z) to the surface ρ_1 makes with the coordinate axes; let a_2, b_2, c_2 be similar quantities with reference to the surface ρ_2; and let a_3, b_3, c_3 be similar quantities with respect to ρ_3. Then

$$\left.\begin{aligned} a_1 &= \frac{1}{h_1}\frac{d\rho_1}{dx}, & b_1 &= \frac{1}{h_1}\frac{d\rho_1}{dy}, & c_1 &= \frac{1}{h_1}\frac{d\rho_1}{dz} \\ a_2 &= \frac{1}{h_2}\frac{d\rho_2}{dx}, & b_2 &= \frac{1}{h_2}\frac{d\rho_2}{dy}, & c_2 &= \frac{1}{h_2}\frac{d\rho_2}{dz} \\ a_3 &= \frac{1}{h_3}\frac{d\rho_3}{dx}, & b_3 &= \frac{1}{h_3}\frac{d\rho_3}{dy}, & c_3 &= \frac{1}{h_3}\frac{d\rho_3}{dz} \end{aligned}\right\} \quad\text{.........(2).}$$

286. Let V denote any function of x, y, z; by substituting for x, y, z their values in terms of ρ_1, ρ_2, ρ_3 from (1), we transform V into a function of ρ_1, ρ_2, ρ_3. Then by the aid of (2) we get

$$\left.\begin{aligned} \frac{dV}{dx} &= \frac{dV}{d\rho_1}a_1h_1 + \frac{dV}{d\rho_2}a_2h_2 + \frac{dV}{d\rho_3}a_3h_3 \\ \frac{dV}{dy} &= \frac{dV}{d\rho_1}b_1h_1 + \frac{dV}{d\rho_2}b_2h_2 + \frac{dV}{d\rho_3}b_3h_3 \\ \frac{dV}{dz} &= \frac{dV}{d\rho_1}c_1h_1 + \frac{dV}{d\rho_2}c_2h_2 + \frac{dV}{d\rho_3}c_3h_3 \end{aligned}\right\} \quad\text{..........(3).}$$

287. Now let us suppose henceforth the three surfaces given by (1) to be *mutually at right angles;* then the nine cosines a_1, b_1, c_1 ... satisfy certain well-known relations, and with the aid of these we deduce from (3) by squaring and adding

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2$$
$$= h_1^2 \left(\frac{dV}{d\rho_1}\right)^2 + h_2^2 \left(\frac{dV}{d\rho_2}\right)^2 + h_3^2 \left(\frac{dV}{d\rho_3}\right)^2 \quad \text{........(4).}$$

288. One of the relations between the nine cosines to which we have just alluded is

$$a_1 (b_2c_3 - b_3c_2) + a_2 (b_3c_1 - b_1c_3) + a_3 (b_1c_2 - b_2c_1) = 1 ;$$

hence by the aid of (2) we have a result which we may express in the notation of determinants, thus:

$$\begin{vmatrix} \frac{d\rho_1}{dx}, & \frac{d\rho_1}{dy}, & \frac{d\rho_1}{dz} \\ \frac{d\rho_2}{dx}, & \frac{d\rho_2}{dy}, & \frac{d\rho_2}{dz} \\ \frac{d\rho_3}{dx}, & \frac{d\rho_3}{dy}, & \frac{d\rho_3}{dz} \end{vmatrix} = h_1h_2h_3.$$

289. From equations (2) we deduce

$$a_1dx + b_1dy + c_1dz = \frac{1}{h_1} d\rho_1,$$
$$a_2dx + b_2dy + c_2dz = \frac{1}{h_2} d\rho_2,$$
$$a_3dx + b_3dy + c_3dz = \frac{1}{h_3} d\rho_3;$$

and from these we deduce

$$\left.\begin{aligned} dx &= \frac{a_1}{h_1} d\rho_1 + \frac{a_2}{h_2} d\rho_2 + \frac{a_3}{h_3} d\rho_3 \\ dy &= \frac{b_1}{h_1} d\rho_1 + \frac{b_2}{h_2} d\rho_2 + \frac{b_3}{h_3} d\rho_3 \\ dz &= \frac{c_1}{h_1} d\rho_1 + \frac{c_2}{h_2} d\rho_2 + \frac{c_3}{h_3} d\rho_3 \end{aligned}\right\} \quad \text{.............(5).}$$

From (5), by squaring and adding, we obtain

$$dx^2 + dy^2 + dz^2 = \frac{1}{h_1^2} d\rho_1^2 + \frac{1}{h_2^2} d\rho_2^2 + \frac{1}{h_3^2} d\rho_3^2 \ldots\ldots (6).$$

The left-hand member may be replaced by ds^2, so that ds denotes the distance between the adjacent points (x, y, z) and $(x+dx,\ y+dy,\ z+dz)$.

290. Three particular cases of (6) deserve special notice. Suppose that the adjacent points both lie on the surface ρ_2, and also both lie on the surface ρ_3; then they both lie on the common intersection of these two surfaces, which by hypothesis is at right angles to the surface ρ_1 at the point (x, y, z). Thus we have $d\rho_2 = 0$, and $d\rho_3 = 0$; so that (6) becomes $ds^2 = \frac{1}{h_1^2} d\rho_1^2$; therefore $\frac{1}{h_1} d\rho_1$ is numerically equal to the distance at the point (x, y, z) between the surface ρ_1 and the adjacent surface $\rho_1 + d\rho_1$.

Similarly we can interpret the special equations

$$ds^2 = \frac{1}{h_2^2} d\rho_2^2, \text{ and } ds^2 = \frac{1}{h_3^2} d\rho_3^2.$$

291. From equations (5), we obtain

$$\frac{dx}{d\rho_1} = \frac{a_1}{h_1}, \quad \frac{dx}{d\rho_2} = \frac{a_2}{h_2}, \quad \frac{dx}{d\rho_3} = \frac{a_3}{h_3},$$

$$\frac{dy}{d\rho_1} = \frac{b_1}{h_1}, \quad \frac{dy}{d\rho_2} = \frac{b_2}{h_2}, \quad \frac{dy}{d\rho_3} = \frac{b_3}{h_3},$$

$$\frac{dz}{d\rho_1} = \frac{c_1}{h_1}, \quad \frac{dz}{d\rho_2} = \frac{c_2}{h_2}, \quad \frac{dz}{d\rho_3} = \frac{c_3}{h_3}.$$

These equations may also be obtained in another way. For if a small change $d\rho_1$ be ascribed to ρ_1 we have $\frac{dx}{d\rho_1} d\rho_1$ for the corresponding change in x. This expression must therefore be equal to the projection on the axis of x of the normal distance between the adjacent surfaces ρ_1 and $\rho_1 + d\rho_1$ at the point (x, y, z). Now this normal distance by Art. 290 is $\frac{1}{h_1}d\rho_1$, and the projection on the axis of x is obtained by multiplying by a_1, which is the cosine of the angle between the normal and the axis of x; so that $\frac{dx}{d\rho_1}d\rho_1 = \frac{a_1}{h_1}d\rho_1$, and therefore $\frac{dx_1}{d\rho_1} = \frac{a_1}{h_1}$.

Similarly the other cases can be established.

By the aid of Art. 285 these become

$$\left.\begin{array}{lll} \frac{dx}{d\rho_1} = \frac{1}{h_1^2}\frac{d\rho_1}{dx}, & \frac{dy}{d\rho_1} = \frac{1}{h_1^2}\frac{d\rho_1}{dy}, & \frac{dz}{d\rho_1} = \frac{1}{h_1^2}\frac{d\rho_1}{dz} \\ \frac{dx}{d\rho_2} = \frac{1}{h_2^2}\frac{d\rho_2}{dx}, & \frac{dy}{d\rho_2} = \frac{1}{h_2^2}\frac{d\rho_2}{dy}, & \frac{dz}{d\rho_2} = \frac{1}{h_2^2}\frac{d\rho_2}{dz} \\ \frac{dx}{d\rho_3} = \frac{1}{h_3^2}\frac{d\rho_3}{dx}, & \frac{dy}{d\rho_3} = \frac{1}{h_3^2}\frac{d\rho_3}{dy}, & \frac{dz}{d\rho_3} = \frac{1}{h_3^2}\frac{d\rho_3}{dz} \end{array}\right\} \ldots\ldots(7).$$

292. From equations (7) we obtain, by the aid of Art. 288, in the notation of determinants

$$\begin{vmatrix} \frac{dx}{d\rho_1}, & \frac{dy}{d\rho_1}, & \frac{dz}{d\rho_1} \\ \frac{dx}{d\rho_2}, & \frac{dy}{d\rho_2}, & \frac{dz}{d\rho_2} \\ \frac{dx}{d\rho_3}, & \frac{dy}{d\rho_3}, & \frac{dz}{d\rho_3} \end{vmatrix} = \frac{1}{h_1h_2h_3}.$$

CHAPTER XXIII.

TRANSFORMATION OF LAPLACE'S PRINCIPAL EQUATION.

293. IN equation (4) of the preceding Chapter a certain expression involving first differential coefficients is transformed from the variables x, y, and z to the variables ρ_1, ρ_2, and ρ_3. It is the object of the present Chapter to effect a similar transformation with respect to the expression $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}$; the expression is very important on account of the well-known equation which Laplace first considered: see Art. 167. The expression is called by Lamé *the parameter of the second order of the function* V.

294. *The parameter of the second order of any function* V *can be expressed in terms of the parameters of the second order of the functions* ρ_1, ρ_2, *and* ρ_3.

For $$\frac{dV}{dx}=\frac{dV}{d\rho_1}\frac{d\rho_1}{dx}+\frac{dV}{d\rho_2}\frac{d\rho_2}{dx}+\frac{dV}{d\rho_3}\frac{d\rho_3}{dx};$$

$$\frac{d^2V}{dx^2}=\frac{d^2V}{d\rho_1^2}\left(\frac{d\rho_1}{dx}\right)^2+\frac{d^2V}{d\rho_2^2}\left(\frac{d\rho_2}{dx}\right)^2+\frac{d^2V}{d\rho_3^2}\left(\frac{d\rho_3}{dx}\right)^2$$
$$+2\frac{d^2V}{d\rho_1 d\rho_2}\frac{d\rho_1}{dx}\frac{d\rho_2}{dx}+2\frac{d^2V}{d\rho_2 d\rho_3}\frac{d\rho_2}{dx}\frac{d\rho_3}{dx}+2\frac{d^2V}{d\rho_3 d\rho_1}\frac{d\rho_3}{dx}\frac{d\rho_1}{dx}$$
$$+\frac{dV}{d\rho_1}\frac{d^2\rho_1}{dx^2}+\frac{dV}{d\rho_2}\frac{d^2\rho_2}{dx^2}+\frac{dV}{d\rho_3}\frac{d^2\rho_3}{dx^2}.$$

Similar expressions hold for $\frac{d^2V}{dy^2}$ and $\frac{d^2V}{dz^2}$; hence by addition, observing that the surfaces of Art. 283 are at right angles, we obtain

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=h_1^2\frac{d^2V}{d\rho_1^2}+h_2^2\frac{d^2V}{d\rho_2^2}+h_3^2\frac{d^2V}{d\rho_3^2}$$
$$+\frac{dV}{d\rho_1}\left(\frac{d^2\rho_1}{dx^2}+\frac{d^2\rho_1}{dy^2}+\frac{d^2\rho_1}{dz^2}\right)+\frac{dV}{d\rho_2}\left(\frac{d^2\rho_2}{dx^2}+\frac{d^2\rho_2}{dy^2}+\frac{d^2\rho_2}{dz^2}\right)$$
$$+\frac{dV}{d\rho_3}\left(\frac{d^2\rho_3}{dx^2}+\frac{d^2\rho_3}{dy^2}+\frac{d^2\rho_3}{dz^2}\right)\ldots\ldots\ldots\ldots(1).$$

Thus the parameter of V of the second order is expressed in terms of ρ_1, ρ_2, and ρ_3 and of the parameters of the second order of ρ_1, ρ_2, and ρ_3: it remains to transform these three parameters.

We shall use the symbol ∇ as an abbreviation of the operation $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$.

295. The relations among the nine cosines to which we have alluded in Art. 287 may be made to give the following results:

$$a_1=b_2c_3-b_3c_2,\quad b_1=c_2a_3-c_3a_2,\quad c_1=a_2b_3-a_3b_2,$$

together with two other similar sets.

Hence by the aid of Art. 285 we obtain

$$\left.\begin{aligned}\frac{d\rho_1}{dx}&=\frac{h_1}{h_2h_3}\left(\frac{d\rho_2}{dy}\frac{d\rho_3}{dz}-\frac{d\rho_3}{dy}\frac{d\rho_2}{dz}\right),\\ \frac{d\rho_1}{dy}&=\frac{h_1}{h_2h_3}\left(\frac{d\rho_2}{dz}\frac{d\rho_3}{dx}-\frac{d\rho_3}{dz}\frac{d\rho_2}{dx}\right),\\ \frac{d\rho_1}{dz}&=\frac{h_1}{h_2h_3}\left(\frac{d\rho_2}{dx}\frac{d\rho_3}{dy}-\frac{d\rho_3}{dx}\frac{d\rho_2}{dy}\right);\end{aligned}\right\}\ldots\ldots\ldots(2)$$

together with two other similar sets.

Differentiate the first of (2) with respect to y, and the second with respect to x, and equate; thus we get

$$\frac{h_1}{h_2h_3}\left(\frac{d^2\rho_2}{dy^2}\frac{d\rho_3}{dz}+\frac{d\rho_2}{dy}\frac{d^2\rho_3}{dydz}-\frac{d^2\rho_3}{dy^2}\frac{d\rho_2}{dz}-\frac{d\rho_3}{dy}\frac{d^2\rho_2}{dydz}\right)$$
$$+\frac{d}{dy}\left(\frac{h_1}{h_2h_3}\right)\left(\frac{d\rho_2}{dy}\frac{d\rho_3}{dz}-\frac{d\rho_3}{dy}\frac{d\rho_2}{dz}\right)$$
$$=\frac{h_1}{h_2h_3}\left(\frac{d^2\rho_2}{dxdz}\frac{d\rho_3}{dx}+\frac{d\rho_2}{dz}\frac{d^2\rho_3}{dx^2}-\frac{d^2\rho_3}{dxdz}\frac{d\rho_2}{dx}-\frac{d\rho_3}{dz}\frac{d^2\rho_2}{dx^2}\right)$$
$$+\frac{d}{dx}\left(\frac{h_1}{h_2h_3}\right)\left(\frac{d\rho_2}{dz}\frac{d\rho_3}{dx}-\frac{d\rho_3}{dz}\frac{d\rho_2}{dx}\right).$$

Re-arranging, and introducing terms for the sake of symmetry, we get

$$\frac{h_1}{h_2h_3}\left(\frac{d\rho_2}{dz}\nabla\rho_3-\frac{d\rho_3}{dz}\nabla\rho_2\right)$$
$$+\frac{h_1}{h_2h_3}\left\{\frac{d\rho_3}{dx}\frac{d}{dx}\left(\frac{d\rho_2}{dz}\right)+\frac{d\rho_3}{dy}\frac{d}{dy}\left(\frac{d\rho_2}{dz}\right)+\frac{d\rho_3}{dz}\frac{d}{dz}\left(\frac{d\rho_2}{dz}\right)\right\}$$
$$-\frac{h_1}{h_2h_3}\left\{\frac{d\rho_2}{dx}\frac{d}{dx}\left(\frac{d\rho_3}{dz}\right)+\frac{d\rho_2}{dy}\frac{d}{dy}\left(\frac{d\rho_3}{dz}\right)+\frac{d\rho_2}{dz}\frac{d}{dz}\left(\frac{d\rho_3}{dz}\right)\right\}$$
$$+\frac{d\rho_2}{dz}\left\{\frac{d\rho_3}{dx}\frac{d}{dx}\left(\frac{h_1}{h_2h_3}\right)+\frac{d\rho_3}{dy}\frac{d}{dy}\left(\frac{h_1}{h_2h_3}\right)+\frac{d\rho_3}{dz}\frac{d}{dz}\left(\frac{h_1}{h_2h_3}\right)\right\}$$
$$-\frac{d\rho_3}{dz}\left\{\frac{d\rho_2}{dx}\frac{d}{dx}\left(\frac{h_1}{h_2h_3}\right)+\frac{d\rho_2}{dy}\frac{d}{dy}\left(\frac{h_1}{h_2h_3}\right)+\frac{d\rho_2}{dz}\frac{d}{dz}\left(\frac{h_1}{h_2h_3}\right)\right\}$$
$$=0\ldots\ldots(3).$$

Now the expression within brackets in the second line, by equations (7) of Art. 291,

$$=h_3^2\left\{\frac{dx}{d\rho_3}\frac{d}{dx}\left(\frac{d\rho_2}{dz}\right)+\frac{dy}{d\rho_3}\frac{d}{dy}\left(\frac{d\rho_2}{dz}\right)+\frac{dz}{d\rho_3}\frac{d}{dz}\left(\frac{d\rho_2}{dz}\right)\right\}$$
$$=h_3^2\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dz}\right)\ldots\ldots\ldots\ldots(4).$$

A similar transformation can be effected of the expressions within brackets in the remaining three lines of (3); and thus (3) becomes

$$H_1\left(\frac{d\rho_2}{dz}\nabla\rho_3-\frac{d\rho_3}{dz}\nabla\rho_2\right)$$
$$+H_1\left\{h_3^2\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dz}\right)-h_2^2\frac{d}{d\rho_2}\left(\frac{d\rho_3}{dz}\right)\right\}$$
$$+\frac{d\rho_2}{dz}h_3^2\frac{dH_1}{d\rho_3}-\frac{d\rho_3}{dz}h_2^2\frac{dH_1}{d\rho_2}=0\text{............}(5),$$

where H_1 stands for $\frac{h_1}{h_2h_3}$.

Divide by H_1, then we obtain

$$\frac{d\rho_2}{dz}\nabla\rho_3-\frac{d\rho_3}{dz}\nabla\rho_2$$
$$+h_3^2\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dz}\right)-h_2^2\frac{d}{d\rho_2}\left(\frac{d\rho_3}{dz}\right)$$
$$+\frac{d\rho_2}{dz}h_3^2\frac{d\log H_1}{d\rho_3}-\frac{d\rho_3}{dz}h_2^2\frac{d\log H_1}{d\rho_2}=0\text{......}(6).$$

Similarly we have

$$\frac{d\rho_2}{dx}\nabla\rho_3-\frac{d\rho_3}{dx}\nabla\rho_2$$
$$+h_3^2\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dx}\right)-h_2^2\frac{d}{d\rho_2}\left(\frac{d\rho_3}{dx}\right)$$
$$+\frac{d\rho_2}{dx}h_3^2\frac{d\log H_1}{d\rho_3}-\frac{d\rho_3}{dx}h_2^2\frac{d\log H_1}{d\rho_2}=0\text{......}(7);$$

and

$$\frac{d\rho_2}{dy}\nabla\rho_3-\frac{d\rho_3}{dy}\nabla\rho_2$$
$$+h_3^2\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dy}\right)-h_2^2\frac{d}{d\rho_2}\left(\frac{d\rho_3}{dy}\right)$$
$$+\frac{d\rho_2}{dy}h_3^2\frac{d\log H_1}{d\rho_3}-\frac{d\rho_3}{dy}h_2^2\frac{d\log H_1}{d\rho_2}=0\text{......}(8).$$

Multiply (6) by $\frac{d\rho_2}{dz}$, (7) by $\frac{d\rho_2}{dx}$, and (8) by $\frac{d\rho_2}{dy}$, and add; then by virtue of the relations alluded to in Art. 287, we obtain

$$h_2^2\nabla\rho_3 + h_3^2\left\{\frac{d\rho_2}{dz}\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dz}\right) + \frac{d\rho_2}{dx}\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dx}\right) + \frac{d\rho_2}{dy}\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dy}\right)\right\}$$

$$-h_3^2\left\{\frac{d\rho_2}{dz}\frac{d}{d\rho_2}\left(\frac{d\rho_3}{dz}\right) + \frac{d\rho_2}{dx}\frac{d}{d\rho_2}\left(\frac{d\rho_3}{dx}\right) + \frac{d\rho_2}{dy}\frac{d}{d\rho_2}\left(\frac{d\rho_3}{dy}\right)\right\}$$

$$+h_2^2h_3^2\frac{d\log H_1}{d\rho_3} = 0 \text{ ....(9).}$$

This may be simplified; for we have

$$\frac{d\rho_2}{dz}\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dz}\right) + \frac{d\rho_2}{dx}\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dx}\right) + \frac{d\rho_2}{dy}\frac{d}{d\rho_3}\left(\frac{d\rho_2}{dy}\right)$$

$$= \frac{1}{2}\frac{d}{d\rho_3}\left\{\left(\frac{d\rho_2}{dz}\right)^2 + \left(\frac{d\rho_2}{dx}\right)^2 + \left(\frac{d\rho_2}{dy}\right)^2\right\} = \frac{1}{2}\frac{d}{d\rho_3}h_2^2 = h_2\frac{dh_2}{d\rho_3};$$

and $\frac{d\rho_2}{dz}\frac{d}{d\rho_2}\left(\frac{d\rho_3}{dz}\right) + \frac{d\rho_2}{dx}\frac{d}{d\rho_2}\left(\frac{d\rho_3}{dx}\right) + \frac{d\rho_2}{dy}\frac{d}{d\rho_2}\left(\frac{d\rho_3}{dy}\right)$

$$= \frac{d\rho_2}{dz}\frac{d}{d\rho_2}\left(h_3^2\frac{dz}{d\rho_3}\right) + \frac{d\rho_2}{dx}\frac{d}{d\rho_2}\left(h_3^2\frac{dx}{d\rho_3}\right) + \frac{d\rho_2}{dy}\frac{d}{d\rho_2}\left(h_3^2\frac{dy}{d\rho_3}\right)$$

$$= h_3^2\left\{\frac{d\rho_2}{dz}\frac{d}{d\rho_2}\left(\frac{dz}{d\rho_3}\right) + \frac{d\rho_2}{dx}\frac{d}{d\rho_2}\left(\frac{dx}{d\rho_3}\right) + \frac{d\rho_2}{dy}\frac{d}{d\rho_2}\left(\frac{dy}{d\rho_3}\right)\right\}$$

$$= h_3^2\left\{\frac{d\rho_2}{dz}\frac{d}{d\rho_3}\left(\frac{dz}{d\rho_2}\right) + \frac{d\rho_2}{dx}\frac{d}{d\rho_3}\left(\frac{dx}{d\rho_2}\right) + \frac{d\rho_2}{dy}\frac{d}{d\rho_3}\left(\frac{dy}{d\rho_2}\right)\right\}$$

$$= h_3^2\left\{\frac{d\rho_2}{dz}\frac{d}{d\rho_3}\left(\frac{1}{h_2^2}\frac{d\rho_2}{dz}\right) + \frac{d\rho_2}{dx}\frac{d}{d\rho_3}\left(\frac{1}{h_2^2}\frac{d\rho_2}{dx}\right) + \frac{d\rho_2}{dy}\frac{d}{d\rho_3}\left(\frac{1}{h_2^2}\frac{d\rho_2}{dy}\right)\right\}$$

$$= h_3^2\left\{-\frac{2}{h_2^3}\frac{dh_2}{d\rho_3}h_2^2 + \frac{1}{h_2^2}h_2\frac{dh_2}{d\rho_3}\right\} = -\frac{h_3^2}{h_2}\frac{dh_2}{d\rho_3}.$$

Hence (9) becomes

$$h_2^2\nabla\rho_3 + 2h_3^2h_2\frac{dh_2}{d\rho_3} + h_2^2h_3^2\frac{d\log H_1}{d\rho_3} = 0;$$

therefore $$\frac{1}{h_3^2}\nabla\rho_3 + \frac{d}{d\rho_3}\log\frac{h_1h_2}{h_3} = 0 \text{ ....(10).}$$

In the same way we have

$$\frac{1}{h_1^2}\nabla\rho_1+\frac{d}{d\rho_1}\log\frac{h_2h_3}{h_1}=0 \text{ ....(11)};$$

and

$$\frac{1}{h_2^2}\nabla\rho_2+\frac{d}{d\rho_2}\log\frac{h_1h_3}{h_2}=0 \text{ ....(12)}.$$

By the aid of (10), (11), and (12) we obtain from (1)

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=$$

$$h_1^2\left\{\frac{d^2V}{d\rho_1^2}+\frac{dV}{d\rho_1}\frac{h_2h_3}{h_1}\frac{d}{d\rho_1}\left(\frac{h_1}{h_2h_3}\right)\right\}+h_2^2\left\{\frac{d^2V}{d\rho_2^2}+\frac{dV}{d\rho_2}\frac{h_3h_1}{h_2}\frac{d}{d\rho_2}\left(\frac{h_2}{h_3h_1}\right)\right\}$$

$$+h_3^2\left\{\frac{d^2V}{d\rho_3^2}+\frac{dV}{d\rho_3}\frac{h_1h_2}{h_3}\frac{d}{d\rho_3}\left(\frac{h_3}{h_1h_2}\right)\right\},$$

that is $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=$

$$h_1h_2h_3\left\{\frac{d}{d\rho_1}\left(\frac{h_1}{h_2h_3}\frac{dV}{d\rho_1}\right)+\frac{d}{d\rho_2}\left(\frac{h_2}{h_3h_1}\frac{dV}{d\rho_2}\right)+\frac{d}{d\rho_3}\left(\frac{h_3}{h_1h_2}\frac{dV}{d\rho_3}\right)\right\} \text{...(13)}.$$

296. Hence we see that the equation $\nabla V=0$ transforms into

$$\frac{d}{d\rho_1}\left(\frac{h_1}{h_2h_3}\frac{dV}{d\rho_1}\right)+\frac{d}{d\rho_2}\left(\frac{h_2}{h_3h_1}\frac{dV}{d\rho_2}\right)+\frac{d}{d\rho_3}\left(\frac{h_3}{h_1h_2}\frac{dV}{d\rho_3}\right)=0 \text{ ...(14)}.$$

As a particular case we may suppose that ρ_1, ρ_2, ρ_3 are respectively the λ, μ, ν of Art. 266; for the equations of that Article theoretically express each of the last three quantities as functions of x, y, and z.

By comparing Art. 274 with Art. 289 we have

$$h_1^2=\frac{(\lambda^2-b^2)(\lambda^2-c^2)}{(\lambda^2-\mu^2)(\lambda^2-\nu^2)},\ h_2^2=\frac{(\mu^2-b^2)(\mu^2-c^2)}{(\mu^2-\nu^2)(\mu^2-\lambda^2)},\ h_3^2=\frac{(\nu^2-b^2)(\nu^2-c^2)}{(\nu^2-\lambda^2)(\nu^2-\mu^2)},$$

and these may be substituted in (14).

But we may make another supposition which will give a still simpler form to (14). We may suppose that ρ_1 is any function we please of λ, that ρ_2 is any function we please of μ, and that ρ_3 is any function we please of ν. Let us put α, β, γ respectively for ρ_1, ρ_2, ρ_3, where

$$\alpha = c\int_c^\lambda \frac{d\lambda}{\sqrt{(\lambda^2-b^2)(\lambda^2-c^2)}}, \quad \beta = c\int_b^\mu \frac{d\mu}{\sqrt{(c^2-\mu^2)(\mu^2-b^2)}},$$

$$\gamma = c\int_0^\nu \frac{d\nu}{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}.$$

Let η_1, η_2, η_3 denote respectively what h_1, h_2, h_3 become when α, β, γ respectively are put for ρ_1, ρ_2, ρ_3.

From Art. 274 we now get

$$ds^2 = (\lambda^2-\mu^2)(\lambda^2-\nu^2)\frac{d\alpha^2}{c^2} + (\lambda^2-\mu^2)(\mu^2-\nu^2)\frac{d\beta^2}{c^2}$$
$$+ (\lambda^2-\nu^2)(\mu^2-\nu^2)\frac{d\gamma^2}{c^2}.$$

Thus

$$\eta_1^2 = \frac{c^2}{(\lambda^2-\mu^2)(\lambda^2-\nu^2)}, \quad \eta_2^2 = \frac{c^2}{(\lambda^2-\mu^2)(\mu^2-\nu^2)}, \quad \eta_3^2 = \frac{c^2}{(\lambda^2-\nu^2)(\mu^2-\nu^2)}.$$

Therefore

$$\frac{\eta_1}{\eta_2\eta_3} = \frac{\mu^2-\nu^2}{c}, \quad \frac{\eta_2}{\eta_3\eta_1} = \frac{\lambda^2-\nu^2}{c}, \quad \frac{\eta_3}{\eta_1\eta_2} = \frac{\lambda^2-\mu^2}{c}.$$

Hence (14) becomes

$$\frac{d}{d\alpha}\left(\frac{\mu^2-\nu^2}{c}\frac{dV}{d\alpha}\right) + \frac{d}{d\beta}\left(\frac{\lambda^2-\nu^2}{c}\frac{dV}{d\beta}\right) + \frac{d}{d\gamma}\left(\frac{\lambda^2-\mu^2}{c}\frac{dV}{d\gamma}\right) = 0,$$

that is

$$(\mu^2-\nu^2)\frac{d^2V}{d\alpha^2} + (\lambda^2-\nu^2)\frac{d^2V}{d\beta^2} + (\lambda^2-\mu^2)\frac{d^2V}{d\gamma^2} = 0 \quad \ldots(15).$$

297. We have obtained equation (13) by the direct processes of the Differential Calculus; we shall now however follow Jacobi in deducing the equation in another way, by the aid of the Calculus of Variations: see *History of the Calculus of Variations*, page 361.

298. Let V be any function of x, y, z; let F be any function of $V, \frac{dV}{dx}, \frac{dV}{dy}, \frac{dV}{dz}$; and for shortness put

$$\frac{dV}{dx}=p_1, \quad \frac{dV}{dy}=p_2, \quad \frac{dV}{dz}=p_3.$$

Consider the triple integral $\iiint F\,dx\,dy\,dz$, which may be supposed to be taken between fixed limits. Let the variables be changed to the ρ_1, ρ_2, ρ_3 of the preceding Chapter. By Art. 290 we have

$$\text{the element of volume } dx\,dy\,dz = \frac{1}{h_1h_2h_3}\,d\rho_1\,d\rho_2\,d\rho_3$$
$$= E\,d\rho_1\,d\rho_2\,d\rho_3 \text{ say.}$$

Hence $$\iiint F\,dx\,dy\,dz = \iiint EF\,d\rho_1\,d\rho_2\,d\rho_3 \quad \text{........(16).}$$

Let V receive a variation δV, then each side of (16) receives a variation which we will now express, beginning with the right-hand side. We have

$$\delta\iiint EF\,d\rho_1\,d\rho_2\,d\rho_3 = \iiint \delta(EF)\,d\rho_1\,d\rho_2\,d\rho_3.$$

For shortness put $\frac{dV}{d\rho_1}=\varpi_1$, $\frac{dV}{d\rho_2}=\varpi_2$, $\frac{dV}{d\rho_3}=\varpi_3$.

Then

$$\delta(EF) = \frac{d(EF)}{dV}\delta V + \frac{d(EF)}{d\varpi_1}\delta\varpi_1 + \frac{d(EF)}{d\varpi_2}\delta\varpi_2 + \frac{d(EF)}{d\varpi_3}\delta\varpi_3$$
$$= E\frac{dF}{dV}\delta V + E\frac{dF}{d\varpi_1}\delta\varpi_1 + E\frac{dF}{d\varpi_2}\delta\varpi_2 + E\frac{dF}{d\varpi_3}\delta\varpi_3.$$

Hence reducing $\iiint \delta(EF)\,d\rho_1\,d\rho_2\,d\rho_3$ in the usual way we find that it becomes $\iiint K\delta V\,d\rho_1\,d\rho_2\,d\rho_3$, where

$$K = E\frac{dF}{dV} - \frac{d}{d\rho_1}\left(E\frac{dF}{d\varpi_1}\right) - \frac{d}{d\rho_2}\left(E\frac{dF}{d\varpi_2}\right) - \frac{d}{d\rho_3}\left(E\frac{dF}{d\varpi_3}\right),$$

together with certain terms in the form of *double* integrals which depend on the limiting values of the variables.

In the same manner if we develope the variation of the left-hand member of (16) we find that it becomes

$$\iiint \left\{ \frac{dF}{dV} - \frac{d}{dx}\left(\frac{dF}{dp_1}\right) - \frac{d}{dy}\left(\frac{dF}{dp_2}\right) - \frac{d}{dz}\left(\frac{dF}{dp_3}\right) \right\} \delta V\, dx\, dy\, dz,$$

together with certain terms in the form of *double* integrals which depend on the limiting values of the variables.

The terms which are in the form of *triple* integrals must agree; and therefore putting $E\, d\rho_1\, d\rho_2\, d\rho_3$ for $dx\, dy\, dz$ in the second we obtain

$$E\left\{ \frac{dF}{dV} - \frac{d}{dx}\left(\frac{dF}{dp_1}\right) - \frac{d}{dy}\left(\frac{dF}{dp_2}\right) - \frac{d}{dz}\left(\frac{dF}{dp_3}\right) \right\}$$

$$= E\frac{dF}{dV} - \frac{d}{d\rho_1}\left(E\frac{dF}{d\varpi_1}\right) - \frac{d}{d\rho_2}\left(E\frac{dF}{d\varpi_2}\right) - \frac{d}{d\rho_3}\left(E\frac{dF}{d\varpi_3}\right);$$

so that
$$\frac{d}{dx}\left(\frac{dF}{dp_1}\right) + \frac{d}{dy}\left(\frac{dF}{dp_2}\right) + \frac{d}{dz}\left(\frac{dF}{dp_3}\right)$$

$$= \frac{1}{E}\left\{ \frac{d}{d\rho_1}\left(E\frac{dF}{d\varpi_1}\right) + \frac{d}{d\rho_2}\left(E\frac{dF}{d\varpi_2}\right) + \frac{d}{d\rho_3}\left(E\frac{dF}{d\varpi_3}\right) \right\} \quad \dots(17).$$

299. As a particular case of the preceding general result suppose we put $\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2$ for F on the left-hand side; then, transferring to the new variables, we see by Art. 287 that we must put $h_1^2\left(\frac{dV}{d\rho_1}\right)^2 + h_2^2\left(\frac{dV}{d\rho_2}\right)^2 + h_3^2\left(\frac{dV}{d\rho_3}\right)^2$ for F on the right-hand side. Hence on the left-hand side $\frac{dF}{dp_1} = 2\frac{dV}{dx}$, and so on; and on the right-hand side $\frac{dF}{d\varpi_1} = 2h_1^2\frac{dV}{d\rho_1}$, and so on. Thus (17) becomes

$$\nabla V = h_1 h_2 h_3 \left\{ \frac{d}{d\rho_1}\left(\frac{h_1}{h_2 h_3}\frac{dV}{d\rho_1}\right) + \frac{d}{d\rho_2}\left(\frac{h_2}{h_3 h_1}\frac{dV}{d\rho_2}\right) + \frac{d}{d\rho_3}\left(\frac{h_3}{h_1 h_2}\frac{dV}{d\rho_3}\right) \right\},$$

which agrees with (13).

300. Another very instructive method of establishing equation (13) is given by Sidler in the treatise mentioned in Art. 4; and is apparently ascribed by him to Dirichlet.

Let V be any function of x, y, z, which together with its first and second differential coefficients with respect to the variables remains finite throughout the space bounded by a given closed surface; then will

$$\iiint \nabla V dx dy dz = -\int \frac{dV}{dn} dS \qquad (18);$$

where the integral on the left-hand side is extended throughout the space, and that on the right-hand side over the whole surface: dS is an element of the surface, and dn an element of the normal to the surface drawn inwards at dS.

The theorem is well known; it may be obtained as a particular case of *Green's Theorem:* see *Statics*, Chapter XV., putting unity for U in the general investigation there given.

Now conceive an infinitesimal element of volume bounded by the three surfaces of Art. 266, and by the three surfaces obtained by changing ρ_1, ρ_2, ρ_3 into $\rho_1 + d\rho_1, \rho_2 + d\rho_2, \rho_3 + d\rho_3$ respectively. To this six-faced element we propose to apply equation (18).

As we have already seen the element of volume $dxdydz$ becomes $\frac{1}{h_1 h_2 h_3} d\rho_1 d\rho_2 d\rho_3$ when expressed in terms of the new variables; hence the left-hand side of (18) becomes $W \frac{1}{h_1 h_2 h_3} d\rho_1 d\rho_2 d\rho_3$, where W is what ∇V becomes when expressed in terms of the new variables.

Now consider the right-hand member of (18). Take first the face which lies on the surface ρ_1. Here $dS = \frac{1}{h_2 h_3} d\rho_2 d\rho_3$, and $dn = \frac{1}{h_1} d\rho_1$; so that $\frac{dV}{dn} dS$ becomes $\frac{h_1}{h_2 h_3} \frac{dV}{d\rho_1} d\rho_2 d\rho_3$. The corresponding value for the opposite face would be found *numerically* from this by changing ρ_1 into $\rho_1 + d\rho_1$; but the sign must be changed because the formula (18) supposes dn always measured *inwards;* hence this value is

$$-\left\{\frac{h_1}{h_2h_3}\frac{dV}{d\rho_1}+\frac{d}{d\rho_1}\left(\frac{h_1}{h_2h_3}\frac{dV}{d\rho_1}\right)d\rho_1\right\}d\rho_2d\rho_3.$$

Hence the balance contributed by these two faces to the right-hand side of (18) is $-\frac{d}{d\rho_1}\left(\frac{h_1}{h_2h_3}\frac{dV}{d\rho_1}\right)d\rho_1d\rho_2d\rho_3$.

Similar expressions arise from the other two pairs of faces of the element considered; and the aggregate is to be put equal to the expression already found for the left-hand side of (18). Therefore

$$W\frac{1}{h_1h_2h_3}d\rho_1d\rho_2d\rho_3$$

$$=\left\{\frac{d}{d\rho_1}\left(\frac{h_1}{h_2h_3}\frac{dV}{d\rho_1}\right)+\frac{d}{d\rho_2}\left(\frac{h_2}{h_3h_1}\frac{dV}{d\rho_2}\right)+\frac{d}{d\rho_3}\left(\frac{h_3}{h_1h_2}\frac{dV}{d\rho_3}\right)\right\}d\rho_1d\rho_2d\rho_3.$$

Then by simplifying we get for W the form already obtained in equation (13).

CHAPTER XXIV.

TRANSFORMATION OF LAPLACE'S SECONDARY EQUATION.

301. WE shall find it useful to transform into Lamé's variables the equation satisfied by Laplace's n^{th} coefficient; this equation we may call Laplace's secondary equation, to distinguish it from that considered in the preceding Chapter.

302. Denoting the n^{th} coefficient by Y, we have by Art. 167 the following equation expressed in terms of the usual variables,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{dY}{d\theta}\sin\theta\right)+\frac{1}{\sin^2\theta}\frac{d^2Y}{d\phi^2}+n(n+1)Y=0\text{......(1)}.$$

Now the following is a very common system of relations connecting polar co-ordinates with rectangular,

$$x=r\cos\theta,\quad y=r\sin\theta\cos\phi,\quad z=r\sin\theta\sin\phi;$$

and by comparing these with Art. 280 the following relations are suggested:

$$\cos\theta=\frac{\mu\nu}{bc},\quad \sin\theta\cos\phi=\frac{\sqrt{(\mu^2-b^2)(b^2-\nu^2)}}{b\sqrt{(c^2-b^2)}},$$

$$\sin\theta\sin\phi=\frac{\sqrt{(c^2-\mu^2)(c^2-\nu^2)}}{c\sqrt{(c^2-b^2)}}\text{.........(2)}.$$

We propose then to transform (1) by the aid of (2); and we shall also introduce auxiliary variables β and γ, which are connected with μ and ν respectively by the equations

$$\beta=c\int_b^\mu\frac{d\mu}{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}},\quad \gamma=c\int_0^\nu\frac{d\nu}{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}\text{...(3)}.$$

We shall shew that the transformed equation is

$$\frac{d^2Y}{d\beta^2}+\frac{d^2Y}{d\gamma^2}+\frac{n(n+1)}{c^2}(\mu^2-\nu^2)\,Y=0 \;\ldots\ldots\ldots\ldots(4).$$

303. From equations (2) we have $\cos\theta$ and $\tan\phi$ expressed as functions of μ and ν. These give

$$\frac{d\theta}{d\mu}=-\frac{\nu}{bc\sin\theta},\quad \frac{d\theta}{d\nu}=-\frac{\mu}{bc\sin\theta};$$

$$\frac{d\phi}{d\mu}=\frac{bc\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}\cdot\frac{\mu}{\mu^2\nu^2-b^2c^2},$$

$$\frac{d\phi}{d\nu}=-\frac{bc\sqrt{(\mu^2-b^2)(c^2-\mu^2)}}{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}\cdot\frac{\nu}{\mu^2\nu^2-b^2c^2};$$

therefore

$$\left.\begin{aligned}
\frac{d\theta}{d\beta}&=-\frac{\nu}{bc^2\sin\theta}\sqrt{(\mu^2-b^2)(c^2-\mu^2)},\\
\frac{d\theta}{d\gamma}&=-\frac{\mu}{bc^2\sin\theta}\sqrt{(b^2-\nu^2)(c^2-\nu^2)},\\
\frac{d\phi}{d\beta}&=\frac{b\mu}{\mu^2\nu^2-b^2c^2}\sqrt{(b^2-\nu^2)(c^2-\nu^2)},\\
\frac{d\phi}{d\gamma}&=-\frac{b\nu}{\mu^2\nu^2-b^2c^2}\sqrt{(\mu^2-b^2)(c^2-\mu^2)},
\end{aligned}\right\}\;\ldots\ldots\ldots\ldots(5).$$

Let us now suppose that $\frac{d^2Y}{d\theta^2}+\cot\theta\frac{dY}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2Y}{d\phi^2}$ transforms into $A_0\frac{d^2Y}{d\beta^2}+A_1\frac{dY}{d\beta}+B_0\frac{d^2Y}{d\gamma^2}+B_1\frac{dY}{d\gamma}+C_0\frac{d^2Y}{d\beta\,d\gamma}$; we want to find A_0, A_1, B_0, B_1 and C_0.

We have $\frac{dY}{d\beta}=\frac{dY}{d\theta}\frac{d\theta}{d\beta}+\frac{dY}{d\phi}\frac{d\phi}{d\beta}$; therefore

$$\frac{d^2Y}{d\beta^2}=\frac{d^2Y}{d\theta^2}\left(\frac{d\theta}{d\beta}\right)^2+\frac{d^2Y}{d\phi^2}\left(\frac{d\phi}{d\beta}\right)^2+2\frac{d^2Y}{d\theta\,d\phi}\frac{d\theta}{d\beta}\frac{d\phi}{d\beta}$$
$$+\frac{dY}{d\theta}\frac{d^2\theta}{d\beta^2}+\frac{dY}{d\phi}\frac{d^2\phi}{d\beta^2};$$

similarly $\frac{dY}{d\gamma}$ and $\frac{d^2Y}{d\gamma^2}$ may be expressed; and

$$\frac{d^2Y}{d\beta d\gamma}=\frac{d^2Y}{d\theta^2}\frac{d\theta}{d\beta}\frac{d\theta}{d\gamma}+\frac{dY}{d\theta}\frac{d^2\theta}{d\beta d\gamma}+\frac{d^2Y}{d\phi^2}\frac{d\phi}{d\beta}\frac{d\phi}{d\gamma}+\frac{dY}{d\phi}\frac{d^2\phi}{d\beta d\gamma}$$
$$+\frac{d^2Y}{d\theta d\phi}\left(\frac{d\theta}{d\beta}\frac{d\phi}{d\gamma}+\frac{d\theta}{d\gamma}\frac{d\phi}{d\beta}\right).$$

Substitute these values in

$$A_0\frac{d^2Y}{d\beta^2}+A_1\frac{dY}{d\beta}+B_0\frac{d^2Y}{d\gamma^2}+B_1\frac{dY}{d\gamma}+C_0\frac{d^2Y}{d\beta d\gamma},$$

and compare the terms with those in the original expression; thus we obtain

$$A_0\left(\frac{d\theta}{d\beta}\right)^2+B_0\left(\frac{d\theta}{d\gamma}\right)^2+C_0\frac{d\theta}{d\beta}\frac{d\theta}{d\gamma}=1\text{.....................}(6);$$

$$A_0\left(\frac{d\phi}{d\beta}\right)^2+B_0\left(\frac{d\phi}{d\gamma}\right)^2+C_0\frac{d\phi}{d\beta}\frac{d\phi}{d\gamma}=\frac{1}{\sin^2\theta}\text{ .... }(7);$$

$$2A_0\frac{d\theta}{d\beta}\frac{d\phi}{d\beta}+2B_0\frac{d\theta}{d\gamma}\frac{d\phi}{d\gamma}+C_0\left(\frac{d\theta}{d\beta}\frac{d\phi}{d\gamma}+\frac{d\theta}{d\gamma}\frac{d\phi}{d\beta}\right)=0\text{...}(8).$$

$$A_1\frac{d\theta}{d\beta}+B_1\frac{d\theta}{d\gamma}+A_0\frac{d^2\theta}{d\beta^2}+B_0\frac{d^2\theta}{d\gamma^2}+C_0\frac{d^2\theta}{d\beta d\gamma}=\cot\theta\text{...}(9).$$

$$A_1\frac{d\phi}{d\beta}+B_1\frac{d\phi}{d\gamma}+A_0\frac{d^2\phi}{d\beta^2}+B_0\frac{d^2\phi}{d\gamma^2}+C_0\frac{d^2\phi}{d\beta d\gamma}=0\text{...}(10).$$

Now equations (5) give

$$\frac{d\phi}{d\beta}=\frac{1}{\sin\theta}\frac{d\theta}{d\gamma}\text{.................................}(11);$$

$$\frac{d\phi}{d\gamma}=-\frac{1}{\sin\theta}\frac{d\theta}{d\beta}\text{ .... }(12).$$

Multiply (7) by $\sin^2\theta$, and subtract from (6); then by (11) and (12) we have

$$(A_0-B_0)\left\{\left(\frac{d\theta}{d\beta}\right)^2-\left(\frac{d\theta}{d\gamma}\right)^2\right\}+2C_0\frac{d\theta}{d\beta}\frac{d\theta}{d\gamma}=0,$$

Also (8) may be written

$$2(A_0-B_0)\frac{d\theta}{d\beta}\frac{d\theta}{d\gamma}-C_0\left\{\left(\frac{d\theta}{d\beta}\right)^2-\left(\frac{d\theta}{d\gamma}\right)^2\right\}=0.$$

The last two equations give

$$A_0-B_0=0, \qquad C_0=0.$$

From (6) we have now $A_0\left\{\left(\frac{d\theta}{d\beta}\right)^2+\left(\frac{d\theta}{d\gamma}\right)^2\right\}=1$, so that

$$A_0\frac{\nu^2(\mu^2-b^2)(c^2-\mu^2)+\mu^2(b^2-\nu^2)(c^2-\nu^2)}{b^2c^4\sin^2\theta}=1,$$

or
$$\frac{A_0(\mu^2-\nu^2)(b^2c^2-\mu^2\nu^2)}{c^2(b^2c^2-\mu^2\nu^2)}=1;$$

therefore A_0 and B_0 each equal $\frac{c^2}{\mu^2-\nu^2}$.

We have still to find A_1 and B_1; the equations for this purpose are (9) and (10).

Now from (5) we get

$$c^2\sin\theta\frac{d^2\theta}{d\beta^2}=-\cos\theta(b^2+c^2-2\mu^2)-\frac{\nu^2\cos\theta}{b^2c^2\sin^2\theta}(\mu^2-b^2)(c^2-\mu^2),$$

$$c^2\sin\theta\frac{d^2\theta}{d\gamma^2}=\quad\cos\theta(b^2+c^2-2\nu^2)-\frac{\mu^2\cos\theta}{b^2c^2\sin^2\theta}(b^2-\nu^2)(c^2-\nu^2);$$

therefore
$$c^2\sin\theta\left(\frac{d^2\theta}{d\beta^2}+\frac{d^2\theta}{d\gamma^2}\right)=(\mu^2-\nu^2)\cos\theta.$$

And from (11) and (12) we have $\frac{d^2\phi}{d\beta^2}+\frac{d^2\phi}{d\gamma^2}=0$.

Hence equations (9) and (10) become

$$A_1\frac{d\theta}{d\beta}+B_1\frac{d\theta}{d\gamma}=0, \qquad A_1\frac{d\phi}{d\beta}+B_1\frac{d\phi}{d\gamma}=0;$$

therefore $A_1=0$ and $B_1=0$.

Thus the truth of (4) is established.

Another investigation of this process of transformation will be found in Liouville's *Journal de Mathématiques* for 1846, pages 458...461.

304. Another transformation of the equation (1) deserves notice. We will express the equation in terms of σ and τ, where

$$\cos\theta = \sigma, \qquad \sin\theta\sin\phi = \tau \quad \text{.............. (13).}$$

From (13) we have

$$\frac{d\sigma}{d\theta} = -\sin\theta,\ \frac{d^2\sigma}{d\theta^2} = -\cos\theta,\ \frac{d\tau}{d\theta} = \cos\theta\sin\phi,\ \frac{d^2\tau}{d\theta^2} = -\sin\theta\sin\phi,$$

$$\frac{d\sigma}{d\phi} = 0,\ \frac{d\tau}{d\phi} = \sin\theta\cos\phi,\ \frac{d^2\tau}{d\phi^2} = -\sin\theta\sin\phi,\ \frac{d^2\tau}{d\theta\, d\phi} = \cos\theta\cos\phi.$$

Now $\dfrac{dY}{d\theta} = \dfrac{dY}{d\sigma}\dfrac{d\sigma}{d\theta} + \dfrac{dY}{d\tau}\dfrac{d\tau}{d\theta} = -\dfrac{dY}{d\sigma}\sin\theta + \dfrac{dY}{d\tau}\cos\theta\sin\phi,$

$$\frac{d^2Y}{d\theta^2} = \frac{d^2Y}{d\sigma^2}\left(\frac{d\sigma}{d\theta}\right)^2 + \frac{d^2Y}{d\tau^2}\left(\frac{d\tau}{d\theta}\right)^2 + \frac{dY}{d\sigma}\frac{d^2\sigma}{d\theta^2} + \frac{dY}{d\tau}\frac{d^2\tau}{d\theta^2} + 2\frac{d^2Y}{d\sigma\, d\tau}\frac{d\sigma}{d\theta}\frac{d\tau}{d\theta}$$

$$= \frac{d^2Y}{d\sigma^2}\sin^2\theta + \frac{d^2Y}{d\tau^2}\cos^2\theta\sin^2\phi - \frac{dY}{d\sigma}\cos\theta - \frac{dY}{d\tau}\sin\theta\sin\phi - 2\frac{d^2Y}{d\sigma\, d\tau}\sin\theta\cos\theta\sin\phi;$$

$$\frac{d^2Y}{d\phi^2} = \frac{d^2Y}{d\tau^2}\left(\frac{d\tau}{d\phi}\right)^2 + \frac{dY}{d\tau}\frac{d^2\tau}{d\phi^2} = \frac{d^2Y}{d\tau^2}\sin^2\theta\cos^2\phi - \frac{dY}{d\tau}\sin\theta\sin\phi.$$

Thus (1) becomes

$$\frac{d^2Y}{d\sigma^2}\sin^2\theta + \frac{d^2Y}{d\tau^2}\cos^2\theta\sin^2\phi - \frac{dY}{d\sigma}\cos\theta - \frac{dY}{d\tau}\sin\theta\sin\phi$$
$$-2\frac{d^2Y}{d\sigma\, d\tau}\sin\theta\cos\theta\sin\phi + \cot\theta\left(-\frac{dY}{d\sigma}\sin\theta + \frac{dY}{d\tau}\cos\theta\sin\phi\right)$$
$$+\frac{d^2Y}{d\tau^2}\cos^2\phi - \frac{dY}{d\tau}\frac{\sin\phi}{\sin\theta} + n(n+1)Y = 0;$$

that is $\frac{d^2Y}{d\sigma^2}\sin^2\theta + \frac{d^2Y}{d\tau^2}(\cos^2\phi + \cos^2\theta\sin^2\phi) - 2\frac{dY}{d\sigma}\cos\theta$

$$-2\frac{dY}{d\tau}\sin\theta\sin\phi - 2\frac{d^2Y}{d\sigma d\tau}\sin\theta\cos\theta\sin\phi + n(n+1)Y = 0,$$

that is $(1-\sigma^2)\frac{d^2Y}{d\sigma^2} + (1-\tau^2)\frac{d^2Y}{d\tau^2} - 2\sigma\tau\frac{d^2Y}{d\sigma d\tau}$

$$-2\sigma\frac{dY}{d\sigma} - 2\tau\frac{dY}{d\tau} + n(n+1)Y = 0 \text{ ...... (14).}$$

305. If now we transform (14) by putting

$$\cos\chi = \sigma, \quad \sin\chi\sin\psi = \tau,$$

we shall obtain an equation like the original with χ instead of θ, and ψ instead of ϕ. But as (14) is symmetrical with respect to σ and τ we shall obtain precisely the same result if we put

$$\cos\chi = \tau, \quad \sin\chi\sin\psi = \sigma.$$

Hence we arrive at the following conclusion: if we transform (1) by supposing

$$\cos\theta = \sin\chi\sin\psi, \quad \sin\theta\sin\phi = \cos\chi \text{ .... (15),}$$

we shall obtain an equation like (1) with χ instead of θ, and ψ instead of ϕ.

From (15) it follows that

$$\sin\theta\cos\phi = \sin\chi\cos\psi.$$

306. From the two preceding Articles we may now draw the following inference: we shall obtain the same result from equation (1) if instead of equations (2) we take any other mode of associating the old and new variables differing from (2) merely in order of arrangement. For instance, instead of (2) we might put

$$\frac{\mu\nu}{bc} = \sin\theta\sin\phi, \quad \frac{\sqrt{(\mu^2-b^2)(b^2-\nu^2)}}{b\sqrt{(c^2-b^2)}} = \sin\theta\cos\phi,$$

$$\frac{\sqrt{(c^2-\mu^2)(c^2-\nu^2)}}{c\sqrt{(c^2-b^2)}} = \cos\theta,$$

CHAPTER XXV.

PHYSICAL APPLICATIONS.

307. ALTHOUGH in the present work we are concerned with pure mathematics, yet it must be remembered that much of the value of the formulæ which are obtained depends upon their application to physics. As we have stated in the beginning, the researches of mathematicians in the theories of the Figure of the Earth and of Attraction first introduced the functions with which we have been occupied. The investigations of Lamé, which we are now more especially considering, were connected mainly with the theory of the propagation of heat, and accordingly we propose to devote a few pages to this subject in order to increase the interest of the subsequent Chapters.

We shall however treat the matter very briefly, as our object is rather to shew the meaning of the symbols employed than to furnish very elaborate demonstration. The reader will see that some of the processes resemble one with which he is probably familiar in the modern treatment of the Equation of Continuity in Hydrostatics.

308. Suppose a homogeneous solid bounded by two parallel planes; let c denote the thickness of the solid. Suppose one face of the solid maintained at the fixed temperature a, and the other at the fixed lower temperature b. Suppose a plane section parallel to the faces, and on this section take an area S. The solid being supposed in a state of equilibrium of temperature there will be a constant transmission of heat from the face which has the higher temperature to that which has the lower.

We take it as a result verified by experiment that the quantity of heat which passes through the area S in a time t is expressed by $St\kappa\frac{a-b}{c}$, where κ is a constant depending on the nature of the substance. If c is the unit of length, S the unit of area, t the unit of time, and $a-b$ the unit of temperature, the expression reduces to κ; and we have thus a definition of what is meant by the *conductivity* of the given substance.

309. *To form the equation for determining the variable state of temperature of a homogeneous body.*

Conceive an elementary rectangular parallelepiped having one corner at the point (x, y, z) and its edges parallel to the coordinate axes: denote the lengths of these edges by δx, δy, and δz respectively.

Let v be the temperature at (x, y, z): then the quantity of heat which passes through the face $\delta y\,\delta z$ into the parallelepiped during the infinitesimal time δt is by the preceding Article ultimately $\kappa\,\delta y\,\delta z\,\frac{v-(v+\delta v)}{\delta x}\,\delta t$, that is $-\kappa\,\delta y\,\delta z\,\frac{dv}{dx}\,\delta t$, where κ is the constant which measures the conductivity of the substance. The quantity which passes out of the parallelepiped during the same time, through the opposite face, will therefore ultimately be

$$-\kappa\,\delta y\,\delta z\,\delta t\left\{\frac{dv}{dx}+\frac{d}{dx}\left(\frac{dv}{dx}\right)\delta x\right\}.$$

Thus the augmentation of heat is $\kappa\,\delta x\,\delta y\,\delta z\,\delta t\,\frac{d^2v}{dx^2}$.

Similarly we may proceed with respect to each of the other pairs of opposite faces. Thus on the whole the augmentation of heat is $\kappa\,\delta x\,\delta y\,\delta z\,\delta t\left\{\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right\}$.

Now let τ be the specific heat, σ the density of the body: then the mass of the element is $\sigma\,\delta x\,\delta y\,\delta z$; and the quantity of heat acquired by the element in the time δt is

$$\sigma\tau\,\frac{dv}{dt}\,\delta t\,\delta x\,\delta y\,\delta z.$$

Thus finally

$$\frac{dv}{dt} = \frac{\kappa}{\sigma\tau}\left\{\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right\} \text{...............(1).}$$

310. If the body is in a state of equilibrium as to temperature, then $\frac{dv}{dt} = 0$, and we obtain

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0 \text{ ....(2).}$$

This important equation coincides with that which we obtain in treating the theory of the *Potential Function*, and which we have already noticed in Art. 167.

311. Besides the general equation (1) or (2) we may have to satisfy special conditions relative to the surface of the body considered.

Thus, for example, the surface of the body may be maintained at a temperature which at any time is an assigned function of the coordinates of that point; and then v must be so taken as to have the assigned value at the surface.

Or the space external to the body may be maintained at a given temperature, say *zero*. Then let δS denote an element of the surface of the body, δn an element of the normal to δS measured outwards. Let η denote the conductivity of the surface of the body. Then the amount of heat which passes through the area δS outwards in an element of time δt is measured by $-\kappa \frac{dv}{dn} \delta S \, \delta t$, and also by $\eta v \, \delta S \, \delta t$. Thus

$$\eta v = -\kappa \frac{dv}{dn} \text{ ....(3).}$$

Equation (3) may be developed. We have

$$\frac{dv}{dn} = \frac{dv}{dx}\frac{dx}{dn} + \frac{dv}{dy}\frac{dy}{dn} + \frac{dv}{dz}\frac{dz}{dn}.$$

Now $\frac{dx}{dn}$ = the cosine of the angle between the normal to the surface at (x, y, z) and the axis of x; so that if $u = 0$ be the equation to the surface, we have

$$\frac{dx}{dn} = \frac{du}{dx}\left\{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2\right\}^{-\frac{1}{2}}.$$

Similar expressions hold for $\frac{dy}{dn}$ and $\frac{dz}{dn}$.

312. The equations (1) and (3) of Arts. 309 and 311 take other forms in special cases, as we will shew in the next two Articles.

313. Suppose we have a right circular cylinder in which the temperature remains unchanged as long as we keep to a straight line parallel to the axis. Take the axis of the cylinder for the axis of z; then $\frac{d^2v}{dz^2}$ is zero, and the equation (1) becomes

$$\frac{dv}{dt} = \frac{\kappa}{\sigma\tau}\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}\right) \quad \text{.....................(4).}$$

We may transform the variables x and y to the usual polar variables r and θ: and thus (4) becomes, if we assume that v is independent of θ,

$$\frac{dv}{dt} = \frac{\kappa}{\sigma\tau}\left(\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr}\right) \quad \text{.....................(5).}$$

The equation (3) will become

$$\eta v + \kappa\frac{dv}{dr} = 0 \quad \text{.........................(6).}$$

This is to hold at the curved *surface* of the cylinder where r has its greatest value.

314. Again let the body be a sphere, and suppose the origin of coordinates at its centre. Assume that v is a function of r the distance from the centre alone; then (1) becomes

$$\frac{dv}{dt} = \frac{\kappa}{\sigma\tau}\left(\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr}\right) \quad \text{..................(7).}$$

This may be written

$$\frac{d}{dt}(vr) = \frac{\kappa}{\sigma\tau}\frac{d^2(vr)}{dr^2}.$$

The equation (3) will become coincident with (6): it is to hold at the surface of the sphere, where r has its greatest value.

315. Let $f(x, y, z)$ denote any value of v which satisfies (2), and let c be any constant; then the surface determined by the equation

$$f(x, y, z) = c$$

is called an *isothermal* surface, being a surface every point of which has the same temperature under the circumstances of the problem.

The constant c is called the *thermometrical parameter* of the surface. If different values are ascribed in succession to c we obtain a *family of isothermal surfaces.*

316. Suppose that the equation $F(x, y, z, \lambda) = 0$ represents a family of isothermal surfaces, by varying the parameter λ. Suppose that two of the surfaces form the boundaries of a solid shell, and that these two surfaces are in contact with constant sources of heat; then the temperature v, and the geometrical parameter λ will have constant values in each individual surface of the family, and will vary from surface to surface. Thus these two quantities will be mutually related; or we may say that v will be a function of λ. Hence we shall be able to find the condition which must hold in order that an assigned family of surfaces may be *isothermal.*

For we have

$$\frac{dv}{dx} = \frac{dv}{d\lambda}\frac{d\lambda}{dx},$$

$$\frac{d^2v}{dx^2} = \frac{dv}{d\lambda}\frac{d^2\lambda}{dx^2} + \frac{d^2v}{d\lambda^2}\left(\frac{d\lambda}{dx}\right)^2;$$

and similar equations hold with respect to y and z.

Thus equation (2) becomes

$$\frac{dv}{d\lambda}\left\{\frac{d^2\lambda}{dx^2}+\frac{d^2\lambda}{dy^2}+\frac{d^2\lambda}{dz^2}\right\}+\frac{d^2v}{d\lambda^2}\left\{\left(\frac{d\lambda}{dx}\right)^2+\left(\frac{d\lambda}{dy}\right)^2+\left(\frac{d\lambda}{dz}\right)^2\right\}=0;$$

therefore
$$\frac{\dfrac{d^2\lambda}{dx^2}+\dfrac{d^2\lambda}{dy^2}+\dfrac{d^2\lambda}{dz^2}}{\left(\dfrac{d\lambda}{dx}\right)^2+\left(\dfrac{d\lambda}{dy}\right)^2+\left(\dfrac{d\lambda}{dz}\right)^2}=-\frac{\dfrac{d^2v}{d\lambda^2}}{\dfrac{dv}{d\lambda}} \quad\text{..........(8).}$$

But the right-hand member is a function of λ only, and therefore the *left-hand member must be a function of* λ *only*.

This is the *necessary* condition in order that a family of surfaces in which λ is the geometrical parameter may be isothermal.

It is also *sufficient;* for when it is satisfied we can determine v as a function of λ from (8), and v will satisfy (2).

317. We shall now investigate by the aid of Art. 316 whether the family of ellipsoids obtained by varying λ in the following equation is isothermal:

$$\frac{x^2}{\lambda^2}+\frac{y^2}{\lambda^2-b^2}+\frac{z^2}{\lambda^2-c^2}=1 \quad\text{................(9).}$$

We have, by differentiating with respect to x,

$$\lambda\frac{d\lambda}{dx}\left\{\frac{x^2}{\lambda^4}+\frac{y^2}{(\lambda^2-b^2)^2}+\frac{z^2}{(\lambda^2-c^2)^2}\right\}=\frac{x}{\lambda^2},$$

$$\text{say } \lambda\frac{d\lambda}{dx}H=\frac{x}{\lambda^2} \quad\text{..................(10);}$$

$$\left\{\frac{x^2}{\lambda^4}+\frac{y^2}{(\lambda^2-b^2)^2}+\frac{z^2}{(\lambda^2-c^2)^2}\right\}\left\{\lambda\frac{d^2\lambda}{dx^2}+\left(\frac{d\lambda}{dx}\right)^2\right\}$$
$$-4\lambda^2\left(\frac{d\lambda}{dx}\right)^2\left\{\frac{x^2}{\lambda^6}+\frac{y^2}{(\lambda^2-b^2)^3}+\frac{z^2}{(\lambda^2-c^2)^3}\right\}+\frac{4\lambda x}{\lambda^4}\frac{d\lambda}{dx}=\frac{1}{\lambda^2},$$

$$\text{say } H\left\{\lambda\frac{d^2\lambda}{dx^2}+\left(\frac{d\lambda}{dx}\right)^2\right\}-4\lambda^2\left(\frac{d\lambda}{dx}\right)^2G+\frac{4\lambda x}{\lambda^4}\frac{d\lambda}{dx}=\frac{1}{\lambda^2} \quad\text{...(11).}$$

Similar formulæ follow from (9) by differentiating with respect to y and z respectively.

Square (10) and the two corresponding equations, and add; thus

$$\lambda^2\left\{\left(\frac{d\lambda}{dx}\right)^2+\left(\frac{d\lambda}{dy}\right)^2+\left(\frac{d\lambda}{dz}\right)^2\right\}H=1 \text{ ....(12).}$$

Again; from (10) we have

$$\lambda\frac{x}{\lambda^4}\frac{d\lambda}{dx}H=\frac{x^2}{\lambda^6} \text{ ....(13).}$$

From this and the two corresponding equations we obtain by addition

$$\lambda\left\{\frac{x}{\lambda^4}\frac{d\lambda}{dx}+\frac{y}{(\lambda^2-b^2)^2}\frac{d\lambda}{dy}+\frac{z}{(\lambda^2-c^2)^2}\frac{d\lambda}{dz}\right\}H=G \text{...(14).}$$

From (11) and the two corresponding equations we obtain by addition and the aid of (12) and (14),

$$\lambda H\left\{\frac{d^2\lambda}{dx^2}+\frac{d^2\lambda}{dy^2}+\frac{d^2\lambda}{dz^2}\right\}=\frac{1}{\lambda^2-b^2}+\frac{1}{\lambda^2-c^2} \text{ ....(15).}$$

From (12) and (15) we have

$$\frac{\dfrac{d^2\lambda}{dx^2}+\dfrac{d^2\lambda}{dy^2}+\dfrac{d^2\lambda}{dz^2}}{\left(\dfrac{d\lambda}{dx}\right)^2+\left(\dfrac{d\lambda}{dy}\right)^2+\left(\dfrac{d\lambda}{dz}\right)^2}=\frac{\lambda}{\lambda^2-b^2}+\frac{\lambda}{\lambda^2-c^2} \text{ ....(16).}$$

The right-hand member is a function of λ only, and thus the condition of Art. 316 is satisfied: hence by varying λ in (9) we obtain a family of isothermal surfaces.

318. If v denote the temperature in the case of the preceding Article, we have by equations (8) and (16)

$$-\frac{\dfrac{d^2v}{d\lambda^2}}{\dfrac{dv}{d\lambda}}=\frac{\lambda}{\lambda^2-b^2}+\frac{\lambda}{\lambda^2-c^2}.$$

Hence, by integration,

$$\log \frac{dv}{d\lambda} = \text{constant} - \frac{1}{2}\log(\lambda^2 - b^2) - \frac{1}{2}\log(\lambda^2 - c^2);$$

therefore
$$v = k_1 \int \frac{d\lambda}{\sqrt{(\lambda^2 - b^2)(\lambda^2 - c^2)}},$$

where k_1 denotes a constant.

319. In the manner of Arts. 317 and 318 we may shew that a family of hyperboloids of one sheet represented by the second equation of Art. 266 is isothermal; and that the temperature v is determined by

$$v = k_2 \int \frac{d\mu}{\sqrt{(c^2 - \mu^2)(\mu^2 - b^2)}},$$

where k_2 denotes a constant.

Also a family of hyperboloids of two sheets represented by the third equation of Art. 266 is isothermal; and the temperature v is determined by

$$v = k_3 \int \frac{d\nu}{\sqrt{(b^2 - \nu^2)(c^2 - \nu^2)}},$$

where k_3 denotes a constant.

320. We will now obtain by a direct process the equation in polar coordinates which corresponds to (2); the result will agree with the well-known transformation of (2): see *Differential Calculus*, Art. 207.

Let r, θ, ϕ be the usual polar coordinates; then the known expression for an element of volume is $r^2 \sin\theta \, dr \, d\theta \, d\phi$.

Put ω_1 for $r d\theta dr$, ω_2 for $r\sin\theta dr d\phi$, ω_3 for $r^2 \sin\theta d\theta d\phi$.

Then $\omega_1, \omega_2, \omega_3$ denote ultimately the areas of the faces of the element of volume which meet at the point (r, θ, ϕ).

Let κ denote the conductivity of the body, v the temperature at the point (r, θ, ϕ).

The quantity of heat which passes through the face ω_1 into the element during the infinitesimal time δt is ultimately $-\kappa\omega_1 \dfrac{dv}{\sin\theta\, r\, d\phi}\,\delta t$, that is $-\dfrac{\kappa\, dr\, d\theta}{\sin\theta}\dfrac{dv}{d\phi}\,\delta t$. The quantity of heat which passes out of the element during the same time through the opposite face will therefore be ultimately $-\dfrac{\kappa\, dr\, d\theta}{\sin\theta}\left\{\dfrac{dv}{d\phi}+\dfrac{d^2v}{d\phi^2}\,d\phi\right\}\delta t$. Thus the augmentation of heat is $\dfrac{\kappa\, dr\, d\theta\, d\phi}{\sin\theta}\dfrac{d^2v}{d\phi^2}\,\delta t$.

Again the quantity of heat which passes through the face ω_2 into the element during the infinitesimal time δt is ultimately $-\kappa\omega_2\dfrac{dv}{r\,d\theta}\,\delta t$, that is $-\kappa\, dr\, d\phi \dfrac{dv}{d\theta}\sin\theta\,\delta t$. The quantity of heat which passes out of the element during the same time through the opposite face will therefore be ultimately $-\kappa\, dr\, d\phi\left\{\dfrac{dv}{d\theta}\sin\theta+\dfrac{d}{d\theta}\left(\dfrac{dv}{d\theta}\sin\theta\right)d\theta\right\}\delta t$. Thus the augmentation of heat is $\kappa\, dr\, d\theta\, d\phi\dfrac{d}{d\theta}\left(\dfrac{dv}{d\theta}\sin\theta\right)\delta t$.

Finally the quantity of heat which passes through the face ω_3 into the element during the infinitesimal time δt is ultimately $-\kappa\omega_3\dfrac{dv}{dr}\,\delta t$, that is $-\kappa r^2\sin\theta\, d\theta\, d\phi\dfrac{dv}{dr}\,\delta t$. The quantity of heat which passes out of the element during the same time through the opposite face will therefore be ultimately $-\kappa\sin\theta\, d\theta\, d\phi\left\{r^2\dfrac{dv}{dr}+\dfrac{d}{dr}\left(r^2\dfrac{dv}{dr}\right)dr\right\}\delta t$. Thus the augmentation of heat is $\kappa\sin\theta\, dr\, d\theta\, d\phi\dfrac{d}{dr}\left(r^2\dfrac{dv}{dr}\right)\delta t$.

Now the sum of the three augmentations must be zero since the temperature is supposed stationary: thus

$$\frac{1}{\sin^2\theta}\frac{d^2v}{d\phi^2}+\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{dv}{d\theta}\sin\theta\right)+\frac{d}{dr}\left(r^2\frac{dv}{dr}\right)=0.$$

321. We shall finally obtain the equation corresponding to (2) when Lamé's elliptical coordinates are employed: see Art. 272.

We suppose that we have a set of variables α, β, γ connected with λ, μ, ν respectively by the relations

$$\alpha = c\int_c^\lambda \frac{d\lambda}{\sqrt{(\lambda^2-b^2)(\lambda^2-c^2)}}, \quad \beta = c\int_b^\mu \frac{d\mu}{\sqrt{(c^2-\mu^2)(\mu^2-b^2)}},$$

$$\gamma = c\int_0^\nu \frac{d\nu}{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}.$$

It may be observed that these relations are not assumed arbitrarily, but are suggested by the process of Arts. 318 and 319. We may consider that these relations give, at least theoretically, α, β, γ in terms of λ, μ, ν respectively; and conversely that they give λ, μ, ν in terms of α, β, γ respectively.

322. Let ds_1 denote the length of the normal to S_1 intercepted between this surface and an adjacent surface of the same family; so that by Arts. 266 and 276 we have

$$ds_1^2 = \frac{d\lambda^2(\lambda^2-\mu^2)(\lambda^2-\nu^2)}{(\lambda^2-b^2)(\lambda^2-c^2)};$$

then from the value of α in Art. 321 we obtain

$$c\,ds_1 = d\alpha\sqrt{(\lambda^2-\mu^2)(\lambda^2-\nu^2)}.$$

Similarly let ds_2 denote the length of the normal to S_2 intercepted between this surface and an adjacent surface of the same family; and let ds_3 denote the length of the normal to S_3 intercepted between this surface and an adjacent surface of the same family. We shall have

$$c\,ds_2 = d\beta\sqrt{(\lambda^2-\mu^2)(\mu^2-\nu^2)},$$

$$c\,ds_3 = d\gamma\sqrt{(\lambda^2-\nu^2)(\mu^2-\nu^2)}.$$

The three normals are all supposed to be drawn from the point (λ, μ, ν).

Put ω_1 for $ds_2\,ds_3$, ω_2 for $ds_3\,ds_1$, ω_3 for $ds_1\,ds_2$.

Take an elementary solid which is ultimately a rectangular parallelepiped having ω_1, ω_2, and ω_3 for adjacent faces.

The quantity of heat which passes through the face ω_1 into the element during the infinitesimal time δt is ultimately $-\kappa\omega_1 \dfrac{dv}{ds_1}\delta t$, that is $-\dfrac{\kappa}{c}\, d\beta\, d\gamma\, (\mu^2-\nu^2)\dfrac{dv}{d\alpha}\delta t$. The quantity of heat which passes out of the element during the same time through the opposite face will therefore be ultimately $-\dfrac{\kappa}{c}\, d\beta\, d\gamma\, (\mu^2-\nu^2)\left\{\dfrac{dv}{d\alpha}+\dfrac{d^2v}{d\alpha^2}\,d\alpha\right\}\delta t$. Thus the augmentation of heat is $\dfrac{\kappa}{c}\, d\alpha\, d\beta\, d\gamma\, (\mu^2-\nu^2)\dfrac{d^2v}{d\alpha^2}\delta t$.

Proceed in the same way for the other pairs of opposite faces; and thus finally we obtain as the condition of stationary temperature

$$(\mu^2-\nu^2)\frac{d^2v}{d\alpha^2}+(\lambda^2-\nu^2)\frac{d^2v}{d\beta^2}+(\lambda^2-\mu^2)\frac{d^2v}{d\gamma^2}=0.$$

CHAPTER XXVI.

LAMÉ'S FUNCTIONS.

323. We are now about to introduce the student to certain functions which we shall call *Lamé's Functions;* their character will be seen more distinctly as we proceed, but in the mean time we may say that they are analogous to Laplace's Functions, only that instead of the variables r, θ, ϕ with which these functions are concerned, we now have Lamé's variables λ, μ, ν involved directly or indirectly.

324. Suppose an ellipsoid of which the semi-axes in descending order of magnitude are r, r', r''. Put $b=\sqrt{(r^2-r'^2)}$, and $c=\sqrt{(r^2-r''^2)}$. It is required to determine V so that for every point within the ellipsoid it shall satisfy the equation

$$(\mu^2-\nu^2)\frac{d^2V}{d\alpha^2}+(\lambda^2-\nu^2)\frac{d^2V}{d\beta^2}+(\lambda^2-\mu^2)\frac{d^2V}{d\gamma^2}=0 \;\ldots\ldots(1),$$

and moreover shall have at the surface an assigned value which is fixed for any point but variable from point to point.

The variables α, β, γ are connected with λ, μ, ν respectively by the equations

$$\alpha=c\int_c^\lambda \frac{d\lambda}{\sqrt{(\lambda^2-b^2)(\lambda^2-c^2)}},\quad \beta=c\int_b^\mu \frac{d\mu}{\sqrt{(\mu^2-b^2)(c^2-\mu^2)}},$$

$$\gamma=c\int_0^\nu \frac{d\nu}{\sqrt{(b^2-\nu^2)(c^2-\nu^2)}}\;\ldots\ldots(2).$$

Thus V may be supposed theoretically to be a function of λ, μ, ν or of α, β, γ. The condition relative to the surface which we are to satisfy may be expressed by saying that V is to be equal to $F(\alpha, \beta)$, where F denotes a given function, when $\lambda = r$, that is when $\alpha = c\int_c^r \frac{d\lambda}{\sqrt{(\lambda^2 - b^2)(\lambda^2 - c^2)}}$.

325. We have in the preceding Article enunciated the problem in a purely mathematical form; but the student who has read Chapter XXV. will readily give to it the additional interest of a physical application, for it amounts to the following: the surface of an ellipsoid is retained at a temperature which is fixed for each point but variable from point to point, and it is required to determine the temperature at any point in the interior of the ellipsoid in the state of equilibrium of temperature.

326. Let us examine whether we can obtain a solution of (1) by taking $V = LMN$, where L involves α alone, M involves β alone, and N involves γ alone.

Substitute in (1), and divide by LMN; thus we get

$$\frac{\mu^2 - \nu^2}{L}\frac{d^2L}{d\alpha^2} + \frac{\lambda^2 - \nu^2}{M}\frac{d^2M}{d\beta^2} + \frac{\lambda^2 - \mu^2}{N}\frac{d^2N}{d\gamma^2} = 0 \quad \text{.....(3).}$$

Now we have identically

$$\mu^2 - \nu^2 + \nu^2 - \lambda^2 + \lambda^2 - \mu^2 = 0,$$

$$(\mu^2 - \nu^2)\lambda^2 + (\nu^2 - \lambda^2)\mu^2 + (\lambda^2 - \mu^2)\nu^2 = 0.$$

Hence if g and h be any constants

$$(\mu^2 - \nu^2)\left(\frac{h\lambda^2}{c^2} - g\right) + (\nu^2 - \lambda^2)\left(\frac{h\mu^2}{c^2} - g\right) + (\lambda^2 - \mu^2)\left(\frac{h\nu^2}{c^2} - g\right) = 0.$$

327. Thus we see that equation (3) will be satisfied if we put

$$\frac{d^2L}{d\alpha^2} = \left(\frac{h\lambda^2}{c^2} - g\right)L, \quad \frac{d^2M}{d\beta^2} = \left(g - \frac{h\mu^2}{c^2}\right)M, \quad \frac{d^2N}{d\gamma^2} = \left(\frac{h\nu^2}{c^2} - g\right)N \text{...(4),}$$

Now from (2), we have

$$c^2\frac{d^2L}{da^2}=\sqrt{(\lambda^2-b^2)(\lambda^2-c^2)}\,\frac{d}{d\lambda}\left\{\sqrt{(\lambda^2-b^2)(\lambda^2-c^2)}\,\frac{dL}{d\lambda}\right\};$$

develope this, putting $b^2+c^2=p$, and $b^2c^2=q$, and treat the other equations (4) in a similar way; thus we obtain

$$\left.\begin{aligned}(\lambda^4-p\lambda^2+q)\frac{d^2L}{d\lambda^2}+(2\lambda^3-p\lambda)\frac{dL}{d\lambda}&=(h\lambda^2-gc^2)\,L\\ (\mu^4-p\mu^2+q)\frac{d^2M}{d\mu^2}+(2\mu^3-p\mu)\frac{dM}{d\mu}&=(h\mu^2-gc^2)\,M\\ (\nu^4-p\nu^2+q)\frac{d^2N}{d\nu^2}+(2\nu^3-p\nu)\frac{dN}{d\nu}&=(h\nu^2-gc^2)\,N\end{aligned}\right\}\ldots(5).$$

These three equations it will be seen are identical in form.

328. It will be seen from the commencement of Art. 326 that we do not profess to investigate the *most general* solution of (1), but only to obtain *a* solution. Thus guided by the analogy of Laplace's Functions, we shall ascribe to the arbitrary constant h of equations (5) the value $n(n+1)$, where n is a positive integer, and then we shall endeavour to find a solution of any one of the three equations (5), involving $2n+1$ terms; and we shall assume the solution to be of the degree n in the independent variable which occurs.

329. Take then the first of equations (5), put $n(n+1)$ for h, and pz for gc^2; thus we have

$$(\lambda^4-p\lambda^2+q)\frac{d^2L}{d\lambda^2}+(2\lambda^3-p\lambda)\frac{dL}{d\lambda}+\{pz-n(n+1)\lambda^2\}L=0\ldots(6).$$

We shall now examine whether this equation has a solution of the form

$$L=\lambda^n+k_1\lambda^{n-2}+k_2\lambda^{n-4}+\ldots+k_{s-1}\lambda^{n-2s+2}+k_s\lambda^{n-2s}+\ldots\ldots.(7).$$

Substitute this value of L in (6); it will be found that the first term, which involves λ^{n+2}, vanishes of itself; and by equating to zero the coefficient of λ^{n-2s+2} we get

$$2s(2n+1-2s)k_s = p\{z-(n-2s+2)^2\}k_{s-1} + q(n-2s+4)(n-2s+3)k_{s-2} \quad \ldots\ldots(8).$$

In this equation put for s in succession the values 1, 2, 3, ...; then observing that $k_0 = 1$, and that k_{-1}, k_{-2}, ... do not exist, we obtain

$$2(2n-1)k_1 = p(z-n^2),$$

$$4(2n-3)k_2 = p\{z-(n-2)^2\}k_1 + qn(n-1),$$

$$6(2n-5)k_3 = p\{z-(n-4)^2\}k_2 + q(n-2)(n-3)k_1,$$

and so on.

The first of these equations gives k_1, and it is of the first degree in z; substitute the value of k_1 in the second equation, and we obtain k_2, which will be of the second degree in z; substitute the values of k_1 and k_2 in the third equation, and we obtain k_3, which will be of the third degree in z; and so on. Thus the coefficient k_s will be of the degree s in z.

But we require the series (7) to be *finite*, and thus the coefficients of which k_s is the type must vanish from and after some certain value of s. This will happen if we can make two consecutive coefficients k_{s-2} and k_{s-1} vanish; for then by means of (8) we have $k_s = 0$, and also all the subsequent coefficients. Thus we have two conditions to satisfy; one may be satisfied by properly choosing the value of z, which is as yet undetermined; the other may be satisfied by taking s equal to $\frac{n+4}{2}$ or $\frac{n+3}{2}$, for in this case the last term of (8) vanishes: the former or the latter value of s must be taken according as n is even or odd.

Let then σ denote the value of s which causes the last term of (8) to vanish; then k_σ is expressed as a multiple of $k_{\sigma-1}$, and therefore if we take z such that $k_{\sigma-1}$ vanishes, then k_σ will also vanish.

The equation $k_{\sigma-1} = 0$ is of the degree $\sigma - 1$ in z, and so has $\sigma - 1$ roots; any one of these roots may be taken: it will be shewn hereafter that these roots are all real.

When n is even the expression (7) contains only *even* powers of λ, and the last term is the constant $k_{\sigma-2}$; when n is odd the expression contains only *odd* powers of λ, and the last term is $k_{\sigma-2}\lambda$.

330. We shall next examine whether the equation (6) has a solution of the form $L = K\sqrt{(\lambda^2 - b^2)}$ where

$$K = \lambda^{n-1} + k_1\lambda^{n-3} + \ldots + k_{s-2}\lambda^{n-2s+3} + k_{s-1}\lambda^{n-2s+1} + \ldots\ldots\ldots \quad (9).$$

Substitute $K\sqrt{(\lambda^2 - b^2)}$ for L in (6); thus we obtain

$$(\lambda^4 - p\lambda^2 + q)\frac{d^2K}{d\lambda^2} + \{4\lambda^3 - (p + 2c^2)\lambda\}\frac{dK}{d\lambda}$$
$$+ \{pz - c^2 - (n-1)(n+2)\lambda^2\}K = 0.$$

Substitute in this the value of K from (9); it will be found that the first term, which involves λ^{n+1}, vanishes of itself; and by equating to zero the coefficient of λ^{n-2s+1} we get

$$2s(2n + 1 - 2s)k_s = \{pz - p(n - 2s + 1)^2 - c^2(2n - 4s + 3)\}k_{s-1}$$
$$+ q(n - 2s + 3)(n - 2s + 2)k_{s-2}.$$

We then proceed to ensure that the series in (9) shall be finite, by a method like that of Art. 329. We take $s = \frac{n+2}{2}$ if n is even, and $= \frac{n+3}{2}$ if n is odd. Let σ denote the value of s thus taken; and let z be found from the equation $k_{\sigma-1} = 0$. Then all the coefficients in (9) from and after $k_{\sigma-1}$ will vanish.

The equation $k_{\sigma-1} = 0$ is of the degree $\sigma - 1$ in z, and so has $\sigma - 1$ roots; any one of these roots may be taken: it will be shewn hereafter that these roots are all real.

When n is even the expression (9) contains only *odd* powers of λ, and the last term is $k_{\sigma-2}\lambda$; when n is odd the expression contains only *even* powers of λ, and the last term is the constant $k_{\sigma-2}$.

331. In the manner of the preceding Article we may also shew that the equation (6) has a solution of the form $L = K\sqrt{(\lambda^2 - c^2)}$, where K is of the same form as in (9). We have only to change b^2 into c^2 in the investigation of the preceding Article.

332. Finally we shall examine whether the equation (6) has a solution of the form $L = K\sqrt{(\lambda^2 - b^2)(\lambda^2 - c^2)}$, where

$$K = \lambda^{n-2} + k_1\lambda^{n-4} + \ldots + k_{s-2}\lambda^{n-2s+2} + k_{s-1}\lambda^{n-2s} + \ldots\ldots(10).$$

Substitute $K\sqrt{(\lambda^2 - b^2)(\lambda^2 - c^2)}$ for L in (6); thus we obtain

$$(\lambda^4 - p\lambda^2 + q)\frac{d^2K}{d\lambda^2} + (6\lambda^3 - 3p\lambda)\frac{dK}{d\lambda} + \{p(z-1) - (n-2)(n+3)\lambda^2\}K = 0.$$

Substitute in this the value of K from (10); it will be found that the first term, which involves λ^n, vanishes of itself; and by equating to zero the coefficient of λ^{n-2s} we get

$$2s(2n+1-2s)k_s = p\{z - 1 - (n-2s)(n-2s+2)\}k_{s-1} + q(n-2s+2)(n-2s+1)k_{s-2}.$$

We then proceed to ensure that the series in (10) shall be finite, by the method already used. We take $s = \frac{n+2}{2}$ if n is even, and $= \frac{n+1}{2}$ if n is odd. Let σ denote the value of s thus taken; and let z be found from the equation $k_{\sigma-1} = 0$. Then all the coefficients in (10) from and after $k_{\sigma-1}$ will vanish.

The equation $k_{\sigma-1} = 0$ is of the degree $\sigma - 1$ in z, and so has $\sigma - 1$ roots; any one of these roots may be taken: it will be shewn hereafter that these roots are all real.

333. We may now sum up the results obtained in Arts. 329...332.

First suppose n *even*, let it be denoted by $2m$. Then L may have $m+1$ values of the form discussed in Art. 329, and m values of each of the forms discussed in Arts. 330, 331, and 332: thus on the whole there are $4m+1$ values, that is $2n+1$ values.

Next suppose n *odd*, let it be denoted by $2m+1$. Then L may have $m+1$ values of each of the forms discussed in Arts. 329, 330, and 331, and m values of the form discussed in Art. 332: thus on the whole there are $4m+3$ values, that is $2n+1$ values.

The values of M and N may be said to be determined by those of L; for by Art. 327 the same form of differential equation applies to all three, and the value of z must be simultaneous for the three.

334. We have still to attend to the condition relative to the surface which is mentioned in Art. 324, and also to shew that the equation $k_{\sigma-1}=0$ which we have used has all its roots real: these points will be considered in the next Chapter.

CHAPTER XXVII.

SEPARATION OF THE TERMS.

335. WHEN a function is expanded in a series of sines or cosines of multiple angles, the coefficient of each term can be found separately; or at least can be expressed in the form of a definite integral: see *Integral Calculus*, Chapter XIII. In like manner when a function of one variable is expanded in a series of Legendre's Coefficients, or a function of two variables is expanded in a series of Laplace's Coefficients, the coefficient of each term can be separately expressed: see Arts. 138 and 204. The object of the present Chapter is to obtain similar results with respect to Lamé's Functions. Lamé does not attempt to give any evidence to shew that an assigned function can be expanded in a series of his functions; but assuming that such expansion is possible he shews in fact how to determine the coefficients. Admitting, however, that the possibility of expansion in a series of Laplace's Functions has been established, we may by the aid of the transformations of Chapter XXIV. grant that a similar proposition holds with respect to Lamé's Functions.

336. In Art. 324 we have defined α, β, γ; we shall now introduce two new symbols connected with β and γ. Let ϖ denote the value of β when $\mu = c$, and ω the value of γ when $\nu = b$; so that

$$\varpi = c\int_b^c \frac{d\mu}{\sqrt{(\mu^2 - b^2)(c^2 - \mu^2)}}, \quad \omega = c\int_0^b \frac{d\nu}{\sqrt{(b^2 - \nu^2)(c^2 - \nu^2)}}.$$

We shall now demonstrate two important propositions relating to the limiting values of μ and ν.

337. At the limits 0 and ϖ for β we have either $M=0$ or $\frac{dM}{d\beta}=0$.

The values of M may be inferred from those of L; and by Arts. 329...332 there are four forms to be considered.

I. See Art. 329. When $\beta=0$ we have $\mu=b$; and therefore $\frac{d\mu}{d\beta}=0$; and since $\frac{dM}{d\beta}=\frac{dM}{d\mu}\frac{d\mu}{d\beta}$ we have $\frac{dM}{d\beta}=0$. Similarly when $\beta=\varpi$ we have $\mu=c$; and therefore $\frac{d\mu}{d\beta}=0$; and therefore $\frac{dM}{d\beta}=0$.

II. See Art. 330. When $\beta=0$ we have $\mu=b$; and therefore $M=0$. When $\beta=\varpi$ we have $\mu=c$; and therefore $\frac{d\mu}{d\beta}=0$; and therefore $\frac{dM}{d\beta}=0$.

III. See Art. 331. Here when $\beta=0$ we have $\frac{dM}{d\beta}=0$, and when $\beta=\varpi$ we have $M=0$.

IV. See Art. 332. Here we have $M=0$, both when $\beta=0$ and when $\beta=\varpi$.

338. At the limits 0 and ω for γ we have either $N=0$ or $\frac{dN}{d\gamma}=0$.

I. See Art. 329. When $\gamma=0$ we have $\nu=0$; and then $N=0$ if n be odd, and $\frac{dN}{d\nu}=0$ if n be even: in the latter case since $\frac{dN}{d\nu}=0$ we have also $\frac{dN}{d\gamma}=0$. When $\gamma=\omega$ we have $\nu=b$; and therefore $\frac{d\nu}{d\gamma}=0$; and therefore $\frac{dN}{d\gamma}=0$.

II. See Art. 330. When $\gamma=0$ we have $N=0$ if n be even, and $\frac{dN}{d\gamma}=0$ if n be odd. When $\gamma=\omega$ we have $N=0$.

III. See Art. 331. When $\gamma = 0$ we have $N = 0$ if n be even, and $\frac{dN}{d\gamma} = 0$ if n be odd. When $\gamma = \omega$ we have $\frac{dN}{d\gamma} = 0$.

IV. See Art. 332. When $\gamma = 0$ we have $N = 0$ if n be odd, and $\frac{dN}{d\gamma} = 0$ if n be even. When $\gamma = \omega$ we have $N = 0$.

339. Let M and M' denote two different expressions of the *same form*, out of the four forms considered in Art. 337; then $M\frac{dM'}{d\beta} - M'\frac{dM}{d\beta}$ vanishes both when $\beta = 0$ and when $\beta = \varpi$. This follows from Art. 337.

340. Let N and N' denote two different expressions of the *same form*, out of the four forms considered in Art. 338; and let n and n' be the corresponding exponents; then $N\frac{dN'}{d\gamma} - N'\frac{dN}{d\gamma}$ vanishes both when $\gamma = 0$ and when $\gamma = \omega$, provided n and n' are both even or both odd. This follows from Art. 338.

341. We can now establish the proposition that the roots of the equations in z obtained in Arts. 329...332 are all real.

For take any one of the equations, and suppose if possible that it has a root $\zeta + \zeta'\sqrt{-1}$; then since the coefficients of the equation are all real, there must also exist the root $\zeta - \zeta'\sqrt{-1}$. We may suppose that in M we put the former root, and in M' the latter root. Suppose then that M takes the form $Z + Z'\sqrt{-1}$, then M' will take the form $Z - Z'\sqrt{-1}$. Substitute these values of M and M' in the expression of Art. 339; then it reduces to $\left(Z'\frac{dZ}{d\beta} - Z\frac{dZ'}{d\beta}\right)\sqrt{-1}$; hence $Z'\frac{dZ}{d\beta} - Z\frac{dZ'}{d\beta}$ must vanish both when $\beta = 0$ and when $\beta = \varpi$.

Now the value of M must verify the second of the differential equations (4) of Art. 327 when we put $n(n+1)$ for h, and $\frac{pz}{c^2}$, that is $\left(1 + \frac{b^2}{c^2}\right)z$, for g. Thus we obtain

$$\frac{d^2Z}{d\beta^2} + \frac{d^2Z'}{d\beta^2}\sqrt{-1}$$
$$= \left\{\left(1 + \frac{b^2}{c^2}\right)(\zeta + \zeta'\sqrt{-1}) - n(n+1)\frac{\mu^2}{c^2}\right\}(Z + Z'\sqrt{-1}).$$

Change the sign of $\sqrt{-1}$ and we obtain the equation which M' must satisfy. Then from the two equations by addition and subtraction we obtain

$$\frac{d^2Z}{d\beta^2} = \left\{\left(1 + \frac{b^2}{c^2}\right)\zeta - n(n+1)\frac{\mu^2}{c^2}\right\}Z - \left(1 + \frac{b^2}{c^2}\right)\zeta' Z',$$

$$\frac{d^2Z'}{d\beta^2} = \left\{\left(1 + \frac{b^2}{c^2}\right)\zeta - n(n+1)\frac{\mu^2}{c^2}\right\}Z' + \left(1 + \frac{b^2}{c^2}\right)\zeta' Z.$$

Multiply the former by Z', and the latter by Z, and subtract; thus

$$Z'\frac{d^2Z}{d\beta^2} - Z\frac{d^2Z'}{d\beta^2} = -\left(1 + \frac{b^2}{c^2}\right)\zeta'(Z'^2 + Z^2).$$

Multiply both members of this equation by $d\beta$, and integrate between the limits 0 and ϖ. Then the left-hand member vanishes, because the indefinite integral $Z'\frac{dZ}{d\beta} - Z\frac{dZ'}{d\beta}$ vanishes at both limits. Therefore

$$-\left(1 + \frac{b^2}{c^2}\right)\zeta'\int_0^{\varpi}(Z'^2 + Z^2)\,d\beta = 0;$$

this is impossible unless $\zeta' = 0$, because every element of the definite integral is positive.

342. We shall now advert to the condition relative to the surface which is mentioned in Art. 324.

The process which we have given leads us to express V by an aggregate of terms each of the type LMN; each term may also be furnished with an arbitrary constant as a multiplier. Now at the surface the value of λ is given, so that the term L becomes constant. Hence in fact we have to satisfy a condition which may be expressed thus

$$F(\beta, \gamma) = CMN + C'M'N' + C''M''N'' + \ldots\ldots\ldots(1),$$

where $M, M', M'', \dots N, N', N'', \dots$ are terms of the nature indicated in Art. 333; and $C, C', C'', \dots$ are arbitrary constants. On the right-hand side of (1) we have $2n+1$ different terms for every value of n.

We shall shew how the values of the arbitrary constants may be determined. The essential part of the process is a proposition analogous to that of Art. 187, which we shall now give.

343. Let M and N be two expressions of the nature indicated in Art. 333, and let them correspond to the values n and z; similarly let M' and N' be two other expressions of the same form as M and N respectively, and let them correspond to the values n' and z'; then will

$$\int_0^{\varpi}\int_0^{\omega}(\mu^2-\nu^2)\,MM'NN'\,d\beta\,d\gamma=0\,;$$

n and n' being supposed both odd or both even.

We have $$\frac{d^2N}{d\gamma^2}=\left\{n\,(n+1)\frac{\nu^2}{c^2}-\left(1+\frac{b^2}{c^2}\right)z\right\}N,$$

$$\frac{d^2N'}{d\gamma^2}=\left\{n'\,(n'+1)\frac{\nu^2}{c^2}-\left(1+\frac{b^2}{c^2}\right)z'\right\}N'\,;$$

hence $$N\frac{d^2N'}{d\gamma^2}-N'\frac{d^2N}{d\gamma^2}$$

$$=\left(1+\frac{b^2}{c^2}\right)(z-z')\,NN'-\{n\,(n+1)-n'\,(n'+1)\}\frac{\nu^2}{c^2}NN'.$$

Multiply by $d\gamma$ and integrate between the limits 0 and ω; the left-hand member vanishes because the indefinite integral $N\dfrac{dN'}{d\gamma}-N'\dfrac{dN}{d\gamma}$ vanishes at both limits by Art. 340. Thus the right-hand member vanishes; and therefore

$$\{n\,(n+1)-n'\,(n'+1)\}\int_0^{\omega}\nu^2NN'd\gamma$$

$$=(b^2+c^2)\,(z-z')\int_0^{\omega}NN'd\gamma\,\dots\dots(2).$$

In a similar manner we may shew that

$$\{n(n+1)-n'(n'+1)\}\int_0^{\varpi}\mu^2 MM'd\beta$$

$$=(b^2+c^2)(z-z')\int_0^{\varpi}MM'd\beta \text{......(3)}.$$

If neither $n(n+1)-n'(n'+1)$ nor $z-z'$ is zero, we obtain by cross multiplication

$$\int_0^{\varpi}\int_0^{\omega}\mu^2 MM'NN'd\beta\, d\gamma=\int_0^{\varpi}\int_0^{\omega}\nu^2 MM'NN'd\beta\, d\gamma;$$

and therefore $\int_0^{\varpi}\int_0^{\omega}(\mu^2-\nu^2)\,MM'NN'd\beta\, d\gamma=0$.........(4).

If however $n(n+1)-n'(n'+1)$ is zero but $z-z'$ is not zero; then we have from (2) and (3)

$$\int_0^{\omega}NN'd\gamma=0,\quad \int_0^{\varpi}MM'd\beta=0.$$

Hence

$$\int_0^{\omega}NN'd\gamma\int_0^{\varpi}\mu^2 MM'd\beta-\int_0^{\varpi}MM'd\beta\int_0^{\omega}\nu^2 NN'd\gamma=0;$$

and thus we again arrive at (4).

Finally, if $z-z'$ is zero but $n(n+1)-n'(n'+1)$ is not zero, we have from (2) and (3)

$$\int_0^{\omega}\nu^2 NN'd\gamma=0,\quad \int_0^{\varpi}\mu^2 MM'd\beta=0;$$

and as before we again arrive at (4).

Thus (4) holds universally except in the case where we have simultaneously $n=n'$, and $z=z'$.

344. It appears from Art. 337 that in two out of the four forms M vanishes when $\beta=0$, and in the other two forms $\frac{dM}{d\beta}$ vanishes when $\beta=0$: in the first case M must be an *odd* function of β, and in the second an *even* function.

It appears also from Art. 338 that in two out of the four forms N vanishes when $\gamma = 0$, and in the other two forms $\frac{dN}{d\gamma}$ vanishes when $\gamma = 0$: in the first case N must be an *odd* function of γ, and in the second an *even* function. If n be odd the forms I. and IV. make N odd, and the forms II. and III. make it even. If n be even this is to be reversed.

This leads us to break up our equation (1) into four parts.

345. Let $F(\beta, \gamma) = f_1(\beta, \gamma) + f_2(\beta, \gamma) + f_3(\beta, \gamma) + f_4(\beta, \gamma)$ where $f_1(\beta, \gamma)$ denotes an expression which is even with respect both to β and γ; $f_2(\beta, \gamma)$ an expression which is even with respect to β and odd with respect to γ; $f_3(\beta, \gamma)$ a function which is odd with respect to β and even with respect to γ; and $f_4(\beta, \gamma)$ a function which is odd with respect both to β and γ.

Then the terms on the right-hand side of (1) must admit of a similar distinction; so that the equation resolves itself into four, of which the type will be

$$f(\beta, \gamma) = CMN + C'M'N' + C''M''N'' + \dots\dots\dots (5),$$

where $f(\beta, \gamma)$ may denote any one of the four terms $f_1(\beta, \gamma)$, $f_2(\beta, \gamma)$, $f_3(\beta, \gamma)$, $f_4(\beta, \gamma)$; and the terms on the right-hand are all of the same kind as $f(\beta, \gamma)$; thus $N, N', N'', \dots$ are all odd or all even functions of γ.

Now to determine C; multiply both sides of (5) by $(\mu^2 - \nu^2) MN d\beta d\gamma$, and integrate between the limits 0 and ϖ for β, and 0 and ω for γ. Then by equation (4) all the terms on the right-hand side vanish except that involving C; and we obtain

$$C \int_0^{\varpi} \int_0^{\omega} M^2 N^2 (\mu^2 - \nu^2) d\beta d\gamma = \int_0^{\varpi} \int_0^{\omega} f(\beta, \gamma) MN (\mu^2 - \nu^2) d\beta d\gamma.$$

This theoretically determines C. In like manner C', C'',... may be determined.

We proceed to discuss the value of

$$\int_0^{\varpi} \int_0^{\omega} M^2 N^2 (\mu^2 - \nu) d\beta d\gamma.$$

346. We have by Art. 324,

$$\left.\begin{aligned} c\frac{d\mu}{d\beta} &= \sqrt{(\mu^2-b^2)(c^2-\mu^2)} \\ c\frac{d\nu}{d\gamma} &= \sqrt{(b^2-\nu^2)(c^2-\nu^2)} \end{aligned}\right\}\text{..............} (6).$$

By differentiating these we get

$$\left.\begin{aligned} c^2\frac{d^2\mu}{d\beta^2} &= -2\mu^3+(b^2+c^2)\mu \\ c^2\frac{d^2\nu}{d\gamma^2} &= 2\nu^3-(b^2+c^2)\nu \end{aligned}\right\}\text{..............} (7).$$

Now, m being any positive integer, multiply the first of equations (7) by $\mu^{2m+1}d\beta$, and integrate between 0 and ϖ; thus

$$c^2\int_0^{\varpi}\mu^{2m+1}\frac{d^2\mu}{d\beta^2}d\beta = -2\int_0^{\varpi}\mu^{2m+4}d\beta + (b^2+c^2)\int_0^{\varpi}\mu^{2m+2}d\beta \text{...(8).}$$

By integration by parts we have

$$\int\mu^{2m+1}\frac{d^2\mu}{d\beta^2}d\beta = \mu^{2m+1}\frac{d\mu}{d\beta} - (2m+1)\int\mu^{2m}\left(\frac{d\mu}{d\beta}\right)^2 d\beta \text{...(9).}$$

When $\beta = 0$ we have $\mu = b$, and when $\beta = \varpi$ we have $\mu = c$; hence we see by (6) that $\frac{d\mu}{d\beta}$ vanishes at both limits, so that from (8) and (9) we get

$$c^2(2m+1)\int_0^{\varpi}\mu^{2m}\left(\frac{d\mu}{d\beta}\right)^2 d\beta =$$
$$2\int_0^{\varpi}\mu^{2m+4}d\beta - (c^2+b^2)\int_0^{\varpi}\mu^{2m+2}d\beta \text{ .... (10).}$$

Substitute for $\left(\frac{d\mu}{d\beta}\right)^2$ its value from (6); thus we get

$$(2m+3)\int_0^{\varpi}\mu^{2m+4}d\beta = (2m+2)\ (c^2+b^2)\int_0^{\varpi}\mu^{2m+2}d\beta$$
$$- (2m+1)\,c^2b^2\int_0^{\varpi}\mu^{2m}d\beta \text{......(11).}$$

Treat the second equation (7) in the same manner as we have treated the first; thus we get if $k=\frac{b}{c}$

$$(2m+3)\int_0^\omega \nu^{2m+4}d\gamma=(2m+2)\,c^2(1+k^2)\int_0^\omega \nu^{2m+2}d\gamma$$
$$-(2m+1)\,c^4k^2\int_0^\omega \nu^{2m}d\gamma\,\ldots\ldots\ldots(12).$$

Put $\int_0^\varpi \mu^2 d\beta=u,\qquad \int_0^\omega \nu^2 d\gamma=v;$

then if we take $m=0$ in (11) and (12), we get

$$\int_0^\varpi \mu^4 d\beta=\frac{2}{3}c^2(1+k^2)\,u-\frac{1}{3}c^4k^2\varpi,$$
$$\int_0^\omega \nu^4 d\gamma=\frac{2}{3}c^2(1+k^2)\,v-\frac{1}{3}c^4k^2\omega.$$

Then in (11) and (12) put for m in succession the values 1, 2, 3, ...; thus we shall obtain

$$\left.\begin{aligned}\int_0^\varpi \mu^{2m}d\beta&=Pc^{2m-2}u+Qc^{2m}\varpi,\\ \int_0^\omega \nu^{2m}d\gamma&=Pc^{2m-2}v+Qc^{2m}\omega;\end{aligned}\right\}\ldots\ldots\ldots\ldots(13),$$

where P and Q are integral functions of k^2.

Now M^2 is some function of μ^2, and N^2 of γ^2; and therefore by the equation (13) we get

$$\left.\begin{aligned}\int_0^\varpi M^2d\beta&=Gu+H\varpi,\\ \int_0^\omega N^2d\gamma&=Gv+H\omega;\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

where G and H are integral functions of k^2 and c^2 and of the coefficients of M or N.

And in the same manner we get

$$\left.\begin{aligned}\int_0^{\varpi} \mu^2 M^2 d\beta &= G_1 u + H_1 \varpi \\ \int_0^{\omega} \nu^2 N^2 d\gamma &= G_1 v + H_1 \omega\end{aligned}\right\} \quad \text{..............} (15).$$

From (14) and (15) we get

$$\int_0^{\varpi}\int_0^{\omega} M^2 N^2 \mu^2 d\beta\, d\gamma = (Gv + H\omega)(G_1 u + H_1 \varpi),$$

$$\int_0^{\varpi}\int_0^{\omega} M^2 N^2 \nu^2 d\beta\, d\gamma = (Gu + H\varpi)(G_1 v + H_1 \omega);$$

then $$\int_0^{\varpi}\int_0^{\omega} M^2 N^2 (\mu^2 - \nu^2)\, d\beta\, d\gamma = (G_1 H - GH_1)(u\omega - v\varpi).$$

But $$u\omega - v\varpi = \int_0^{\varpi}\int_0^{\omega} (\mu^2 - \nu^2)\, d\beta\, d\gamma;$$

and by Art. 277 this $= c^2 \dfrac{\pi}{2}$.

Thus finally

$$\int_0^{\varpi}\int_0^{\omega} M^2 N^2 (\mu^2 - \nu^2)\, d\beta\, d\gamma = \frac{\pi}{2}(G_1 H - GH_1)\, c^2,$$

where the multiplier of $\dfrac{\pi}{2}$ is an integral function of c^2, k^2, and the coefficients of M or N.

CHAPTER XXVIII.

SPECIAL CASES.

347. It must be observed that the formulæ which we gave in Chapter XXI. are not applicable to the case in which $b = c$, nor to the case in which $b = 0$.

For since μ^2 is supposed to lie between b^2 and c^2, when $b = c$ the values of y and z take the form $\frac{0}{0}$. And if $b = 0$ then ν also $= 0$, and the values of x and y take the form $\frac{0}{0}$.

Now the advantage of the formulæ already used is that they enable us to solve problems in which the general enunciation is accompanied by some special condition which is to hold at the surface of an *ellipsoid;* but when that ellipsoid becomes one of revolution, we have either $b = c$, or $b = 0$; and hence the investigations hitherto given become inadmissible.

Lamé accordingly supplies special investigations, which are applicable to the case in which the problems are modified by reference to an ellipsoid of revolution instead of a general ellipsoid: these special cases are also treated by Mathieu in the work cited in Art. 265; his method is not identical with Lamé's. These special investigations however add nothing of importance to the analytical results already given; and we shall accordingly confine ourselves to a few paragraphs giving the method of Mathieu.

348. In order to obtain formulæ which shall be universally applicable, let us introduce two angles ϕ and ψ, connected with Lamé's variables by the relations

$$\mu = \sqrt{(c^2 \sin^2 \phi + b^2 \cos^2 \phi)}, \quad \nu = b \cos \psi ;$$

hence $\sqrt{(b^2 - \nu^2)} = b \sin \psi$, $\sqrt{(\mu^2 - b^2)} = \sqrt{(c^2 - b^2)} \sin \phi$,

$$\sqrt{(c^2 - \mu^2)} = \sqrt{(c^2 - b^2)} \cos \phi, \quad \sqrt{(c^2 - \nu^2)} = \sqrt{(c^2 - b^2 \cos^2 \psi)}.$$

Thus we have for x, y, z by Art. 271 the expressions

$$x = \frac{\lambda \cos \psi}{c} \sqrt{(c^2 \sin^2 \phi + b^2 \cos^2 \phi)},$$

$$y = \sqrt{(\lambda^2 - b^2)} \sin \psi \sin \phi,$$

$$z = \sqrt{(\lambda^2 - c^2)} \cos \phi \frac{\sqrt{(c^2 - b^2 \cos^2 \psi)}}{c}.$$

These formulæ are universally applicable.

If $b = c$, they become

$$x = \lambda \cos \psi, \quad y = \sqrt{(\lambda^2 - c^2)} \sin \psi \sin \phi, \quad z = \sqrt{(\lambda^2 - c^2)} \sin \psi \cos \phi.$$

If $b = 0$, they become

$$x = \lambda \cos \psi \sin \phi, \quad y = \lambda \sin \psi \sin \phi, \quad z = \sqrt{(\lambda^2 - c^2)} \cos \phi.$$

349. It is easy to transform the differential equations which are given in Art. 327 for M and N.

The equation for M may be written

$$\sqrt{\mu^2 - b^2} \sqrt{c^2 - \mu^2} \frac{d}{d\mu} \left\{ \sqrt{\mu^2 - b^2} \sqrt{c^2 - \mu^2} \frac{dM}{d\mu} \right\} + (h\mu^2 - gc^2) M = 0.$$

We have $\sqrt{\mu^2 - b^2} \sqrt{c^2 - \mu^2} \dfrac{dM}{d\mu} = \sqrt{\{c^2 - (c^2 - b^2) \cos^2 \phi\}} \dfrac{dM}{d\phi}$;

and thus we shall obtain the equation

$$\{c^2 - (c^2 - b^2) \cos^2 \phi\} \frac{d^2M}{d\phi^2} + (c^2 - b^2) \sin \phi \cos \phi \frac{dM}{d\phi}$$
$$- \{h (c^2 - b^2) \cos^2 \phi - (h - g) c^2\} M = 0.$$

In like manner the equation for N may be transformed into

$$(c^2 - b^2 \cos^2\psi) \frac{d^2N}{d\psi^2} + b^2 \sin\psi \cos\psi \frac{dN}{d\psi} - (hb^2 \cos^2\psi - gc^2) N = 0.$$

The simpler forms which these equations assume when $b = c$, and when $b = 0$, can now be readily obtained.

350. We will give finally an investigation which indirectly establishes the transformation of Art. 303, though not in a very rigorous manner.

We have by Art. 327

$$\frac{d^2M}{d\beta^2} = \left(g - \frac{h\mu^2}{c^2}\right) M, \quad \frac{d^2N}{d\gamma^2} = \left(\frac{h\nu^2}{c^2} - g\right) N.$$

Multiply the first equation by N, and the second by M, and add; then putting F for MN we have

$$\frac{d^2F}{d\beta^2} + \frac{d^2F}{d\gamma^2} + \frac{h}{c^2}(\mu^2 - \nu^2) F = 0 \quad \text{............(1)}.$$

Here β and γ are known theoretically as functions of μ and ν respectively; we propose to transform (1) by the relations

$$\left.\begin{aligned} \frac{\mu\nu}{bc} &= \cos\phi \sin\theta \\ \frac{\sqrt{\mu^2 - b^2}\sqrt{b^2 - \nu^2}}{b\sqrt{c^2 - b^2}} &= \sin\phi \sin\theta \\ \frac{\sqrt{c^2 - \mu^2}\sqrt{c^2 - \nu^2}}{c\sqrt{c^2 - b^2}} &= \cos\theta \end{aligned}\right\} \quad \text{............(2)}.$$

Instead however of effecting the transformation directly as in Chapter XXIV. we will adopt an indirect process.

Let us suppose that instead of the variables which occur in (1) we substitute a corresponding system in which accented letters are used to denote quantities analogous to those in (1). Moreover, let us assume consistently with (2) that

$$\frac{\mu\nu}{bc}=\frac{\mu'\nu'}{b'c'}, \quad \text{and} \quad \frac{\sqrt{\mu^2-b^2}\sqrt{b^2-\nu^2}}{b\sqrt{c^2-b^2}}=\frac{\sqrt{\mu'^2-b'^2}\sqrt{b'^2-\nu'^2}}{b'\sqrt{c'^2-b'^2}} \quad \ldots\ldots(3).$$

We have also

$$\frac{c\,d\mu}{\sqrt{\mu^2-b^2}\sqrt{c^2-\mu^2}}=d\beta, \quad \frac{c\,d\nu}{\sqrt{b^2-\nu^2}\sqrt{c^2-\nu^2}}=d\gamma,$$

$$\frac{c'\,d\mu'}{\sqrt{\mu'^2-b'^2}\sqrt{c'^2-\mu'^2}}=d\beta', \quad \frac{c'\,d\nu'}{\sqrt{b'^2-\nu'^2}\sqrt{c'^2-\nu'^2}}=d\gamma'.$$

Now suppose b' and c' proportional to b and c, so that

$$\frac{b'}{b}=\frac{c'}{c}.$$

Denote this ratio by σ; then

$$\sigma=\frac{b'}{b}=\frac{c'}{c}=\frac{\sqrt{c'^2-b'^2}}{\sqrt{c^2-b^2}}.$$

Hence we shall get from (3)

$$\mu\nu=\frac{1}{\sigma^2}\mu'\nu',$$

$$\text{and} \quad \mu^2+\nu^2=\frac{1}{\sigma^2}(\mu'^2+\nu'^2);$$

therefore $$\mu=\frac{1}{\sigma}\mu', \quad \nu=\frac{1}{\sigma}\nu';$$

$$d\beta'=d\beta, \quad d\gamma'=d\gamma.$$

Hence (1) becomes

$$\frac{d^2F}{d\beta'^2}+\frac{d^2F}{d\gamma'^2}+\frac{h}{c'^2}(\mu'^2-\nu'^2)F=0 \quad \ldots\ldots\ldots\ldots (4).$$

Hence, without interfering with the final transformation by the aid of (2), we may change b and c respectively into b' and c', where b' and c' may be as small as we please, provided only $\frac{b'}{c'}=\frac{b}{c}$. So that we may ultimately suppose b and c to vanish.

Thus the transformation of (1) by the aid of (2) will be fully effected if we ascertain what (1) becomes consistently with (2) when b and c are supposed ultimately to vanish.

Now we have in general

$$\beta = c\int_b^\mu \frac{d\mu}{\sqrt{\mu^2 - b^2}\sqrt{c^2 - \mu^2}};$$

put $\eta = c\int_\mu^c \frac{d\mu}{\sqrt{\mu^2 - b^2}\sqrt{c^2 - \mu^2}}$, $\quad k = c\int_b^c \frac{d\mu}{\sqrt{\mu^2 - b^2}\sqrt{c^2 - \mu^2}}$;

then $\beta = k - \eta$, and k is a constant, so that

$$\frac{d^2F}{d\beta^2} = \frac{d^2F}{d\eta^2}.$$

When b is very small we may assume consistently with (2)

$$\nu = b\cos\phi, \quad \mu = c\sin\theta;$$

therefore $$\eta = c\int_\theta^{\frac{\pi}{2}} \frac{c\cos\theta\, d\theta}{c^2\sin\theta\cos\theta} = \int_\theta^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}.$$

Hence $$\frac{d\eta}{d\theta} = -\frac{1}{\sin\theta}.$$

Also $$\gamma = c\int_0^\nu \frac{d\nu}{\sqrt{b^2 - \nu^2}\sqrt{c^2 - \nu^2}} = -\int_{\frac{\pi}{2}}^\phi d\phi = \frac{\pi}{2} - \phi.$$

Thus $$\frac{d^2F}{d\beta^2} = \sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dF}{d\theta}\right),$$

and $$\frac{d^2F}{d\gamma^2} = \frac{d^2F}{d\phi^2};$$

and therefore when b is indefinitely small (1) becomes

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dF}{d\theta}\right) + \frac{d^2F}{d\phi^2} + hF\sin^2\theta = 0.$$

This equation does not involve c, and therefore remains the same when we suppose $c = 0$. Thus we have the required transformation.

CHAPTER XXIX.

MISCELLANEOUS PROPOSITIONS.

351. In Art. 296 we introduced certain auxiliary variables α, β, γ, connected respectively with the original variables λ, μ, ν. We may observe that these auxiliary variables can be made to depend upon elliptic functions.

352. For begin with γ; we have

$$\gamma = c\int_0^\nu \frac{d\nu}{\sqrt{b^2-\nu^2}\sqrt{c^2-\nu^2}}.$$

Assume $\nu = b\sin\psi$; thus

$$\gamma = c\int_0^\psi \frac{d\psi}{\sqrt{c^2-b^2\sin^2\psi}}$$

$$= \int_0^\psi \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}},$$

where $k = \frac{b}{c}$.

Thus k is the *modulus*, and ψ the *amplitude* of γ, which is an elliptic function of the first kind; see *Integral Calculus*, Chapter x.

Let ω have the meaning assigned in Art. 336; so that ω is the value of γ when ν has the value b. Then, as ν is supposed to vary between $-b$ and b, we have γ varying between $-\omega$ and ω.

353. In the notation of elliptic functions the relation $\nu = b \sin \psi$ may be expressed thus

$$\nu = b \sin am \left(\gamma, \frac{b}{c}\right),$$

that is, $\frac{\nu}{b}$ is the sine of the angle which is the amplitude of γ corresponding to the modulus $\frac{b}{c}$.

Then $\sqrt{(b^2-\nu^2)} = b \cos am \left(\gamma, \frac{b}{c}\right)$; and

$$\sqrt{c^2-\nu^2} = c\sqrt{1-\frac{\nu^2}{c^2}} = c\sqrt{1-\frac{b^2}{c^2}\frac{\nu^2}{b^2}} = c\sqrt{1-\frac{b^2}{c^2}\sin^2\psi}:$$

the last result is usually expressed thus

$$\sqrt{c^2-\nu^2} = c\Delta\, am \left(\gamma, \frac{b}{c}\right).$$

354. Next consider the equation

$$\beta = c\int_b^\mu \frac{d\mu}{\sqrt{\mu^2-b^2}\sqrt{c^2-\mu^2}}.$$

Assume $\sqrt{c^2-\mu^2} = \sigma$, $\sqrt{c^2-b^2} = h$; and then we shall have

$$\sqrt{\mu^2-b^2} = \sqrt{h^2-\sigma^2}, \quad \mu = \sqrt{c^2-\sigma^2};$$

therefore $$\frac{d\mu}{\sqrt{\mu^2-b^2}\sqrt{c^2-\mu^2}} = -\frac{d\sigma}{\sqrt{h^2-\sigma^2}\sqrt{c^2-\sigma^2}}.$$

Hence, by integration,

$$c\int_b^\mu \frac{d\mu}{\sqrt{\mu^2-b^2}\sqrt{c^2-\mu^2}} + c\int_0^\sigma \frac{d\sigma}{\sqrt{h^2-\sigma^2}\sqrt{c^2-\sigma^2}} = \text{constant}.$$

To determine the constant, we observe that for $\mu = c$ we have $\sigma = 0$; so that the constant becomes the ϖ of Art. 336. Hence from the preceding equation we have

$$\varpi - \beta = c\int_0^\sigma \frac{d\sigma}{\sqrt{h^2-\sigma^2}\sqrt{c^2-\sigma^2}};$$

and then as in Arts. 352 and 353 we get

$$\sigma = h \sin am \left(\varpi - \beta, \frac{h}{c}\right).$$

Thus σ may be considered known in terms of β; and then μ, $\sqrt{\mu^2 - b^2}$, and $\sqrt{c^2 - \mu^2}$ may also be considered known. For we have

$$\sqrt{c^2 - \mu^2} = h \sin am \left(\varpi - \beta, \frac{h}{c}\right),$$

$$\sqrt{\mu^2 - b^2} = h \cos am \left(\varpi - \beta, \frac{h}{c}\right),$$

$$\mu = c \,\Delta\, am \left(\varpi - \beta, \frac{h}{c}\right).$$

355. Finally, consider the equation

$$\alpha = c \int_c^\lambda \frac{d\lambda}{\sqrt{\lambda^2 - b^2}\sqrt{\lambda^2 - c^2}}.$$

Assume $\lambda = \frac{bc}{\tau}$; then we shall have

$$\sqrt{\lambda^2 - b^2} = \frac{b}{\tau}\sqrt{c^2 - \tau^2}, \qquad \sqrt{\lambda^2 - c^2} = \frac{c}{\tau}\sqrt{b^2 - \tau^2};$$

therefore $$\frac{d\lambda}{\sqrt{\lambda^2 - b^2}\sqrt{\lambda^2 - c^2}} = -\frac{d\tau}{\sqrt{b^2 - \tau^2}\sqrt{c^2 - \tau^2}}.$$

Hence, by integration,

$$c\int_c^\lambda \frac{d\lambda}{\sqrt{\lambda^2 - b^2}\sqrt{\lambda^2 - c^2}} + c\int_0^\tau \frac{d\tau}{\sqrt{b^2 - \tau^2}\sqrt{c^2 - \tau^2}} = \text{constant}.$$

To determine the constant put $\lambda = c$, then the first integral vanishes and the second becomes ω; so that the constant is equal to ω. Hence

$$\alpha = \omega - c\int_0^\tau \frac{d\tau}{\sqrt{b^2 - \tau^2}\sqrt{c^2 - \tau^2}}.$$

From this formula we deduce

$$\tau = b \sin am \left(\omega - \alpha, \frac{b}{c}\right).$$

Hence
$$\lambda = \frac{c}{\sin am\,(\omega - \alpha,\ k)},$$

$$\sqrt{\lambda^2 - b^2} = \frac{c\,\Delta\, am\,(\omega - \alpha,\ k)}{\sin am\,(\omega - \alpha,\ k)},$$

$$\sqrt{\lambda^2 - c^2} = \frac{c\cos am\,(\omega - \alpha,\ k)}{\sin am\,(\omega - \alpha,\ k)};$$

where k is put for $\frac{b}{c}$.

356. The results of Art. 354 and 355 may be put in a more convenient form by the aid of certain elementary formulæ in elliptic integrals. Thus take the notation of Art. 355, and assume that the modulus is k throughout, which will save the trouble of repeating it. We have

$$\left.\begin{aligned} \sin am\,(\omega - \alpha) &= \frac{\cos am\,\alpha}{\Delta\, am\,\alpha} \\ \cos am\,(\omega - \alpha) &= \sqrt{1-k^2}\,\frac{\sin am\,\alpha}{\Delta\, am\,\alpha} \\ \Delta\, am\,(\omega - \alpha) &= \frac{\sqrt{1-k^2}}{\Delta\, am\,\alpha} \end{aligned}\right\}\ldots\ldots\ldots\ldots(1).$$

Thus the results of Art. 355 may be written

$$\lambda = \frac{c\,\Delta\, am\,\alpha}{\cos am\,\alpha},$$

$$\sqrt{\lambda^2 - b^2} = \frac{\sqrt{c^2 - b^2}}{\cos am\,\alpha},$$

$$\sqrt{\lambda^2 - c^2} = \frac{\sqrt{c^2 - b^2}\sin am\,\alpha}{\cos am\,\alpha}.$$

357. To prove the formulæ (1) we observe that by the fundamental property of Elliptic Functions explained in the *Integral Calculus*, Chap. x., if we have

$$\int_0^\theta \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} + \int_0^\phi \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} = \int_0^\mu \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}\ldots(2),$$

then θ, ϕ and μ are connected in the manner which may be expressed by any one of the following equations,

$$\left.\begin{aligned}\cos\theta\cos\phi - \sin\theta\sin\phi\sqrt{1-k^2\sin^2\mu} &= \cos\mu,\\ \cos\phi\cos\mu + \sin\phi\sin\mu\sqrt{1-k^2\sin^2\theta} &= \cos\theta,\\ \cos\theta\cos\mu + \sin\theta\sin\mu\sqrt{1-k^2\sin^2\phi} &= \cos\phi.\end{aligned}\right\}\ldots\ldots(3).$$

The modulus being supposed to be k throughout, let

$$\theta = am\, u, \text{ and } \phi = am\, v;$$

then (2) gives

$$\mu = am\,(u+v).$$

Thus equations (3) may be expressed as follows,

$$\left.\begin{aligned}\cos am(u+v) &= \cos am\,u \cos am\,v - \sin am\,u \sin am\,v\,\Delta\, am\,(u+v)\\ \cos am\,u &= \cos am\,v \cos am(u+v) + \sin am\,v \sin am(u+v)\,\Delta\, am\,u\\ \cos am\,v &= \cos am\,u \cos am(u+v) + \sin am\,u \sin am(u+v)\,\Delta\, am\,v\end{aligned}\right\}$$
$$\ldots\ldots(4).$$

Suppose that $\mu = \frac{\pi}{2}$, then $\int_0^\mu \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$ becomes the ω of Art. 336; also $\sin am\,(u+v) = 1$, and $\cos am\,(u+v) = 0$. Thus the second of equations (4) gives

$$\cos am\,u = \sin am\,v\,\Delta\, am\,u \ldots\ldots\ldots\ldots\ldots(5).$$

This coincides with the first of equations (1), for we may put α for u, and then (2) gives $v = \omega - \alpha$.

Again, supposing still that $\mu = \frac{\pi}{2}$, the first of equations (4) gives

$$\cos am\,u \cos am\,v = \sin am\,u \sin am\,v\sqrt{1-k^2};$$

divide this by (5); thus

$$\cos am\,v = \frac{\sin am\,u\sqrt{1-k^2}}{\Delta\, am\,u}\ldots\ldots\ldots\ldots\ldots(6).$$

This coincides with the second of equations (2).

Finally, square the first of equations (1), and multiply by $1-k^2$; then square the second of equations (1); add the two results and extract the square root, and we obtain the third of equations (1).

358. In like manner the results of Art. 354 may be modified in form by the use of equations like (1) of Art. 356.

359. In the results of Art. 353 we see that ν is expressed in terms of a sine, and so may be regarded as an *odd* function of γ; while $\sqrt{b^2-\nu^2}$ and $\sqrt{c^2-\nu^2}$ may be regarded as *even* functions of γ. Again, in the results of Art. 354 when we use equations like (1), we shall see in like manner that $\sqrt{\mu^2-b^2}$ may be regarded as an *odd* function of β; while $\sqrt{c^2-\mu^2}$ and μ may be regarded as *even* functions of β. Finally in the results of Art. 355, as modified in Art. 356, we see that $\sqrt{\lambda^2-c^2}$ may be regarded as an *odd* function of α, while λ and $\sqrt{\lambda^2-b^2}$ may be regarded as *even* functions of α.

360. As an example of the values of the auxiliary variables α, β, γ at special points, consider the ellipsoid represented by the first equation of Art. 266. At all points of the surface of the ellipsoid λ has the same value, and so α will have the same *numerical* value.

At the ends of the major axis we have $\beta=\pm\varpi$, and $\gamma=\pm\omega$; the upper signs belonging to one end and the lower to the other. At the ends of the mean axis we have $\beta=\pm\varpi$, and $\gamma=0$; the upper signs belonging to one end, and the lower to the other. At the ends of the least axis we have $\beta=0$ and $\gamma=0$. See Art. 267.

361. We shall not enter here further into the consideration of Elliptic Functions; we may observe that the first of Lamé's works, cited in Art. 266, is much concerned with this department of analysis, but by no means supersedes the necessity of studying the systematic treatises on the subject.

362. In Art. 326 we do not profess to obtain the *most general* solution of a certain differential equation, but only *a* solution. Also when we treated one of the differential equations of Art. 327 we did not seek the *most general*

solution, but only obtained *a* solution. In this latter case however it is easy to complete the process, at least theoretically, and thus to obtain the most general solution.

For let L denote one solution of the differential equation

$$\frac{d^2L}{d\alpha^2} = \left\{n(n+1)\frac{\lambda^2}{c^2} - g\right\} L \quad \text{..............} (7),$$

and let S denote a second solution; so that

$$\frac{d^2S}{d\alpha^2} = \left\{n(n+1)\frac{\lambda^2}{c^2} - g\right\} S \quad \text{..............} (8).$$

From (7) and (8) we obtain

$$L\frac{d^2S}{d\alpha^2} - S\frac{d^2L}{d\alpha^2} = 0;$$

therefore, by integration,

$$L\frac{dS}{d\alpha} - S\frac{dL}{d\alpha} = C_1, \text{ a constant.}$$

Divide by L^2, and integrate; thus

$$S = C_1 L \int \frac{d\alpha}{L^2} \quad \text{........................} (9).$$

Thus the solution of (7) may be given in the form $C_1 L \int \frac{d\alpha}{L^2} + C_2 L$, where C_2 is another constant; and as there are here two arbitrary constants this is the general solution.

363. Lamé tacitly assumes that for the solution of his problem we must put $C_1 = 0$. Mathieu gives on his page 255 a reason for this. We have

$$\int \frac{d\alpha}{L^2} = \int \frac{c\,d\lambda}{L^2\sqrt{(\lambda^2 - b^2)(\lambda^2 - c^2)}} \quad \text{............} (10).$$

Now corresponding to $\alpha = 0$ we have $\lambda = c$; and then the first surface of Art. 266 degenerates from an ellipsoid to the area on the plane of (x, y) bounded by the ellipse

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - b^2} = 1 \quad \text{....................} (11).$$

The value of V ought to differ very little for two points which are very near the area bounded by the ellipse (11), one point being on one side of the plane (x, y), and the other on the other. But the formula in (10) changes sign with α, for it changes sign with $\sqrt{\lambda^2-c^2}$; and thus V would differ to a finite extent for two such points though indefinitely close.

Hence for the solution of Lamé's problem we must put $C_1=0$.

364. But for the solution of other problems it might happen that we must put $C_2=0$. Suppose for instance we want to find the potential of the ellipsoid defined by the first equation of Art. 266 for all external points. Then for all such points the equation (1) of Art. 324 must hold with respect to the potential V. Moreover for points at an indefinitely great distance from the ellipsoid the potential must *vanish*. Now when λ is very great we find that the L of equation (9) or (10) varies approximately as λ^n, and then $L\int\frac{d\alpha}{L^2}$ will vary approximately as $\frac{1}{\lambda^{n+1}}$. It is obvious therefore that the potential cannot involve the term C_2L, though it may involve the term $C_1L\int\frac{d\alpha}{L^2}$.

365. In Art. 341 we have shewn that all the values of z are real; this result can also be deduced readily from equation (4) of Art. 343, as by Mathieu on his page 265.

For if possible let $\zeta+\zeta'\sqrt{-1}$ denote a value of z; let $M_1+M_2\sqrt{-1}$ denote the corresponding value of M, and $N_1+N_2\sqrt{-1}$ that of N. Then there must also be a value $\zeta-\zeta'\sqrt{-1}$ of z, and we may take for M' the value $M_1-M_2\sqrt{-1}$, and for N' the value $N_1-N_2\sqrt{-1}$. Thus (4) of Art. 343 becomes

$$\int_0^{\varpi}\int_0^{\omega}(M_1^2+M_2^2)(N_1^2+N_2^2)(\mu^2-\nu^2)\,d\beta d\gamma=0;$$

but this is obviously impossible, for μ^2 is greater than ν^2, so that every element of the integral is positive.

366. If we compare equation (4) of Art. 343 with the corresponding equation respecting Laplace's coefficients, which is given in Art. 187, we shall be led to anticipate that $(\mu^2 - \nu^2)\, d\beta d\gamma$ is the variable part of the transformation of $\sin\theta\, d\theta\, d\phi$. This is easily verified. For we know by the *Integral Calculus*, Art. 246, that $d\theta\, d\phi$ transforms to

$$\left(\frac{d\theta}{d\gamma}\,\frac{d\phi}{d\beta} - \frac{d\theta}{d\beta}\,\frac{d\phi}{d\gamma}\right) d\beta\, d\gamma.$$

Now by Art. 303 we have $\dfrac{d\theta}{d\gamma}\,\dfrac{d\phi}{d\beta} - \dfrac{d\theta}{d\beta}\,\dfrac{d\phi}{d\gamma}$

$$= -\frac{1}{c^2 \sin\theta\,(\mu^2\nu^2 - b^2c^2)} \left\{\mu^2 (b^2 - \nu^2)(c^2 - \nu^2) + \nu^2 (\mu^2 - b^2)(c^2 - \mu^2)\right\}$$

$$= -\frac{\mu^2\nu^4 - \nu^2\mu^4 + b^2c^2(\mu^2 - \nu^2)}{c^2 \sin\theta\,(\mu^2\nu^2 - b^2c^2)} = \frac{\mu^2 - \nu^2}{c^2 \sin\theta},$$

so that $c^2 \sin\theta\, d\theta\, d\phi$ is equivalent to $(\mu^2 - \nu^2)\, d\beta\, d\gamma$.

367. It ought to be remarked that the notation of the present volume is not coincident with Lamé's; for English readers would be displeased with his neglect of symmetry. The following table will exhibit the principal changes which have been made; the first column contains the symbols of the present volume, and the second column Lamé's corresponding symbols,

$\lambda,\ \mu,\ \nu$	$\rho,\ \mu,\ \nu$
$\alpha,\ \beta,\ \gamma$	$\gamma,\ \beta,\ \alpha$
$L,\ M,\ N$	$R,\ M,\ N$

368. In Chapter XXVI. we have investigated Lamé's results independently as he does himself; they might however have been derived from Laplace's results, by the aid of the transformation of Chapter XXIV. Heine pays some attention to this mode of derivation; I may remark that he states on his page 207 the result obtained in Chapter XXIV. without reference to a place where it is worked out, or any warning of the length of the necessary process.

Lamé says on his page 196 with respect to his independent treatment: Facilement applicable à tout autre système de coordonnées curvilignes, cette méthode directe a l'inappréciable avantage d'éviter tout passage par l'antique système des coordonnées rectilignes: instrument désormais impuissant et stérile, dont l'emploi abusif sera plutôt un obstacle qu'un secours pour les progrès futurs des diverses branches de la physique mathématique. It may however be doubted whether Lamé's opinion of his own methods as compared with those of his predecessors is not too favourable.

CHAPTER XXX.

DEFINITION OF BESSEL'S FUNCTIONS.

369. THE functions we are now about to consider were formally introduced to the attention of mathematicians by the distinguished astronomer Bessel, in a memoir published in 1824 in the Transactions of the Berlin Academy. They have since been the subject of investigations in various memoirs, and have been discussed in two special treatises which have the following titles: *Theorie der Bessel'schen Functionen* ... von Carl Neumann, Leipzig 1867; *Studien über die Bessel'schen Functionen*, von Dr Eugen Lommel, Leipzig 1868. These two treatises supply references to various memoirs on the subject.

In the present and following Chapters we shall give all the most important theorems relating to these functions.

370. If we seek for a series proceeding according to ascending powers of x, which satisfies the differential equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left(1 - \frac{n^2}{x^2}\right)u = 0 \quad \text{.................} \quad (1),$$

we obtain

$$u = Cx^n\left\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \frac{x^6}{2.4.6(2n+2)(2n+4)(2n+6)} + \dots\right\},$$

where C is an arbitrary constant.

If we suppose n a positive integer, and ascribe to C the value $\frac{1}{2^n \lfloor n}$, the expression is called *Bessel's Function*, and is denoted by $J_n(x)$, so that

$$J_n(x) = \frac{x^n}{2^n \lfloor n} \left\{ 1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \frac{x^6}{2.4.6(2n+2)(2n+4)(2n+6)} + \ldots \right\} \ldots (2).$$

The series within the brackets is always *convergent;* see *Algebra*, Art. 559.

Or, taking a somewhat more general view, let us ascribe to the constant C the value $\frac{1}{2^n \Gamma(n+1)}$; this will agree with the former when n is a positive integer, and will be real and finite, whatever n may be, provided $n+1$ be positive. Thus we have

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left\{ 1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \frac{x^6}{2.4.6(2n+2)(2n+4)(2n+6)} \ldots \right\} \ldots\ldots (3).$$

This then is the definition of Bessel's Function, n being any real quantity algebraically greater than -1, and x any real quantity.

The student is supposed to be acquainted with the properties of the *Gamma Function:* see *Integral Calculus*, Chapter XII.

371. We may also express Bessel's Function in the following manner by a definite integral for *any value of n which is algebraically greater than* $-\frac{1}{2}$:

$$J_n(x) = \frac{x^n}{\sqrt{\pi}\, 2^n \Gamma\left(n+\frac{1}{2}\right)} \int_0^\pi \cos(x \cos\phi) \sin^{2n}\phi \, d\phi \ldots (4).$$

For $\cos(x\cos\phi) = 1 - \frac{x^2}{\lfloor 2}\cos^2\phi + \frac{x^4}{\lfloor 4}\cos^4\phi - \frac{x^6}{\lfloor 6}\cos^6\phi + \ldots$;
and thus the general term under the integral sign may be denoted by

$$\frac{(-1)^m}{\lfloor 2m} x^{2m} \int_0^\pi \cos^{2m}\phi \sin^{2n}\phi \, d\phi.$$

Put $\cos^2\phi = t$; thus we get

$$\int_0^\pi \cos^{2m}\phi \sin^{2n}\phi \, d\phi = 2\int_0^{\frac{\pi}{2}} \cos^{2m}\phi \sin^{2n}\phi \, d\phi$$

$$= \int_0^1 t^{\frac{2m-1}{2}} (1-t)^{\frac{2n-1}{2}} dt = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+m+1)}$$

$$= \frac{(2m-1)(2m-3)\ldots 1\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{2^m(n+m)(n+m-1)\ldots(n+1)\,\Gamma(n+1)}.$$

Thus the general term on the, right-hand side of (4) becomes

$$\frac{x^n}{2^n\, 2.4\ldots 2m} \times \frac{(-1)^m x^{2m}}{2^m(n+m)(n+m-1)\ldots(n+1)\,\Gamma(n+1)};$$

and this coincides with the general term in (3).

372. We may also express Bessel's Function in another manner by a definite integral, for any *positive integral value* of n, thus:

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - x\sin\phi)\, d\phi \;\ldots\ldots\ldots\ldots (5).$$

For this expression

$$= \frac{1}{\pi}\int_0^\pi \{\cos n\phi \cos(x\sin\phi) + \sin n\phi \sin(x\sin\phi)\}\, d\phi;$$

it is necessary to treat separately the cases of n odd and n even.

First suppose n *even;* then $\int_0^\pi \sin n\phi \sin(x\sin\phi)\,d\phi$ vanishes. For by changing ϕ into $\pi - \phi'$ we have

$$\int_0^\pi \sin n\phi \sin(x\sin\phi)\,d\phi = \int_0^\pi \sin n(\pi - \phi')\sin(x\sin\phi')\,d\phi'$$

$$= -\cos n\pi \int_0^\pi \sin n\phi' \sin(x\sin\phi')\,d\phi'$$

$$= -\int_0^\pi \sin n\phi \sin(x\sin\phi)\,d\phi;$$

thus $$2\int_0^\pi \sin n\phi \sin(x\sin\phi)\,d\phi = 0.$$

Hence the proposed definite integral reduces to

$$\frac{1}{\pi}\int_0^\pi \cos n\phi \cos(x\sin\phi)\,d\phi,$$

and this $$= \frac{1}{\pi}\int_0^\pi \cos n\phi \left\{1 - \frac{x^2\sin^2\phi}{\lfloor 2} + \frac{x^4\sin^4\phi}{\lfloor 4} - \dots + \frac{(-1)^m x^{2m}\sin^{2m}\phi}{\lfloor 2m} - \dots\right\} d\phi.$$

Now let the powers of $\sin\phi$ be expressed in terms of cosines of multiples of ϕ by the formula

$$2^{2m-1}(-1)^m \sin^{2m}\phi = \cos 2m\phi - 2m\cos(2m-2)\phi + \frac{2m(2m-1)}{\lfloor 2}\cos(2m-4)\phi - \dots;$$

then if there be a term which involves $\cos n\phi$ there will be a corresponding term in $\int_0^\pi \cos n\phi \sin^{2m}\phi\,d\phi$, and no other. In this way we obtain the required result.

Next suppose n *odd;* then $\int_0^\pi \cos n\phi \cos(x\sin\phi)\,d\phi$ vanishes. For by changing ϕ into $\pi - \phi'$ we have

$$\int_0^\pi \cos n\phi \cos(x \sin\phi)\, d\phi = \int_0^\pi \cos n(\pi - \phi') \cos(x \sin\phi')\, d\phi'$$

$$= \cos n\pi \int_0^\pi \cos n\phi' \cos(x\sin\phi')\, d\phi'$$

$$= -\int_0^\pi \cos n\phi \cos(x \sin\phi)\, d\phi;$$

thus $$2\int_0^\pi \cos n\phi \cos(x\sin\phi)\, d\phi = 0.$$

Hence the proposed definite integral reduces to

$$\frac{1}{\pi}\int_0^\pi \sin n\phi \sin(x\sin\phi)\, d\phi,$$

and this $$= \frac{1}{\pi}\int_0^\pi \sin n\phi \left\{x\sin\phi - \frac{x^3\sin^3\phi}{\lfloor 3} + \dots + \frac{(-1)^m x^{2m+1}\sin^{2m+1}\phi}{\lfloor 2m+1} + \dots\right\} d\phi.$$

Now let the powers of $\sin\phi$ be expressed in terms of sines of multiples of ϕ by the formula

$$2^{2m}(-1)^m \sin^{2m+1}\phi = \sin(2m+1)\phi - (2m+1)\sin(2m-1)\phi + \dots;$$

then proceeding as before we obtain the required result.

373. We may observe that for the case in which *n is a positive integer* the formula of Art. 371 may be deduced from that of Art. 372.

First suppose *n even;* then by Art. 372

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos n\phi \cos(x\sin\phi)\, d\phi.$$

Change ϕ into $\frac{\pi}{2} + \phi'$; thus we get

$$J_n(x) = \frac{1}{\pi}\cos\frac{n\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos n\phi' \cos(x\cos\phi')\, d\phi'$$

$$=\frac{2}{\pi}\cos\frac{n\pi}{2}\int_0^{\frac{\pi}{2}}\cos n\phi'\cos(x\cos\phi')\,d\phi'$$

$$=\frac{1}{\pi}\cos\frac{n\pi}{2}\int_0^{\pi}\cos n\phi'\cos(x\cos\phi')\,d\phi';$$

see *Integral Calculus*, Art. 42.

But by Jacobi's Formula, given in *Differential Calculus*, Art. 370, if $t=\cos\phi'$, then

$$\cos n\phi'\,d\phi'=\frac{(-1)^{n-1}}{1.3.5\ldots(2n-1)}\frac{d^n}{dt^n}(1-t^2)^{\frac{2n-1}{2}}\,dt.$$

Therefore if $f(\cos\phi')$ denote any function of $\cos\phi'$, we have

$$\int_0^{\pi}f(\cos\phi')\cos n\phi'\,d\phi'$$

$$=\frac{(-1)^{n-1}}{1.3.5\ldots(2n-1)}\int_1^{-1}f(t)\frac{d^n}{dt^n}(1-t^2)^{\frac{2n-1}{2}}dt;$$

integrate by parts n times in succession, and we finally obtain

$$\int_0^{\pi}f(\cos\phi')\cos n\phi'\,d\phi'=\frac{1}{1.3.5\ldots(2n-1)}\int_0^{\pi}f^{(n)}(t)\sin^{2n}\phi'\,d\phi'.$$

Put $f(\cos\phi')=\cos(x\cos\phi')$; then

$$f^{(n)}(t)=x^n\cos\left(x\cos\phi'+\frac{n\pi}{2}\right)=x^n\cos\frac{n\pi}{2}\cos(x\cos\phi').$$

Thus $J_n(x)=\dfrac{x^n}{\pi.1.3.5\ldots(2n-1)}\displaystyle\int_0^{\pi}\cos(x\cos\phi')\sin^{2n}\phi'\,d\phi'$, which agrees with equation (4).

Next suppose n odd; then by Art. 372

$$J_n(x)=\frac{1}{\pi}\int_0^{\pi}\sin n\phi\sin(x\sin\phi)\,d\phi.$$

Change ϕ into $\frac{\pi}{2}+\phi'$; thus we get

$$J_n(x) = \frac{1}{\pi} \sin \frac{n\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos n\phi' \sin (x \cos \phi')\, d\phi'$$

$$= \frac{2}{\pi} \sin \frac{n\pi}{2} \int_0^{\frac{\pi}{2}} \cos n\phi' \sin (x \cos \phi')\, d\phi'$$

$$= \frac{1}{\pi} \sin \frac{n\pi}{2} \int_0^{\pi} \cos n\phi' \sin (x \cos \phi')\, d\phi'.$$

Then use Jacobi's formula as before, and we arrive at the same result.

374. In equation (4) put z for $\cos \phi$; thus we obtain

$$J_n(x) = \frac{1}{\sqrt{\pi}} \frac{x^n}{2^n \Gamma\left(n + \frac{1}{2}\right)} \int_{-1}^{1} \cos (zx)\, (1 - z^2)^{n-\frac{1}{2}}\, dz \ldots (6).$$

375. In the expression just obtained put $1 - v$ for z; thus

$$J_n(x) = \frac{1}{\sqrt{\pi}} \frac{x^n}{2^n \Gamma\left(n + \frac{1}{2}\right)} \int_0^2 \cos \{x(1 - v)\} \{v (2 - v)\}^{n-\frac{1}{2}}\, dv;$$

now $\quad \int_0^2 \cos \{x(1-v)\} \{v (2-v)\}^{n-\frac{1}{2}}\, dv =$

$$\cos x \int_0^2 \cos (xv) \{v (2-v)\}^{n-\frac{1}{2}}\, dv + \sin x \int_0^2 \sin (xv) \{v (2-v)\}^{n-\frac{1}{2}}\, dv.$$

If we expand $\cos (xv)$ and $\sin (xv)$ in powers of xv we obtain expressions to integrate of which the general type is

$$\int_0^2 v^m \{v (2-v)\}^{n-\frac{1}{2}}\, dv.$$

Put $2y$ for v; thus we get $2^{m+2n} \int_0^1 y^{m+n-\frac{1}{2}} (1 - y)^{n-\frac{1}{2}}\, dy$,

that is $\quad \dfrac{2^{m+2n} \Gamma\left(m + n + \frac{1}{2}\right) \Gamma\left(n + \frac{1}{2}\right)}{\Gamma (m + 2n + 1)}.$

Thus $J_n(x) = \dfrac{1}{\sqrt{\pi}} \dfrac{x^n}{2^n \Gamma\left(n+\frac{1}{2}\right)} (C\cos x + S\sin x)$, where

$$C = \frac{2^{2n}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)} - \frac{x^2}{\lfloor 2} \frac{2^{2n+2}\Gamma\left(2+n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2+2n+1)}$$

$$+ \frac{x^4}{\lfloor 4} \frac{2^{2n+4}\Gamma\left(4+n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(4+2n+1)} - \ldots,$$

$$S = x\frac{2^{2n+1}\Gamma\left(1+n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(1+2n+1)}$$

$$- \frac{x^3}{\lfloor 3} \frac{2^{2n+3}\Gamma\left(3+n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(3+2n+1)}$$

$$+ \frac{x^5}{\lfloor 5} \frac{2^{2n+5}\Gamma\left(5+n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(5+2n+1)} - \ldots$$

We may change the expressions for C and S, since

$$\frac{\Gamma\left(m+n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(m+2n+1)}$$

$$= \frac{\left(m+n-\frac{1}{2}\right)\left(m+n-\frac{3}{2}\right)\ldots\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{(m+2n)(m+2n-1)\ldots(2n+1)\Gamma(2n+1)},$$

and $\Gamma(2n+1) = \dfrac{2^{2n}}{\sqrt{\pi}} \Gamma\left(n+\frac{1}{2}\right)\Gamma(n+1)$, (*Integral Calculus*, Art. 267).

Thus $$J_n(x) = \frac{x^n \cos x}{2^n \Gamma(n+1)} \left\{1 - \frac{2n+3}{2n+2} \frac{x^2}{\lfloor 2} + \frac{(2n+5)(2n+7)}{(2n+2)(2n+4)} \frac{x^4}{\lfloor 4} - \frac{(2n+7)(2n+9)(2n+11)}{(2n+2)(2n+4)(2n+6)} \frac{x^6}{\lfloor 6} + \ldots\right\}$$
$$+ \frac{x^n \sin x}{2^n \Gamma(n+1)} \left\{x - \frac{2n+5}{2n+2} \frac{x^3}{\lfloor 3} + \frac{(2n+7)(2n+9)}{(2n+2)(2n+4)} \frac{x^5}{\lfloor 5} - \ldots\right\} \ldots (7).$$

The series within the brackets are always convergent.

376. Suppose $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$ to be expanded in powers of z. Since this is the same as $e^{\frac{xz}{2}} e^{-\frac{x}{2z}}$, we obtain

$$\left\{1 + \frac{xz}{2} + \frac{x^2 z^2}{2^2 \lfloor 2} + \frac{x^3 z^3}{2^3 \lfloor 3} + \ldots\right\} \left\{1 - \frac{x}{2z} + \frac{x^2}{2^2 \lfloor 2\, z^2} - \frac{x^3}{2^3 \lfloor 3\, z^3} + \ldots\right\}.$$

Multiply out and arrange in powers of z; and then according to the notation of Art. 370 we obtain

$$e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = J_0(x) + z\, J_1(x) + z^2 J_2(x) + z^3 J_3(x) + \ldots$$
$$- \frac{J_1(x)}{z} + \frac{J_2(x)}{z^2} - \frac{J_3(x)}{z^3} + \ldots\ldots\ldots\ldots (8).$$

Thus we see that for positive integral values of n we have $J_n(x)$ equal to the coefficient of z^n in the expansion of $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$ in powers of z.

377. It should be remarked that the definition of the Functions has been slightly modified by Hansen who is followed by Schlömilch; see *Zeitschrift für Mathematik*, Vol. II. page 145: according to these mathematicians we should have $2x$ instead of x in the various expressions which we have given for $J_n(x)$, so that for instance they put

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\phi - 2x \sin\phi)\, d\phi.$$

We mention this in order that the student may be prepared for the diversity if it should occur in other works; but we shall adhere to the definition we have formerly given.

378. As a simple example, we observe that by Art. 371 we have

$$J_1(x) = \frac{x}{\pi}\int_0^\pi \cos(x\cos\phi)\sin^2\phi\, d\phi;$$

by changing ϕ into $\frac{\pi}{2} - \phi$ we obtain

$$J_1(x) = \frac{x}{\pi}\int_0^\pi \cos(x\sin\phi)\cos^2\phi\, d\phi.$$

By integrating the following expressions by parts, we see that each of them is also equal to $J_1(x)$;

$$\frac{1}{\pi}\int_0^\pi \sin(x\cos\phi)\cos\phi\, d\phi, \quad \frac{1}{\pi}\int_0^\pi \sin(x\sin\phi)\sin\phi\, d\phi:$$

either of these may be obtained from the other by changing ϕ into $\frac{\pi}{2} - \phi$.

Again, by comparing the equation (3) with the known expressions for $\cos z$ and $\sin z$, it is easy to see that

$$\text{when } n = \frac{1}{2}, \text{ we have } J_n(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$

$$\text{and when } n = \frac{3}{2}, \text{ we have } J_n(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right).$$

CHAPTER XXXI.

PROPERTIES OF BESSEL'S FUNCTIONS.

379. DIFFERENTIATE both sides of equation (8) of Art. 376 with respect to z: thus

$$\frac{x}{2}\left(1+\frac{1}{z^2}\right)e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_1(x) + 2zJ_2(x) + 3z^2J_3(x) + \dots$$

$$+\frac{J_1(x)}{z^2} - \frac{2J_2(x)}{z^3} + \frac{3J_3(x)}{z^4} - \dots$$

Hence if we multiply the series on the right hand of equation (8) of Art. 376 by $\frac{x}{2}\left(1+\frac{1}{z^2}\right)$ the result must be equal to the series on the right hand of the equation just given. Thus we obtain for any positive integral value of n,

$$\frac{x}{2}\{J_{n-1}(x) + J_{n+1}(x)\} = nJ_n(x) \quad \dots\dots\dots\dots (1).$$

380. The equation (8) of Art. 376 can be made in this manner to furnish various formulæ, which may if we please be verified by the use of some of the other expressions given for $J_n(x)$. Thus for instance we may obtain (1) by the aid of the expression of Art. 372. For let $\psi = n\phi - x\sin\phi$, so that

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos\psi\, d\phi,$$

$$J_{n-1}(x) = \frac{1}{\pi}\int_0^\pi \cos(\psi-\phi)\, d\phi,$$

$$J_{n+1}(x) = \frac{1}{\pi}\int_0^\pi \cos(\psi+\phi)\, d\phi;$$

therefore $\quad J_{n-1}(x) + J_{n+1}(x) = \frac{2}{\pi}\int_0^\pi \cos\psi\cos\phi\, d\phi.$

Now $\quad d \sin\psi = \cos\psi\, d\psi = \cos\psi\,(n\,d\phi - x\cos\phi\, d\phi)$;
integrate between the limits 0 and π for ϕ: thus

$$0 = \int_0^\pi \cos\psi\,(n - x\cos\phi)\,d\phi$$

$$= n\int_0^\pi \cos\psi\, d\phi - x\int_0^\pi \cos\psi \cos\phi\, d\phi;$$

therefore $\qquad 0 = nJ_n(x) - \frac{x}{2}\{J_{n-1}(x) + J_{n+1}(x)\}.$

This investigation, like that of Art. 379, applies to the case in which n is a positive integer; but we may verify the equation by means of equation (3) of Art. 370, and thus it will be seen to hold for *every positive* value of n.

381. Differentiate both sides of equation (8) of Art. 376 with respect to x; thus

$$\frac{1}{2}\left(z - \frac{1}{z}\right) e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = \frac{dJ_0(x)}{dx} + z\frac{dJ_1(x)}{dx} + z^2\frac{dJ_2(x)}{dx} + z^3\frac{dJ_3(x)}{dx} + \dots$$
$$- \frac{1}{z}\frac{dJ_1(x)}{dx} + \frac{1}{z^2}\frac{dJ_2(x)}{dx} - \frac{1}{z^3}\frac{dJ_3(x)}{dx} + \dots$$

Hence if we multiply the series on the right-hand side of equation (8) of Art. 376 by $\frac{1}{2}\left(z - \frac{1}{z}\right)$ the result must be equal to the series on the right-hand side of the equation just given. Thus we obtain for any positive integral value of n,

$$\frac{dJ_n(x)}{dx} = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\} \quad\dots\dots\dots\dots (2);$$

and we have also the special result

$$\frac{dJ_0(x)}{dx} = -J_1(x) \quad\dots\dots\dots\dots\dots\dots\dots\dots (3).$$

The equation (2) may also be obtained by the aid of the expression of Art. 372. For as in Art. 380 we have

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos\psi\, d\phi,$$

therefore $\frac{dJ_n(x)}{dx} = \frac{1}{\pi}\int_0^\pi \frac{d\cos\psi}{dx}\,d\phi = \frac{1}{\pi}\int_0^\pi \sin\psi\sin\phi\,d\phi$;

also $J_{n-1}(x) - J_{n+1}(x) = \frac{1}{\pi}\int_0^\pi \{\cos(\psi-\phi) - \cos(\psi+\phi)\}\,d\phi$

$$= \frac{2}{\pi}\int_0^\pi \sin\psi\sin\phi\,d\phi;$$

therefore $$\frac{dJ_n(x)}{dx} = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}.$$

Similarly (3) may be obtained, observing that we have by Art. 372

$$J_1(x) = \frac{1}{\pi}\int_0^\pi \sin\phi\sin(x\sin\phi)\,d\phi.$$

We may also verify equation (2) by means of equation (3) of Art. 370, and then it will be seen to hold for *every positive* value of n.

382. From (1) and (2) we obtain

$$\frac{2n}{x}\frac{d}{dx}\{J_n(x)\}^2 = \{J_{n-1}(x)\}^2 - \{J_{n+1}(x)\}^2 \;\ldots\ldots.\; (4).$$

383. We have by Art. 376

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0(x) + zJ_1(x) + z^2J_2(x) + \ldots - \frac{J_1(x)}{z} + \frac{J_2(x)}{z^2} - \ldots$$

Change the sign of z; thus

$$e^{-\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0(x) - zJ_1(x) + z^2J_2(x) - \ldots + \frac{J_1(x)}{z} + \frac{J_2(x)}{z^2} + \ldots$$

Hence since $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} \times e^{-\frac{x}{2}\left(z-\frac{1}{z}\right)} = 1$ we have unity for the product of the two series just written; and this gives rise to various results by equating to zero the coefficients of various powers of z. By considering the terms independent of z we obtain

$$1 = \{J_0(x)\}^2 + 2\{J_1(x)\}^2 + 2\{J_2(x)\}^2 + 2\{J_3(x)\}^2 + \ldots\ldots (5).$$

384. Multiply both sides of equation (4) by n; thus

$$\frac{2n^2}{x}\frac{d}{dx}\{J_n(x)\}^2 = n\,[\{J_{n-1}(x)\}^2 - \{J_{n+1}(x)\}^2].$$

Ascribe to n in succession all positive integral values 1, 2, 3, ... and add; then the terms on the right-hand side reduce to the series which occurs in (5), and thus

$$\frac{2}{x}\Sigma n^2 \frac{d}{dx}\{J_n(x)\}^2 = 1.$$

385. Differentiate (2); thus

$$2\frac{d^2J_n(x)}{dx^2} = \frac{dJ_{n-1}(x)}{dx} - \frac{dJ_{n+1}(x)}{dx};$$

substitute for the differential coefficients on the right-hand side their values from (2); thus

$$2^2\frac{d^2J_n(x)}{dx^2} = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x).$$

Similarly $2^3\frac{d^3J_n(x)}{dx^3} = J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)$; and so on.

These formulæ must be understood with the conditions which follow from (2); thus in the last which is expressed n may be any positive quantity greater than 3.

Thus the successive differential coefficients of any one of Bessel's Functions can be expressed in terms of Functions of higher and lower orders.

386. From (1) and (2) we have

$$J_{n+1}(x) = \frac{n}{x}J_n(x) - \frac{dJ_n(x)}{dx} \qquad (6),$$

and

$$J_{n-1}(x) = \frac{n}{x}J_n(x) + \frac{dJ_n(x)}{dx} \qquad (7).$$

Then from (3) and (6) we obtain in succession

$$J_1(x) = -\frac{dJ_0(x)}{dx},$$

$$J_2(x) = -\frac{1}{x}\frac{dJ_0(x)}{dx} + \frac{d^2J_0(x)}{dx^2},$$

$$J_3(x) = -\frac{3}{x^2}\frac{dJ_0(x)}{dx} + \frac{3}{x}\frac{d^2J_0(x)}{dx^2} - \frac{d^3J_0(x)}{dx^3},$$

and so on.

Thus for a positive integral value of n we can express $J_n(x)$ in terms of $J_0(x)$ and the differential coefficients of $J_0(x)$.

387. From (6) we have

$$\frac{dJ_n(x)}{dx} = \frac{n}{x}J_n(x) - J_{n+1}(x);$$

therefore $\dfrac{d^2J_n(x)}{dx^2} = -\dfrac{n}{x^2}J_n(x) + \dfrac{n}{x}\dfrac{dJ_n(x)}{dx} - \dfrac{dJ_{n+1}(x)}{dx}$

$$= -\frac{n}{x^2}J_n(x) + \frac{n}{x}\left\{\frac{n}{x}J_n(x) - J_{n+1}(x)\right\} + \frac{n+1}{x}J_{n+1}(x) - J_n(x),$$

by (6) and (7).

Thus

$$\frac{d^2J_n(x)}{dx^2} = \left\{\frac{n(n-1)}{x^2} - 1\right\}J_n(x) + \frac{1}{x}J_{n+1}(x).$$

We may now differentiate again, and thus obtain $\dfrac{d^3J_n(x)}{dx^3}$ in terms of $J_n(x)$ and $J_{n+1}(x)$; and so on.

388. Let $Q_n(x)$ stand for $\dfrac{xJ_{n+1}(x)}{J_n(x)}$. From (1) we have

$$\frac{J_{n-1}(x)}{J_n(x)} + \frac{J_{n+1}(x)}{J_n(x)} = \frac{2n}{x};$$

therefore $$\frac{x}{Q_{n-1}(x)} + \frac{Q_n(x)}{x} = \frac{2n}{x};$$

therefore $$Q_n(x) = 2n - \frac{x^2}{Q_{n-1}(x)};$$

therefore $$Q_{n+1}(x) = 2(n+1) - \frac{x^2}{Q_n(x)};$$

therefore $$Q_n(x) = \frac{x^2}{2(n+1) - Q_{n+1}(x)}.$$

Hence, continuing the process, we have

$$Q_n(x) = \cfrac{x^2}{2(n+1) - \cfrac{x^2}{2(n+2) - Q_{n+2}(x)}};$$

and so on.

Moreover we can shew that $Q_{n+m}(x)$ vanishes when m is indefinitely great; for, by Art. 370,

$$Q_{n+m}(x) = \frac{xJ_{n+m+1}(x)}{J_{n+m}(x)} = \frac{x^2}{2(n+m+1)} \frac{1 - \dfrac{x^2}{4(n+m+2)} + \dots}{1 - \dfrac{x^2}{4(n+m+1)} + \dots};$$

the first factor vanishes, while the second factor is finite when m is indefinitely great.

Hence our process develops $Q_n(x)$ into an infinite convergent fraction of the second class, in which the first component is $\dfrac{x^2}{2(n+1)}$, and the r^{th} component is $\dfrac{x^2}{2(n+r+1)}$: see *Algebra*, Art. 778.

389. Various interesting theorems have been obtained with respect to Bessel's Functions when the variable is not x but $\sqrt{x}$; with some of these we shall close the present Chapter.

390. To shew that

$$\frac{d^m}{dx^m}\{x^{-\frac{n}{2}}J_n(\sqrt{x})\} = \left(-\frac{1}{2}\right)^m x^{-\frac{n+m}{2}} J_{n+m}(\sqrt{x}).$$

By Art. 371 we have

$$x^{-\frac{n}{2}}J_n(\sqrt{x}) = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2^n \Gamma\left(n+\frac{1}{2}\right)} \int_0^\pi \cos(\sqrt{x}\cos\phi)\sin^{2n}\phi\, d\phi;$$

thus $\dfrac{d}{dx}\{x^{-\frac{n}{2}}J_n(\sqrt{x})\}$

$$= -\frac{1}{2\sqrt{\pi x}} \cdot \frac{1}{2^n \Gamma\left(n+\frac{1}{2}\right)} \int_0^\pi \sin(\sqrt{x}\cos\phi)\sin^{2n}\phi\cos\phi\, d\phi;$$

but by integration by parts

$$\int \sin(\sqrt{x}\cos\phi)\sin^{2n}\phi\cos\phi\, d\phi$$

$$= \frac{1}{2n+1}\sin^{2n+1}\phi\sin(\sqrt{x}\cos\phi) + \frac{\sqrt{x}}{2n+1}\int\cos(\sqrt{x}\cos\phi)\sin^{2n+2}\phi\, d\phi.$$

Thus, taking the integrals between the limits we have

$\dfrac{d}{dx}\{x^{-\frac{n}{2}}J_n(\sqrt{x})\}$

$$= -\frac{1}{2\sqrt{\pi}} \cdot \frac{1}{2^n(2n+1)\Gamma\left(n+\frac{1}{2}\right)} \int_0^\pi \cos(\sqrt{x}\cos\phi)\sin^{2n+2}\phi\, d\phi$$

$$= -\frac{1}{2\sqrt{\pi}} \frac{1}{2^{n+1}\Gamma\left(n+1+\frac{1}{2}\right)} \int_0^\pi \cos(\sqrt{x}\cos\phi)\sin^{2n+2}\phi\, d\phi$$

$$= -\frac{1}{2}x^{-\frac{n+1}{2}} J_{n+1}(\sqrt{x}).$$

Then differentiating again we have

$$\frac{d^2}{dx^2}\{x^{-\frac{n}{2}} J_n(\sqrt{x})\} = -\frac{1}{2}\frac{d}{dx}\{x^{-\frac{n+1}{2}} J_{n+1}(\sqrt{x})\}$$

$$= \left(-\frac{1}{2}\right)^2 x^{-\frac{n+2}{2}} J_{n+2}(\sqrt{x}).$$

In this way we obtain the proposed theorem.

391. To shew that

$$\frac{d^m}{dx^m}\{x^{\frac{n}{2}} J_n(\sqrt{x})\} = \left(\frac{1}{2}\right)^m x^{\frac{n-m}{2}} J_{n-m}(\sqrt{x}).$$

We have $\frac{d}{dx}\{x^{\frac{n}{2}} J_n(\sqrt{x})\} = \frac{d}{dx}\{x^n x^{-\frac{n}{2}} J_n(\sqrt{x})\}$

$$= x^n \frac{d}{dx}\left\{x^{-\frac{n}{2}} J_n(\sqrt{x})\right\} + x^{-\frac{n}{2}} J_n(\sqrt{x}) \frac{d}{dx} x^n$$

$$= -\frac{1}{2} x^{\frac{n-1}{2}} J_{n+1}(\sqrt{x}) + n x^{\frac{n-2}{2}} J_n(\sqrt{x}).$$

But by (1) we have $J_{n+1}(\sqrt{x}) = \frac{2n}{\sqrt{x}} J_n(\sqrt{x}) - J_{n-1}(\sqrt{x})$; hence by substitution we get

$$\frac{d}{dx}\{x^{\frac{n}{2}} J_n(\sqrt{x})\} = \frac{1}{2} x^{\frac{n-1}{2}} J_{n-1}(\sqrt{x}).$$

Then differentiating again, we have

$$\frac{d^2}{dx^2}\{x^{\frac{n}{2}} J_n(\sqrt{x})\} = \frac{1}{2}\frac{d}{dx}\{x^{\frac{n-1}{2}} J_{n-1}(\sqrt{x})\}$$

$$= \left(\frac{1}{2}\right)^2 x^{\frac{n-2}{2}} J_{n-2}(\sqrt{x}).$$

In this way we obtain the proposed theorem.

392. By Taylor's Theorem we have

$$(x+h)^{-\frac{n}{2}} J_n(\sqrt{x+h}) = x^{-\frac{n}{2}} J_n(\sqrt{x})$$
$$+ h\frac{d}{dx}\{x^{-\frac{n}{2}} J_n(\sqrt{x})\} + \frac{h^2}{\underline{|2}}\frac{d^2}{dx^2}\{x^{-\frac{n}{2}} J_n(\sqrt{x})\}$$
$$+ \ldots + \frac{h^r}{\underline{|r}}\frac{d^r}{dx^r}\{x^{-\frac{n}{2}} J_n(\sqrt{x})\} + \frac{h^{r+1}}{\underline{|r+1}}\frac{d^{r+1}}{dx^{r+1}}\{\xi^{-\frac{n}{2}} J_n(\sqrt{\xi})\},$$

where ξ is put for $x+\theta h$, and θ denotes a proper fraction.

The differential coefficients which occur on the right-hand side of this formula may be conveniently expressed by the theorem of Art. 390.

Similarly by Taylor's Theorem we have

$$(x+h)^{\frac{n}{2}} J_n(\sqrt{x+h}) = x^{\frac{n}{2}} J_n(\sqrt{x})$$
$$+ h\frac{d}{dx}\{x^{\frac{n}{2}} J_n(\sqrt{x})\} + \frac{h^2}{\underline{|2}}\frac{d^2}{dx^2}\{x^{\frac{n}{2}} J_n(\sqrt{x})\}$$
$$+ \ldots + \frac{h^r}{\underline{|r}}\frac{d^r}{dx^r}\{x^{\frac{n}{2}} J_n(\sqrt{x})\} + \frac{h^{r+1}}{\underline{|r+1}}\frac{d^{r+1}}{dx^{r+1}}\{\xi^{\frac{n}{2}} J_n(\sqrt{\xi})\};$$

and the differential coefficients which occur on the right-hand side of this formula may be conveniently expressed by the theorem of Art. 391.

393. In the theorem of Art. 391, change n into $n+m$; thus

$$\left(\frac{1}{2}\right)^m x^{\frac{n}{2}} J_n(\sqrt{x}) = \frac{d^m}{dx^m}\{x^{\frac{n+m}{2}} J_{n+m}(\sqrt{x})\}$$
$$= \frac{d^m}{dx^m}\{x^{n+m} \cdot x^{-\frac{n+m}{2}} J_{n+m}(\sqrt{x})\};$$

by the theorem of Leibnitz, the right-hand member is equal to

$$x^{n+m}\frac{d^m}{dx^m}\{x^{-\frac{n+m}{2}} J_{n+m}(\sqrt{x})\} + m\frac{d}{dx}(x^{n+m})\frac{d^{m-1}}{dx^{m-1}}\{x^{-\frac{n+m}{2}} J_{n+m}(\sqrt{x})\}$$
$$+ \frac{m(m-1)}{\underline{|2}}\frac{d^2}{dx^2}(x^{n+m})\frac{d^{m-2}}{dx^{m-2}}\left\{x^{-\frac{n+m}{2}} J_{n+m}(\sqrt{x})\right\} + \ldots;$$

and by Art. 390, this

$$= \left(-\frac{1}{2}\right)^m x^{\frac{n}{2}} J_{2m+n}(\sqrt{x}) + m(m+n)\left(-\frac{1}{2}\right)^{m-1} x^{\frac{n-1}{2}} J_{2m-1+n}(\sqrt{x})$$

$$+ \frac{m(m-1)}{\underline{|2}}(m+n)(m+n-1)\left(-\frac{1}{2}\right)^{m-2} x^{\frac{n-2}{2}} J_{2m-2+n}(\sqrt{x}) + \ldots$$

Thus

$$J_n(\sqrt{x}) = (-1)^m \left\{ J_{2m+n}(\sqrt{x}) - \frac{2m(m+n)}{\sqrt{x}} J_{2m+n-1}(\sqrt{x}) \right.$$

$$\left. + \frac{2^2 m(m-1)(m+n)(m+n-1)}{\underline{|2}\,(\sqrt{x})^2} J_{2m+n-2}(\sqrt{x}) - \ldots \right\}.$$

Then putting x^2 for x, we have

$$J_n(x) = (-1)^m \left\{ J_{2m+n}(x) - \frac{2m(m+n)}{x} J_{2m+n-1}(x) \right.$$

$$\left. + \frac{2^2 m(m-1)(m+n)(m+n-1)}{\underline{|2}\, x^2} J_{2m+n-2}(x) - \ldots \right\}.$$

In this theorem m may be any positive integer, and n any quantity which is algebraically greater than -1. The demonstration, as it rests on Art. 371, would require n to be algebraically greater than $-\frac{1}{2}$; but from the form of the result it is easily seen, by the aid of Art. 370, that n may be any quantity which is algebraically greater than -1.

Lommel proposes to *define* $J_n(x)$ for negative values of x so as to make this theorem always hold. Thus, for example, suppose n a negative integer, and put it equal to $-m$; then we have by this theorem

$$J_{-m}(x) = (-1)^m J_m(x).$$

CHAPTER XXXII.

FOURIER'S EXPRESSION.

394. SUPPOSE $n=0$ in the equation (2) of Art. 370; thus

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\dots.$$

This expression had been studied by Fourier before Bessel brought forward his general Functions: see Fourier's *Théorie de la Chaleur*, Chapitre VI. We will reproduce Fourier's results, and then shew that they may be extended to Bessel's general Functions.

395. Put θ for $\frac{x^2}{2^2}$ in the preceding series, and denote the expression then by $f(\theta)$; thus

$$f(\theta)=1-\theta+\frac{\theta^2}{2^2}-\frac{\theta^3}{2^2.3^2}+\frac{\theta^4}{2^2.3^2.4^2}-\dots.$$

We shall first shew that the equation $f(\theta)=0$ *has an infinite number of roots, all real and positive.*

In treating this proposition, and that of the next Article, it is really assumed that $f(\theta)$ may be considered to be a *finite* algebraical expression; the justification of this assumption must be found in the fact that $f(\theta)$ is a rapidly convergent series, and thus whatever may be the value of θ, and whatever may be the closeness of approximation we desire, the terms in $f(\theta)$ after some finite number of them may be neglected.

It is easily seen by two differentiations that

$$f(\theta)+f'(\theta)+\theta f''(\theta)=0;$$

or this may be obtained from the general differential equation of Art. 370 by changing the independent variable.

By successive differentiation we now obtain

$$f'(\theta) + 2f''(\theta) + \theta f'''(\theta) = 0,$$
$$f''(\theta) + 3f'''(\theta) + \theta f''''(\theta) = 0,$$
$$f'''(\theta) + 4f''''(\theta) + \theta f'''''(\theta) = 0,$$

and so on.

These equations shew that when any one of the derived functions, $f'(\theta)$, $f''(\theta)$, ... vanishes, the preceding and following functions have *contrary* signs, if θ be positive.

Now suppose we consider $f(\theta)$ to be of the m^{th} degree in θ, where m may be as large as we please. Take the series of functions

$$f(\theta),\ f'(\theta),\ f''(\theta), \ldots f^m(\theta);$$

this series may be called *Fourier's Functions*, and the student may be assumed to be acquainted with their importance; see *Theory of Equations*, Chapter XV.

No change of sign in the series can be lost by the passage of θ through a value which makes any of the derived functions vanish; for as we have just seen when any derived function vanishes the preceding and following functions have contrary signs. Hence a change of sign in the series can be lost only when θ passes through a value which makes $f(\theta)$ vanish. But m changes of sign in the series are lost as θ passes from 0 to $+\infty$. Hence the equation $f(\theta) = 0$ has m real positive roots; that is all its roots are real and positive.

We may remark that it is obvious that $f(\theta)$ cannot vanish when θ is negative.

396. *If λ be any given positive quantity the following equation has an infinite number of roots, all real and positive:*

$$\lambda + \frac{\theta f'(\theta)}{f(\theta)} = 0 \quad \text{.....................(1).}$$

Let a and c denote two consecutive roots of $f(\theta) = 0$; by the Theory of Equations $f'(\theta) = 0$ has a root between a and c: denote it by b.

Then as θ changes from a to b the numerical value of $\frac{\theta f'(\theta)}{f(\theta)}$ diminishes from ∞ to 0, while the sign remains unchanged. As θ changes from b to c the numerical value of $\frac{\theta f'(\theta)}{f(\theta)}$ increases from 0 to ∞, while the sign remains unchanged, but contrary to what it was before. Hence $\frac{\theta f'(\theta)}{f(\theta)}$ takes, once at least, any specified value as θ changes from a to c. Therefore (1) has a root between a and c. In this way we see that there is a root of (1) between every two consecutive roots of $f(\theta)=0$. And since λ is positive there will be one root of (1) between 0 and the least root of $f(\theta)=0$. Thus all the roots of (1) are real and positive. Moreover only a single root can lie within each interval which we have considered.

397. The equations of Art. 395 which connect the successive derived functions may be put in the form

$$\frac{f'(\theta)}{f(\theta)} = -\frac{1}{1+\dfrac{\theta f''(\theta)}{f'(\theta)}},$$

$$\frac{f''(\theta)}{f'(\theta)} = -\frac{1}{2+\dfrac{\theta f'''(\theta)}{f''(\theta)}},$$

$$\frac{f'''(\theta)}{f''(\theta)} = -\frac{1}{3+\dfrac{\theta f''''(\theta)}{f'''(\theta)}},$$

and so on.

Thus $$\frac{f'(\theta)}{f(\theta)} = -\frac{1}{1-\dfrac{\theta}{2-\dfrac{\theta}{3-\dots}}},$$

and $$\lambda = \frac{\theta}{1-\dfrac{\theta}{2-\dfrac{\theta}{3-\dots}}}.$$

Thus λ is exhibited as an infinite continued fraction of the second class in which the r^{th} component is $\frac{\theta}{r}$; see *Algebra*, Art. 778.

398. The results obtained by Fourier admit of easy extension to Bessel's general Functions, as we shall now shew.

We have by Art. 370,

$$\frac{2^n \Gamma(n+1)}{x^n} J_n(x) = 1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \dots$$

Put θ for $\frac{x^2}{2^2}$ in the preceding series, and denote the expression then by $F(\theta)$; then

$$F(\theta) = 1 - \frac{\theta}{n+1} + \frac{\theta^2}{1.2(n+1)(n+2)} - \frac{\theta^3}{1.2.3(n+1)(n+2)(n+3)} + \dots$$

It is easily shewn by two differentiations that

$$F(\theta) + (n+1) F'(\theta) + \theta F''(\theta) = 0;$$

or this may be obtained from the general differential equation of Art. 370 by first putting vx^n for u, and then changing the independent variable from x to θ.

By successive differentiation we now obtain

$$F'(\theta) + (n+2) F''(\theta) + \theta F'''(\theta) = 0,$$

$$F''(\theta) + (n+3) F'''(\theta) + \theta F''''(\theta) = 0,$$

and so on.

399. *The equation $F(\theta) = 0$ has an infinite number of roots all real and positive.*

The demonstration is precisely like that of Art. 395.

400. *If λ be any given positive quantity, the following equation has an infinite number of roots all real and positive:*

$$\lambda + \frac{\theta F'(\theta)}{F(\theta)} = 0.$$

The demonstration is precisely like that of Art. 396.

401. From the preceding equation, by a process precisely like that of Art. 397, we deduce the following expression for λ as an infinite continued fraction:

$$\lambda = \cfrac{\theta}{n+1-\cfrac{\theta}{n+2-\cfrac{\theta}{n+3-\dots}}}$$

CHAPTER XXXIII.

LARGE ROOTS OF FOURIER'S EQUATION.

402. Poisson has shewn how to determine the large roots of the equation $J_0(x)=0$: see *Journal de l'École Polytechnique*, Cahier 19, pages 349...353. We will give his principal results though not altogether according to his method.

Let y stand for $\pi J_0(x)$, so that

$$y=\int_0^\pi \cos(x\cos\phi)\,d\phi \qquad \text{.................(1)};$$

we have

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0 \qquad \text{..................(2)};$$

this may be written

$$\frac{d^2(y\sqrt{x})}{dx^2}+\left(\frac{1}{4x^2}+1\right)y\sqrt{x}=0 \qquad \text{...............(3)}.$$

This suggests that when x is very great, so that $\frac{1}{4x^2}$ may be neglected in comparison with unity, we shall have very approximately

$$y\sqrt{x}=A_0\cos x+B_0\sin x \qquad \text{................(4)},$$

where A_0 and B_0 are constants.

403. Poisson assumes that the integral of (3) can be put in the form

$$y\sqrt{x}=\left(A_0+\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}+\dots\right)\cos x$$
$$+\left(B_0+\frac{B_1}{x}+\frac{B_2}{x^2}+\frac{B_3}{x^3}+\dots\right)\sin x,$$

where $A_1, A_2, \dots B_1, B_2, \dots$ are constants to be determined.

Substitute in (3) and equate the coefficients of distinct terms to zero; thus we obtain the following equations for expressing the constants $A_1, A_2, \ldots B_1, B_2, \ldots$ in terms of A_0 and B_0.

$2A_1+\frac{1}{4}B_0=0,$	$-2B_1+\frac{1}{4}A_0=0,$
$2.2A_2+\left\{1.2+\frac{1}{4}\right\}B_1=0,$	$-2.2B_2+\left\{1.2+\frac{1}{4}\right\}A_1=0,$
....	....
$2rA_r+\left\{(r-1)r+\frac{1}{4}\right\}B_{r-1}=0,$	$-2rB_r+\left\{(r-1)r+\frac{1}{4}\right\}A_{r-1}=0,$
....	....

But the series we thus obtain are divergent for any assigned value of x.

404. Let us however assume that (4) is admissible when x is very large; thus

$$y=\frac{A_0\cos x+B_0\sin x}{\sqrt{x}},\quad \frac{dy}{dx}=\frac{B_0\cos x-A_0\sin x}{\sqrt{x}},$$

approximately.

Therefore y vanishes when $\tan x=-\frac{A_0}{B_0}$; so that $x=n\pi+\alpha$, where $\tan\alpha=-\frac{A_0}{B_0}$, and n is any integer. In like manner $\frac{dy}{dx}$ vanishes when $x=m\pi+\beta$, where β is such that $\tan\beta=\frac{B_0}{A_0}$, and m is any integer. But m and n must be supposed very large integers, as we are concerned only with very large values of x.

405. It is natural to conjecture that $A_0=B_0$; for then the large roots of $\frac{dy}{dx}=0$ are midway between those of $y=0$.

This conjecture may be verified. We have

$$y=\int_0^\pi \cos\left(2x\cos^2\frac{\phi}{2}-x\right)d\phi$$

$$=\cos x\int_0^\pi \cos\left(2x\cos^2\frac{\phi}{2}\right)d\phi+\sin x\int_0^\pi \sin\left(2x\cos^2\frac{\phi}{2}\right)d\phi\ldots(5).$$

We shall investigate the value of y when $x=2r\pi$, where r is a large integer. We have then from (5)

$$y=\int_0^\pi \cos\left(2x\cos^2\frac{\phi}{2}\right)d\phi.$$

Put $2x\cos^2\frac{\phi}{2}=t$; thus

$$y=\int_0^{2x}\frac{\cos t\,dt}{\sqrt{t}\sqrt{(2x-t)}}=\int_0^x\frac{\cos t\,dt}{\sqrt{t}\sqrt{(2x-t)}}+\int_x^{2x}\frac{\cos t\,dt}{\sqrt{t}\sqrt{(2x-t)}}.$$

In the second of the two integrals put $t=2x-\tau$; then observing that $\cos(2x-\tau)=\cos\tau$, it becomes $\int_0^x\frac{\cos\tau\,d\tau}{\sqrt{\tau}\sqrt{(2x-\tau)}}$, so that we have

$$y=2\int_0^x\frac{\cos t\,dt}{\sqrt{t}\sqrt{(2x-t)}}=\frac{2}{\sqrt{2x}}\int_0^x\frac{\cos t\,dt}{\sqrt{t}\sqrt{\left(1-\frac{t}{2x}\right)}}.$$

This integral when x is very large may be replaced by $\frac{2}{\sqrt{2x}}\int_0^x\frac{\cos t\,dt}{\sqrt{t}}$; for $1-\frac{t}{2x}$ may be taken as unity so long as t is not large, and when t is large the corresponding elements of the integral are of no account because then $\frac{\cos t}{\sqrt{t}}$ is very small.

Hence we may say that $y=\frac{2}{\sqrt{2x}}\int_0^\infty\frac{\cos t}{\sqrt{t}}dt$, and this $=\frac{\sqrt{\pi}}{\sqrt{x}}$; see *Integral Calculus*, Art. 303.

Comparing this result with the value given by (4) when $x=2r\pi$, we see that $A=\sqrt{\pi}$.

Similarly by finding the value of the right-hand member of (5) when $x=\left(2r+\frac{1}{2}\right)\pi$, we shall see that $B=\sqrt{\pi}$.

406. The method of the preceding Article admits of extension to Bessel's general Functions.

Let u stand for $J_n(x)$; we know that

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{n^2}{x^2}\right)u=0,$$

thus $$\frac{d^2(u\sqrt{x})}{dx^2}+u\sqrt{x}-\frac{u}{x^2}\left(n^2-\frac{1}{4}\right)\sqrt{x}=0;$$

and when x is very great, provided n be finite we have approximately

$$\frac{d^2(u\sqrt{x})}{dx^2}+u\sqrt{x}=0,$$

so that $$u\sqrt{x}=A_0\cos x+B_0\sin x \text{ ....(6).}$$

Now by Art. 371, adopting the same method as in Art. 405, we have

$$u=\frac{x^n\cos x}{\sqrt{\pi}\,2^n\Gamma\left(n+\frac{1}{2}\right)}\int_0^\pi\cos\left(2x\cos^2\frac{\phi}{2}\right)\sin^{2n}\phi\,d\phi$$

$$+\frac{x^n\sin x}{\sqrt{\pi}\,2^n\Gamma\left(n+\frac{1}{2}\right)}\int_0^\pi\sin\left(2x\cos^2\frac{\phi}{2}\right)\sin^{2n}\phi\,d\phi \text{ ...(7).}$$

Suppose $x=2r\pi$, where r is a large integer; then we have from (7)

$$u=\frac{x^n}{\sqrt{\pi}\,2^n\Gamma\left(n+\frac{1}{2}\right)}\int_0^\pi\cos\left(2x\cos^2\frac{\phi}{2}\right)\sin^{2n}\phi\,d\phi.$$

Put $2x\cos^2\frac{\phi}{2}=t$; then proceeding as in Art. 405, we get

$$u=\frac{2}{\sqrt{2\pi x}\,\Gamma\left(n+\frac{1}{2}\right)}\int_0^\infty\cos t\,.\,t^{n-\frac{1}{2}}\left(1-\frac{t}{2x}\right)^{n-\frac{1}{2}}dt \text{ .....(8).}$$

The equation (8) is exact. If we continue as in Art. 405, we should first suppose that $\left(1-\frac{t}{2x}\right)^{n-\frac{1}{2}}$ may be replaced by unity; and thus the integral reduces to $\int_0^x \cos t \,.\, t^{n-\frac{1}{2}} dt$. Then replacing the upper limit by ∞, and using Art. 302 of the *Integral Calculus*, we obtain $\Gamma\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)\frac{\pi}{2}$. Thus finally

$$u=\frac{2}{\sqrt{2\pi x}}\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right).$$

Hence by comparison with (6) we have

$$A_0=\frac{\sqrt{2}}{\sqrt{\pi}}\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right).$$

Similarly by finding the value of the right-hand member of (7) when $x=\left(2r+\frac{1}{2}\right)\pi$, we get

$$B_0=\frac{\sqrt{2}}{\sqrt{\pi}}\sin\left(\frac{n\pi}{2}+\frac{\pi}{4}\right).$$

Hence by (6),

$$u=\frac{\sqrt{2}}{\sqrt{\pi x}}\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}-x\right) \quad\text{...............(9).}$$

407. The approximations which we employed after obtaining the exact equation (8) are not very satisfactory for every value of n; but at least they involve little difficulty so long as n is less than $\frac{1}{2}$. The formula of Art. 371 from which we started supposes n to be algebraically greater than $-\frac{1}{2}$. Hence we may consider that (9) is fairly established for any value of n between $-\frac{1}{2}$ and $\frac{1}{2}$. Then we infer that it will hold generally by the aid of equation (6) of Art. 386; for when x is *very great* we obtain from that formula $J_{n+1}(x)=-\frac{dJ_n(x)}{dx}$; and from (9) we have approximately

$$\frac{du}{dx}=\frac{\sqrt{2}}{\sqrt{\pi x}}\sin\left(\frac{n\pi}{2}+\frac{\pi}{4}-x\right)=-\frac{\sqrt{2}}{\sqrt{\pi x}}\cos\left(\frac{n+1}{2}\pi+\frac{\pi}{4}-x\right);$$

so that if (9) holds for any value of n it *holds when that value is increased by unity.* Hence since it holds when n lies between $-\frac{1}{2}$ and $\frac{1}{2}$, it holds when n lies between $\frac{1}{2}$ and $\frac{3}{2}$, then it holds when n lies between $\frac{3}{2}$ and $\frac{5}{2}$, and so on.

408. Another method of obtaining the result in Art. 405 has been given. We continue to use y for $\pi J_0(x)$.

Thus $y=\int_0^\pi \cos(x\cos\phi)\,d\phi=2\int_0^{\frac{\pi}{2}}\cos(x\cos\phi)\,d\phi$;

put $1-z$ for $\cos\phi$; then $y=2\int_0^1 \frac{\cos x(1-z)}{\sqrt{z(2-z)}}dz$

$$=\sqrt{2}\cos x\int_0^1\frac{\cos(xz)\,dz}{\sqrt{z\left(1-\frac{1}{2}z\right)}}+\sqrt{2}\sin x\int_0^1\frac{\sin(xz)\,dz}{\sqrt{z\left(1-\frac{1}{2}z\right)}}$$

$$=\sqrt{2}\cos x\int_0^1\frac{\cos(xz)}{\sqrt{z}}\left\{1+\frac{1}{2}\cdot\frac{z}{2}+\frac{1.3}{2.4}\left(\frac{z}{2}\right)^2+\dots\right\}dz$$

$$+\sqrt{2}\sin x\int_0^1\frac{\sin(xz)}{\sqrt{z}}\left\{1+\frac{1}{2}\cdot\frac{z}{2}+\frac{1.3}{2.4}\left(\frac{z}{2}\right)^2+\dots\right\}dz.$$

As soon as the values of $\int_0^1\frac{\cos(xz)}{\sqrt{z}}dz$ and $\int_0^1\frac{\sin(xz)}{\sqrt{z}}dz$ are known we can obtain by differentiation with respect to x the values of the other integrals which occur in the expression for y. Thus denote the former by P, and the latter by Q; then we have

$$\int_0^1\frac{z\cos(xz)}{\sqrt{z}}dz=\frac{dQ}{dx},\qquad \int_0^1\frac{z\sin(xz)}{\sqrt{z}}dz=-\frac{dP}{dx},$$

$$\int_0^1 \frac{z^2 \cos(xz)}{\sqrt{z}}\, dz = -\frac{d^2P}{dx^2}, \quad \int_0^1 \frac{z^2 \sin(xz)}{\sqrt{z}}\, dz = -\frac{d^2Q}{dx^2},$$

and so on.

Thus we find that

$$y = \sqrt{2}\left\{P\cos x + Q\sin x - \frac{1.3}{4.8}(P''\cos x + Q''\sin x) + \dots\right.$$
$$\left. + \frac{1}{4}(Q'\cos x - P'\sin x) - \frac{1.3.5}{4.8.12}(Q'''\cos x - P'''\sin x) + \dots\right\} \quad \dots\dots\dots(10),$$

where the accents denote differentiation with respect to x.

$$\text{Now } \int_0^1 \frac{\cos(xz)}{\sqrt{z}}\, dz = \int_0^\infty \frac{\cos(xz)}{\sqrt{z}}\, dz - \int_1^\infty \frac{\cos(xz)}{\sqrt{z}}\, dz$$
$$= \frac{\sqrt{\pi}}{\sqrt{2x}} - \int_1^\infty \frac{\cos(xz)}{\sqrt{z}}\, dz, \text{ by } \textit{Integral Calculus}, \text{ Art. } 303.$$

By integration by parts we have

$$\int \frac{\cos(xz)}{\sqrt{z}}\, dz = \frac{\sin(xz)}{x\sqrt{z}} + \frac{1}{2x}\int \frac{\sin(xz)}{z^{\frac{3}{2}}}\, dz$$
$$= \frac{\sin(xz)}{x\sqrt{z}} - \frac{\cos(xz)}{2x^2 z^{\frac{3}{2}}} - \frac{1.3}{2^2x^2}\int \frac{\cos(xz)}{z^{\frac{5}{2}}}\, dz.$$

In this way we find that

$$P = \frac{\sqrt{\pi}}{\sqrt{2x}} + \sin x\left\{\frac{1}{x} - \frac{1.3}{2^2x^3} + \frac{1.3.5.7}{2^4x^5} - \dots\right\}$$
$$- \cos x\left\{\frac{1}{2x^2} - \frac{1.3.5}{2^3x^4} + \frac{1.3.5.7.9}{2^5x^6} - \dots\right\};$$

we will denote this result thus,

$$P = \frac{\sqrt{\pi}}{\sqrt{2x}} + \phi(x)\sin x - \psi(x)\cos x.$$

In the same manner we may shew that

$$Q = \frac{\sqrt{\pi}}{\sqrt{2x}} - \phi(x)\cos x - \psi(x)\sin x.$$

Hence we find that

$$P\cos x + Q\sin x = \frac{\sqrt{\pi}}{\sqrt{x}}\cos\left(x - \frac{\pi}{4}\right) - \psi(x).$$

Also

$$\frac{dP}{dx} = -\frac{\sqrt{\pi}}{(2x)^{\frac{3}{2}}} + \phi(x)\cos x + \phi'(x)\sin x + \psi(x)\sin x - \psi'(x)\cos x,$$

$$\frac{dQ}{dx} = -\frac{\sqrt{\pi}}{(2x)^{\frac{3}{2}}} + \phi(x)\sin x - \phi'(x)\cos x - \psi(x)\cos x - \psi'(x)\sin x;$$

therefore $\cos x\dfrac{dQ}{dx} - \sin x\dfrac{dP}{dx} = \dfrac{\sqrt{\pi}}{2x^{\frac{3}{2}}}\sin\left(x - \dfrac{\pi}{4}\right) - \phi'(x) - \psi(x).$

Therefore if we stop at this stage of approximation, we get from (10)

$$y = \sqrt{2}\left\{\frac{\sqrt{\pi}}{\sqrt{x}}\cos\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{\pi}}{8x^{\frac{3}{2}}}\sin\left(x - \frac{\pi}{4}\right) - \frac{1}{4}\phi'(x) - \frac{5}{4}\psi(x)\right\} \quad \text{...(11)}.$$

Thus as far as we have gone we see two classes of terms in y; one class involves fractional powers of x with trigonometrical functions, and the other class involves whole powers of x without trigonometrical functions. We shall shew however that the latter class of terms will disappear as the process is continued.

I. We shall shew that $\phi(x)$ and $\psi(x)$ and their differential coefficients will occur, as they do in (11), free from $\sin x$ and $\cos x$ as multipliers. For we have

$$\left.\begin{aligned} P\cos x + Q\sin x &= -\psi(x) \\ P\sin x - Q\cos x &= \phi(x) \end{aligned}\right\} \quad \text{............(12)},$$

omitting the terms which are multiplied by $\dfrac{\sqrt{\pi}}{\sqrt{2x}}$, for we are not concerned with them here.

Then, by differentiating,

$$P'\cos x + Q'\sin x - P\sin x + Q\cos x = -\psi'(x),$$
$$P'\sin x - Q'\cos x + P\cos x + Q\sin x = \phi'(x).$$

From these and (12) we obtain

$$\left.\begin{aligned} P'\cos x + Q'\sin x &= -\psi'(x) + \phi(x) \\ P'\sin x - Q'\cos x &= \phi'(x) + \psi(x) \end{aligned}\right\} \text{......(13)}.$$

In like manner from (13) and its derived equations we obtain

$$P''\cos x + Q''\sin x = \varpi_2(x),$$
$$P''\sin x - Q''\cos x = \chi_2(x),$$

where $\varpi_2(x)$ and $\chi_2(x)$ involve only $\phi(x)$ and $\psi(x)$ and their derivatives.

Then again we obtain

$$P'''\cos x + Q'''\sin x = \varpi_3(x),$$
$$P'''\sin x - Q'''\cos x = \chi_3(x),$$

and so on.

Then substituting in (10), we see that in the value of y we shall have $\phi(x)$ and $\psi(x)$ and their derived functions free from $\sin x$ and $\cos x$ as multipliers.

II. But on the whole the terms involving $\phi(x)$ and $\psi(x)$ and their derived functions must adjust themselves so as to cancel and disappear. For if they did not suppose $\frac{A}{x^a}$ the first term which remained in y; substitute in the differential equation $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$, then as none of the terms involving fractional powers of x and trigonometrical functions can combine with this, we see that the differential equation will not be satisfied unless $A = 0$.

Thus omitting all the terms which depend on $\phi(x)$ and $\psi(x)$ we obtain finally

$$y = \frac{\sqrt{2\pi}}{\sqrt{x}} \cos\left(x - \frac{1}{4}\pi\right)\left\{1 - \frac{1^2.3^2}{4.8}\left(\frac{1}{2x}\right)^2 + \frac{1^2.3^2.5^2.7^2}{4.8.12.16}\left(\frac{1}{2x}\right)^4 - \dots\right\}$$
$$+ \frac{\sqrt{2\pi}}{\sqrt{x}} \sin\left(x - \frac{1}{4}\pi\right)\left\{\frac{1^2}{4}\left(\frac{1}{2x}\right) - \frac{1^2.3^2.5^2}{4.8.12}\left(\frac{1}{2x}\right)^3 + \dots\right\}.$$

This will be found to agree with the result obtained by Poisson, when in his result we take $A_0 = B_0 = \sqrt{\pi}$. The series within the brackets are divergent; but we may in our process instead of infinite series use finite series with symbols for the remainders. Thus when we apply integration by parts to $\int \frac{\cos(xz)}{\sqrt{z}} dz$, we may, as we have seen, denote the remainder by an integral after any number of terms we please. So in the expansion of $\left(1 - \frac{1}{2}z\right)^{-\frac{1}{2}}$ which we have used we may express the remainder after any number of terms in the method given by the modern investigations of Maclaurin's Theorem.

CHAPTER XXXIV.

EXPANSIONS IN SERIES OF BESSEL'S FUNCTIONS.

409. We shall in the present Chapter give examples of the expansion of various functions in infinite series of Bessel's Functions.

410. We know by the Integral Calculus that

$$\cos(x \sin\phi) = a_0 + a_1 \cos\phi + a_2 \cos 2\phi + a_3 \cos 3\phi + \ldots,$$

where $$a_n = \frac{2}{\pi}\int_0^\pi \cos(x\sin\phi)\cos n\phi d\phi,$$

except when $n = 0$, and then we must take half this value.

Hence, as we have shewn in Art. 372, we have $a_n = 0$ when n is odd, and $a_n = 2J_n(x)$ when n is even; except when $n = 0$, and then $a_0 = J_0(x)$. Therefore

$$\cos(x\sin\phi) = J_0(x) + 2J_2(x)\cos 2\phi + 2J_4(x)\cos 4\phi + \ldots.$$

411. In the manner of the preceding Article we can shew that

$$\sin(x\sin\phi) = 2J_1(x)\sin\phi + 2J_3(x)\sin 3\phi + 2J_5(x)\sin 5\phi + \ldots$$

412. As particular cases we have

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + \ldots,$$

$$x = 2.1J_1(x) + 2.3J_3(x) + 2.5J_5(x) + \ldots;$$

the former is obtained from Art. 410 by putting $\phi = 0$, and the latter is obtained from Art. 411 by dividing by ϕ and then putting $\phi = 0$.

413. In Art. 410 change ϕ into $\frac{\pi}{2}+\phi$; thus

$$\cos(x\cos\phi)=J_0(x)-2J_2(x)\cos 2\phi+2J_4(x)\cos 4\phi-\ldots.$$

Similarly from Art. 411 we get

$$\sin(x\cos\phi)=2J_1(x)\cos\phi-2J_3(x)\cos 3\phi+2J_5(x)\cos 5\phi+\ldots.$$

Various particular cases may be deduced. Thus putting $\phi=0$, we have

$$\cos x=J_0(x)-2J_2(x)+2J_4(x)-\ldots,$$
$$\sin x=2J_1(x)-2J_3(x)+2J_5(x)-\ldots.$$

Again differentiate these two formulæ twice with respect to ϕ, and then put $\phi=0$; thus we get

$$x\sin x=2\{2^2J_2(x)-4^2J_4(x)+6^2J_6(x)-\ldots\},$$
$$x\cos x=2\{1^2J_1(x)-3^2J_3(x)+5^2J_5(x)-\ldots\}.$$

414. In Art. 410 we have shewn that

$$\cos(x\sin\phi)=J_0(x)+2J_2(x)\cos 2\phi+2J_4(x)\cos 4\phi+\ldots.$$

Now we know by *Plane Trigonometry*, Art. 287, that if n be *even*,

$$\cos n\phi=1-\frac{n^2}{\underline{|2}}\sin^2\phi+\frac{n^2(n^2-2^2)}{\underline{|4}}\sin^4\phi-\ldots;$$

and $$\cos(x\sin\phi)=1-\frac{x^2\sin^2\phi}{\underline{|2}}+\frac{x^4\sin^4\phi}{\underline{|4}}-\ldots.$$

Hence equating the coefficients of the powers of $\sin\phi$ we have the following results in which Σ denotes summation with respect to *even* values of n from 2 to infinity:

$$1=J_0(x)+2\Sigma J_n(x),$$
$$x^2=2\Sigma n^2J_n(x),$$
$$x^4=2\Sigma n^2(n^2-2^2)J_n(x),$$
$$x^6=2\Sigma n^2(n^2-2^2)(n^2-4^2)J_n(x),$$

and so on.

415. In Art. 411 we have shewn that

$$\sin(x\sin\phi) = 2J_1(x)\sin\phi + 2J_3(x)\sin 3\phi + 2J_5(x)\sin 5\phi + \dots$$

Now we know by *Plane Trigonometry*, Art. 287, that if n be *odd*,

$$\sin n\phi = n\sin\phi - \frac{n(n^2-1^2)}{\lfloor 3}\sin^3\phi + \frac{n(n^2-1^2)(n^2-3^2)}{\lfloor 5}\sin^5\phi - \dots;$$

and $$\sin(x\sin\phi) = x\sin\phi - \frac{x^3\sin^3\phi}{\lfloor 3} + \frac{x^5\sin^5\phi}{\lfloor 5} - \dots.$$

Hence equating the coefficients of the powers of $\sin\phi$ we have the following results in which Σ denotes summation with respect to *odd* values of n from 1 to infinity,

$$x = 2\Sigma nJ_n(x),$$
$$x^3 = 2\Sigma n(n^2-1^2)J_n(x),$$
$$x^5 = 2\Sigma n(n^2-1^2)(n^2-3^2)J_n(x),$$

and so on.

416. Suppose n an *even* number. If we combine two of the results obtained in Art. 414 we deduce the following:

$$2\Sigma n^4J_n(x) = x^4 + 4x^2.$$

In like manner we see that $2\Sigma n^6J_n(x)$ can be expressed in terms of x^6, x^4, and x^2. Thus we are naturally led to conjecture that $2\Sigma n^{2m}J_n(x)$ can be expressed in terms of $x^{2m}, x^{2m-2}, \dots x^4, x^2$. To shew the truth of this conjecture take the expansion given in equation (2) of Art. 370, and substitute in every term of $2\Sigma n^{2m}J_n(x)$; then picking out the coefficient of x^{2r} we shall find it to be

$$\frac{2}{2^{2r}\lfloor 2r}\left\{(2r)^{2m} - 2r(2r-2)^{2m} + \frac{2r(2r-1)}{\lfloor 2}(2r-4)^{2m} - \dots\right\},$$

that is

$$\frac{2}{2^{2r-2m}\lfloor 2r}\left\{r^{2m} - 2r(r-1)^{2m} + \frac{2r(2r-1)}{\lfloor 2}(r-2)^{2m} - \dots\right\};$$

the series within the brackets is to continue until 1^{2m} occurs, so that there will be r terms. When m is specified the value of this expression can be calculated for any value of r; and it will be found to vanish when r is greater than m, and to be equal to unity when r is equal to m. To shew the truth of these statements it is convenient to put the expression in the form

$$\frac{1}{2^{2r-2m} \lfloor 2r} \left\{ r^{2m} - 2r(r-1)^{2m} + \frac{2r(2r-1)}{\lfloor 2}(r-2)^{2m} - \dots \right\},$$

where the series within the brackets is now to be continued until it ends with $-2r(r-2r+1)^{2m} + (r-2r)^{2m}$, that is with $-2r(-r+1)^{2m} + (-r)^{2m}$; thus there are now $2r+1$ terms, of which the middle one is zero. With the notation of *Finite Differences* the expression becomes $\frac{1}{2^{2r-2m} \lfloor 2r} \Delta^{2r} x^{2m}$, where we are to put $-r$ for x after the operation denoted by Δ^{2r} has been performed. Then it is known by the theory of *Finite Differences* that the expression vanishes when r is greater than m, and is equal to unity when r is equal to m.

417. Suppose n an *odd* number. If we combine two of the results obtained in Art. 415 we deduce the following:

$$2\Sigma n^3 J_n(x) = x^3 + x.$$

In like manner we see that $2\Sigma n^5 J_n(x)$ can be expressed in terms of x^5, x^3, and x. From this we are naturally led to conjecture that $2\Sigma n^{2m+1} J_n(x)$ can be expressed in terms of $x^{2m+1}, x^{2m-1}, \dots x^3, x$. To shew the truth of this conjecture take the expression given in equation (2) of Art. 370, and substitute in every term of $2\Sigma n^{2m+1} J_n(x)$; then picking out the coefficient of x^{2r+1} we shall find it to be

$$\frac{2}{2^{2r+1} \lfloor 2r+1} \left\{ (2r+1)^{2m+1} - (2r+1)(2r-1)^{2m+1} + \frac{(2r+1)\,2r}{\lfloor 2}(2r-3)^{2m+1} - \dots \right\};$$

the series within the brackets is to continue until 1^{2m+1} occurs, so that there will be $r+1$ terms. When m is specified the value of this expression can be calculated for any value of r; and it will be found to vanish when r is greater than m, and to be equal to unity when r is equal to m. To shew the truth of these statements it is convenient to put the expression in the form

$$\frac{1}{2^{2r-2m}\,\lfloor 2r+1}\left\{\left(r+\frac{1}{2}\right)^{2m+1}-(2r+1)\left(r+\frac{1}{2}-1\right)^{2m+1}\right.$$
$$\left.+\frac{(2r+1)\,2r}{\lfloor 2}\left(r+\frac{1}{2}-2\right)^{2m+1}-\ldots\right\},$$

where the series within the brackets is now to be continued until it ends with $+(2r+1)\left(-r+1-\frac{1}{2}\right)^{2m+1}-\left(-r-\frac{1}{2}\right)^{2m+1}$; thus there are now $2r+2$ terms. With the notation of *Finite Differences* the expression becomes $\frac{1}{2^{2r-2m}\,\lfloor 2r+1}\Delta^{2r+1}x^{2m+1}$, where we are to put $-r-\frac{1}{2}$ for x after the operation denoted by Δ^{2r+1} has been performed. Then it is known by the theory of *Finite Differences* that the expression vanishes when r is greater than m, and is equal to unity when r is equal to m.

418. From Art. 376 we have

$$e^{\frac{xz}{2}}=e^{\frac{x}{2z}}\left\{J_0+zJ_1(x)+z^2J_2(x)+\ldots-\frac{1}{z}J_1(x)+\frac{1}{z^2}J_2(x)-\ldots\right\}.$$

Expand the exponential functions; and then equate the coefficients of z^r; thus

$$\frac{1}{\lfloor r}\left(\frac{x}{2}\right)^r=J_r(x)+\frac{x}{2}J_{r+1}(x)+\left(\frac{x}{2}\right)^2\frac{1}{\lfloor 2}J_{r+2}(x)+\left(\frac{x}{2}\right)^3\frac{1}{\lfloor 3}J_{r+3}(x)+\ldots$$

Equate the coefficients of $\frac{1}{z^r}$; thus

$$0=\left(\frac{x}{2}\right)^r\frac{1}{\lfloor r}J_0(x)+\left(\frac{x}{2}\right)^{r+1}\frac{1}{\lfloor r+1}J_1(x)+\left(\frac{x}{2}\right)^{r+2}\frac{1}{\lfloor r+2}J_2(x)+\ldots$$
$$-\left(\frac{x}{2}\right)^{r-1}\frac{1}{\lfloor r-1}J_1(x)+\left(\frac{x}{2}\right)^{r-2}\frac{1}{\lfloor r-2}J_2(x)-\left(\frac{x}{2}\right)^{r-3}\frac{1}{\lfloor r-3}J_3(x)-\ldots$$

419. From the latter formula, by putting for r in succession the values 1, 2, ... we obtain

$$\frac{x}{2} J_0(x) + \frac{x^2}{2.4} J_1(x) + \frac{x^3}{2.4.6} J_2(x) + \ldots = J_1(x),$$

$$\frac{x^2}{2.4} J_0(x) + \frac{x^3}{2.4.6} J_1(x) + \frac{x^4}{2.4.6.8} J_2(x) + \ldots = \frac{x}{2} J_1(x) - J_2(x),$$

and so on.

420. From the two expressions just given, we obtain

$$J_2(x) = \frac{x^2}{2.4} J_0(x) + \frac{2x^3}{2.4.6} J_1(x) + \frac{3x^4}{2.4.6.8} J_2(x) + \ldots$$

In like manner by proceeding to a third expression in Art. 419, and combining with the other two, we can deduce a formula for $J_3(x)$; and so on. The general formula is

$$J_r(x) = \frac{x^r}{2^r \lfloor r} J_0(x) + \frac{x^{r+1}}{2^{r+1} \lfloor r+1} rJ_1(x) + \frac{x^{r+2}}{2^{r+2} \lfloor r+2} \frac{r(r+1)}{\lfloor 2} J_2(x)$$

$$+ \frac{x^{r+3}}{2^{r+3} \lfloor r+3} \frac{r(r+1)(r+2)}{\lfloor 3} J_3(x) + \ldots (1).$$

This may be established by induction. For assume that (1) is true, divide by x^r and differentiate; then by equations (6) and (7) of Art. 386 we obtain

$$-x^{-r} J_{r+1}(x) = \frac{1}{2^r \lfloor r} \frac{d}{dx} J_0(x) + \frac{x}{2^{r+1} \lfloor r+1} rJ_0(x)$$

$$+ \frac{x^2}{2^{r+2} \lfloor r+2} \frac{r(r+1)}{\lfloor 2} J_1(x)$$

$$+ \frac{x^3}{2^{r+3} \lfloor r+3} \frac{r(r+1)(r+2)}{\lfloor 3} J_2(x) + \ldots$$

Now by Art. 381 we have $\frac{dJ_0(x)}{dx} = -J_1(x)$; substitute from Art. 419; thus

$$-x^{-r}J_{r+1}(x) = -\frac{1}{2^r\,\lfloor r}\left\{\frac{x}{2}J_0(x) + \frac{x^2}{2.4}J_1(x) + \frac{x^3}{2.4.6}J_2(x) + \ldots\right\}$$

$$+\frac{x}{2^{r+1}\,\lfloor r+1}\,rJ_0(x) + \frac{x^2}{2^{r+2}\,\lfloor r+2}\,\frac{r(r+1)}{\lfloor 2}J_1(x)$$

$$+\frac{x^3}{2^{r+3}\,\lfloor r+3}\,\frac{r(r+1)(r+2)}{\lfloor 3}J_2(x) + \ldots$$

so that finally $J_{r+1}(x) =$

$$\frac{x^{r+1}}{2^{r+1}\,\lfloor r+1}J_0(x) + \frac{x^{r+2}}{2^{r+2}\,\lfloor r+2}(r+1)J_1(x)$$

$$+\frac{x^{r+3}}{2^{r+3}\,\lfloor r+3}\,\frac{(r+1)(r+2)}{\lfloor 2}J_2(x) + \ldots \quad (2).$$

Thus (2) is the same formula as we should get by changing r into $r+1$ in (1). But we have seen that (1) is true when $r=1$, and when $r=2$, hence it is true when $r=3$, and when $r=4$, and so on.

421. In equation (8) of Art. 376 change x into kx; thus

$$e^{\frac{kx}{2}\left(z-\frac{1}{z}\right)} = J_0(kx) + zJ_1(kx) + z^2J_2(kx) + \ldots$$
$$-\frac{1}{z}J_1(kx) + \frac{1}{z^2}J_2(kx)\ldots(3).$$

Again, in equation (8) of Art. 376 change z into kz; thus

$$e^{\frac{x}{2}\left(kz-\frac{1}{kz}\right)} = J_0(x) + kz\,J_1(x) + k^2z^2J_2(x) + \ldots$$
$$-\frac{1}{kz}J_1(x) + \frac{1}{k^2z^2}J_2(x)\ldots(4).$$

But $e^{\frac{x}{2}\left(kz-\frac{1}{kz}\right)} = e^{\frac{kx}{2}\left(z-\frac{1}{z}\right)} \times e^{\frac{x}{2}\left(k-\frac{1}{k}\right)\frac{1}{z}}$, so that the product of the right-hand side of (4) into $e^{-\frac{x}{2}\left(k-\frac{1}{k}\right)\frac{1}{z}}$ must be equal to the right-hand side of (3). Thus putting μ for $k-\frac{1}{k}$, we have

$$J_0(kx) + zJ_1(kx) + z^2J_2(kx) + \ldots - \frac{1}{z}J_1(kx) + \frac{1}{z^2}J_2(kx) - \ldots$$

$$= e^{-\frac{\mu x}{2z}}\{J_0(x) + kzJ_1(x) + k^2z^2J_2(x) + \ldots$$

$$\left. - \frac{1}{kz}J_1(x) + \frac{1}{k^2z^2}J_2(x) - \ldots\right\}.$$

Expand the exponential and equate the coefficients of z^r; thus

$$J_r(kx) = k^rJ_r(x) - xk^{r+1}\frac{\mu}{2}J_{r+1}(x) + \frac{x^2k^{r+2}}{\lfloor 2}\left(\frac{\mu}{2}\right)^2 J_{r+2}(x) - \ldots$$

For a particular case we may suppose $k = \sqrt{2}$, and then $\mu = \frac{1}{\sqrt{2}}$.

422. Take equation (8) of Art. 376, and suppose both sides integrated m times with respect to x; the integration can be effected on the left-hand side, and may be denoted by the symbol S^m on the right-hand side. Thus we have

$$2^m\left(z - \frac{1}{z}\right)^{-m} e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = S^mJ_0(x) + zS^mJ_1(x) + z^2S^mJ_2(x) + \ldots$$

$$- \frac{1}{z}S^mJ_1(x) + \frac{1}{z^2}S^mJ_2(x) - \ldots;$$

and therefore

$$2^mz^{-m}\left(1 - \frac{1}{z^2}\right)^{-m}\left\{J_0(x) + zJ_1(x) + z^2J_2(x) + \ldots - \frac{1}{z}J_1(x) + \ldots\right\}$$

$$= S^mJ_0(x) + zS^mJ_1(x) + z^2S^mJ_2(x) + \ldots$$

$$- \frac{1}{z}S^mJ_1(x) + \frac{1}{z^2}S^mJ_2(x) - \ldots \quad (5).$$

From (5) we may deduce various formulæ. Thus for example equating the terms which are independent of z, we have

$$S^mJ_0(x) = 2^m\left\{J_m(x) + \frac{m}{1}J_{m+2}(x) + \frac{m(m+1)}{\lfloor 2}J_{m+4}(x) + \ldots\right\}$$

so that

$$\underline{|m-1}\, S^m J_0(x) = 2^m \left\{ \underline{|m-1}\, J_m(x) + \frac{\underline{|m}}{1} J_{m+2}(x) + \frac{\underline{|m+1}}{\underline{|2}} J_{m+4}(x) + \dots \right\} \text{.....(6).}$$

Particular cases of (6) may be obtained by putting for m in succession the values 1, 2, 3, ...

In the same way as (6) is obtained we may by equating the coefficients of z^r in (5) obtain a formula which differs from (6) in having the order of every Bessel's Function advanced by r; so that

$$\underline{|m-1}\, S^m J_r(x) = 2^m \left\{ \underline{|m-1}\, J_{m+r}(x) + \frac{\underline{|m}}{1} J_{m+r+2}(x) + \frac{\underline{|m+1}}{\underline{|2}} J_{m+r+4}(x) + \dots \right\}.$$

CHAPTER XXXV.

GENERAL THEOREMS WITH RESPECT TO EXPANSIONS.

423. In the preceding Chapter we have given various examples of the expansion of functions in infinite series of Bessel's Functions; in the present Chapter we shall give some general theorems relating to the subject.

424. We know that the function $J_0(x)$ satisfies the differential equation

$$\frac{d^2J_0(x)}{dx^2}+\frac{1}{x}\frac{dJ_0(x)}{dx}+J_0(x)=0.$$

Let α be a constant, and put u for $J_0(\alpha x)$; thus

$$\alpha^2u+\frac{1}{x}\frac{du}{dx}+\frac{d^2u}{dx^2}=0 \quad\text{..................(1).}$$

Let β be another constant, and put v for $J_0(\beta x)$; thus

$$\beta^2v+\frac{1}{x}\frac{dv}{dx}+\frac{d^2v}{dx^2}=0 \quad\text{..................(2).}$$

Let ξ be any assigned quantity; then we shall shew that

$$(\beta^2-\alpha^2)\int_0^\xi xuv\,dx=\xi\left[v\frac{du}{dx}-u\frac{dv}{dx}\right] \quad\text{.........(3);}$$

where the *square brackets* denote that for x we are to put ξ after the operations indicated have been performed; we shall employ *square brackets* throughout the Chapter in this sense.

For by the aid of (1) we have

$$\int xuv\,dx = -\frac{1}{\alpha^2}\int v\left(\frac{du}{dx} + x\frac{d^2u}{dx^2}\right)dx$$

$$= -\frac{1}{\alpha^2}\int v\frac{d}{dx}\left(x\frac{du}{dx}\right)dx$$

$$= -\frac{1}{\alpha^2}\left\{vx\frac{du}{dx} - \int x\frac{du}{dx}\frac{dv}{dx}dx\right\}$$

$$= -\frac{1}{\alpha^2}\left\{vx\frac{du}{dx} - xu\frac{dv}{dx} + \int u\frac{d}{dx}\left(x\frac{dv}{dx}\right)dx\right\}$$

$$= -\frac{x}{\alpha^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right) + \frac{\beta^2}{\alpha^2}\int xuv\,dx, \text{ by (2).}$$

Thus if we integrate between limits 0 and ξ, we have

$$(\beta^2-\alpha^2)\int_0^\xi xuv\,dx = \xi\left[v\frac{du}{dx} - u\frac{dv}{dx}\right].$$

425. We shall next determine the value of $\int_0^\xi xu^2dx$. We have shewn that

$$\int_0^\xi xuv\,dx = \frac{\xi\left[v\frac{du}{dx} - u\frac{dv}{dx}\right]}{\beta^2-\alpha^2}.$$

Now let us suppose that β approaches α as a limit, then the expression on the right-hand side takes the form $\frac{0}{0}$; and hence its limit found in the usual way is

$$\frac{\xi}{2\alpha}\left[\frac{dv}{d\beta}\frac{du}{dx} - u\frac{d^2v}{d\beta\,dx}\right],$$

where β is to be made equal to α ultimately.

Now $v=J_0(\beta x)$; thus $\frac{dv}{d\beta} = \frac{x}{\beta}\frac{dv}{dx}$, and

$$\frac{d^2v}{d\beta\,dx} = \frac{1}{\beta}\frac{dv}{dx} + \frac{x}{\beta}\frac{d^2v}{dx^2}.$$

Thus $\frac{dv}{d\beta}\frac{du}{dx} - u\frac{d^2v}{d\beta\, dx} = \frac{x}{\beta}\frac{du}{dx}\frac{dv}{dx} - \frac{u}{\beta}\left(\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right)$

$$= \frac{x}{\beta}\frac{du}{dx}\frac{dv}{dx} + \beta xuv, \text{ by (2).}$$

When β is made equal to α this becomes $\frac{x}{\alpha}\left(\frac{du}{dx}\right)^2 + \alpha xu^2$; so that finally

$$\int_0^\xi xu^2 dx = \frac{\xi^2}{2\alpha^2}\left[\alpha^2 u^2 + \left(\frac{du}{dx}\right)^2\right] \text{.............(4).}$$

426. We are about to particularise the values of α and β. Suppose p and q two roots of the equation (1) of Art. 396; and let α and β be determined by

$$p = \frac{\alpha^2\xi^2}{2^2}, \quad q = \frac{\beta^2\xi^2}{2^2} \text{..................(5).}$$

then will $$\int_0^\xi xuvdx = 0.$$

For we have $$\lambda + \frac{pf'(p)}{f(p)} = 0.$$

Now $f(p)$ is the value of u when we put ξ for x; so that $f(p) = [u]$. And $\frac{\alpha^2\xi}{2} f'(p)$ is the value of $\frac{du}{dx}$ when we put ξ for x; so that $\frac{\alpha^2\xi}{2} f'(p) = \left[\frac{du}{dx}\right]$.

Therefore $$\frac{pf'(p)}{f(p)} = \frac{\frac{\alpha^2\xi^2}{2^2} f'(p)}{f(p)} = \frac{\xi}{2}\left[\frac{1}{u}\frac{du}{dx}\right];$$

so that $$\lambda = -\frac{\xi}{2}\left[\frac{1}{u}\frac{du}{dx}\right].$$

In the same way we obtain $\lambda = -\frac{\xi}{2}\left[\frac{1}{v}\frac{dv}{dx}\right]$.

Hence the right-hand side of (3) vanishes, and therefore

$$\int_0^\xi xuv\, dx = 0 \text{......................(6).}$$

427. With the value of α assigned in the preceding Article we shall have

$$\int_0^\xi x u^2 \, dx = \frac{1}{2}\left(\frac{4\lambda^2}{\alpha^2} + \xi^2\right)[u^2] \text{..............} (7).$$

For, as we have just shewn, $\lambda = -\frac{\xi}{2}\left[\frac{1}{u}\frac{du}{dx}\right]$; and therefore $\left[\frac{du}{dx}\right] = -\frac{2\lambda}{\xi}[u]$. Hence substituting in (4) we obtain (7).

428. Suppose now that any function, as $\phi(x)$, can be expanded in the following form

$$\phi(x) = AJ_0(\alpha x) + BJ_0(\beta x) + CJ_0(\gamma x) + \text{.........} (8),$$

where $\alpha, \beta, \gamma, \ldots$ are constants determined by (5) and other similar equations, and $A, B, C, \ldots$ are constant coefficients, then the preceding theorems enable us to find the values of these constant coefficients.

Suppose for instance we wish to find the value of A; multiply both sides of (8) by $xJ_0(\alpha x)$ and integrate between 0 and ξ; then by (6) we have

$$\int_0^\xi x\phi(x)\, J_0(\alpha x)\, dx = A\int_0^\xi x\, \{J_0(\alpha x)\}^2\, dx;$$

and by (7) the value of the right-hand side is

$$\frac{A}{2}\left(\frac{4\lambda^2}{\alpha^2} + \xi^2\right)\{J_0(\alpha\xi)\}^2,$$

thus A is known, or at least its value depends only on the single definite integral

$$\int_0^\xi x\,\phi(x)\, J_0(\alpha x)\, dx.$$

Similarly $B, C, \ldots$ can be found.

429. It will be seen that in the preceding Article we do not undertake to shew that $\phi(x)$ *can* be always expanded in the assigned form, but *assuming* that it can be so expanded we find the values of the constant coefficients.

The fact is that the solutions of various physical problems lead to such processes as we have given, and the nature of the problems themselves may perhaps give some evidence of the possibility of the expansion: writers for the most part content themselves with finding the values of certain coefficients, as in Art. 428. Thus for instance Fourier discusses in the Chapter cited in Art. 394 a problem respecting the propagation of heat in a cylinder. He arrives at the general equation

$$\frac{dv}{dt} = k\left(\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx}\right) \quad\text{..................(9)};$$

this is to be satisfied consistently with the following special equation which is to hold when x has its greatest value ξ,

$$hv + \frac{dv}{dx} = 0 \quad\text{....................(10)}:$$

v is the temperature, t is the time, x is the distance from the axis of the cylinder. Assume $v = e^{-mt}u$; then if we put q for $\frac{m}{k}$ we obtain

$$qu + \frac{1}{x}\frac{du}{dx} + \frac{d^2u}{dx^2} = 0 \quad\text{..................(11)}.$$

The constant q will have various values to be found by the aid of (10). The general solution of (9) is taken to be $v = \Sigma e^{-mt}u$, where Σ refers to the different values of m.

The mathematical investigations which Fourier gives are equivalent to those of Arts. 395...397 and 424...428.

430. Suppose that $\alpha, \beta, \gamma, \ldots$ instead of being determined as in Art. 428 are such that

$$J_0(\alpha\xi) = 0, \quad J_0(\beta\xi) = 0, \quad J_0(\gamma\xi) = 0, \ldots\ldots,$$

and that any function, as $\phi(x)$, can be expanded in the form

$$\phi(x) = AJ_0(\alpha x) + BJ_0(\beta x) + CJ_0(\gamma x) + \ldots\ldots(12);$$

then we may find the values of the constant coefficients $A, B, C, \ldots$ by a process like that applied in Art. 428.

For equation (3) holds as before; and then since in the present case $[u] = 0$, and $[v] = 0$, we should obtain equation (6) as before.

Also equation (4) holds as before; and then since in the present case $[u] = 0$, it reduces to

$$\int_0^\xi x u^2 \, dx = \frac{\xi^2}{2\alpha^2} \left[\left(\frac{du}{dx}\right)^2\right] \text{ .... (13).}$$

Moreover $\dfrac{du}{dx} = \dfrac{dJ_0(\alpha x)}{dx} = \alpha \dfrac{dJ_0(\alpha x)}{d(\alpha x)} = -\alpha J_1(\alpha x)$ by Art. 381.

Hence we may if we please put (13) in the form

$$\int_0^\xi x \{J_0(\alpha x)\}^2 dx = \frac{1}{2} \xi^2 \{J_1(\alpha\xi)\}^2 \text{ .... (14).}$$

Hence by (6) and (14) we have

$$\int_0^\xi x\phi(x) \, J_0(\alpha x) \, dx = \frac{A}{2} \xi^2 \{J_1(\alpha\xi)\}^2.$$

Similarly B, C, ... can be found.

431. The process of Art. 430 may be regarded as an easy modification of Fourier's, and by several German writers is stated to be given in the Chapter of Fourier which we have cited: but what Fourier really gives is that which we have ascribed to him in Art. 429.

432. The investigations of the present Chapter admit of obvious extension, as we will now briefly indicate.

433. Let α and β be constants. Let $J_n(\alpha x) = (\alpha x)^n u$ and $J_n(\beta x) = (\beta x)^n v$. We shall find from Art. 370 that

$$\alpha^2 u + \frac{2n+1}{x}\frac{du}{dx} + \frac{d^2u}{dx^2} = 0 \text{ .... (15),}$$

$$\beta^2 v + \frac{2n+1}{x}\frac{dv}{dx} + \frac{d^2v}{dx^2} = 0 \text{ .... (16).}$$

434. Let ξ be any assigned quantity; then we shall have

$$(\beta^2-\alpha^2)\int_0^{\xi} x^{2n+1}uv\,dx=\xi^{2n+1}\left[v\frac{du}{dx}-u\frac{dv}{dx}\right]\ldots(17).$$

The demonstration is precisely like that of Art. 424.

435. Also

$$\int_0^{\xi} x^{2n+1}u^2dx=\frac{\xi^{2n+1}}{2\alpha^2}\left[xu^2\alpha^2+2nu\frac{du}{dx}+x\left(\frac{du}{dx}\right)^2\right]\ldots(18).$$

The demonstration is like that of Art. 425.

436. Let p and q be roots of the equation of Art. 400; and let α and β be determined by

$$p=\frac{\alpha^2\xi^2}{2^2},\quad q=\frac{\beta^2\xi^2}{2^2}\ldots\ldots\ldots\ldots\ldots(19);$$

then will

$$\int_0^{\xi} x^{2n+1}uv\,dx=0\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

The demonstration is like that of Art. 426.

437. With the value of α assigned in the preceding Article we shall have

$$\int_0^{\xi} x^{2n+1}u^2dx=\frac{\xi^{2n}}{2\alpha^2}(\alpha^2\xi^2-4n\lambda+4\lambda^2)[u^2]\ldots\ldots(21).$$

The demonstration is like that of Art. 427.

438. Suppose then that any function, as $\phi(x)$, can be expressed in the following form

$$\phi(x)=Au+Bv+Cw+\ldots,$$

where u, and v are as already stated, w is similarly related to $J_n(\gamma x)$, where γ is of the same nature as α and β, and so on; then the constant coefficients $A, B, C, \ldots$ may be found.

For by (20) we have

$$\int_0^{\xi} x^{2n+1}u\phi(x)\,dx=A\int_0^{\xi} x^{2n+1}u^2dx,$$

and the integral occurring on the right-hand side is known by (21). Similarly $B, C, \ldots$ can be found.

439. Suppose now that $\alpha, \beta, \gamma, \ldots$ instead of being determined as in Art. 436 are such that

$$\frac{1}{\xi^n} J_n(\alpha\xi) = 0, \quad \frac{1}{\xi^n} J_n(\beta\xi) = 0, \quad \frac{1}{\xi^n} J_n(\gamma\xi) = 0, \ldots,$$

then if any function, as $\phi(x)$, can be expanded in the form

$$\phi(x) = Au + Bv + Cw + \ldots,$$

we may find the value of the constant coefficients $A, B, C, \ldots$ by a process like that applied in Art. 438.

For equation (17) holds as before; and then since in the present case $[u] = 0$ and $[v] = 0$, we should obtain equation (20) as before.

Also equation (18) holds as before; and then since in the present case $[u] = 0$, it reduces to

$$\int_0^\xi x^{2n+1} u^2 dx = \frac{\xi^{2n+2}}{2\alpha^2} \left[\left(\frac{du}{dx}\right)^2\right] \ldots\ldots\ldots\ldots\ldots(22).$$

Thus as before we can find $A, B, C, \ldots$.

440. If in Art. 434 we put for u and v their values in terms of Bessel's Functions we shall find that equation (17) becomes

$$(\beta^2 - \alpha^2) \int_0^\xi x J_n(\alpha x) J_n(\beta x)\, dx$$
$$= \xi \left[J_n(\beta x) \frac{d}{dx} J_n(\alpha x) - J_n(\alpha x) \frac{d}{dx} J_n(\beta x) \right];$$

and by equation (6) of Art. 386 the right-hand member may be transformed into

$$\xi \{\beta J_n(\alpha\xi) J_{n+1}(\beta\xi) - \alpha J_n(\beta\xi) J_{n+1}(\alpha\xi)\}.$$

In like manner equation (22) becomes

$$\int_0^\xi x \{J_n(\alpha x)\}^2 dx = \frac{\xi^{2n+2}}{2\alpha^2} \left[\left\{\frac{d}{dx} x^{-n} J_n(\alpha x)\right\}^2\right]$$
$$= \frac{\xi^2}{2\alpha^2} \left[\left\{\frac{d}{dx} J_n(\alpha x)\right\}^2\right]$$
$$= \frac{\xi^2}{2} \left\{J_{n+1}(\alpha\xi)\right\}^2, \text{ by equation (6) of Art. 386.}$$

441. We shall now give a remarkable theorem due to Schlömilch by which any function is expressed in an infinite series of Bessel's Functions.

We know that if $F(z)$ denote any function of z, then for any value of z which lies between 0 and h, we have

$$F(z) = \frac{1}{2}A_0 + A_1 \cos\frac{\pi z}{h} + A_2 \cos\frac{2\pi z}{h} + A_3 \cos\frac{3\pi z}{h} + \dots (23),$$

where $$A_n = \frac{2}{h}\int_0^h F(u) \cos\frac{n\pi u}{h}\, du.$$

For h put $\frac{1}{2}\pi$, and for z put μx; thus

$$F(\mu x) = \frac{1}{2}A_0 + A_1 \cos 2\mu x + A_2 \cos 4\mu x + A_3 \cos 6\mu x + \dots,$$

where $$A_n = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} F(u) \cos 2nu\, du.$$

Multiply each side of this equation by $\frac{2}{\pi}\frac{d\mu}{\sqrt{(1-\mu^2)}}$, and integrate between the limits 0 and 1; this gives, by Art. 374,

$$\frac{2}{\pi}\int_0^1 \frac{F(\mu x)\, d\mu}{\sqrt{(1-\mu^2)}} = \frac{1}{2}A_0 + A_1 J_0(2x) + A_2 J_0(4x) + A_3 J_0(6x) + \dots \quad \dots\dots (24),$$

the relation holds for values of x between 0 and $\frac{1}{2}\pi$, because μ is never greater than unity.

Now suppose that

$$\frac{2}{\pi}\int_0^1 \frac{F(x\mu)\, d\mu}{\sqrt{(1-\mu^2)}} = f(x) \dots\dots (25);$$

differentiate with respect to x, thus

$$\frac{2}{\pi}\int_0^1 \frac{\mu F'(\mu x)\, d\mu}{\sqrt{(1-\mu^2)}} = f'(x).$$

In this equation write $k\xi$ instead of x, multiply both sides by $\frac{kd\xi}{\sqrt{(1-\xi^2)}}$, and integrate between the limits 0 and 1 for ξ: thus

$$\frac{2k}{\pi}\int_0^1 \frac{1}{\sqrt{(1-\xi^2)}}\left\{\int_0^1 \frac{\mu F'(\mu k\xi)\,d\mu}{\sqrt{(1-\mu^2)}}\right\}d\xi = k\int_0^1 \frac{f'(k\xi)\,d\xi}{\sqrt{(1-\xi^2)}}.$$

Hence by a theorem due to Abel, which will be established in the next Article, we shall have

$$F(k) - F(0) = k\int_0^1 \frac{f'(k\xi)\,d\xi}{\sqrt{(1-\xi^2)}}.$$

When $x=0$ we have $F(0)=f(0)$ from (25);

hence $$F(k) = f(0) + k\int_0^1 \frac{f'(k\xi)\,d\xi}{\sqrt{(1-\xi^2)}} \quad\text{............(26).}$$

Equation (26) involves the solution of (25), when in (25) we regard f as a given, and F as an unknown form.

Substitute in (24) for F in terms of f: thus

$$f(x) = \frac{1}{2}A_0 + A_1J_0(2x) + A_2J_0(4x) + A_3J_0(6x) + \ldots,$$

where $A_n = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\left\{f(0) + u\int_0^1 \frac{f'(u\xi)\,d\xi}{\sqrt{(1-\xi^2)}}\right\}\cos 2nu\,du$:

for every value of n *except zero* the last equation reduces to

$$A_n = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} u\cos 2nu\left\{\int_0^1 \frac{f'(u\xi)\,d\xi}{\sqrt{(1-\xi^2)}}\right\}du,$$

but in the case of $n=0$ we must add $2f(0)$.

Thus $f(x)$ is expanded in an infinite series of Bessel's Functions.

442. It remains to establish the theorem due to Abel.

It is immediately obvious that

$$\int_0^k\int_0^{\sqrt{(k^2-x^2)}} \frac{F'(x)\,dx\,dy}{\sqrt{(k^2-x^2-y^2)}} = \frac{\pi}{2}\{F(k) - F(0)\}.$$

Transform the definite double integral by the use of polar coordinates; then it becomes

$$\int_0^k \int_0^{\frac{1}{2}\pi} \frac{F'(r\cos\theta)\, r\, dr\, d\theta}{\sqrt{(k^2 - r^2)}}.$$

Put $\cos\theta = \xi$, and $r = k\mu$; then the definite double integral becomes $k\int_0^1 \int_0^1 \frac{F'(\mu k\xi)\, \mu d\mu d\xi}{\sqrt{(1-\xi^2)}\sqrt{(1-\mu^2)}}$. Hence we have

$$k\int_0^1 \frac{1}{\sqrt{(1-\xi^2)}}\left\{\int_0^1 \frac{F'(\mu k\xi)\,\mu d\mu}{\sqrt{(1-\mu^2)}}\right\} d\xi = \frac{\pi}{2}\{F(k) - F(0)\}:$$

this is the theorem which was to be established.

443. Differentiate with respect to x the result obtained in Art. 441; and put $\phi(x)$ for $f'(x)$; then since $\frac{dJ_0(x)}{dx} = -J_1(x)$, we have

$$\phi(x) = B_1 J_1(2x) + B_2 J_1(4x) + B_3 J_1(6x) + \ldots,$$

where $$B_n = -\frac{8}{\pi} n \int_0^{\frac{1}{2}\pi} u \cos 2nu \left\{\int_0^1 \frac{\phi(u\xi)}{\sqrt{(1-\xi^2)}}\, d\xi\right\} du.$$

444. If we put $h = \pi$ instead of $h = \frac{1}{2}\pi$ in equation (23) and proceed as in Art. 441, instead of the result of that Article we shall obtain the following:

$$f(x) = \frac{1}{2}a_0 + a_1 J_0(x) + a_2 J_0(2x) + a_3 J_0(3x) + \ldots,$$

where $$a_n = \frac{2}{\pi}\int_0^{\pi} u\cos nu \left\{\int_0^1 \frac{f'(u\xi)\, d\xi}{\sqrt{(1-\xi^2)}}\right\} du,$$

for every value of n except zero, and when n is zero we must add $2f(0)$. The formula holds for values of x between 0 and π.

By differentiating this, as in Art. 443, we obtain

$$\phi(x) = b_1 J_1(x) + b_2 J_1(2x) + b_3 J_1(3x) + \ldots,$$

where $$b_n = -\frac{2n}{\pi}\int_0^{\pi} u\cos nu\, du \int_0^1 \frac{\phi(u\xi)\, d\xi}{\sqrt{(1-\xi^2)}}.$$

The formulæ of the present Article might also be deduced from those of Arts. 441 and 443 by putting in them $x = \frac{x'}{2}$.

445. In the first formula of the preceding Article change x into $\sqrt{x}$; thus

$$f(\sqrt{x}) = \frac{1}{2} a_0 + \Sigma a_n J_0 (n \sqrt{x}),$$

where Σ denotes summation with respect to n from 1 to infinity.

Differentiate both sides m times with respect to x; then since by Art. 390 we have

$$\frac{d^m J_0 (n \sqrt{x})}{dx^m} = \left(-\frac{1}{2}\right)^m n^m x^{-\frac{m}{2}} J_m (n \sqrt{x}),$$

we obtain

$$\frac{d^m f(\sqrt{x})}{dx^m} = \left(-\frac{1}{2}\right)^m x^{-\frac{m}{2}} \Sigma a_n n^m J_m (n \sqrt{x}),$$

where a_n has the value assigned in Art. 444.

CHAPTER XXXVI.

MISCELLANEOUS PROPOSITIONS.

446. In this Chapter we shall collect some miscellaneous propositions which involve the use of Bessel's Functions.

447. Having given $y = z + x \sin y$ it is required to express y in terms of z.

This problem may be stated in Astronomical language thus; to express the eccentric anomaly in terms of the mean anomaly: see Hymers's *Astronomy*, Art. 315, or Godfray's *Astronomy*, Art. 179.

When $y=0$ we have $z=0$, and when $y=\pi$ we have $z=\pi$. Thus $y-z$ vanishes both when $z=0$ and when $z=\pi$; and we may therefore expand $y-z$ in the following series:

$$y - z = C_1 \sin z + C_2 \sin 2z + C_3 \sin 3z + \ldots,$$

where
$$C_n = \frac{2}{\pi}\int_0^\pi (y-z)\sin nz\, dz$$
$$= \frac{2}{\pi}\int_0^\pi y \sin nz\, dz + \frac{2}{n}\cos n\pi.$$

By integration by parts we have

$$\int y \sin nz\, dz = -\frac{y}{n}\cos nz + \frac{1}{n}\int \cos nz\, dy$$
$$= -\frac{y}{n}\cos nz + \frac{1}{n}\int \cos n\,(y - x\sin y)\, dy;$$

therefore

$$\int_0^\pi y \sin nz\, dz = -\frac{\pi}{n}\cos n\pi + \frac{1}{n}\int_0^\pi \cos n\,(y - x\sin y)\, dy.$$

Thus

$$C_n = \frac{2}{\pi n}\int_0^\pi \cos n\,(y - x\sin y)\,dy = \frac{2}{n} J_n(nx), \text{ by Art. 372.}$$

Therefore

$$y - z = 2\left\{J_1(x)\sin z + \frac{1}{2} J_2(2x)\sin 2z + \frac{1}{3} J_3(3x)\sin 3z + \dots\right\}\dots(1).$$

448. In like manner we may find expressions for $\cos ky$ and $\sin ky$, where k is any integer.

For we may put

$$\cos ky = A_0 + A_1\cos z + A_2\cos 2z + A_3\cos 3z + \dots$$

Then $A_0 = \frac{1}{\pi}\int_0^\pi \cos ky\,dz = \frac{1}{\pi}\int_0^\pi \cos ky\,(1 - x\cos y)\,dy$;

this vanishes if k is not unity, and is equal to $-\frac{x}{2}$ if k is unity.

Moreover $\quad A_n = \frac{2}{\pi}\int_0^\pi \cos ky \cos nz\,dz$

$$= \frac{2k}{\pi n}\int_0^\pi \sin ky \sin nz\,dy, \text{ by integration by parts,}$$

$$= \frac{2k}{\pi n}\int_0^\pi \sin ky \sin n(y - x\sin y)\,dy$$

$$= \frac{k}{\pi n}\int_0^\pi \cos(ny - ky - nx\sin y)\,dy$$

$$- \frac{k}{\pi n}\int_0^\pi \cos(ny + ky - nx\sin y)\,dy$$

$$= \frac{k}{n}\{J_{n-k}(nx) - J_{n+k}(nx)\}.$$

In like manner we may put

$$\sin ky = B_1\sin z + B_2\sin 2z + B_3\sin 3z + \dots,$$

and proceeding as before we shall find that

$$B_n = \frac{k}{n}\{J_{n-k}(nx) + J_{n+k}(nx)\}.$$

Suppose for example that $k=1$; then by Art. 379

$$B_n = \frac{1}{n} \cdot \frac{2n}{xn} J_n(nx) = \frac{2}{nx} J_n(nx);$$

therefore

$$x \sin y = 2\left\{J_1(x)\sin z + \frac{1}{2}J_2(2x)\sin 2z + \frac{1}{3}J_3(3x)\sin 3z + \dots\right\} \dots (2).$$

Thus (2) agrees with (1).

449. Let r denote the radius vector from the focus in the ellipse corresponding to the eccentric anomaly y, and suppose the semi-axis major to be unity; then $r = 1 - x\cos y$, and this can be expressed in terms of z, since the series for $\cos y$ is known by Art. 448.

Also we have $\frac{dy}{dz}(1 - x\cos y) = 1$; therefore $\frac{1}{r} = \frac{dy}{dz}$: and finding $\frac{dy}{dz}$ from (1), we have

$$\frac{1}{r} = 1 + 2\{J_1(x)\cos z + J_2(2x)\cos 2z + J_3(3x)\cos 3z + \dots\}.$$

450. To shew that

$$\int_0^\infty e^{-bx} J_0(ax)\,dx = \frac{1}{\sqrt{(a^2+b^2)}}.$$

We have

$$J_0(ax) = \frac{1}{\pi}\int_0^\pi \cos(ax\cos\phi)\,d\phi;$$

therefore

$$\int_0^\infty e^{-bx} J_0(ax)\,dx = \frac{1}{\pi}\int_0^\infty \int_0^\pi e^{-bx}\cos(ax\cos\phi)\,dx\,d\phi.$$

Integrate with respect to x first; thus we get

$$\frac{1}{\pi}\int_0^\pi \frac{b\,d\phi}{b^2 + a^2\cos^2\phi}; \text{ and this} = \frac{2b}{\pi}\int_0^{\frac{1}{2}\pi} \frac{d\phi}{b^2+a^2\cos^2\phi} = \frac{1}{\sqrt{(a^2+b^2)}}.$$

Put $b=0$ in the preceding result; thus

$$\int_0^\infty J_0(ax)\,dx = \frac{1}{a}.$$

451. To shew that

$$\int_0^\infty x^{m-1} J_0(ax)\,dx = \frac{\Gamma(m)\,\Gamma\left(\frac{1-m}{2}\right)\Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{3}{2}}a^m}\sin m\pi,$$

where m denotes a positive proper fraction.

We have

$$J_0(ax) = \frac{1}{\pi}\int_0^\pi \cos(ax\cos\phi)\,d\phi;$$

hence the proposed definite integral

$$= \frac{1}{\pi}\int_0^\infty \int_0^\pi x^{m-1}\cos(ax\cos\phi)\,dx\,d\phi.$$

But $\int_0^\pi \cos(ax\cos\phi)\,d\phi = 2\int_0^{\frac{1}{2}\pi}\cos(ax\cos\phi)\,d\phi$; thus the proposed definite integral

$$= \frac{2}{\pi}\int_0^\infty \int_0^{\frac{1}{2}\pi} x^{m-1}\cos(ax\cos\phi)\,dx\,d\phi.$$

Integrate with respect to x first; then by *Integral Calculus*, Art. 302, we get

$$\frac{2\Gamma(m)\cos\frac{m\pi}{2}}{\pi a^m}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\cos^m\phi}.$$

But $$2\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\cos^m\phi} = \int_0^1 \frac{2dz}{z^m(1-z^2)^{\frac{1}{2}}} = \int_0^1 \frac{2z\,dz}{z^{m+1}(1-z^2)^{\frac{1}{2}}}$$

$$= \int_0^1 \frac{dy}{y^{\frac{m+1}{2}}(1-y)^{\frac{1}{2}}} = \int_0^1 y^{-\frac{m+1}{2}}(1-y)^{-\frac{1}{2}}dy = \frac{\Gamma\left(\frac{1-m}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1-\frac{m}{2}\right)};$$

and $$\Gamma\left(\frac{m}{2}\right)\Gamma\left(1-\frac{m}{2}\right) = \frac{\pi}{\sin\frac{m\pi}{2}}:$$

thus the definite integral

$$= \frac{\Gamma(m)\,\Gamma\left(\frac{1-m}{2}\right)\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\pi^2 a^m}\cos\frac{m\pi}{2}\sin\frac{m\pi}{2}$$

$$= \frac{\Gamma(m)\,\Gamma\left(\frac{1-m}{2}\right)\Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{3}{2}} a^m}\sin m\pi.$$

452. We have, by Art. 371,

$$J_0(x+y) = \frac{1}{\pi}\int_0^\pi \cos(x\cos\phi + y\cos\phi)\,d\phi$$

$$= \frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\cos(y\cos\phi)\,d\phi - \frac{1}{\pi}\int_0^\pi \sin(x\cos\phi)\sin(y\cos\phi)\,d\phi.$$

But, by Art. 413,

$$\cos(x\cos\phi) = J_0(x) - 2J_2(x)\cos 2\phi + 2J_4(x)\cos 4\phi - \ldots,$$

$$\cos(y\cos\phi) = J_0(y) - 2J_2(y)\cos 2\phi + 2J_4(y)\cos 4\phi - \ldots;$$

therefore $\frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\cos(y\cos\phi)\,d\phi$

$$= J_0(x)\,J_0(y) + 2J_2(x)\,J_2(y) + 2J_4(x)\,J_4(y) + \ldots$$

Also, by Art. 413,

$$\sin(x\cos\phi) = 2J_1(x)\cos\phi - 2J_3(x)\cos 3\phi + 2J_5(x)\cos 5\phi - \ldots,$$

$$\sin(y\cos\phi) = 2J_1(y)\cos\phi - 2J_3(y)\cos 3\phi + 2J_5(y)\cos 5\phi - \ldots;$$

therefore $\frac{1}{\pi}\int_0^\pi \sin(x\cos\phi)\sin(y\cos\phi)\,d\phi$

$$= 2J_1(x)\,J_1(y) + 2J_3(x)\,J_3(y) + 2J_5(x)\,J_5(y) + \ldots$$

Hence finally $J_0(x+y) =$

$$J_0(x)\,J_0(y) - 2J_1(x)\,J_1(y) + 2J_2(x)\,J_2(y) - 2J_3(x)\,J_3(y) + \ldots$$

453. Let P, Q, R, k be constants which satisfy the relations

$$R\cos k = P, \quad R\sin k = Q;$$

so that
$$R^2 = P^2 + Q^2.$$

By Art. 372 we have

$$J_0(R) = \frac{1}{\pi}\int_0^\pi \cos(R\sin\phi)\,d\phi.$$

Now obviously

$$\int_0^\pi \cos(R\sin\phi)\,d\phi = \int_{-k}^{\pi-k} \cos\{R\sin(\phi+k)\}\,d\phi;$$

and by differentiating the last expression with respect to k so far as depends on the limits of the integration we obtain zero for the result, so that the value is independent of the value which we ascribe to k in the *limit*, and we may consequently put zero for k in the *limit*. Thus

$$J_0(R) = \frac{1}{\pi}\int_0^\pi \cos\{R\sin(\phi+k)\}\,d\phi$$

$$= \frac{1}{\pi}\int_0^\pi \cos R\{\sin\phi\cos k + \cos\phi\sin k\}\,d\phi$$

$$= \frac{1}{\pi}\int_0^\pi \cos(P\sin\phi + Q\cos\phi)\,d\phi \quad\ldots\ldots\ldots\ldots(3).$$

In precisely the same manner we may shew that

$$J_0(R) = \frac{1}{\pi}\int_0^\pi \cos(P\sin\phi - Q\cos\phi)\,d\phi \quad\ldots\ldots\ldots(4).$$

From (3) and (4) by addition and subtraction

$$J_0(R) = \frac{2}{\pi}\int_0^\pi \cos(P\sin\phi)\cos(Q\cos\phi)\,d\phi \quad\ldots\ldots(5),$$

$$0 = \frac{1}{\pi}\int_0^\pi \sin(P\sin\phi)\sin(Q\cos\phi)\,d\phi \quad\ldots\ldots(6).$$

Now let R denote the distance of two points determined by polar coordinates, so that we may put

$$R^2 = r^2 + r_1^2 - 2rr_1 \cos\theta,$$

or

$$R^2 = (r - r_1\cos\theta)^2 + r_1^2 \sin^2\theta.$$

Then by (4) we have

$$J_0(R) = \frac{1}{\pi}\int_0^\pi \cos\{(r - r_1\cos\theta)\sin\phi - r_1\sin\theta\cos\phi\}\, d\phi,$$

that is

$$J_0(R) = \frac{1}{\pi}\int_0^\pi \cos\{r\sin\phi - r_1\sin(\phi+\theta)\}\, d\phi \quad \text{.......(7).}$$

But by Arts. 410 and 411,

$$\cos(r\sin\phi) = J_0(r) + 2J_2(r)\cos 2\phi + 2J_4(r)\cos 4\phi + \dots,$$

$$\sin(r\sin\phi) = 2J_1(r)\sin\phi + 2J_3(r)\sin 3\phi + 2J_5(r)\sin 5\phi + \dots;$$

and two other formulæ may be expressed by changing r into r_1, and ϕ into $\phi + \theta$.

Thus we obtain

$$\int_0^\pi \cos(r\sin\phi)\cos\{r_1\sin(\phi+\theta)\}\, d\phi$$

$$= \pi\{J_0(r)J_0(r_1) + 2J_2(r)J_2(r_1)\cos 2\theta + 2J_4(r)J_4(r_1)\cos 4\theta + \dots\},$$

and $$\int_0^\pi \sin(r\sin\phi)\sin\{r_1\sin(\phi+\theta)\}\, d\phi$$

$$= 2\pi\{J_1(r)J_1(r_1)\cos\theta + J_3(r)J_3(r_1)\cos 3\theta + J_5(r)J_5(r_1)\cos 5\theta + \dots\}.$$

Add the last two results, and thus we obtain from (7)

$$J_0(R) = J_0(r)\,J_0(r_1) + 2\Sigma J_n(r)\,J_n(r_1)\cos n\theta \quad \text{.........(8),}$$

where Σ denotes summation with respect to n from 1 to infinity.

If we suppose $\theta = \pi$ the result agrees with that obtained in Art. 452 for $J_0(x+y)$.

454. As a particular case of (8) suppose $r_1 = r$, so that $R = 2r \sin \frac{\theta}{2}$; then

$$J_0\left(2r \sin \frac{\theta}{2}\right) = \{J_0(r)\}^2 + 2\Sigma \{J_n(r)\}^2 \cos n\theta \;\ldots\ldots(9).$$

But by Art. 372 we have

$$J_0\left(2r \sin \frac{\theta}{2}\right) = \frac{1}{\pi}\int_0^\pi \cos\left(2r \sin \frac{\theta}{2} \sin \phi\right) d\phi,$$

and, by Art. 410,

$$\cos\left(2r \sin \phi \sin \frac{\theta}{2}\right) = J_0(2r \sin \phi) + 2J_2(2r \sin \phi) \cos \theta$$
$$+ 2J_4(2r \sin \phi) \cos 2\theta + \ldots,$$

therefore $J_0\left(2r \sin \frac{\theta}{2}\right) =$

$$\frac{1}{\pi}\int_0^\pi \{J_0(2r \sin \phi) + 2J_2(2r \sin \phi) \cos \theta$$
$$+ 2J_4(2r \sin \phi) \cos 2\theta + \ldots\} \, d\phi \ldots (10).$$

Hence comparing (9) and (10) we have

$$\{J_n(r)\}^2 = \frac{1}{\pi}\int_0^\pi J_{2n}(2r \sin \phi) \, d\phi.$$

455. The equation of Art. 376, as we have seen in the preceding Chapters, easily leads to various theorems respecting Bessel's Functions when the number expressing the order of the function is a *positive integer*. And, as we have seen in Arts. 380 and 381, it is sometimes easy to extend these theorems to the case in which the number expressing the order is *not restricted* to be a positive integer. As another example of such extension we may take the last formula of Art. 422, which has been established on the supposition that r is a *positive integer*, and shew that this restriction may be removed.

The first term in $J_r(x)$ is $\dfrac{1}{2^r\Gamma(r+1)}x^r$; and when this is integrated m times we obtain $\dfrac{1}{2^r\Gamma(m+r+1)}x^{m+r}$, and thus we easily see that the lowest term on the left-hand side is identical with the lowest on the right-hand side.

In like manner the other terms will be identical. For multiply both sides by $\Gamma(r+1)$, and then when the appropriate reductions are made which the properties of the Gamma functions allow, we shall obtain for the coefficients of any assigned power of x, *definite algebraical functions of* r; and as we know already that they coincide for every integral value of r it follows that they are identically equal.

456. Both Neumann and Lommel have introduced functions to which they give the name of Bessel's Functions of *the second order;* the two functions are not the same, but for them the reader is referred to the original works.

We may observe that equation (1) of Art. 370 remains unchanged when the sign of n is changed; this suggests that a second integral of the equation will be given by the following series when n is not a positive integer:

$$Bx^{-n}\left\{1+\frac{x^2}{2(2n-2)}+\frac{x^4}{2.4(2n-2)(2n-4)}\right.$$
$$\left.+\frac{x^6}{2.4.6(2n-2)(2n-4)(2n-6)}+\dots\right\},$$

and this may be easily verified.

In Lommel's work will also be found tables of the numerical values of $J_0(x)$ and $J_1(x)$ and of some others of the functions.

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