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KEY

TO

TODHUNTER'S

DIFFERENTIAL CALCULUS.

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BY

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PREFACE.

THE solutions in the *Key* will, I think, be found sufficiently full, and no pains have been spared to ensure their accuracy; they are also so worded that figures can be easily supplied by the student to enable him more readily to follow the reasoning, when geometrical. Figures are however given to the curves in Chaps. XXVI. and XXVIII.

To make the book more useful and complete, I have, in some instances, adopted improved methods—chiefly in the Solutions to some of the examples in Chaps. XI., XIII., XV., XX. and XXII.; and I am bound to acknowledge some obligations to the last chapter of Mr Turnbull's *Analytical Plane Geometry*.

H. ST. J. H.

The references, unless otherwise specified, are to the *tenth* edition of the *Differential Calculus*.

KEY TO DIFFERENTIAL CALCULUS.

CHAPTER IV.

1. See Arts. 29, 33, and 47. *Or*, from definition $\frac{dy}{dx} = lt.$ of

$$\frac{c}{h}(\sqrt{x+h} - \sqrt{x}) = c \div (\sqrt{x+h} + \sqrt{x}) = c \div 2\sqrt{x}.$$

2. $$y = \frac{a}{x} - 1, \;\therefore\text{ by Art. 47, } \frac{dy}{dx} = -\frac{a}{x^2}.$$

3. By Arts. 31 and 44, $\frac{dy}{dx} = \frac{1}{1+x^2} - (1+x) \cdot \frac{2x}{(1+x^2)^2} = \frac{1-2x-x^2}{(1+x^2)^2}.$

4. By Arts. 29 and 50, log meaning to the base e, $\frac{dy}{dx} = \log x + \frac{x}{x}.$

5. By Arts. 50, 54 and 63, $\frac{dy}{dx} = \tan x \,.\, (-\operatorname{cosec}^2 x) = -\frac{2}{\sin 2x}.$

6. Arts. 63, 47, and 31 lead to $\frac{dy}{dx} = \frac{1}{(a^2-x^2)^{\frac{1}{2}}} + \frac{x \,.\, x}{(a^2-x^2)^{\frac{3}{2}}} = \frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}.$

7. So $$\frac{dy}{dx} = \frac{3x^2}{(1-x^2)^{\frac{3}{2}}} + \frac{x^3 \,.\, \left(-\frac{3}{2}\right)(-2x)}{(1-x^2)^{\frac{5}{2}}} = \frac{3x^2}{(1-x^2)^{\frac{5}{2}}}.$$

8. From Arts. 29 and 49, $\frac{dy}{dx} = e^x(1-x^3) + e^x(-3x^2)$, &c.

9. So $$\frac{dy}{dx} = 1 \,.\, e^{2x} + 2e^{2x}(x-3) + 4e^x + 4xe^x + 1,$$
$$= e^{2x}(2x-5) + 4e^x(x+1) + 1.$$

10. Here $\frac{dy}{dx} = 2 \,.\, e^{2x} + 2e^{2x}(2x-5) + 4e^x + 4(x+1)e^x$, &c.

11. $\log y = nx \,.\, \log\frac{x}{n}$, $\therefore$ (Art. 72) $\frac{1}{y}\,.\,\frac{dy}{dx} = n\,.\,\log\frac{x}{n} + nx\,.\,\frac{n}{x}\,.\,\frac{1}{n}$

$$= n\log\frac{x}{n} + n, \;\therefore\; \frac{dy}{dx} = n\left(\log\frac{x}{n}+1\right).\,y, \text{ \&c.}$$

12. $$\frac{dy}{dx} = \frac{n\,.\,x^{n-1}}{(1+x)^n} - \frac{x^n\,.\,n}{(1+x)^{n+1}} = \frac{nx^{n-1}}{(1+x)^{n+1}}.$$

13. $$\frac{dy}{dx} = \frac{e^x+e^{-x}}{e^x+e^{-x}} - \frac{(e^x-e^{-x})\,(e^x-e^{-x})}{(e^x+e^{-x})^2} = \frac{4}{(e^x+e^{-x})^2}.$$

14. $$\frac{dy}{dx} = \frac{1}{e^x+e^{-x}}\,.\,(e^x-e^{-x}), \text{ by Arts. 50, and 63.}$$

15. $$\frac{dy}{dx} = \frac{2y}{x} + \frac{3y}{a+x} - \frac{4y}{b-x}$$

$$= x\,(a+x)^2\,(b-x)^3\,(2\overline{a+x}\;\overline{b-x} + 3x\,\overline{b-x} - 4x\,\overline{a+x}), \text{ \&c.}$$

16. $$\frac{dy}{dx} = m\,(a+x)^{m-1}\,(b+x)^n + (a+x)^m\,.\,n\,(b+x)^{n-1}, \text{ \&c.}$$

17. $$\frac{dy}{dx} = -\frac{m}{(a+x)^{m+1}\,.\,(b+x)^n} - \frac{n}{(a+x)^m\,.\,(b+x)^{n+1}} = \text{\&c.}$$

18. $$\frac{dy}{dx} = \tan^2 x\,.\,\sec^2 x - \sec^2 x + 1 = \tan^2 x\,\sec^2 x - \tan^2 x = \tan^4 x.$$

19. $$\frac{dy}{dx} = -\frac{1}{(x+\sqrt{1-x^2})^2}\left\{1-\frac{1}{2}\,.\,\frac{2x}{\sqrt{1-x^2}}\right\} = \frac{-\sqrt{1-x^2}+x}{\sqrt{1-x^2}\,(x+\sqrt{1-x^2})^2}, \text{ \&c.}$$

20. Art. 69 leads to $\frac{dy}{dx} = 2x\,.\,\tan^{-1}\frac{x}{a} + (a^2+x^2)\,.\,\frac{1}{1+\frac{x^2}{a^2}}\,.\,\frac{1}{a}$, $\therefore$ &c.

21. $y = \frac{1}{x}\sqrt{ax^2+bx+c}$, $\therefore$ $\frac{dy}{dx} = -\frac{1}{x^2}\sqrt{ax^2+bx+c}$

$$+\frac{1}{x}\,.\,\frac{1}{2}\,.\,\frac{2ax+b}{\sqrt{ax^2+bx+c}} = \frac{-2\,(ax^2+bx+c)+x\,(2ax+b)}{2x^2\sqrt{ax^2+bx+c}} = \frac{+bx+2c}{-2x^2\sqrt{ax^2+bx+c}}.$$

22. $$\frac{dy}{dx} = \frac{1}{\log\,(a+bx^n)}\,.\,\frac{1}{(a+bx^n)}\,.\,nbx^{n-1}, \text{ by an extension of Art. 63.}$$

23. $$\frac{dy}{dx} = \cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right).\,\frac{1}{2} = \frac{1}{\sin\left(\frac{\pi}{2}+x\right)} = \sec x.$$

24. $$\frac{dy}{dx} = \cos x\,.\,e^{(a+x)^2} + \sin x\,.\,e^{(a+x)^2}\,.\,2\,(a+x).$$

25. $y(\sqrt{x}+\sqrt{a})=\sqrt{x+a}$, $\therefore\ \frac{dy}{dx}(\sqrt{x}+\sqrt{a})=-y\cdot\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{x+a}}$

$$=-\frac{\sqrt{x+a}}{2\sqrt{x}(\sqrt{x}+\sqrt{a})}+\frac{1}{2\sqrt{x+a}}=\frac{\sqrt{a}(\sqrt{x}-\sqrt{a})}{2\sqrt{x}(\sqrt{x}+\sqrt{a})\sqrt{x+a}},\ \&c.$$

26. $2\log y=\log\left(\frac{1+x}{1-x}\right)$, $\therefore\ \frac{2}{y}\cdot\frac{dy}{dx}=\frac{1}{1+x}+\frac{1}{1-x}$,

$$\therefore\ \frac{dy}{dx}=\frac{1}{1-x^2}\cdot\sqrt{\frac{1+x}{1-x}},\ \&c.$$

27. $\text{Log}\, y=\frac{1}{2}\log(1-x^2)-\frac{3}{2}\log(1+x^2)$;

$$\therefore\ \frac{1}{y}\frac{dy}{dx}=\frac{-x}{1-x^2}-\frac{3x}{1+x^2}=\frac{-x(4-2x^2)}{1-x^4},\ \&c.$$

28. $\frac{dy}{dx}=\frac{1}{e^x-1}-\frac{x\,.\,e^x}{(e^x-1)^2}=\&c.$

29. $y=\frac{e^{2x}(x-2)+e^x(x+2)}{(e^x-1)^3}$.

$$\therefore\ \frac{dy}{dx}=\frac{e^{2x}(\overline{2x-4+1})+e^x(\overline{x+2+1})}{(e^x-1)^3}-\frac{3e^x}{(e^x-1)^4}\{e^{2x}(x-2)+e^x(x+2)\}$$

$$=\frac{e^x}{(e^x-1)^4}\{(e^x-1)(e^x\,\overline{2x-3}+x+3)-3e^{2x}(x-2)-3e^x(x+2)\}$$

$$=\frac{e^x}{(e^x-1)^4}\{e^{2x}(-x+3)+e^x(-4x)-x-3\},\ \&c.$$

30. $y=\log\frac{1+\sqrt{1-x^2}}{x}$, $\therefore\ \frac{dy}{dx}=\frac{1}{1+\sqrt{1-x^2}}\cdot\frac{-x}{\sqrt{1-x^2}}-\frac{1}{x}$

$$=\frac{1-\sqrt{1-x^2}}{-x\sqrt{1-x^2}}-\frac{1}{x}=-\frac{1}{x\sqrt{1-x^2}}.$$

31. $\frac{d}{dx}(x+\sqrt{1-x^2})=1-\frac{x}{\sqrt{1-x^2}}$,

$$\therefore\ \frac{dy}{dx}=n\{x+\sqrt{1-x^2}\}^{n-1}\,.\,\frac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}}.$$

32. $\frac{1}{n}\log y=\log\frac{1-\sqrt{1-x^2}}{x}$, $\therefore\ \frac{1}{ny}\cdot\frac{dy}{dx}=\frac{1}{1-\sqrt{1-x^2}}\cdot\frac{x}{\sqrt{1-x^2}}-\frac{1}{x}$

$$=\frac{1+\sqrt{1-x^2}}{x\sqrt{1-x^2}}-\frac{1}{x}=\frac{1}{x\sqrt{1-x^2}},\ \therefore\ \&c.$$

33. $\log y = (n+1)\log x - \frac{1}{2}\log(1-x^2) - n\log(1+\sqrt{1-x^2})$;

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{n+1}{x} + \frac{x}{1-x^2} + \frac{n}{1+\sqrt{1-x^2}} \cdot \frac{x}{\sqrt{1-x^2}}$$

$$= \frac{n+1}{x} + \frac{x}{1-x^2} + \frac{n(1-\sqrt{1-x^2})}{x \cdot \sqrt{1-x^2}} = \frac{n}{x\sqrt{1-x^2}} + \frac{1}{x(1-x^2)}$$

$$= \frac{1+n\sqrt{1-x^2}}{x(1-x^2)}, \therefore \text{ \&c.}$$

34. $\frac{dy}{dx} = y \cdot \log_e a \cdot \frac{d}{dx}\{(a^2-x^2)^{-\frac{1}{2}}\} = y \cdot \log_e a \cdot \frac{x}{(a^2-x^2)^{\frac{3}{2}}}$.

35. $$\frac{dy}{dx} = \sec^2 a^{\frac{1}{x}} \cdot a^{\frac{1}{x}} \log_e a \cdot \left(-\frac{1}{x^2}\right).$$

36. $y = \frac{1}{2}\log 2(1+\sqrt{1-x^4})$, $\therefore \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{1+\sqrt{1-x^4}} \cdot \frac{-2x^3}{\sqrt{1-x^4}}$

$$= -\frac{x^3}{\sqrt{1-x^4}} \cdot \frac{1-\sqrt{1-x^4}}{x^4}, \text{ \&c.}$$

37. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\sqrt{a^{\frac{1}{2}}+x^{\frac{1}{2}}} + \frac{(2a^{\frac{1}{2}}+x^{\frac{1}{2}})}{2\sqrt{a^{\frac{1}{2}}+x^{\frac{1}{2}}}} \cdot \frac{1}{2\sqrt{x}}$

$$= \frac{1}{4\sqrt{x}} \cdot \frac{4a^{\frac{1}{2}}+3x^{\frac{1}{2}}}{(a^{\frac{1}{2}}+x^{\frac{1}{2}})^{\frac{1}{2}}}.$$

38. $\frac{dy}{dx} = 1 + \frac{1}{\cos\left(\frac{\pi}{4}-x\right)}\left\{-\sin\left(\frac{\pi}{4}-x\right)\right\}(-1)$

$$= 1 + \tan\left(\frac{\pi}{4}-x\right) = 1 + \frac{1-\tan x}{1+\tan x}, \text{ \&c.}$$

39. $y = \frac{1+\sqrt{1-x^4}}{x^2}$, $\therefore \frac{dy}{dx} = \frac{1}{x^2} \cdot \frac{-2x^3}{\sqrt{1-x^4}} - (1+\sqrt{1-x^4}) \cdot \frac{2}{x^3}$

$$= -\frac{2}{x^3} \cdot \frac{\sqrt{1-x^4}+x^4+1-x^4}{\sqrt{1-x^4}}, \text{ \&c.}$$

40. Arts. 29 and 65.

41. Arts. 29, 53 and 67.

42. $\frac{dy}{dx}=n\cos nx\sin^n x+\sin nx\,.\,n\sin^{n-1}x\,.\,\cos x=$&c.

43. $\frac{dy}{dx}=\frac{m(\sin nx)^{m-1}n\,.\cos nx}{(\cos mx)^n}-\frac{(\sin nx)^m\,.\,n(-m\sin mx)}{(\cos mx)^{n+1}}=$&c.

44. $\frac{dy}{dx}=e^{-a^2x^2}(-2a^2x\cos rx)-r\sin rx\,.\,e^{-a^2x^2}$, $\therefore$ &c.

45. $\frac{dy}{dx}=\frac{1-\frac{1}{\sqrt{1-x^2}}}{\sin^3 x}-(x-\sin^{-1}x)\,.\,\frac{3\cos x}{\sin^4 x}$, &c.

46. $y=\log\left(a+b\tan\frac{x}{2}\right)-\log\left(a-b\tan\frac{x}{2}\right)$;

$$\therefore \frac{dy}{dx}=\frac{1}{a+b\tan\frac{x}{2}}\,.\,b\sec^2\frac{x}{2}\,.\,\frac{1}{2}+\frac{1}{a-b\tan\frac{x}{2}}\,.\,b\sec^2\frac{x}{2}\,.\,\frac{1}{2}$$

$$=\frac{ab\sec^2\frac{x}{2}}{a^2-b^2\tan^2\frac{x}{2}}=\frac{ab}{a^2\cos^2\frac{x}{2}-b^2\sin^2\frac{x}{2}}.$$

47. Cf. Art. 73. Thus $\frac{dy}{dx}=x^x\,.\log x+x\,.\,x^{x-1}$, &c.

48. Cf. Art. 73, and $\frac{dy}{dx}=x^{\frac{1}{x}}\,.\log x\,.\left(-\frac{1}{x^2}\right)+\frac{1}{x}\,.\,x^{\frac{1}{x}-1}=$&c.

Aliter: $\log y=\frac{1}{x}\log x$, &c.

49. $\log y=\sin^{-1}x\,.\log x$, $\therefore\ \frac{1}{y}\,.\,\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}\,.\log x+\frac{1}{x}\sin^{-1}x$, &c.

50. $\log y=e^x$, $\therefore\ \frac{1}{y}\,.\,\frac{dy}{dx}=e^x$, $\therefore$ &c.

51. $\log y=x^x$, $\therefore\ \frac{1}{y}\,.\,\frac{dy}{dx}=x^x\,.\log x+x\,.\,x^{x-1}$ (Art. 73), &c.

52. $\log y=x^x\,.\log x$, $\therefore\ \frac{1}{y}\,.\,\frac{dy}{dx}=(x^x\log x+x\,.\,x^{x-1})\log x+\frac{1}{x}\,.\,x^x$, &c.

53. $\log y=e^x\,.\log x$, $\therefore\ \frac{1}{y}\,.\,\frac{dy}{dx}=e^x\,.\log x+e^x\,.\,\frac{1}{x}$, &c.

54. $\tan y = \dfrac{2x}{1+x^2}$, $\therefore\ \sec^2 y \cdot \dfrac{dy}{dx} = \dfrac{2}{1+x^2} - \dfrac{2x \cdot 2x}{(1+x^2)^2}$,

or $\dfrac{dy}{dx} \cdot \dfrac{(1+x^2)^2+4x^2}{(1+x^2)^2} = \dfrac{2(1-x^2)}{(1+x^2)^2}$, $\therefore$ &c.

55. $$\frac{dy}{dx} = \frac{1}{\sqrt{1-\frac{(x+1)^2}{2}}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1-2x-x^2}}.$$

56. $$\frac{dy}{dx} = \sec^2 \sqrt{1-x}\left(\frac{-1}{2\sqrt{1-x}}\right).$$

57. $$y = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \sin^{-1} x, \ \therefore \text{ \&c.}$$

58. $$\frac{dy}{dx} = \frac{1}{1+n^2\tan^2 x} \cdot (n\sec^2 x), \text{ \&c.}$$

59. $y = \cos^{-1}\sqrt{1-\dfrac{x^2}{a^2}} = \sin^{-1}\dfrac{x}{a}$, $\therefore\ \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\dfrac{x^2}{a^2}}} \cdot \dfrac{1}{a}$, &c.

60. $\dfrac{dy}{dx} = \tan^{-1}\sqrt{\dfrac{x}{a}} + (x+a)\dfrac{1}{1+\dfrac{x}{a}} \cdot \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{ax}} - \dfrac{1}{2} \cdot \sqrt{\dfrac{a}{x}}$, &c.

61. $\dfrac{dy}{dx} = \dfrac{a}{a^2+x^2} + \dfrac{1}{2}\left(\dfrac{1}{x-a} - \dfrac{1}{x+a}\right) = \dfrac{a}{a^2+x^2} - \dfrac{a}{a^2-x^2}$, &c.

62. $\dfrac{dy}{dx} = \dfrac{1}{1-\sin x} \cdot \dfrac{1}{2} \cdot \dfrac{1}{\sqrt{\sin x}} \cdot \cos x = \dfrac{1}{2}\,\dfrac{\sqrt{1+\sin x}}{\sqrt{\sin x}}$, &c.

63. $$\frac{y}{2} = \tan^{-1} x, \ \therefore\ \frac{dy}{dx} = \frac{2}{1+x^2}.$$

64. $\sin y = \dfrac{ax}{b+cx^2}$, $\therefore\ \cos y \cdot \dfrac{dy}{dx} = \dfrac{a}{b+cx^2} - \dfrac{2acx^2}{(b+cx^2)^2}$,

or $\dfrac{dy}{dx} \cdot \dfrac{\sqrt{(b+cx^2)^2 - a^2x^2}}{b+cx^2} = \dfrac{ab-acx^2}{(b+cx^2)^2}$, $\therefore$ &c.

65. $$\frac{dy}{dx} = \frac{-x\sin^{-1} x}{\sqrt{1-x^2}} + 1 - 1, \text{ \&c.}$$

66. $\dfrac{dy}{dx} = \sin^{-1} x\left(\dfrac{1}{\sqrt{1-x^2}} + \dfrac{x^2}{(1-x^2)^{\frac{3}{2}}}\right) + \dfrac{x}{1-x^2} + \dfrac{1}{2}\left(\dfrac{1}{1-x^2}\right)(-2x) = $ &c.

67. $\tan y = x + \sqrt{1-x^2}$, $\therefore \sec^2 y \cdot \frac{dy}{dx} = 1 - \frac{x}{\sqrt{1-x^2}}$,

or $\frac{dy}{dx}(2 + 2x\sqrt{1-x^2}) = (\sqrt{1-x^2} - x) \div \sqrt{1-x^2}$, &c.

68. $$\sin y = \frac{x \tan a}{\sqrt{a^2 - x^2}}, \ \therefore \cos y \frac{dy}{dx}$$

$$= \tan a \left\{ \frac{1}{(a^2 - x^2)^{\frac{1}{2}}} + \frac{x^2}{(a^2 - x^2)^{\frac{3}{2}}} \right\} = \frac{a^2 \tan a}{(a^2 - x^2)^{\frac{3}{2}}}$$

$$= \frac{dy}{dx} \cdot \frac{\sqrt{a^2 - x^2 \sec^2 a}}{(a^2 - x^2)^{\frac{1}{2}}}, \ \therefore \text{ \&c.}$$

69. $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \frac{a^2 - x^2}{b^2 - x^2}}} \cdot \frac{1}{2} \sqrt{\frac{b^2 - x^2}{a^2 - x^2}} \left\{ \frac{-2x}{b^2 - x^2} + \frac{2x(a^2 - x^2)}{(b^2 - x^2)^2} \right\}$$

$$= \frac{-(b^2 - x^2)\, x\, (b^2 - a^2)}{(b^2 - a^2)^{\frac{1}{2}} \sqrt{a^2 - x^2} \cdot (b^2 - x^2)^2}, \text{ \&c.}$$

70. $$y = \tan^{-1}\left(\tan \frac{x}{2}\right) = \frac{x}{2}, \ \therefore \text{ \&c.}$$

71. $$\sin y = \frac{b + a\cos x}{a + b\cos x} = \frac{a\cos x + \frac{a^2}{b} - \frac{a^2 - b^2}{b}}{a + b\cos x} = \frac{a^2 - b^2}{b} \cdot \frac{1}{a + b\cos x} + \frac{a}{b};$$

$$\therefore \cos y \frac{dy}{dx} = -\frac{a^2 - b^2}{b} \cdot \frac{b \sin x}{(a + b\cos x)^2} = \frac{dy}{dx} \frac{\sqrt{a^2 - b^2} \cdot \sin x}{a + b\cos x};$$

$\therefore$ &c.

72. $$\therefore \frac{1}{\sqrt{a^2 - b^2}} \tan y = \frac{\sin x}{b + a\cos x},$$

$$\therefore \frac{\sec^2 y \cdot \frac{dy}{dx}}{\sqrt{a^2 - b^2}} = \frac{\cos x\,(b + a\cos x) + a\sin^2 x}{(b + a\cos x)^2},$$

or $$\frac{dy}{dx} \left\{ \frac{(b + a\cos x)^2 + (a^2 - b^2)\sin^2 x}{(b + a\cos x)^2} \right\} = \frac{\sqrt{a^2 - b^2}\,(a + b\cos x)}{(b + a\cos x)^2}, \text{ \&c.}$$

73. $$\cos y = 1 - \frac{2}{x^{2n} + 1};$$

$$\therefore -\sin y \frac{dy}{dx} = \frac{2 \cdot 2nx^{2n-1}}{(x^{2n} + 1)^2} = -\frac{dy}{dx} \cdot \frac{2x^n}{x^{2n} + 1}, \text{ \&c.}$$

74. $\cos y = 2x^2 - 1$, $\therefore\ -\sin y \cdot \dfrac{dy}{dx} = 4x = -\dfrac{dy}{dx} \cdot 2x\sqrt{1-x^2}$, &c.

75. $\dfrac{dy}{dx} = \dfrac{x^2}{x^2+2+x^2-2\sqrt{1+x^2}} \cdot \left\{\dfrac{1}{\sqrt{1+x^2}} - \dfrac{\sqrt{1+x^2}-1}{x^2}\right\} =$ &c.

76. $$\frac{dy}{dx} = \frac{2x+\sqrt{2}}{1+x^2+x\sqrt{2}} - \frac{2x-\sqrt{2}}{1+x^2-x\sqrt{2}}$$

$$+2 \cdot \frac{1}{1+\dfrac{2x^2}{(1-x^2)^2}} \left\{\frac{\sqrt{2}}{1-x^2} + \frac{2\sqrt{2}x^2}{(1-x^2)^2}\right\},$$

$$= \frac{2\sqrt{2}(1+x^2) - 4\sqrt{2}x^2}{1+x^4} + \frac{2\sqrt{2}}{1+x^4}\{1+x^2\} = \text{\&c.}$$

77. $$\frac{du}{dy} = \frac{1}{3(y+1)} - \frac{1}{6} \cdot \frac{2y-1}{y^2-y+1} - \frac{1}{\sqrt{3}} \cdot \frac{3}{3+(2y-1)^2} \cdot \frac{2}{\sqrt{3}}$$

$$= \frac{1}{3(y+1)} - \frac{1}{6} \cdot \frac{2y-1+3}{y^2-y+1} = -\frac{y}{y^3+1},$$

and $y^3 = \dfrac{1}{x^3} + \dfrac{3}{x^2} + \dfrac{3}{x}$, $\therefore\ 3y^2\dfrac{dy}{dx} = -\dfrac{3}{x^4} - \dfrac{6}{x^3} - \dfrac{3}{x^2}$,

or $y^2\dfrac{dy}{dx} = -\dfrac{(1+x)^2}{x^4}$;

$\therefore\ \dfrac{du}{dx} = \dfrac{du}{dy} \cdot \dfrac{dy}{dx} = \dfrac{y(1+x)^2}{y^2x^4(y^3+1)}$, but $y^3+1 = \dfrac{(1+x)^3}{x^3}$;

$\therefore\ \dfrac{du}{dx} = \dfrac{1}{xy(1+x)}$.

78. Hence the sum

$$= \left(\frac{n+1}{2}\cos\frac{n+1}{2}x\sin\frac{nx}{2} + \frac{n}{2}\sin\frac{n+1}{2}x\cos\frac{nx}{2}\right) \div \sin\frac{x}{2}$$

$$-\frac{1}{2}\cos\frac{x}{2}\sin\frac{n+1}{2}x\sin\frac{nx}{2} \div \sin^2\frac{x}{2}$$

$$= \frac{n+1}{2}\sin\frac{2n+1}{2}x \div \sin\frac{x}{2} - \frac{1}{2}\sin\frac{n+1}{2}x \cdot \sin\frac{n+1}{2}x \div \sin^2\frac{x}{2}$$

= &c.

79. Take the logarithm of each side and then differentiate with regard to x.

80. Differentiate the result of Ex. 79 with regard to x.

CHAPTER V.

1. $y=\frac{1+\sin x}{\cos x}$, $\therefore\ \frac{dy}{dx}=\frac{\cos x}{\cos x}+\frac{(1+\sin x)\sin x}{\cos^2 x}=\frac{1+\sin x}{\cos^2 x}=\frac{1}{1-\sin x}$,

$$\therefore\ \frac{d^2y}{dx^2}=-\frac{1}{(1-\sin x)^2}(-\cos x)=\frac{\cos x}{(1-\sin x)^2}.$$

2. Cf. Art. 78.

3. $\frac{dy}{dx}=2x\log x+x$, $\therefore\ \frac{dy^2}{dx^2}=2\log x+2+1$, $\therefore\ \frac{d^3y}{dx^3}=\frac{2}{x}$.

4. $\frac{dy}{dx}=3x^2\log x+x^2$, $\therefore\ \frac{d^2y}{dx^2}=6x\log x+3x+2x$, $\therefore\ \frac{d^3y}{dx^3}=6\log x+11$,

$$\therefore\ \frac{d^4y}{dx^4}=\frac{6}{x}.$$

5. $\frac{dy}{dx}=2x\tan^{-1}\frac{x}{a}+a$, $\therefore\ \frac{d^2y}{dx^2}=2\tan^{-1}\frac{x}{a}+\frac{2ax}{a^2+x^2}$,

$$\therefore\ \frac{d^3y}{dx^3}=\frac{2a}{a^2+x^2}+\frac{2a}{a^2+x^2}-\frac{2ax}{(a^2+x^2)^2}\cdot 2x=\frac{4a\cdot a^2}{(a^2+x^2)^2}.$$

6. $\frac{dy}{dx}=-e^{-x}\cdot\cos x-e^{-x}\cdot\sin x$,

$$\therefore\ \frac{d^2y}{dx^2}=e^{-x}\cos x+2e^{-x}\cdot\sin x-e^{-x}\cos x=2e^{-x}\sin x;$$

$$\therefore\ \frac{d^3y}{dx^3}=-2e^{-x}\cdot\sin x+2e^{-x}\cos x,$$

$$\therefore\ \frac{d^4y}{dx^4}=2e^{-x}\sin x-4e^{-x}\cdot\cos x-2e^{-x}\cdot\sin x=-4y.$$

7. $y=x^{\frac{3}{2}}\div(x-a)^{\frac{1}{2}}$, $\therefore\ \frac{dy}{dx}=\frac{3}{2}\frac{x^{\frac{1}{2}}}{\sqrt{x-a}}-\frac{1}{2}\frac{x^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}$;

$$\therefore\ \frac{d^2y}{dx^2}=\frac{3}{4}\cdot\frac{x^{-\frac{1}{2}}}{\sqrt{x-a}}-\frac{3}{4}\frac{x^{\frac{1}{2}}}{(x-a)^{\frac{3}{2}}}-\frac{3}{4}\frac{x^{\frac{1}{2}}}{(x-a)^{\frac{3}{2}}}+\frac{3}{4}\cdot\frac{x^{\frac{3}{2}}}{(x-a)^{\frac{5}{2}}}$$

$$=\frac{3}{4}\cdot\frac{a}{\sqrt{x}\,(x-a)^{\frac{3}{2}}}-\frac{3}{4}\cdot\frac{a\sqrt{x}}{(x-a)^{5}}=\frac{3a^2}{4\sqrt{x}\,(x-a)^{\frac{5}{2}}}.$$

8. $\frac{dy}{dx}=n\{x+\sqrt{x^2-1}\}^{n-1}\cdot\left\{1+\frac{x}{\sqrt{x^2-1}}\right\}=\frac{ny}{\sqrt{x^2-1}}$;

$$\therefore\ \frac{d^2y}{dx^2}=\frac{n}{\sqrt{x^2-1}}\cdot\frac{dy}{dx}-\frac{nxy}{(x^2-1)^{\frac{3}{2}}}=\frac{n^2y}{x^2-1}-\frac{x}{x^2-1}\cdot\frac{dy}{dx};$$

$$\therefore\ (x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y.$$

9. By Arts. 79 and 80, or thus: $\because \frac{d^n}{dx^n}(x^{n-1})=0$,

$$\frac{d^ny}{dx^n}=\frac{d^{n-1}}{dx^{n-1}}(\overline{n-1}\,x^{n-2}\,.\log x+x^{n-2})=\frac{d^{n-1}}{dx^{n-1}}\{(n-1)\,x^{n-2}\,.\log x\},$$

$$\text{so } \frac{d^ny}{dx^n}=(n-1)(n-2)\frac{d^{n-2}}{dx^{n-2}}\{x^{n-3}\log x\}, \text{ and so on,}$$

$$=\lfloor n-1\,.\frac{d}{dx}(\log x)=\frac{\lfloor n-1}{x}.$$

10.
$$y=\frac{1-x}{1+x}=\frac{2}{1+x}-1,$$

$$\therefore \frac{dy}{dx}=\frac{2}{1}\cdot\frac{1}{(1+x)^2}(-1), \text{ and so on, } \frac{d^ny}{dx^n}=\frac{2(-1)^n\lfloor n}{(1+x)^{n+1}}.$$

11. $\frac{du_n}{dx}=n\,(e^x+e^{-x})^{n-1}\,.\,(e^x-e^{-x})$,

$$\therefore \frac{d^2u_n}{dx^2}=n\,(n-1)\,(e^x+e^{-x})^{n-2}\,.\,(e^x-e^{-x})^2$$

$$+n\,(e^x+e^{-x})^{n-1}\,.\,(e^x+e^{-x})=n\,(n-1)\,(e^x+e^{-x})^{n-2}\,(\overline{e^x+e^{-x}}|^2-4)+nu_n$$
$$=n\,(n-1)\,u_n-4n\,(n-1)\,u_{n-2}+nu_n=n^2u_n-4n\,(n-1)\,u_{n-2}.$$

12. $\log y=2\sqrt{x}$, $\therefore \frac{1}{y}\frac{dy}{dx}=\frac{1}{\sqrt{x}}$, $\therefore -\frac{1}{y^2}\cdot\overline{\frac{dy}{dx}}\Big|^2+\frac{1}{y}\frac{d^2y}{dx^2}=-\frac{1}{2x^{\frac{3}{2}}}$;

$$\therefore \frac{d^2y}{dx^2}=y\left\{\frac{1}{x}-\frac{1}{2x^{\frac{3}{2}}}\right\}=e^{2\sqrt{x}}\left\{\frac{2\sqrt{x}-1}{2x\sqrt{x}}\right\}.$$

13. $y=\frac{x^3}{1-x}=\frac{x^3-x^2+x^2-x+x-1+1}{1-x}=-x^2-x-1+\frac{1}{1-x}$,

$$\therefore \frac{d^4y}{dx^4}=\frac{d^4}{dx^4}\left(\frac{1}{1-x}\right)=\frac{\lfloor 4\,(-1)^4}{(1-x)^5}.$$

14. By Art. 55, $2y\frac{dy}{dx}=\sec 2x\,.\tan 2x\,.\,2$, $\therefore$ (Art. 53),

$$\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=2\sec 2x\,.\tan^2 2x+2\sec^3 2x, \text{ and } \overline{\frac{dy}{dx}}\Big|^2=\frac{\sec^2 2x\,.\tan^2 2x}{\sec 2x};$$

$$\therefore y\frac{d^2y}{dx^2}=2\sec^3 2x+\sec 2x\,(\sec^2 2x-1)=3y^6-y^2, \text{ and, } \therefore \text{ \&c.}$$

15. $y=\frac{x^2-x+1}{(x^2+1)^{\frac{1}{2}}}$, $\therefore \frac{dy}{dx}=\frac{2x-1}{(x^2+1)^{\frac{1}{2}}}-\frac{x\,(x^2-x+1)}{(x^2+1)^{\frac{3}{2}}}=\frac{x^3+x-1}{(x^2+1)^{\frac{3}{2}}}$;

$$\therefore \frac{d^2y}{dx^2}=\frac{(3x^2+1)(x^2+1)-3x\,(x^3+x-1)}{(x^2+1)^{\frac{5}{2}}}=\frac{x^2+3x+1}{(x^2+1)^{\frac{5}{2}}}.$$

16. $y = \frac{ax+b}{x^2-c^2} = \frac{p}{x-c} + \frac{q}{x+c}$ suppose, $\therefore p+q=a,\ p-q=\frac{b}{c}$;

$$\therefore y = \frac{1}{2c}\left\{\frac{ac+b}{x-c} + \frac{ac-b}{x+c}\right\},\ \text{and}\ \frac{d^n}{dx^n}\left(\frac{1}{x-c}\right) = (-1)^n \lfloor n \div (x-c)^{n+1},\ \&c.$$

17. If $y = \sin x \,.\, x^n$, by Arts. 78 and 80,

$$\frac{d^n y}{dx^n} = \sin x \,.\, \lfloor n + n \sin\left(x+\frac{\pi}{2}\right) . \frac{\lfloor n}{\lfloor 1} . x + \frac{n\,\overline{n-1}}{\lfloor 2} \sin\left(x+\frac{2\pi}{2}\right) \frac{\lfloor n}{\lfloor 2} . x^2$$

$$+ \frac{n(n-1)(n-2)}{\lfloor 3} . \sin\left(x+\frac{3\pi}{2}\right) . \frac{\lfloor n}{\lfloor 3} . x^3 + \ldots .$$

21. If $y = \frac{1}{e^x-1}$, $\frac{dy}{dx} = \frac{-e^x}{(e^x-1)^2}$, $\therefore \frac{d^2y}{dx^2} = \frac{-e^x}{(e^x-1)^2} + \frac{2e^{2x}}{(e^x-1)^3}$,

$$\therefore \frac{d^3y}{dx^3} = \frac{-e^x}{(e^x-1)^2} + \frac{6e^{2x}}{(e^x-1)^3} - \frac{6e^{3x}}{(e^x-1)^4},$$

$$\therefore \frac{d^4y}{dx^4} = \frac{-e^x}{(e^x-1)^2} + \frac{14e^{2x}}{(e^x-1)^3} - \frac{(6\,.\,3+6\,.\,3)\,e^{3x}}{(e^x-1)^4} + \frac{6\,.\,4\,.\,e^{4x}}{(e^x-1)^5}$$

$$= \{-e^x(e^x-1)^3 + 14e^{2x}(e^x-1)^2 - 36e^{3x}(e^x-1) + 24e^{4x}\} \div (e^x-1)^5$$

$$= \frac{1}{(e^x-1)^5}\{24e^{4x} - 36e^{4x} + 36e^{3x} + 14e^{4x} - 28e^{3x} + 14e^{2x} - e^{4x} + 3e^{3x} - 3e^{2x} + e^x\}$$

$$= \frac{e^x + 11e^{2x} + 11e^{3x} + e^{4x}}{(e^x-1)^5}.$$

If $y = e^{-\frac{1}{x^2}}$, $\frac{dy}{dx} = e^{-\frac{1}{x^2}} . \frac{2}{x^3}$, $\therefore \frac{d^2y}{dx^2} = e^{-\frac{1}{x^2}}\left(\frac{4}{x^6} - \frac{6}{x^4}\right)$;

$$\therefore \frac{d^3y}{dx^3} = e^{-\frac{1}{x^2}}\left\{\frac{2}{x^3}\left(\frac{4}{x^6} - \frac{6}{x^4}\right) - \frac{24}{x^7} + \frac{24}{x^5}\right\};$$

$$\therefore \frac{d^4y}{dx^4} = e^{-\frac{1}{x^2}}\left\{\frac{2}{x^3}\left(\frac{8}{x^9} - \frac{12}{x^7} - \frac{24}{x^7} + \frac{24}{x^5}\right) - \frac{72}{x^{10}} + \frac{36\,.\,7}{x^8} - \frac{120}{x^6}\right\}$$

$$= e^{-\frac{1}{x^2}}\left\{\frac{16}{x^{12}} - \frac{144}{x^{10}} + \frac{48+252}{x^8} - \frac{120}{x^6}\right\} = \&c.$$

22. By Arts. 79 and 80, $\frac{d^n . (x^2 . a^x)}{dx^n} = x^2 . \frac{d^n(a^x)}{dx^n} + n \,.\, 2x \frac{d^{n-1}(a^x)}{dx^{n-1}}$

$$+ \frac{2n\,\overline{n-1}}{\lfloor 2} . \frac{d^{n-2}(a^x)}{dx^{n-2}} = x^2 . c^n . a^x + 2nxc^{n-1} . a^x + n(n-1) . a^x . c^{n-2}.$$

23. $$y=\sin(m\sin^{-1}x),\ \therefore\ \frac{dy}{dx}=\cos(m\sin^{-1}x)\cdot\frac{m}{\sqrt{1-x^2}};$$

$$\therefore\ \frac{d^2y}{dx^2}=-\sin(m\sin^{-1}x)\cdot\frac{m^2}{1-x^2}+\cos(m\sin^{-1}x)\cdot\frac{mx}{(1-x^2)^{\frac{3}{2}}},$$

$$\text{or } (1-x^2)\frac{d^2y}{dx^2}=-m^2y+x\frac{dy}{dx}.$$

Then by Art. 80, differentiating n times with regard to x,

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}-2x\cdot n\frac{d^{n+1}y}{dx^{n+1}}-2\cdot\frac{n(n-1)}{2}\frac{d^ny}{dx^n}=-m^2\cdot\frac{d^ny}{dx^n}$$

$$+x\frac{d^{n+1}y}{dx^{n+1}}+n\cdot\frac{d^ny}{dx^n},\ \text{or }\ (1-x^2)\frac{d^{n+2}y}{dx^{n+2}}=(2n+1)x\frac{d^{n+1}y}{dx^{n+1}}+(n^2-m^2)\frac{d^ny}{dx^n}.$$

24. $$\frac{dy}{dx}=-\frac{a}{x}\sin(\log x)+\frac{b}{x}\cos(\log x),$$

$$\text{or } x\frac{dy}{dx}=-a\sin(\log x)+b\cos(\log x);$$

$$\therefore\ \frac{dy}{dx}+x\frac{d^2y}{dx^2}=-\frac{a}{x}\cos(\log x)-\frac{b}{x}\sin(\log x)=-\frac{y}{x},$$

$$\text{or } x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$

Then by Leibnitz's Theorem, differentiating n times,

$$x^2\cdot\frac{d^{n+2}y}{dx^{n+2}}+2x\cdot n\frac{d^{n+1}y}{dx^{n+1}}+2\cdot\frac{n(n-1)}{2}\cdot\frac{d^ny}{dx^n}$$

$$+x\cdot\frac{d^{n+1}y}{dx^{n+1}}+n\cdot\frac{d^ny}{dx^n}+\frac{d^ny}{dx^n}=0,$$

$$\text{or } x^2\frac{d^{n+2}y}{dx^{n+2}}+(2n+1)x\frac{d^{n+1}y}{dx^{n+1}}+(n^2+1)\frac{d^ny}{dx^n}=0.$$

CHAPTER VI.

1. If $$\frac{a+bx}{b-ax}=z,\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{1}{1+z^2}\cdot\frac{dz}{dx},$$

and $$\frac{dz}{dx}=\frac{b}{b-ax}-\frac{a+bx}{(b-ax)^2}(-a)=\frac{a^2+b^2}{(b-ax)^2},$$

and $$1+z^2=\frac{(a^2+b^2)(1+x^2)}{(b-ax)^2},\ \therefore\ \frac{dy}{dx}=\frac{1}{1+x^2}.$$

2. $$\frac{dy}{dx}=\tan^{-1}\left(\frac{1}{x}\right)+x\cdot\frac{1}{1+\frac{1}{x^2}}\cdot\left(\frac{-1}{x^2}\right)=\tan^{-1}\left(\frac{1}{x}\right)-\frac{x}{1+x^2}.$$

3. $$y=\log\left(\sqrt{x^2+a^2}+\sqrt{x^2+b^2}\right)^2-\log\left(a^2-b^2\right),$$

$$\therefore\ \frac{dy}{dx}=\frac{2}{\sqrt{x^2+a^2}+\sqrt{x^2+b^2}}\left\{\frac{x}{\sqrt{x^2+a^2}}+\frac{x}{\sqrt{x^2+b^2}}\right\}=\frac{2x}{\sqrt{x^2+a^2}\cdot\sqrt{x^2+b^2}}.$$

4. $$\frac{dy}{dx}=\left\{\frac{-x}{\sqrt{1-x^2}}\cdot e^{\sin^{-1}x}+\sqrt{1-x^2}\cdot e^{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}\right\}\div\left(\sqrt{1-x^2}+x\right)$$

$$-\frac{\sqrt{1-x^2}\cdot e^{\sin^{-1}x}}{\left(\sqrt{1-x^2}+x\right)^2}\cdot\left\{\frac{-x}{\sqrt{1-x^2}}+1\right\}$$

$$=e^{\sin^{-1}x}\cdot\left\{-\frac{x}{\sqrt{1-x^2}}+1\right\}\cdot\left\{\frac{1}{\sqrt{1-x^2}+x}\right\}^2\left\{\sqrt{1-x^2}+x-\sqrt{1-x^2}\right\}$$

$$=e^{\sin^{-1}x}\left(\sqrt{1-x^2}-x\right)x\div\sqrt{1-x^2}\left(\sqrt{1-x^2}+x\right)^2.$$

5. $$\operatorname{Log}(y)=\frac{x}{\sin x}\cdot\log\frac{\sin x}{x};$$

$$\therefore\ \frac{1}{y}\cdot\frac{dy}{dx}=\left(\frac{1}{\sin x}-\frac{x\cos x}{\sin^2 x}\right)\log\frac{\sin x}{x}+\frac{x^2}{\sin^2 x}\left(\frac{\cos x}{x}-\frac{\sin x}{x^2}\right)$$

$$=\frac{\sin x-x\cos x}{\sin^2 x}\left\{\log\left(\frac{\sin x}{x}\right)-1\right\},\ \text{and}\ \log(e)=1,$$

$$\therefore\ \frac{1}{y}\cdot\frac{dy}{dx}=\frac{x\cos x-\sin x}{\sin^2 x}\log\left(\frac{ex}{\sin x}\right).$$

6. $$\operatorname{Log}\{f(x)\}=(a+b+2x)\{\log(a+x)-\log(b+x)\};$$

$$\therefore\ \frac{1}{f(x)}\cdot f'(x)=2\log\left(\frac{a+x}{b+x}\right)+(a+b+2x)\left\{\frac{1}{a+x}-\frac{1}{b+x}\right\},$$

and putting $x=0$, $$f'(0)=f(0)\left\{2\log\frac{a}{b}+(a+b)\left(\frac{1}{a}-\frac{1}{b}\right)\right\},$$

$$\text{or } f'(0)=\left(\frac{a}{b}\right)^{a+b}\cdot\left\{2\log\left(\frac{a}{b}\right)+\frac{b^2-a^2}{ab}\right\}.$$

7. $$y=(x-a)^{\frac{2}{3}}\cdot(x-c)^{\frac{1}{3}},\ \therefore\ \frac{dy}{dx}=\frac{2}{3}(x-a)^{-\frac{1}{3}}\cdot(x-c)^{\frac{1}{3}}+\frac{1}{3}(x-a)^{\frac{2}{3}}\cdot(x-c)^{-\frac{2}{3}};$$

$$\therefore\ \frac{d^2y}{dx^2}=-\frac{2}{9}(x-a)^{-\frac{4}{3}}\cdot(x-c)^{\frac{1}{3}}+\frac{4}{9}(x-a)^{-\frac{1}{3}}\cdot(x-c)^{-\frac{2}{3}}-\frac{2}{9}(x-a)^{\frac{2}{3}}\cdot(x-c)^{-\frac{5}{3}}$$

$$=-\frac{2y}{9}\left\{\frac{1}{(x-a)^2}-\frac{2}{(x-a)(x-c)}+\frac{1}{(x-c)^2}\right\}=\frac{-2y(a-c)^2}{9(x-a)^2(x-c)^2}.$$

8. If $a=\kappa\cos\alpha$ and $b=\kappa\sin\alpha$, then $\kappa^2=a^2+b^2$,

$$\tan\alpha=\frac{b}{a},\quad x=\kappa\,.\cos(\theta-\alpha),\quad y=\kappa\,.\sin(\theta-\alpha),$$

$$\therefore \text{ (Art. 78) } \frac{d^mx}{d\theta^m}=\kappa\cos\left(\theta-\alpha+\frac{m\pi}{2}\right),\quad \frac{d^ny}{d\theta^n}=\kappa\sin\left(\theta-\alpha+\frac{n\pi}{2}\right);$$

thus $\dfrac{d^mx}{d\theta^m}\cdot\dfrac{d^ny}{d\theta^n}-\dfrac{d^nx}{d\theta^n}\cdot\dfrac{d^my}{d\theta^m}$

$$=\kappa^2\,.\sin\left\{\left(\theta-\alpha+\frac{n\pi}{2}\right)-\left(\theta-\alpha+\frac{m\pi}{2}\right)\right\}=\kappa^2\,.\sin(n-m)\frac{\pi}{2}=\pm\kappa^2 \text{ or } 0,$$

according to the value of $n-m$, and κ is independent of θ.

9. $\cos^{-1}\dfrac{y}{a}=n\log x-n\log b$, $\therefore\ -\dfrac{1}{\sqrt{a^2-y^2}}\cdot\dfrac{dy}{dx}=\dfrac{n}{x}$,

$$\text{or } x\frac{dy}{dx}=-n\sqrt{a^2-y^2},\quad \therefore\ \frac{dy}{dx}+x\frac{d^2y}{dx^2}=\frac{ny}{\sqrt{a^2-y^2}}\cdot\frac{dy}{dx}=-\frac{n^2y}{x},$$

$$\text{or } x^2\,.\frac{d^2y}{dx^2}+x\frac{dy}{dx}+n^2y=0;$$

hence (Art. 80), differentiating n times,

$$x^2\frac{d^{n+2}y}{dx^{n+2}}+n\,.\,2x\frac{d^{n+1}y}{dx^{n+1}}+\frac{n\,\overline{n-1}}{2}\,.\,2\frac{d^ny}{dx^n}+x\frac{d^{n+1}y}{dx^n}+n\,.\frac{d^ny}{dx^n}$$

$$+n^2\frac{d^ny}{dx^n}=0=x^2\frac{d^{n+2}y}{dx^{n+2}}+x\frac{d^{n+1}y}{dx^{n+1}}(2n+1)+\frac{d^ny}{dx^n}\,.\,2n^2.$$

10. If $u=(x-2)e^x+x+2$, then $u=0$ when $x=0$, and

$$\frac{du}{dx}=e^x+(x-2)e^x+1=(x-1)e^x+1,$$

$\therefore\ \dfrac{du}{dx}=0$ when $x=0$, and $\dfrac{d^2u}{dx^2}=e^x+(x-1)e^x=xe^x$, $\therefore\ \dfrac{d^2u}{dx^2}$ is always positive when x is so, $\therefore\ \dfrac{du}{dx}$ must increase with x (Art. 89) when x is positive, and $\dfrac{du}{dx}=0$ when $x=0$, $\therefore\ \dfrac{du}{dx}$ is always positive when x is so, $\therefore\ u$ increases with x; but $u=0$ when $x=0$, $\therefore\ u$ must be positive for all positive values of x.

CHAPTER VII.

1. If $f(x)=e^{2x}.(3-x)-4x.e^{x}-x-3,\ f(0)=0;$

$$f'(x)=e^{2x}(6-2x-1)-4e^{x}(1+x)-1,\ \therefore f'(0)=0;$$
$$f''(x)=e^{2x}(10-4x-2)-4e^{x}(2+x),\ \therefore f''(0)=0;$$
$$f'''(x)=e^{2x}(16-8x-4)-4e^{x}(3+x),\ \therefore f'''(0)=0;$$
$$f^{\text{iv}}(x)=e^{2x}(24-16x-8)-4e^{x}(4+x),\ \therefore f^{\text{iv}}(0)=0;$$
$$f^{\text{v}}(x)=e^{2x}(32-32x-16)-4e^{x}(5+x),\ \therefore f^{\text{v}}(0)=-4;$$
$$\therefore \text{ first term is } \frac{-4x^5}{\lfloor 5}.$$

2. If $f(x)=\log(1+e^{x}),\ f(0)=\log 2,$

$$f'(x)=\frac{e^x}{1+e^x}=1-\frac{1}{1+e^x},\ f'(0)=\frac{1}{2},\ f''(x)=\frac{e^x}{(1+e^x)^2},$$
$$f''(0)=\frac{1}{4};\text{ and } f''(x)=\frac{1}{1+e^x}-\frac{1}{(1+e^x)^2},$$
$$\therefore f'''(x)=\frac{-e^x}{(1+e^x)^2}+\frac{2e^x}{(1+e^x)^3}=\frac{-2}{(1+e^x)^3}+\frac{3}{(1+e^x)^2}-\frac{1}{1+e^x};$$
$$\therefore f'''(0)=0;\text{ and } f^{\text{iv}}(x)=\frac{6e^x}{(1+e^x)^4}-\frac{6e^x}{(1+e^x)^3}+\frac{e^x}{(1+e^x)^2};$$
$$\therefore f^{\text{iv}}(0)=-\frac{1}{2^3},\ \therefore \text{ \&c. by Maclaurin's Theorem.}$$

3. If $f(x)=e^{x\sin x},\ f(0)=1,$ and $f(x)=f(-x),\ \therefore$ there are no odd powers of x, $\therefore f'(0)=0=f'''(0)=$ &c., and $f'(x)=f(x)(\sin x+x\cos x)$;

$$\therefore f''(x)=f'(x)(\sin x+x\cos x)+f(x)(2\cos x-x\sin x) \ \ldots\ldots (1);$$
$$\therefore f''(0)=2,\text{ and differentiating (1) twice by Art. 80,}$$
$$f^{\text{iv}}(x)=3f''(x)(2\cos x-x\sin x)-f(x)\frac{d}{dx}(3\sin x+x\cos x)$$
$$+\text{terms in } f'(x) \text{ and } f'''(x),$$
$$\therefore f^{\text{iv}}(0)=12-(4\cos x-x\sin x)\text{ when } x=0,\ \text{ or } f^{\text{iv}}(0)=8,$$
$$\therefore e^{x\sin x}=1+\frac{2x^2}{\lfloor 2}+\frac{8x^4}{\lfloor 4}+\ldots$$

4. If $f(x)=e^{x}.\sec x,\ f(0)=1,$

$$f'(x)=e^{x}(\sec x+\sec x\tan x)=f(x)(1+\tan x),$$
$$\therefore f'(0)=1,\ f''(x)=f'(x)(1+\tan x)+f(x).\sec^2 x;$$
$$\therefore f''(0)=2;\ f'''(x)=f''(x)(1+\tan x)+f'(x).2\sec^2 x+f(x).2\sec^2 x\tan x,$$
$$\therefore f'''(0)=4,\ \therefore \text{ \&c.}$$

5. If $f(x)=\left(\frac{1+e^x}{2}\right)^n$, $f(0)=1$; $f'(x)=\frac{n}{2}e^x\cdot\left(\frac{1+e^x}{2}\right)^{n-1}$;

$\therefore f'(0)=\frac{n}{2}$, and $f''(x)=\frac{n}{2}\cdot e^x\left(\frac{1+e^x}{2}\right)^{n-1}+\frac{n(n-1)e^{2x}}{4}\left(\frac{1+e^x}{2}\right)^{n-2}$;

$$\therefore f''(0)=\frac{n}{2}+\frac{n(n-1)}{4}=\frac{n(n+1)}{2.2}, \text{ \&c.}$$

6. If
$$f(x)=+\sqrt{(1+4x+12x^2)},\ f(0)=1,$$
$$f'(x)=\frac{2+12x}{\sqrt{(1+4x+12x^2)}},\ \therefore f'(0)=2,$$
and
$$f'(x)(1+4x+12x^2)=f(x)(2+12x);$$
$$\therefore f''(x)(1+4x+12x^2)+f'(x)(4+24x)=f'(x)(2+12x)+f(x).12;$$
$$\therefore f''(0)=-8+4+12=8, \text{ \&c.}$$

7. Let $f(x, n)=(e^x+e^{-x})^n$, then $f(x, n)=f(-x, n)$, $\therefore$ there are no odd powers of x, and $f(0, n)=2^n$,
$$f'(x, n)=n(e^x-e^{-x})(e^x+e^{-x})^{n-1},$$
$$f''(x, n)=n(e^x+e^{-x})^n+n(n-1)(e^x-e^{-x})^2(e^x+e^{-x})^{n-2}$$
$$=n^2.f(x, n)-4n(n-1)f(x, n-2);$$
$$\therefore f''(0, n)=n^2.2^n-4n(n-1).\frac{2^n}{4}=n.2^n,$$
and
$$f^{iv}(x, n)=n^2f''(x, n)-4n(n-1)f''(x, n-2)$$
$$\therefore f^{iv}(0, n)=n^3.2^n-4n(n-1)(n-2)\frac{2^n}{4}=2^n.n(3n-2), \text{ \&c.}$$

8. If $f(x, n)=\cos^n x$, $f(x, n)=f(-x, n)$, $\therefore$ there are no odd powers of x;
$$f(0, n)=1,\ f'(x, n)=-n\cos^{n-1}x.\sin x,$$
$$f''(x, n)=n(n-1)\cos^{n-2}x.\sin^2 x-n\cos^n x$$
$$=n(n-1)f(x, n-2)-n^2.f(x, n),\ \therefore f''(0, n)=-n,$$
and
$$f^{iv}(x, n)=n(n-1)f''(x, n-2)-n^2f''(x, n);$$
$$\therefore f^{iv}(0, n)=-n(n-1)(n-2)+n^3=n(3n-2),$$
and $f^{vi}(x, n)=n(n-1)f^{iv}(x, n-2)-n^2f^{iv}(x, n)$, which gives
$$f^{vi}(0, n)=-n\{15(n-1)^2+1\},\ \therefore \text{ \&c.}$$

9. If $f(x)=-\log(\cos x)$, $f(x)=f(-x)$, $\therefore$ there are no odd powers of x, and $f(0)=0$, $f'(x)=\tan x$, $f''(x)=\sec^2 x$,
$$f''(0)=1,\ f'''(x)=2\sec^2 x\tan x,$$

or $$\tfrac{1}{2}f'''(x)=f''(x)\,.\,f'(x) \text{ .... } (1);$$

$\therefore\ \tfrac{1}{2}f^{\text{iv}}(x)=f'''(x)\,.\,f'(x)+\{f''(x)\}^2$, $\therefore\ f^{\text{iv}}(0)=2$, and by Art. 80 from (1),

$$\tfrac{1}{2}f^{\text{vi}}(x)=f^{\text{v}}(x)\,.\,f'(x)+3f^{\text{iv}}(x)\,.\,f''(x)+3\{f'''(x)\}^2+f''(x)\,.\,f^{\text{iv}}(x);$$

$$\therefore\ f^{\text{vi}}(0)=16,$$

and $\tfrac{1}{2}f^{\text{viii}}(x)=f^{\text{vii}}(x)\,.\,f'(x)+5f^{\text{vi}}(x)f''(x)+10f^{\text{v}}(x)f'''(x)$
$$+10\{f^{\text{iv}}(x)\}^2+5f'''(x)f^{\text{v}}(x)+f''(x)\,.\,f^{\text{vi}}(x);$$

$$\therefore\ f^{\text{viii}}(0)=2(5\,.\,16+10\,.\,4+16)=16\,.\,17, \text{ \&c.}$$

10. If $$f(x)=e^{\cos x},\ f(0)=e,\ f(x)=f(-x),$$
$$-f'(x)=+f(x)\,.\,\sin x \text{ .... } (1),$$

$-f''(x)=+f'(x)\sin x+f(x)\cos x$, $\therefore\ f''(0)=-e$, and from (1) by Art. 80

$$-f^{\text{iv}}(x)=f'''(x)\sin x+3f''(x)\,.\,\cos x-3f'(x)\sin x-f(x)\cos x;$$

$$\therefore\ f^{\text{iv}}(0)=4e,$$

and $-f^{\text{vi}}(x)=f^{\text{v}}(x)\,.\,\sin x+5f^{\text{iv}}(x)\cos x-10f'''(x)\sin x-10f''(x)\cos x$
$$+5f'(x)\sin x+f(x)\cos x;$$

$$\therefore\ f^{\text{vi}}(0)=-31e,\ \therefore\ \text{\&c.}$$

11. By Taylor's Theorem if $f(x)=\sin^{-1}x$,

$$\sin^{-1}(x+h)=\sin^{-1}x+h\,.\,f'(x)+\frac{h^2}{\underline{|2}}f''(x)+\dots$$

and here $$f'(x)=\frac{1}{\sqrt{1-x^2}},\ f''(x)=\frac{x}{(1-x^2)^{\frac{3}{2}}},$$

$$f'''(x)=\frac{1}{(1-x^2)^{\frac{3}{2}}}+\frac{3x^2}{(1-x^2)^{\frac{5}{2}}}=\frac{1+2x^2}{(1-x^2)^{\frac{5}{2}}},$$

and $$f^{\text{iv}}(x)=\frac{4x}{(1-x^2)^{\frac{5}{2}}}+\frac{5x(1+2x^2)}{(1-x^2)^{\frac{7}{2}}}=\frac{3x(3+2x^2)}{(1-x^2)^{\frac{7}{2}}},\ \therefore\ \text{\&c.}$$

12. Let $$f(x)=\log(1-x+x^2), \text{ then } f(0)=0,$$

and $$f'(x)(x^2-x+1)=2x-1,\ \therefore\ f'(0)=-1,$$

and $$f''(x)(x^2-x+1)+f'(x)(2x-1)=2,\ \therefore\ f''(0)=1,$$

$$f'''(x)(x^2-x+1)+2f''(x)(2x-1)+2f'(x)=0,\ \therefore\ f'''(0)=4,$$

$$f^{\text{iv}}(x)(x^2-x+1)+3f'''(x)(2x-1)+6f''(x)=0,\ \therefore\ f^{\text{iv}}(0)=6,$$

$$f^{\text{v}}(x)(x^2-x+1)+4f^{\text{iv}}(x)(2x-1)+12f'''(x)=0,\ \therefore\ f^{\text{v}}(0)=-24.$$

13. Let $f(x)=\log\{\sqrt{a^2+x^2}+x\}$, then $f(x)+f(-x)=2\log a$,
$\therefore$ there are no even powers of x in the expansion,

and $f(0)=\log a$, $f'(x)=\left(1+\dfrac{x}{\sqrt{a^2+x^2}}\right)\div(\sqrt{a^2+x^2}+x)=\dfrac{1}{\sqrt{a^2+x^2}}$,

$$\therefore f'(0)=\frac{1}{a},$$

and $$f''(x)=\frac{-x}{(a^2+x^2)^{\frac{3}{2}}}, \quad \therefore (a^2+x^2)\,.\,f''(x)+x\,.\,f'(x)=0,$$

$$\therefore (a^2+x^2)f'''(x)+f''(x)\,.\,3x+f'(x)=0, \quad \therefore f'''(0)=-\frac{1}{a^3};$$

and $$(a^2+x^2)f^{\text{iv}}(x)+5x\,.\,f'''(x)+4f''(x)=0,$$

$$\therefore (a^2+x^2)f^{\text{v}}(x)+7x\,.\,f^{\text{iv}}(x)+9f'''(x)=0;$$

$$\therefore f^{\text{v}}(0)=\frac{9}{a^5}, \quad \therefore \text{ \&c.}$$

14. If $f(x)=\log(1+\sin x)$, $f(0)=0$, $f'(x)=\dfrac{\cos x}{1+\sin x}$, $f'(0)=1$,

$$f''(x)=\frac{-\sin x}{1+\sin x}-\frac{\cos^2 x}{(1+\sin x)^2}=\frac{-1}{1+\sin x}, \quad \therefore f''(0)=-1,$$

and $$f'''(x)=\frac{\cos x}{(1+\sin x)^2}, \quad \therefore f'''(0)=1, \quad \therefore \text{ \&c.}$$

15. If $f(x)=e^{\tan^{-1}x}$, $f(0)=1$, $f'(x)(1+x^2)=f(x)$, $\therefore f'(0)=1$,

and $$f''(x)(1+x^2)+2x\,.\,f'(x)=f'(x), \quad \therefore f''(0)=1,$$

and $$f'''(x)(1+x^2)+4xf''(x)+2f'(x)=f''(x), \quad \therefore f'''(0)=-1,$$

and $$f^{\text{iv}}(x)(1+x^2)+6xf'''(x)+6f''(x)=f'''(x), \quad \therefore f^{\text{iv}}(0)=-7, \quad \therefore \text{ \&c.}$$

16. $y=(x-a)^{\frac{7}{5}}\,.\,(x-b)^2(x-c)^{-\frac{2}{5}}$; $\therefore$ if $x=c$, y and all its differential coefficients are infinite; if $x=a$, all the differential coefficients of y beginning with the 2nd are infinite.

17. Here if $y=\tan^{-1}x=\dfrac{\pi}{2}-\theta$, $\dfrac{d^ny}{dx^n}=\underline{|n-1}\cos\left\{ny+(n-1)\dfrac{\pi}{2}\right\}\cos^n y$

$$=\underline{|n-1}\cos\left(n\pi-\frac{\pi}{2}-n\theta\right)\cos^n\left(\frac{\pi}{2}-\theta\right) \text{ (Ex. 18, Ch. V.)},$$

$$=\underline{|n-1}\sin n(\pi-\theta)\sin^n\theta, \ \therefore, \text{ by Taylor's Theorem,}$$

$$\tan^{-1}(x+h)=\tan^{-1}x+h\sin\theta\,.\,\sin\theta-\frac{h^2}{\underline{|2}}\sin 2\theta\,.\,\sin^2\theta$$
$$+\frac{h^3}{3}\sin 3\theta\sin^3\theta+\ldots\ldots+\text{\&c.}$$

18. $\tan^{-1} x = \frac{\pi}{2} - \theta$, $\because x = \cot\theta$, $\therefore$ by Ex. 17 if $h = -x$,

$$0 = \frac{\pi}{2} - \theta - \cot\theta \,.\, \sin^2\theta - \frac{\cot^2\theta}{2}\sin^2\theta\sin 2\theta - \frac{\cot^3\theta}{3}\sin 3\theta\sin^3\theta$$
$$- \frac{\cot^4\theta}{4}\sin^4\theta\sin 4\theta - \ldots\ldots$$

or $$\frac{\pi}{2} - \theta = \sin\theta\cos\theta + \frac{\cos^2\theta\sin 2\theta}{2} + \frac{\cos^3\theta\sin 3\theta}{3} + \ldots\ldots$$

19. $$x = \cot\theta, \; \therefore \; h = -x - \frac{1}{x} = -\cot\theta - \tan\theta = \frac{-1}{\sin\theta\cos\theta},$$

and $\therefore$, by Ex. 17, $\tan^{-1}(x+h) - \tan^{-1} x = -\theta - \left(\frac{\pi}{2} - \theta\right) = -\frac{\pi}{2}$

$$= -\frac{\sin^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta\sin 2\theta}{2\sin^2\theta\cos^2\theta} - \ldots\ldots$$

or $$\frac{\pi}{2} = \frac{\sin\theta}{\cos\theta} + \frac{\sin 2\theta}{2\cos^2\theta} + \frac{\sin 3\theta}{3\cos^3\theta} + \ldots\ldots$$

20. $$h = -\sqrt{1+x^2} = -\operatorname{cosec}\theta,$$

$$\therefore \tan^{-1} x - \tan^{-1}(x+h) = \frac{\pi}{2} - \theta - \tan^{-1}\left(\frac{\cos\theta - 1}{\sin\theta}\right)$$

$$= \frac{\pi}{2} - \theta + \frac{\theta}{2} = \frac{\pi - \theta}{2} = \operatorname{cosec}\theta\sin^2\theta + \frac{1}{2}\operatorname{cosec}^2\theta\sin^2\theta\sin 2\theta + \ldots\ldots$$

$$= \sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \ldots\ldots$$

CHAPTER VIII.

1. $$\frac{du}{dy} = \frac{x^2}{a^2 - z^2}, \; \therefore \; \frac{d^2u}{dxdy} = \frac{2x}{a^2 - z^2};$$

$$\frac{du}{dz} = \frac{2x^2yz}{(a^2 - z^2)^2}, \; \therefore \; \frac{d^2u}{dydz} = \frac{2x^2z}{(a^2 - z^2)^2}.$$

2. (1) $$\frac{du}{dy} = x\cos y + \sin x, \; \therefore \; \frac{d^2u}{dxdy} = \cos y + \cos x;$$

$$\frac{du}{dx} = \sin y + y\cos x, \; \therefore \; \frac{d^2u}{dydx} = \cos y + \cos x.$$

$$(2)\quad \frac{du}{dy}=\frac{x}{y},\ \therefore\ \frac{d^2u}{dxdy}=\frac{1}{y};$$

$$\frac{du}{dx}=\log y,\ \therefore\ \frac{d^2u}{dydx}=\frac{1}{y}.$$

$$(3)\quad \frac{du}{dy}=x^y.\log x,\ \therefore\ \frac{d^2u}{dxdy}=yx^{y-1}.\log x+x^{y-1};$$

$$\frac{du}{dx}=y\,.\,x^{y-1},\ \therefore\ \frac{d^2u}{dydx}=x^{y-1}+y\,.\,x^{y-1}\log x.$$

$$(4)\quad \frac{du}{dy}=\cot\frac{y}{x}\,.\,\frac{1}{x},\ \therefore\ \frac{d^2u}{dxdy}=-\operatorname{cosec}^2.\left(\frac{y}{x}\right).\left(-\frac{y}{x^3}\right)-\frac{1}{x^2}.\cot\frac{y}{x}$$

$$=\frac{y}{x^3}.\operatorname{cosec}^2\left(\frac{y}{x}\right)-\frac{1}{x^2}\cot\left(\frac{y}{x}\right);$$

and $$\frac{du}{dx}=-\cot\left(\frac{y}{x}\right).\frac{y}{x^2},\ \therefore\ \frac{d^2u}{dydx}=\operatorname{cosec}^2\left(\frac{y}{x}\right).\frac{y}{x^3}-\frac{1}{x^2}\cot\left(\frac{y}{x}\right).$$

$$(5)\quad u\,(by-ax)=ay-bx,$$

$$\therefore\ \frac{du}{dy}(by-ax)+ub=a,\ \therefore\ \frac{d^2u}{dxdy}(by-ax)=a\frac{du}{dy}-b\frac{du}{dx};$$

and $$\frac{du}{dx}(by-ax)=au-b,\ \therefore\ \frac{d^2u}{dydx}(by-ax)=-b\frac{du}{dx}+a\frac{du}{dy}.$$

$$(6)\quad \frac{du}{dy}=\log(1+xy)+\frac{yx}{1+yx}=\log(1+xy)+1-\frac{1}{1+xy};$$

$$\therefore\ \frac{d^2u}{dxdy}=\frac{y}{1+xy}+\frac{y}{(1+xy)^2};\ \text{and}\ \frac{du}{dx}=\frac{y^2}{1+xy},$$

$$\therefore\ \frac{d^2u}{dydx}=\frac{2y}{1+xy}-\frac{y^2x}{(1+xy)^2}=\frac{y}{1+xy}+\frac{y}{(1+xy)^2}=\frac{d^2u}{dxdy}.$$

3. $$x\frac{du}{dx}=\alpha Ax^{\alpha}y^{\alpha'}+\ldots+,\ y\frac{du}{dy}=\alpha'.Ax^{\alpha}y^{\alpha'}+\ldots+$$

$$\therefore\ x\frac{du}{dx}+y\frac{du}{dy}=n.Ax^{\alpha}y^{\alpha'}+\ldots+=nu.$$

4. $\dfrac{du}{dx}$ is a homogeneous function of $(n-1)$ dimensions,

$\therefore$ by Ex. 3, $$x\frac{d}{dx}\left(\frac{du}{dx}\right)+y\frac{d}{dy}\left(\frac{du}{dx}\right)=(n-1)\frac{du}{dx},$$

or $$x\frac{d^2u}{dx^2}+y\frac{d^2u}{dxdy}=(n-1)\frac{du}{dx}.$$ So for $\dfrac{du}{dy}$.

5. Multiplying the results in Ex. 4 by x and y and adding,

$$x^2\frac{d^2u}{dx^2}+2xy\frac{d^2u}{dxdy}+y^2\frac{d^2u}{dy^2}=(n-1)\left(x\frac{du}{dx}+y\frac{du}{dy}\right)=n(n-1)u.$$

6. (1) $\frac{du}{dx}=2(x+y)=\frac{du}{dy}$, $\therefore \left(x\frac{du}{dx}+y\frac{du}{dy}\right)=2(x+y)^2=2u.$

Also
$$\frac{d^2u}{dx^2}=2=\frac{d^2u}{dxdy}=\frac{d^2u}{dy^2};$$

$$\therefore x\frac{d^2u}{dx^2}+y\frac{d^2u}{dxdy}=2(x+y)=\frac{du}{dx}, \text{ \&c.}$$

(2) $\frac{du}{dx}=\frac{y}{x+y}-\frac{xy}{(x+y)^2}$, $\frac{du}{dy}=\frac{x}{x+y}-\frac{xy}{(x+y)^2}$;

$$\therefore x\frac{du}{dx}+y\frac{du}{dy}=\frac{2xy}{x+y}-\frac{xy}{x+y}=u, \text{ and } n=1.$$

Also
$$\frac{du}{dx}=\frac{y^2}{(x+y)^2}, \quad \therefore \frac{d^2u}{dx^2}=-\frac{2y^2}{(x+y)^3},$$

and
$$\frac{d^2u}{dxdy}=\frac{2y}{(x+y)^2}-\frac{2y^2}{(x+y)^3}=\frac{2xy}{(x+y)^3};$$

$$\therefore x\frac{d^2u}{dx^2}+y\frac{d^2u}{dxdy}=0, \text{ and } n \text{ here}=1; \text{ \&c.}$$

(3) $\frac{du}{dx}=\frac{2}{3}\cdot\frac{x}{(x^2+y^2)^{\frac{2}{3}}}$, $\frac{du}{dy}=\frac{2}{3}\cdot\frac{y}{(x^2+y^2)^{\frac{2}{3}}}$, and $n=\frac{2}{3}$,

thus
$$x\frac{du}{dx}+y\frac{du}{dy}=\frac{2}{3}(x^2+y^2)^{\frac{1}{3}}=nu.$$

Also
$$\frac{d^2u}{dx^2}=\frac{2}{3}\cdot\frac{1}{(x^2+y^2)^{\frac{2}{3}}}-\frac{8}{9}\cdot\frac{x^2}{(x^2+y^2)^{\frac{5}{3}}},$$

$$\frac{d^2u}{dxdy}=-\frac{8}{9}\cdot\frac{xy}{(x^2+y^2)^{\frac{5}{3}}};$$

$$\therefore x\frac{d^2u}{dx^2}+y\cdot\frac{d^2u}{dxdy}=\frac{2}{3}\cdot\frac{x}{(x^2+y^2)^{\frac{2}{3}}}-\frac{8}{9}\frac{x}{(x^2+y^2)^{\frac{2}{3}}}=-\frac{2x}{9(x^2+y^2)^{\frac{2}{3}}}$$

$$=-\frac{1}{3}\cdot\frac{du}{dx}=(n-1)\frac{du}{dx}, \text{ \&c.}$$

7. $\frac{du}{dy}=2e^x.yz^3+2x^2yz^2$; $\therefore \frac{d^2u}{dydz}=6e^xyz^2+4x^2yz$;

$$\therefore \frac{d^4u}{dx^2dydz}=6e^xyz^2+8yz.$$

8. $$\frac{du}{dz}=xye^{xyz},\ \therefore\ \frac{d^2u}{dydz}=xe^{xyz}+x^2.yze^{xyz},$$

and $$\frac{d^3u}{dxdydz}=e^{xyz}(1+xyz+2xyz+x^2y^2z^2),\text{ or \&c.}$$

9. $$\frac{du}{dx}=\frac{-xy}{\sqrt{a^2-x^2}}+\sqrt{a^2-y^2},\text{ so }\frac{du}{dy}=-\frac{xy}{\sqrt{a^2-y^2}}+\sqrt{a^2-x^2},$$

and $$\frac{d^2u}{dxdy}=\frac{-x}{\sqrt{a^2-x^2}}-\frac{y}{\sqrt{a^2-y^2}},\ \therefore\ \frac{du}{dx}\cdot\frac{du}{dy}+\sqrt{a^2-x^2}.\sqrt{a^2-y^2}.\left(\frac{d^2u}{dxdy}\right)^2$$

$$=\frac{\{\sqrt{(a^2-x^2)(a^2-y^2)}-xy\}^2}{\sqrt{(a^2-x^2)(a^2-y^2)}}+\frac{(x\sqrt{a^2-y^2}+y\sqrt{a^2-x^2})^2}{\sqrt{(a^2-x^2)(a^2-y^2)}}$$

$$=\frac{a^4}{\sqrt{(a^2-x^2)(a^2-y^2)}}.$$

10. $$\frac{du}{dx}=\frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2}\left\{\frac{y}{\sqrt{1+x^2+y^2}}-\frac{x^2.y}{(1+x^2+y^2)^{\frac{3}{2}}}\right\}$$

$$=\frac{y}{1+x^2}\cdot\frac{1}{(1+x^2+y^2)^{\frac{1}{2}}};$$

$$\therefore\ \frac{d^2u}{dxdy}=\frac{1}{1+x^2}\left\{\frac{1}{(1+x^2+y^2)^{\frac{1}{2}}}-\frac{y^2}{(1+x^2+y^2)^{\frac{3}{2}}}\right\}$$

$$=\frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}.$$

Also $$\frac{d^3u}{dxdy^2}=\frac{-3y}{(1+x^2+y^2)^{\frac{5}{2}}},\ \therefore\ \frac{d^4u}{dx^2dy^2}=\frac{15xy}{(1+x^2+y^2)^{\frac{7}{2}}}.$$

11. Here $$\sqrt{a^2-x^2}.\frac{du}{dx}=\sqrt{a^2-x^2}\sqrt{a^2-y^2}\sqrt{a^2-z^2}-xy\sqrt{a^2-z^2}$$

$$-zx\sqrt{a^2-y^2}-yz\sqrt{a^2-x^2}$$

$$=\sqrt{a^2-y^2}.\frac{du}{dy}=\sqrt{a^2-z^2}.\frac{du}{dz}\text{ by symmetry.}$$

Also, $\therefore$ $$\sqrt{a^2-y^2}.\sqrt{a^2-z^2}.\frac{d^2u}{dydz}=-y\sqrt{a^2-x^2}.\sqrt{a^2-z^2}$$

$$-x\sqrt{a^2-y^2}.\sqrt{a^2-z^2}+xyz-z\sqrt{a^2-x^2}.\sqrt{a^2-y^2},$$

and $\therefore$ $$\sqrt{a^2-x^2}\sqrt{a^2-y^2}\sqrt{a^2-z^2}\frac{d^3u}{dxdydz}$$

$$=xy\sqrt{a^2-z^2}-\sqrt{a^2-x^2}.\sqrt{a^2-y^2}\sqrt{a^2-z^2}+yz\sqrt{a^2-x^2}$$

$$+zx\sqrt{a^2-y^2}=-\sqrt{a^2-x^2}.\frac{du}{dx},\ \therefore\ \text{\&c.}$$

12. (1) $e^u = x^3 + y^3 + z^3 - 3xyz$, and $\frac{du}{dx} = 3(x^2 - yz)e^{-u}$,

$$\therefore \frac{du}{dx} \cdot \frac{du}{dy} \cdot \frac{du}{dz} = 27(x^2 - yz)(y^2 - zx)(z^2 - xy)e^{-3u},$$

and $\frac{d^2u}{dydx} = -3z \cdot e^{-u} - 9(x^2 - yz)(y^2 - zx)e^{-2u}$;

$$\therefore \frac{d^3u}{dzdydx} = -3e^{-u} + 9z(z^2 - xy)e^{-2u} + 9(y^3 - 2xyz + x^3)e^{-2u}$$
$$+ 54(x^2 - yz)(y^2 - zx)(z^2 - xy)e^{-3u}$$
$$= 2\frac{du}{dx} \cdot \frac{du}{dy} \cdot \frac{du}{dz} + 6e^{-u}, \ \ldots \ \&c.$$

(2) $\frac{du}{dx} + \frac{du}{dy} + \frac{du}{dz} = 3e^{-u}(x^2 - yz + y^2 - zx + z^2 - xy) = \frac{3}{x + y + z}$.

(3) Differentiating the result of (2) with respect to x, y, and z and adding,

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2u}{dydz} + 2\frac{d^2u}{dzdx} + 2\frac{d^2u}{dxdy} = -\frac{9}{(x + y + z)^2}.$$

(4) From (2) $\frac{d^6u}{dx^2dy^2dz^2} + \frac{d^6u}{dx^3dy^2dz} + \frac{d^6u}{dx^2dy^3dz}$

$$= \frac{d^5}{dx^2dy^2dz}\left(\frac{du}{dz} + \frac{du}{dx} + \frac{du}{dy}\right) = \frac{d^5}{dx^2dy^2dz}\left(\frac{3}{x + y + z}\right)$$
$$= \frac{-3\,\underline{|5}}{(x + y + z)^6} = -\frac{360}{(x + y + z)^6}.$$

(5) $\frac{du}{dx} = 3(x^2 - yz)e^{-u}$;

$$\therefore \frac{d^2u}{dx^2} = 6xe^{-u} - 9(x^2 - yz)^2 e^{-2u};$$

$$\therefore -\frac{e^{2u}}{3}\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) = 3\{(x^2 - yz)^2 + (y^2 - zx)^2 + (z^2 - xy)^2\}$$
$$- 2(x + y + z)(x^3 + y^3 + z^3 - 3xyz)$$
$$= x^4 + y^4 + z^4 + 3(y^2z^2 + z^2x^2 + x^2y^2) - 2x(y^3 + z^3) - 2y(z^3 + x^3) - 2z(x^3 + y^3)$$
$$= (x^2 + y^2 + z^2)^2 + (yz + zx + xy)^2 - 2x^2(yz + zx + xy)$$
$$- 2y^2(yz + zx + xy) - 2z^2(yz + zx + xy)$$
$$= (x^2 + y^2 + z^2 - yz - zx - xy)^2;$$
$$\therefore \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = -\frac{3}{(x + y + z)^2}.$$

(6) $\frac{d^5u}{dx^3dydz} + \ldots = \frac{d^3u}{dxdydz}\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right)$;

$$\therefore = \frac{3 \cdot 2 \cdot 3 \cdot 4}{(x + y + z)^5} = \frac{72}{(x + y + z)^5}.$$

CHAPTER IX.

6. $y=a+x\log y$, $\therefore$ in Lagrange's Theorem $z=a$, and

$$\phi(y)=\log y,\ f(y)=\sin(y);$$

$$\therefore \sin y=\sin a+x\,.\cos a\,.\log a+\frac{x^2}{\lfloor 2}\frac{d}{da}\{\cos a\,(\log a)^2\}+$$

$$+\frac{x^n}{\lfloor n}\frac{d^{n-1}}{da^{n-1}}\{\cos a\,(\log a)^n\}+.$$

7. $$\phi(y)=y^p\,.\,e^{qy},\ f(y)=y^m\,.\,e^{ny};$$

$$\therefore y^m\,.\,e^{ny}=z^m\,.\,e^{nz}+x\,.\,z^p\,.\,e^{qz}\,.\frac{d}{dz}(z^m\,.\,e^{nz})+\ldots\ldots$$

$$+\frac{x^r}{\lfloor r}\,.\frac{d^{r-1}}{dz^{r-1}}\left\{z^{rp}\,.\,e^{qrz}\,.\,z^m\,.\,e^{nz}\left(\frac{m}{z}+n\right)\right\}+\ldots\ \&c.$$

8. (1) $f(y)=\sin y=\phi(y)$, $\therefore \sin y=\sin z+x\,.\sin z\,.\cos z$

$$+\frac{x^2}{\lfloor 2}\frac{d}{dz}(\sin^2 z\,.\cos z)++\frac{x^n}{\lfloor n}\,.\frac{d^{n-1}}{dz^{n-1}}(\sin^n z\cos z)+.$$

(2) $f(y)=\sin 2y,\ \phi(y)=\sin y,$

$$\therefore \sin 2y=\sin 2z+x\,.\sin z\,.\,2\cos 2z++\frac{x^n}{\lfloor n}\frac{d^{n-1}}{dz^{n-1}}(\sin^n z\,.\,2\cos 2z)+.$$

9. Here in Laplace's Theorem $f(y)=e^y$, $\phi(y)=\cos y$, and $F(z)=\log z$,

$$\therefore f\{F(z)\}=z,$$

and $$\therefore e^y=z+x\cos(\log z)++\frac{x^n}{\lfloor n}\frac{d^{n-1}}{dz^{n-1}}\{\cos^n(\log z)\}+.$$

10. $y=-\frac{1}{2}-\frac{x}{2}(y^4+2y^3+3y^2)$; $\therefore$ to apply Lagrange's Theorem,

$$f(y)=y,\ \phi(z)=-\frac{1}{2}(z^4+2z^3+3z^2),\text{ and } z=-\frac{1}{2}$$

after differentiation, and $f'(z)=1$; thus

$$y=-\frac{1}{2}+x\,.\,\phi(z)+\frac{x^2}{\lfloor 2}\frac{d}{dz}\{\phi(z)^2\}+\frac{x^3}{\lfloor 3}\frac{d^2}{dz^2}\{\phi(z)^3\}+$$

when z is put $=-\frac{1}{2}$ after the differentiations have been performed. Now

when $$z=-\frac{1}{2},\ \phi(z)=-\frac{1}{8}\left(\frac{1}{4}-1+3\right)=-\frac{9}{32},$$

$$\phi'\,.(z)=-(2z^3+3z^2+3z)=\frac{1}{2}\left(\frac{1}{2}-\frac{3}{2}+3\right)=1,$$

and $$\phi''(z) = -3(2z^2+2z+1) = -3\left(\frac{1}{2}-1+1\right) = -\frac{3}{2};$$

thus $$\frac{d}{dz}\{\phi(z)^2\} = 2\phi(z) \cdot \phi'(z) = -\frac{9}{16};$$

$$\frac{d^2}{dz^2}\{\phi(z)^3\} = \frac{d}{dz}\{3\phi(z)^2 \cdot \phi'(z)\}$$

$$= 6\phi(z) \cdot \{\phi'(z)\}^2 + 3\{\phi(z)\}^2 \cdot \phi''(z)$$

$$= -\frac{6 \cdot 9}{32} - \frac{9}{2} \cdot \frac{9^2}{32^2} = -\frac{6 \cdot 9}{64^2}(128+27) = -6 \cdot \frac{1395}{4096}, \text{ and } \therefore \text{ \&c.}$$

11. Put $\log y = y'$, then $y' = \frac{\pi}{4} + x \sin y'$, and $f(y') = \cos y'$.

$$\phi(y') = \sin y', \text{ and } z = \frac{\pi}{4},$$

$$\therefore \cos \log y = \cos y' = \cos \frac{\pi}{4}$$

$$+ x \cdot \sin z\,(-\sin z) - \frac{x^2}{\lfloor 2}\frac{d}{dz}(\sin^3 z) - \frac{x^3}{\lfloor 3} \cdot \frac{d^2}{dz^2}(\sin^4 z) + \ldots\ldots$$

when z is put $= \frac{\pi}{4}$ after the differentiations. Now

$$-\sin^2\frac{\pi}{4} = -\frac{1}{2}, \ \frac{d}{dz}(\sin^3 z) = 3\sin^2 z \cdot \cos z = \frac{3}{2\sqrt{2}},$$

and $$\frac{d}{dz}(4\sin^3 z \cos z) = 12\sin^2 z \cos^2 z - 4\sin^4 z = \frac{8}{4} = 2; \ \therefore \text{ \&c.}$$

12. $y = 0 + x(y^2 + my + n)^{-1}$, thus by Lagrange's Theorem

$$y = z + \frac{x}{z^2+mz+n} + \frac{x^2}{2} \cdot \frac{d}{dz} \cdot \frac{1}{(z^2+mz+n)^2} + \ldots +$$

when z is put $= 0$ after the differentiations: thus the coefficient of x is $\frac{1}{n}$,

$$\frac{d}{dz}\frac{1}{(z^2+mz+n)^2} = -\frac{2(2z+m)}{(z^2+mz+n)^3} = -\frac{2m}{n^3};$$

$$\frac{d^2}{dz^2}\frac{1}{(z^2+mz+n)^3} = -\frac{d}{dz} \cdot \frac{3(2z+m)}{(z^2+mz+n)^4} = -\frac{6}{(z^2+mz+n)^4} + \frac{12(2z+m)^2}{(z^2+mz+n)^5}$$

$$= -\frac{6}{n^4} + \frac{12m^2}{n^5} = \frac{6(2m^2-n)}{n^5};$$

and $\frac{d^3}{dz^3}\left(\frac{1}{z^2+mz+n}\right)^4=-\frac{d^2}{dz^2}\frac{4(2z+m)}{(z^2+mz+n)^5}=\frac{d}{dz}\left\{\frac{-8}{(z^2+mz+n)^5}\right.$

$\left.+\frac{20(2z+m)^2}{(z^2+mz+n)^6}\right\}=+\frac{120(2z+m)}{(z^2+mz+n)^6}-\frac{120(2z+m)^3}{(z^2+mz+n)^7}=-\frac{\lfloor 5 . m(m^2-n)}{n^7}$, $\therefore$ &c.,

which can only apply to one of the values of y in the given cubic, if they be all different.

CHAPTER X.

1. Lt. $=$ lt. of $\frac{1}{x}\div 1=1$.

2. $=$ lt. $1\div nx^{n-1}=\frac{1}{n}$, when $x=1$.

3. $=$ lt. of $(e^x+e^{-x})\div\cos x=2$.

4. $=$ lt. $(e^x+e^{-x}-2)\div(1-\cos x)=$ lt. $(e^x-e^{-x})\div\sin x$
$=$ lt. $(e^x+e^{-x})\div\cos x=2$.

5. $=$ lt. $\left(1-\frac{1}{\sqrt{1-x^2}}\right)\div 3\sin^2 x\cos x=\left\{1-\left(1+\frac{1}{2}x^2\right)\right\}\div 3x^2$,

neglecting higher powers of x, $=-\frac{1}{6}$.

6. $=$ lt. of $a^x\log a-b^x . \log b=\log a-\log b$ when $x=0$.

7. $=$ lt. of $(\sec^2 x-1)\div(1-\cos x)=$ lt. $2\sec^2 x\tan x\div\sin x=2\sec^3 x=2$.

8. Expand $\sin x$; or thus: lt. $=$ lt. of $(1-\cos x)\div 3x^2=$ lt. $\sin x\div 6x$
$=$ lt. $\cos x\div 6=\frac{1}{6}$.

9. $=$ lt. of $3\cos 3x\div(1-3\cos 2x)=-\frac{3}{2}$.

10. $=$ lt. of $\left(-1+\frac{1}{x}\right)\div\frac{(x-1)}{\sqrt{2x-x^2}}=-\frac{\sqrt{2x-x^2}}{x}=-1$.

11. Lt. of $\frac{1-x}{\log x}=$ lt. $-1\div\frac{1}{x}=-1$.

12. Lt. $=(e^x-e^{-x}+2\sin x)\div(\sin x+x\cos x)=$ lt. of $\frac{(e^x+e^{-x}+2\cos x)}{2\cos x-x\sin x}$
$=\frac{4}{2}=2$.

13. $\text{Lt.} = \dfrac{2\cos 2x + 2\sin 2x - 2\cos x}{\sin 2x - \sin x} = \dfrac{-4\sin 2x + 4\cos 2x + 2\sin x}{2\cos 2x - \cos x} = 4.$

14. $\text{Lt. of } \left(x\sin x - \dfrac{\pi}{2}\right) \div \cos x = \text{lt. } (\sin x + x\cos x) \div (-\sin x) = -1.$

15. $\text{Lt.} = \{e^x(x-1)+1\} \div 3e^x(e^x-1)^2 = \text{lt. of } \{e^x(x-1)+1\} \div 3(e^x-1)^2$

$$= \text{lt. of } xe^x \div 6e^x(e^x-1) = \frac{x}{3} \div 2(e^x-1) = \frac{1}{3} \div 2e^x = \frac{1}{6},$$

or expand e^x.

16. Algebraically, expunging the common factor x^2-9, the quotient reduces to that of 2 quadratic expressions, and the results are obtained by making $x=3$ and -3 successively.

Differentiating, lt. (when $x=3$) $=$ lt. of $\dfrac{4x^3+9x^2-14x-27}{4x^3-9x^2-14x+27}$

$$= (162-42) \div (135-123) = 10,$$

and when $x=-3$, this fraction $= (-135+123) \div (-162+42) = \dfrac{1}{10}$.

17. $= \text{lt. } \left\{\sqrt{3x-2x^4} + \dfrac{x}{2} \cdot \dfrac{(3-8x^3)}{\sqrt{3x-2x^4}} - \dfrac{6}{5} \cdot x^{\frac{1}{5}}\right\} \div \left(-\dfrac{2}{3}x^{-\frac{1}{3}}\right)$

$$= -\frac{3}{2}\left\{1 - \frac{5}{2} - \frac{6}{5}\right\} = \frac{81}{20}.$$

18. $= \text{lt. of } \left\{\dfrac{3}{2}\sqrt{x} + \dfrac{3}{2}(x-1)^{\frac{1}{2}}\right\} \div \{3x(x^2-1)^{\frac{1}{2}} - 1\} = -\dfrac{3}{2}.$

19. $= \text{lt. of } \left\{\dfrac{3}{2}x^{\frac{1}{2}} + \dfrac{3}{2}(x-1)^{\frac{1}{2}}\right\} \div \dfrac{x}{\sqrt{x^2-1}} = \dfrac{3\sqrt{x^2-1}}{2x}(x^{\frac{1}{2}} + \overline{x-1}|^{\frac{1}{2}}) = 0.$

20. Expanding $\sin x$, &c., the lt. is

$$\left\{m\left(x - \frac{x^3}{\underline{|3}}\right) - \left(mx - \frac{m^3x^3}{\underline{|3}}\right)\right\} \div x\left\{1 - \frac{x^2}{2} - \left(1 - \frac{m^2x^2}{2}\right)\right\}$$

$$= (m^3 - m)\frac{x^3}{\underline{|3}} \div (m^2-1)\frac{x^3}{2} = \frac{m}{3},$$ or, differentiate thrice.

21. $= \text{lt. of } \dfrac{2x}{m\sin mx} = \text{lt. } \dfrac{2}{m^2\cos mx} = \dfrac{2}{m^2}.$

22. The fraction $= \cos\alpha \cdot \sin x \div (-\sin\alpha \cdot \sin x) = -\cot\alpha$: or differentiate.

23. $= \text{lt. of } \dfrac{n\sec^2 nx - n\sec^2 x}{n\cos x - n\cos nx} = \dfrac{\cos x + \cos nx}{\cos^2 x \cdot \cos^2 nx} = 2.$

24. The fraction $=(\sqrt{a+x}+\sqrt{a-x})\div\frac{1}{\sqrt{a}}(\sqrt{a^2+ax+x^2}+\sqrt{ax})$

$$=\sqrt{2}\div(\sqrt{3}+1).$$

25. $=\text{lt.}\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{x-a}}\right)\div\frac{2x}{\sqrt{x^2-a^2}}$

$$=\frac{\sqrt{x+a}}{2x^{\frac{3}{2}}}(\sqrt{x-a}+\sqrt{x})=\frac{\sqrt{2}}{2\sqrt{a}}=\frac{1}{\sqrt{2a}}.$$

26. Lt. of $\frac{2+\cos 2x-\sin x}{x\sin 2x+x\cos x}=\text{lt.}\frac{-2\sin 2x-\cos x}{\sin 2x+\cos x+x(2\cos 2x-\sin x)}=0,$

and lt. of $\frac{\pi-2x}{2\sin 2x}=$ lt. of $-2\div 4\cos 2x=\frac{1}{2}$, $\therefore$ lt. reqd. $=-\frac{1}{4}$.

27. If 2^x be very large, $\sin\left(\frac{a}{2^x}\right)\div\frac{1}{2^x}=\frac{a}{2^x}\div\frac{1}{2^x}=a$:

or, differentiating, lt. $=$ that of $\cos\frac{a}{2^x}\,.\,a\,.\,2^{-x}\log\frac{1}{2}\div\left(2^{-x}\,.\log\frac{1}{2}\right)=a.$

28. If $x=\frac{1}{y}$, $(a^{\frac{1}{x}}-1)\,.\,x=\frac{a^y-1}{y}$, $\therefore$ lt. when $y=0$, is

$$a^y\log a\div 1=\log a.$$

29. $\left(\frac{a}{x}+1\right)^x=e^{x\log\left(\frac{a}{x}+1\right)}$, and lt. when $x=\infty$ of

$$\log\left(\frac{a}{x}+1\right)\div\frac{1}{x}=\text{lt. of }-\frac{a}{x^2}\,.\,\frac{1}{\frac{a}{x}+1}\div\left(-\frac{1}{x^2}\right)=\frac{a}{\frac{a}{x}+1}=a,$$

$$\therefore\ \text{lt. of }\left(\frac{a}{x}+1\right)^x=e^a.$$

30. (1) $=$ lt. of $\{m^x\log(m)\,.\,\sin(nx)-n^x\log(n)\sin(mx)$
$+n\,.\,m^x\cos nx-m\,.\,n^x\cos mx\}\div(n\sec^2 nx-m\sec^2 mx)=1.$

(2) x is arbitrary, and differentiating with respect to m, the lt. $=(x\,.\,m^{x-1}\,.\sin nx-x\,.\,n^x\cos mx)\div(-x\sec^2 mx)$, or when $m=n$, lt. $=n^{x-1}(n\cos nx-\sin nx)\cos^2 nx$.

31. $\left(1+\frac{1}{x^2}\right)^x=e^{x\log\left(1+\frac{1}{x^2}\right)}$, and lt. of $\log\left(1+\frac{1}{x^2}\right)\div\frac{1}{x}$

$$=\text{lt.}\ \frac{1}{1+\frac{1}{x^2}}\,.\left(-\frac{2}{x^3}\right)\div\left(-\frac{1}{x^2}\right)=\frac{2}{x}=0,\ \therefore\ \text{lt. required}$$

$$=e^0=1.$$

32. $\left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = e^{\frac{1}{x}\log\frac{\tan x}{x}}$, and lt. of $\frac{\log\tan x - \log x}{x}$

$$= \text{lt. of } \frac{\cot x \sec^2 x - \frac{1}{x}}{1} = \frac{2x - \sin 2x}{x \sin 2x} = \text{lt. of } \frac{2 - 2\cos 2x}{\sin 2x + 2x\cos 2x}$$

$$= \text{lt. of } \frac{4\sin 2x}{4\cos 2x - 4\sin 2x \,.\, x} = 0, \ \therefore \ \text{lt. reqd.} = e^0 = 1.$$

33. $\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{x^2}\log\frac{\tan x}{x}}$, and lt. of $(\log\tan x - \log x) \div x^2$

$$= \text{lt. of } \left(\cot x \sec^2 x - \frac{1}{x}\right) \div 2x = \frac{2x - \sin 2x}{2x^2 \sin 2x}$$

$$= \text{lt. of } \frac{2 - 2\cos 2x}{4x\sin 2x + 4x^2\cos 2x} = \text{lt. of } \frac{4\sin 2x}{4\sin 2x + 16x\cos 2x - 8x^2\sin 2x}$$

$$= \text{lt. of } \frac{8\cos 2x}{24\cos 2x + \text{vanishing terms}} = \frac{1}{3},$$

$$\therefore \ \text{lt. reqd.} = e^{\frac{1}{3}}.$$

34. $\left(\frac{\tan x}{x}\right)^{\frac{1}{x^3}} = e^{\frac{1}{x^3}\log\frac{\tan x}{x}}$, and lt. of $(\log\tan x - \log x) \div x^3$

$$= \text{lt. of } \left(\cot x \sec^2 x - \frac{1}{x}\right) \div 3x^2 = \frac{2x - \sin 2x}{3x^3 \sin 2x},$$

and $\sin 2x = 2x - \frac{(2x)^3}{\lfloor 3}$, $\therefore$ lt. reqd. $= e^{\frac{8x^3}{6} \div 3x^3 . 2x}$ approximately

$$= e^{\frac{2}{9x}} = \infty.$$

35. $(\cos mx)^{\frac{n}{x}} = e^{n\log(\cos mx) \div x}$,

and lt. of $n\log(\cos mx) \div x = \text{lt. of} - mn\tan mx = 0$,

$$\therefore \ \text{lt. reqd.} = e^0 = 1.$$

36. $(\cos mx)^{\frac{n}{x^2}} = e^{n\log(\cos mx) \div x^2}$, and lt. of $n\log(\cos mx) \div x^2$

$$= \text{lt. of} - mn\tan(mx) \div 2x = \text{lt. of} - m^2 n \sec^2(mx) \div 2$$

$$= -\frac{m^2 n}{2}, \ \therefore \ \text{reqd. lt.} = e^{-\frac{m^2 n}{2}}.$$

37. $(\cos mx)^{\frac{n}{x^3}} = e^{\frac{n}{x^3}\log(\cos mx)}$, and lt. of $n \log(\cos mx) \div x^3$

$$= \text{lt. of } -\frac{mn \tan(mx)}{2x^2} = -\frac{m^2 nx}{2x^2} \text{ when } x \text{ is very small;}$$

$\therefore$ in the lt. when $x=0$, lt. reqd. $= e^{-\frac{m^2 n}{2x}} = e^{-\infty} = 0$,

n being supposed positive. If n be negative the lt. $= \infty$.

38. $\dfrac{x^3 \cot^2 x + \sin x}{x} = \text{lt. of } 3x^2 \cot^2 x - 2x^3 \cot x \operatorname{cosec}^2 x + \cos x$

$$= 3\cos^2 x \cdot \frac{x^2}{\sin^2 x} - 2\cos x \cdot \frac{x^3}{\sin^3 x} + \cos x = 3\cos^2 x - 2\cos x + \cos x = 2,$$

$$\because \text{ lt. of } x \div \sin x = 1.$$

39. Differentiate 4 times; *or*, expanding up to terms in x^4,

$$\text{lt.} = \text{that of } \frac{1}{x^4}\left(2x + \frac{2x^3}{\underline{|3}} + \right)^2 - \frac{4x^2}{x^4}\left(1 + \frac{x^2}{\underline{|2}} + \right)$$

$$= \text{lt. of } \frac{1}{x^4}\left(\frac{8x^4}{6} - \frac{4x^4}{2}\right) = -\frac{2}{3}.$$

40. $= \text{lt. of } \dfrac{1}{2\sqrt{1-x}} \div \left\{\dfrac{1}{2} \cdot \dfrac{1}{\sqrt{1+x}} - \dfrac{x}{\sqrt{1+x^2}}\right\} = \dfrac{1}{2} \div \dfrac{1}{2} = 1.$

41. $(\sin x)^{\tan x} = e^{\tan x \log \sin x}$, and lt. of $\dfrac{\log(\sin x)}{\cot x}$

$$= \text{lt. of } \cot x \div -\operatorname{cosec}^2 x = -\sin x \cos x = 0,$$

$$\therefore \text{ reqd. lt.} = e^0 = 1.$$

42. $= \text{lt. of } \dfrac{\sin x - \cos x}{2 \cot 2x}$ or $(\because \sin 2x = 1) = \dfrac{\sin x - \cos x}{2 \cos 2x}$

$$= \text{lt. of } -(\sin x + \cos x) \div 4 \sin 2x = -\frac{1}{2\sqrt{2}}.$$

43. Lt. of $\dfrac{\tan\theta}{\theta}$ is 1 when $\theta = 0$, $\left(= \dfrac{\sec^2\theta}{1}\right)$,

and $\sqrt{a^2 - x^2} \cdot \cot\left\{\dfrac{\pi}{2}\sqrt{\dfrac{a-x}{a+x}}\right\} = \dfrac{\pi}{2} \cdot \sqrt{\dfrac{a-x}{a+x}} \cdot \cot\left\{\dfrac{\pi}{2}\sqrt{\dfrac{a-x}{a+x}}\right\} \cdot \dfrac{2(a+x)}{\pi}$,

$$\therefore \text{ its lt.} = \text{that of } \frac{2}{\pi}(a+x) = \frac{4a}{\pi}.$$

44. $=\text{lt. of } (1-x)\sin\frac{\pi x}{2}\div\cos\frac{\pi x}{2}=\dfrac{\left(-\sin\frac{\pi x}{2}+\overline{1-x}\,\frac{\pi}{2}\cos\frac{\pi x}{2}\right)}{-\frac{\pi}{2}\sin\frac{\pi x}{2}}=\frac{2}{\pi}.$

45. $x^{\frac{1}{1-x}}=e^{\frac{\log x}{1-x}}$, and lt. of $\log x\div(1-x)$

$$=\text{lt. of } -\frac{1}{x}=-1,\ \therefore\ \text{lt. reqd.}=e^{-1}=\frac{1}{e}.$$

46. $x^{x^{-a}}=e^{x^{-a}.\log x}$, and lt. of $\log x\div x^a$, when $x=0$,

$$=-\infty\div(+0)=-\infty,\ \therefore\ x^{x^{-a}}=e^{-\infty}=0.$$

But if a be negative,

$$\log x\div x^a=\frac{-\infty}{\infty}.\ =\frac{1}{ax^a}=0,\ \text{and}\ \therefore\ x^{x^{-a}}=1.$$

47. $=\text{lt. of}\frac{\pi}{2}\sec\left(\frac{\pi}{2}x\right)\tan\left(\frac{\pi}{2}x\right)\div\left(-\frac{2}{1-x}\right)=-\frac{\pi}{4}.\dfrac{\sin\frac{\pi}{2}x.(1-x)}{\cos^2\frac{\pi}{2}x}$

$$=-\frac{\pi}{4}(1-x)\div\cos^2\frac{\pi}{2}x=-\frac{\pi}{4}\div\frac{\pi}{2}\sin\pi x=\infty.$$

48. $(Ax^m++)^{\frac{1}{x}}=e^{\frac{1}{x}\log(Ax^m++)}$, and lt. of the power of e

$=\text{lt. of }\dfrac{mAx^{m-1}++}{Ax^m++}=0$ when $x=\infty$, m being a positive integer, $\therefore$ required lt.$=e^0=1$.

49. $=\text{lt. of } 2\sqrt{x}\left\{\left(2+\frac{a+b+2x}{\sqrt{ax+bx+x^2}}\right)\frac{1}{2x+b+2\sqrt{ax+bx+x^2}}\right.$

$$\left.-\frac{a}{\sqrt{ax}}.\frac{1}{b+2\sqrt{ax}}\right\}$$

$$=\frac{4\sqrt{x}}{b}+\frac{2(a+b)}{\sqrt{a+b}.b}-\frac{2\sqrt{a}}{b}=\frac{2}{b}(\sqrt{a+b}-\sqrt{a}).$$

50. The expression$=\dfrac{1}{2x}\left\{-1+\sqrt{\dfrac{1-5x}{1-x}}\right\}=\dfrac{1}{2x}\{-1+(1-5x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}\}$,

$\therefore$ expanding by Bin. Theor.$=\dfrac{1}{2x}\left(-\frac{5}{2}x+\frac{x}{2}\right)=-1.$

Or differentiate the form $\{\sqrt{1-5x}-\sqrt{1-x}\}\div 2x\sqrt{1-x}$, &c.

51. Lt. = that of $-\theta\sin(x\theta)\div\{-2x\theta e^{-x^2\theta}\}=\dfrac{\sin a\theta\,.\,e^{a^2\theta}}{2a}$.

52. Lt. = that of $\left(e^x-\dfrac{1}{1-x}\right)\div\tan^2 x=\dfrac{\left(1+x+\dfrac{x^2}{\lfloor 2}+\right)-(1+x+x^2+)}{x^2+}$

$=-\dfrac{1}{2}$ when $x=0$.

53. Lt. = that of $\left\{e^x(\sin x+\cos x)-e^a\sqrt{2}\cos\left(a-\dfrac{\pi}{4}\right)\right\}\div(e^x-e^a)$

= lt. of $e^x\,.\,2\cos x\div e^x=2\cos a$ when $x=a$.

54. = lt. of $e^{nx\log\left(\frac{a_1^{\frac{1}{x}}++}{n}\right)}$, and lt. of $\{\log(a_1^{\frac{1}{x}}++)-\log n\}\div\dfrac{1}{nx}$

= lt. of $\dfrac{1}{a_1^{\frac{1}{x}}++}\left\{a_1^{\frac{1}{x}}\log a_1\left(-\dfrac{1}{x^2}\right)++\right\}\div-\dfrac{1}{nx^2}$, and $a_1^{\frac{1}{x}}=1$ when $x=\infty$,

$\therefore$ lt. reqd. $=e^{\log a_1+\log a_2++}$

$=a_1a_2\ldots a_n$.

55. Expanding, the lt. = that of $\left\{x+x-\dfrac{x^3}{\lfloor 3}+-4\left(\dfrac{x}{2}-\dfrac{x^3}{8\lfloor 3}+\right)\right\}^4$

$\div\left\{3+1-\dfrac{x^2}{2}+\dfrac{x^4}{\lfloor 4}-4\left(1-\dfrac{x^2}{4\,.\,2}+\dfrac{x^4}{2^4\lfloor 4}-\right)\right\}^3$

$=\left(-\dfrac{x^3}{2\lfloor 3}+\right)^4\div\left(\dfrac{3}{4}\,.\,\dfrac{x^4}{\lfloor 4}+\right)^3=32^3\div12^4=\dfrac{128}{81}$.

56. $=e^y$ say, where $y=\dfrac{1}{x}\log\left(\dfrac{\log x}{x}\right)=\{\log(\log x)-\log x\}\div x$

= lt. of $\dfrac{1}{x\log x}-\dfrac{1}{x}=0$, $\therefore$ lt. reqd. $=e^0=1$.

57. If $x=1-z$ where z is small and ultimately $=0$,

the lt. $=\left\{\left(-z-\dfrac{z^2}{2}-\right)^{\frac{2}{3}}+(2z-z^2)^{\frac{3}{4}}\right\}\div\left(-z+\dfrac{z^3}{\lfloor 3}-\right)^{\frac{2}{3}}$

$=(z^{\frac{2}{3}}+\text{higher powers})\div(z^{\frac{2}{3}}+\text{higher powers})=1$.

58. If $u=a^{x^m}\div b^{x^n}$, $\log u=x^m.\log a-x^n\log b$, and as a, b are each >1, $\log a$, $\log b$ are both positive, $\therefore \log u$ is ∞ or $-\infty$ as $m>$ or $<n$, when $x=\infty$, and accordingly $u=\infty$ or 0.

59. If $x=\frac{1}{y}$, $x-x^2\log\left(1+\frac{1}{x}\right)=\frac{y-\log(1+y)}{y^2}=\text{lt.}\left(1-\frac{1}{1+y}\right)\div 2y$

$$=\frac{1}{(1+y)^2}\cdot\frac{1}{2}=\frac{1}{2}\text{ when } y=0.$$

60. Let $\frac{a\sqrt{x}}{\sqrt{c}}=y$, and $\frac{cu}{a}=v$, then $vy=\tan^{-1}y+\log(y+\sqrt{1+y^2})$;

$$\therefore v=\text{lt.}\ \frac{1}{1+y^2}+\frac{1}{y+\sqrt{1+y^2}}\left(1+\frac{y}{\sqrt{1+y^2}}\right)=\frac{1}{1+y^2}+\frac{1}{\sqrt{1+y^2}};$$

$$\therefore \text{when } y=0,\ v=\frac{cu}{a}=2 \text{ or } u=\frac{2a}{c}, \text{ and when } y=\infty,\ v=0=u.$$

Also $$x=\frac{cy^2}{a^2},\ \therefore \frac{dx}{dy}=\frac{2cy}{a^2},$$

and $$y\frac{dv}{dy}=\frac{c}{a}y\frac{du}{dx}\cdot\frac{2cy}{a^2}=-v+\frac{1}{1+y^2}+\frac{1}{\sqrt{1+y^2}};$$

$$\therefore \text{when } y=\infty,\ \frac{du}{dx}=\frac{0}{y^2}=0, \text{ and when } y=0,$$

$$\frac{2c^2}{a^3}\cdot\frac{du}{dx}\cdot y^3=-vy+\frac{y}{1+y^2}+\frac{y}{\sqrt{1+y^2}}$$

$$=-\tan^{-1}y-\log(y+\sqrt{1+y^2})+\frac{y}{1+y^2}+\frac{y}{\sqrt{1+y^2}};$$

$$\therefore \frac{2c^2}{a^3}\frac{du}{dx}=\left\{-\frac{1}{1+y^2}-\frac{1}{\sqrt{1+y^2}}+\frac{1}{1+y^2}-\frac{2y^2}{(1+y^2)^2}+\frac{1}{\sqrt{1+y^2}}-\frac{y^2}{(1+y^2)^{\frac{3}{2}}}\right\}\div 3y^2$$

$$=-\left\{\frac{2}{(1+y^2)^2}+\frac{1}{(1+y^2)^{\frac{3}{2}}}\right\}\div 3=-1,\text{ when } y=0;$$

$$\therefore \frac{du}{dx}=-\frac{a^3}{2c^2}.$$

CHAPTER XI.

1. $\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}+\frac{du}{dz}\cdot\frac{dz}{dx}$, and $2\log u=\log(z^2-y^2)-\log(z^2+y^2)$;

$$\therefore \frac{2}{u}\cdot\frac{du}{dz}=\frac{2z}{z^2-y^2}-\frac{2z}{z^2+y^2},\ \frac{2}{u}\cdot\frac{du}{dy}=-\frac{2y}{z^2-y^2}-\frac{2y}{z^2+y^2},$$

thus $\frac{du}{dx}=\frac{2u}{z^4-y^4}\left\{y^2z\frac{dz}{dx}-yz^2\frac{dy}{dx}\right\}=2yz\left(y\frac{dz}{dx}-z\frac{dy}{dx}\right)\cdot\frac{1}{\sqrt{z^2-y^2}\,(z^2+y^2)^{\frac{3}{2}}}$.

2. $\sin u = \frac{z}{y}$; $\therefore \cos u \cdot \frac{du}{dx} = \frac{1}{y} \cdot \frac{dz}{dx} - \frac{z}{y^2} \cdot \frac{dy}{dx}$,

$$\text{and } \therefore \frac{du}{dx} = \left(y\frac{dz}{dx} - z\frac{dy}{dx}\right) \div y \cdot \sqrt{y^2 - z^2}.$$

3. $e^{ny}(1+ny)\frac{dy}{dx} = max^{m-1} = \frac{m}{x} \cdot e^{ny} \cdot y$;

$$\therefore \frac{dy}{dx} = my \div x(1+ny).$$

4. $y \log x = x \log y$, $\therefore \left(\log x - \frac{x}{y}\right)\frac{dy}{dx} = \log y - \frac{y}{x}$;

$$\therefore \frac{dy}{dx} = \frac{y}{x}(x \log y - y) \div (y \log x - x).$$

5. $(x+a)^2 y^2 = (y+a)^2 (y^2 - b^2)$;

$$\therefore (x+a)y^2 + (x+a)^2 y \frac{dy}{dx} = \{(y+a)(y^2-b^2) + (y+a)^2 y\}\frac{dy}{dx};$$

$$\therefore \frac{dy}{dx} = (x+a)y^3 \div \{-(y+a)^2(y^2-b^2) + (y^2+ay)(y^2-b^2) + y^2(y+a)^2\}$$
$$= y^3(x+a) \div \{(\overline{y+a}|^2 - \overline{y^2+ay}|)b^2 + y^2(y^2+ay)\}$$
$$= y^3(x+a) \div (y+a)(y^3+ab^2) = \pm y^2\sqrt{y^2-b^2} \div (y^3+ab^2).$$

6. $\cos(xy)\left(y + x\frac{dy}{dx}\right) = m$, $\therefore \frac{dy}{dx} = \frac{m - y\cos(xy)}{x\cos(xy)}$

$$= \left\{m - \frac{1}{x}\sin^{-1}(mx)\sqrt{1-m^2x^2}\right\} \div x\sqrt{1-m^2x^2}.$$

7. $\therefore \frac{dy}{dx}(y^2 - ax) + x^2 - ay = 0$,

$$\therefore \frac{d^2y}{dx^2}(y^2 - ax) + \frac{dy}{dx}\left(2y\frac{dy}{dx} - a\right) + 2x - a\frac{dy}{dx} = 0,$$

$$\therefore -\frac{d^2y}{dx^2}(y^2-ax)^3 = 2y(x^2-ay)^2 + 2a(x^2-ay)(y^2-ax) + 2x(y^2-ax)^2 = 2a^3xy$$

from the given equation.

8. $4x^3 + 4axy = \frac{dy}{dx}(3ay^2 - 2ax^2)$ which gives $\frac{dy}{dx}$;

$$\therefore 12x^2 + 4ay + 4ax\frac{dy}{dx} = \frac{d^2y}{dx^2}(3ay^2 - 2ax^2) + \frac{dy}{dx}\left(6ay\frac{dy}{dx} - 4ax\right),$$

$$\therefore \frac{d^2y}{dx^2}(3ay^2 - 2ax^2)^3 = -6ay \cdot 16x^2(x^2+ay)^2 + 8ax(3ay^2-2ax^2)4x(x^2+ay)$$
$$+ 4(3x^2+ay)(3ay^2-2ax^2)^2.$$

Also $$12x^2+4ay+\frac{dy}{dx}(8ax)=6ay\left.\frac{dy}{dx}\right|^2+\frac{d^2y}{dx^2}(3ay^2-2ax^2);$$

$$\therefore\ 24x+12a\frac{dy}{dx}+12ax\frac{d^2y}{dx^2}=\frac{d^3y}{dx^3}(3ay^2-2ax^2)+6a\left(\frac{dy}{dx}\right)^3+18ay\frac{d^2y}{dx^2}\cdot\frac{dy}{dx}.$$

9. $$\frac{dy}{dx}=\frac{d\phi}{dx}+\frac{d\phi}{dy}\cdot\frac{dy}{dx}+\frac{d\phi}{du}\cdot\frac{du}{dx}\text{ and }\frac{d\psi}{dx}+\frac{d\psi}{dy}\cdot\frac{dy}{dx}+\frac{d\psi}{du}\cdot\frac{du}{dx}=0;$$

$$\therefore\ \frac{du}{dx}\left\{\frac{d\phi}{du}\cdot\frac{d\psi}{dy}-\frac{d\psi}{du}\left(\frac{d\phi}{dy}-1\right)\right\}+\frac{d\phi}{dx}\cdot\frac{d\psi}{dy}-\frac{d\psi}{dx}\left(\frac{d\phi}{dy}-1\right)=0,\ \&c.$$

10. x and y are connected by the equation $\phi(x,\ y)=\chi(x)$; and $\frac{du}{dy}$, $\frac{d\phi}{dy}$ mean total and partial differential coefficients. Taking y as independent variable, then,
$$\frac{du}{dy}=\chi'(x)\cdot\frac{dx}{dy}=\frac{d\phi}{dy}+\frac{d\phi}{dx}\cdot\frac{dx}{dy};$$

$$\therefore\ \frac{du}{dy}\left(\frac{d\phi}{dx}-\chi'(x)\right)=-\chi'(x)\cdot\frac{d\phi}{dy}.$$

11. (1) $$\frac{du}{dx}=(a)^{x^y}\cdot yx^{y-1}\cdot\log a+\frac{y}{2}\sqrt{\sec xy}\cdot\tan xy.$$

(2) $$\frac{dy}{dx}=-1,\ \therefore\ \frac{du}{dx}=a^{x^y}\cdot yx^{y-1}\cdot\log a+\frac{y}{2}\sqrt{\sec xy}\cdot\tan xy$$
$$-a^{x^y}\cdot\log a\cdot x^y\cdot\log x-\frac{x}{2}\sqrt{\sec xy}\cdot\tan xy.$$

12. From Ex. 11, $$0=a^{x^y}\cdot yx^{y-1}\cdot\log a+\frac{y}{2}\sqrt{\sec xy}\cdot\tan xy$$
$$+\frac{dy}{dx}\left(a^{x^y}\cdot x^y\log a\ \log x+\frac{x}{2}\sqrt{\sec xy}\cdot\tan xy\right)\ \&c.$$

13. Here when $x=0=y$, $(\Delta x)^4+2a\Delta x^2\cdot\Delta y-a\Delta y^3=0$,
$$\therefore\ \Delta x+2a\cdot\frac{\Delta y}{\Delta x}-a\left(\frac{\Delta y}{\Delta x}\right)^3=0,$$

$\therefore$ in the limit $$2\frac{dy}{dx}=\left(\frac{dy}{dx}\right)^3,\text{ i.e. }\frac{dy}{dx}=0,\text{ or }\pm\sqrt{2}.$$

14. Here when $x=0=y$, $\Delta x^4-a\Delta y^3+2a\Delta x\Delta y^2+3a\Delta x^2\cdot\Delta y=0$;
$$\therefore\ \Delta x-a\left(\frac{\Delta y}{\Delta x}\right)^3+2a\left(\frac{\Delta y}{\Delta x}\right)^2+3a\frac{\Delta y}{\Delta x}=0,\ \therefore\text{ in the limit}$$
$$\left(\frac{dy}{dx}\right)^3-2\left(\frac{dy}{dx}\right)^2-3\frac{dy}{dx}=0,\text{ i.e. }\frac{dy}{dx}=0,\text{ or }\left(\frac{dy}{dx}-3\right)\left(\frac{dy}{dx}+1\right)=0;$$
$\therefore$ &c.

15. When $x=0=y$, $a\Delta x^3+\Delta x^3\,.\,\Delta y-a\Delta y^3=0$, whence $1=\left(\frac{dy}{dx}\right)^3$,

i.e. the real value of $\frac{dy}{dx}$ is 1.

16. If $y+b=z$, $\frac{dy}{dx}=\frac{dz}{dx}$, and

$$x^2(z-b)^2=z^2(a^2-\overline{z-b}|^2)=z^2(a^2-b^2+2bz-z^2),$$

$$\therefore \text{ when } x=0=z,\ \Delta x^2(\Delta z-b)^2=\Delta z^2(a^2-b^2+2b\Delta z-\Delta z^2),$$

and in the limit $b^2=(a^2-b^2)\left(\frac{dz}{dx}\right)^2$, $\therefore \frac{dy}{dx}=\pm\frac{b}{\sqrt{a^2-b^2}}$.

17. When $x=0=y$, in the limit, $\frac{3}{2}(\Delta y^2-\Delta x^2)=8\Delta x^2$, $\therefore \frac{dy}{dx}=\pm\sqrt{\frac{19}{3}}$.

If $x-1=x'$, $y-1=y'$, $\Delta x=\Delta x'$, $\Delta y=\Delta y'$, $\therefore \frac{dy}{dx}=\frac{dy'}{dx'}$,

and $\{\overline{y'+1}|^2-(x'+1)^2\}\,x'\left(x'-\frac{1}{2}\right)=2\,\{(y'+1)^2+(x'+1)^2-2\,(x'+1)\}^2$,

$$\therefore \text{ approximately } -\Delta x'(\Delta y'-\Delta x')=8\Delta y'^2,$$

$$\therefore 8\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}-1=0, \text{ or } \frac{dy}{dx}=-\frac{1\pm\sqrt{33}}{16}.$$

18. If $x=0$, $y=0$ or ± 1. When $y=0$, $-\Delta y^2+3\Delta x\Delta y-2\Delta x^2=0$,

and $\therefore \left(\frac{dy}{dx}\right)^2-3\frac{dy}{dx}+2=0$, i.e. $\frac{dy}{dx}=1$ or 2.

If $y\mp 1=z$, $\frac{dy}{dx}=\frac{dz}{dx}$, and $(z\pm 1)^4-(z\pm 1)^2+3x\,(z\pm 1)=2x^2$;

$$\therefore \text{ when } z=0=x,\ \pm 2\Delta z\pm 3\Delta x=0, \text{ and } \therefore \frac{dy}{dx}=-\frac{3}{2}.$$

19. y, z are functions of x,

$$\therefore \frac{1}{xy}\left(y+x\frac{dy}{dx}\right)+\frac{1}{x}\cdot\frac{dy}{dx}-\frac{y}{x^2}=0,$$

$$\therefore \frac{dy}{dx}=\frac{y-x}{x}\cdot\frac{y}{y+x},$$

$$\frac{1}{z}\cdot\frac{dz}{dx}-\frac{1}{x}+x\frac{dz}{dx}+z=0,\quad \therefore \frac{dz}{dx}=\frac{1-xz}{1+xz}\cdot\frac{z}{x},$$

and
$$u\frac{du}{dx}+x+y\frac{dy}{dx}+z\frac{dz}{dx}=0;$$

$$\therefore u\frac{du}{dx}=-x+\frac{y^2(x-y)}{x(x+y)}+\frac{z^2}{x}\,\frac{(xz-1)}{xz+1}.$$

20. Here z is taken as a function of the independent variables x and y.

Thus $$\frac{x}{a^2}+\frac{z}{c^2}\cdot\frac{dz}{dx}=0,\ \therefore\ \frac{1}{a^2}+\frac{1}{c^2}\cdot\left(\frac{dz}{dx}\right)^2+\frac{z}{c^2}\cdot\frac{d^2z}{dx^2}=0;$$

$$\therefore\ \frac{d^2z}{dx^2}=-\frac{c^2}{z}\left\{\frac{1}{a^2}+\frac{1}{c^2}\cdot\frac{c^4}{a^4}\cdot\frac{x^2}{z^2}\right\}=-\frac{c^4}{a^2z^3}\left(1-\frac{y^2}{b^2}\right).$$

So $$\frac{d^2z}{dy^2}=-\frac{c^4}{b^2z^3}\left(1-\frac{x^2}{a^2}\right).\quad \text{Also } \frac{y}{b^2}+\frac{z}{c^2}\cdot\frac{dz}{dy}=0,$$

$$\therefore\ \frac{dz}{dx}\cdot\frac{dz}{dy}+z\frac{d^2z}{dxdy}=0,$$

$$\therefore\ \frac{d^2z}{dxdy}=-\frac{1}{z}\cdot\frac{c^4xy}{a^2b^2z^2}.$$

CHAPTER XII.

1. $x=e^y,\ \therefore\ \dfrac{dx}{dy}=e^y=x$, and $\therefore\ \dfrac{du}{dy}=x\dfrac{du}{dx}$,

$$\therefore\ \frac{d^2u}{dy^2}=x\frac{d}{dx}\left(x\frac{du}{dx}\right)=x^2\frac{d^2u}{dx^2}+x\frac{du}{dx},\ \therefore\ \frac{d^2u}{dy^2}+u=0.$$

2. $\dfrac{dy}{dx}=\dfrac{dy}{d\theta}\cdot\dfrac{d\theta}{dx}=\dfrac{1}{(1+x^2)}\cdot\dfrac{dy}{d\theta}$, or $(1+x^2)\dfrac{dy}{dx}=\dfrac{dy}{d\theta}$;

$$\therefore\ (1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=\frac{d^2y}{d\theta^2}\cdot\frac{1}{(1+x^2)};$$

$$\therefore\ (1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx}+y=\frac{d^2y}{d\theta^2}+y=0.$$

3. $x=2\dfrac{dt}{dx},\ \therefore\ \dfrac{dy}{dx}=\dfrac{dy}{dt}\cdot\dfrac{x}{2},\ \therefore\ \dfrac{d^2y}{dx^2}=\dfrac{1}{2}\cdot\dfrac{dy}{dt}+\dfrac{x^2}{4}\cdot\dfrac{d^2y}{dt^2}$,

$$\therefore\ \frac{x^2}{4}\cdot\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=0=t\frac{d^2y}{dt^2}+\frac{dy}{dt}+y.$$

4. $(e^x+e^{-x})^2=\left(\dfrac{t}{\sqrt{1-t^2}}+\dfrac{\sqrt{1-t^2}}{t}\right)^2=\dfrac{1}{t^2(1-t^2)}$,

and $$\frac{dx}{dt}=\frac{d}{dt}\left\{\log t-\frac{1}{2}\log(1-t^2)\right\}=\frac{1}{t}+\frac{t}{1-t^2}=\frac{1}{t(1-t^2)};$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dt}\cdot t(1-t^2),$$

and $$\frac{d^2y}{dx^2}=t(1-t^2)\left\{\frac{d^2y}{dt^2}t(1-t^2)+\frac{dy}{dt}(1-3t^2)\right\}=yt^2(1-t^2);$$

$$\therefore\ t(1-t^2)\frac{d^2y}{dt^2}+\frac{dy}{dt}(1-3t^2)=ty.$$

5. $1=-\sin t\,.\,\frac{dt}{dx}$, $\therefore\ \frac{dy}{dx}=-\frac{dy}{dt}\,.\,\text{cosec}\,t,$

$$\therefore\ \frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}\text{cosec}^2 t-\frac{dy}{dt}\text{cosec}^2 t\,.\,\cot(t);$$

$$\therefore\ \sin^2 t\,.\,\frac{d^2y}{dx^2}-\cos t\frac{dy}{dx}=0=\frac{d^2y}{dt^2},\text{ or }\frac{d^2y}{dt^2}=0.$$

6. $\frac{dy}{dx}=\left(\sin\theta\frac{dr}{d\theta}+r\cos\theta\right)\div\left(\cos\theta\frac{dr}{d\theta}-r\sin\theta\right);$

$$\therefore\ \frac{x\frac{dy}{dx}-y}{\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}}=\frac{\left(r\sin\theta\cos\theta\frac{dr}{d\theta}+r^2\cos^2\theta-r\sin\theta\cos\theta\frac{dr}{d\theta}+r^2\sin^2\theta\right)}{\left\{\left(\sin\theta\frac{dr}{d\theta}+r\cos\theta\right)^2+\left(\cos\theta\frac{dr}{d\theta}-r\sin\theta\right)^2\right\}^{\frac{1}{2}}}$$

$$=\frac{r^2}{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}}.$$

7. $$x+y\frac{dy}{dx}=\frac{r\cos\theta\left(\cos\theta\frac{dr}{d\theta}-r\sin\theta\right)+r\sin\theta\left(\sin\theta\frac{dr}{d\theta}+r\cos\theta\right)}{\cos\theta\frac{dr}{d\theta}-r\sin\theta}$$

$$=r\frac{dr}{d\theta}\div\left(\cos\theta\,.\,\frac{dr}{d\theta}-r\sin\theta\right),$$

and $$x\frac{dy}{dx}-y\ (\text{as in Ex. 6})=\frac{r^2}{\left(\cos\theta\frac{dr}{d\theta}-r\sin\theta\right)};$$

$$\therefore\ \left(x+y\frac{dy}{dx}\right)\div\left(x\frac{dy}{dx}-y\right)=\frac{1}{r}\frac{dr}{d\theta}.$$

8. $\frac{dx}{dt}=a\sin t,\ \frac{dy}{dt}=a(n+\cos t)$

$$\therefore\ \frac{dy}{dx}=\frac{n+\cos t}{\sin t},\quad\therefore\ \frac{d^2y}{dx^2}=\frac{dt}{dx}\cdot\frac{d}{dt}\left(\frac{n+\cos t}{\sin t}\right)$$

$$=\frac{1}{a\sin t}\left\{-1-\frac{(n+\cos t)\cos t}{\sin^2 t}\right\}=-\frac{n\cos t+1}{a\,.\,\sin^3 t}.$$

9. $r\dfrac{dr}{dx_1}=x_1$ &c., $\therefore\ \dfrac{du}{dx_1}=\dfrac{du}{dr}\cdot\dfrac{x_1}{r}$;

$$\therefore\ \frac{d^2u}{dx_1^{\,2}}=\frac{d^2u}{dr^2}\cdot\frac{x_1^{\,2}}{r^2}+\frac{du}{dr}\cdot\frac{1}{r}-\frac{du}{dr}\cdot\frac{x_1}{r^2}\cdot\frac{x_1}{r};$$

$$\therefore\ \frac{d^2u}{dx_1^{\,2}}+\ldots+\frac{d^2u}{dx_n^{\,2}}=0=\frac{d^2u}{dr^2}+\frac{du}{dr}\cdot\frac{n}{r}-\frac{du}{dr}\cdot\frac{r^2}{r^3},$$

or
$$\frac{d^2u}{dr^2}+(n-1)\frac{du}{dr}=0.$$

10. $\dfrac{dy}{d\phi}=b\cos\phi,\ \dfrac{dx}{d\phi}=-a\sin\phi,\ \therefore\ \dfrac{dy}{dx}=-\dfrac{b}{a}\cot\phi$;

$$\therefore\ 1+\left(\frac{dy}{dx}\right)^2=(a^2\sin^2\phi+b^2\cos^2\phi)\div a^2\sin^2\phi;$$

and $\dfrac{d^2y}{dx^2}=\dfrac{d\phi}{dx}\cdot\dfrac{d}{d\phi}\left(-\dfrac{b}{a}\cot\phi\right)=-\dfrac{b}{a}\cdot\dfrac{1}{a\sin\phi}(\operatorname{cosec}^2\phi)=-b\div a^2\sin^3\phi,$

$$\therefore\ \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}\div\left(-\frac{d^2y}{dx^2}\right)=\frac{(a^2\sin^2\phi+b^2\cos^2\phi)^{\frac{3}{2}}}{ab}.$$

11. $e^{2x}=\tan(t),\ \therefore\ \dfrac{dt}{dx}=2e^{2x}\div\sec^2t=\dfrac{2}{e^{-2x}+e^{-2x}},$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{2}{e^{2x}+e^{-2x}},$$

$$\therefore\ \frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}\cdot\frac{4}{(e^{2x}+e^{-2x})^2}-4\frac{dy}{dt}\cdot\frac{e^{2x}-e^{-2x}}{(e^{2x}+e^{-2x})^2},$$

$$\therefore\ \frac{d^2y}{dx^2}+\frac{2(e^{2x}-e^{-2x})}{e^{2x}+e^{-2x}}\cdot\frac{dy}{dx}+\frac{4n^2y}{(e^{2x}+e^{-2x})^2}=4\left(\frac{d^2y}{dt^2}+n^2y\right)\div(e^{2x}+e^{-2x})^2,$$

$$\text{i. e. } \frac{d^2y}{dt^2}+n^2y=0.$$

12. $\dfrac{dx}{dy}=\sec^2y=1+x^2,\ \therefore\ \dfrac{du}{dy}=\dfrac{du}{dx}(1+x^2),$

$$\frac{d^2u}{dy^2}=\frac{d^2u}{dx^2}(1+x^2)^2+2x\frac{du}{dx}(1+x^2),$$

$$\frac{d^3u}{dy^3}=\frac{d^3u}{dx^3}(1+x^2)^3+6x(1+x^2)^2\frac{d^2u}{dx^2}+\frac{du}{dx}(1+x^2)\,2(1+3x^2);$$

$\therefore$ original expression $=(1+x^2)^3\dfrac{d^3u}{dx^3}+2x(1+x^2)^2\dfrac{d^2u}{dx^2}$

$$+\frac{du}{dx}(1+x^2)\{2+6x^2-8x^2+2x^2\}=0,$$

$$\therefore\ \frac{d^3u}{dx^3}(1+x^2)^2+2x(1+x^2)\frac{d^2u}{dx^2}+2\frac{du}{dx}=0.$$

13. As in Art. 198, $\frac{d^2y}{dx^2} = -\frac{d^2x}{dy^2} \div \left(\frac{dx}{dy}\right)^3$,

$$\therefore \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \left(\frac{dx}{dy} - \frac{d^2x}{dy^2}\right) \div \left(\frac{dx}{dy}\right)^3.$$

14. $\frac{dx}{dt} = 1 + 2t$, $\therefore \frac{du}{dt} = \frac{du}{dx}(1+2t)$,

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2}(1+2t)^2 + 2\frac{du}{dx} = \frac{d^2u}{dx^2}(1+4x) + 2\frac{du}{dx}.$$

15. $\frac{dz}{du} = 1 - e\cos u$, $\frac{1-\cos u}{1+\cos u} = \frac{1-e}{1+e} \cdot \frac{1-\cos v}{1+\cos v}$;

$$\therefore \cos u = \frac{e+\cos v}{1+e\cos v}, \text{ and } \frac{du}{dv} \cdot \frac{1}{1+\cos u} = \sqrt{\frac{1-e}{1+e}} \cdot \frac{1}{1+\cos v};$$

$$\therefore \frac{dz}{dv} = \frac{1-e^2}{1+e\cos v} \cdot \sqrt{\frac{1-e}{1+e}} \cdot \frac{(1+e)(1+\cos v)}{(1+\cos v)(1+e\cos v)}$$

$$= (1-e^2)^{\frac{3}{2}} \div (1+e\cos v)^2.$$

16. $\frac{dz}{dy} = \frac{dz}{dx} \cdot \frac{dx}{dy} = -\frac{dz}{dx} \cdot \frac{y}{x}$;

$$\therefore \frac{d^2z}{dy^2} = +\frac{d^2z}{dx^2} \cdot \frac{y^2}{x^2} - \frac{dz}{dx} \cdot \left(\frac{1}{x} + \frac{y}{x^2} \cdot \frac{y}{x}\right);$$

$$\therefore x^2 \frac{d^2z}{dy^2} = (a^2 - x^2)\frac{d^2z}{dx^2} - \frac{a^2}{x} \cdot \frac{dz}{dx} = z.$$

17. $b = e^t \cdot \frac{dt}{dx}$, $\therefore \frac{dt}{dx} = \frac{b}{a+bx}$,

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{b}{a+bx}, \text{ or } (a+bx)\frac{dy}{dx} = b\frac{dy}{dt};$$

$$\therefore (a+bx)\frac{d^2y}{dx^2} + b\frac{dy}{dx} = b\frac{d^2y}{dt^2} \cdot \frac{b}{a+bx},$$

$$\therefore (a+bx)^2 \cdot \frac{d^2y}{dt^2} + A(a+bx)\frac{dy}{dx} + By - F(x)$$

$$= b^2\frac{d^2y}{dt^2} + (A-b)\,b\frac{dy}{dt} + By - F\left(\frac{e^t-a}{b}\right) = 0.$$

18. By a continuation of the process in Arts. 202—205.

Thus by Art. 205 (1) $\frac{dz}{dx} = \cos\theta\frac{dz}{dr} - \frac{\sin\theta}{r} \cdot \frac{dz}{d\theta}$, and $$\frac{d^2z}{dx^2}=\left(\cos\theta\frac{d}{dr}-\frac{\sin\theta}{r}\frac{d}{d\theta}\right)\left(\cos\theta\frac{dz}{dr}-\frac{\sin\theta}{r}\frac{dz}{d\theta}\right)$$

$$=\cos^2\theta\frac{d^2z}{dr^2}+\frac{\sin\theta\cos\theta}{r^2}\cdot\frac{dz}{d\theta}-\frac{\sin\theta\cos\theta}{r}\cdot\frac{d^2z}{drd\theta}$$

$$+\frac{\sin^2\theta}{r}\cdot\frac{dz}{dr}-\frac{\sin\theta\cos\theta}{r}\cdot\frac{d^2z}{drd\theta}+\frac{\sin\theta\cos\theta}{r^2}\cdot\frac{dz}{d\theta}$$

$$+\frac{\sin^2\theta}{r^2}\cdot\frac{d^2z}{d\theta^2}=(A+B)\cos^2\theta+\sin 2\theta\,(-C)+\sin^2\theta\,(A-B)$$

$$=A+B\cos 2\theta-C\sin 2\theta.$$

Putting $\frac{\pi}{2}-\theta$ for θ, $\frac{d^2z}{dy^2}=A-B\cos 2\theta-\sin 2\theta\,(-C)$;

$\because$ C changes sign with θ.

Also $$\frac{d^2z}{dydx}=\left(\sin\theta\cdot\frac{d}{dr}+\frac{\cos\theta}{r}\cdot\frac{d}{d\theta}\right)\left(\cos\theta\frac{dz}{dr}-\frac{\sin\theta}{r}\cdot\frac{dz}{d\theta}\right)$$

$$=\frac{\sin 2\theta}{2}\cdot\frac{d^2z}{dr^2}+\frac{\sin^2\theta}{r^2}\cdot\frac{dz}{d\theta}-\frac{\sin^2\theta}{r}\cdot\frac{d^2z}{drd\theta}-\frac{\sin 2\theta}{2r}\cdot\frac{dz}{dr}$$

$$+\frac{\cos^2\theta}{r}\cdot\frac{d^2z}{drd\theta}-\frac{\cos^2\theta}{r^2}\cdot\frac{dz}{d\theta}-\frac{\sin 2\theta}{2r^2}\cdot\frac{d^2z}{d\theta^2}$$

$$=\sin 2\theta\left(\frac{A+B}{2}-\frac{A-B}{2}\right)+C\cos 2\theta,\ \&c.$$

19. $\xi=f+a_1x+a_2y+a_3z$, $\eta=g+b_1x+b_2y+b_3z$, $\zeta=h+c_1x+c_2y+c_3z$; and $a_1{}^2+a_2{}^2+a_3{}^2=1$, and 2 similar equations hold, also

$$a_1b_1+a_2b_2+a_3b_3=0,$$

and 2 similar equations. See Frost's *Solid Geometry*.

Hence $$\frac{d\phi}{dx}=a_1\cdot\frac{d\phi}{d\xi}+b_1\frac{d\phi}{d\eta}+c_1\frac{d\phi}{d\zeta},$$

and $\frac{d^2\phi}{dx^2}=\left(a_1\frac{d}{d\xi}++\right)\frac{d\phi}{dx}=a_1{}^2\frac{d^2\phi}{d\xi^2}+b_1{}^2\frac{d^2\phi}{d\eta^2}+c_1{}^2\frac{d^2\phi}{d\zeta^2}+2b_1c_1\frac{d^2\phi}{d\eta d\zeta}+2$ similar quantities, whence $\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}$ reduces, by virtue of the relations among the constants, to $\frac{d^2\phi}{d\xi^2}+\frac{d^2\phi}{d\eta^2}+\frac{d^2\phi}{d\zeta^2}$, $\therefore$ &c.

20. By the 2 given equations, x and z are functions of y,

$$\therefore\ \frac{dx}{dy}=e^z\left(1+y\frac{dz}{dy}\right),\ \therefore\ 1+y\frac{dz}{dy}=\frac{y}{x\frac{dy}{dx}};$$

$$\therefore\ y\frac{d^2z}{dy^2}+\frac{dz}{dy}=\frac{1}{x\frac{dy}{dx}}+y\frac{dx}{dy}\cdot\frac{d}{dx}\left(\frac{1}{x\frac{dy}{dx}}\right)$$

$$=\frac{1}{x\frac{dy}{dx}}-\frac{y}{\frac{dy}{dx}}\left(\frac{1}{x^2\frac{dy}{dx}}+\frac{\frac{d^2y}{dx^2}}{x\left(\frac{dy}{dx}\right)^2}\right)$$

$$=\frac{y}{x^2\left(\frac{dy}{dx}\right)^3}\left\{\frac{x}{y}\cdot\left(\frac{dy}{dx}\right)^2-\frac{dy}{dx}-x\frac{d^2y}{dx^2}\right\}=0.$$

21. If x and y be each a function of t, so is s, and $\frac{dx}{dt}=\frac{dx}{ds}\cdot\frac{ds}{dt}$;

thus $$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{ds}{dt}\right)^2,$$

and $$\frac{d^2x}{dt^2}=\frac{dx}{ds}\cdot\frac{d^2s}{dt^2}+\left(\frac{ds}{dt}\right)^2\cdot\frac{d^2x}{ds^2},$$

and from the 2nd given equation $\frac{dx}{ds}\cdot\frac{d^2x}{ds^2}+\frac{dy}{ds}\cdot\frac{d^2y}{ds^2}=0,$

$$\therefore\ \left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2=\left(\frac{d^2s}{dt^2}\right)^2\left\{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right\}+\left(\frac{ds}{dt}\right)^4\cdot u, \text{ \&c.}$$

22. $e^x=\sec\theta$, $\therefore\ \frac{dx}{d\theta}=\tan\theta$, and $\frac{dy}{d\theta}=\tan\theta\cdot\frac{dy}{dx}$;

$$\therefore\ \frac{d^2y}{d\theta^2}=\sec^2\theta\frac{dy}{dx}+\tan^2\theta\cdot\frac{d^2y}{dx^2};$$

$$\therefore\ 0=\tan^2\theta\frac{d^2y}{dx^2}+\sec^2\theta\frac{dy}{dx}-\sec\theta\operatorname{cosec}\theta\cdot\tan\theta\frac{dy}{dx}+n^2y\tan^2\theta,$$

or $$\frac{d^2y}{dx^2}+n^2y=0.$$

23. $\frac{dy}{d\theta}=-e^{-\theta}$, and $\frac{d\theta}{dx}=\sec\theta$, $\therefore\ \frac{dy}{dx}=-e^{-\theta}\cdot\sec\theta$;

$$\therefore\ \frac{d^2y}{dx^2}=\sec\theta\frac{d}{d\theta}(-e^{-\theta}\sec\theta)=e^{-\theta}\sec^2\theta\,(1-\tan\theta);$$

$$\therefore\ \frac{d^3y}{dx^3}=\sec\theta e^{-\theta}\left\{-\sec^2\theta\,(1-\tan\theta)+2\sec^2\theta\tan\theta\,(1-\tan\theta)-\sec^4\theta\right\}$$

$$=e^{-\theta}\cdot\sec^3\theta\,(-1+3\tan\theta-2\tan^2\theta-\sec^2\theta)$$

$$=e^{-\theta}\cdot\sec^5\theta\,(3\sin\theta\cos\theta-\sin^2\theta-2).$$

24. Here $\frac{ds}{dx}=e^x$; $\frac{dt}{dx}=-e^{-x}$, $\therefore\ \frac{du}{dx}=e^x\cdot\frac{du}{ds}-e^{-x}\cdot\frac{du}{dt}$;

$$\therefore\ \frac{d^2u}{dx^2}=e^x\cdot\frac{du}{ds}+e^{-x}\cdot\frac{du}{dt}+\left(e^{2x}\cdot\frac{d^2u}{ds^2}-\frac{d^2u}{dsdt}\right)-e^{-x}\left(e^x\frac{d^2u}{dsdt}-e^{-x}\frac{d^2u}{dt^2}\right)$$

$$=e^{2x}\frac{d^2u}{ds^2}-2\frac{d^2u}{dsdt}+e^{-2x}\frac{d^2u}{dt^2}+e^x\cdot\frac{du}{ds}+e^{-x}\frac{du}{dt};$$ so for $\frac{d^2u}{dy^2}$ writing y for x;

also
$$\frac{d^2u}{dxdy}=\frac{d}{dx}\left(e^y\cdot\frac{du}{ds}-e^{-y}\cdot\frac{du}{dt}\right)$$

$$=e^{x+y}\frac{d^2u}{ds^2}-e^{+x-y}\cdot\frac{d^2u}{dsdt}-e^{-x+y}\cdot\frac{d^2u}{dsdt}+e^{-(x+y)}\cdot\frac{d^2u}{dt^2};$$

thus
$$\frac{d^2u}{dx^2}+2\frac{d^2u}{dxdy}+\frac{d^2u}{dy^2}=\frac{d^2u}{ds^2}(e^{2x}+2e^{x+y}+e^{2y})-2\frac{d^2u}{dsdt}(2+e^{x-y}+e^{-x+y})$$

$$+\frac{d^2u}{dt^2}(e^{-2x}+e^{-2y}+2e^{-(x+y)})+s\cdot\frac{du}{ds}+t\cdot\frac{du}{dt}=\&c.$$

25. $\frac{dx}{d\theta}=ae^\theta\cdot\cos\phi=x$, $\frac{dx}{d\phi}=-ae^\theta\cdot\sin\phi=-y$, $\frac{dy}{d\theta}=y$, $\frac{dy}{d\phi}=x$,

$$\therefore\ \frac{du}{d\theta}=x\frac{du}{dx}+y\frac{du}{dy},\quad \frac{du}{d\phi}=-y\frac{du}{dx}+x\frac{du}{dy};$$

$$\therefore\ \frac{d^2u}{d\phi^2}=-y\left(-y\frac{d^2u}{dx^2}+\frac{du}{dy}+x\frac{d^2u}{dxdy}\right)+x\left(-\frac{du}{dx}-y\frac{d^2u}{dxdy}+x\frac{d^2u}{dy^2}\right);$$

$$\therefore\ y^2\frac{d^2u}{dx^2}-2xy\frac{d^2u}{dxdy}+x^2\frac{d^2u}{dy^2}=\frac{d^2u}{d\phi^2}+x\frac{du}{dx}+y\frac{du}{dy}=\frac{d^2u}{d\phi^2}+\frac{du}{d\theta}.$$

CHAPTER XIII.

1. Let $u=x^5-5x^4+5x^3-1$, then $\frac{du}{dx}=0$ gives $x^4-4x^3+3x^2=0$, i.e. $x=0$ or $x^2-4x+3=0$, i.e. $x=0$ or 3 or 1: again $\frac{d^2u}{dx^2}=5(4x^3-12x^2+6x)$; $\therefore$ when $x=0$, $\frac{d^2u}{dx^2}=5\cdot 0=0$, which is neither positive nor negative, and $\frac{d^3u}{dx^3}=5\cdot 6$, when $x=0$, which does not vanish; $\therefore$ $x=0$ gives neither a maximum nor minimum.

When $x=3$, $\frac{d^2u}{dx^2}=5(4\cdot 27-12\cdot 9+6\cdot 3)=15(36-36+6)$, which is positive, $\therefore$ $x=3$ gives a minimum.

When $x=1$, $\frac{d^2u}{dx^2}=5(4-12+6)$, which is negative, $\therefore$ $x=1$ gives a maximum.

If (compare Art. 222) x be itself a maximum or minimum, that is $+\infty$ or $-\infty$, then u is accordingly a maximum or minimum.

2. If $u=x^3-3x^2+3x+7$, $\frac{du}{dx}=3x^2-6x+3$, which is satisfied by $x=1$, and then $\frac{d^2u}{dx^2}=6x-6=0$, and $\frac{d^3u}{dx^3}=6$, $\therefore$ neither maximum nor minimum; of course u is a maximum when $x=+\infty$, and a minimum when $x=-\infty$.

3. $\frac{du}{dx}=3x^2-6x+6=3(\overline{x-1}|^2+3)$ which cannot vanish, but $x=\pm\infty$ give maxima and minima values.

4. $\frac{du}{dx}=3x^2-18x+15$, $\therefore$ for maxima or minima values besides ∞ values of x, $x^2-6x+5=0$, i.e. $x=5$ or 1, and $\frac{d^2u}{dx^2}=6x-18=6(x-3)$; $\therefore$ $x=5$ gives a minimum value $\left(\text{making } \frac{d^2u}{dx^2} \text{ positive}\right)$, and $x=1$ gives a maximum.

5. $$\frac{du}{dx}=u\left(\frac{4}{x-1}+\frac{3}{x+2}\right)=(7x+5)(x-1)^3(x+2)^2;$$

$$\therefore \text{ for maxima or minima } x=-\frac{5}{7} \text{ or } 1 \text{ or } -2;$$

and $$\frac{d^2u}{dx^2}=7(x-1)^3(x+2)^2+(7x+5)\,.\,3(x-1)^2(x+2)^2$$
$$+(7x+5)(x-1)^3\,.\,2(x+2);$$

$\therefore$ $x=-\frac{5}{7}$ makes $\frac{d^2u}{dx^2}=7(x-1)^3(x+2)^2=7(x+2)^2\left(-\frac{12}{7}\right)^3$ which is negative, $\therefore$ $x=-\frac{5}{7}$ gives a maximum.

Also $x=1$ makes $\frac{d^2u}{dx^2}=0$,

$$\frac{d^3u}{dx^3}=0, \text{ and } \frac{d^4u}{dx^4}=6(x+2)^2(7x+5)+\text{terms which vanish},$$

$$\therefore \frac{d^4u}{dx^4} \text{ is positive, and } \therefore x=1 \text{ gives a minimum.}$$

When $x=-2$, $\frac{d^2u}{dx^2}=0$, and $\frac{d^3u}{dx^3}=(7x+5)(x-1)^3\,.\,2+\text{terms which vanish}$, $\therefore$ $\frac{d^3u}{dx^3}$ does not vanish, and $\therefore$ there is neither maximum nor minimum.

6. $\frac{du}{dx}=\frac{2}{3}\,.\,x^{-\frac{1}{3}}(7-x)^2-2(1+x^{\frac{2}{3}})(7-x)$, $\therefore$ $\frac{du}{dx}=0$ gives

$$x=7 \text{ or } 7-x=3x^{\frac{1}{3}}+3x, \text{ i.e. } 4\overline{x-1}+3(x^{\frac{1}{3}}-1)=0,$$

i.e. $x=1$ or imaginary values: also $\frac{du}{dx}=\infty$, if $x=0$.

Thus $$\frac{d^2u}{dx^2} = -\frac{2}{9}x^{-\frac{4}{3}}(7-x)^2 - \frac{8}{3}x^{-\frac{1}{3}}(7-x) + 2(1+x^{\frac{2}{3}});$$

$\therefore$ $x=7$ makes $\frac{d^2u}{dx^2}$ positive, or gives a minimum;

$x=1$, makes $\frac{d^2u}{dx^2} = -8-16+4$ which is negative, and $\therefore$ gives a maximum;

and $x=0$, makes $\frac{d^2u}{dx^2}=2$, and $\therefore$ gives a minimum.

7. $\frac{du}{dx}=0$, $\therefore$ $3x^4-75x^2+432=0$ or $x^4-25x^2+144=0$;

$\therefore$ $x^2=9$ or 16; and $\frac{d^2u}{dx^2}=15\,(4x^3-50x)=30\,.\,x\,(2x^2-25)$;

thus $x=3$ makes $\frac{d^2u}{dx^2}$ negative, and gives a maximum,

...... $x=-3$.............. positive, ... minimum,

...... $x=4$ .... positive, ... minimum,

...... $x=-4$.............. negative, ... maximum.

8. $u=\frac{2}{1+x-x^2}-1$, $\therefore$ is a maximum or minimum as $1+x-x^2=v$, say, is a minimum or maximum; and $\frac{dv}{dx}=1-2x$, $\frac{d^2v}{dx^2}=-2$, $\therefore$ $x=\frac{1}{2}$ makes u a minimum.

9. $u=\frac{(x-10)(x+3)+36}{x-10}=x+3+\frac{36}{x-10}$, $\therefore$ $\frac{du}{dx}=1-\frac{36}{(x-10)^2}$, $\therefore$ $\frac{du}{dx}=0$ gives $x=16$ or 4, and $\frac{d^2u}{dx^2}=\frac{72}{(x-10)^3}$; thus $x=16$ gives a minimum and $x=4$ a maximum.

10. $x=0, 1, 2$ or 3; and as in one differentiation of $\frac{du}{dx}$, no factor will disappear, $\frac{d^2u}{dx^2}=0$ for each of the values of x: and when $x=0$, $\frac{d^3u}{dx^3}=0$ and $\frac{d^4u}{dx^4}=6\,(x-1)^2(x-2)^3(x-3)^4$ when $x=0$, $\therefore$ $\frac{d^4u}{dx^4}=-6\,.\,3^4\,.\,2^3$, or $x=0$ gives a maximum. When $x=1$, $\frac{d^3u}{dx^3}$ has no factor $(x-1)$ in one term, $\therefore$ is not zero, and $\therefore$ there is no maximum or minimum. When $x=2$, $\frac{d^3u}{dx^3}=0$, and $\frac{d^4u}{dx^4}=$ terms which vanish $+\,6x^3(x-1)^2(x-3)^4$, $\therefore$ there is a minimum. When $x=3$,

the 1st differential coefficient which does not vanish is $\frac{d^5u}{dx^5}$, $\therefore$ there is neither maximum nor minimum.

11. $x=1$, 2, or 3: and when $x=1$, $\frac{d^2u}{dx^2}=(1-2)^3(1-3)^2=$ a negative quantity, $\therefore$ there is a maximum. When $x=2$, $\frac{d^3u}{dx^3}$ is not $=0$, $\therefore$ neither maximum nor minimum. When $x=3$, $\frac{d^3u}{dx^3}=0$, and $\frac{d^4u}{dx^4}=6\,(3-1)\,(3-2)^2$, $\therefore$ a minimum.

12. $$\frac{du}{dx}=u\left(\frac{1}{x}+\frac{2}{a+x}-\frac{3}{a-x}\right)=(a+x)\,(a-x)^2\,\{a^2-x^2-x\,(5x+a)\}$$
$$=(a+x)\,(a-x)^2\,(a+2x)\,(a-3x).$$

When $x=-a$, $\frac{d^2u}{dx^2}=(2a)^2\,(-a)\,.\,4a$, $\therefore$ a maximum. When $x=a$, $\frac{d^2u}{dx^2}=0$, but $\frac{d^3u}{dx^3}$ is not 0, $\therefore$ neither. When $x=-\frac{a}{2}$, $\frac{d^2u}{dx^2}=2\,.\,\frac{a}{2}\,.\left(\frac{3a}{2}\right)^2.\,\frac{5a}{2}$, $\therefore$ a minimum. When $x=\frac{a}{3}$, $\frac{d^2u}{dx^2}=-3\,.\,\frac{4a}{3}\,.\left(\frac{2a}{3}\right)^2.\,\frac{5a}{3}$, $\therefore$ a maximum.

13. $\frac{du}{dx}=u\left(\frac{-3}{a-x}+\frac{2}{a-2x}\right)$, $\therefore 2\,(x-a)=3\,(2x-a)$ or $x=\frac{a}{4}$, and then

$$\frac{d^2u}{dx^2}=\frac{du}{dx}\cdot\left(-\frac{3}{a-x}+\frac{2}{a-2x}\right)-u\left\{\frac{3}{(a-x)^2}-\frac{4}{(a-2x)^2}\right\}=-\frac{u}{a^2}\left\{\frac{3}{\left(\frac{3}{4}\right)^2}-\frac{4}{\left(\frac{1}{2}\right)^2}\right\}$$
$$=-\frac{u}{a^2}\left(\frac{16}{3}-16\right),\ \therefore \text{ a minimum.}$$

14. If x be changed from a to $a\pm h$, $u=b+ch^{\frac{2}{3}}$, $\therefore u$ is altered by a quantity whose sign is the same as that of c, $\therefore x=a$ gives a minimum or maximum of u as c is positive or negative.

15. $\frac{du}{dx}=-\frac{a^2}{x^2}+\frac{b^2}{(a-x)^2}$, $\therefore a\,(a-x)=\pm bx$, and $x=\frac{a^2}{a\pm b}$.

Also $$\frac{d^2u}{dx^2}=\frac{2a^2}{x^3}+\frac{2b^2}{(a-x)^3}=\frac{2\,(a\pm b)^3}{a^4}\pm\frac{2b^2\,(a\pm b)^3}{a^3b^3}=\frac{2\,(a\pm b)^3\,(b\pm a)}{a^4b},$$

$\therefore x=\frac{a^2}{a+b}$ gives a minimum or maximum as b is positive or negative; and $x=\frac{a^2}{a-b}$, a maximum or minimum as b is positive or negative.

16. If $x^2=z$, and $a^2=c$, $u=\frac{3z-c}{(z+c)^3}$, $\therefore \frac{du}{dz}=\frac{3}{(z+c)^3}-\frac{3(3z-c)}{(z+c)^4}$;

$$\therefore \frac{du}{dz}=0 \text{ when } z+c=3z-c, \text{ or } z=c, \text{ and } \therefore x=\pm a; \text{ and}$$

$$\frac{d^2u}{dz^2}=-\frac{9 \cdot 2}{(z+c)^4}+\frac{12(3z-c)}{(z+c)^5}, \therefore \text{if } z=c,\ \frac{d^2u}{dz^2}=\frac{1}{(2c)^4}(-18+12),$$

$\therefore x=\pm a$ gives a maximum. Also z has a minimum value $=0$, and if z be small (it must be positive), $u=\frac{3z-c}{c^3}\left(1-\frac{3z}{c}\right)$ nearly $=\frac{-c+6z}{c^3}$, or u is increased, and $\therefore x=0$ gives a minimum.

17. $\frac{du}{dx}=(m+n)\,m\,(mx+na)^{m+n-1}-(m+n)^{m+n}\,.\,mx^{m-1}\,.\,a^n$,

$$\therefore 0=m(m+n)\left\{(mx+na)^{m+n-1}-(ma+na)^{m+n-1}\right\} + m(m+n)^{m+n}\,.\,a^n(a^{m-1}-x^{m-1});$$

$\therefore x=a$ is a solution, and gives $\frac{d^2u}{d^2x}$

$$=m^2(m+n)(m+n-1)(m+n)^{m+n-2}\,.\,a^{m+n-2}-(m+n)^{m+n}\,.\,m(m-1)\,a^{m+n-2}$$

$$=a^{m+n-2}\,.\,m(m+n)^{m+n-1}\,.\,\{m\,\overline{m-1}+mn-m\,\overline{m-1}-n\,\overline{m-1}\}$$

$$=mn(m+n)^{m+n-1}\,.\,a^{m+n-2};$$

$\therefore$ if all the quantities are positive, $x=a$ gives a minimum.

18. If $u=\frac{1}{x}+\tan x$, for maximum or minimum $\frac{du}{dx}=0=-\frac{1}{x^2}+\sec^2 x$,

$\therefore x=\pm\cos x$, and $\frac{d^2u}{dx^2}=\frac{2}{x^3}+2\sec^2 x\,.\,\tan x$,

$$\therefore \text{when } x=\cos x,\ \frac{d^2u}{dx^2}=\frac{2}{x^3}(1+\sqrt{1-x});$$

$\therefore u$ is a minimum or maximum as x is positive or negative, and $\frac{x}{1+x\tan x}$ a maximum or minimum, and clearly $x=\cos x$ is only true for some positive value of x less than the unit of circular measure, $\therefore$ &c.

So $x=-\cos x$ gives a minimum.

19. If $u=x^{\frac{1}{x}}$, $\log u=\frac{1}{x}\log x$, $\therefore \frac{1}{u}\cdot\frac{du}{dx}=-\frac{1}{x^2}\,.\,\log x+\frac{1}{x^2}$,

$$\therefore \log x=1 \text{ or } x=e, \text{ and } -\frac{1}{u^2}\cdot\left(\frac{du}{dx}\right)^2+\frac{1}{u}\cdot\frac{d^2u}{dx^2}=\frac{2}{x^3}\log x-\frac{3}{x^3}=-\frac{1}{e^3},$$

$\therefore x=e$ gives a maximum.

20. Put $u=\frac{\tan^3 x}{\tan 3x}=z^3\div\frac{3z-z^3}{1-3z^2}=\frac{z^2(3z^2-1)}{z^2-3}=\frac{3y^2-y}{y-3}$

$$=\frac{3y^2-9y+8y-24+24}{y-3}=3y+8+\frac{24}{y-3},$$

$$\therefore \frac{du}{dy}=3-\frac{24}{(y-3)^2}, \therefore \text{ for maximum } y-3=\pm 2\sqrt{2}, \text{ and } \frac{d^2u}{dy^2}=\frac{24\,.\,2}{(y-3)^3},$$

$$\therefore y=3-2\sqrt{2} \text{ gives a maximum, } \therefore z^2=3-2\sqrt{2},$$

and $$z=\sqrt{2}-1=\tan\frac{\pi}{8}, \text{ or } x=\frac{\pi}{8}.$$

21. If $u=\sin x\,(1+\cos x), \dfrac{du}{dx}=\cos x\,(1+\cos x)-\sin^2 x$, $\therefore$ for maximum

$$2\cos^2 x+\cos x-1=0=(2\cos x-1)(\cos x+1), \therefore x=\frac{\pi}{3} \text{ or } \pi,$$

and $$\frac{d^2u}{dx^2}=-\sin x-2\sin 2x, \therefore x=\frac{\pi}{3} \text{ makes } u \text{ a maximum.}$$

22. $xy^2-x^2y=2a^3$, and $\dfrac{dy}{dx}=0$; thus $y^2-2xy=0$, $\therefore y=2x=2a$,

and $$y^2-2xy+(2xy-x^2)\frac{dy}{dx}=0, \therefore -2y+(2xy-x^2)\frac{d^2y}{dx^2}=0,$$

$$\therefore \frac{d^2y}{dx^2}=\frac{4a}{3a^2}, \therefore x=a \text{ gives a minimum.}$$

23. $y^3+12ax^2+(6a^2y+3xy^2)\dfrac{dy}{dx}=0$, $\therefore y^3+12ax^2=0$, $\therefore 3a^2y^2=8ax^3$,

$$\therefore 27a^3\,.\,y^6=27a^3\,.\,3^2\,.\,4^2\,.\,a^2x^4=2^9\,.\,x^9, \therefore (2x)^5=(3a)^5 \text{ or } x=\frac{3a}{2},$$

and $$\therefore y=-(27a^3)^{\frac{1}{3}}=-3a, \text{ and } 24ax+(6a^2y+3xy^2)\frac{d^2y}{dx^2}=0,$$

$$\therefore \frac{d^2y}{dx^2}=\frac{-24a\,.\,\frac{3a}{2}}{\left(-18+27\,.\,\frac{3}{2}\right)a^3}=-\frac{8}{5a}, \therefore x=\frac{3a}{2} \text{ gives a maximum of } y.$$

24. $4x^3+4axy+(2ax^2-3ay^2)\dfrac{dy}{dx}=0$, $\therefore x^2+ay=0$,

$$\therefore a^2y^2-2a^2y^2-ay^3=0, \therefore y=0 \text{ or } -a, \text{ and if } y=-a,\ x=\pm a;$$

and $$12x^2+4ay+(2ax^2-3ay^2)\frac{d^2y}{dx^2}=0, \therefore \frac{d^2y}{dx^2}=\frac{-8a^2}{-a^3}, \therefore y \text{ is a minimum.}$$

(2) When x is a maximum, $(2ax^2-3ay^2)+(4x^3+4axy)\dfrac{dx}{dy}=0$, $\therefore 2x^2=3y^2$,

$$\therefore \frac{9}{4}y^4+3ay^3=ay^3, \therefore y=0 \text{ or } -\frac{8}{9}a, \text{ and if } y=-\frac{8}{9}a,$$

$$x=\pm y\sqrt{\frac{3}{2}}=\mp\frac{4a\sqrt{6}}{9};$$

also $-6ay+(4x^3+4axy)\dfrac{d^2x}{dy^2}=0$, $\therefore\ \dfrac{d^2x}{dy^2}=-\dfrac{16}{3}a^2\div x\left(6\,.\,\overline{\tfrac{8}{9}}\big|^2\,.\,a^2-\dfrac{32}{9}a^2\right)$,
and $8^2\,.\,6-9\,.\,32=3\,.\,32$; $\therefore$ x is a maximum or minimum as it is positive or negative, when $y=-\dfrac{8}{9}a$.

25. $10x^4-2xy^3+(12ay^3-3x^2y^2)\dfrac{dy}{dx}=0$, $\therefore\ x=0$ or $y=x\,.\,5^{\frac{1}{3}}$, and

$$\therefore\ 2x^5-x^5\,.\,5+3ax^4\,.\,5^{\frac{4}{3}}=0,\ \therefore\ x=a\,.\,5^{\frac{4}{3}},\text{ and } y=a\,.\,5^{\frac{5}{3}}.$$

Also $$40x^3-2y^3+(12ay^3-3x^2y^2)\frac{d^2y}{dx^2}=0,$$

$\therefore$ when $x=a\,.\,5^{\frac{4}{3}}$, $\dfrac{d^2y}{dx^2}=-30x^3\div(60ax^3-3x^4\,.\,5^{\frac{2}{3}})=-\dfrac{10}{a}\div(20-5^2)$;
$\therefore$ a minimum.

26. $4(y^3-c^2x)\dfrac{dy}{dx}-4(c^2y-x^3)=0$, $\therefore\ c^2y=x^3$,
$\therefore\ y^4-4(c^2y)^{\frac{4}{3}}+(c^2y)^{\frac{4}{3}}=0$, $\therefore\ y^8=27c^8$, i.e. $y=\pm c\sqrt[8]{27}$, and $\therefore x=\pm c\sqrt[8]{3}$.

Also $(y^3-c^2x)\dfrac{d^2y}{dx^2}+3x^2=0$, $\therefore\ \dfrac{d^2y}{dx^2}=-3x^2\div c^3(\pm\sqrt[8]{(27)^3}\mp\sqrt[8]{3})$;
$\therefore\ x=c\sqrt[8]{3}$ gives a maximum, and $x=-c\sqrt[8]{3}$ gives a minimum.

27. If he be at A, B the nearest point of the beach (straight), C the point to be reached, X the landing-place, $BX=x$, t the time along AX and XC, then $t=\dfrac{AX}{4}+\dfrac{CX}{5}=\dfrac{1}{4}\cdot\sqrt{x^2+9}+1-\dfrac{x}{5}$, $\therefore$ for minimum of t,

$$\frac{1}{4}\cdot\frac{x}{\sqrt{x^2+9}}=\frac{1}{5},\text{ i.e. } 25x^2=16(x^2+9),\ \therefore\ x=\pm4,$$

corresponding to the positive and negative signs of $\sqrt{x^2+9}$, $\therefore$ $x=4$, only, applies; and this gives $CX=1$ mile.

Also $4\dfrac{d^2t}{dx^2}=\dfrac{1}{\sqrt{x^2+9}}-\dfrac{x^2}{(x^2+9)^{\frac{3}{2}}}=\dfrac{9}{(x^2+9)^{\frac{3}{2}}}$, which is positive as the positive sign of the radical has to be taken, and $\therefore$ the solution gives a minimum.

28. If a side of the circumscribing rectangle make an angle θ with a side a of the given rectangle, the sides of the circumscribing rectangle are $(a\cos\theta+b\sin\theta)$ and $(a\sin\theta+b\cos\theta)$, $\therefore$ the maximum is required of $(a\cos\theta+b\sin\theta)(a\sin\theta+b\cos\theta)$ which $=(a^2+b^2)\,.\,\sin\theta\,.\,\cos\theta+ab$, and this is greatest when $2\theta=\dfrac{\pi}{2}$, and $\therefore$ the sides of the outer rectangle are $\dfrac{a+b}{\sqrt{2}}$ each, and it is $\therefore$ a square. When $\theta=0$, there is a minimum, i.e. the rectangle itself.

29. If x = the side, the content of the box (of depth x, and without top) is $x(a-2x)(b-2x)=u$ say,

$$\therefore \frac{du}{dx}=(a-2x)(b-2x)-x\{2(b-2x)+2(a-2x)\}=12x^2-4x(a+b)+ab,$$

$$\therefore \text{ for maximum } \left(x-\frac{a+b}{6}\right)^2=\left(\frac{a+b}{6}\right)^2-\frac{ab}{12}, \text{ and } x=\frac{a+b\pm\sqrt{a^2-ab+b^2}}{6}.$$

Also $\frac{d^2u}{dx^2}=4(6x-\overline{a+b})$, $\therefore$ the negative sign of the radical gives the maximum of u, i.e. $x=\frac{a+b-\sqrt{a^2-ab+b^2}}{6}$.

30. If r = radius of semi-circle, h the height of the rectangle, a = the perimeter, and u = the size of the window; then $a=2h+2r+\pi r$,

$$u=\frac{\pi r^2}{2}+2rh, \quad \therefore u=\frac{\pi r^2}{2}+r(a-2r-\pi r),$$

$$\therefore \text{ for maximum } 0=\pi r+a-4r-2\pi r, \quad \therefore r=\frac{a}{\pi+4},$$

$$\therefore 2h=a-\frac{\pi+2}{\pi+4}\cdot a=\frac{2a}{\pi+4}, \quad \therefore h=r. \quad \text{Also } \frac{d^2u}{dr^2}=-4-\pi,$$

$\therefore$ the result is a maximum.

31. If PQR be a circumscribing equilateral triangle, P corresponding to A &c., and $\angle ACQ=\theta$, u = height of PQR; then

$$CQ=\frac{b\sin\left(\theta+\frac{\pi}{3}\right)}{\sin\frac{\pi}{3}}, \quad CP.\sin\frac{\pi}{3}=a\sin\left(\frac{\pi}{3}+\pi-\theta-C\right),$$

and $$\therefore u=(CP+CQ).\sin\frac{\pi}{3}=b\sin\left(\theta+\frac{\pi}{3}\right)+a\sin\left(\theta+C-\frac{\pi}{3}\right),$$

$$\therefore \text{ for maximum } \frac{du}{d\theta}=0=b.\cos\left(\theta+\frac{\pi}{3}\right)+a\cos\left(\theta+C-\frac{\pi}{3}\right),$$

$$\therefore \text{ squaring and adding, } u^2=b^2+a^2+2ab\cos\left(C-\frac{2\pi}{3}\right)$$

$$=a^2+b^2-2ab\cos\left(C+\frac{\pi}{3}\right).$$

Also $\frac{d^2u}{d\theta^2}=-b\sin\left(\theta+\frac{\pi}{3}\right)-a\sin\left(\theta+C-\frac{\pi}{3}\right)=-u$, and is $\therefore$ negative, $\therefore$ the result is a maximum.

There is a minimum of u when one side of the equilateral triangle is wholly or partly coincident with a side of the given triangle, θ then having its greatest or least value consistent with the question and the nature of the given triangle.

32. The axes are supposed rectangular, and OA, OB both positive, as also a and b.

(1) See p. 198. If OB might be negative (or OA) then OP would give a minimum.

(2) $OA+OB=a+b\cot\theta+b+a\tan\theta$, and clearly $\theta=0$ or $\frac{\pi}{2}$ would give a maximum, $\therefore$ for minimum $b\,\text{cosec}^2\theta=a\sec^2\theta$ and $\tan\theta=\sqrt{\frac{b}{a}}$, the negative sign being inapplicable.

(3) $OA\times OB=(a+b\cot\theta)^2\,.\tan\theta=a^2\tan\theta+2ab+b^2\cot\theta$; $\therefore$ for minimum $a^2\sec^2\theta=b^2\,\text{cosec}^2\theta$, and $\tan\theta=\frac{b}{a}$, as in (2), gives the minimum value: and $\theta=0$ and $\frac{\pi}{2}$ give maxima.

(4) $OA+OB+AB=a+b\cot\theta+b+a\tan\theta+\frac{b}{\sin\theta}+\frac{a}{\cos\theta}$; $\therefore$ for minimum $a(\sec^2\theta+\sin\theta\sec^2\theta)=b(\text{cosec}^2\theta+\cos\theta\,.\,\text{cosec}^2\theta)$,

or
$$a\sin^2\theta\,(1+\sin\theta)=b\cos^2\theta\,(1+\cos\theta),$$
$$\therefore\ a(1-\cos\theta)=b(1-\sin\theta),\ \therefore\ (a-b)^2(\tan^2\theta+1)=(a-b\tan\theta)^2,$$
or
$$\tan^2\theta\,(a^2-2ab)+2ab\tan\theta+b^2-2ab=0,$$
$$\therefore\ \tan\theta\,(a^2-2ab)=-ab\pm\sqrt{2ab}\,.\,(a-b)$$
$$=(a\mp\sqrt{2ab})(b\pm\sqrt{2ab}),$$
the ambiguities corresponding, $\therefore$ $\tan\theta=\frac{b\pm\sqrt{2ab}}{a\pm\sqrt{2ab}}$, and the lower signs may make $\tan\theta$ negative, $\therefore$ for minimum $\tan\theta=\frac{b+\sqrt{2ab}}{a+\sqrt{2ab}}$: $\theta=0$ or $\frac{\pi}{2}$ correspond to maxima values.

(5) $OA\times OB\times AB=(a+b\cot\theta)^3\,.\tan\theta\sec\theta$; $\therefore$ for minimnm
$$(a+b\cot\theta)(\sec^3\theta+\sec\theta\tan^2\theta)=3b\,\text{cosec}^2\theta\tan\theta\sec\theta,$$
or
$$(a\tan\theta+b)(2\tan^2\theta+1)=3b\tan^2\theta\,\text{cosec}^2\theta=3b(\tan^2\theta+1),$$
or
$$2a\tan^3\theta-b\tan^2\theta+a\tan\theta-2b=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

The roots of (1) are all positive, if real, and $\theta=0$ or $\frac{\pi}{2}$ gives a maximum. Thus there is one minimnm, *or* there are 2 minima values and another maximum, *as* one or 3 roots of (1) are real.

(6) Here for a minimum

$$(a+b\,.\cot\theta)^{n-1}\,b\,.\,\text{cosec}^2\theta=(a\tan\theta+b)^{n-1}\,.\,a\sec^2\theta;$$

$$\therefore\ \tan^{n+1}\theta=\frac{b}{a},\text{ and the real value of }\tan\theta=\left(\frac{b}{a}\right)^{\frac{1}{n+1}},$$

$$\theta=0\text{ and }\frac{\pi}{2}\text{ give maxima values.}$$

33. If ABP be the triangle, the angle APB being constant, P lies on a fixed circle through A, B; and AB being constant the area is a maximum when P is at the points C and D where CD bisects AB at right angles, and C is equidistant from A and B; so also is D. Practically if C be farther than D is from AB, the maximum area is ACB.

34. Let the quadrilateral be $ABCP$, where $\angle APC=\alpha$, $AB=a$, $BC=c$, and from Ex. 33, $CP=AP=x$ say, and let $\angle ABC=\theta$.

Then $AC^2=2x^2(1-\cos\alpha)=a^2+c^2-2ac\cos\theta$, and twice the area

$$=x^2\sin\alpha+ac\sin\theta=ac\sin\theta+\frac{\sin\alpha}{2(1-\cos\alpha)}(a^2+c^2-2ac\cos\theta);$$

$$\therefore\ \text{for maximum}\ \cos\theta+\frac{\sin\alpha}{1-\cos\alpha}\sin\theta=0,\ \text{or}\ \tan\theta=-\tan\frac{\alpha}{2},\ \therefore\ \theta=\pi-\frac{\alpha}{2},$$

$$\therefore\ 2\theta=2\pi-\alpha=\theta+\angle BAP+BCP,\ \text{or}\ ABC=BAP+BCP.$$

If now $BP>AP$, $BP>CP$, thus $\angle BAP+BCP>ABC$. So if $BP<AP$, $\angle ABC>BAP+BCP$, $\therefore$ BP must $=AP=CP$, i.e. P is equidistant from A, B, C.

35. If any circumscribing ellipse be projected orthogonally into a circle on the minor axis as diameter, the parallelogram must project into a rectangle, and the area of the ellipse is to that of the parallelogram as the area of the circle is to that of the rectangle, $\therefore$ this latter ratio is least when the ellipse is least, but the area of a circle to an inscribed rectangle is clearly least when the rectangle is a square: let r = radius of the circle in this case, then the ratio required is $\pi r^2 : 2r^2=\pi : 2$.

Aliter: analogously by aid of the auxiliary circle. Also the diagonals of the parallelogram bisect each other, $\therefore$ they intersect in the centre of the ellipse, and the diagonals of the square are at right angles, thus the diagonals of the parallelogram are conjugate diameters of the minimum ellipse, whence it can be constructed; for if $2r$, $2r'$ be the diagonals, α the angle between them; $2a$, $2b$ the axes of the ellipse, θ the $\angle$ between $2a$ and $2r$, then a, b are known, and $\dfrac{r^2\cos^2\theta}{a^2}+\dfrac{r^2\sin^2\theta}{b^2}=1$ gives θ.

36. Let C be the centre of the ellipse, and the tangent at P (eccentric angle ϕ) meet the axes of x and y in Q and R; then the equation of QR is $\dfrac{x\cos\phi}{a}+\dfrac{y\sin\phi}{b}=1$, $\therefore$ $CQ=a\sec\phi$, $CR=b\,\text{cosec}\,\phi$ (or from the auxiliary

circle), $\therefore\ QR^2=a^2\sec^2\phi+b^2\operatorname{cosec}^2\phi$, and as QR is a maximum when $\phi=0$ or $\frac{\pi}{2}$, for a minimum $a^2\sec^2\phi\,.\,\tan\phi=b^2\operatorname{cosec}^2\phi\,.\,\cot\phi$;

$$\therefore\ \frac{\sin^2\phi}{b}=\frac{\cos^2\phi}{a}=\frac{1}{a+b},\ \text{and } QR^2=a(a+b)+b(a+b),\ \therefore\ QR=a+b;$$

and $\quad PR:PQ::a\cos\phi:a(\sec\phi-\cos\phi)::\cos^2\phi:\sin^2\phi::a:b,$

$$\therefore\ PR=a,\ PQ=b.$$

37. If QR be the chord the vertex P is clearly such that the tangent to the parabola at P is parallel to QR. If S be the focus, M the centre of QR, and a the angle between it and the axis, the area of the triangle

$$=QM\,.\,PM\,.\sin a,\ \text{and}\ QM^2=4SP\,.\,PM,$$

$$\therefore\ \text{area}=QM^3\,.\,\sin a\div 4SP=QR^3\sin a\div 32\,.\,SP.$$

38. The corresponding minimum triangle on the circle on the minor axis (as diameter) is equilateral, and the heights are the same, as corresponding tangents meet on the minor axis, $\therefore$ height required is x, where

$$\frac{x}{3}=b\ \text{or}\ x=3b.$$

39. If r be the radius and θ the angle of the sector, the perimeter $=r\theta+2r=a$ say, and the area $=\frac{r^2\theta}{2}=\frac{r}{2}(a-2r)$; $\therefore$ for maximum or minimum $\frac{a}{2}=2r$, and then $r\theta=a-2r=2r$. Also r itself may be a maximum or minimum. If θ be restricted to 2π, the least value of $r=\frac{a}{2(\pi+1)}$, and then area $=\frac{\pi a^2}{4(\pi+1)^2}$, but when $\frac{a}{2}=2r$, the area $=r^2=\frac{a^2}{16}$ which $>\frac{\pi a^2}{4(\pi+1)^2}$, $(\pi>3)$; and when r is greatest $\theta=0$, and area $=0$; thus $r=\frac{a}{4}$ gives the greatest area. Negative values are excluded.

40. If PSp make an angle θ with the major axis, $SP=l-eSP\cos\theta$, $Pp=\frac{2l}{1-e^2\cos^2\theta}$, and $\therefore$ area $=\frac{1}{2}HS\,.\,Pp\sin\theta\propto\frac{\sin\theta}{1-e^2\cos^2\theta}$.

If then $u=\frac{\sin\theta}{1-e^2\cos^2\theta}$, for maximum or minimum

$$\frac{du}{d\theta}=0=\frac{\cos\theta}{1-e^2\cos^2\theta}-\frac{\sin\theta\,.\,2e^2\sin\theta\cos\theta}{(1-e^2\cos^2\theta)^2}$$

$$=\frac{\cos\theta\,(1+e^2\cos^2\theta-2e^2)}{(1-e^2\cos^2\theta)^2},\ \therefore\ \theta=\frac{\pi}{2}\ \text{or}\ \cos^2\theta=2-\frac{1}{e^2},$$

the latter of which is only possible if $2e^2>1$.

Now as θ increasing passes through the value $\frac{\pi}{2}$, $\frac{du}{d\theta}$ changes sign from negative to positive or the reverse as $2e^2>$ or not >1, and $\therefore\ \theta=\frac{\pi}{2}$ gives a minimum or maximum accordingly.

Also if $2e^2 > 1$, as θ passes through the least value given by

$$\cos\theta = +\sqrt{2-\frac{1}{e^2}},\ \frac{du}{d\theta} \text{ becomes negative,}$$

$\therefore$ there is a maximum. So by symmetry if $\cos\theta = -\sqrt{2-\frac{1}{e^2}}$. And if $\theta = 0$ or π there is a minimum.

41. Cf. Todhunter's *Conics*, Ch. VIII. Ex. 20. Thus the length of the normal chord at $(x, y) = \frac{2}{y^2}(4a^2+y^2)^{\frac{3}{2}}$; $\therefore$ for minimum, as the vertex and points at an ∞ distance give maximum lengths,

$$\frac{2}{y^3}(4a^2+y^2) = \frac{3y}{y^2},\ \therefore\ y^2 = 8a^2,$$

and $\therefore$ the length $= \frac{1}{4a^2}(12a^2)^{\frac{3}{2}} = \frac{a}{4}\,.\,8\,.\,3\sqrt{3} = 6a\sqrt{3}$.

Also the distance of the point of intersection from the vertex is

$$\left(\frac{8a^2+y^2}{y}\right)\cdot\left\{\frac{1}{16a^2}\left(\frac{8a^2+y^2}{y}\right)^2+1\right\}^{\frac{1}{2}},$$

and such distance is a maximum when (x, y) is the vertex or infinitely distant; $\therefore$ for a minimum of $\frac{8a^2+y^2}{4ay^2}\{(y^2+8a^2)^2+16a^2y^2\}^{\frac{1}{2}}$

$$-\frac{16a^2}{y^3}\{(y^2+8a^2)^2+16a^2y^2\}+\frac{8a^2+y^2}{y^2}\,.\,\{(y^2+8a^2)\,.\,2y+16a^2y\}=0,$$

or $$c^2\{(y^2+c^2)^2+2c^2y^2\}=(y^2+c^2)\,y^2\,(y^2+2c^2),$$

or $$(y^2+c^2)^2(c^2-y^2)+c^2y^2(c^2-y^2)=0, \text{ i.e. } y^2=c^2=8a^2$$

is the only real solution, and $\therefore$ &c.

It may be easily seen from a figure that the shortest normal chord intersects an adjacent normal chord on the curve, and in one of the 2 points where the curve meets its evolute: and the evolute being convex to the axis, any other normal chord meets the curve in a point farther from the vertex. See Chs. XXIV. and XXV. Ex. 5, and cf. Art. 337. Now (Todhunter's *Conics*, Ch. VIII.) the normal at (x', y') is $y = -\frac{y'}{2a}x + y' + \frac{y'^3}{8a^2}$, and if this pass through (h, k) a point on the parabola,

$$k - y' = \frac{y'^3}{8a^2} - \frac{y'}{2a}\cdot\frac{k^2}{4a};$$

thus one normal from (h, k) being that at (h, k), the equation $-1 = \frac{y'(y'+k)}{8a^2}$ has equal roots in y', but $y'^2 + y'k + 8a^2 = 0$, $\therefore$ $k^2 = 32a^2$, or $$k = \pm 4a\sqrt{2}, \text{ and } y' = -\frac{k}{2} = \mp 2a\sqrt{2},$$

and $\therefore$ if the length of the shortest chord be z,

$$z^2 = (k-y')^2 + (h-x')^2$$

$$= (k-y')^2\left\{1 + \left(\frac{k+y'}{4a}\right)^2\right\},$$

or $$z = \frac{6a\sqrt{2}}{4a}\{16a^2 + 8a^2\}^{\frac{1}{2}}$$

$$= 6a\sqrt{2}\,.\sqrt{\frac{3}{2}} = 6a\sqrt{3}.$$

42. If D be the distance of the ships at any (say subsequent) time $+t$,

$$D^2 = (a+ut)^2 + (b+vt)^2 - 2(a+ut)(b+vt)\cos\theta,$$

$\therefore$ for minimum of D, $\because t = \pm\infty$ gives a maximum,

$$(a+ut)u + (b+vt)v = u(b+vt)\cos\theta + v(a+ut)\cos\theta,$$

or $$\frac{a+ut}{u\cos\theta - v} = \frac{b+vt}{u - v\cos\theta}\left(= \frac{av-bu}{2uv\cos\theta - v^2 - u^2}\right),$$

and $\therefore$ $$\pm D = \frac{av-bu}{2uv\cos\theta - v^2 - u^2}$$

$$\times\{(u\cos\theta - v)^2 + (u - v\cos\theta)^2 - 2\cos\theta(u\cos\theta - v)(u - v\cos\theta)\}^{\frac{1}{2}}$$

$$= \frac{-(av-bu)}{u^2+v^2-2uv\cos\theta}\{u^2\sin^2\theta + v^2\sin^2\theta - 2uv\cos\theta(2-\cos^2\theta-1)\}^{\frac{1}{2}},$$

or D, being considered positive,

$$D = \frac{(av \sim bu)\sin\theta}{(u^2+v^2-2uv\cos\theta)^{\frac{1}{2}}}.$$

Aliter: if at the given time the ships be at A, B, and the velocity of the ship at B relative to that at A make an angle ϕ with AB, then obviously the least distance $= AB\sin\phi$ &c.

43. Let A be the vertex $(a, 0)$ say, APQ the straight line meeting the ellipse in P and the circle in Q, $PQ = r$; then $AQ = 2a\cos\theta$,

$$a^2\,.\,AP^2\sin^2\theta + b^2(a - AP\cos\theta)^2 = a^2b^2,$$

or $$AP(a^2\sin^2\theta + b^2\cos^2\theta) = 2ab^2\cos\theta,$$

and $$\therefore r(a^2\sin^2\theta + b^2\cos^2\theta) = 2a\cos\theta(a^2\sin^2\theta - b^2\sin^2\theta)$$

$$= 2a^3e^2\sin^2\theta\cos\theta;$$

$\therefore$ for maximum of r, $r\,.\,a^2e^2\,.\,2\sin\theta\cos\theta = 2a^3e^2(2\sin\theta\cos^2\theta - \sin^3\theta)$,

or $$r\cos\theta = a(2\cos^2\theta - \sin^2\theta);$$

$$\therefore 2a^3\,.\,e^2\sin^2\theta\cos^2\theta = a^3(2\cos^2\theta - \sin^2\theta)(1 - e^2\cos^2\theta),$$

or $$2e^2(\cos^4\theta - \cos^2\theta) = (e^2\cos^2\theta - 1)(3\cos^2\theta - 1),$$

or $e^2\cos^4\theta - \cos^2\theta(3-e^2)+1=0$; now the product of the values of $\cos^2\theta$ from this $=\frac{1}{e^2}$ which >1, $\therefore$ the greater value is impossible, and the lesser value, given by $2e^2\cos^2\theta = 3-e^2-\sqrt{(3-e^2)^2-4e^2} = (3-e^2)-\sqrt{(9-e^2)(1-e^2)}$, is possible if $9(1-e^2)^2 < (3-e^2)^2 - 4e^2$, $\therefore$ if $8e^4 < 8e^2$, which is true. When $\theta=0$ or $\frac{\pi}{2}$, $r=0$, and is $\therefore$ a minimum. Thus the above equation must give the 2 maxima values, there being 2 positions of APQ equally inclined to the major axis.

44. If the ellipse be $a^2y^2+b^2x^2=a^2b^2$, and the circle $x^2+(y+b)^2=r^2$, then $a^2y^2-b^2(y+b)^2+b^2r^2-a^2b^2=0$ has equal roots,

$$\therefore\ b^6=(a^2-b^2)\,b^2(r^2-a^2-b^2),$$

or

$$0=a^2(r^2-a^2)-b^2r^2, \text{ i. e. } b^2=a^2-\frac{a^4}{r^2},$$

and area $=\pi ab$; if then $u=a^2b^2=a^4-\frac{a^6}{r^2}$, for a maximum area, $4a^3=\frac{6a^5}{r^2}$, $\therefore a=0$ which gives a line ellipse area $=0$, *or* $a^2=\frac{2r^2}{3}$, and $\therefore b^2=r^2\left(\frac{2}{3}-\frac{4}{9}\right)$,

and

$$\therefore\ \text{area}=\pi r^2\sqrt{\frac{2}{3}\cdot\frac{2}{9}}=\frac{2\pi r^2}{3\sqrt{3}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also a may be a maximum, and $\therefore =r$, and the ellipse a line ellipse area $=0$, thus (1) gives the maximum area.

45. If a triangle be drawn with the same vertex, and base in the same straight line as the base of the given triangle ABC, and circumscribing the circle on the axis of the ellipse which is perpendicular to the base, then the area of the circle is to that of the described triangle as that of the ellipse is to the area of ABC, and the first ratio is least when the described triangle is equilateral, and then its height which equals that of ABC is 3 times the $\frac{1}{2}$ axis, $\therefore$ &c. The ellipse is a minimum when it is a line-ellipse, and the finite axis coincides with either the base or axis of ABC. Cf. Ex. 38.

46. Let a = radius of the circle, 2θ = angle of sector, r = radius of base of cone, 2ϕ = its vertical angle; then a = slant height $= r\operatorname{cosec}\phi$, surface of cone $=\pi r^2\operatorname{cosec}\phi = a^2(\pi-\theta)$, and volume of cone $= u$ say $=\frac{1}{3}\pi r^3 . \cot\phi$. Thus $r=a\sin\phi$, $\pi-\theta=\pi\sin\phi$, and $\frac{3u}{\pi a^3}=\sin^2\phi . \cos\phi$, $\therefore$ for maximum or minimum of u, $2\sin\phi\cos^2\phi=\sin^3\phi$, $\therefore \phi=0$, when u must $=0$, a minimum, or $3\cos^2\phi=1$, and then $\theta=\pi(1-\sin\phi)=\pi\left(1-\sqrt{\frac{2}{3}}\right)$,

$$\text{i.e. } 2\theta=\frac{2\pi}{\sqrt{3}}(\sqrt{3}-\sqrt{2}).$$

Also $\theta=\pi$ gives surface and $\therefore$ the volume $=0$, a minimum.

47. If PSp, QSq be the chords, and PSp make an angle θ with the major axis, $Pp=\frac{l}{1+e\cos\theta}+\frac{l}{1-e\cos\theta}=\frac{2l}{1-e^2\cos^2\theta}$,

$\therefore\ Qq=\frac{2l}{1-e^2\sin^2\theta}$, $\therefore$ the sum $=\frac{2l(2-e^2)}{1-e^2+e^4\sin^2\theta\cos^2\theta}$,

is a maximum or minimum as $\sin\theta\cos\theta$ is a minimum or maximum, i.e. as $\theta=0$ or $\frac{\pi}{4}$.

Thus when the chords are parallel to the axes their sum is a maximum, and when they make equal angles with each axis their sum is a minimum.

48. Consider a plane section through the axis of the cone and cylinder. If x be the radius of the base and y the height of the cylinder, $y:b::a-x:a$, and $\therefore$ the vol. of the cylinder $=u=\pi x^2\cdot\frac{b}{a}(a-x)$ which is a minimum when $x=a$ or 0, and $\therefore$ for a maximum of u, $2x(a-x)=x^2$, i. e. $x=\frac{2}{3}a$,

$$\text{and } \therefore \text{ vol.}=\frac{\pi b}{a}\cdot\frac{4}{9}a^3\cdot\frac{1}{3}=\frac{4}{27}\pi ba^2.$$

49. The convex surface $=2\pi x\cdot y=2\pi\cdot\frac{b}{a}\cdot x(a-x)$, $\therefore$ for maximum $a-x=x$, i. e. $x=\frac{a}{2}$, and surface $=2\pi\frac{b}{a}\cdot\frac{a^2}{4}=\frac{\pi ba}{2}$: the surface being least when $x=0$ or a.

50. Whole surface $=2\pi xy+2\pi x^2=2\pi x^2+2\pi\frac{b}{a}x(a-x)=u$,

$$\therefore \text{ for maximum } \frac{2ax}{b}+a-x-x=0,$$

$\therefore\ x=\frac{ab}{2(b-a)}$; but $x<a$, and $b>a$ or x is negative. If $ab\div 2(b-a)<a$, $b<2(b-a)$ or $b>2a$, which includes the other condition. If then $b>2a$ there is a maximum, for $\frac{d^2u}{dx^2}=4\pi\frac{a-b}{a}$: also $x=0$ and a give minima values, x being then a minimum or maximum.

If $b<2a$ the solution does not apply (there being discontinuity owing to the restriction of x being not $>a$), but then the solution indicates that u increases with x, so that the real maximum is when $x=a$; the minimum being, as before, when $x=0$. Also here the sign of $\frac{d^2u}{dx^2}$ is no guide, as the variable x is limited and $\frac{du}{dx}$ is *not* $=0$. In this case the maximum and minimum values of x itself correspond to those of u.

51. Consider a plane section through the axis of the cylinder (passing through the centre of the sphere). If $2x$ be the height of the cylinder, its volume $=2x \cdot \pi (r^2-x^2)$, and the minima values are when $x=0$ and r, $\therefore$ for the maximum $r^2-x^2=2x^2$ or the height $=\dfrac{2r}{\sqrt{3}}$.

52. If height $=2x$, convex surface $=2x \cdot 2\pi \sqrt{r^2-x^2}=4\pi\sqrt{u}$ say, the surface is a minimum when $x=0$ or r, and for a maximum of

$$u=x^2(r^2-x^2),\ x(r^2-x^2)=x^3 \text{ or } x^2=\frac{r^2}{2},\ \therefore \text{ height}=r\sqrt{2}.$$

53. The whole surface $=4\pi x\sqrt{r^2-x^2}+2\pi(r^2-x^2)=2\pi u$ say,

$$\therefore \text{ for maximum } 0=2\sqrt{r^2-x^2}-\frac{2x^2}{\sqrt{r^2-x^2}}-2x,$$

or
$$r^2-2x^2=x\sqrt{r^2-x^2} \quad\text{.............................(1)},$$

$$\text{and } \therefore\ 5x^4-5r^2x^2+r^4=0,$$

or
$$x^2=\frac{r^2}{10}(5\pm\sqrt{5}) \quad\text{.............................(2)},$$

and substituting in (1), we get $\mp\dfrac{\sqrt{5}}{5}=+\left(\dfrac{5\pm\sqrt{5}}{10}\cdot\dfrac{5\mp\sqrt{5}}{10}\right)^{\frac{1}{2}}$, $\because$ x must be positive, $\therefore$ the lesser value of x^2 in (2) alone satisfies (1), the other value being introduced in the process of rationalization. Thus

$$\frac{4x^2}{r^2}=2\left(1-\frac{1}{\sqrt{5}}\right),$$

and $\therefore$ the height $=r\cdot\left\{2\left(1-\dfrac{1}{\sqrt{5}}\right)\right\}^{\frac{1}{2}}$. This gives a maximum as $x=0$ or r give minima.

54. Consider a plane section of the cone through its axis, which passes through the centre of the sphere. If x be the height of the cone, 2θ its vertical angle, then $x\sec\theta=2r\cos\theta$, and the volume

$$=\frac{1}{3}\pi x^3\cdot\tan^2\theta=\frac{8}{3}\pi r^3\sin^2\theta\cos^4\theta,$$

and there is a minimum when $x=0$ or $2r$, $\therefore$ for maximum $\sin\theta\cos^2\theta$ is a maximum, $\therefore$ $\cos^3\theta=2\sin^2\theta\cos\theta$ or $\cos^2\theta=\dfrac{2}{3}$, and $\therefore$ $x=2r\cos^2\theta=\dfrac{4r}{3}$.

55. (See Ex. 54). The convex surface
$=\operatorname{cosec}\theta\cdot\pi x^2\tan^2\theta=4\pi r^2\sin\theta\cos^2\theta$,

$$\therefore \text{ for maximum } \cos^3\theta=2\sin^2\theta\cos\theta, \text{ and } x=\frac{4}{3}r,$$

as $x=0$ or $2r$ gives a minimum.

56. The whole surface

$$=\pi x^2 \tan^2\theta\,(\operatorname{cosec}\theta+1)=4\pi r^2(\sin\theta\cos^2\theta+\sin^2\theta\cos^2\theta),$$

and $\theta=0$ or $\frac{\pi}{2}$ give minima values, $\therefore$ for a maximum,

$$0=\cos^3\theta-2\sin^2\theta\cos\theta+2\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta),$$

or if $\quad x(=2r\cos^2\theta)=2rz,\ 3z-2=-2\sqrt{1-z}\,(2z-1)\ \ldots\ldots\ldots\ldots (1),$

$\therefore\ 16z^3-23z^2+8z=0$, and $32z=23\pm\sqrt{17}$, and substituting in (1), and observing that $\sin\theta$ and $\therefore\ \sqrt{1-z}$ is positive,

$$69\pm3\sqrt{17}-64=-2\sqrt{1-z}\,(46\pm2\sqrt{17}-32),$$

and the upper sign is $\therefore$ inapplicable, as it does not satisfy (1), while the lower sign gives $3\sqrt{17}-5=4(7-\sqrt{17})\sqrt{\dfrac{9+\sqrt{17}}{32}}$,

$$\text{or } 6\sqrt{17}-10=(7-\sqrt{17})(\sqrt{17}+1) \text{ which is true,}$$

$$\therefore \text{ for a maximum } x=2rz$$

$$=\frac{r}{16}(23-\sqrt{17}).$$

57. If height $=y$ and $x=$ radius, $\pi x^2y=V$, and $u=$ sum of the areas of convex surface and one end $=2\pi xy+\pi x^2$,

$$\therefore\ u=\frac{2V}{x}+\pi x^2,\ \frac{du}{dx}=-\frac{2V}{x^2}+2\pi x,$$

$$\therefore \text{ for minimum } x=\left(\frac{V}{\pi}\right)^{\frac{1}{3}},\ \therefore\ x^2y=\frac{V}{\pi}=x^3 \text{ and } x=y;$$

also $\dfrac{d^2u}{dx^2}=\dfrac{4V}{x^3}+2\pi$ is positive, $\therefore$ result is a minimum, u is clearly a maximum when $x=\infty$, or $y=\infty$.

58. If 2θ be the vertical angle, h the height of the cone, and a the radius of the sphere, then $h=a+a\operatorname{cosec}\theta$, and volume

$$=\frac{\pi}{3}\cdot a^3(1+\operatorname{cosec}\theta)^3\cdot\tan^2\theta, \text{ and for maximum } \theta=0 \text{ or } \frac{\pi}{2},$$

$\therefore$ for minimum $-3\operatorname{cosec}\theta\cot\theta\cdot\tan^2\theta+2(1+\operatorname{cosec}\theta)\tan\theta\sec^2\theta=0$,

or $\quad 3\sec\theta=2\sec^3\theta(1+\sin\theta)$, i.e. $3(1-\sin\theta)=2$,

and $\quad \therefore\ \theta=\sin^{-1}\left(\frac{1}{3}\right)$.

59. Let 2θ be the vertical angle, l the slant side of any one, then volume $=\frac{\pi}{3}l^3\cdot\sin^2\theta\cos\theta$, and for minimum clearly $\theta=0$ or $\frac{\pi}{2}$; $\therefore$ for maximum $\quad 0=2\sin\theta\cos^2\theta-\sin^3\theta$, and $\therefore\ \tan\theta=\sqrt{2}$.

60. If O be the given point, POp the required chord, QOq an adjacent position, for a minimum ultimately the area $POQ=pOq$ when Q moves up to P, $\therefore \dfrac{OP^2}{Op^2}=1$ ultimately, i.e. POp is bisected at O. Hence if RO be the diameter through O meeting the parabola in R, POp is parallel to the tangent at R. If POp be drawn parallel to the axis the area is a maximum.

61. Let PCP', QCQ' be the 2 conjugate diameters, α the eccentric angle of P, $\dfrac{\pi}{2}+\alpha$ that of Q, and R any point $(a\cos\theta, b\sin\theta)$ on the ellipse; then CP, CQ are $xb\sin\alpha-ay\cos\alpha=0$, and $xb\cos\alpha+ay\sin\alpha=0$, $\therefore$ sum of perpendiculars from R squared

$$=\frac{a^2b^2(\cos\theta\sin\alpha-\sin\theta\cos\alpha)^2}{a^2\cos^2\alpha+b^2\sin^2\alpha}+\frac{a^2b^2(\cos\alpha\cos\theta+\sin\alpha\sin\theta)^2}{a^2\sin^2\alpha+b^2\cos^2\alpha};$$

let then $u=\sin^2(\theta-\alpha)(a^2\sin^2\alpha+b^2\cos^2\alpha)+\cos^2\overline{\theta-\alpha}\,(a^2\cos^2\alpha+b^2\sin^2\alpha)$,

then for maximum, $\dfrac{du}{d\theta}=0=-\sin 2(\theta-\alpha)(a^2-b^2)\cos 2\alpha$,

$\therefore \theta-\alpha=0, \dfrac{\pi}{2}, \pi$ or 3π, i.e. R is an end of one of the conjugate diameters;

and $\dfrac{d^2u}{d\theta^2}=-2(a^2-b^2)\cos 2\alpha\cos 2(\theta-\alpha)$, $\therefore$ when $\theta-\alpha=0$ or π, $\dfrac{d^2u}{d\theta^2}$ is negative, and when $\theta-\alpha=\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$, $\dfrac{d^2u}{d\theta^2}$ is positive; thus P, P' are positions of R for a maximum, and Q, Q' for a minimum, α being $<\dfrac{\pi}{2}$. θ itself has no proper maximum or minimum value: $\therefore$ the above forms the complete solution.

62. (1) For a small change in x, $f(x)$ is unaltered, and $\therefore$ also $\phi\{f(x)\}$; and on a nearer approximation $f(x)$ is diminished whether x increase or decrease; $\therefore$ the corresponding change in $\phi\{f(x)\}$ is of the same sign whether x is increased or decreased, i.e. $\phi\{f(x)\}$ is either a maximum or a minimum. Similarly in the other case.

CHAPTER XIV.

1. If the given expression $=u$, $\dfrac{du}{dx}=2\left(\dfrac{1}{x}-\dfrac{1}{c+x}\right)-\dfrac{c}{x^2}-\dfrac{c}{(c+x)^2}$ which reduces to $-c^3\div x^2(c+x)^2$, $\therefore \dfrac{du}{dx}$ is negative, c being positive, and $\therefore$ &c.

2. If $e^u=\left(\dfrac{x}{c+x}\right)^{c+2x}$, $u=(c+2x)\log\dfrac{x}{c+x}$;

$$\therefore \frac{du}{dx}=2\log\frac{x}{c+x}+(c+2x)\left(\frac{1}{x}-\frac{1}{c+x}\right)=2\log\frac{x}{c+x}+\frac{c}{x}+\frac{c}{c+x},$$

∴ by Ex. 1, $\frac{du}{dx}$ diminishes as x increases, but when $x=\infty$, $\frac{du}{dx}=0$, ∴ $\frac{du}{dx}$ is always positive, x and c being so, ∴ $\frac{d}{dx}(e^u)$ which $=e^u \cdot \frac{du}{dx}$ is always positive, and ∴ e^u increases with x.

3. $\frac{du}{dx}=e^{2x}(1+2\overline{x-3})+4e^x(1+x)+1$,

$$\therefore \frac{d^2u}{dx^2}=e^{2x}(2\overline{2x-5}+2)+4e^x(2+x)=4e^x\{e^x(x-2)+x+2\},$$

∴ by Ex. 10 p. 86, $\frac{d^2u}{dx^2}$ is positive for all positive values of x; ∴ $\frac{du}{dx}$ increases with x, but when $x=0$, $\frac{du}{dx}=0$, ∴ $\frac{du}{dx}$ is positive for all positive values of x; and ∴ u increases with x, and when $x=0$, $u=0$, ∴ u is positive for all positive values of x.

4. If $w=$the given expression,

$$\frac{dw}{dx}=\frac{e^{2x}\{2(x-2)+1\}+e^x(x+2+1)}{(e^x-1)^3}-\frac{3e^x}{(e^x-1)^4}\{e^{2x}(x-2)+e^x(x+2)\},$$

thus $\frac{dw}{dx}=\frac{e^x}{(e^x-1)^4}\{e^{2x}(3-x)-4x\,e^x-x-3\}$, ∴, by Ex. 3, $\frac{dw}{dx}$ is negative for positive values of x.

Also $x=0$ gives by Ex. 15, p. 144, $w=\frac{1}{6}$, which is, ∴, the maximum of w for positive values of x.

5. If $u=(1+x)^{\frac{1}{x}}$, $\log u=\frac{1}{x}\log(1+x)=1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots$ by Art. 117,

∴ (Art. 115) $\qquad u=e^{1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots}$

$$=e\left\{1+\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}\right)+\frac{1}{\underline{|2}}\left(-\frac{x}{2}+\frac{x^2}{3}-\right)^2+\frac{1}{\underline{|3}}\left(-\frac{x}{2}+\dots\right)^3+\dots\right\}$$

$$=e\left\{1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\frac{x^2}{8}-\frac{x^3}{6}-\frac{x^3}{48}+\dots\right\}$$

$$=e\left\{1-\frac{x}{2}+\frac{11}{24}x^2-\frac{21x^3}{48}\right\}, \therefore \text{ \&c.}$$

6. By the method of Ex. 5, $(1+x)^{\frac{1}{x}}-e=e\left(1-\frac{x}{2}\right)-e$ approximately;

$$\therefore \frac{(1+x)^{\frac{1}{x}}-e}{x}=-\frac{e}{2}, \text{ in the limit when } x=0.$$

7. If $x=\frac{1}{y}$, the expression $=u=\frac{1}{y}(1+y)^{\frac{1}{y}}-\frac{e}{y^2}\log(1+y)$ and as in Ex. 6, $\frac{1}{y}(1+y)^{\frac{1}{y}}=\frac{e}{y}-\frac{e}{2}$ neglecting higher powers of y than y, and

$$\frac{e}{y^2}\log(1+y)=\frac{e}{y^2}\left(y-\frac{y^2}{2}+\dots\right);$$

thus $u=\frac{e}{y}-\frac{e}{2}-\frac{e}{y}+\frac{e}{2}+$ positive powers of y, $\therefore$ when $x=\infty$, and $y=0$, $u=0$.

8. If the given expression $=u$, and $y=\frac{1}{x}$,

$$u=\frac{8}{y^2}(1+y)^{\frac{1}{y}}-\frac{8e}{y^3}\log(1+y),$$

$$\therefore \text{ by Ex. 5 } u=\frac{8}{y^2}\cdot e\left\{1-\frac{y}{2}+\frac{11}{24}y^2-\dots\right\}-\frac{8e}{y^3}\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\dots\right)$$

$$=\frac{11}{3}e-\frac{8}{3}e+\text{positive powers of } y,$$

$$\therefore \text{ when } x=\infty \text{ and } y=0,\ u=e.$$

9. $$\text{Lt.}=\text{lt. of}-\frac{1}{1+x^2}\div\left\{nx^{n-1}-e^{\sin\log x}\cdot\cos(\log x)\cdot\frac{1}{x}\right\}$$

$$=-\frac{1}{2}\div\{n-e^0\}=\frac{-1}{2(n-1)}.$$

10. $$\text{Lt.}=\text{lt. of}-\tan\frac{x}{2}\cdot\operatorname{cosec}^2\frac{x}{2}\cdot\frac{1}{2}\div\left(-\operatorname{cosec}^2 x+\frac{1}{x}\right)$$

$$=+\frac{x}{\sin x}\div\frac{x-\sin^2 x}{\sin^2 x}=\frac{\sin^2 x}{x-\sin^2 x}=\text{lt. of }\frac{x^2}{x}=0.$$

11. If $u^n=\frac{\sec^n x}{e^{\tan x}}$, $u=\frac{\sec x}{e^{\frac{1}{n}\tan x}}=$ lt. of

$$\sec x\tan x\div e^{\frac{1}{n}\tan x}\left(\frac{1}{n}\sec^2 x\right)=n\sin x\div e^{\frac{1}{n}\tan x}=0 \text{ when } x=\frac{\pi}{2},$$

$\therefore$ $u^n=0$, n being finite.

12. $$u=(\tan nx-\tan mx)\div\sin(n^2x-m^2x)=\frac{\sin(n-m)x\cdot\sec nx\sec mx}{\sin\{x(n^2-m^2)\}},$$

$\therefore$ (1) $u=\frac{n-m}{n^2-m^2}=\frac{1}{m+n}$:

(2) $u=\frac{n-m}{n^2-m^2}\cdot\sec^2 mx=\frac{1}{2m}\sec^2 mx$. Also by differentiation.

13. The given equation shows that θ is in general a function of x and h, and if $f''(x)$ be not zero, Taylor's Theorem leads to

$$f(x+h)-f(x)=hf'(x)+\frac{h^2}{\underline{|2}}\cdot f''(x+\theta_1 h)$$

$$=hf'(x+\theta h)=h\,.\,f'(x)+h\,.\,\theta h\,.\,f''(x+\theta_2 h);$$

θ_1, θ_2 being proper fractions; $\therefore$ when h becomes 0, $\theta=\frac{1}{2}$.

So if $f^r(x)$ be the 1st of the differential coefficients, after $f'(x)$, which is not zero,

$$hf'(x)+\frac{h^r}{\underline{|r}}f^r(x+\theta_1 h)=hf'(x)+h\,.\,\frac{(\theta h)^{r-1}}{\underline{|r-1}}\cdot f^r(x+\theta_2 h),$$

whence $$\theta=\left(\frac{1}{r}\right)^{\frac{1}{r-1}}.$$

14. If θ be the same for all values of h, θ must $=\frac{1}{2}$ by Ex. 13, and extending the expansions to $f'''(x)$, it then follows that $f'''(x)=0$, $\therefore f''(x)$ must be constant with reference to x. This would make $f(x)$ of the form

$$a+bx+cx^2.$$

15. $\operatorname{Log} z=\sin x$, $\therefore z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}=\sec^2 x$,

and $$\frac{dx}{dz}=\frac{1}{z}\sec x,\ \therefore\ z\frac{dy}{dz}=\sec x\frac{dy}{dx};$$

$$\therefore\ z\frac{d^2y}{dz^2}+\frac{dy}{dz}=\frac{\sec x}{z}\left(\sec x\frac{d^2y}{dx^2}+\sec x\tan x\frac{dy}{dx}\right),$$

and $$\therefore\ 1=\frac{d^2y}{dx^2}+\tan x\,.\,\frac{dy}{dx}.$$

16. If $x=\rho\cos\phi$, $y=\rho\sin\phi$; $z=r\cos\theta$, $\rho=r\sin\theta$, then by Art. 205 (1),

$$\frac{du}{dx}=\cos\phi\frac{du}{d\rho}-\frac{1}{\rho}\sin\phi\frac{du}{d\phi},$$

$$\frac{du}{dy}=\sin\phi\frac{du}{d\rho}+\frac{1}{\rho}\cos\phi\frac{du}{d\phi};$$

$$\therefore\ \left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2=\left(\frac{du}{d\rho}\right)^2+\frac{1}{\rho^2}\cdot\left(\frac{du}{d\phi}\right)^2;$$

so $$\left(\frac{du}{dz}\right)^2+\left(\frac{du}{d\rho}\right)^2=\left(\frac{du}{dr}\right)^2+\frac{1}{r^2}\cdot\left(\frac{du}{d\theta}\right)^2;$$

$$\therefore\ \left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2+\left(\frac{du}{dz}\right)^2=\left(\frac{du}{dr}\right)^2+\frac{1}{r^2}\cdot\left(\frac{du}{d\theta}\right)^2+\frac{1}{r^2\sin^2\theta}\cdot\left(\frac{du}{d\phi}\right)^2.$$

Also $x\dfrac{du}{dx}+y\dfrac{du}{dy}=\rho\cos^2\phi\dfrac{du}{d\rho}-\sin\phi\cos\phi\dfrac{du}{d\phi}+\rho\sin^2\phi\dfrac{du}{d\rho}+\sin\phi\cos\phi\dfrac{du}{d\phi}$

$$=\rho\frac{du}{d\rho},$$

so
$$z\frac{du}{dz}+\rho\frac{du}{d\rho}=r\frac{du}{dr},$$

$\therefore$ the given expression

$$=\left\{\left(\frac{du}{dr}\right)^2+\frac{1}{r^2}\cdot\left(\frac{du}{d\theta}\right)^2+\frac{1}{r^2\sin^2\theta}\cdot\left(\frac{du}{d\phi}\right)^2\right\}\div r^2\left(\frac{du}{dr}\right)^2.$$

17. If (x, y) be P, each co-ordinate plane is supposed to pass through P, and $\therefore$ x, y being variable, the co-ordinate planes are coincident, and by ordinary formulæ, $\xi=a+x\cos\alpha+y\sin\alpha$, and $\eta=b-x\sin\alpha+y\cos\alpha$, where a, b, α are constants.

Thus
$$\frac{d\xi}{dx}=\cos\alpha,\ \frac{d\eta}{dx}=-\sin\alpha,\ \frac{d\xi}{dy}=\sin\alpha,\ \frac{d\eta}{dy}=\cos\alpha,$$

and
$$\therefore\ \frac{d^2\phi}{dx^2}=\left(\cos\alpha\frac{d}{d\xi}-\sin\alpha\frac{d}{d\eta}\right)^2(\phi),\ \text{(vide Art. 226)}$$

$$=\cos^2\alpha\frac{d^2\phi}{d\xi^2}-\sin 2\alpha\frac{d^2\phi}{d\xi d\eta}+\sin^2\alpha\frac{d^2\phi}{d\eta^2},\ \text{so putting } \frac{\pi}{2}+\alpha \text{ for } \alpha,$$

$$\frac{d^2\phi}{dy^2}=\sin^2\alpha\frac{d^2\phi}{d\xi^2}+\sin 2\alpha\,.\frac{d^2\phi}{d\xi d\eta}+\cos^2\alpha\frac{d^2\phi}{d\eta^2},$$

and
$$\frac{d^2\phi}{dxdy}=\left(\cos\alpha\frac{d}{d\xi}-\sin\alpha\frac{d}{d\eta}\right)\left(\sin\alpha\frac{d}{d\xi}+\cos\alpha\frac{d}{d\eta}\right)(\phi)$$

$$=\sin\alpha\cos\alpha\left(\frac{d^2\phi}{d\xi^2}-\frac{d^2\phi}{d\eta^2}\right)+\cos 2\alpha\frac{d^2\phi}{d\xi d\eta};$$

from which $\dfrac{d^2\phi}{dx^2}\cdot\dfrac{d^2\phi}{dy^2}-\left(\dfrac{d^2\phi}{dxdy}\right)^2$ reduces to

$$\frac{d^2\phi}{d\xi^2}\cdot\frac{d^2\phi}{d\eta^2}-\left(\frac{d^2\phi}{d\xi\,.\,d\eta}\right)^2,\ \text{or, \&c.}$$

18. If the given expression $=u$, $\dfrac{du}{dx}=2x+x\cos x-3\sin x$,

$$\therefore\ \frac{du}{dx}=0,\ \text{when } x=0,\ \text{and then } \frac{d^2u}{dx^2}=2-x\sin x-2\cos x=0,$$

$$\frac{d^3u}{dx^3}=-x\cos x+\sin x=0,\ \frac{d^4u}{dx^4}=x\sin x=0,$$

$$\frac{d^5u}{dx^5}=x\cos x+\sin x=0,\ \frac{d^6u}{dx^6}=-x\sin x+2\cos x=2,$$

when $x=0$, $\therefore$ u is a minimum when $x=0$.

Aliter, by expansion.

19. If $CP=r$, $CQ=p$, and $2a$, $2b$ be the axes of the ellipse,

$$u^2=PQ^2=r^2-p^2=r^2-\frac{a^2b^2}{a^2+b^2-r^2},$$

$\therefore$ for a maximum of u, $r=\frac{a^2b^2r}{(a^2+b^2-r^2)^2}$, $\therefore$ $a^2+b^2-r^2=\pm ab$, but r^2 is not $>a^2$, $\therefore$ $r^2=a^2-ab+b^2$, and $\therefore$ $u^2=a^2-ab+b^2-ab$ and $PQ=a-b$. PQ is a minimum when r is itself a maximum or minimum for then $PQ=0$.

20. Suppose PQ to pass through B' (the negative end of the minor axis), to make PN positive, if possible, and let P be (x, y), the ellipse being $a^2y^2+b^2x^2=a^2b^2$. Then $\frac{QN}{y}=\frac{x}{b+y}$,

$$\therefore\ \Delta PQN=\frac{y^2}{2}\cdot\frac{x}{b+y}, \text{ and } x=\frac{a}{b}\cdot\sqrt{b^2-y^2},$$

$$\therefore\ \Delta PQN\propto y^2\cdot\sqrt{\frac{b-y}{b+y}}=u \text{ say},$$

$$\therefore\ \log u=2\log y+\frac{1}{2}\log\left(\frac{b-y}{b+y}\right), \text{ and for maximum of } u$$

$$0=\frac{2}{y}-\frac{1}{2}\left(\frac{1}{b-y}+\frac{1}{b+y}\right),$$

or $2(y^2-b^2)+by=0$, $\therefore$ $\frac{4y}{b}=-1\pm\sqrt{17}$, but $y<b$ in magnitude, whether positive or negative, $\therefore$ the positive sign is alone suitable,

$$\text{i.e. } y=\frac{b}{4}(\sqrt{17}-1).$$

Symmetrically PN has the same numerical value if PQ pass through the other end of the minor axis, or if x be negative in either case. Also PQN is a minimum if $y=0$ or b, i.e. if Q be the centre or an end of the major axis, whence it is inferrible that the first value of y gives a maximum.

CHAPTER XV.

7. If $u=6x^3y^2-x^4y^2-x^3y^3$, $\frac{du}{dx}=x^2y^2(18-4x-3y)$,

$$\frac{du}{dy}=x^3y(12-2x-3y);\ \frac{d^2u}{dx^2}=36xy^2-12x^2y^2-6xy^3=6xy^2(6-2x-y),$$

$$\frac{d^2u}{dxdy}=x^2y(36-8x-9y),\ \frac{d^2u}{dy^2}=x^3(12-2x-6y).$$

Now $\frac{du}{dx}=0$ when $x=0$, or $y=0$, or $18-4x-3y=0$,

and $$\frac{du}{dy}=0 \text{ when } x=0, \text{ or } y=0 \text{ or } 12-2x-3y=0.$$

(1) If $18=4x+3y$, and $12=2x+3y$, $x=3$ and $\therefore y=2$,

and $$\frac{d^2u}{dx^2}=-6xy^2 \,.\, 2,\ \frac{d^2u}{dxdy}=-x^2y \,.\, 6,\ \frac{d^2u}{dy^2}=-x^3 \,.\, 6;$$

$\therefore$ A is negative, and $AC-B^2=x^4y^2(72-36)$ is positive, $\therefore$ there is a maximum of u.

If either x or $y=0$, A, B, C all vanish; and (cf. Art. 230)

$$\frac{d^3u}{dx^3}=36y^2-24xy^2-6y^3,\ \frac{d^3u}{dx^2dy}=72xy-24x^2y-18xy^2,$$
$$\frac{d^3u}{dxdy^2}=36x^2-8x^3-18x^2y,\ \frac{d^3u}{dy^3}=-6x^3,$$

$\therefore$ when $y=0$ but x is not zero, $P=0=Q$ but S and T do not vanish, and when $x=0$ and y is not zero, P does not vanish, $\therefore$ in neither case is there a maximum or minimum. If $x=0=y$; P, Q, S, T all vanish, and then

$$\frac{d^4u}{dx^4}=-24y^2=0,\ \frac{d^4u}{dx^3dy}=72y-48xy=0,$$

$$\frac{d^4u}{dx^2dy^2}=72x-24x^2-36xy=0,\ \frac{d^4u}{dxdy^3}=-18x^2=0,\ \frac{d^4u}{dy^4}=0;$$

then $$\frac{d^5u}{dx^5}=0,\ \frac{d^5u}{dx^4dy}=-48y=0,\ \frac{d^5u}{dx^3dy^2}=72,$$

which does not vanish, $\therefore$ by an extension of Art. 230, there can be neither maximum nor minimum.

8. $\dfrac{du}{dx}=2(a-x)(2by-y^2)$, $\dfrac{du}{dy}=2(b-y)(2ax-x^2)$, $\therefore$ for maximum or minimum $x=a$ or $y=0$ or $2b=y$; and $y=b$, or $x=0$ or $2a$; and

$$\frac{d^2u}{dx^2}=-2(2by-y^2),\ \frac{d^2u}{dy^2}=-2(2ax-x^2),\ \frac{d^2u}{dxdy}=4(a-x)(b-y).$$

(1) If $x=a$ and $y=b$, $B=0$, $A=-2b^2$, $C=-2a^2$, $\therefore$ there is a maximum.

(2) If $y=0$ and $x=0$, $A=0=C$ and $B=4ab$, $\therefore$ neither maximum nor minimum. So if $y=0$ and $x=2a$, and symmetrically if $y=2b$ and $x=0$; and also if $y=2b$ and $x=2a$ similar results follow.

9. $\dfrac{du}{dx}=4(x^3-x+y)$, $\dfrac{du}{dy}=4(y^3-y+x)$;

$$\frac{d^2u}{dx^2}=12x^2-4,\ \frac{d^2u}{dxdy}=4,\ \frac{d^2u}{dy^2}=12y^2-4;$$

and $$\frac{du}{dx}=0=\frac{du}{dy} \text{ give } x^3=x-y=-y^3,\ \therefore\ x=-y \text{ and } x^3=2x,$$
$$\therefore\ x=\pm\sqrt{2} \text{ and } y=\mp\sqrt{2}, \text{ or } x=0=y.$$

(1) If $x=\pm\sqrt{2}=-y$, $A=20$, $B=4$, $C=20$, $\therefore$ there is a minimum.

(2) If $x=0=y$, $A=-4=C$, and $B=4$, $\therefore AC=B^2$ and the expansion in Art. 228 reduces to $-2(h^2-2hk+k^2)+R_1$, and when h, k are small enough this is negative, unless $h=k$; now

$$\frac{d^3u}{dx^3}=24x=0,\ \frac{d^3u}{dx^2dy}=0=\frac{d^3u}{dxdy^2},\ \frac{d^3u}{dy^3}=24y=0,\ \text{and}\ \frac{d^4u}{dx^4}=24=\frac{d^4u}{dy^4}$$

and the other terms in R_2 vanish, $\therefore$ when $h=k$, the expansion reduces to $h^4+k^4=2h^4$, which is positive, $\therefore$ there is neither a maximum nor minimum.

If x or y be ∞, u is a maximum.

10. $\frac{du}{dx}=3(x^2-2x-1)$, $\frac{du}{dy}=4(y^3-6y^2+9y-2)$,

$$\frac{d^2u}{dx^2}=6(x-1),\ \frac{d^2u}{dxdy}=0,\ \frac{d^2u}{dy^2}=12(y^2-4y+3)=12(y-1)(y-3);$$

$$\therefore \frac{du}{dx}=0 \text{ gives } x=1\pm\sqrt{2},\ \frac{du}{dy}=0 \text{ gives } (y-2)(y^2-4y+1)=0.$$

(1) If $y=2$ and $x=1\pm\sqrt{2}$, $A=\pm 6\sqrt{2}$, $B=0$ and $C=-12$, $\therefore y=2$ and $x=1-\sqrt{2}$ give a maximum and then

$$u=(y^2-4y)^2+2(y^2-4y)+x(x^2-2x-1)-(x^2-2x-1)-4x-1$$
$$=16-8-1-4+4\sqrt{2}=3+4\sqrt{2}.$$

(2) If $x=1\pm\sqrt{2}$, and $y=2\pm\sqrt{3}$, $A=\pm 6\sqrt{2}$, and

$$C=12(1\pm\sqrt{3})(-1\pm\sqrt{3})=24,\ \therefore x=1+\sqrt{2} \text{ and } y=2\pm\sqrt{3}$$

give minima values, and then $u=1-2-1-4(1+\sqrt{2})=-6-4\sqrt{2}$.

11. $\frac{du}{dx}=2x+y-a$, $\frac{du}{dy}=2y+x-b$, $\frac{d^2u}{dx^2}=2=\frac{d^2u}{dy^2}$, $\frac{d^2u}{dxdy}=1$, $\therefore$ there is a minimum when $2x+y=a$ and $2y+x=b$,

and $\qquad \therefore x+y=\frac{a+b}{3}$, and $x=\frac{2a-b}{3}$, $y=\frac{2b-a}{3}$,

and $\therefore u=\frac{1}{4}\{3(x+y)^2+(x-y)^2\}-ax-by$

$$=\frac{1}{4}\left\{\frac{1}{3}(a+b)^2+(a-b)^2\right\}-\frac{2}{3}(a^2+b^2-ab)=-\frac{1}{3}(a^2+b^2-ab).$$

12. Here $u=\frac{xy}{2}+\left(\frac{x}{3}+\frac{y}{4}\right)(n-x-y)=n\left(\frac{x}{3}+\frac{y}{4}\right)-\frac{x^2}{3}-\frac{y^2}{4}-\frac{xy}{12}$;

$$\therefore \frac{du}{dx}=\frac{n}{3}-\frac{2x}{3}-\frac{y}{12},\ \frac{du}{dy}=\frac{n}{4}-\frac{y}{2}-\frac{x}{12},\ \frac{d^2u}{dx^2}=-\frac{2}{3},$$

$\frac{d^2u}{dxdy}=-\frac{1}{12},\ \frac{d^2u}{dy^2}=-\frac{1}{2}$, $\therefore$ there is a maximum when

$$4n=8x+y \text{ and } 3n=6y+x, \text{ i. e. } 12n=24x+3y=24y+4x,$$

and $$\therefore \frac{x}{21}=\frac{y}{20}=\frac{4n}{188}=\frac{x+y+z}{47}=\frac{z}{6}.$$

13. $\frac{du}{dx}=3(x^2+ay),\ \frac{du}{dy}=3(y^2+ax),\ \therefore \frac{d^2u}{dx^2}=6x,\ \frac{d^2u}{dxdy}=3a,\ \frac{d^2u}{dy^2}=6y$:

and for maximum or minimum $x^2+ay=0=y^2+ax$, and $\therefore$ $x=y$ or $x+y=a$, $\therefore$ $x^2-ax+a^2=0$, which is imaginary; or $x=y=0$, which give $A=0=C$, and $\therefore$ neither maximum nor minimum; *or* $x=y=-a$, and $\therefore$ $A=-6a=C$ and there is a maximum or minimum as a is positive or negative; and then $u=a^3$. Also x or y may $=\pm\infty$, &c.

14. If $u=x(x^2+y^2)-3axy,\ \frac{du}{dx}=3x^2+y^2-3ay,\ \frac{du}{dy}=2xy-3ax,$

$\frac{d^2u}{dx^2}=6x,\ \frac{d^2u}{dxdy}=2y-3a,\ \frac{d^2u}{dy^2}=2x$: and for maximum or minimum

(1) $x=0=y$, and $\therefore$ $A=0=C$, $B=-3a$, $\therefore$ neither:

(2) $x=0$ and $y=3a$, $\therefore$ $A=0=C$, $B=3a$, $\therefore$ neither:

(3) $2y=3a$ and $3x^2=\left(\frac{3a}{2}\right)^2$, i. e. $x=\pm\frac{a\sqrt{3}}{2}$, and then

$$A=\pm a3\sqrt{3},\ C=\pm a\sqrt{3} \text{ and } B=0,$$

$\therefore$ a minimum or maximum as $x=\frac{a\sqrt{3}}{2}$ or $-\frac{a\sqrt{3}}{2}$,

and $$u=\pm\frac{a^3\sqrt{3}}{2}(3)\mp 3a^3\cdot\frac{3\sqrt{3}}{4}=\mp\frac{3}{4}a^3\sqrt{3}.$$

15. If $u=$ the given expression and $v=1-ax-by$,

$$\frac{du}{dx}=\frac{2x}{v}+\frac{a}{v^2}(1+x^2+y^2),\ \frac{du}{dy}=\frac{2y}{v}+\frac{b}{v^2}(1+x^2+y^2),$$

$\therefore \frac{x}{a}=\frac{y}{b}=r$ say, and for maximum or minimum

$$2ra\{1-r(a^2+b^2)\}+a\{1+r^2(a^2+b^2)\}=0,$$

i.e. $r^2(a^2+b^2)-2r-1=0$ and $\therefore$ $r(a^2+b^2)=1\pm\sqrt{1+a^2+b^2}$, and $$\frac{d^2u}{dx^2}=\frac{2}{v}+\frac{4ax}{v^2}+\frac{2a^2}{v^3}(1+x^2+y^2)=\frac{2}{v},\ \because\ \frac{du}{dx}=0,$$

$$\therefore\ A=C=\frac{2}{1-r(a^2+b^2)}=\mp\frac{2}{\sqrt{1+a^2+b^2}},\text{ and}$$

$$\frac{d^2u}{dxdy}=\frac{2bx}{v^2}+\frac{2ay}{v^2}+\frac{2ab}{v^3}(1+x^2+y^2)=\frac{2}{v^2}(ay-bx),\ \therefore\ B=0;$$

thus $\frac{x}{a}=\frac{y}{b}=\frac{1\pm\sqrt{1+a^2+b^2}}{a^2+b^2}$ give a maximum with the upper sign and a minimum with the lower.

16. When y is constant $u^2 \propto (c-x)(x+y-c)$ but $(c-x)+(x+y-c)=y$, $\therefore$ u is greatest by variation of x when $c-x=x+y-c$; so by variation of y when $c-y=x+y-c$, and $\therefore$ u is a maximum when $x=y=\frac{2}{3}c$.

17. If $x=r\cos\theta$ and $y=r\sin\theta$, $u=\dfrac{a+r(b\cos\theta+c\sin\theta)}{(1+r^2)^{\frac{1}{2}}}$,

and $$\frac{du}{dr}=\frac{-ar+(b\cos\theta+c\sin\theta)}{(1+r^2)^{\frac{3}{2}}},\quad \frac{du}{d\theta}=(c\cos\theta-b\sin\theta)\frac{r}{(1+r^2)^{\frac{1}{2}}},$$

$\therefore$ for maximum $\tan\theta=\frac{c}{b}$ and $ar=\sqrt{b^2+c^2}$, and $\therefore$ $x=\frac{b}{a}$, $y=\frac{c}{a}$;
also $B=0$ clearly,

$$\frac{d^2u}{d\theta^2}=-(c\sin\theta+b\cos\theta)\frac{r}{\sqrt{1+r^2}}=\frac{-(b^2+c^2)}{\sqrt{a^2+b^2+c^2}},$$

and $$\frac{d^2u}{dr^2}=\frac{-3r(b\cos\theta+c\sin\theta)-a(1+r^2)+3ar^2}{(1+r^2)^{\frac{5}{2}}}$$

$$=\frac{-a}{\sqrt{1+r^2}}\left(\because\ \frac{du}{dr}=0\right),\ =\frac{-a^2}{a^2+b^2+c^2};$$

$$\therefore\ \text{there is a maximum when } \frac{x}{b}=\frac{y}{c}=\frac{1}{a}.$$

18. If $u=xe^{y+x\sin y}$, $\frac{du}{dx}=\frac{u}{x}(1+x\sin y)$, $\frac{du}{dy}=u(1+x\cos y)$,

$\therefore$ for maximum or minimum $\sin y=\cos y=\pm\frac{1}{\sqrt{2}}$,

and $$x=\mp\sqrt{2};\ \text{also}\ \frac{d^2u}{dx^2}=\frac{u}{x}(2\sin y+x\sin^2 y);$$

$$\therefore\ A=e^{y_0-1}\left(\pm\sqrt{2}\mp\frac{1}{\sqrt{2}}\right),\ \frac{d^2u}{dxdy}=\frac{u}{x}(1+x\cos y)(2+x\sin y);$$

$$\therefore\ B=0 \text{ and } \frac{d^2u}{dy}=u\,\{(1+x\cos y)^2-x\sin y\}\,;$$

$$\therefore\ C=-e^{y_0-1}\,.\,x^2\sin y=\mp e^{y_0-1}\,.\,\sqrt{2}\,;$$

but $A=\pm\dfrac{e^{y_0-1}}{\sqrt{2}}$, $\therefore$ A and C are of opposite signs, and u has neither a maximum nor a minimum: (y_0 means any value of y corresponding to the case in question where $\sin y=\cos y=\pm\dfrac{1}{\sqrt{2}}$). The general value of y is $2n\pi+\dfrac{\pi}{4}$, n being integral, when $\dfrac{du}{dx}=0=\dfrac{du}{dy}$.

See Ch. XVI. Art. 239, for some of the remaining examples.

19. $0=dx+dy+dz=\dfrac{a}{x^2}\,.\,dx+\dfrac{b}{y^2}\,.\,dy+\dfrac{c}{z^2}\,.\,dz\,;$

$$\therefore\ 1+\frac{\lambda a}{x^2}=0=1+\frac{\lambda b}{y^2}=1+\frac{\lambda c}{z^2}\,;$$

$$\therefore\ \frac{x}{\sqrt{a}}=\sqrt{-\lambda}=\frac{y}{\sqrt{b}}=\frac{z}{\sqrt{c}}\,;$$

but
$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1,$$

$\therefore$ $\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{-\lambda}$, $\therefore$ &c. clearly a minimum, as for maximum one or two of the variables is ∞.

20. $0=\dfrac{p}{x}\,.\,dx+\dfrac{q}{y}\,.\,dy+\dfrac{r}{z}\,.\,dz=\dfrac{a}{x^2}\,.\,dx+\dfrac{b}{y^2}\,.\,dy+\dfrac{c}{z^2}\,.\,dz\,;$

$$\therefore\ \frac{p}{a}x=\lambda=\frac{q}{b}y=\frac{r}{c}z,\ \therefore\ \frac{p+q+r}{\lambda}=1,$$

and
$$\therefore\ \frac{px}{a}=\frac{qy}{b}=\frac{rz}{c}=p+q+r.$$

21. If a, b, c be the sides of the Δ; x, y, z the distances from them of any point within it, xyz is to be a maximum and $ax+by+cz=2$. area of Δ; hence
$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0=adx+bdy+cdz,$$

$\therefore$ $ax=by=cz=\dfrac{2}{3}$ area of Δ. For minimum $x=0$, &c. If this point be P, and ABC the triangle, the area $PBC=\dfrac{ax}{2}$, . . &c.

22. If $u=xyz$, $0=\dfrac{dx}{x}++=\dfrac{xdx}{a^2}++$, hence

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{1}{3}\text{, and }\therefore\ u=\frac{abc}{3\sqrt{3}}.$$

For minimum $x=0$, &c.

23. $u=p^2+q^2+r^2$, $pa+qb+rc=2$. area of Δ,

$$\therefore\ pdp+\,+\,=0,\quad adp+\,+\,=0,$$

and
$$\therefore\ \frac{p}{a}=\frac{q}{b}=\frac{r}{c}=\frac{pa+\,+}{a^2+\,+}=\frac{2\,.\,\text{area of }\Delta}{a^2+b^2+c^2}.$$

24. If ABC be the Δ, D the middle of BC and P the point required, for a slight change in P's position when AP remains constant, P must describe a small arc of the circle centre A and radius AP, and then BP^2+CP^2 is constant and $\therefore$ also DP, $\therefore$ the circle centre D and radius DP must touch the 1st circle, $\therefore$ APD is a straight line. So if point E be the middle of CA, P lies on BE; thus P must be at the centre of gravity of the Δ. If AP, BP be given, CP is known, $\therefore$ only 2 of the distances are independent.

25. If ABC be the Δ; x, y, z the distances of the point P from the sides, the Δ with one side on $BC=\frac{1}{2}\,\frac{x^2\sin A}{\sin B\sin C}$, hence $a^2x^2+b^2y^2+c^2z^2$ is to be a minimum, and $ax+by+cz=2$ area of Δ, thus $ax=by=cz$, $\therefore\ x=\frac{1}{3}$ height of A above BC, and so on, and P is $\therefore$ the centre of gravity.

26. If ABC be the Δ; $r_1r_2r_3$ the radii of the fences, $r_1A+r_2B+r_3C=$ constant, and $r_1^2\,.\,A+\,+$ is to be a minimum, $\therefore\ \frac{r_1\,.\,A}{A}=r_1=r_2=r_3$.

27. If x, y, z be the edges, $x+y+z=a$, and $yz+zx+xy$ is a maximum,

$$\therefore\ dx+\,+\,=0=dx\,(y+z)+\,+,$$

$$\therefore\ y+z=z+x=x+y,\ \text{and}\ \therefore\ x=y=z,\ \text{a cube},\ \therefore\,.$$

28. If the edges be $2x$, $2y$, $2z$, and a the radius of the sphere

$$x^2+y^2+z^2=a^2,$$

and volume$=8xyz$, $\therefore$ for a maximum volume, $xdx+ydy+zdz=0$ and

$$\frac{1}{x}\,dx+\frac{1}{y}\,dy+\frac{1}{z}\,dz=0,$$

and $\therefore$ $x^2=y^2=z^2$, i.e. the parallelepiped is a cube. If any one of the edges$=0$ the volume is a minimum.

The surface$=8\,(yz+zx+xy)$, $\therefore$ for maximum

$$xdx+ydy+zdz=0,\ \text{and}\ (y+z)\,dx+(z+x)\,dy+(x+y)\,dz=0,$$

$$\therefore\ \frac{y+z}{x}+1=\frac{z+x}{y}+1=\frac{x+y}{z}+1,$$

or $x=y=z$; a cube. $y=0=z$ give a minimum, &c.

29. Let the side fixed in position be of length x; y, z the lengths of the other two, p be the distance from x of the vertex opposite to it; then the volume of the double cone is $\frac{x}{3}\cdot\pi p^2$. Hence for a given value of x, the volume is greatest when p is, and $\therefore$ ($y+z$ being then constant) when $y=z$, the vertex then lying on an ellipse of which the foci are the extremities of the base x. Thus, subject to $x+2y=a$, the greatest volume is the maximum value of $$\frac{\pi x}{3}\left(y^2-\frac{x^2}{4}\right)=\frac{\pi x}{12}\{(a-x)^2-x^2\}=\frac{\pi x}{12}(a^2-2ax),$$

and $\therefore$ for a maximum $a^2-2ax-2ax=0$, or $x=\frac{a}{4}$,

and $$\therefore\ y=\frac{a}{2}-\frac{a}{8}=\frac{3a}{8},\ \text{i.e. } x=\frac{2}{3}y=\frac{2}{3}z.$$

The minimum volume is when $x=0$ or $\frac{a}{2}$, which are the maximum and minimum values of x.

30. If a, b, c be the co-ordinates of the given point; p, q, r the edges of the parallelepiped, the plane containing the point is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=1$,

and $$\therefore\ \frac{a}{p}+\frac{b}{q}+\frac{c}{r}=1,\ \text{and the diagonal}=u=(p^2+q^2+r^2)^{\frac{1}{2}},$$

$\therefore$ as a maximum of u^2 would be given by $p=\infty$ &c.,

for a minimum, $$pdp+qdq+rdr=0,$$

and $$\frac{a}{p^2}dp+\frac{b}{q^2}\cdot dq+\frac{c}{r^2}dr=0;$$

$$\therefore\ \frac{a}{p^2}=\lambda p,\ \frac{b}{q^2}=\lambda q,\ \frac{c}{r^2}=\lambda r,$$

and $$\therefore\ 1=\lambda(p^2+q^2+r^2)=\lambda u^2;$$

$$\therefore\ u^2=(a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}})\frac{1}{\lambda^{\frac{2}{3}}},$$

or $$u=(a^{\frac{2}{3}}+b^{\frac{2}{3}}+c^{\frac{2}{3}})^{\frac{3}{2}}.$$

CHAPTER XVI.

6. By Art. 239, $0=\frac{2u}{x}Dx+\frac{3u}{y}\cdot Dy+\frac{4u}{z}Dz$,

and $$2Dx+3Dy+4Dz=0;\ \text{thus } x=y=z=\frac{a}{9},$$

and $\therefore u=\left(\frac{a}{9}\right)^9$. If x, y, z be all positive, u is least when any one of them $=0$, and the maximum is $\therefore$ that found above; if one or two of the variables may be negative, u may $=\pm\infty$. Analytically, the method of Chap. XV. is most easy, i.e. reducing u to 2 variables.

7. $0 = xdx + ydy + +$, and $adx + bdy + + = 0$,

thus $$\frac{x}{a} = \frac{y}{b} = = = \frac{k}{a^2 + b^2 + +},$$

and $$\therefore\ u = x^2 + y^2 + + = \frac{k^2}{a^2 + b^2 + +}.$$

If any one of the variables as x be $\pm\infty$, $u = \infty$, $\therefore$ there must be some finite value of x corresponding to a minimum of u; from which it may be inferred that the value above is a minimum.

8. If $u =$ the given expression, $\frac{du}{dx} = 2x + 1 - y$, $\frac{du}{dy} = 2y - x$, $\frac{du}{dz} = 2(z-1)$, $\therefore$ for maximum or minimum $z = 1$, $x = 2y$, and $\therefore$ $y = -\frac{1}{3}$, $x = -\frac{2}{3}$; also (Art. 236) $A = 2$, $B = 2$, $C = 2$, $A' = 0 = B'$, $C' = -1$; $\therefore$ $AB - C'^2 = 3$ which is positive, and $(AB - C'^2)(AC - B'^2) - (AA' - B'C')^2 = 3 \,.\, 4$ which is positive; $\therefore$ there is a minimum, *or*, $\because$ when x is $\pm\infty$, $u = \infty$, there must be some intermediate value of x for which u is a minimum, &c.

9. $$x^3 \,.\, dx + y^3 dy + z^3 dz = 0 = c^3\left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}\right);$$

$$\therefore\ x^4 = y^4 = z^4, \ \therefore\ x = \pm y = \pm z = \pm c,$$

and $\therefore$ $u = x^4 + y^4 + z^4 = 3c^4$. If any one of the variables be $\pm\infty$, $u = \infty$, and $\therefore$ $3c^4$ must be a minimum.

10. Let $u = \sin^m x \,.\, \sin^n y \,.\, \sin^p z$, where $x + y + z = 2\pi$; then $\sin x$, $\sin y$, $\sin z$ are all positive, and $u = 0$ if any one of the variables $= 0$ or π; thus for a maximum $$0 = mu \cot x dx + nu \cot y dy + pu \cot z dz,$$

and $$dx + dy + dz = 0;$$

$$\therefore\ \frac{\tan x}{m} = \frac{\tan y}{n} = \frac{\tan z}{p} = \frac{\tan x \,.\, \tan y \,.\, \tan z}{m+n+p} = k,$$

where $$\tan x \,.\, \tan y \,.\, \tan z = k^3 \,.\, mnp, \text{ or } k^2 = \frac{m+n+p}{mnp},$$

and $$\therefore\ x = \tan^{-1}\left\{m\left(\frac{m+n+p}{mnp}\right)^{\frac{1}{2}}\right\} \text{ \&c.}$$

This assumes that m, n, p are all positive.

11. If $u = x^p \,.\, y^q \,.\, z^r$, $0 = \frac{pu}{x}dx + \frac{qu}{y} \,.\, dy + \frac{ru}{z} \,.\, dz$,

and $$0 = ldx + mdy + ndz,$$

and $$\therefore\ \frac{lx}{p} = \frac{my}{q} = \frac{nz}{r} = \frac{a}{p+q+r},$$

and $$\therefore\ u = \left(\frac{a}{p+q+r}\right)^{p+q+r} \,.\, \left(\frac{p}{l}\right)^p \,.\, \left(\frac{q}{m}\right)^q \,.\, \left(\frac{r}{n}\right)^r :$$

if p, q, r be all positive, u is then a maximum as in Ex. 6.

For the parallelepiped, $p=1=q=r$, and $u=\left(\frac{a}{3}\right)^3 \cdot \frac{1}{lmn}$ (taken positively, if one or three of the quantities l, m, n, a be negative and the remainder positive).

Here $u=xyz$ and $lx+my+nz=a$, and if any one of the quantities x, y, z vanish. $u=0$, but if they be all positive u is positive, $\therefore$ the value found above is a maximum: u may also be $\pm\infty$ for infinite values of 2 or more of the variables.

Here and in Ex. 10 when one or more of the quantities p, q, r is negative, u may be correspondingly a maximum a minimum or indeterminate. Similarly in Ex. 10, u may be impossible.

12. $0=xdx+ydy$, and $0=(2ax+by)\,dx+(2cy+bx)\,dy$;

$$\therefore\ 2ax+by=\lambda x,\ \ 2cy+bx=\lambda y,$$

and multiplying these 2 equations by x and y and adding,

$$2f=\lambda r^2, \text{ and } \therefore\ 2x\left(\frac{f}{r^2}-a\right)=by,$$

and
$$2y\left(\frac{f}{r^2}-c\right)=bx, \text{ and } \therefore\ 4\left(\frac{f}{r^2}-a\right)\left(\frac{f}{r^2}-c\right)=b^2,$$

i.e. $(b^2-4ac)\,r^4+4f\,(a+c)\,r^2-4f^2=0$ gives the maxima and minima values of r; which are the $\frac{1}{2}$ axes to the conic $ax^2+bxy+cy^2=f$. If $b^2>4ac$, x and y may be ∞, and $\therefore$ r corresponding to asymptotic directions.

13. If $u=$ the given expression

$$\frac{du}{dx}=e^{-a^2x^2-\beta^2y^2-\gamma^2z^2}\cdot\{a-2a^2x\,(ax+by+cz)\},$$

thus
$$\frac{a^2x}{a}=\frac{\beta^2y}{b}=\frac{\gamma^2z}{c}=\frac{1}{2\,(ax+by+cz)}=\mu \text{ say};$$

$$\therefore\ x=\frac{\mu a}{a^2}\ \&\text{c., and }\therefore\ \frac{1}{2\mu}=\mu\left(\frac{a^2}{a^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right),$$

or
$$\frac{1}{\mu}=\pm\sqrt{2}\left(\frac{a^2}{a^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)^{\frac{1}{2}}.$$

Now when any one of the quantities x, y, $z=\pm\infty$, u is easily seen to $=0$, and if $x=0=y=z$, again $u=0$, $\therefore$ there must be a maximum for positive values of x, y, z and a maximum for negative values, and these correspond to the 2 values of μ, and then

$$u=\pm\frac{1}{\sqrt{2}}\left(\frac{a^2}{a^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)^{\frac{1}{2}}\cdot e^{-\frac{1}{2}}.$$

The upper sign gives the absolute maximum.

14. Let the capacity $=u=(x-2a)(y-2a)(z-a)$, then

$$V=xyz-u=xyz-(x-2a)(y-2a)(z-a);$$

$$\therefore\ 0=(y-2a)(z-a)\,dx+(x-2a)(z-a)\,dy+(x-2a)(y-2a)\,dz,$$

and $dx\,\{yz-(y-2a)(z-a)\}+dy\,\{zx-(x-2a)(z-a)\}$

$$+dz\,\{xy-(x-2a)(y-2a)\}=0;$$

$$\therefore\ \lambda yz=(y-2a)(z-a),\ \lambda zx=(x-2a)(z-a),$$

and $\lambda xy=(x-2a)(y-2a)$; thus $\dfrac{x}{y}=\dfrac{x-2a}{y-2a}=\dfrac{2a}{2a}$, $\therefore\ x=y$,

and $$\frac{z}{x}=\frac{z-a}{x-2a}=\frac{a}{2a},\ \therefore\ z=\frac{x}{2};$$

$$\therefore\ V=\frac{x^3}{2}-\frac{1}{2}(x-2a)^3=a\,(x^2+x\,\overline{x-2a}+\overline{x-2a}|^2)=3a\,(x-a)^2+a^3,$$

$$\text{i.e. } x=a+\left(\frac{V-a^3}{3a}\right)^{\frac{1}{2}},\ \text{(the upper sign} \because x>2a).$$

For minimum $x=2a$, or $y=2a$ or $z=a$, or two of these cases.

15. $xdx+ydy+zdz=0=ldx+mdy+ndz$,

and $dx\,(ax+c'y+b'z)+dy\,(by+a'z+c'x)+dz\,(cz+b'x+a'y)=0$;

$\therefore\ x=\lambda l+\mu\,(ax+c'y+b'z)$ and 2 similar equations, and multiplying by x, y, z and adding, $r^2=\mu$, $\therefore\ x\left(a-\dfrac{1}{r^2}\right)+yc'+zb'+\dfrac{\lambda l}{r^2}=0$,

so $$x\,.\,c'+y\left(b-\frac{1}{r^2}\right)+za'+\frac{\lambda m}{r^2}=0,$$

and $$x\,.\,b'+ya'+z\left(c-\frac{1}{r^2}\right)+\frac{\lambda n}{r^2}=0,$$

also $$lx+my+nz=0;$$

hence, eliminating x, y, z, λ,

$$\begin{vmatrix} l, & m, & n, & 0 \\ a-\frac{1}{r^2}, & c', & b', & l \\ c', & b-\frac{1}{r^2}, & a', & m \\ b', & a', & c-\frac{1}{r^2}, & n \end{vmatrix}=0;$$

in this every term clearly involves l, m, n in the 2nd degree, and the coefficient of l^2 is $\left(b-\dfrac{1}{r^2}\right)\left(c-\dfrac{1}{r^2}\right)-a'$; and that of lm is

$$-c'\left(c-\frac{1}{r^2}\right)+a'b'-c'\left(c-\frac{1}{r^2}\right)+a'b';$$

and thus by symmetry the result follows.

CHAPTER XVII.

1. $\therefore\ x\dfrac{dy}{dx}+y=\left(1+\dfrac{dy}{dx}\right)(c-1)$, but $c-1=\dfrac{xy-1}{x+y+1}$;

$$\therefore\ \left(x\frac{dy}{dx}+y\right)(x+y+1)=\left(\frac{dy}{dx}+1\right)(xy-1),\ \therefore\ \&c.$$

2. $y=e^x\,.\cos x,\ \therefore\ \dfrac{dy}{dx}=e^x(\cos x-\sin x)$;

$$\therefore\ \frac{d^2y}{dx^2}=e^x(\cos x-\sin x-\sin x-\cos x)=-2e^x\sin x=2\left(\frac{dy}{dx}-y\right),\ \therefore\ \&c.$$

3. $\therefore\ x=a\dfrac{dy}{dx},\ \therefore\ 1=a\dfrac{d^2y}{dx^2}$, and $\therefore\ x\dfrac{d^2y}{dx^2}=\dfrac{dy}{dx}$.

4. $y\,.\,e^{-mx}=a\sin nx,\ \therefore\ -n^2a\sin nx=\dfrac{d^2}{dx^2}(y\,.\,e^{-mx})$

$$=e^{-mx}\left(\frac{d^2y}{dx^2}-2m\frac{dy}{dx}+m^2y\right)=-n^2\,.\,ye^{-mx},\ \therefore\ \&c.$$

5. $\therefore\ y=k\,.\sin(x+a),\ \therefore\ \dfrac{d^2y}{dx^2}=-k\sin(x+a)=-y,\ \therefore\ \&c.$

6. Here $\dfrac{d^2}{dx^2}(xy)=ae^x+be^{-x}=xy=x\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}$ by Art. 80.

7. $y\dfrac{dy}{dx}+bx=0,\ \therefore\ y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=-b=+\dfrac{y}{x}\,.\,\dfrac{dy}{dx}$ &c.

8. $\therefore\ (ae^y-be^{-y})\dfrac{dy}{dx}=fe^x-ge^{-x}$,

$$\therefore\ (ae^y+be^{-y})\left(\frac{dy}{dx}\right)^2+(ae^y-be^{-y})\frac{d^2y}{dx^2}=fe^x+ge^{-x}=ae^y+be^{-y},$$

or $$(ae^y+be^{-y})\left\{\left(\frac{dy}{dx}\right)^2-1\right\}+(ae^y-be^{-y})\frac{d^2y}{dx^2}=0\ \ldots\ldots\ldots\ (\alpha),$$

$$\therefore\ (ae^y-be^{-y})\left\{\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}\right\}+(ae^y+be^{-y})\,.\,3\frac{dy}{dx}\,.\,\frac{d^2y}{dx^2}=0\ \ldots.\ (\beta),$$

$\therefore$ &c.

9. $c+\log x=e^{\frac{y}{x}}\cdot\left(\frac{x}{x+y}\right);$

$$\therefore\ \frac{1}{x}=e^{\frac{y}{x}}\left\{\frac{x}{x+y}\left(\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}\right)+\frac{1}{x+y}-\frac{x}{(x+y)^2}-\frac{x}{(x+y)^2}\cdot\frac{dy}{dx}\right\}$$

$$=\frac{e^{\frac{y}{x}}}{(x+y)^2}\left\{(x+y)\frac{dy}{dx}-\frac{y}{x}(x+y)+x+y-x-x\frac{dy}{dx}\right\},$$

or
$$(x+y)^2\cdot e^{-\frac{y}{x}}=xy\frac{dy}{dx}-y^2.$$

10. $y\sqrt{x}=a\cos\left(\frac{\sqrt{7}}{2}\cdot\log x+b\right);$

$$\therefore\ x^{\frac{1}{2}}\cdot\frac{dy}{dx}+\frac{y}{2\sqrt{x}}=-a\sin\left(\frac{\sqrt{7}}{2}\log x+b\right)\cdot\frac{\sqrt{7}}{2x};$$

$$\therefore\ \frac{2}{\sqrt{7}}\cdot x^{\frac{3}{2}}\frac{dy}{dx}+\frac{y}{\sqrt{7}}\sqrt{x}=-a\sin\left(\frac{\sqrt{7}}{2}\log x+b\right);$$

$$\therefore\ \frac{2}{\sqrt{7}}x^{\frac{3}{2}}\frac{d^2y}{dx^2}+\frac{3}{\sqrt{7}}x^{\frac{1}{2}}\cdot\frac{dy}{dx}+\frac{x^{\frac{1}{2}}}{\sqrt{7}}\cdot\frac{dy}{dx}+\frac{1}{\sqrt{x}}\cdot\frac{y}{2\sqrt{7}}$$

$$=-a\cos\left(\frac{\sqrt{7}}{2}\log x+b\right)\cdot\frac{\sqrt{7}}{2x}=-\frac{y\sqrt{7}}{2\sqrt{x}},$$

and
$$\therefore\ \frac{4x^2}{7}\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{8x}{7}+\frac{y}{7}=-y;$$

$$\therefore\ x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+2y=0.$$

11. $0=ax+b\left(y+x\frac{dy}{dx}\right)+cy\frac{dy}{dx}$ ....(1);

$$\therefore\ 0=a+b\left(2\frac{dy}{dx}+x\frac{d^2y}{dx^2}\right)+c\left\{\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right\},$$

and
$$0=b\left(3\frac{d^2y}{dx^2}+x\frac{d^3y}{dx^3}\right)+c\left(3\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+y\frac{d^3y}{dx^3}\right).$$

Eliminating a, $0=b\left(y-x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}\right)+c\left\{y\frac{dy}{dx}-x\left|\overline{\frac{dy}{dx}}\right|^2-xy\frac{d^2y}{dx^2}\right\},$

and
$$\therefore\ \left(3\frac{d^2y}{dx^2}+x\cdot\frac{d^3y}{dx^3}\right)\left(y\frac{dy}{dx}-x\left|\overline{\frac{dy}{dx}}\right|^2-xy\frac{d^2y}{dx^2}\right)$$

$$=\left(3\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+y\frac{d^3y}{dx^3}\right)\left(y-x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}\right);$$

$$\therefore \left(y - x\frac{dy}{dx}\right)\frac{d^3y}{dx^3}\left(x\frac{dy}{dx} - y\right) = x\frac{d^2y}{dx^2}\left(3y \cdot \frac{d^2y}{dx^2} - 3x\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\right);$$

$\therefore \left(y - x\frac{dy}{dx}\right)\frac{d^3y}{dx^3} + 3x\left(\frac{d^2y}{dx^2}\right)^2 = 0$, $\because$ the other result $x\frac{dy}{dx} - y = 0$ does not agree with equation (1).

12. Differentiating with regard to x,

$$-\frac{1}{z^2}\cdot\frac{dz}{dx} + \frac{1}{x^2} = f'\left(\frac{1}{y} - \frac{1}{z}\right)\cdot\left(\frac{1}{x^2}\right),$$

and with regard to y, $\quad -\frac{1}{z^2}\cdot\frac{dz}{dy} = f'\left(\frac{1}{y} - \frac{1}{z}\right)\left(-\frac{1}{y^2}\right);$

$\therefore$ eliminating f', $\quad -\frac{x^2}{z^2}\cdot\frac{dz}{dx} + 1 - \frac{y^2}{z^2}\frac{dz}{dy} = 0$, $\therefore$ &c.

13. $\frac{1}{z}\cdot\frac{dz}{dx} = b\,\{\phi'(ay+bx) - \psi'(ay-bx)\}$,

$$\frac{1}{z}\cdot\frac{dz}{dy} = a\,\{\phi'(ay+bx) + \psi'(ay-bx)\};$$

$$\therefore\ -\frac{1}{z^2}\cdot\left|\frac{dz}{dx}\right|^2 + \frac{1}{z}\frac{d^2z}{dx^2} = b^2\{\phi''(ay+bx) + \psi''(ay-bx)\},$$

and $\quad -\frac{1}{z^2}\cdot\left|\frac{dz}{dy}\right|^2 + \frac{1}{z}\cdot\frac{d^2z}{dy^2} = a^2\{\phi''(ay+bx) + \psi''(ay-bx)\}$, $\therefore$ &c.

14. $\frac{dz}{dx} = e^{\frac{x}{x+y}}\cdot\left\{\phi(x+y)\cdot\left(\frac{1}{x+y} - \frac{x}{(x+y)^2}\right) + \phi'(x+y)\right\}$,

$\frac{dz}{dy} = e^{\frac{x}{x+y}}\cdot\left\{\phi(x+y)\cdot\left(-\frac{x}{\overline{x+y}|^2}\right) + \phi'(x+y)\right\};$

$$\therefore\ \frac{dz}{dx} - \frac{dz}{dy} = \frac{z}{x+y}.$$

15. $\frac{dz}{dx} = e^x \sin y\,.\,\phi'(e^x\sin y)$, $\frac{dz}{dy} = e^x\cos y\,\phi'\,.\,(e^x\sin y)$,

and $\quad \therefore \cos y\frac{dz}{dx} = \sin y\,.\,\frac{dz}{dy}.$

16. $-f(z) = \frac{dz}{dx} \div \frac{dz}{dy}$,

$$\therefore\ -f'(z)\cdot\frac{dz}{dx} = \left(\frac{d^2z}{dx^2}\cdot\frac{dz}{dy} - \frac{dz}{dx}\cdot\frac{d^2z}{dxdy}\right) \div \left(\frac{dz}{dy}\right)^2,$$

and $\quad -f'(z)\frac{dz}{dy} = \left(\frac{d^2z}{dxdy}\cdot\frac{dz}{dy} - \frac{dz}{dx}\cdot\frac{d^2z}{dy^2}\right) \div \left(\frac{dz}{dy}\right)^2,$

and result follows on eliminating $f'(z)$.

17. $\dfrac{dz}{dx}=f'\cdot\left\{-n\dfrac{dz}{dx}(x-mz)-(y-nz)\left(1-m\dfrac{dz}{dx}\right)\right\}\div(x-mz)^2,$

$$\frac{dz}{dy}=f'\cdot\left\{\left(1-n\frac{dz}{dy}\right)(x-mz)+(y-nz)\,m\frac{dz}{dy}\right\}\div(x-mz)^2;$$

$$\therefore\ \frac{dz}{dx}\div\frac{dz}{dy}=-\frac{(y-nz)+(nx-my)\dfrac{dz}{dx}}{(x-mz)+(my-nx)\dfrac{dz}{dy}},$$

$$\therefore\ (x-mz)\frac{dz}{dx}+(y-nz)\frac{dz}{dy}=0.$$

18. $\dfrac{dz}{dx}=\phi+ax\phi'+ay\psi'$, so $\dfrac{dz}{dy}=\psi+by\,.\,\psi'+bx\phi'$;

$$\therefore\ \frac{d^2z}{dx^2}=2a\phi'+a^2x\phi''+a^2y\psi'',$$

$$\frac{d^2z}{dxdy}=b\phi'+abx\phi''+a\psi'+aby\psi'',$$

$\dfrac{d^2z}{dy^2}=2b\psi'+b^2y\psi''+b^2x\phi''$, and the result follows.

19. $\dfrac{du}{dx}=e^{nx}(nF+F')+e^{-nx}(-nf+f'),$

$$\frac{d^2u}{dx^2}=e^{nx}(n^2F+2nF'+F'')+e^{-nx}(n^2f-2nf'+f'')$$

$$=n^2u+2n\{e^{nx}\,.\,F'-e^{-nx}f'\}+e^{nx}F''+e^{-nx}f$$

and $\dfrac{du}{dy}=e^{nx}\,.\,F'-e^{-nx}\,.\,f',\ \therefore\ \dfrac{d^2u}{dy^2}=e^{nx}\,.\,F''+e^{-nx}\,.\,f'',$

and $\therefore\ \dfrac{d^2u}{dx^2}=n^2u+2n\dfrac{du}{dy}+\dfrac{d^2u}{dy^2}.$

20. (1) $x\dfrac{dy}{dx}=\cos(\log x),\ \therefore\ \dfrac{dy}{dx}+x\dfrac{d^2y}{dx^2}=-\sin(\log x)\,.\,\dfrac{1}{x},$

or $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0.$

(2) $\dfrac{dy}{dx}=\cot x,\ \therefore\ \dfrac{d^2y}{dx^2}=-\operatorname{cosec}^2x=-(1+\cot^2x),\ \therefore$ &c.

21. $\dfrac{dz}{dx}=-\dfrac{1}{x^2}\,.\,\phi',\ \dfrac{dz}{dy}=y+\dfrac{1}{y}\,.\,\phi';$

$$\therefore\ y\frac{dz}{dy}+x^2\frac{dz}{dx}=y^2.$$

22. $0=f(z)+xf'(z)\cdot\frac{dz}{dx}+\phi'(z)\cdot\frac{dz}{dx}$,

$$1=xf'(z)\cdot\frac{dz}{dy}+\phi'(z)\cdot\frac{dz}{dy},\ \therefore\ \frac{dz}{dx}\div\frac{dz}{dy}=-f(z),\ \therefore\ \text{\&c. by Ex. 16.}$$

23. $\frac{dz}{dx}+m=2f'\cdot\left\{(x-a)+(z-c)\frac{dz}{dx}\right\}$,

$$\frac{dz}{dy}+n=2f'\cdot\left\{y-b+(z-c)\frac{dz}{dy}\right\};$$

then eliminate f'.

24. $\frac{1}{x}\frac{dz}{dx}=3ax+2(\phi'-\psi')$,

$$\therefore\ \frac{1}{x^2}\cdot\frac{d^2z}{dx^2}-\frac{1}{x^3}\frac{dz}{dx}=\frac{3a}{x}+4(\phi''+\psi''),$$

$$\frac{1}{y}\cdot\frac{dz}{dy}=\frac{bx^2}{y}+2(\phi'+\psi'),$$

$$\therefore\ \frac{1}{y^2}\cdot\frac{d^2z}{dy^2}-\frac{1}{y^3}\cdot\frac{dz}{dy}=-\frac{bx^2}{y^3}+4(\phi''+\psi''),$$ whence the result follows.

25. $\frac{dz}{dx}=\phi',\ \frac{dz}{dy}=\phi'\cdot f'(y),\ \frac{d^2z}{dx^2}=\phi''$,

$$\frac{d^2z}{dxdy}=f'(y)\cdot\phi''=\frac{d^2z}{dx^2}\cdot\frac{dz}{dy}\div\frac{dz}{dx},\ \therefore\ \text{\&c.}$$

26. $\frac{1}{z}\cdot\frac{dz}{dx}=\frac{f'}{f}\cdot\frac{1}{y}+\frac{\phi'}{\phi}\cdot\frac{4xy^2}{(x^2+y^2)^2}+\frac{\chi'}{\chi}\cdot y$,

$$\frac{1}{z}\cdot\frac{dz}{dy}=-\frac{f'}{f}\cdot\frac{x}{y^2}-\frac{\phi'}{\phi}\cdot\frac{4x^2y}{(x^2+y^2)^2}+\frac{\chi'}{\chi}\cdot x;$$

$$\therefore\ \frac{1}{z}\left(x\frac{dz}{dx}-y\frac{dz}{dy}\right)=\frac{2x}{y}\cdot\frac{f'}{f}+\frac{8x^2y^2}{(x^2+y^2)^2}\cdot\frac{\phi'}{\phi}\ \ldots\ldots\ldots\ldots\ (1),$$

and $$\frac{1}{z}\cdot\frac{d^2z}{dx^2}-\frac{1}{z^2}\cdot\left(\frac{dz}{dx}\right)^2=\frac{1}{y^2}\left\{\frac{f''}{f}-\left(\frac{f'}{f}\right)^2\right\}+\frac{16x^2y^4}{(x^2+y^2)^4}\left\{\frac{\phi''}{\phi}-\left(\frac{\phi'}{\phi}\right)^2\right\}$$
$$+y^2\left\{\frac{\chi''}{\chi}-\left(\frac{\chi'}{\chi}\right)^2\right\}+4y^2\cdot\frac{\phi'}{\phi}\left\{\frac{1}{(x^2+y^2)^2}-\frac{4x^2}{(x^2+y^2)^3}\right\},$$

and $$\frac{1}{z}\cdot\frac{d^2z}{dy^2}-\frac{1}{z^2}\cdot\left(\frac{dz}{dy}\right)^2=\frac{x^2}{y^4}\left\{\frac{f''}{f}-\left(\frac{f'}{f}\right)^2\right\}+\frac{16x^4y^2}{(x^2+y^2)^4}\left\{\frac{\phi''}{\phi}-\left(\frac{\phi'}{\phi}\right)^2\right\}$$
$$+x^2\left\{\frac{\chi''}{\chi}-\left(\frac{\chi'}{\chi}\right)^2\right\}+\frac{2x}{y^3}\cdot\frac{f'}{f}-4x^2\cdot\frac{\phi'}{\phi}\left\{\frac{1}{(x^2+y^2)^2}-\frac{4y^2}{(x^2+y^2)^3}\right\};$$

$$\therefore\ \frac{1}{z}\left(x^2\frac{d^2z}{dx^2}-y^2\frac{d^2z}{dy^2}\right)=\frac{1}{z^2}\left\{x^2\left(\frac{dz}{dx}\right)^2-y^2\left(\frac{dz}{dy}\right)^2\right\}+4x^2y^2\cdot\frac{\phi'}{\phi}\left\{\frac{-2}{(x^2+y^2)^2}\right\}-\frac{2x}{y}\cdot\frac{f'}{f};$$

$$\therefore\ \text{by (1)}\ z\left(x^2\frac{d^2z}{dx^2}-y^2\frac{d^2z}{dy^2}\right)=x^2\left(\frac{dz}{dx}\right)^2-y^2\left(\frac{dz}{dy}\right)^2-z\left(x\frac{dz}{dx}-y\frac{dz}{dy}\right),\ \therefore\ \text{\&c.}$$

27. Taking x, y, z as the independent variables, if

$$x(u-y)=s, \text{ and } x(y-z)=t,$$

$$\frac{du}{dx}=2xf+x^2\frac{df}{ds}\cdot\left(u-y+x\frac{du}{dx}\right)+x^2\frac{df}{dt}(y-z),$$

or
$$x\frac{du}{dx}-2(u+y+z)=x^3\frac{df}{ds}\left(x\frac{du}{dx}+u-y\right)+x^3\cdot\frac{df}{dt}(y-z) \text{ ...... (1)},$$

$$\frac{du}{dy}+1=x^2\frac{df}{ds}\left(x\frac{du}{dy}-x\right)+x^2\frac{df}{dt}\cdot x \text{ .... (2)},$$

$$\frac{du}{dz}+1=x^2\frac{df}{ds}\cdot x\frac{du}{dz}+x^2\frac{df}{dt}(-x) \text{ .... (3)}:$$

from (2) and (3) $\frac{du}{dy}+\frac{du}{dz}+2=x^3\frac{df}{ds}\left(\frac{du}{dy}+\frac{du}{dz}-1\right)$ and by (1) and (3),

$$x\frac{du}{dx}-2(u+y+z)+(y-z)\left(\frac{du}{dz}+1\right)=x^3\frac{df}{ds}\cdot\left\{x\frac{du}{dx}+u-y+(y-z)\frac{du}{dz}\right\};$$

$$\therefore\ \left(\frac{du}{dy}+\frac{du}{dz}+2\right)\left\{x\frac{du}{dx}+u-y+(y-z)\frac{du}{dz}\right\}$$

$$=\left(\frac{du}{dy}+\frac{du}{dz}-1\right)\left\{x\frac{du}{dx}+(y-z)\frac{du}{dz}-2u-y-3z\right\},$$

and
$$\therefore\ \left(\frac{du}{dy}+\frac{du}{dz}\right)(3u+3z)+3x\frac{du}{dx}+3(y-z)\frac{du}{dz}=3y+3z,$$

and $\therefore$ &c.

28. $\frac{du}{dy}=\frac{d\phi}{dF}\cdot 2y+\frac{d\phi}{df}\cdot\left(\frac{3z}{x}-\frac{6y^2}{x^2}\right),$

$$\frac{du}{dz}=\frac{d\phi}{dF}\cdot(-x)+\frac{d\phi}{df}\left(\frac{3y}{x}\right), \text{ and } \frac{du}{dt}=\frac{d\phi}{df}\cdot(-1);$$

$$\therefore\ x\frac{du}{dy}+2y\frac{du}{dz}=-\frac{du}{dt}\cdot 3z, \ \therefore \text{ \&c.}$$

29. $\frac{1}{u}\cdot\frac{du}{dx}=\frac{1}{x}+\frac{1}{F}\left\{\frac{dF}{df_1}\cdot 2x+\frac{dF}{df_2}\cdot(y+z)\right\},$

and 2 similar equations, $\therefore$ eliminating $\frac{dF}{df_1}$ and $\frac{dF}{df_2}$,

$$\frac{y-z}{u}\cdot\frac{du}{dx}+\frac{z-x}{u}\cdot\frac{du}{dy}+\frac{x-y}{u}\cdot\frac{du}{dz}=\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z} \text{ or, \&c.}$$

30. Taking x as independent variable, as in Arts. 196 and 197,

$$\frac{d^2x}{dz^2}=\phi=-\frac{d^2z}{dx^2}\div\left(\frac{dz}{dx}\right)^3 \quad\ldots\ldots\ldots\ldots (1),$$

and
$$\frac{d^2y}{dz^2}=\psi=\left(\frac{d^2y}{dx^2}\cdot\frac{dz}{dx}-\frac{dy}{dx}\cdot\frac{d^2z}{dx^2}\right)\div\left(\frac{dz}{dx}\right)^3 \quad\ldots\ldots\ldots (2),$$

$$\therefore \text{ by (1) } \psi\cdot\left(\frac{dz}{dx}\right)^2=\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\phi\cdot\left(\frac{dz}{dx}\right)^2,$$

$$\therefore \frac{1}{\left(\frac{dz}{dx}\right)^2}=\left(\psi-\phi\cdot\frac{dy}{dx}\right)\div\frac{d^2y}{dx^2},$$

$\therefore$ differentiating, by (1), $2\phi=\frac{d}{dx}\left\{\left(\psi-\phi\cdot\frac{dy}{dx}\right)\div\frac{d^2y}{dx^2}\right\}$.

31. $\frac{dz}{dx}=nx^{n-1}\cdot f-f'\cdot y\cdot x^{n-2}+\phi'\cdot\frac{1}{y^{n+1}},$

$$\frac{dz}{dy}=x^{n-1}\cdot f'-\frac{n}{y^{n+1}}\cdot\phi-\frac{x}{y^{n+2}}\cdot\phi'.$$

$$\therefore x\frac{dz}{dx}+y\frac{dz}{dy}=n\cdot x^n\cdot f-\frac{n}{y^n}\cdot\phi.$$

So
$$\left(x\frac{d}{dx}+y\frac{d}{dy}\right)\left(x\frac{dz}{dx}+y\frac{dz}{dx}\right)=n^2\cdot x^n\cdot f+\frac{n^2}{y^n}\cdot\phi,$$

$$\text{i.e. } n^2z=x\frac{d^2z}{dx^2}+2xy\frac{d^2z}{dxdy}+y^2\frac{d^2z}{dy^2}+x\frac{dz}{dx}+y\frac{dz}{dy}.$$

32. $\frac{dz}{dx}=\phi_2+2x\phi_3+ + (n-1)x^{n-2}\cdot\phi_n-\frac{y}{x^2}(\phi_1'+x\phi_2'+ + x^{n-1}\cdot\phi_n'),$

and
$$\frac{dz}{dy}=\frac{1}{x}(\phi_1'+x\phi_2'+ + x^{n-1}\cdot\phi_n');$$

$$\therefore \frac{dz}{dx}+\frac{y}{x}\cdot\frac{dz}{dy}=\phi_2+2x\phi_3+ + (n-1)x^{n-2}\cdot\phi_n,$$

where ϕ_1 is eliminated, cf. Art. 251;

so $\left(\frac{d}{dx}+\frac{y}{x}\cdot\frac{d}{dy}\right)\left(\frac{dz}{dx}+\frac{y}{x}\cdot\frac{dz}{dy}\right)=2\phi_3+3\cdot 2\phi_4+ + (n-1)(n-2)x^{n-3}\cdot\phi_n,$

and thus, expressed symbolically,

$$\left(\frac{d}{dx}+\frac{y}{x}\cdot\frac{d}{dy}\right)^n\cdot(z)=0.$$

This is equivalent to $\left(x\frac{d}{dx}+y\frac{d}{dy}\right)^n(z)=0.$

CHAPTER XVIII.

1. $\frac{dy}{dx}(x-1)(x-2)+\frac{(2x-3)(x-3)}{(x-1)(x-2)}=1$, and $\frac{dy}{dx}=0$,

$$\therefore\ (2x-3)(x-3)=(x-1)(x-2) \text{ or } x^2-6x+7=0;$$

$$\therefore\ x=3\pm\sqrt{2}.$$

2. $2y^2.\frac{dy}{dx}=(x-a)^2(x-c)\left\{\frac{2}{x-a}+\frac{1}{x-c}\right\}$, $\therefore\ \frac{dy}{dx}=0$,

when $$3x=a+2c.$$

3. When x and y are very small, the curve is coincident with $x+y=0$, i.e. &c.

4. When x and y are both very small, the curve coincides with $x-y=0$, i.e. &c.

5. (Cf. Art. 262) the tangent at (x, y) is $(x'-x)\,x^{-\frac{1}{3}}+(y'-y)\,y^{-\frac{1}{3}}=0$, and $\therefore$ (1) the perpendicular from the origin

$$=(x^{\frac{2}{3}}+y^{\frac{2}{3}})\div\sqrt{x^{-\frac{2}{3}}+y^{-\frac{2}{3}}}=a^{\frac{2}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\div a^{\frac{1}{3}}=\&c.$$

(2) the tangent is $\frac{x'}{x^{\frac{1}{3}}}+\frac{y'}{y^{\frac{1}{3}}}=a^{\frac{2}{3}}$, and $\therefore$ the length between the axes

$$=a^{\frac{2}{3}}.\sqrt{x^{\frac{2}{3}}+y^{\frac{2}{3}}}=a.$$

6. The tangent at (x, y) is $\frac{(x'-x)}{(a^2x)^{\frac{1}{3}}}+\frac{y'-y}{(b^2y)^{\frac{1}{3}}}=0$;

$$\therefore\ x_1=(a^2x)^{\frac{1}{3}},\ y_1=(b^2y)^{\frac{1}{3}},\ \therefore\ \left(\frac{x_1}{a}\right)^2+\left(\frac{y_1}{b}\right)^2=\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$$

7. The tangent at (a, b) is

$$(x'-a).\frac{na^{n-1}}{a^n}+(y'-b)\frac{nb^{n-1}}{b^n}=0, \text{ or } \frac{x'-a}{a}+\frac{y'-b}{b}=0,$$

which is the same for all values of n.

8. $y'-y=(x'-x)\frac{a^{n-1}}{ny^{n-1}}=(x'-x)\frac{y}{nx}$, or $\frac{y'}{y}-\frac{x'}{nx}=\frac{n-1}{n}$.

Hence the area $$=-\frac{1}{2}\left(\frac{n-1}{n}\right)^2.nxy=-\frac{1}{2}(n-1)^2.\frac{1}{n}.\frac{y^{n+1}}{a^{n-1}},$$

which is constant for all values of y, if $n+1=0$, i.e. if the curve be $a^2=xy$; (a rectangular hyperbola).

9. Here $\cot\phi = -\frac{dy}{dx} = \left(\frac{y}{x}\right)^{\frac{1}{3}}$, $\therefore\ \frac{\cos\phi}{y^{\frac{1}{3}}} = \frac{\sin\phi}{x^{\frac{1}{3}}} = \frac{1}{a^{\frac{1}{3}}}$,

and $\therefore$ the normal is $y'\cos\phi - x'\sin\phi = y\cos\phi - x\sin\phi$

$$= \cos\phi \,.\, a\cos^3\phi - \sin\phi \,.\, a\sin^3\phi = a\cos 2\phi,$$

or dropping the dashes, $y\cos\phi - x\sin\phi = a\cos 2\phi$.

10. Both curves pass through the origin, and their tangents there are given by the terms of the first degree; thus the parabola touches one branch of the cubic and cuts the other branch at right angles at the origin. The curves also meet where $x^3 = axy$, i.e. $x^2 = ay = \sqrt{2a^3x}$ and $\therefore\ x^3 = a^3 \,.\, 2$ or $x = a\sqrt[3]{2}$ and $y = a\sqrt[3]{4}$: and if at this point, θ, θ' be the angles their tangents make with the axis of x, $\tan\theta = \frac{a}{y} = \frac{1}{\sqrt[3]{4}}$; and

$$3(x^2 - ay) + 3(y^2 - ax)\tan\theta' = 0, \ \therefore\ \tan\theta' = 0,$$

and $\therefore$ the required angle $= \tan^{-1}\left(\frac{1}{\sqrt[3]{4}}\right)$.

11. Let the eccentric angle of the point on the ellipse be θ, then the 2 points are $(a\cos\theta,\ b\sin\theta)$, $(a\cos\theta,\ a\sin\theta)$, and the 2 tangents make angles $\tan^{-1}m$, $\tan^{-1}m'$ with the axis, where $-m = \frac{b^2}{a^2} \,.\, \frac{a}{b}\cot\theta$, $-m' = \cot\theta$,

$$\therefore\ \tan\phi = \cot\theta\left(1 - \frac{b}{a}\right) \div \left(1 + \frac{b}{a}\cot^2\theta\right),$$

or $(a-b)\cot\phi = a\tan\theta + b\cot\theta$, and for a maximum of ϕ,

$$a\sec^2\theta = b\,\mathrm{cosec}^2\,\theta, \text{ and } \therefore\ \tan\theta = \sqrt{\frac{b}{a}},$$

thus $\tan\phi = (a-b) \div \{2\sqrt{ab}\}$. When $\theta = 0$ or $\frac{\pi}{2}$, $\phi = 0$.

12. $(h-x)\frac{x^2}{a^3} + (k-y)\frac{y^2}{b^3} = 0$, $\therefore\ \frac{hx^2}{a^3} + \frac{ky^2}{b^3} = 1$,

i.e. $(x,\ y)$ lies on &c.

13. At any such point $0 = 4a\left(1 + \cos\frac{x}{a}\right)$, or $\cos\frac{x}{a} = -1$,

and $\therefore$ at all such points $y^2 = 4ax + 4a \,.\, 0$, i.e. $y^2 = 4ax$, $\therefore$ &c.

14. The tangent and normal at $(x,\ y)$ are $y'y = 2a(x'+x)$ and

$$(y'-y)\,2a + (x'-x) \,.\, y = 0, \ \therefore \text{ at } Q,\ x' = -a \text{ and } y' \,.\, 2a = y(x+3a),$$

and at R $y' \,.\, y = 2a(x-a)$, $\therefore\ QR = \frac{1}{y} \,.\, 2(x+a)^2 = \frac{(x+a)^2}{\sqrt{ax}}$,

$\therefore$ for minimum $2(x+a) = \frac{(x+a)^2}{2x}$, $\therefore\ x = \frac{a}{3}$.

(2) $$PQR=\frac{1}{2}QR(x+a)=(x+a)^3\div 2\sqrt{ax},$$

∴ for minimum $$3(x+a)^2=\frac{(x+a)^3}{2x}, \text{ and } \therefore x=\frac{a}{5}.$$

CHAPTER XIX.

1. If $y=mx+c$ be an asymptote, $(mx+c)^2(x-2a)=x^3-a^3$ when x is very great, thus $m^2=1$, and the coefficient of x^2 gives $2mc=2am^2$, ∴ $c=\pm a$ and $m=\pm 1$ and $y=\pm(x+a)$ are asymptotes; also the coefficient of y^2 gives $x=2a$ when y is ∞ (Art. 277).

2. If $y=mx+c$ be an asymptote $(mx+c)^3=x^2(2a-x)$ when $x=\infty$; thus $m^3+1=0$, ∴ m has only one real value, viz. -1, and $3m^2c=2a$, ∴ the real asymptote is $$y+x=\frac{2a}{3}.$$

3. If $y=mx+c$, $m^3=0$ and $c=0$, ∴ $y=0$ is an asymptote. This is also given by the coefficient of x^2, and that of y gives 2 imaginary asymptotes, and there can be but 3 in all to a cubic.

4. If $y=mx+c$, $(mx+c)^2(\overline{am+b}\,x+ac)=a^2(mx+c)^2+b^2x^2$, ∴ for asymptotes $m^2(am+b)=0$, i.e. $m=0$ or $-\frac{b}{a}$,

and $$2cm(am+b)+m^2ac=m^2a^2+b^2,$$

∴ $m=0$ requires $c=\infty$, and $am+b=0$ gives $c=\dfrac{2b^2}{\frac{b^2}{a}}=2a$, ∴ the finite asymptote is $y=-\frac{b}{a}x+2a$, the other 2 being at ∞.

5. If $y=mx+n$ be an asymptote, $m^3=1$, ∴ $m=1$ is the only real value, and $$3m^2\,.\,n=-(2a+c),\ \therefore n=-\frac{2a+c}{3}.$$

Aliter: $$y=x\left(1-\frac{a}{x}\right)^{\frac{2}{3}}\left(1-\frac{c}{x}\right)^{\frac{1}{3}}=x\left(1-\frac{2a}{3x}-\frac{c}{3x}\right),$$

when $$x=\infty,\ \therefore y=x-\frac{2a+c}{3}.$$

Aliter: by last paragraph of Art. 274.

6. The highest powers of x and y give the axes as asymptotes. Also $y=mx+c$ gives $m^2+m=0$, i.e. $m=-1$ for the 3rd asymptote, and

$$2mc+c=0 \text{ or } c=0, \text{ i.e. } x+y=0.$$

7. The coefficient of x^3 gives $y=\pm a$ for asymptotes, and the coefficient of y^2 gives 2 imaginary asymptotes, and there can be but 4 to a quartic.

8. The coefficient of y^2 gives $3x+a=0$ for one asymptote: for the other 2, let $y=mx+c$ be one, then $x^3=ax^2+(a+3x)(mx+c)^2$, and $\therefore\ 1=3m^2$ or $m=\pm\frac{1}{\sqrt{3}}$, and $0=a+am^2+6mc$, i.e. $c=-a\,.\,\frac{4}{3}\div\pm 2\sqrt{3}=\mp\frac{2a}{3\sqrt{3}}$, and the other two asymptotes are, $\therefore\ y=\pm\left(\frac{x}{\sqrt{3}}-\frac{2a}{3\sqrt{3}}\right)$.

9. The coefficients of x^2 and y^2 give $x+a=0$ and $y+b=0$ for 2 asymptotes, and for the remaining one, $y=mx+c$ gives $(x+a)(mx+c)^2=x^2(mx+c+b)$, and $\therefore\ m^2=m$ or $m=1$, and $am^2+2mc=c+b$; $\therefore\ c=b-a$, and the asymptote is $y=x+b-a$.

10. If $y=mx+c$ be an asymptote,

$$\{(m-2)x+c\}\{(mx+c)^2-x^2\}-a\{(m-1)x+c\}^2+4a^2(x+y)=a^3,$$

$$\therefore\ (m-2)(m^2-1)=0,\ \text{i.e. } m=\pm 1 \text{ or } 2,$$

and $c(m^2-1)+2mc(m-2)=a(m-1)^2$; thus if $m=1$, $-2c=0$, and the asymptote is $y=x$; if $m=-1$, $+6c=4a$, and the asymptote is

$$x+y=\frac{2}{3}a;\ \text{and if } m=2,\ 3c=a \text{ and the asymptote is } y=2x+\frac{a}{3}.$$

11. If $y=mx+c$ be an asymptote,

$$(mx+c)^2(\overline{m-1}\,x+c)^2-ax^2(\overline{m-1}\,x+c)-3a^2y^2=a^4;$$

thus $m^2(m-1)^2=0$ or $m=0$ or 1, and $2mc(m-1)^2+2(m-1)c\,.\,m^2=a(m-1)$, $\therefore$ if $m=0$, c is ∞, and the 2 corresponding asymptotes are at ∞, but if $m=1$, c is indeterminate, and proceeding to the coefficient of x^2, and neglecting terms involving $(m-1)$, $m^2c^2-ac-3a^2m^2=0$ or $c^2-ac-3a^2=0$, and $\therefore\ \frac{2c}{a}=1\pm\sqrt{13}$, and $\therefore$ the asymptotes are $y=x+\frac{a}{2}(1\pm\sqrt{13})$.

12. If $y=mx+c$ be an asymptote,

$$x^3-2(mx+c)^3-3x(mx+c)^2-a^2x+2a^2y=a^3,$$

and $\therefore\ 1-2m^3-3m^2=0$, i.e. $1-m^2=2m^2(m+1)$, and $\therefore\ m=-1$,

or $$2m^2+m-1=0,\ \text{i.e. } m=\frac{1}{2} \text{ or } -1;$$

and $6m^2c+6mc=0$, $\therefore$ if $m=\frac{1}{2}$, $c=0$ and one asymptote is $2y=x$; but if $m=-1$ the 2 other asymptotes are indeterminate, and proceeding to the coefficient of x, $-6c^2m-3c^2-a^2=-2a^2m$, i.e. $3c^2=3a^2$ and $c=\pm a$, and the 2 asymptotes are $y+x=\pm a$.

13. The coefficient of y^2 gives $x=\pm a$ for 2 of the asymptotes; and if $y=mx+c$, $x^2\{(m-1)x+c\}^2=a^2\{(mx+c)^2+x^2\}$, and $\therefore\ (m-1)^2=0$, and $2c(m-1)=0$ which makes c indeterminate, and proceeding to x^2,

$$c^2=a^2(m^2+1)=2a^2,$$

and thus the 2 asymptotes are $y=x\pm a\sqrt{2}$.

14. The coefficient of y^2 gives $x = \pm a$ for 2 asymptotes; and for the other 2, if $y = mx + c$, $\{(m-1)x + c\}^2 (x^2 - a^2) = a^4$, $\therefore (m-1)^2 = 0$, $2c(m-1) = 0$, and $c^2 = a^2 (m-1)^2$, $\therefore c = 0$, and there are 2 coincident asymptotes $y = x$.

15. If $y = mx + c$ be an asymptote,

$$(mx+c)^3 - 3x(mx+c)^2 + 4x^3 + a(mx+c)^2 + ax(mx+c) - 6ax^2 + 2b^2x - b^2y + c^3 = 0,$$

$$\therefore m^3 - 3m^2 + 4 = 0, \text{ i.e. } m + 1 = 0 \text{ or } m^2 - m + 1 - 3m + 3 = 0,$$

$$\text{i.e. } m^2 - 4m + 4 = (m-2)^2 = 0; \text{ and } 3m^2c - 6mc + am^2 + am = 6a,$$

$\therefore$ if $m = -1$, $9c = 6a$ or an asymptote is $x + y = \dfrac{2a}{3}$; if $m = 2$, $c\,(0) = 0$, and c is indeterminate, and then $3c^2m - 3c^2 + 2amc + ac + 2b^2 - b^2m = 0$, or

$$3c^2 + 5c \,.\, a = 0, \text{ i.e. } c = 0 \text{ or } -\frac{5}{3}a,$$

and the other 2 asymptotes are $y = 2x$ and $y = 2x - \dfrac{5}{3}a$.

16. By Art. 274, if $y = mx + n$ be an asymptote, on substituting this value of y in the equation to the curve, the terms of the 3rd and 2nd degrees must vanish; $\therefore$ if $y = 0$ be an asymptote, these terms must be divisible by y; similarly they must be divisible by x, if $x = 0$ be an asymptote; and $\therefore$ the equation is of the form $xy(ax + by + c) + a'x + b'y + c' = 0$: also if $ax + by + c = 0$, the same terms vanish, and $\therefore$ $ax + by + c = 0$ is the 3rd asymptote.

CHAPTER XX.

1. $\tan\phi = \dfrac{r\,d\theta}{dr} = \dfrac{r}{a\cos\theta} = \tan\theta$, $\therefore$ with the fig. on p. 304, $\phi = \theta$.

2. y is a maximum or minimum at such points, $\therefore \dfrac{d}{d\theta}(r\sin\theta) = 0$

$$= r\cos\theta + \sin\theta\,\frac{dr}{d\theta}, \ \therefore \cos\theta\,(1 + \cos\theta) + \cos^2\theta - 1 = 0,$$

$= 2\cos^2\theta + \cos\theta - 1$, $\therefore 4\cos\theta = -1 \pm 3$, $\therefore \cos\theta = \dfrac{1}{2}$ or -1, and $\theta = \pm\dfrac{\pi}{3}$ or π.

3. $r^2\dfrac{d\theta}{dr} = -\dfrac{a}{r^2}\,.\,r^2 = -a =$ constant. T in the fig. on p. 308, will fall on the opposite side of S in this spiral.

4. $\dfrac{ab}{r} = ae^{\theta} + be^{-\theta}$, $\therefore -\dfrac{ab}{r^2} = (ae^{\theta} - be^{-\theta})\dfrac{d\theta}{dr}$, $\therefore$ &c.

5. Comparing the 1st figure to the ellipse in Todhunter's *Conic Sections*, the equation being $\frac{l}{r}=1+e\cos\theta$, if P, T in Art. 285 be (r, θ) and (ρ, ϕ);

$$\rho=r^2\frac{d\theta}{dr}, \text{ and } \phi=\theta-\frac{\pi}{2}, \therefore \rho=\frac{l}{e\sin\theta}=\frac{l}{e\cos\phi},$$

$\therefore$ the locus of T may be written $x=\frac{l}{e}$, $\therefore$ &c.

6. (1) If $\frac{l}{r}=1+e\cos\theta$, $\tan\phi=\frac{rd\theta}{dr}=\frac{l}{re\sin\theta}=\frac{1+e\cos\theta}{e\sin\theta}$.

Also $e\sec^2\phi\,.\,\frac{d\phi}{d\theta}=\frac{d}{d\theta}(\operatorname{cosec}\theta+e\cot\theta)=-\operatorname{cosec}\theta\cot\theta-e\operatorname{cosec}^2\theta$;

$\therefore$ for maximum of ϕ, $\cos\theta=-e$, and then $e\sec^2\phi\,.\,\frac{d^2\phi}{d\theta^2}$

$$=\operatorname{cosec}\theta\,.\,\cot^2\theta+\operatorname{cosec}^3\theta+2e\operatorname{cosec}^2\theta\,.\,\cot\theta=\frac{e^2+1-2e^2}{\sin^3\theta},$$

$\therefore$ for a maximum $\sin\theta$ is negative and $\therefore$ $\theta=\pi+\cos^{-1}e$, and

$$\tan\phi=-\frac{\sqrt{1-e^2}}{e} \text{ or } \phi=\pi-\cos^{-1}e, \text{ and } r=\frac{l}{1-e^2}=a,$$

or the point is the negative end of the minor axis. The *obtuse* angle between the tangent and radius vector has the same value at the other end of the minor axis.

(2) The ellipse is $a^2\sin^2\theta+b^2\cos^2\theta=\frac{a^2b^2}{r^2}$,

$$\therefore (a^2-b^2)\sin\theta\cos\theta\,.\,\frac{d\theta}{dr}=-\frac{a^2b^2}{r^3};$$

$$\therefore (a^2-b^2)\tan\phi=-\frac{a^2\sin^2\theta+b^2\cos^2\theta}{\sin\theta\cos\theta}=-(a^2\tan\theta+b^2\cot\theta),$$

$$\therefore \text{ for maximum of } \phi,\ -(a^2-b^2)\sec^2\phi\,.\,\frac{d\phi}{d\theta}=0=a^2\sec^2\theta-b^2\operatorname{cosec}^2\theta,$$

and $\therefore$ $\tan\theta=\pm\frac{b}{a}$, and then $-(a^2-b^2)\sec^2\phi\,.\,\frac{d^2\phi}{d\theta^2}$

$$=2a^2\sec^2\theta\,.\,\tan\theta+2b^2\operatorname{cosec}^2\theta\,.\,\cot\theta,$$

$$\therefore \text{ for a maximum } \tan\theta=\frac{b}{a}, \text{ and } \frac{\sin\theta}{b}=\frac{\cos\theta}{a}=\frac{1}{\sqrt{a^2+b^2}};$$

$\therefore$ $r^2=\frac{a^2+b^2}{2}$, or the point is an extremity of an equi-conjugate diameter, and the *obtuse* angle between the tangent and radius vector has the same value at all 4 of such points.

7. $\text{Tan}\,\phi = r\dfrac{d\theta}{dr} = \dfrac{r}{a\sin\theta} = \tan\dfrac{\theta}{2}$, $\therefore\ \phi = \dfrac{\theta}{2}$ (cf. a figure); and

$$\therefore\ p = r\sin\phi = a(1-\cos\theta)\sin\frac{\theta}{2} = 2a\sin^3\left(\frac{\theta}{2}\right):$$

and polar snbtangent $= r\tan\phi = a(1-\cos\theta)\,.\,\tan\dfrac{\theta}{2} = 2a\sin^2\dfrac{\theta}{2}\,.\,\tan\dfrac{\theta}{2}$.

8. $\cos 2\theta = \dfrac{a^2}{r^2}$, $\therefore\ \tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a^2}{r^2\sin 2\theta}$,

and $\sin\phi = \dfrac{a^2}{\sqrt{a^4 + r^2\sin^2 2\theta}} = \dfrac{a^2}{r^2}$ by the given equation.

9. $r = -a^2\sin 2\theta\,.\,\dfrac{d\theta}{dr}$, $\therefore\ \tan\phi = -\dfrac{r^2}{a^2\sin 2\theta} = -\cot 2\theta = -\tan\left(\dfrac{\pi}{2} - 2\theta\right)$,

$$\therefore\ \phi = \pi - \left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{2} + 2\theta.$$

The general formula gives $\phi = n\pi + \dfrac{\pi}{2} + 2\theta$, but when θ is small $\phi < \pi$ obviously from a figure, . . $n = 0$ throughout the curve.

10. If Y be (r', θ'), $\theta' + \theta = \dfrac{\pi}{2} - \phi$, and $r' = r\sin\phi$:

also $$1 = a\sec^3\frac{\theta}{3}\tan\frac{\theta}{3}\,.\,\frac{d\theta}{dr},\ \therefore\ \tan\phi = \cot\frac{\theta}{3},$$

and $$\phi = \frac{\pi}{2} - \frac{\theta}{3},\ \therefore\ r' = r\cos\frac{\theta}{3} = a\sec^2\frac{\theta}{3},\ \text{and}\ \frac{\theta}{3} = -\frac{\theta'}{2};$$

$\therefore\ r' = a\sec^2\dfrac{\theta'}{2}$, i.e. $r'(1+\cos\theta') = 2a$, a parabola.

11. If Y be (r', θ'), $\theta' + \theta = \dfrac{\pi}{2} - \phi$, and $r' = r\sin\phi$;

also $$1 = -a\sin\theta\,.\,\frac{d\theta}{dr},\ \therefore\ \tan\phi = -\cot\frac{\theta}{2},$$

and $$\phi = \frac{\pi}{2} + \frac{\theta}{2},\ \therefore\ \theta' = -\frac{3\theta}{2},\ \text{and}\ r' = r\cos\frac{\theta}{2}$$

$$= 2a\cos^3\frac{\theta}{2} = 2a\cos^3\frac{\theta'}{3},\ \text{i.e. \&c.}$$

12. If Y be (r', θ'), $\theta' + \theta = \dfrac{\pi}{2} - \phi$, $r' = r\sin\phi$;

and (cf. Ex. 9) $$\phi = \frac{\pi}{2} + 2\theta,\ \therefore\ \theta' = -3\theta,$$

and $$r' = r\cos 2\theta\ \text{or}\ r'^2 = a^2\cos^3 2\theta = a^2\cos^3\frac{2\theta'}{3}.$$

13. r is ∞ when $\cos\theta=0$, $\therefore$ the asymptotes if any are parallel to $x=0$, and of the form $x=c$, where c is the limit when $r=\infty$, of $r\cos\theta$, which $=$ lt. of $a\cos 2\theta$ when $\cos\theta=0$, i.e. the asymptote is $r\cos\theta=a(2\cos^2\theta-1)=-a$.

14. When $r=\infty$, $\sin\theta=0$, and the asymptote is $r\sin\theta=$ lt. of $b+a\sin\theta$ when $\sin\theta=0$, i.e. $r\sin\theta=b$.

15. Here $r(\cos^2\theta-\sin^2\theta)=a$, $\therefore$ the asymptotes are $r(\cos\theta\pm\sin\theta)=$ lt. of $a\div(\cos\theta\mp\sin\theta)$ when $\cos\theta\pm\sin\theta=0$, and when

$$\cos\theta+\sin\theta=0,\ \cos\theta=-\sin\theta=\pm\frac{1}{\sqrt{2}},$$

$\therefore\ r(\cos\theta+\sin\theta)=\pm\dfrac{a}{\sqrt{2}}$; so the other asymptotes are

$$r(\cos\theta-\sin\theta)=\pm\frac{a}{\sqrt{2}}.$$

16. Here $4r\cos\theta\cos 2\theta=a(\cos 2\theta+2\cos^2\theta)$, $\therefore$ one asymptote is

$$4r\cos\theta=\text{lt. of } a(4\cos^2\theta-1)\div\cos 2\theta,$$

when $\cos\theta=0$, or $4r\cos\theta=a$. For the other asymptotes

$$\cos\theta=\pm\sin\theta=\pm\frac{1}{\sqrt{2}},$$

the ambiguities being independent; thus if $\cos\theta=\sin\theta$,

$$4r(\cos\theta-\sin\theta)=a(2-1)\div(\pm\sqrt{2})\left(\pm\frac{1}{\sqrt{2}}\right)=a;$$

so if $\cos\theta=-\sin\theta$, $4r(\cos\theta+\sin\theta)=a$.

CHAPTER XXI.

1. Near the origin the curve approximates to the axis of x, and ultimately when $y=0$ that axis and the curve coincide, $\therefore$ $y=0$ is the tangent at the origin, and y and x are of the same sign, $\therefore$ there is a point of inflexion such as Q on p. 315, at the origin.

Generally,
$$\frac{dy}{dx}=\frac{3x^2}{a^2+x^2}-\frac{2x^4}{(a^2+x^2)^2},$$

$$\therefore\ (a^2+x^2)^2\frac{dy}{dx}=x^2(3a^2+x^2);$$

$$\therefore\ (a^2+x^2)^2\cdot\frac{d^2y}{dx^2}+4x(a^2+x^2)\frac{dy}{dx}=6a^2x+4x^3,$$

and, when $\dfrac{d^2y}{dx^2}=0$, $(a^2+x^2)^2\cdot\dfrac{d^3y}{dx^3}+(4a^2+12x^2)\dfrac{dy}{dx}=6a^2+12x^2$,

$\therefore$ for points of inflexion (1) $x=0$, and then $a^4\dfrac{d^3y}{dx^3}=6a^2$, or $\dfrac{d^3y}{dx^3}$ is finite and $y=0$, $\therefore$ the origin is a point of inflexion:

$$(2)\quad 0=x\left\{6a^2+4x^2-\frac{4x^2(3a^2+x^2)}{a^2+x^2}\right\},$$

$$\therefore 6a^4=2a^2x^2 \text{ and } x=\pm a\sqrt{3};$$

and then $(4a^2)^2\cdot\dfrac{d^3y}{dx^3}=42a^2-40a^2\cdot\dfrac{3\cdot 6}{16}=-3a^2$, $\therefore \dfrac{d^3y}{dx^3}$ is finite, and there are 2 more points of inflexion, y being possible when $x=\pm a\sqrt{3}$.

2. $ay(x-a)=x^3+ax^2$, $\therefore a\dfrac{dy}{dx}(x-a)+ay=3x^2+2ax$,

$$a\frac{d^2y}{dx^2}(x-a)+2a\frac{dy}{dx}=6x+2a,$$

$\therefore$ for points of inflexion,

$$2a(x-a)\frac{dy}{dx}=(x-a)(6x+2a)=2(3x^2+2ax)-2\frac{x^3+ax^2}{x-a},$$

or $$(x-a)^2(3x+a)=(x-a)(3x^2+2ax)-(x^3+ax^2),$$

which gives $(x-a)^3=-2a^3$ and $\therefore x=a(1-\sqrt[3]{2})$: and then

$$a\frac{d^3y}{dx^3}(x-a)=6 \text{ and } \frac{d^3y}{dx^3} \text{ is finite.}$$

3. $(a^4-b^4)\dfrac{d^2y}{dx^2}=8(x-a)^3+12x(x-a)^2$, $\therefore$ for points of inflexion

$$x=a, \text{ or } 20x=8a, \text{ and } \therefore x=\frac{2a}{5};$$

and $(a^4-b^4)\dfrac{d^3y}{dx^3}=36(x-a)^2+24x(x-a)$, $\therefore \dfrac{d^3y}{dx^3}=0$ when $x=a$, and

$$=\frac{a^2}{a^4-b^4}\left\{36\cdot\frac{9}{25}-24\cdot\frac{6}{25}\right\}$$

which is not zero, when $x=\dfrac{2a}{5}$, which, $\therefore$, gives a point of inflexion, unless $a=b$ when the curve reduces to straight lines and there is no inflexion. For

$$x=a,\ (a^4-b^4)\frac{d^4y}{dx^4}=96(x-a)+24x$$

which does not vanish when $x=a$, $\therefore$ there is no point of inflexion then.

4. $\frac{y^2}{a^2}=\frac{a}{x}-1$, $\therefore\ \frac{2y}{a^2}\cdot\frac{dy}{dx}=-\frac{a}{x^2}$, $\frac{2y}{a^2}\frac{d^2y}{dx^2}+\frac{2}{a^2}\cdot\left(\frac{dy}{dx}\right)^2=\frac{2a}{x^3}$;

$\therefore$, for points of inflexion, $\left(\frac{dy}{dx}\right)^2=\frac{a^3}{x^3}=\left(\frac{a^3}{2x^2}\right)^2\cdot\frac{x}{a^2(a-x)}$,

and $\qquad \therefore\ 4(a-x)=a$ or $x=\frac{3a}{4}$, and $\therefore\ y=\frac{a}{\sqrt{3}}$;

and then $\qquad \frac{2y}{a^2}\cdot\frac{d^3y}{dx^3}=-\frac{6a}{x^4}$ or $\frac{d^3y}{dx^3}$ is finite.

5. $\frac{1}{a}\cdot\frac{dy}{dx}=\frac{2x}{a^2}+\frac{3}{5}a^{-\frac{3}{5}}\cdot(x-a)^{-\frac{2}{5}}$,

$$\frac{1}{a}\cdot\frac{d^2y}{dx^2}=\frac{2}{a^2}-\frac{6}{25}a^{-\frac{3}{5}}(x-a)^{-\frac{7}{5}},$$

$\therefore$ there is a point of inflexion when $x=a\left\{1+\left(\frac{3}{25}\right)^{\frac{5}{7}}\right\}$.

For $x=a$, changing the origin to (a, a), $\frac{y}{a}=\frac{x^2}{a^2}+\frac{2x}{a}+\left(\frac{x}{a}\right)^{\frac{3}{5}}$, or approximately $\frac{y}{a}=\left(\frac{x}{a}\right)^{\frac{3}{5}}$, $\therefore$ the curve touches the new axis of y, and x and y change signs together, and $\therefore$ there is a point of inflexion as at R in the figure to Art. 291.

6. $\frac{dy}{dx}=y\cdot\frac{1}{3}x^{-\frac{2}{3}}$, $\therefore\ \frac{d^2y}{dx^2}=y\left(\frac{1}{9}x^{-\frac{4}{3}}-\frac{2}{9}x^{-\frac{5}{3}}\right)$,

$\frac{d^3y}{dx^3}=\frac{y}{27}(x^{-2}-4x^{-\frac{7}{3}}+10x^{-\frac{8}{3}})$; thus for points of inflexion either (1) $y=0$, and then $\frac{d^3y}{dx^3}$ and clearly all succeeding differential coefficients would vanish; or (2) $x^{\frac{1}{3}}=2$, i.e. $x=8$ and then $27\frac{d^3y}{dx^3}=e^2\left(\frac{1}{64}-\frac{1}{32}+\frac{5}{128}\right)=e^2\cdot\frac{3}{128}$, and $\therefore$ there is a point of inflexion.

7. $\frac{y}{a}=\frac{x^2}{x^2+a^2}=1-\frac{a^2}{x^2+a^2}$, $\therefore\ \frac{1}{a}\cdot\frac{dy}{dx}=\frac{2a^2x}{(x^2+a^2)^2}$,

and $\frac{1}{2a^3}\cdot\frac{d^2y}{dx^2}=\frac{1}{(x^2+a^2)^2}-\frac{4x^2}{(x^2+a^2)^3}$, $\frac{1}{2a^3}\cdot\frac{d^3y}{dx^3}=-\frac{12x}{(x^2+a^2)^3}+\frac{24x^3}{(x^2+a^2)^4}$;

$\therefore$ when $\frac{d^2y}{dx^2}=0$, $3x^2=a^2$ or $x=\pm\frac{a}{\sqrt{3}}$, and then $\frac{1}{2a^3}\cdot\frac{d^3y}{dx^3}=\frac{12x}{(x^2+a^2)^4}(x^2-a^2)$ which is negative or positive as $x=\pm\frac{a}{\sqrt{3}}$, $\therefore$ there are 2 points of inflexion.

8. $y^2(2a-x)=a^2x$, $\therefore\ 2y\frac{dy}{dx}(2a-x)-y^2=a^2$,

$$\therefore\ 2y(2a-x)\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2(2a-x)-4y\frac{dy}{dx}=0;$$

$$\therefore\ \text{when } \frac{d^2y}{dx^2}=0,\ 2y(2a-x)\frac{d^3y}{dx^3}=6\left(\frac{dy}{dx}\right)^2,$$

and
$$(2a-x)\frac{dy}{dx}=2y,\ \therefore\ 4y^2=a^2+y^2,$$

and $\therefore\ (2a-x)=3x$ and $x=\frac{a}{2}$ and $\frac{d^3y}{dx^3}$ does not vanish, $\therefore$ &c.

9. $x\frac{dy}{dx}+y=\frac{a^2}{x}$, $\therefore\ x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=-\frac{a^2}{x^2}$, and $\therefore$ when

$$\frac{d^2y}{dx^2}=0,\ 2x\frac{dy}{dx}=-\frac{a^2}{x}=\frac{2a^2}{x}-2y,\ \therefore\ xy=\frac{3a^2}{2},$$

and $\therefore\ x=ae^{\frac{3}{2}}$; also $x\frac{d^3y}{dx^3}=\frac{2a^2}{x^3}$ or $\frac{d^3y}{dx^3}$ is finite, $\therefore$ &c.

10. $y=2a^{\frac{2}{3}}\cdot x^{\frac{1}{3}}\pm 2\sqrt{ax}$, $\therefore\ \frac{dy}{dx}=\frac{2}{3}a^{\frac{2}{3}}x^{-\frac{2}{3}}\pm a^{\frac{1}{2}}\cdot x^{-\frac{1}{2}}$,

$$\frac{d^2y}{dx^2}=-\frac{4}{9}a^{\frac{2}{3}}x^{-\frac{5}{3}}\mp\frac{1}{2}a^{\frac{1}{2}}x^{-\frac{3}{2}},\ \frac{d^3y}{dx^3}=\frac{20}{27}a^{\frac{2}{3}}x^{-\frac{8}{3}}\pm\frac{3}{4}a^{\frac{1}{2}}\cdot x^{-\frac{5}{2}};$$

hence for points of inflexion $8a^{\frac{2}{3}}-9a^{\frac{1}{2}}x^{\frac{1}{6}}=0$;

$$\therefore\ x=a\left(\frac{8}{9}\right)^6,\ \text{and}\ \frac{d^3y}{dx^3}$$

vanishes when $80a^{\frac{2}{3}}-81a^{\frac{1}{2}}x^{\frac{1}{6}}=0$, and $\therefore$ is finite for the value above found.

11. $\frac{y}{a^2}=\frac{a-x}{x^2+a^2}$, $\therefore\ \frac{1}{a^2}\cdot\frac{dy}{dx}=\frac{-1}{x^2+a^2}+\frac{2x(x-a)}{(x^2+a^2)^2}$,

$$\frac{1}{a^2}\cdot\frac{d^2y}{dx^2}=\frac{6x-2a}{(x^2+a^2)^2}-\frac{8x^2(x-a)}{(x^2+a^2)^3},$$

$$\frac{1}{a^2}\cdot\frac{d^3y}{dx^3}=\frac{6}{(x^2+a^2)^2}-\frac{48x^2-24ax}{(x^2+a^2)^3}+\frac{48x^3(x-a)}{(x^2+a^2)^4};$$

$\therefore$ for points of inflexion $(x^2+a^2)(3x-a)=4x^2(x-a)$,

i.e. $x^3-3ax^2-3a^2x+a^3=0=(x+a)(x^2-4ax+a^2)$ .... (1),

and $\frac{d^3y}{dx^3}=0$ when $(x^2+a^2)^2-4(x^2+a^2)(2x^2-ax)+8x^3(x-a)=0$,

or
$$x^4-4ax^3-6a^2x^2+4a^3x+a^4=0,$$
$$=x^4+ax^3-5ax^2(x+a)-a^2x(x+a)+5a^3(x+a)-4a^4,$$

and
$$=x^2(x^2-4ax+a^2)-7a^2x^2+4a^3x+a^4$$
$$=(x^2-4ax+a^2)(x^2+a^2)-8a^2x^2+8a^3x,$$

$\therefore$ the values of x in (1) do not make $\frac{d^3y}{dx^3}$ vanish, and $\therefore$ they give 3 points of inflexion, for which $x=-a$ and $\therefore y=a$, and $x=a(2\pm\sqrt{3})$,

and $\therefore \frac{y}{a}=(-1\mp\sqrt{3})\div(1+7\pm4\sqrt{3})=-\frac{1}{4}\frac{(1\pm\sqrt{3})}{2\pm\sqrt{3}}$

$$=-\frac{1}{4}(-1\pm\sqrt{3}) \text{ or } y=\frac{a}{4}(1\mp\sqrt{3}),$$

and the straight line joining the 2 last is $\frac{x-a(2+\sqrt{3})}{2a\sqrt{3}}=\frac{y-\frac{a}{4}(1-\sqrt{3})}{-\frac{a}{2}\sqrt{3}}$,

or
$$x-a(2+\sqrt{3})+4y-a(1-\sqrt{3})=0,$$
i.e. $x+4y=3a$ which is satisfied by $(-a, a)$, and $\therefore$ the 3 points of inflexion lie on a straight line.

N.B. *Generally* if there be 3 points of inflexion on a cubic curve, they lie on a straight line. Cf. Ex. 1.

12. Here $au=1-\frac{1}{\theta^2}$, $\therefore a\left(u+\frac{d^2u}{d\theta^2}\right)=1-\frac{1}{\theta^2}-\frac{2.3}{\theta^4}$, $\therefore$ for a point of inflexion $\theta^4-\theta^2-6=0=(\theta^2-3)(\theta^2+2)$, $\therefore \theta^2=3$ and $r=a.\frac{3}{2}$.

13. $bu=\theta^{-n}$, $\therefore b\left(u+\frac{d^2u}{d\theta^2}\right)=\theta^{-n}+n(n+1)\,\theta^{-(n+2)}$, and $\therefore$ for a point of inflexion $\theta^2=-n(n+1)$, and $r=b.\theta^n=b\{-n(n+1)\}^{\frac{n}{2}}$. For real values of θ, n lies between 0 and -1.

14. $\frac{dx}{d\phi}=a\sin\phi$, $\frac{dy}{d\phi}=a(n+\cos\phi)$,

$$\therefore \frac{d^2y}{dx^2}=\frac{1}{a\sin\phi}\cdot\frac{d}{d\phi}(n\operatorname{cosec}\phi+\cot\phi);$$

$\therefore$ for a point of inflexion $n\operatorname{cosec}\phi.\cot\phi+\operatorname{cosec}^2\phi=0$, or $n\cos\phi+1=0$, and

$$\frac{d^3y}{dx^3}=-\frac{1}{a^2\sin\phi}\frac{d}{d\phi}(n\operatorname{cosec}^2\phi\cot\phi+\operatorname{cosec}^3\phi)$$

$$=\frac{1}{a^2\sin\phi}(2n\operatorname{cosec}^2\phi.\cot^2\phi+n\operatorname{cosec}^4\phi+3\operatorname{cosec}^3\phi\cot\phi)$$

$$=\frac{1}{a^2\sin^5\phi}(2n\cos^2\phi+3\cos\phi+n)$$

which does not vanish when $\cos\phi=-\frac{1}{n}$, and $\therefore$ there is a point of inflexion. If $n=1$, the differential coefficients are indeterminate; the curve is then a cycloid and there is no point of inflexion.

CHAPTER XXII.

1. Near the origin $a^2y^2=a^2x^2$, $\therefore$ there is a double point the tangents to the 2 branches being $y=\pm x$. Cf. Art. 347.

2. (1) $\frac{d^2y}{dx^2}=-\sin x=0$ at $(0, 0)$, and $\frac{d^3y}{dx^3}=-\cos x=-1$, $\therefore$ there is a point of inflexion.

(2) $\frac{dy}{dx}=\cos x-x\sin x$, $\therefore$ $\frac{d^2y}{dx^2}=-2\sin x-x\cos x=0$ at $(0, 0)$,

and $\frac{d^3y}{dx^3}=-3\cos x+x\sin x=-3$, $\therefore$ a point of inflexion.

(3) $\frac{dy}{dx}=\sec^2 x$, $\therefore$ $\frac{d^2y}{dx^2}=2\sec^2 x\,.\,\tan x=0$ at $(0, 0)$,

and $\frac{d^3y}{dx^3}=4\sec^2 x\tan^2 x+2\sec^4 x=2$ at $(0, 0)$, $\therefore$ a point of inflexion.

(4) $\frac{dy}{dx}=2x\tan x+x^2\sec^2 x$,

$\therefore$ $\frac{d^2y}{dx^2}=2\tan x+4x\sec^2 x+2x^2\sec^2 x\tan x=0$ at $(0, 0)$,

and $\frac{d^3y}{dx^3}=6\sec^2 x+\text{vanishing terms}=6$, $\therefore$ there is a point of inflexion at the origin.

3. (1) To the 2nd order of small quantities $y^2=0$, $\therefore$ there is a cusp, the 2 tangents being coincident, x being only positive and y of either sign. First kind.

(2) Similarly $y-x=0$ is the tangent to a cusp at the origin of the first kind, for $y-x=\pm x^{\frac{3}{2}}$, so that x must be positive.

(3) So $y=0$ is the tangent to a cusp at $(0, 0)$, and $y=x^2\pm x^{\frac{5}{2}}$, $\therefore$ near the origin, $\because$ $x^{\frac{5}{2}}<x^2$, the cusp is of the 2nd kind.

4. When $x=a$, $y=\phi(a)$ and changing the origin to $(a, \phi(a))$ the curve is $y+\phi(a)=\phi(x+a)\pm x^{\frac{2p+1}{2q}}\,.\,F(x+a)$, $\therefore$ x must be positive, and expanding, $y=x\phi'(a)+\frac{x^2}{2}\phi''(a)\pm x^{\frac{2p+1}{2q}}\,.\,\{F(a)+x\,.\,F'(a)\}$, $\therefore$ there is a cusp at the new origin the tangent at which is $y=x\,.\,\phi'(a)$, and the cusp is of the

1st kind if $x^{\frac{2p+1}{2q}} \cdot F(a)$ is of a lower order of small quantities than

$$\frac{x^2}{2} \cdot \phi''(a) \text{ i. e. if } \frac{2p+1}{2q} < 2,$$

assuming that $x^{\frac{2p+1}{2q}} \cdot F(a)$ is of a higher order than $x \cdot \phi'(a)$, i. e. $\frac{2p+1}{2q} > 1$.

If $\frac{2p+1}{2q} > 2$, the cusp is of the 2nd kind (cf. Ex. 3 (3)).

5. Making $(a, 0)$ origin the curve is $y^3 = x^3 + (a-c)x^2$, $\therefore x=0$ is a double tangent, and near the origin $x^3 < x^2$, $\therefore x = \pm\left(\frac{y^3}{a-c}\right)^{\frac{1}{2}}$; $\therefore y$ is of the same sign as $(a-c)$ and there is a cusp of 1st kind.

6. Where $x=1$, $(y+1)^2=0$, $\therefore y=-1$ twice, $\therefore$ there are 2 branches, and with $(1, -1)$ as origin the curve is

$$\{(x+1)(y-1)+1\}^2 = x^3 - x^4 = (xy+y-x)^2,$$

$\therefore$ the 2 branches have the common tangent $y=x$; and near the origin

$$x^4 < x^3, \quad \therefore y - x = \pm x^{\frac{3}{2}} - xy = \pm x^{\frac{3}{2}} \text{ nearly;}$$

$\therefore x$ must be positive, and the cusp is of the 1st kind, the 2 branches being on opposite sides of $y-x=0$.

7. With (a, b) as origin the curve is $y = x^{\frac{1}{3}} + x^{\frac{3}{4}}$, and on rationalising this the term of lowest dimensions will give $x=0$ as a tangent. Also as $y = x^{\frac{1}{3}} \pm x^{\frac{3}{4}}$, x must be positive, and $\because x^{\frac{3}{4}} < x^{\frac{1}{3}}$, the 2 values of y near the origin are both positive; thus there is a cusp of the 2nd kind.

8. The lowest terms give for the tangents at $(0, 0)$, $y^2=0$, $\therefore$ there are 2 branches with a common tangent. Also $ay^2(a-x) = 2ax^2y - x^4$, $\therefore$ near the origin y must be of the same sign as a, and for a given value of x there are 2 real values of y if $a^2x^4 > x^4(a^2 - ax)$, i. e. if x be of the same sign as a, $\therefore$ the 2 branches are on the positive or negative sides only of the axes, as a is positive or negative, and thus there is a cusp of the 2nd kind.

9. The tangents at the origin are given by $y^2=0$, the homogeneous terms of lowest degree, $\therefore$ there are 2 branches which are real or imaginary as $a^2x^4 >$ or $< 4x^4(a^2 - ax)$, or *near* the origin as $a^2 >$ or $< 4a^2$, $\therefore$ the branches are imaginary and $(0, 0)$ is a conjugate point.

10. Here $(x^2-a^2)^2 = 2ay^3 + 3a^2y^2$, $\therefore$ making $(\pm a, 0)$ origin, the curve is $(x^2 \pm 2ax)^2 = 2ay^3 + 3a^2y^2$, $\therefore$ tangents are given by

$$4a^2x^2 = 3a^2y^2, \text{ or } \frac{y}{x} = \pm\sqrt{\frac{4}{3}} = \frac{dy}{dx} \text{ here:}$$

and there is a double point. When $y=-a$, $(x^2-a^2)^2 = a^4$, or $x^2=0$ or $2a^2$. Changing the origin to $(\pm a\sqrt{2}, -a)$ curve is

$$(x^2 \pm 2ax\sqrt{2} + a^2)^2 = 2a(y-a)^3 + 3a^2(y-a)^2;$$

$\therefore$ terms of lowest degree give $\pm 4a^3x\sqrt{2}=a^3y(+6-6)=0$, or $x=0$ is a tangent. Also changing origin to $(0, -a)$, curve is

$$(x^2-a^2)^2=2a(y-a)^3+3a^2(y-a)^2,$$

and tangents at new origin are given by

$$-2a^2x^2=a^2y^2(-6+3) \text{ or } \frac{y^2}{x^2}=\frac{2}{3}, \ \therefore \ \frac{dy}{dx}=\pm\sqrt{\frac{2}{3}}.$$

11. Making $(a, 0)$ the origin, the curve is $ay^2=x^2(x+a-b)$, $\therefore$ the tangents are given by $ay^2=(a-b)x^2$, and they are impossible or the point is conjugate, if $a<b$ numerically and a and b of same sign. The tangents are real and different, if $a>b$, and are coincident or there is a cusp if $a=b$, viz. $ay^2=x^3$, of 1st kind.

12. At $(0, 0)$ the tangents are given by $ay^2+bx^2=0$, $\therefore$ conjugate point if a and b have same sign, otherwise a double point or a cusp if either a or $b=0$.

Also for point of inflexion

$$2ay\frac{dy}{dx}=3x^2-2bx, \ \therefore \ 2a\left(\frac{dy}{dx}\right)^2=6x-2b, \ \left(\because \frac{d^2y}{dx^2}=0\right);$$

$$\therefore \ \frac{3x-b}{a}=\frac{x^2(3x-2b)^2}{4(x^3-bx^2)a}, \ \therefore \ 4(3x-b)(x-b)=(3x-2b)^2;$$

$$\therefore \ (12-9)x=(16-12)b \text{ or } x=\frac{4b}{3}.$$

For $\frac{d^3y}{dx^3}$, $2a\left(\frac{dy}{dx}\right)^2+2ay\frac{d^2y}{dx^2}=6x-2b$,

$$\therefore \ 6a\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+2ay\frac{d^3y}{dx^3}=6, \ \therefore \text{ when } x=\frac{4b}{3},$$

and $\therefore$ $ay^2=x^2\cdot\frac{b}{3}$, $\frac{d^3y}{dx^3}=\frac{3}{ay}$ and does not vanish; thus there are 2 points of inflexion when $x=\frac{4b}{3}$, or none, as a and b are of the same sign or not.

13. (1) $\pm y=\frac{x^2-x+1}{(x^2+1)^{\frac{1}{2}}}$, $\therefore$ $\pm\frac{dy}{dx}=\frac{2x-1}{(x^2+1)^{\frac{1}{2}}}-\frac{x(x^2-x+1)}{(x^2+1)^{\frac{3}{2}}}=\frac{x^3+x-1}{(x^2+1)^{\frac{3}{2}}}$;

$$\therefore \ \pm\frac{d^2y}{dx^2}=\frac{3x^2+1}{(x^2+1)^{\frac{3}{2}}}-\frac{3x(x^3+x-1)}{(x^2+1)^{\frac{5}{2}}}=\frac{x^2+3x+1}{(x^2+1)^{\frac{5}{2}}}.$$

$\therefore$ for points of inflexion $x^2+3x+1=0$, and then $\pm\frac{d^3y}{dx^3}=\frac{2x+3}{(x^2+1)^{\frac{5}{2}}}+$ a vanishing term, $\therefore$ as $x=-\frac{3\pm\sqrt{5}}{2}$, $\frac{d^3y}{dx^3}$ does not vanish; also

$y^2=\frac{(-4x)^2}{-3x}=-\frac{16}{3}x$, and $\therefore$ $y=\pm\frac{2}{\sqrt{3}}(\sqrt{5}\mp 1)$, and there are 4 points of inflexion.

(2) Here $a^2u^2=\theta$, $\therefore\ 2a^2u\dfrac{du}{d\theta}=1$, $\therefore\ 2a^2\left(\dfrac{du}{d\theta}\right)^2+2a^2u\dfrac{d^2u}{d\theta^2}=0$ ...... (a);

$$\therefore\ u+\frac{d^2u}{d\theta^2}=0=u-\frac{1}{u}\cdot\left(\frac{du}{d\theta}\right)^2=u-\frac{1}{4a^4u^3},\ \therefore\ r^2=2a^2,$$

$$\text{and } r=\pm a\sqrt{2},\text{ and } \theta=\frac{1}{2}.$$

(3) $au=\theta\sin\theta$, $\therefore\ a\left(u+\dfrac{d^2u}{d\theta^2}\right)=\theta\sin\theta-\theta\sin\theta+2\cos\theta$,

$$\therefore\ \frac{dp}{dr}=0,\text{ if }\cos\theta=0,\text{ and }\therefore\ \theta=2n\pi\pm\frac{\pi}{2}.$$

14. (1) If $(1, -2)$ be made origin, the curve is $(y+x)^2=-x^5$, $\therefore\ y+x=0$ is a double tangent, and x must be negative, $\therefore$ there is a cusp; and, $\because\ y+x=\pm(-x)^{\frac{5}{2}}$, the curve lies on opposite sides of $y+x=0$, $\therefore$ the cusp is of the 1st kind. Hence there is a cusp of the 1st kind at $(1, -2)$. The equations of Art. 298 give no other singular points. Also $\dfrac{d^2y}{dx^2}=0$, gives $x=1$ with the original origin, and there cannot be a point of inflexion at this point, and $\dfrac{d^3y}{dx^3}=\infty$.

(2) At the origin the tangents are $xy^2=0$, $\therefore$ there are a branch touching $x=0$, and a cusp the tangent at which is $y=0$; and

$$2y^2=ax\pm\sqrt{a^2x^2-4x^4},$$

thus x must be of the same sign as a, and the upper sign of the radical corresponds to the branch touching $x=0$, the lower to the cusp-branch, and the 2 values of y there for a given value of x being of opposite sign, the cusp is of the 1st kind.

(3) $y=0$ is a double tangent at the origin, and $y=\pm x^{\frac{3}{2}}\sqrt{1-x}$, $\therefore\ x$ lies between 0 and 1, and there is a cusp at the origin of the 1st kind.

For points of inflexion, $\pm\dfrac{dy}{dx}=\dfrac{3}{2}x^{\frac{1}{2}}\cdot(1-x)^{\frac{1}{2}}-\dfrac{1}{2}\cdot\dfrac{x^{\frac{3}{2}}}{(1-x)^{\frac{1}{2}}}$;

$$\therefore\ \pm\frac{d^2y}{dx^2}=\frac{3}{4}\frac{(1-x)^{\frac{1}{2}}}{\sqrt{x}}-\frac{6}{4}\cdot\frac{\sqrt{x}}{(1-x)^{\frac{1}{2}}}-\frac{1}{4}\cdot\frac{x^{\frac{3}{2}}}{(1-x)^{\frac{3}{2}}},$$

$$\therefore\ 3(1-x)^2-6x(1-x)-x^2=0=8x^2-12x+3,$$

$\therefore\ x=\dfrac{6-2\sqrt{3}}{8}$ (<1), and y has 2 corresponding values, or there are 2 points of inflexion.

(4) The tangents at the origin are $x^2=0$ and $ay-bx=0$; $\therefore$ there is a branch touching $ay=bx$, and a cusp the tangent at which is $x=0$. To determine its kind, near $x=0$, x is much smaller than y, $\therefore$ $xy^3<y^4$, and $bx<ay$, $\therefore$ the equation is approximately $y^4+ayx^2=0$, or when y is not zero, $y^3+ax^2=0$, $\therefore$ y is of the opposite sign to a, and the values of x being of opposite signs for a given value of y, the cusp is of the 1st kind.

CHAPTER XXIII.

1. From $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $\dfrac{dy}{dx}=-\dfrac{b^2}{a^2}\cdot\dfrac{x}{y}$,

$$\therefore \left(\frac{ds}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2=\frac{a^4y^2+b^4x^2}{a^4y^2}=\frac{a^2b^2(a^2-x^2)+b^4x^2}{a^2b^2(a^2-x^2)}$$

$$=\frac{a^2-x^2+(1-e^2)x^2}{a^2-x^2}=\frac{a^2-e^2x^2}{a^2-x^2}\text{, i.e. \&c.}$$

If $x=a\sin\phi$, $y=\pm b\cos\phi$, $\therefore$ $\dfrac{dx}{d\phi}=a\cos\phi$, $\dfrac{dy}{d\phi}=\mp b\sin\phi$,

and $$\therefore \frac{ds}{d\phi}=\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}=a\sqrt{1-e^2\sin^2\phi}.$$

2. $\dfrac{dy}{dx}=\dfrac{2a}{y}$. $\therefore$ $\left(\dfrac{ds}{dx}\right)^2=1+\dfrac{4a^2}{4ax}=\dfrac{x+a}{x}$, &c.

3. $x^2+y^2=a^2$, $\therefore$ $x+y\dfrac{dy}{dx}=0$, and $\left(\dfrac{ds}{dx}\right)^2=1+\dfrac{x^2}{y^2}=\dfrac{a^2}{y^2}$, &c.

4. $e^y\left\{(e^x-1)\dfrac{dy}{dx}+e^x\right\}=e^x$,

$$\therefore \frac{dy}{dx}=e^x\left(1-\frac{e^x+1}{e^x-1}\right)\div(e^x+1)=\frac{-2e^x}{e^{2x}-1}, \quad \therefore \frac{ds}{dx}=\frac{e^{2x}+1}{e^{2x}-1}.$$

5. $x^{-\frac{1}{3}}+y^{-\frac{1}{3}}\cdot\dfrac{dy}{dx}=0$, $\therefore$ $\left(\dfrac{ds}{dx}\right)^2=1+\dfrac{x^{-\frac{2}{3}}}{y^{-\frac{2}{3}}}=\dfrac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}$, &c.

6. $\dfrac{dr}{d\theta}=-a\sin\theta$, $\therefore$ $\left(\dfrac{ds}{d\theta}\right)^2=a^2(1+\cos\theta)^2+a^2\sin^2\theta=2a^2(1+\cos\theta)$,

and $$\therefore \frac{ds}{d\theta}=2a\cos\frac{\theta}{2}.$$

7. $\frac{dr}{d\theta}=a^\theta . \log a = r\log a$, $\therefore$ $\left(\frac{ds}{d\theta}\right)^2 = r^2+r^2(\log a)^2$, &c.

8. $r\frac{dr}{d\theta}=-a^2\sin 2\theta$, $\therefore$ $r^2\left(\frac{dr}{d\theta}\right)^2+r^4=a^4$, or $\frac{ds}{d\theta}=\frac{a^2}{r}$.

9. $1=a\frac{d\theta}{dr}$, $\therefore$ $\left(\frac{ds}{dr}\right)^2=1+\frac{r^2}{a^2}$, &c.

10. $e^{-y}.\frac{dy}{dx}=\sin x$, $\therefore$ $\frac{dy}{dx}=\tan x$, and $\frac{ds}{dx}=\sec x$, or $\frac{dx}{ds}=\cos x$.

CHAPTER XXIV.

1. $\frac{dy}{dx}=\frac{1}{2}(e^{\frac{x}{c}}-e^{-\frac{x}{c}})$, $\therefore$ $\frac{d^2y}{dx^2}=\frac{y}{c^2}$,

$$\therefore \rho=\left\{1+\left(\frac{e^{\frac{x}{c}}-e^{-\frac{x}{c}}}{2}\right)^2\right\}^{\frac{3}{2}}\div\frac{y}{c^2}=\left(\frac{y}{c}\right)^3\div\frac{y}{c^2}=\frac{y^2}{c},$$

or $y^2=c\rho$, and at the lowest point $x=0$, $\therefore$ $y=c$, and $\therefore$ $\rho=c$ or &c.

2. The tangent at the origin to the curve is given by the terms of lowest degree, i.e. $y=0$, and the curve approximates to $y=-18x^2$, $\therefore$ y is negative, and if (x, y) be also a point on the circle of curvature at $(0, 0)$, from a figure it will be seen that the limit of $\frac{x^2}{-y}=2\rho$, and $\therefore$ $\rho=\frac{1}{36}$.

3. $\frac{dy}{dx}=6+10x+3x^2$, $\frac{d^2y}{dx^2}=10+6x$,

$$\therefore \text{ when } x=0,\ \rho=\frac{(1+6^2)^{\frac{3}{2}}}{10}, \therefore \text{ \&c. by logarithms.}$$

Also generally $\rho=\{1+(6+10x+3x^2)^2\}^{\frac{3}{2}}\div(10+6x)$, and is $\therefore$ ∞ when $x=\infty$ or $-\frac{5}{3}$.

4. $\frac{d\phi}{dx}+\frac{d\phi}{dy}\cdot\frac{dy}{dx}=0$, and $\frac{d^2\phi}{dx^2}+2\frac{d^2\phi}{dxdy}\cdot\frac{dy}{dx}+\frac{d^2\phi}{dy^2}\cdot\left(\frac{dy}{dx}\right)^2+\frac{d\phi}{dy}\cdot\frac{d^2y}{dx^2}=0$,

$$\therefore -\frac{d^2y}{dx^2}\cdot\left(\frac{d\phi}{dy}\right)^3=\frac{d^2\phi}{dx^2}\cdot\left(\frac{d\phi}{dy}\right)^2-2\frac{d^2\phi}{dxdy}\cdot\frac{d\phi}{dx}\cdot\frac{d\phi}{dy}+\frac{d^2\phi}{dy^2}\cdot\left(\frac{d\phi}{dx}\right)^2,$$

and $\therefore$ as in Art. 320, or substituting in the result there, the expression for ρ follows.

5. Let the parabola be $(x-h)^2=m(y-k)$, then it must pass through (a, a), $\therefore (a-h)^2=m(a-k)$, and for the parabola $2(x-h)=m\dfrac{dy}{dx}$, and for the curve $\dfrac{dy}{dx}=\dfrac{3x^2}{a^2}$, $\therefore$ when $x=a$, $2(a-h)=3m$, and again $2=m\dfrac{d^2y}{dx^2}=\dfrac{6m}{a}$, $\therefore m=\dfrac{a}{3}$, $h=\dfrac{a}{2}$, and $k=a-\dfrac{a^2}{4}\cdot\dfrac{3}{a}=\dfrac{a}{4}$, thus the parabola is determined, being $\left(x-\dfrac{a}{2}\right)^2=\dfrac{a}{3}\left(y-\dfrac{a}{4}\right)$ and it has contact of the 2nd order with the given curve.

6. $\tan\phi=r\dfrac{d\theta}{dr}$, and here $\dfrac{dr}{d\theta}=a\sin\theta$, $\therefore \tan\phi=\dfrac{1-\cos\theta}{\sin\theta}=\tan\dfrac{\theta}{2}$,

$$\therefore \phi=\frac{\theta}{2} \text{ and } p=r\sin\phi=2a\sin^3\frac{\theta}{2},$$

and $$\rho=r\frac{dr}{dp}=r\frac{dr}{d\theta}\div\frac{dp}{d\theta}=a\,.\,2\sin^2\frac{\theta}{2}\sin\theta\div3\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}=\frac{4a}{3}\sin\frac{\theta}{2}.$$

7. By Art. 323 or thus: $r\dfrac{d\theta}{dr}=\tan\phi=\dfrac{2\cos\theta-1}{-2\sin\theta}$.

$$\therefore \sin\phi=\frac{2\cos\theta-1}{\sqrt{5-4\cos\theta}}=\frac{r}{\sqrt{(3a-2r)\,a}},\text{ and } p=\frac{r^2}{\sqrt{(3a-2r)\,a}},$$

$$\therefore \frac{r}{\rho}=\frac{dp}{dr}=\frac{1}{\sqrt{a}}\,\frac{2r(3a-2r)+r^2}{(3a-2r)^{\frac{3}{2}}};$$

$$\therefore \rho=\frac{\sqrt{a}\,.\,(3a-2r)^{\frac{3}{2}}}{6a-3r}=\frac{a\,.\,(5-4\cos\theta)^{\frac{3}{2}}}{9-6\cos\theta}.$$

8. If they touch at (x, y), $\dfrac{dy}{dx}$ is the same for each, and

$$\frac{df}{dx}+\frac{df}{dy}\cdot\frac{dy}{dx}=0=\frac{d\phi}{dx}+\frac{d\phi}{dy}\cdot\frac{dy}{dx},$$

and eliminating $\dfrac{dy}{dx}$ the result follows, for the particular values of x and y.

9. $\dfrac{df}{dx}=\dfrac{1}{a}$, $\dfrac{df}{dy}=\dfrac{1}{b}$, $\dfrac{d\phi}{dx}=\dfrac{2}{3}x^{-\frac{1}{3}}$, $\dfrac{d\phi}{dy}=\dfrac{2}{3}y^{-\frac{1}{3}}$, $\therefore$ they touch at (x, y) if $ay^{\frac{1}{3}}=bx^{\frac{1}{3}}$; and on the straight line where $\dfrac{x}{a^3}=\dfrac{y}{b^3}$ each $=\dfrac{1}{a^2+b^2}$; and on the curve where $\dfrac{x^{\frac{2}{3}}}{a^2}=\dfrac{y^{\frac{2}{3}}}{b^2}$ each equals $(a^2+b^2)^{-\frac{2}{3}}$, and $\therefore \dfrac{x}{a^3}=\dfrac{y}{b^3}=\dfrac{1}{a^2+b^2}$, or the same values of x and y satisfy the line and curve, and $\therefore$ they touch.

10. ϕ is a minimum or maximum and $p = r\sin\phi$,

$$\therefore \frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr} = \sin\phi = \frac{p}{r},$$

$$\therefore \frac{p}{r} = \frac{r}{\rho} \text{ or \&c.}$$

11. $\rho = \pm\frac{ds}{d\psi}$, $\cos\psi = \frac{dx}{ds}$, $\sin\psi = \frac{dy}{ds}$; $\therefore$ $2a\cos\psi = \frac{b^2}{y} + y$,

$$\therefore -2a\sin\psi \cdot \frac{d\psi}{ds} = \left(1 - \frac{b^2}{y^2}\right)\frac{dy}{ds}, \therefore \mp\frac{2a}{\rho} = \frac{y^2 - b^2}{y^2},$$

and $\rho = \frac{2ay^2}{y^2 \sim b^2}$. Also the part of the normal $= y \div \frac{dx}{ds}$; $\therefore$ $\frac{1}{n} = \frac{b^2 + y^2}{2ay^2}$,

and $\therefore \frac{1}{n} \pm \frac{1}{\rho} = \frac{1}{a}$, $\pm$ as $y^2 >$ or $< b^2$:

a being taken positive, if a be negative the signs of n and ρ must be altered.

12. $r = a\cos\theta$ is a circle of radius $\frac{a}{2} = \rho$.

13. $\frac{dx}{ds} = \frac{s}{x} = \cos\psi$, $\therefore$ differentiating as to s, $-\sin\psi \cdot \left(\pm\frac{1}{\rho}\right) = \frac{c^2 + s^2 - s^2}{x^3}$;

$$\therefore \mp\rho = \frac{x^3}{c^2} \cdot \frac{c}{x} \text{ or, taken positively, } \rho = \frac{x^2}{c}.$$

14. From Art. 320, $1 + \overline{\frac{dy}{dx}}\Big|^2 + y\frac{d^2y}{dx^2} = b\frac{d^2y}{dx^2}$,

and $$\frac{dr^2}{dx} = 2x + 2y\frac{dy}{dx}, \therefore \frac{d^2r^2}{dx^2} = 2 + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2};$$

$$\therefore 2b\frac{d^2y}{dx^2} = \frac{d^2r^2}{dx^2}. \quad \text{By symmetry}$$

$$2a\frac{d^2x}{dy^2} = \frac{d^2r^2}{dy^2}.$$

15. $(x-a)^2 + (y-b)^2 = \rho^2$, $\therefore$ $(x-a)\frac{dx}{dy} + y - b = 0$, and

$$(x-a)\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 + 1 = 0; \text{ and } y = 2m\frac{dx}{dy},\ 1 = 2m\frac{d^2x}{dy^2},$$

$$\therefore \frac{a}{2m} = \frac{x}{2m} + \frac{x}{m} + 1 \text{ or } a = 2m + 3x;$$

and $$b = y + (x-a)\frac{y}{2m} = \frac{y}{2m}(-2x) = -\frac{2x^{\frac{3}{2}}}{\sqrt{m}},$$

and $$\therefore \rho^2 = (x-a)^2\left(1 + \frac{y^2}{4m^2}\right) = 4(m+x)^2\frac{m+x}{m} \text{ or } \rho = \frac{2(m+x)^{\frac{3}{2}}}{\sqrt{m}}.$$

Hence the circle of curvature at (x, y) on $y^2=4mx$ being

$$(x'-a)^2+(y'-b)^2=\rho^2$$

cuts the axis of x in points given by $x'^2-2ax'+a^2+b^2-\rho^2=0$, but

$$a^2+b^2-\rho^2=(2m+3x)^2+\frac{4x^3}{m}-\frac{4(m+x)^3}{m}=\frac{1}{m}(-3mx^2),$$

which is negative unless $x=0$, ∴ &c.

16. $x-a+(y-b)\frac{dy}{dx}=0$, ∴ $1+\overline{\frac{dy}{dx}}\Big|^2+(y-b)\frac{d^2y}{dx^2}=0$;

but $$Ax+By\frac{dy}{dx}=0, \therefore A+B\left(\frac{dy}{dx}\right)^2+By\frac{d^2y}{dx^2}=0,$$

and $$\therefore b\frac{d^2y}{dx^2}.B=B+B\left(\frac{dy}{dx}\right)^2+By\frac{d^2y}{dx^2}=B-A,$$

and $$\therefore b=\frac{(A-B)y}{A+B\left(\frac{dy}{dx}\right)^2}=\frac{(A-B)y^3}{Ay^2+\frac{A^2x^2}{B}}=\frac{(B-A)By^3}{AC}.$$

So $$a=\frac{(A-B)Ax^3}{BC}.$$

17. ∴ $\tan\psi=\frac{a}{s}$, ∴ $\sec^2\psi.\frac{d\psi}{ds}=-\frac{a}{s^2}$,

$$\text{or } \rho=\frac{s^2}{a}.\sec^2\psi=\frac{s^2}{a}\left(1+\frac{a^2}{s^2}\right)=\frac{a^2+s^2}{a}.$$

18. When $y=0$, $x=0$ or $3a$, and the curve is $y^2(4a-x)=ax(3a-x)$, ∴ at the origin the tangent is $x=0$, and (cf. Ex. 2)

$$2\rho=\text{lt.}\frac{y^2}{x}=\frac{3a^2}{4a}, \therefore \rho=\frac{3a}{8}.$$

If the origin be changed to $(3a, 0)$ the curve is given by

$$y^2(a-x)=ax(3a+x),$$

and at the new origin $2\rho=\text{lt. of }\frac{y^2}{x}=\frac{3a^2}{a}$, ∴ $\rho=\frac{3a}{2}$.

19. $\frac{ds}{d\psi}=\rho=ns.\cot\psi$, which can be expressed in terms of s or ψ by the given equation.

20. $x=0$ is a double tangent, and the 2 branches are $y^2=\pm x\sqrt{4a^2-x^2}$, and $2\rho=\text{lt. of }\frac{y^2}{x}=\pm 2a$, and thus the 2 circles of curvature at the origin are $x^2-2\rho x+y^2=0$, i. e. $x^2\mp 2ax+y^2=0$.

21. $1=\frac{dx}{dy}\cdot e^{-\frac{x}{a}}$, or $-\frac{a}{y}=\frac{dx}{dy}$, $\therefore\ \frac{d^2x}{dy^2}=\frac{a}{y^2}$,

and $$\therefore\ \rho=\frac{(a^2+y^2)^{\frac{3}{2}}}{y^3}\div\frac{a}{y^2}=\frac{(a^2+y^2)^{\frac{3}{2}}}{ay}.$$

22. For the circle

$$x-\frac{3a}{4}+\left(y-\frac{3a}{4}\right)\frac{dy}{dx}=0,\ 1+\left(\frac{dy}{dx}\right)^2+\left(y-\frac{3a}{4}\right)\frac{d^2y}{dx^2}=0,$$

and $$3\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+\left(y-\frac{3a}{4}\right)\frac{d^3y}{dx^3}=0,\ \therefore\ \text{at}\ \left(\frac{a}{4},\ \frac{a}{4}\right),$$

$$\frac{dy}{dx}=-1,\ \frac{d^2y}{dx^2}=\frac{4}{a},\ \text{and}\ \frac{d^3y}{dx^3}=-\frac{24}{a^2}.$$

For the parabola $\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}\cdot\frac{dy}{dx}=0$, or $\frac{dy}{dx}=1-\frac{\sqrt{a}}{\sqrt{x}}$;

$$\therefore\ \frac{d^2y}{dx^2}=\frac{\sqrt{a}}{2x^{\frac{3}{2}}},\ \frac{d^3y}{dx^3}=-\frac{3}{4}\frac{\sqrt{a}}{x^{\frac{5}{2}}},\ \text{and}\ \therefore\ \text{at}\ \left(\frac{a}{4},\ \frac{a}{4}\right),$$

$$\frac{dy}{dx}=-1,\ \frac{d^2y}{dx^2}=\frac{4}{a}\ \text{and}\ \frac{d^3y}{dx^3}=-\frac{3\cdot 8}{a^2},\ \text{and}\ \therefore\ \&c.$$

23. $r=a\sec^2\frac{\theta}{2}$, $\therefore\ \frac{dr}{d\theta}=a\sec^2\frac{\theta}{2}\cdot\tan\frac{\theta}{2}$,

$$\therefore\ \tan\phi=r\frac{d\theta}{dr}=\cot\frac{\theta}{2}\ \text{and}\ \phi=\frac{\pi}{2}-\frac{\theta}{2},\ \therefore\ p=r\sin\phi=a\sec\frac{\theta}{2},$$

$$\therefore\ \rho=r\frac{dr}{dp}=a\sec^2\frac{\theta}{2}\cdot a\sec^2\frac{\theta}{2}\cdot\tan\frac{\theta}{2}\div\frac{dp}{d\theta}$$

$$=a\sec^4\frac{\theta}{2}\cdot\tan\frac{\theta}{2}\div\frac{1}{2}\sec\frac{\theta}{2}\cdot\tan\frac{\theta}{2}=2a\sec^3\frac{\theta}{2}.$$

24. For the circle, at $(a,\ 2a)$, $x+y\frac{dy}{dx}=0$, $\therefore\ \frac{dy}{dx}=-\frac{1}{2}$,

and $$1+\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=0,\ \therefore\ \frac{d^2y}{dx^2}=-\frac{5}{8a}.$$

Let the parabolas be $(y-k)^2+m(x-h)=0$, $(x-f)^2+n(y-g)=0$:

then $$(2a-k)^2+m(a-h)=0,\ 2(y-k)\frac{dy}{dx}+m=0,$$

$$\therefore\ m=(2a-k),\ \text{and}\ 2\left(\frac{dy}{dx}\right)^2+2(y-k)\frac{d^2y}{dx^2}=0,$$

$$\therefore\ 4(2a-k)=\frac{8a}{5} \text{ or } k=\frac{8a}{5},\ \therefore\ m=\frac{2a}{5}.$$

and $h=a+\left(\frac{2a}{5}\right)^2\cdot\frac{5}{2a}=\frac{7a}{5}$, $\therefore$ the 1st parabola is

$$\left(y-\frac{8a}{5}\right)^2=\frac{2a}{5}\left(\frac{7a}{5}-x\right). \quad \text{Also } (a-f)^2+n(2a-g)=0,$$

$$2(x-f)+n\frac{dy}{dx}=0 \text{ or } n=4(a-f), \text{ and } 2+n\frac{d^2y}{dx^2}=0,$$

$$\therefore\ n=\frac{16a}{5},\ \therefore\ f=\frac{a}{5}, \text{ and } \therefore\ g=2a+\frac{n}{16}=\frac{11a}{5},$$

and the 2nd parabola is $\left(x-\frac{a}{5}\right)^2=\frac{16a}{5}\left(\frac{11a}{5}-y\right)$.

25. $\frac{dy}{dx}=\frac{1}{2}(e^{\frac{x}{c}}-e^{-\frac{x}{c}})=\sqrt{\frac{y^2}{c^2}-1},\ \frac{d^2y}{dx^2}=\frac{1}{c}\cdot\frac{y}{c};$

$$\therefore\ Y=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}=y+\frac{y^2}{c^2}\div\frac{y}{c^2}=2y;$$

and
$$X=x-y\cdot\frac{dy}{dx}=x-y\sqrt{\frac{y^2}{c^2}-1}.$$

For the evolute $y=\frac{1}{2}\cdot Y$, and $e^{\frac{x}{c}}=\frac{y}{c}+\sqrt{\frac{y^2}{c^2}-1}$;

$\therefore\ x=c\log\frac{y+\sqrt{y^2-c^2}}{c}$, and the evolute is given by

$$X=c\log\frac{Y+\sqrt{Y^2-4c^2}}{2c}-\frac{Y}{4c}\sqrt{Y^2-4c^2}.$$

26. If (h, k) be the centre of curvature at (x, y) on $a^2y^2+b^2x^2=a^2b^2$,

$$k=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}, \text{ and } a^2y\frac{dy}{dx}+b^2x=0,$$

and
$$a^2y\frac{d^2y}{dx^2}+a^2\left(\frac{dy}{dx}\right)^2+b^2=0,\ \therefore\frac{dy}{dx}=-\frac{b^2x}{a^2y},$$

and
$$-\frac{d^2y}{dx^2}=\frac{1}{a^2y}\left\{b^2+\frac{b^4x^2}{a^2y^2}\right\}=\frac{b^4}{a^2y^3}, \text{ and } 1+\left(\frac{dy}{dx}\right)^2=\frac{a^2y^2+b^4-b^2y^2}{a^2y^2};$$

$$\therefore\ k=y-\frac{(a^2-b^2)y^2+b^4}{b^4}\cdot y=-(a^2-b^2)\frac{y^3}{b^4}.$$

So $\quad h=(a^2-b^2)\dfrac{x^3}{a^4}$, and $\therefore (ah)^{\frac{2}{3}}+(bk)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$, $\therefore$ &c.

Also cf. (Art. 331) the length of the evolute, by symmetry, $=4$ times the difference of the radii of curvature at an end of the minor and an end of the major axis; and referring the ellipse to $(a, 0)$ its equation is

$$a^2y^2+b^2x^2+2ab^2x=0,$$

and $\therefore$ at $(a, 0)$, $\rho=$ lt. of $\quad \dfrac{y^2}{-2x}=\dfrac{ab^2}{a^2}=\dfrac{b^2}{a}$,

and $\therefore$ the length $=4\left(\dfrac{a^2}{b}-\dfrac{b^2}{a}\right)$. *Or* from the values of h and k, these radii of curvature may be found.

27. If for Y (Art. 284) r, p be r', p', by Art. 329,

$$p'r=p^2, \text{ and } \because p=r',\ p'=\frac{r'^2}{f(r')}.$$

If $p^2=\dfrac{b^2r}{2a-r}$, the curve is an ellipse with a focus for pole, and $\therefore$ the locus is the auxiliary circle; but, analytically,

$$p=r', \text{ and } r=\frac{r'^2}{p'},\ \therefore\ 2ar'^2=\frac{r'^2}{p'}(b^2+r'^2),$$

or $2ap'=b^2+r'^2$, and then $\rho'=\dfrac{r'dr'}{dp'}=a$, .. the locus is a circle, consecutive radii of curvature intersecting in the centre of curvature.

28. $\because r^2-p^2=a^2$ the perpendicular from the pole on the tangent to the evolute $=a$, and $\therefore$ the evolute is the circle of radius a and centre at the pole.

29. Comparing the figures in Arts. 313 and 331, $\dfrac{dA}{d\psi}=\dfrac{\rho^2}{2}$,

$$\text{or } \frac{dA}{d\psi}=\frac{\rho}{2}\cdot\frac{ds}{d\psi},\ \therefore\ \frac{dA}{dx}=\frac{\rho}{2}\cdot\frac{ds}{dx}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^2\div 2\,\frac{d^2y}{dx^2}.$$

30. As $\rho=\dfrac{ds}{d\psi}$, so $\rho'=$ radius of curvature at the corresponding point of the evolute

$$=\frac{ds'}{d\psi}=\frac{d\rho}{d\psi}=\frac{d\rho}{ds}\cdot\frac{ds}{d\psi}=\rho\,\frac{d\rho}{ds}.$$

31. $3y^2=a^2\dfrac{dx}{dy}$, $6y=a^2\dfrac{d^2x}{dy^2}$, $\therefore$ as in Art. 320,

$$x'=x+\frac{1+\dfrac{9y^4}{a^4}}{\dfrac{6y}{a^2}}=\frac{y^3}{a^2}+\frac{9y^4+a^4}{6a^2y}=\frac{a^4+15y^4}{6a^2y},$$

and

$$y'=y-\frac{3y^2}{a^2}\left(\frac{a^4+9y^4}{6a^2y}\right)=\frac{y}{2a^4}(a^4-9y^4).$$

32. As in Ex. 15, $a-x=2x+2m$, and $\therefore$ if n be the part of the normal in question, $\rho : n :: 2x+2m : x+m :: 2 : 1$.

33. (1) $\rho = r\dfrac{dr}{dp}$, $\dfrac{p^2}{r^2}=\dfrac{b^2}{rr'}$ or $p^2=\dfrac{b^2r}{2a-r}$,

$$\therefore\ 2p\frac{dp}{dr}=\frac{2ab^2}{(2a-r)^2}, \text{ and } \therefore\ \rho=\frac{pr\,.\,r'^2}{ab^2}=\frac{(rr')^{\frac{3}{2}}}{ab}.$$

(2) $\dfrac{dy}{dx}=-\cot\phi$, and $a^2y\dfrac{dy}{dx}+b^2x=0$,

$$\therefore\ \frac{bx}{a\cos\phi}=\frac{ay}{b\sin\phi}=\frac{ab}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}, \text{ and } a^2y\frac{d^2y}{dx^2}+a^2\cot^2\phi+b^2=0;$$

$$\therefore\ \rho=\frac{\operatorname{cosec}^3\phi\,.\,a^2y}{a^2\cot^2\phi+b^2}=\frac{a^2y\operatorname{cosec}\phi}{a^2\cos^2\phi+b^2\sin^2\phi}=\frac{a^2b^2}{(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{3}{2}}}$$

$$=\frac{b^2}{a(1-e^2\sin^2\phi)^{\frac{3}{2}}}.$$

34. Let the given point be origin, and the axes in the directions of those of the ellipses; and for the curve at (0, 0) let $\dfrac{dy}{dx}=m$, $\dfrac{d^2y}{dx^2}=n$. Then if one of the ellipses be

$$a^2(y-k)^2+b^2(x-h)^2=a^2b^2,\ a^2(y-k)\frac{dy}{dx}+b^2(x-h)=0,$$

$$a^2(y-k)\frac{d^2y}{dx^2}+a^2\left(\frac{dy}{dx}\right)^2+b^2=0, \text{ and } \therefore \text{ at the origin,}$$

$$-a^2k\,.\,n+a^2m^2+b^2=0,\ a^2k\,.\,m+b^2h=0,$$

and $\therefore\ \dfrac{m^2-nk}{mk}=\dfrac{1}{h}$ or the locus of (h, k) is $nxy+my=m^2x$, a rectangular hyperbola through the origin.

35. From the figure in p. 349, the radius of curvature being a tangent to the evolute, there is an asymptote to it when the radius of curvature is ∞, for a finite point on the given curve, and $\therefore\ \dfrac{d^2y}{dx^2}=0$, and $\therefore$ in general the asymptotes to the evolute correspond to the points of inflexion on the given curve. Here $\dfrac{dy}{dx}=a\sec^2 x$, $\dfrac{d^2y}{dx^2}=2a\sec^2x\,.\tan x$, $\therefore\ \tan x=0$ corresponds to the asymptotes, i.e. $y=0$ and $x=n\pi$, and the asymptotes are $ay\mp x=\pm n\pi$, n being integral.

36. $a^{\frac{3}{2}}\,.\,y=\pm x^{\frac{5}{2}}$, and the 2 branches are similar; for the positive value of y, $a^{\frac{3}{2}}\,.\,\dfrac{dy}{dx}=\dfrac{5}{2}x^{\frac{3}{2}}$, $a^{\frac{3}{2}}\,.\,\dfrac{d^2y}{dx^2}=\dfrac{15}{4}x^{\frac{1}{2}}$, $\therefore$ if (x', y') on the evolute correspond

to (x, y) $$y'=y+\frac{1}{a^{\frac{3}{2}}}\frac{\left(a^3+\frac{25}{4}x^3\right)}{\frac{15}{4}x^{\frac{1}{2}}}=\frac{x^{\frac{5}{2}}}{a^{\frac{3}{2}}}+\frac{4}{15a^{\frac{3}{2}}}\left(\frac{a^3}{\sqrt{x}}+\frac{25}{4}x^{\frac{5}{2}}\right),$$

or approximately $$y'=\frac{4a^{\frac{3}{2}}}{15\sqrt{x}};$$

and $$x'=x-\frac{5x^{\frac{3}{2}}}{2a^{\frac{3}{2}}}\cdot\frac{a^3+\frac{25}{4}x^3}{\frac{15}{4}a^{\frac{3}{2}}\cdot\sqrt{x}}=x-\frac{2}{3}x \text{ nearly};$$

$$\therefore\ 3x'=x=\left(\frac{4a^{\frac{3}{2}}}{15y'}\right)^2 \text{ or } x'y'^2=c^3.$$

This as y' may be negative includes the evolute of the other branch: and a may be negative and then x is negative, and c^3 also.

37. If $a^{\frac{1}{2}}=\frac{1}{c}$, $y=c^2x^2+c^3x^{\frac{5}{2}}$, $\therefore \frac{dy}{dx}=2c^2x+\frac{5}{2}cx^{\frac{3}{2}}$,

and $$\frac{d^2y}{dx^2}=2c^2+\frac{15}{4}c^3x^{\frac{1}{2}};$$

hence, if (x', y') be the corresponding point on the evolute,

$$y'=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}=c^2x^2+c^3x^{\frac{5}{2}}+\frac{1+4c^4x^2}{2c^2}\left(1-\frac{15}{8}cx^{\frac{1}{2}}\right)$$

approximately $=\frac{1}{2c^2}-\frac{15}{16}\frac{x^{\frac{1}{2}}}{c}$ nearly; and approximately,

$$x'=x-\left(2c^2x+\frac{5}{2}c^3x^{\frac{3}{2}}\right)(1+4c^4x^2)\left(1-\frac{15}{8}cx^{\frac{1}{2}}\right)\frac{1}{2c^2},$$

or $2c^2x'=-\frac{5}{2}c^3x^{\frac{3}{2}}+\frac{15}{4}c^3x^{\frac{3}{2}}$ nearly,

$$\therefore\ x'=\frac{5}{8}cx^{\frac{3}{2}}=\frac{5c^4}{8}\left\{\frac{16}{15}\left(\frac{1}{2c^2}-y'\right)\right\}^3,$$

$$\therefore\ (y'-\alpha)^3+\beta^2x=0.$$

38. Drawing the circle of curvature at (x, y) in a fig. such as that on p. 328, $\frac{1}{2}$ the chord is easily seen to $=x\sim h$, where (h, k) is the centre of

curvature, $\therefore$ $\frac{1}{2}$ chord $$=\frac{\frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}},$$

and here $$\cos\frac{y}{a}=e^{-\frac{x}{a}},\ \therefore\ \sin\frac{y}{a}\cdot\frac{dy}{dx}=e^{-\frac{x}{a}},$$

or $$\frac{dy}{dx}=\cot\frac{y}{a},\text{ and }\therefore\ \frac{d^2y}{dx^2}=-\cot\frac{y}{a}\cdot\frac{1}{a}\cdot\operatorname{cosec}^2\frac{y}{a},$$

and $$\therefore\ x-h=-\frac{a\cot\frac{y}{a}}{\cot\frac{y}{a}\cdot\operatorname{cosec}^2\frac{y}{a}}\left(1+\cot^2\frac{y}{a}\right)=-a,$$

and the chord $$=2a.$$

For the evolute $$x=h-a,$$

and $$k=y-a\tan\frac{y}{a}=y-a\left(\frac{y}{a}-\frac{y^3}{6a^3}\right)\div\left(1-\frac{y^2}{2a^2}\right)$$

approximately $=-\dfrac{y^3}{3a^2}$ nearly, and $\therefore\ -\dfrac{y}{a}=\left(\dfrac{3k}{a}\right)^{\frac{1}{3}}$; hence $\sec\left(\dfrac{3k}{a}\right)^{\frac{1}{3}}=e^{\frac{h-a}{a}}$,

or the evolute is approximately $\sec\left(\dfrac{3y}{a}\right)^{\frac{1}{3}}=e^{\frac{x-a}{a}}$.

39. Let the common tangent and normal be taken as axes of x and y, then at the point of contact $\dfrac{dx}{ds}=1$, $\dfrac{dy}{ds}=0$, and by Art. 95, if on the curve

$$x=f(\delta s),\text{ and } y=\phi(\delta s),$$

$$x=f(0)+\delta s\,.\,f'(0)+\frac{(\delta s)^2}{\underline{|2}}\,.\,f''(0)+$$

$$=0+\delta s\cdot\frac{dx}{ds}+\frac{\delta s^2}{\underline{|2}}\cdot\frac{d^2x}{ds^2}+\frac{\delta s^3}{\underline{|3}}\cdot\frac{d^3x}{ds^3}+\frac{\delta s^4}{\underline{|4}}\cdot\frac{d^4x}{ds^4}+\dots$$

and so $$y=0+\delta s\cdot\frac{dy}{ds}+\frac{\delta s^2}{\underline{|2}}\cdot\frac{d^2y}{ds^2}+\frac{\delta s^3}{\underline{|3}}\cdot\frac{d^3y}{ds^3}+;$$

but $$\frac{dx}{ds}\cdot\frac{d^2x}{ds^2}+\frac{dy}{ds}\cdot\frac{d^2y}{ds^2}=0,\ \therefore\ \frac{d^2x}{ds^2}=0,$$

and $$\frac{1}{\rho}=\frac{d^2y}{ds^2}\div\frac{dx}{ds}=\frac{d^2y}{ds^2},\text{ and }-\frac{1}{\rho}=\frac{d^2x}{ds^2}\div\frac{dy}{ds},$$

$$\therefore\ -\frac{1}{\rho^2}\cdot\frac{d\rho}{ds}=\frac{d^3y}{ds^3}\div\frac{dx}{ds}-\frac{d^2y}{ds^2}\cdot\frac{d^2x}{ds^2}\div\left(\frac{dx}{ds}\right)^2=\frac{d^3y}{ds^3},$$

and $$\frac{1}{\rho^2}\cdot\frac{d\rho}{ds}=\frac{d^3x}{ds^3}\div\frac{dy}{ds}-\frac{d^2x}{ds^2}\cdot\frac{d^2y}{ds^2}\div\left(\frac{dy}{ds}\right)^2,\ \therefore\ \frac{d^3x}{ds^3}=0,$$

and $$\therefore\ x=\delta s+\frac{\delta s^4}{\underline{|4}}\cdot\frac{d^4x}{ds^4}+\dots,\ y=\frac{\delta s^2}{2}\cdot\frac{1}{\rho}-\frac{\delta s^3}{6}\cdot\frac{1}{\rho^2}\cdot\frac{d\rho}{ds}+\dots,$$

and if (x', y') be the corresponding point on the circle of curvature, to the 3rd power of δs, $x'-x=0$, and

$$y'=\frac{1}{2\rho}\delta s^2,\ \because\ \frac{d\rho}{ds}=0,\ \therefore\ y'-y=\frac{1}{6\rho^2}\cdot\frac{d\rho}{ds}\cdot\delta s^3,\text{ and }\therefore\ \&c.$$

40. The rectangular equation of any conic involves in general 5 independent constants, and it may $\therefore$ be made to satisfy 5 conditions, and thus by Art. 316, it can in general have a contact of the 4th order with the given curve at any proposed point thereon. If the given curve be $s=f(\phi)$, and $\phi=a$ at the proposed point, ω the angle the normal to the conic at that point makes with its axis, ψ an adjacent value of ω, then $\psi-\phi=$constant, $\because$ the conic touches the curve; $\therefore\ \dfrac{ds}{d\phi}=\dfrac{ds}{d\psi},\ \dfrac{d^2s}{d\phi^2}=\dfrac{d^2s}{d\psi^2}$, and $\dfrac{d^3s}{d\phi^3}=\dfrac{d^3s}{d\psi^3}$, $\because$ the contact is of the 4th order (cf. Art. 325), *when* $\phi=a$. Now $\dfrac{ds}{d\psi}=\rho$, and by Ex. 33 (2), $\rho=\dfrac{b^2}{a(1-e^2\sin^2\psi)^{\frac{3}{2}}}$, which holds for the hyperbola when $e>1$, and for the parabola when $e=1$ and $\dfrac{b^2}{a}$ is finite. Hence the equations for determining a, e $\left(\text{and} \therefore b \text{ or } \dfrac{b^2}{a}\right)$ and ω, i.e. the dimensions, nature and position of the conic, are

$$\frac{b^2}{a(1-e^2\sin^2\omega)^{\frac{3}{2}}}=f'(a) \quad\quad (1),$$

$$f''(a)=\frac{3b^2e^2\sin\omega\cos\omega}{a(1-e^2\sin^2\omega)^{\frac{5}{2}}} \quad\quad (2),$$

$$f'''(a)=\frac{3b^2e^2}{a}\left\{\frac{\cos 2\omega(1-e^2\sin^2\omega)+5e^2\sin^2\omega\cos^2\omega}{(1-e^2\sin^2\omega)^{\frac{7}{2}}}\right\} \quad\quad (3).$$

If the curve be (Art. 354) $r=ce^{\theta\cot a}$,

$$\frac{dr}{ds}=\sin a, \text{ and } \theta=\phi-a,$$

if ϕ be angle between the tangent to the spiral and the initial line, and

$$\therefore\ \rho=\frac{ds}{d\phi}=\frac{dr}{d\phi}\cdot\frac{ds}{dr}=\kappa\cot a\,.\,e^{\phi\cot a},$$

κ being constant, and equations (1) (2) (3) become, writing in them γ instead of a, $\kappa\cot a\,.\,e^{\gamma\cot a}=\dfrac{b^2}{a}\cdot\dfrac{1}{(1-\epsilon^2\sin^2\omega)^{\frac{3}{2}}}$, ϵ being the eccentricity,

$$\kappa\cot^2 a\,.\,e^{\gamma\cot a}=\frac{3b^2\epsilon^2\sin\omega\cos\omega}{a(1-\epsilon^2\sin^2\omega)^{\frac{5}{2}}},$$

and $$\kappa\cot^3 a\,.\,e^{\gamma\cot a}=\frac{3b^2\epsilon^2}{a}\cdot\frac{\cos 2\omega(1-\epsilon^2\sin^2\omega)+5\epsilon^2\sin^2\omega\cos^2\omega}{(1-\epsilon^2\sin^2\omega)^{\frac{7}{2}}}.$$

Hence $$\cot a(1-\epsilon^2\sin^2\omega)=3\epsilon^2\sin\omega\cos\omega, \text{ and}$$

$$\cot a(1-\epsilon^2\sin^2\omega)\sin\omega\cos\omega=\cos 2\omega(1-\epsilon^2\sin^2\omega)+5\epsilon^2\sin^2\omega\cos^2\omega,$$

$$\therefore\ 3\epsilon^2\sin^2\omega\cos^2\omega=\cos 2\omega(1-\epsilon^2\sin^2\omega)+5\epsilon^2\sin^2\omega\cos^2\omega,$$

or $$\cos 2\omega=\epsilon^2\sin^2\omega(\cos 2\omega-2\cos^2\omega)=-\epsilon^2\sin^2\omega,$$

and $\therefore$ from the former equation

$$\cot a(1+\cos 2\omega)+3\cos 2\omega\,.\,\cot\omega=0,$$

or $$\cot a \,.\, 2\sin\omega\cos\omega+3\cos 2\omega=0,$$

and $$\therefore\ \tan 2\omega+3\tan a=0.$$

Also $\epsilon^2=\dfrac{-\cos 2\omega}{\sin^2\omega}=2-\dfrac{1}{\sin^2\omega}$, $\therefore\ \epsilon^2<1$ and the conic is an ellipse.

CHAPTER XXV.

1. $$\frac{x}{a^2}\,.\,da+\frac{y}{b^2}\,.\,db=0,\ \text{and}\ \frac{da}{\sqrt{a}}+\frac{db}{\sqrt{b}}=0,\ \therefore\ \frac{x}{a^2}=\frac{\lambda}{\sqrt{a}},$$

$$\frac{y}{b^2}=\frac{\lambda}{\sqrt{b}},\ \text{and}\ \therefore\ \frac{x}{a}+\frac{y}{b}=\lambda\,(\sqrt{a}+\sqrt{b})\ \text{or}\ 1=\lambda\sqrt{\kappa},$$

and $$\therefore\ \frac{1}{a}=\frac{1}{(x\sqrt{\kappa})^{\frac{2}{3}}},\ \frac{1}{b}=\frac{1}{(y\sqrt{\kappa})^{\frac{2}{3}}},\ \text{and}\ \therefore\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=\kappa^{\frac{1}{3}}.$$

2. $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \text{and}\ a+b=c;\ \therefore\ \frac{x^2}{a^3}\,.\,da+\frac{y^2}{b^3}\,.\,db=0,$$

and $$da+db=0;\ \therefore\ \frac{x^2}{a^3}=\lambda=\frac{y^2}{b^3},$$

$$\therefore\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=\lambda\,(a+b),\ \text{i.e.}\ 1=\lambda c,$$

$$\therefore\ c=a+b=(cx^2)^{\frac{1}{3}}+(cy^2)^{\frac{1}{3}},\ \therefore\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}.$$

3. $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \text{and}\ \pi ab=\pi c^2;$$

$$\therefore\ \frac{x^2}{a^3}\,.\,da+\frac{y^2}{b^3}\,.\,db=0,\ \text{and}\ \frac{da}{a}+\frac{db}{b}=0;$$

$$\therefore\ \frac{\lambda x^2}{a^3}=\frac{1}{a},\ \frac{\lambda y^2}{b^3}=\frac{1}{b},\ \text{and}\ \therefore\ \lambda=2;$$

and $$\therefore\ a^2=2x^2,\ b^2=2y^2,\ \text{and}\ \therefore\ c^4=4x^2y^2.$$

4. If the line be $\dfrac{x}{a}+\dfrac{y}{b}=1$, $na+b=c$,

$$\therefore\ \frac{x}{a^2}\,da+\frac{y}{b^2}\,db=0,\ \text{and}\ nda+db=0;$$

$$\therefore\ \frac{\lambda x}{a^2}=n,\ \frac{\lambda y}{b^2}=1;\ \therefore\ \lambda=na+b=c;$$

$$\therefore\ a^2=\frac{cx}{n},\ b^2=cy,\ \text{and}\ \therefore\ \sqrt{cnx}+\sqrt{cy}=c,$$

or $(nx+y-c)^2=4nxy$, a parabola.

5. $y=m(x-2a)-am^3$, and m being the parameter,

$$0=x-2a-3am^2; \therefore, \text{ eliminating } m,$$

$$y=\pm\sqrt{\frac{x-2a}{3a}}(x-2a)\cdot\frac{2}{3}, \quad \therefore\ 27ay^2=4(x-2a)^3.$$

6. The equation to the normal may be written

$$y\cos\phi - x\sin\phi = a\cos 2\phi,$$

$\therefore$ ϕ being the parameter, $y\sin\phi + x\cos\phi = 2a\sin 2\phi$; hence

$$y=a(\sin\phi\sin 2\phi+\cos\phi),\quad x=a(\sin 2\phi\cos\phi+\sin\phi),$$

$$\therefore\ x+y=a(\sin\phi+\cos\phi)^3,\quad x-y=a(\sin\phi-\cos\phi)^3,$$

or
$$\left(\frac{x+y}{a}\right)^{\frac{1}{3}}=\sin\phi+\cos\phi,\quad \left(\frac{x-y}{a}\right)^{\frac{1}{3}}=\sin\phi-\cos\phi;$$

squaring and adding $\left(\frac{x+y}{a}\right)^{\frac{2}{3}}+\left(\frac{x-y}{a}\right)^{\frac{2}{3}}=2$ or &c.

7. $\dfrac{x^{\frac{1}{2}}}{a^{\frac{3}{2}}}\cdot da+\dfrac{y^{\frac{1}{2}}}{b^{\frac{3}{2}}}db=0$, and $\dfrac{da}{a}+\dfrac{db}{b}=0$;

$$\therefore\ \sqrt{\frac{x}{a}}=\lambda=\sqrt{\frac{y}{b}},\quad \therefore\ \lambda=\frac{1}{2},\quad \therefore\ c^2=\frac{xy}{\lambda^2}=\frac{xy}{4},$$

an hyperbola with the axes for asymptotes (rectangular if the axes be so).

8. The normal at (x, y) meets the axis in $(x+2a, 0)$, $\therefore$ the straight line perpendicular to the normal is $y'=m(x'-x-2a)$ where $m=\dfrac{2a}{y}$, or $4ax=\dfrac{4a^2}{m^2}$, $\therefore$ the line is $y'=m(x'-2a)-\dfrac{a}{m}$, $\therefore$ for the locus

$$0=x'-2a+\frac{a}{m^2}, \text{ and } \therefore\ x'-2a=\frac{y'}{m}+\frac{a}{m^2},$$

or
$$2(x'-2a)=y'\cdot\sqrt{\frac{2a-x'}{a}}, \text{ and } \therefore\ 4a(x'-2a)+y'^2=0.$$

9. The parabola being $y^2=4ax$, and the given straight line $x=c$, any tangent is $y=mx+\dfrac{a}{m}$, which meets $x=c$ where $y=mc+\dfrac{a}{m}$, and $\therefore$ the perpendicular line is

$$m\left(y-mc-\frac{a}{m}\right)+x-c=0, \text{ or } my+x-m^2c-a-c=0,$$

and $\therefore$ for the envelope which this line touches $y=2mc$, $\therefore$ eliminating m,

$$\frac{y^2}{2c}+x-\frac{y^2}{4c}=a+c \text{ or } y^2=-4c(x-a-c),$$

a parabola the vertex of which is at $(a+c, 0)$ and $\therefore$ the focus at $(a, 0)$ i.e. a parabola confocal with $y^2=4ax$, being co-axial.

If $c=0$, for the envelope $y=0$ and $x=a$, and the straight lines all pass through the focus.

10. $(x-a)\frac{da}{h^2}+(y-b)\frac{db}{k^2}=0$, and $\frac{ada}{h^2}+\frac{bdb}{k^2}=0$,

$$\therefore\ \frac{x-a}{h^2}=\frac{\lambda a}{h^2},\ \frac{y-b}{k^2}=\frac{\lambda b}{k^2};\ \therefore\ \frac{a}{x}=\frac{b}{y}=\mu \text{ say},$$

$$\therefore\ \mu^2\left(\frac{x}{h}\right)^2+\mu^2\left(\frac{y}{k}\right)^2=1, \text{ and } \frac{x^2}{h^2}+\frac{y^2}{k^2}=\frac{2ax}{h^2}+\frac{2by}{k^2}=2\mu\left(\frac{x^2}{h^2}+\frac{y^2}{k^2}\right),$$

$$\therefore\ \mu=\frac{1}{2}, \text{ and } \therefore\ \frac{x^2}{h^2}+\frac{y^2}{k^2}=\frac{1}{\mu^2}=4.$$

11. If a double ordinate of $y^2=4ax$ be given by $x=x'$, the corresponding circle is

$$(x-x')^2+y^2=y'^2 \quad\ldots\ldots\ldots\ldots(1),$$

or $(x-x')^2+y^2=4ax'$, $\therefore$ for the envelope $x-x'+2a=0$, $\therefore$, eliminating x', $4a^2+y^2=4a(x+2a)$ which is an equal parabola.

Also by (1) $4a^2+y^2=y'^2$, $\therefore$ y' is not $<2a$, i.e. the circles on the ordinates between the focus and vertex do not meet.

12. If (x', y') be a point on the parabola, the corresponding circle is

$$(x-x')^2+(y-y')^2=x'^2+y'^2,$$

or $x^2+y^2-x\cdot\frac{y'^2}{2a}-2yy'=0$, $\therefore$ for the envelope $\frac{xy'}{a}+2y=0$, $\therefore$ the envelope is

$$x^2+y^2=x\cdot\frac{2ay^2}{x^2}-\frac{4ay^2}{x} \text{ or } x(x^2+y^2)+2ay^2=0, \text{ i.e., \&c.}$$

13. (x', y') being a point on the given parabola $y^2=l'x$, the corresponding described parabola is

$$(y-y')^2=l(x-x'),\ \therefore\ 2(y-y')=l\frac{dx'}{dy'}=\frac{l}{l'}2y',$$

$$\therefore\ y'=\frac{l'y}{l+l'}, \text{ and } x'=\frac{y'^2}{l'}=\frac{l'y^2}{(l+l')^2},\ \therefore\ \left(\frac{ly}{l+l'}\right)^2=l\left\{x-\frac{l'y^2}{(l+l')^2}\right\},$$

or

$$lx=\frac{y^2l}{(l+l')^2}(l+l'), \text{ i.e. } y^2=(l+l')x,\ \therefore \text{ \&c.}$$

14. The axes of the ellipses must be fixed in direction, or any ellipse as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ would have its major and minor auxiliary circles for envelope by rotation round its centre, and by varying a and b indefinitely subject to $a^2+b^2=c^2$, the envelope would be the plane of xy.

Hence with the above restriction (equivalent to 2 extra conditions),

$$\frac{x^2}{a^3}\cdot da+\frac{y^2}{b^3}db=0=ada+bdb,\ \therefore\ \frac{\lambda x^2}{a^3}=a,\ \frac{\lambda y^3}{b^3}=b,$$

$$\therefore\ \lambda=c^2,\ \therefore\ a^2=\pm cx,\ b^2=\pm cy, \text{ and } \therefore\ \pm cx\pm cy=c^2,$$

or

$$x\pm y=\pm c.$$

Any 2 other independent conditions would bring the question within the definition of Art. 334.

15. If $(h,\ k)$ be a point on the ellipse, the corresponding line is $\frac{x}{h}+\frac{y}{k}=1$; thus for its envelope $\frac{x}{h^2}\cdot dh+\frac{y}{k^2}dk=0$,

and
$$\frac{hdh}{a^2}+\frac{kdk}{b^2}=0,\ \therefore\ \frac{\lambda x}{h^2}=\frac{h}{a^2},\ \frac{\lambda y}{k^2}=\frac{k}{b^2};$$

$$\therefore\ \lambda\left(\frac{x}{h}+\frac{y}{k}\right)=\lambda=1,\ \therefore\ \frac{1}{h}=\frac{1}{(a^2x)^{\frac{1}{3}}},\ \frac{1}{k}=\frac{1}{(b^2y)^{\frac{1}{3}}},$$

and
$$\therefore\ \left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1.$$

16. If $(x',\ y')$ be on the 1st ellipse, the corresponding chord of contact is $\frac{xx'}{a^2}+\frac{yy'}{b^2}=1$, $\therefore$ for the locus

$$\frac{x}{a^2}\cdot dx'+\frac{y}{b^2}\cdot dy'=0=\frac{x'}{h^2}dx'+\frac{y'dy'}{k^2};$$

$$\therefore\ \frac{\lambda x}{a^2}=\frac{x'}{h^2},\ \frac{\lambda y}{b^2}=\frac{y'}{k^2},\ \therefore\ \lambda=1,\ \text{and}\ \therefore\ x'=\frac{h^2}{a^2}x,$$

$$y'=\frac{k^2}{b^2}y,\ \text{and}\ \therefore\ \text{the locus is}\ \frac{x^2h^2}{a^4}+\frac{y^2k^2}{b^4}=1.$$

17. If $(h,\ k)$ be its centre, such a circle is $x^2+y^2=2hx+2ky$ where
$$a^2k^2+b^2h^2-2ab^2h=0,$$

$\therefore$ for the locus
$$xdh+ydk=0=a^2kdk+b^2(h-a)\,dh;$$

$$\therefore\ a^2k=\lambda y,\ b^2(h-a)=\lambda x,$$

and
$$\therefore\ x^2+y^2=\frac{2\lambda y^2}{a^2}+2\frac{\lambda x^2}{b^2}+2ax,$$

or
$$\lambda=a^2b^2(x^2+y^2-2ax)\div 2(a^2x^2+b^2y^2),$$

and
$$\frac{\lambda^2y^2}{a^2}+\frac{\lambda^2x^2}{b^2}=a^2b^2,$$

$$\therefore\ a^4b^4(x^2+y^2-2ax)^2=4a^4b^4(a^2x^2+b^2y^2),$$

or
$$(x^2+y^2-2ax)^2=4(a^2x^2+b^2y^2).$$

18. If $(h,\ k)$ be the centre of the 2nd circle $\frac{h^2}{\alpha^2}+\frac{k^2}{\beta^2}=1$, and its equation is $x^2+y^2-2hx-2ky=0$, $\therefore$ the common chord is $x(h+a)+y(k+b)+c=0$, $\therefore$, for its envelope,
$$xdh+ydk=0=\frac{hdh}{\alpha^2}+\frac{kdk}{\beta^2};$$

$$\therefore\ \lambda x=\frac{h}{\alpha^2},\ \lambda y=\frac{k}{\beta^2},$$

and
$$\therefore\ \lambda^2(x^2\alpha^2+y^2\beta^2)=1\ \text{and}\ \lambda(x^2\alpha^2+y^2\beta^2)+ax+by+c=0;$$

$$\therefore\ (ax+by+c)^2=x^2\alpha^2+y^2\beta^2.$$

19. $-y\cos\theta+x\sin\theta+c=c\sin\theta\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$ .... (1);

$$\therefore\ y\sin\theta+x\cos\theta=c\cos\theta\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+\frac{c}{2}\sin\theta\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\sec^2\overline{\frac{\pi}{4}+\frac{\theta}{2}}$$

$$=c\cos\theta\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+c\frac{\sin\theta}{\cos\theta};$$

$\therefore$, eliminating the log.

$$y=c\left(\cos\theta+\frac{\sin^2\theta}{\cos\theta}\right)\text{ or }y\cos\theta=c,$$

$$\therefore\text{ by (1) }\frac{x}{c}=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right),\ \therefore\ e^{\frac{x}{c}}=\frac{\cos\theta}{1-\sin\theta};$$

$$\therefore\ e^{\frac{x}{c}}+e^{-\frac{x}{c}}=\frac{\cos^2\theta+(1-\sin\theta)^2}{\cos\theta\,(1-\sin\theta)}=\frac{2}{\cos\theta}=\frac{2y}{c}.$$

20. If (see fig. p. 304) S be the pole, and P, Q two adjacent points on the spiral, and the perpendiculars to SP, SQ meet in O, O is ultimately a point on the envelope, and SO is the diameter of a circle through P, Q, $\therefore$ if P be (r,θ) and O be (r',θ'), $\angle\,OSP+\frac{\pi}{2}+\phi=\pi$, i.e. $OSP=\frac{\pi}{2}-\phi$,

and $$\therefore\ \theta'=\theta+\frac{\pi}{2}-\phi,\text{ and }r=r'\cos\left(\frac{\pi}{2}-\phi\right);$$

but $$r^{n-1}\,.\cos n\theta=r^n\,.\sin n\theta\frac{d\theta}{dr},\ \therefore\ \tan\phi=\cot n\theta;$$

$$\therefore\ \frac{\pi}{2}-\phi=n\theta,\ \therefore\ \theta'=(n+1)\,\theta,$$

and $$r=r'\cos n\theta=r'\cos m\theta'=\frac{a}{(\cos n\theta)^{\frac{1}{n}}};$$

$$\therefore\ r'(\cos m\theta')^{1+\frac{1}{n}}=a,$$

or $$r'^m\cos(m\theta')=a^m.$$

21. If an ellipse be $y^2+x^2(1-e^2)=a^2(1-e^2)$, and the given directrix be $x=c$, then $a=ec$, and the ellipse is

$$y^2+x^2(1-e^2)=c^2e^2(1-e^2),$$

$\therefore$ for the envelope $-x^2e=c^2e\{1-e^2-e^2\}$; $\therefore$ for real values of x, $e^2>\frac{1}{2}$, i.e. the envelope does not meet those ellipses in which $e<\frac{1}{\sqrt{2}}$.

If $e^2 > \frac{1}{2}$, $y^2 = (1-e^2)(c^2e^2 - x^2) = \left(1 - \frac{c^2+x^2}{2c^2}\right)\left(\frac{c^2+x^2}{2} - x^2\right)$;

$$\therefore\ y = \pm \frac{c^2 - x^2}{2c}, \text{ i.e. 2 parabolas.}$$

22. If the given line and point be axis of x and origin, the ellipse is $a^2(y \pm b)^2 + b^2x^2 = a^2b^2$, as the ellipse is on the negative or positive side of the axis of x, and the equation is

$$y^2 \pm 2by + (1-e^2)x^2 = 0,$$

subject to $a = ce$, or
$$b^2 = c^2e^2(1-e^2),$$

$$\therefore\ y^2 \pm 2yce\sqrt{1-e^2} + (1-e^2)x^2 = 0,$$

and
$$\pm 2yc\left(\frac{1-e^2-e^2}{\sqrt{1-e^2}}\right) - 2ex^2 = 0,$$

and with the upper sign y is negative, $\therefore$ for real values of x, $2e^2 > 1$, i.e. $e > \frac{1}{\sqrt{2}}$; and so for the lower sign by symmetry.

Also when $2e^2 > 1$, $y^2 + (1-e^2)x^2 = -\frac{2e^2x^2(1-e^2)}{1-2e^2}$.

or
$$y^2(2e^2-1) = x^2(1-e^2),\ \therefore\ e^2 = \frac{x^2+y^2}{x^2+2y^2}, \text{ and } 1 - e^2 = \frac{y^2}{x^2+2y^2};$$

$$\therefore\ \left(y^2 + \frac{x^2y^2}{x^2+2y^2}\right)^2 = 4c^2y^2 \cdot \frac{(x^2+y^2)\,y^2}{(x^2+2y^2)^2},\ \therefore\ x^2 + y^2 = c^2.$$

23. If S be the focus and P, Q be 2 adjacent points on the conic, S is one point of intersection of the circles on SP, SQ and the other point is the foot of the perpendicular from S on PQ, and $\therefore$ ultimately the locus is that of the foot of the perpendicular from S on the tangent at P. Hence the locus is a circle or a straight line, as the conic is a central conic or a parabola.

24. If a pair of tangents at right angles be drawn from (h, k) to

$$a^2y^2 + b^2x^2 = a^2b^2,\ h^2 + k^2 = a^2 + b^2,$$

and the chord of contact is $a^2yk + b^2xh = a^2b^2$, $\therefore$ for the envelope of the chord, $a^2 \cdot ydk + b^2 \cdot xdh = 0$, and $hdh + kdk = 0$; $\therefore\ a^2y = \lambda k$, $b^2x = \lambda h$;

$$\therefore\ a^2b^2 = \lambda(a^2+b^2),\ \therefore\ h = \frac{x}{a^2}(a^2+b^2),\ k = \frac{y(a^2+b^2)}{b^2}.$$

and $\therefore\ \frac{a^2y^2}{b^2} + \frac{b^2x^2}{a^2} = \frac{a^2b^2}{a^2+b^2}$; if this be written

$$p^2y^2 + q^2x^2 = p^2q^2,\ p^2 - q^2 = \frac{a^4}{a^2+b^2} - \frac{b^4}{a^2+b^2} = a^2 - b^2;$$

$$\therefore \text{ the envelope is a confocal ellipse.}$$

25.
$$x\cos 3\theta + y\sin 3\theta = a(\cos 2\theta)^{\frac{3}{2}};$$
$$\therefore\ -x\sin 3\theta + y\cos 3\theta = -a(\sin 2\theta)(\cos 2\theta)^{\frac{1}{2}};$$
$$\therefore\ x^2+y^2=a^2\cos 2\theta,$$
and
$$x = a\sqrt{\cos 2\theta}\,(\cos\theta),\quad y = a\sqrt{\cos 2\theta}\sin\theta;$$
$$\therefore\ x^2-y^2=a^2\cos^2 2\theta,\quad \therefore\ a^2(x^2-y^2)=(x^2+y^2)^2.$$

26. As in Ex. 23, the locus is that of the foot of the perpendicular from the centre on the tangent; and if ϕ be the eccentric angle of any point, the tangent is $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$, and the perpendicular on it from the centre is
$$\frac{x\sin\phi}{b}=\frac{y\cos\phi}{a}.\quad \therefore\ \frac{\cos\phi}{ax}=\frac{\sin\phi}{by}=\frac{1}{\sqrt{a^2x^2+b^2y^2}},$$
and $\therefore$ the envelope is $\quad x^2+y^2=\sqrt{a^2x^2+b^2y^2}$ or &c.

27. As in Ex. 23, the envelope is the locus of the foot of the perpendicular from the pole on the tangent: and if y (fig. on p. 308) be (r', θ'),
$$\theta' = -\left(\frac{\pi}{2}-\phi-\theta\right),\ r'=r\sin\phi,\ \text{but}\ \frac{dr}{d\theta}=c\sec^n\frac{\theta}{n}\cdot\tan\frac{\theta}{n},$$
$$\therefore\ \tan\phi = r\frac{d\theta}{dr}=\cot\frac{\theta}{n}\ \text{ or }\ \phi=\frac{\pi}{2}-\frac{\theta}{n},$$
and
$$\therefore\ -\theta'=\frac{\theta}{n}-\theta\ \text{ or }\ \theta=\frac{n\theta'}{n-1},\ \text{and}\ r'=r\cos\frac{\theta}{n},$$
$$\therefore\ r'=c\sec^{n-1}\frac{\theta}{n}=c\sec^{n-1}\frac{\theta'}{n-1}.$$

28. If a be the angle of the spiral, its equation is (Art. 354) $r=be^{\theta\cot a}$, and if (r', θ') be any point on the parabola which corresponds to (r, θ), then
$$r\sin a = r' + r'\cos\left(\theta'+\frac{\pi}{2}-\theta-a\right),$$
or $r'+r'\sin(\theta+a-\theta')=b\sin a e^{\theta\cot a}$, is the equation of the parabola, θ being the parameter, $\therefore$, for the envelope, $r'\cos(\theta+a-\theta')=b\cos a\,.\,e^{\theta\cot a}$,

and
$$\therefore\ \cos a=\sin(\theta'-\theta),\quad \therefore\ \theta'-\theta=\frac{\pi}{2}-a,$$
and
$$\therefore\ r'\,.\,\sin 2a = b\cos a\,.\,e^{\cot a\left(\theta'+a-\frac{\pi}{2}\right)},$$
or
$$r'=\frac{b}{2}\operatorname{cosec} a\,.\,e^{\left(a-\frac{\pi}{2}\right)\cot a}\,.\,e^{\theta'\cot a}=ce^{\theta'\cot a},$$

i.e. a similar equiangular spiral.

29. As in Ex. 23, the envelope is the locus of the foot of the perpendicular from the pole S on the tangent at Y (Art. 284) to the locus of Y. If the locus of Y be given by p', r' and $p=r\sin\alpha$ be the equation to the spiral, by Art. 329
$$p'=\frac{p^2}{r},\text{ and } p=r',$$
$\therefore\ p'=r'^2\div r'\operatorname{cosec}\alpha=r'\sin\alpha$, $\therefore$ the locus of Y and similarly the envelope is a similar equiangular spiral.

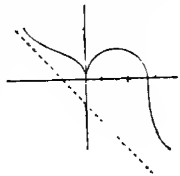
30. If the parabola be $y^2=4ax$, and (h, k) any point on it, the corresponding circle is $(x-h)^2+(y-k)^2=(h+a+c)^2$, subject to $k^2=4ah$, $\therefore$ for the envelope $\{(x-h)+(h+a+c)\}\,dh+(y-k)\,dk=0$, and $kdk=2adh$,
$$\therefore\ k(x+a+c)+2a(y-k)=0,$$
or
$$k=-\frac{2ay}{x-a+c},\ \therefore\ y-k=\frac{y(x+a+c)}{x-a+c},$$
and
$$h=\frac{ay^2}{(x-a+c)^2},\text{ but } (y-k)^2+x^2-2xh=(a+c)^2+2h(a+c);$$
$$\therefore y^2\frac{(x+a+c)^2}{(x-a+c)^2}+x^2-(a+c)^2=2\frac{(x+a+c)\,ay^2}{(x-a+c)^2},$$
or
$$\frac{(x+a+c)\,y^2}{(x-a+c)^2}(x-a+c)+x^2-(a+c)^2=0,$$
or
$$(x+a+c)\,\{y^2+(x-a)^2-c^2\}=0.$$

31. For the envelope $ax\sec\theta\tan\theta+by\operatorname{cosec}\theta\cot\theta=0$;
$$\therefore\ \frac{\sin\theta}{(by)^{\frac{1}{3}}}=-\frac{\cos\theta}{(ax)^{\frac{1}{3}}}=\frac{\pm 1}{\{(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\}^{\frac{1}{2}}};$$
$$\therefore\ \mp\{(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\}^{\frac{3}{2}}=a^2-b^2,\ \therefore\ (ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$$

32. If $AP=2a\cos\theta$, P is $(2a\cos^2\theta,\ 2a\sin\theta\cos\theta)$, and $\therefore$ the circle on HP (A being the origin) is
$$\left(x-\frac{3}{2}a\cos^2\theta\right)^2+\left(y-\frac{3}{2}a\sin\theta\cos\theta\right)^2=\frac{a^2}{4}\cos^2\theta,$$
or
$$x^2+y^2-3ax\cos^2\theta-\frac{3ay}{2}\sin 2\theta+2a^2\cos^2\theta=0,$$
or
$$2(x^2+y^2)-3ay\sin 2\theta-3ax(1+\cos 2\theta)+2a^2(1+\cos 2\theta)=0;$$
$\therefore$ for the envelope $\quad 3ay\cos 2\theta=(3ax-2a^2)\sin 2\theta$;
$$\therefore\ \frac{\sin 2\theta}{3ay}=\frac{\cos 2\theta}{3ax-2a^2}=\frac{\pm 1}{\{(3ay)^2+(3ax-2a^2)^2\}^{\frac{1}{2}}};$$
$$\therefore\ 2(x^2+y^2)-3ax+2a^2=\pm\{(3ay)^2+(3ax-2a^2)^2\}^{\frac{1}{2}};$$
$$\therefore\ \{2(x^2+y^2)-3ax\}^2=9a^2(x^2+y^2)-12a^3x-8a^2\{x^2+y^2\}+12a^3x$$
$$=a^2(x^2+y^2).$$

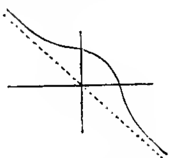
CHAPTER XXVI.

1. The *direction* of the asymptote is given by $x^3+y^3=0$, $\therefore$, if its equation be $x+y=c$, $-3c=a$, and the asymptote is $x+y=-\frac{a}{3}$. At the origin approximately $y^3=ax^2$, $\therefore$ there is a cusp of the 1st kind, $x=0$ being the tangent. Also for any value of x there is only one real value of y, and $\because$ $y^3=x^2(a-x)$, y is positive when x lies between a and $-\infty$, and when $x>a$, y is negative; and the curve meets $y=0$ when $x=0$ or a and changing the origin to $(a, 0)$ clearly the curve cuts the axis of x at right angles. Hence (a being positive) the general description of the curve is that it has a cusp at (0, 0) where the 2 branches touch the axis of y above $y=0$, the branch to the right of $x=0$ cutting $y=0$ again, and extending unbroken to touch the asymptote at ∞; the other branch extends to touch the asymptote at ∞ at the other end. If a be negative the curve is the same as the first turned round the origin in its plane through 2 right angles, the equation being the same in form as when a is positive, on changing the signs of both x and y.

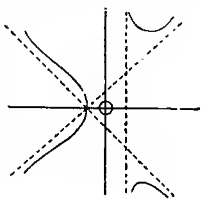


As a test of the curve it may be noticed that for any value of y there are one or three real values of x.

2. If $x+y=b$ be the asymptote, $-3b=0$, $\therefore$ the asymptote is $x+y=0$. Where the curve meets $x=y$, $x=y=\frac{a}{\sqrt[3]{2}}=c$ say, and changing the origin to this point, the curve is $(x+c)^3+(y+c)^3=2c^3$, and approximately near the new origin $x^2+y^2+c(x+y)=0$, $\therefore$ the tangent is $x+y=0$, and $x+y$ is of the opposite sign to c; thus the curve is symmetrical as to $x=y$ and lies between $x+y=0$ and the asymptote $x+y+2c=0$, and there being but one real value of x or y corresponding to any value of y or x respectively, the curve spreads from the origin to ∞ to touch the asymptote at both ends, and having necessarily 2 points of inflexion equidistant from $x=y$. Turning it through 2 right angles, the curve is seen to be the same whether c and $\therefore$ a be positive or negative.



3. If $y=mx+n$ be an asymptote, $(mx+n)^2(x-a)=x^2(x+a)$ has 2 ∞ roots in x, $\therefore$ $m^2=1$ and $2mn-m^2a=a$, $\therefore$ $n=\frac{a}{m}$, and $\therefore$ $y=\pm(x+a)$ are asymptotes, as also $x=a$. Also if x be positive $x>a$, and if negative $x<-a$ for real values of y, but when $y=0$, $x=0$, or $-a$. Hence the origin is a conjugate point, and the curve, which is symmetrical as to the axis of x, cuts it at $(-a, 0)$. Changing the origin to this point the equation is $y^2(x-2a)=x(x-a)^2$, $\therefore$ near this origin $2ay^2+a^2x=0$, or the curve is nearly parabolic, being concave towards the left of $x=0$, and this branch extends with 2 points of inflexion symmetrically as to $y=0$ to touch both



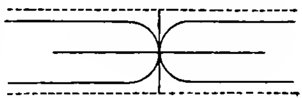
$$y=\pm(x+a) \text{ at } \infty.$$

When $x>a$, $\frac{y^2}{(x+a)^2}=\frac{x^2}{x^2-a^2}$, $\therefore$ taking the positive value of y, $\frac{y}{x+a}$ on the curve >1, $\therefore$ the curve lies *above* $y=x+a$: thus there are 2 branches touching the 3 asymptotes at ∞ as in the fig. p. 372, but on the right of $x=0$.

If a be negative and the signs of x and y be altered, the equation is of the same form, $\therefore$ the curve is the same as before but turned through 2 right angles.

4. The real asymptotes are $y=\pm a$, the curve is symmetrical as to both axes, and passes through the origin where the tangents are $x=\pm y$. Also $y^2=\frac{a^2x^2}{a^2+x^2}$, $\therefore$ y is possible for all values of x; so $x^2=\frac{a^2y^2}{a^2-y^2}$, $\therefore$ y lies between a and $-a$, and for any value of x there is but one positive value of y. Hence the curve consists of 2 branches undulating through the origin, and extending the one to

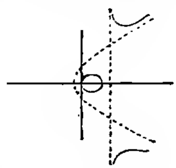
$(+\infty, a)$ and $(-\infty, -a)$,



the other to the other ends of the asymptotes.

5. The asymptote is $x=4a$, the curve is symmetrical as to $y=0$, and when $y=0$, $x=0$ or $3a$. If x be negative (supposing a positive) y is impossible. For each value of x from 0 to $3a$, there is one positive value of y, thus there is a loop joining (0, 0) and ($3a$, 0): x cannot lie between $3a$ and $4a$. Also $\frac{y^2}{a}=x\left(1-\frac{3a}{x}\right)\left(1-\frac{4a}{x}\right)^{-1}$, $\therefore$ when x is large

$$\frac{y^2}{a}=x\left(1+\frac{a}{x}\right)+\text{terms which vanish when } x=\infty,$$

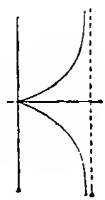


$\therefore$ there is a parabolic asymptote $y^2=a(x+a)$, and when $x=4a$, $y=\infty$, $\therefore$ there are 2 branches on the right of $x=4a$ lying between $x=4a$ and the parabola and touching both at ∞.

Turning the curve through an angle π in its own plane round the origin, the curve is that given when a is negative.

6. Changing to polar co-ordinates $r=\pm a\sin 2\theta$, and the curve is easily found to consist of 4 equal and similar loops through the origin, symmetrically arranged with $x=\pm y$ for the axis of each.

7. The real asymptote is $x=2a$, and for real values of y, x must lie between 0 and $2a$; and near the origin $y^2 \cdot 2a=x^3$, $\therefore$ there is a cusp of the 1st kind there, and the curve being symmetrical as to $y=0$, consists of 2 branches starting from the origin and touching the axis of x to the right of $x=0$, and extending to infinity on the left side of $x=2a$ to touch that asymptote, one branch at either end.



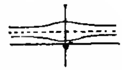
8. With $(0, -b)$ as origin the equation is

$$x^2(y-b)^2=\{a^2-(y-b)^2\}\cdot y^2,$$

or $(r\sin\theta - b)^2 = a^2\sin^2\theta$, and $\therefore r = b\,\text{cosec}\,\theta \pm a$. Thus if O be the origin, Q the point on the line $y=b$ where OQ makes an angle θ with the axis of x, the corresponding points P, P' on the curve are given by

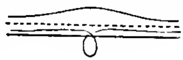
$$OP = OQ + a,\quad OP' = OQ - a.$$

When $\theta = 0$, P, P' pass to ∞, and $y=b$ is the asymptote; when $\theta = \frac{\pi}{2}$, r is a minimum; thus from $\theta=0$ to π, P, P' trace out 2 ∞ branches symmetrical as to $x=0$ and on either side of and concave to $y=b$, supposing that $b>a$. Also when $\theta' = \theta + \pi$, $r' = -b\,\text{cosec}\,\theta \pm a$, and $\therefore$ the points P, P' are reproduced in reverse order, and $\therefore$ the curve is composed of the 2 branches, while $r=0$ is impossible on the lower branch, $\because b>a$: but r was a factor in the polar equation, $\therefore O$ is a conjugate point.



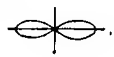
If $b=a$, the lower branch passes through the origin, and the rectangular equation reduces to $x^2(y-b)^2 = y^2(2by-y^2)$, $\therefore x=0$ is a double tangent at O, and the equation is approximately $bx^2 = 2y^3$, $\therefore O$ is a cusp of the 1st kind.

If $a>b$, let $a = b\,\text{cosec}\,\alpha$, then from $\theta=\alpha$ to $\pi-\alpha$, P' will trace out a loop on the left of $x=0$, which will pass through O, $\because OP'=0$ when $\theta=\alpha$, and in this case at O the equation is approximately $b^2x^2 = y^2(a^2-b^2)$, and $\therefore O$ is a double point: *or* the directions of the curve at O are given by $\theta=\alpha$ and $\theta=\pi-\alpha$, $\therefore$ there are 2 branches at O.

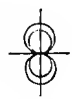


In fig. (1) $b>a$; in (2) $b<a$.

9. The origin is a double point the tangents there being $x=\pm y$. In polar co-ordinates the curve is $r^2 = a^2\cos 2\theta$, $\therefore$ when $\theta=0$, $r=\pm a$, and when $\theta=\frac{\pi}{4}$, $r=0$, and the curve is symmetrical as to each axis, and thus takes the shape of a figure of 8. The origin is also a point of double inflexion, as each tangent there cuts the curve as well as touches it.



10. As $\theta \propto$ from 0 to π, the end of the radius vector traces out a loop through the pole and above $y=0$ and touching it: as $\theta \propto$ from π to 2π, another outer loop is traced similarly and so on *ad infinitum*. To negative values of θ, corresponds a similar series of loops below $y=0$.



11. The curve starts from the pole touching $y=0$, and the radius vector is ever increasing with θ, $\because \frac{dr}{d\theta} = a(1+\cos\theta)$ which is never negative: thus the curve roughly resembles the *whole* of the fig. to Art. 354. Negative values of θ give an exactly similar curve, the reverse way round the pole.

12. When $\theta=0$, $r=\infty$, and then r diminishes till when $\theta=\frac{\pi}{2}$, $r=0$; from $\theta=\frac{\pi}{2}$ to π, r increases from 0 to ∞; and when r is ∞, $r\sin\theta = a$ ultimately, thus $y=a$ is an asymptote. Also any value α of θ gives the

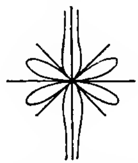
same point as $a+\pi$, $\therefore$ the whole curve consists of 2 branches through the pole, and is symmetrical as to $x=0$, extends to ∞ above $y=0$, and is bounded by $y=a$; and by varying θ slightly from $\frac{\pi}{2}$, it will be seen that each branch touches $x=0$ at the pole, which is, $\therefore$, a cusp (of the 1st kind) on the curve. Negative values of θ only produce the same curve. If a be negative the curve is turned round $y=0$ through an angle π on to its plane again.

13. When $\theta=0$, $r=-\infty$; $\theta=\frac{\pi}{2}$, $r=0$, and the *direction* of the curve there is along $x=0$; when $\theta=\pi$, $r=-\infty$ again; and negative values of $\sin\theta$ are inadmissible; thus r being always negative, the curve consists of 2 branches forming a cusp at the pole (of the 1st kind) below $y=0$, and spreading to ∞ one on either side of $x=0$ below $y=0$: when $\theta=0$, the limit of $r\sin\theta =$ that of $\frac{\log\sin\theta}{\operatorname{cosec}\theta}$,

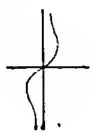
$$=\cot\theta\div(-\operatorname{cosec}\theta\cot\theta)=-\sin\theta=0,$$

$\therefore$ $y=0$ is an asymptote: $2\pi+\theta$ or $-(\pi+\theta)$ &c. give only the same curve, which is symmetrical as to $x=0$.

14. Corresponding to each value of θ there are 2 values of r, subject to $\cos\theta$ being always positive. From $\theta=0$ to $\frac{\pi}{6}$, $r^2 \propto$ from 0 to $\frac{2a^2}{\sqrt{3}}$; and from $\theta=\frac{\pi}{6}$ to $\frac{\pi}{3}$, $r^2\propto$ from $\frac{2a^2}{\sqrt{3}}$ to 0; these values give 2 equal and similar loops through the pole. So, the values of θ from 0 to $-\frac{\pi}{3}$ give 2 more equal loops. Also, as $\theta \propto$ from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, $r^2\propto$ from 0 to ∞, and when $\theta=\frac{\pi}{2}$, the limit of $r\cos\theta=0$, $\therefore$ there is a branch through the pole touching 2 of the loops and having $x=0$ for asymptote on opposite sides. Values of θ from $-\frac{\pi}{3}$ to $-\frac{\pi}{2}$ give a similar branch with $x=0$ for asymptote and on opposite sides of it. Other admissible values of θ only give points on the above loops and branches.



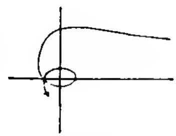
15. For real values of r, $\sin\theta$ and $\cos\theta$ must have the same sign, $\therefore$ only values of θ need be considered which lie between 0 and $\frac{\pi}{2}$, and between π and $\frac{3\pi}{2}$; for the former series $r^2\propto$ from 0 to ∞, and the 2 values of r give a curve touching the axis of x at the pole (the *direction* being given there by $\theta=0$) and passing to ∞ at each end of $x=0$, on opposite sides, $x=0$ being an asymptote, the limit of $r\cos\theta$ when $\theta=\pi$ being 0. The other values of θ give the same curve.



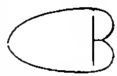
16. If the sign of θ be changed r has still the same value, $\therefore$ the curve is symmetrical as to the axis of x. When $\theta=0$; $\frac{r}{a}=$ the limit of

$$(\theta+\theta)\div\left(\theta-\theta+\frac{\theta^3}{6}\right)=\frac{12}{\theta^2}=\infty,$$

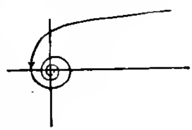
and the limit of $\frac{r}{a}\sin\theta=\frac{12\theta}{\theta^2}=\infty$, $\therefore$ there is no rectilinear asymptote, as no other value of θ makes $r=\infty$. If n be a positive integer, and $\theta=\phi+n\pi$, then as $\phi \propto$ from 0 to $\frac{\pi}{2}$ and then to π, $\frac{r}{a}\propto$ from 1 to $\dfrac{\left(n+\frac{1}{2}\right)\pi+(-1)^n}{\left(n+\frac{1}{2}\right)\pi-(-1)^n}$ and then to 1 again. Thus, for positive values of θ, the curve stretches to ∞ above the axis of x to the right of $x=0$, and, as θ increases, curves round the origin *ad infinitum*, cutting the axis of x in the same 2 points at each revolution, the part above that axis being greater than the part below, but tending ultimately to the form of a circle, as n increases indefinitely. Similarly for negative values of θ.



17. The equation gives the same value of r for $\pm\theta$, $\therefore$ the curve is symmetrical as to $y=0$, and r cannot be negative (a being supposed positive, and if a be negative each value of r is negative, and $\therefore$ the curve may be supposed turned over the axis of y on to its original plane). When $\theta=0$, $r=0$, and as r cannot be negative, there is a cusp at the pole. When $\theta=\pm\frac{\pi}{2}$, $r=a$, and when $\theta=\pi$, $r=2a$, and thus the curve is in the shape of a heart, as $\cos\theta$ can but vary from -1 to 1.



18. If $-\theta$ be put for θ, r becomes $-r$, $\therefore$ the curve is symmetrical as to $x=0$; r continually decreases as θ increases, and thus the curve for positive values of $\theta>\frac{\pi}{2}$ forms an infinite coil round the pole, which it continually approaches and ultimately reaches. When $\theta=0$, $r=\infty$, and $r\sin\theta=$ the limit of $\frac{a\sin\theta}{\theta}=a$, $\therefore$ $y=a$ is an asymptote: thus from $\theta=0$ to $\frac{\pi}{2}$, the curve stretches from ∞ to $r=\frac{2a}{\pi}$. The negative values give a similar curve, also having $y=a$ for an asymptote, but ultimately touching it at the opposite end. If a were negative the asymptote would be below $y=0$ &c.



19. If P be (x, y) the tangent is $(x'-x)\dfrac{dy}{d\theta}=(y'-y)\dfrac{dx}{d\theta}$ .... (1),

and the normal is $$(x'-x)\frac{dx}{d\theta}+(y'-y)\frac{dy}{d\theta}=0 \text{ .... (2);}$$

where $$\frac{dx}{d\theta}=(a+b)\left(\sin\frac{a+b}{b}\theta-\sin\theta\right),$$

and $$\frac{dy}{d\theta}=-(a+b)\left(\cos\frac{a+b}{b}\theta-\cos\theta\right);$$

$$\therefore \text{ (1) becomes } x'\left(\cos\frac{a+b}{b}\theta-\cos\theta\right)+y'\left(\sin\frac{a+b}{b}\theta-\sin\theta\right)$$

$$=\left(\sin\frac{a+b}{b}\theta-\sin\theta\right)\left\{(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta\right\}$$

$$+\left(\cos\frac{a+b}{b}\theta-\cos\theta\right)\left\{(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta\right\}$$

$$=(a+b)\cos\frac{a}{b}\theta-b-(a+b)+b\cos\frac{a}{b}\theta,$$

or $$x'.\sin\frac{a+2b}{2b}\theta.\sin\frac{a}{2b}\theta-y'.\cos\frac{a+2b}{2b}\theta\sin\frac{a}{2b}\theta$$

$$=(a+2b)\sin^2\frac{a\theta}{2b},$$

$$\therefore \text{ the tangent is } x'.\sin\frac{a+2b}{2b}\theta-y'.\cos\frac{a+2b}{2b}\theta=(a+2b)\sin\frac{a\theta}{2b}.$$

$$\text{Also (2) becomes } x'\left(\sin\frac{a+b}{b}\theta-\sin\theta\right)-y'\left(\cos\frac{a+b}{b}\theta-\cos\theta\right)$$

$$=-\left(\sin\frac{a+b}{b}\theta-\sin\theta\right)\left\{b\cos\frac{a+b}{b}\theta-(a+b)\cos\theta\right\}$$

$$+\left(\cos\frac{a+b}{b}\theta-\cos\theta\right)\left\{b\sin\frac{a+b}{b}\theta-(a+b)\sin\theta\right\}$$

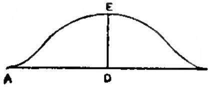
$$=+(a+b)\sin\frac{a}{b}\theta-b\sin\frac{a}{b}\theta,$$

or $$x'\cos\frac{a+2b}{2b}\theta+y'\sin\frac{a+2b}{2b}\theta=a\cos\frac{a\theta}{2b};$$

which is satisfied by $(a\cos\theta, a\sin\theta)$. In fact P is turning round B and $\therefore$ the tangent at P is perpendicular to BP and $\therefore$ &c.

20. Interchanging x and y, with the notation and figure of Art. 356, the curve required is clearly the locus of the foot of the perpendicular from M on BC, and is $\therefore$ a curve within the cycloid but meeting it at A and E, and symmetrical as to DE. Also $\frac{dy}{dx}=\frac{d}{d\phi}(1-\cos\phi)=\sin\phi$, $\therefore$ at A, where $\phi=0$, the curve touches AD; and at E, where $\phi=\pi$, $\frac{dy}{dx}=0$, $\therefore$ the curve

touches the cycloid at E. Further, $\frac{d^2y}{dx^2}=\frac{1}{a}\cdot\frac{d}{d\phi}(\sin\phi)=\frac{1}{a}\cos\phi$, $\therefore$ the curve is convex to AD from $\phi=0$ to $\frac{\pi}{2}$; and from $\phi=\frac{\pi}{2}$ to π, the curve is concave to AD, so that when $\phi=\frac{\pi}{2}$ there is a point of inflexion.



Aliter: the curve may be traced from the equation $x=a-a\cos\frac{y}{a}$.

21. As in Art. 360, $x=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta$,

and $$y=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta,\ \therefore \text{ when } a=2b,$$

$$x=3b\cos\theta-b\cos 3\theta,\ y=3b\sin\theta-b\sin 3\theta,$$

or $$x=b(6\cos\theta-4\cos^3\theta),\ \text{and } y=b(4\sin^3\theta);$$

$$\therefore \sin^2\theta=\left(\frac{y}{4b}\right)^{\frac{2}{3}},\ \text{and } \therefore x=2b\cos\theta\,.\left\{1+2\left(\frac{y}{4b}\right)^{\frac{2}{3}}\right\},$$

or $$\left(\frac{x}{2b}\right)^2=\left\{1-\left(\frac{y}{4b}\right)^{\frac{2}{3}}\right\}\left\{1+2\left(\frac{y}{4b}\right)^{\frac{2}{3}}\right\}^2$$

$$=\left\{1-\left(\frac{y}{4b}\right)^{\frac{2}{3}}\right\}\left\{1+4\left(\frac{y}{4b}\right)^{\frac{2}{3}}+4\left(\frac{y}{4b}\right)^{\frac{4}{3}}\right\}$$

$$=1+3\left(\frac{y}{4b}\right)^{\frac{2}{3}}-4\left(\frac{y}{4b}\right)^2,$$

or $$\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2-1=3\left(\frac{y}{2a}\right)^{\frac{2}{3}},\ \text{and } \therefore 4(x^2+y^2-a^2)^3=27a^4y^2.$$

22. When $a=b$ in Art. 360, $x=2a\cos\theta-a\cos 2\theta$,

and $$y=2a\sin\theta-a\sin 2\theta,$$

$$\therefore \frac{x^2+y^2}{a^2}=5-4\cos\theta,\ \therefore \frac{x}{a}=\frac{5a^2-x^2-y^2}{2a^2}+1-\frac{(5a^2-x^2-y^2)^2}{8a^4},$$

or, changing the origin to $(a, 0)$,

$$\frac{x}{a}=\frac{5a^2-(x+a)^2-y^2}{2a^2}-\{5a^2-(x+a)^2-y^2\}^2\div 8a^4,$$

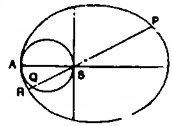
or $(x^2+y^2+2ax-4a^2)^2+4a^2(x^2+y^2+2ax-4a^2)+8a^3x=0$, or in polar co-ordinates, $(r^2+2ar\cos\theta-4a^2)^2+4a^2(r^2+2ar\cos\theta-4a^2)+8a^3r\cos\theta=0$,

or, $r^4+4ar^3\cos\theta+4a^2r^2\cos^2\theta-8a^2r^2-16a^3r\cos\theta+4a^2r^2+16a^3r\cos\theta=0$,

or $$r^2+4ar\cos\theta+4a^2\cos^2\theta=4a^2,$$

$\therefore r+2a\cos\theta=\pm 2a$, which represents 2 cardioids which are identical, as by writing $-r$ for r and $\pi+\theta$ for θ in either equation, it is transformed into the other, and the point $(-r, \pi+\theta)$ is the same as (r, θ).

23. As $\cos\left(-\frac{\theta}{3}\right)=\cos\frac{\theta}{3}$, the curve is symmetrical as to $\theta=0$. When $\theta=0$, $r=a$, and r diminishes till $\frac{\theta}{3}=\frac{\pi}{2}$, when $r=0$. From $\theta=0$ to $-\frac{3\pi}{3}$, a similar arc is formed, thus the curve consists of 2 loops, one within the other and touching it; and, as any line through S meets it in 3 points besides S (in general), and for every value of θ there are (in general) 3 values not zero of $\cos\frac{\theta}{3}$, $\therefore$ there can be no other branch of the curve. When $\theta=\pi$, $r=SA=\frac{a}{2}$.



If θ, at P, $=3\alpha$, then at Q, $\theta=3\alpha+\pi$, and at R, $-\theta=-(\pi-3\alpha)$; thus

$$\tan PAS=a\cos\alpha\,.\,\sin 3\alpha\div\left(a\cos\alpha\,.\,\cos 3\alpha+\frac{a}{2}\right)$$

$$=(\sin 4\alpha+\sin 2\alpha)\div(2\cos^2 2\alpha+\cos 2\alpha)=\tan 2\alpha,\ \therefore PAS=2\alpha,$$

$$\therefore APS=\alpha \text{ and } ASQ=3\alpha=3\,.\,APS.$$

Also
$$\tan SAQ=SQ\sin 3\alpha\div\left(\frac{a}{2}-SQ\,.\,\cos 3\alpha\right)$$

$$=a\cos\left(\alpha+\frac{\pi}{3}\right)\sin 3\alpha\div\left\{\frac{a}{2}-a\cos\left(\alpha+\frac{\pi}{3}\right)\cos 3\alpha\right\}$$

$$=\left\{\sin\left(4\alpha+\frac{\pi}{3}\right)+\sin\left(2\alpha-\frac{\pi}{3}\right)\right\}\div\left\{1-\cos\left(4\alpha+\frac{\pi}{3}\right)-\cos\left(2\alpha-\frac{\pi}{3}\right)\right\}$$

$$=\left\{\sin\left(4\alpha-\frac{2\pi}{3}+\pi\right)+\sin\left(2\alpha-\frac{\pi}{3}\right)\right\}\div\left\{1-\cos\left(2\alpha-\frac{\pi}{3}\right)+\cos\left(4\alpha-\frac{2\pi}{3}\right)\right\}$$

$$=-\tan\left(2\alpha-\frac{\pi}{3}\right),\ \therefore\ SAQ=\frac{\pi}{3}-2\alpha,\ \therefore\ PAQ=\frac{\pi}{3}.$$

$$\text{Tan } SAR=\cos\left(\frac{\pi}{3}-\alpha\right)\sin 3\alpha\div\left\{-\cos\left(\frac{\pi}{3}-\alpha\right)\cos 3\alpha+\frac{1}{2}\right\}$$

$$=\left\{\sin\left(2\alpha+\frac{\pi}{3}\right)+\sin\left(4\alpha-\frac{\pi}{3}\right)\right\}\div\left\{1-\cos\left(2\alpha+\frac{\pi}{3}\right)-\cos\left(4\alpha-\frac{\pi}{3}\right)\right\}$$

$$=\left\{-\sin\left(2\alpha-\frac{2\pi}{3}\right)-\sin\left(4\alpha-\frac{4\pi}{3}\right)\right\}\div\left\{1+\cos\left(2\alpha-\frac{2\pi}{3}\right)+\cos\left(4\alpha-\frac{4\pi}{3}\right)\right\}$$

$$=\sin\left(\frac{2\pi}{3}-2\alpha\right)\left\{1+2\cos\left(\frac{2\pi}{3}-2\alpha\right)\right\}\div\left\{1+2\cos\left(\frac{2\pi}{3}-2\alpha\right)\right\}.\cos\left(\frac{2\pi}{3}-2\alpha\right);$$

$$\therefore\ SAR=\frac{2\pi}{3}-2\alpha,\ \therefore\ QAR=\frac{\pi}{3}=PAQ.$$

Incidentally the 3 values of $\cos\frac{\theta}{3}$ are given above in terms of θ. The *figure* assumes that $3\alpha<\frac{\pi}{2}$, whence the particular solutions.

24. In $2a = r(1 - \tan\theta)$, putting $\theta - \frac{\pi}{4}$ for θ, i.e. turning the initial line through an angle $\frac{\pi}{4}$, the equation becomes $2a = r\left(1 - \frac{\tan\theta - 1}{\tan\theta + 1}\right)$,

or $$r = a(\tan\theta + 1),$$

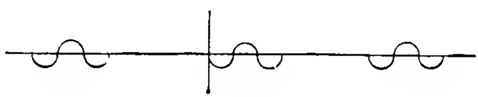
and now changing the sign of θ, $r = a(1 - \tan\theta)$, and $\therefore$ the 2 equations represent the same curve in different positions. This may also be arrived at by tracing the curves.

From the above it appears that to move the curve from the position given by the 2nd equation to that given by the 1st, it has to be turned round the pole in its plane through an angle $\frac{\pi}{4}$, and then turned round the prime radius on to its plane again. Hence any radius vector to a point of intersection is symmetrically situated as to the 2 curves, and must $\therefore$ bisect the angle between the tangents at such point of intersection.

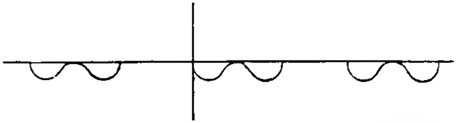
25. For simplicity the equation may be written $y = \sin x \log(m \sin x)$, which is only altering all the ordinates in the same finite ratio, and so of the abscissae. If m be positive, $\sin x$ cannot be negative, and if m were negative, by changing the signs of x and y the equation would be the same in fact, and $\therefore$ the curve is the same for a particular numerical value of m, whether m be positive or negative. Taking m then as positive, x must lie between $2p\pi$ and $2p\pi + \pi$, p being any integer: $\frac{dy}{dx} = \cos x \log(m \sin x) + \cos x$, $\therefore$ the curve is parallel to $y = 0$ when $\cos x = 0$ *or* $m \sin x = e^{-1}$ ....(a); and when $x = 2p\pi$ or $2(p+1)\pi$, $y = \frac{\log(m \sin x)}{\operatorname{cosec} x}$ = the limit of

$$\frac{\cot x}{-\operatorname{cosec} x \cot x} = -\sin x = 0.$$

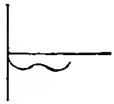
(1) Thus, when $m > 1$, y is negative from $x = 0$ till $m \sin x = 1$, and then $y = 0$ again, and as x increases to $\pi - \sin^{-1}\left(\frac{1}{m}\right)$ y is positive, and $= 0$ when $x = \pi - \sin^{-1}\left(\frac{1}{m}\right)$, and then becomes negative till $x = \pi$. Then there is a break in the curve till either $x = 2\pi$ or $-\pi$, and so on; i.e. the curve consists of an ∞ number of separated equal and similar branches. If m were now negative and of the same numerical value, the breaks would all be filled up, but the curve would not be continuous.



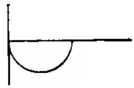
(2) If $m=1$, $m \sin x=1$ gives $x=2p\pi+\frac{\pi}{2}$, and thus there are no positive values of y.



(3) If $m<1$ and $>\frac{1}{e}$, both solutions of (a) are possible, but when $x=\frac{\pi}{2}$, $y=\log m=$ a negative quantity, and each branch is of the type in the adjoining fig.



(4) If $m<\frac{1}{e}$, the 2nd solution of (a) is impossible, and $\therefore$ each branch is parallel to $y=0$ only once, as here.



CHAPTER XXVII.

1. Here $\frac{du}{dv}=\frac{du}{dx}\div\frac{dv}{dx}$;

and
$$\frac{du}{dx}=\frac{1}{\sqrt{(1-x)}}\cdot\frac{1}{2\sqrt{x}}-\frac{1-2x}{2\sqrt{(x-x^2)}}=\frac{\sqrt{x}}{\sqrt{(1-x)}};$$

and
$$\frac{dv}{dx}=-\frac{1}{(1-x^{\frac{2}{3}}a^{\frac{1}{3}})^{\frac{1}{2}}}\cdot\frac{1}{3}x^{-\frac{2}{3}}a^{\frac{1}{6}}-\frac{1}{2}\cdot\frac{1}{(x^{\frac{2}{3}}a^{\frac{1}{3}}-x^{\frac{4}{3}}\cdot a^{\frac{2}{3}})^{\frac{1}{2}}}\left(\frac{2}{3}x^{-\frac{1}{3}}a^{\frac{1}{3}}-\frac{4}{3}x^{\frac{1}{3}}a^{\frac{2}{3}}\right)$$
$$=-\frac{a^{\frac{1}{6}}\cdot x^{-\frac{2}{3}}}{3(1-x^{\frac{2}{3}}a^{\frac{1}{3}})^{\frac{1}{2}}}(1+1-2x^{\frac{2}{3}}a^{\frac{1}{3}})=-\frac{2}{3}a^{\frac{1}{6}}x^{-\frac{2}{3}}(1-x^{\frac{2}{3}}a^{\frac{1}{3}})^{\frac{1}{2}},$$
$$\therefore \frac{du}{dv}=-\frac{3}{2}\frac{x^{(\frac{1}{2}+\frac{2}{3})}}{a^{\frac{1}{6}}}\cdot\frac{1}{\sqrt{1-x}\sqrt{1-x^{\frac{2}{3}}a^{\frac{1}{3}}}}.$$

2. If $y=(\sin x)^{\sin x}$, $\log y=\sin x\log(\sin x)$,
$$\therefore \frac{1}{y}\cdot\frac{dy}{dx}=\cos x\log(\sin x)+\cos x,$$
$\therefore$ for maxima or minima values of y, $\cos x=0$ or $\sin x=\frac{1}{e}$, or $y=0$,

or
$$\frac{dy}{dx}=\infty, \text{ and } \therefore \sin x=0;$$

also when $\frac{dy}{dx}=0,\ \frac{1}{y}\frac{d^2y}{dx^2}=-\sin x\log(\sin x)+\frac{\cos^2 x}{\sin x}-\sin x,$

$\therefore$ when $\sin x=\frac{1}{e},\ \frac{1}{y}\cdot\frac{d^2y}{dx^2}=-\frac{1}{e}\log\left(\frac{1}{e}\right)+e-\frac{2}{e}=\frac{1}{e}+e-\frac{2}{e}$ which is positive,

and $\therefore$ there is a minimum value of $y=\left(\frac{1}{e}\right)^{\frac{1}{e}}$. When $\cos x=0$ and $y=1$, $\frac{d^2y}{dx^2}=-1$, and $\therefore$ y is a maximum (as is otherwise obvious).

When $y=0$, $\log y=-\infty=\sin x.\log(\sin x)$, and the only value of $\sin x$ which might apply would be $\sin x=0$, but the limit of $\sin x\log\sin x$ is 0, thus $y=0$ gives no solution.

When $\sin x=0$, $y=0^0=1$, which is clearly a maximum.

When $\cos x=0$ and $\sin x=-1$, $y=-1$ and $\frac{dy}{dx}$ is imaginary, but $y=-1$ is really a maximum, though not according to the definition in Chap. XIII. If $\sin x$ be negative, the differential coefficients of y are all imaginary, and if the curve $y=(\sin x)^{\sin x}$ be traced, negative values of $\sin x$ will give an ∞ number of conjugate points, for if $-\sin x$ be any proper fraction, in its lowest terms, with an odd number for its denominator, then y is possible. If $-x$ be put for $\sin x$ for possible values of y, $y=\pm\left(\frac{1}{x}\right)^x$ as the numerator of x is even or odd, x being positive, but <1; and if x be very small and give a possible value of y, in the limit $\pm y=(x)^{-x}=e^{-x\log x}=e^0=1$, or $y=\pm 1$. As for positive values of $\sin x, y=\pm(e)^{\frac{1}{e}}$ would correspond to a maximum or minimum if such a value made y possible. Hence it may be conjectured that fractions (in their lowest terms) with an odd denominator will give an algebraically greater or smaller value of y, the nearer they approach $\frac{1}{e}$ in value, as the numerators are even or odd. Thus $\sin x=-\frac{1}{3}$ will give a larger negative value of y than $\sin x=-\frac{1}{5}$; so $(2{\cdot}7)^{\frac{10}{27}}>(2{\cdot}9)^{\frac{10}{29}}$ (by logarithms), &c.

3. The ellipse being $a^2y^2+b^2x^2=a^2b^2$, let one of the base angles be $(x,\ y)$; then the area $=y\,(a-x)$; this is clearly a minimum when $y=0$ (and $\therefore$ $x=\pm a$), and $\therefore$ for a maximum

$$a^2y\,dy+b^2x\,dx=0 \text{ and } (a-x)\,dy-y\,dx=0,$$

$$\therefore\ \frac{a^2y}{a-x}+\frac{b^2x}{y}=0 \text{ or } a^2-x^2+x\,(a-x)=0,$$

$$\text{i.e. } 2x^2-ax-a^2=0=(2x+a)\,(x-a);$$

$$\therefore\ x=-\frac{a}{2} \text{ for the maximum value, and then } \frac{y^2}{b^2}=1-\frac{1}{4},$$

and $\therefore$ the greatest area is $\frac{3ab}{4}\cdot\sqrt{3}$.

4. If PQ be at a distance y from AB, and $AB=2a$, $PQ=2x$, the equation of AQ is $\frac{x'-a}{a+x}=\frac{y'}{-y}$, and at R, by symmetry $x'=0$, $\therefore\ y'=\frac{ay}{x+a}$; hence the $\triangle PQR=x\left\{y-\frac{ay}{x+a}\right\}=\frac{x^2y}{x+a}$, subject to $x^2+y^2=a^2$; $\therefore$ for maximum

$$xdx+ydy=0 \text{ and } \{2x(x+a)-x^2\}\frac{ydx}{(x+a)^2}+\frac{x^2dy}{x+a}=0,$$

or

$$(x+2a)\,ydx+x\,(x+a)\,dy=0,$$

$$\therefore\ (x+2a)\,y^2=x^2(x+a)=(x+2a)\,(a^2-x^2);$$

$\therefore$ either $x=-a$ which gives a minimum, or $x^2+(x-a)(x+2a)=0$.

$$\text{i.e. } 2x^2+ax-2a^2=0, \text{ and } \therefore \frac{4x}{a}=-1\pm\sqrt{17};$$

the negative sign would make $x>a$ numerically, $\therefore$ for the maximum value of PQR, $\frac{2x}{2a}=\frac{PQ}{AB}=\frac{\sqrt{17}-1}{4}$.

5. If a be the radius of the $\frac{1}{2}$ circle, its base axis of x, the vertex of the $\triangle$ at $(0, a)$ and its base angles at $(\pm x, y)$, the area of the figure is

$$2xy+x\,(a-y)=x\,(y+a),$$

subject to $x^2+y^2=a^2$, $\therefore$ for the maximum value,

$$xdx+ydy=0, \text{ and } (y+a)\,dx+xdy=0,$$

$$\therefore\ x^2=y\,(y+a)=a^2-y^2;$$

but y cannot be negative here, $\therefore\ y=\frac{a}{2}$, i.e. &c. For minimum $x=0$.

6. If $2\theta=$ vertical angle of the cone, r the radius of the base, and a that of the sphere, the slant surface of the cone $=\pi r^2\operatorname{cosec}\theta$; and from a figure of a principal section of the sphere and cone it will be seen that

$$r\cot\theta=a\,(1+\operatorname{cosec}\theta) \text{ or } r=a\,(\tan\theta+\sec\theta);$$

$\therefore$ for a maximum surface

$$2r\operatorname{cosec}\theta\,dr=r^2\operatorname{cosec}\theta\cot\theta\,d\theta, \text{ and } dr=a\,(\sec^2\theta+\sec\theta\tan\theta)\,d\theta,$$

and,

$$\therefore\ \frac{r}{2}\cot\theta=a\sec\theta\,(\sec\theta+\tan\theta)=\frac{a}{2}(1+\operatorname{cosec}\theta);$$

$$\therefore\ 2\sin\theta\,(1+\sin\theta)=(1+\sin\theta)\cos^2\theta,$$

or

$$\sin^2\theta+2\sin\theta-1=0, \text{ and } \therefore \sin\theta=-1+\sqrt{2},$$

the negative values being inapplicable.

7. Here, from Ex. 6, the surface $=\pi r^2(1+\operatorname{cosec}\theta)$ subject to

$$r\cot\theta=a\,(1+\operatorname{cosec}\theta), \ \therefore \text{ the surface}=\frac{\pi r^3\cot\theta}{a};$$

hence for maximum

$$3r^2 \cot\theta\, dr = r^3 \operatorname{cosec}^2\theta\, d\theta \text{ and } dr = a\,d\theta\,(\sec^2\theta + \sec\theta\tan\theta),$$

and $$\therefore \frac{r\operatorname{cosec}^2\theta}{3\cot\theta} = a\sec\theta\,(\sec\theta+\tan\theta) = \frac{a}{3}\,\frac{(1+\operatorname{cosec}\theta)\operatorname{cosec}^2\theta}{\cot^2\theta},$$

or $$3\,(1+\sin\theta) = 1+\operatorname{cosec}\theta \text{ and } \therefore \sin\theta = \frac{1}{3}.$$

8. If $u = \cos\theta\,.\,\cos\phi\cos\psi$,

$$\tan\theta\,.\,d\theta + \tan\phi\,d\phi + \tan\psi\,d\psi = 0 = d\theta + d\phi + d\psi;$$

$$\therefore \tan\theta = \tan\phi = \tan\psi, \text{ and } \tan\theta + \tan(\phi+\psi) = 0;$$

$\therefore \tan\theta\,(1-\tan^2\theta) + 2\tan\theta = 0$, $\therefore \tan\theta = 0$, which would make

$$\tan\phi = 0 = \tan\psi, \text{ and } \therefore u = -1;$$

or else $\tan^2\theta = 3$ and $\theta = \phi = \psi = \dfrac{\pi}{3}$ and then $u = \dfrac{1}{8}$, clearly $\therefore$ a maximum.

9. Here $\dfrac{dx'}{dx} = l_1$, $\dfrac{dy'}{dx} = l_2$, $\therefore \dfrac{du}{dx} = l_1\dfrac{du}{dx'} + l_2\dfrac{du}{dy'}$;

$$\therefore \frac{d^2u}{dx^2} = \left(l_1\frac{d}{dx'} + l_2\frac{d}{dy'}\right)^2 u = l_1^2\frac{d^2u}{dx'^2} + 2l_1l_2\frac{d^2u}{dx'dy'} + l_2^2\frac{d^2u}{dy'^2} \text{ (cf. Art. 226);}$$

so $$\frac{d^2u}{dy^2} = m_1^2\frac{d^2u}{dx'^2} + 2m_1m_2\frac{d^2u}{dx'dy'} + m_2^2\frac{d^2u}{dy'^2},$$

and $\therefore$ the result follows.

10. If it be necessary to differentiate m times, then as in Art. 250, there will be obtained $\frac{1}{2}(m+1)(m+2)$ equations and there will be $(m+1)\,n$ arbitrary functions and the $4n^2-n-1$ arbitrary constants to be eliminated; $\therefore \frac{1}{2}(m+1)(m+2) = (m+1)\,n + 4n^2 - n - 1 + 1$ at least to give an independent eqnation, i.e. the least value of m is given by

$$m^2+3m+2 = 2mn+2n+8n^2-2n \text{ or } m^2 - m\,(2n-3) + 2 - 8n^2 = 0,$$

$$\therefore 2m - (2n-3) = \pm\{(2n-3)^2 - 8 + 32n^2\}^{\frac{1}{2}} = \pm(36n^2-12n+1)^{\frac{1}{2}} = \pm(6n-1),$$

$\therefore 2m = 8n-4$ or $-4n-2$, and the latter value being inapplicable,

$$m = 4n-2.$$

11. $$\frac{dV'}{du} = \frac{dV}{dx}\cdot\frac{dx}{du} + \frac{dV}{dy}\cdot\frac{dy}{du} + \frac{dV}{dz}\cdot\frac{dz}{du} = \frac{dV}{dy}\cdot w + \frac{dV}{dz}\cdot v;$$

$$\therefore \frac{d^2V'}{du^2} = \left(w\frac{d}{dy} + v\frac{d}{dz}\right)^2 . V,$$

$$\therefore u^2 . \frac{d^2V'}{du^2} = y^2\frac{d^2V}{dy^2} + 2yz\frac{d^2V}{dydz} + z^2\frac{d^2V}{dz^2},$$

&c., and the result follows.

12. If $zx = -u$, $y = 2\left(1 + \frac{u^2}{\lfloor 2} + \frac{u^4}{\lfloor 4} + \ldots + \frac{u^{2p}}{\lfloor 2p} + \ldots\right)$, and $u = x^2 \cdot e^{2u}$: now to expand u^{2p} in powers of x^2 by Lagrange's Theorem, $u = z + x^2 \cdot e^{2u}$, z being put $=0$ after differentiation, $\phi(u) = e^{2u}$, and $f(u) = u^{2p}$, $\therefore$ the term in the expansion of u^{2p} involving x^{2n} is $\frac{x^{2n}}{\lfloor n} \cdot \frac{d^{n-1}}{dz^{n-1}} \{e^{2nz} \cdot 2pz^{2p-1}\}$, and $\therefore$ the term in the expansion of y which involves x^{2n} is $\frac{Ax^{2n}}{\lfloor n}$, where

$$A = 2\Sigma \left\{ \frac{1}{\lfloor 2p} \cdot \frac{d^{n-1}}{dz^{n-1}} (e^{2nz} \cdot 2pz^{2p-1}) \right\},$$

from $\quad p = 1$ to ∞, $= \frac{d^{n-1}}{dz^{n-1}} \left\{ e^{2nz} \cdot \Sigma \frac{2z^{2p-1}}{\lfloor 2p-1} \right\}$

$$= \frac{d^{n-1}}{dz^{n-1}} \{e^{2nz}(e^z - e^{-z})\} = \frac{d^{n-1}}{dz^{n-1}} \{e^{(2n+1)z} - e^{(2n-1)z}\} = (2n+1)^{n-1} - (2n-1)^{n-1},$$

when $z = 0$.

13. If $\beta = \kappa a$ and so on, $y = x + a\{\psi(y) + \kappa\phi(y) + \ldots\}$ where κ is a constant, and thus by Lagrange's Theorem,

$$F(y) = F(x) + \ldots + \frac{a^n}{\lfloor n} \frac{d^{n-1}}{dx^{n-1}} [F'(x)\{\psi(y) + \kappa\phi(y) + \ldots\}^n] + \ldots$$

$$= F(x) + \ldots + \frac{1}{\lfloor n} \frac{d^{n-1}}{dx^{n-1}} [F'(x)\{a\psi(y) + \beta\phi(y) + \ldots\}^n] + \ldots,$$

a being a constant.

14. By Lagrange's Theorem, the general term of $f(y, y')$ in powers of x is $\frac{x^m}{\lfloor m} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ \overline{\phi(z)}^m \frac{df(z, y')}{dz} \right\}$, and $\therefore$ the general term in powers and products of x and x' is $\frac{x^m}{\lfloor m} \cdot \frac{x'^n}{\lfloor n} \frac{d^{m+n-2}}{dz^{m-1}\, dz'^{n-1}} \left\{ \phi(z)^m \cdot \psi(z')^n \frac{d^2 f(z, z')}{dz dz'} \right\}$.

If $m = 2$, $n = 1$, $f(y, y') = \cos(ay + a'y')$, $\phi(y) = \sin y$, and

$$\psi(y') = \sin y';$$

then the coefficient of $x^2 \cdot x'$ is $\frac{1}{2} \frac{d}{dz} \left\{ \sin^2 z \cdot \sin z' \cdot \frac{d^2}{dz dz'} \cos(az + a'z') \right\}$

$$= \frac{-\sin z'}{2} \cdot \frac{d}{dz} \{aa' \cdot \cos(az + a'z') \sin^2 z\}$$

$$= \frac{aa'}{2} \cdot \sin z \cdot \sin z' \{a \cdot \sin(az + a'z') \sin z - 2\cos z \cdot \cos(az + a'z')\}.$$

15. It equals $r \cos\phi$ where $\cos\phi = \frac{dr}{ds}$, $\therefore$ &c.

16. The equation may be written (if the co-ordinates be rectangular)

$$(y'-y)\frac{dy}{dx}+x'-x=0, \text{ and } \frac{dx}{ds}\cdot\frac{d^2x}{ds^2}+\frac{dy}{ds}\cdot\frac{d^2y}{ds^2}=0,$$

and $$\frac{dy}{dx}=\frac{dy}{ds}\div\frac{dx}{ds}, \therefore \frac{x'-x}{\frac{d^2x}{ds^2}}=\frac{y'-y}{\frac{d^2y}{ds^2}}.$$

If $PQ=\delta s$, T is $\left(x+\delta s\cdot\frac{dx}{ds},\ y+\delta s\cdot\frac{dy}{ds}\right)$, P being (x, y), and Q is

$$\left(x+\delta s\cdot\frac{dx}{ds}+\frac{\delta s^2}{2}\cdot\frac{d^2x}{ds^2}+\ldots,\ y+\delta s\cdot\frac{dy}{ds}+\frac{ds^2}{2}\cdot\frac{d^2y}{ds^2}+\ldots\right),$$

by Maclaurin's Theorem, $\therefore$ the equation of QT is approximately

$$\frac{\left(x'-x-\delta s\cdot\frac{dx}{ds}\right)}{\frac{\delta s^2}{2}\cdot\frac{d^2x}{ds^2}}=\frac{y'-y-\delta s\cdot\frac{dy}{ds}}{\frac{ds^2}{2}\cdot\frac{d^2y}{ds^2}};$$

$\therefore$ ultimately QT is $\frac{x'-x}{\frac{d^2x}{ds^2}}=\frac{y'-y}{\frac{d^2y}{ds^2}}$, and $\therefore$ QT is ultimately perpendicular to PT, as may be easily proved geometrically.

17. The axes of the ellipse being axes of co-ordinates, and ϕ_1, ϕ_2, ϕ_3 the three angles, as in Ex. 6, Chap. XX.,

$$\tan\phi_1=\frac{l}{re\sin\theta}=\frac{l}{ey}=\frac{b^2}{aey};$$

and $$\tan\phi_2=\frac{-2a^2b^2}{r^2(a^2-b^2)\sin 2\theta}=\frac{-a^2b^2}{xy\cdot a^2\cdot e^2}; \text{ and } \tan\phi_3=-\frac{b^2x}{a^2y};$$

$$\therefore \tan\phi_2\cdot\tan\phi_3=\frac{b^4}{a^2y^2e^2}=\tan^2\phi_1.$$

18. Here $\frac{dy}{dx}=\frac{\sin\psi}{\sin(\alpha-\psi)}$, $\therefore \tan\psi=\sin\alpha\cdot\frac{dy}{dx}\div\left(\cos\alpha\,\frac{dy}{dx}+1\right)$, and as in Art. 307, $\frac{ds}{dx}=\sqrt{\left\{1+2\cos\alpha\cdot\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2\right\}}$.

$$\text{Thus } \sec^2\psi\cdot\frac{d\psi}{dx}=\frac{\sin\alpha\cdot\frac{d^2y}{dx^2}\left(\cos\alpha\,\frac{dy}{dx}+1\right)-\sin\alpha\cdot\frac{dy}{dx}\cdot\cos\alpha\,\frac{d^2y}{dx^2}}{\left(1+\cos\alpha\cdot\frac{dy}{dx}\right)^2};$$

and $$\therefore \frac{d\psi}{dx}=\sin\alpha\cdot\frac{d^2y}{dx^2}\div\left\{\left(\sin\alpha\cdot\frac{dy}{dx}\right)^2+\left(\cos\alpha\,\frac{dy}{dx}+1\right)^2\right\}$$

$$=\sin\alpha\cdot\frac{d^2y}{dx^2}\div\left\{1+2\cos\alpha\cdot\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2\right\},$$

and $$\therefore\ \rho=\frac{ds}{dx}\div\frac{d\psi}{dx}=\frac{\left\{1+2\cos\alpha\,\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\sin\alpha\,\frac{d^2y}{dx^2}}.$$

The sign of this is positive or negative, with $\sin\alpha\,.\,\frac{d^2y}{dx^2}$, to make ρ positive.

19. Here $\frac{1}{\sqrt{ax}}+\frac{1}{\sqrt{by}}\,.\,\frac{dy}{dx}=0$,

$$\frac{a}{2\,(ax)^{\frac{3}{2}}}+\frac{b}{2\,(by)^{\frac{3}{2}}}\,.\left(\frac{dy}{dx}\right)^2-\frac{1}{\sqrt{by}}\,.\,\frac{d^2y}{dx^2}=0;$$

$$\therefore\ \frac{dy}{dx}=-\sqrt{\frac{by}{ax}},$$

and $$\frac{d^2y}{dx^2}=\frac{1}{2}\sqrt{by}\left\{\frac{a}{(ax)^{\frac{3}{2}}}+\frac{b}{ax\sqrt{by}}\right\}=\frac{1}{2\,(ax)^{\frac{3}{2}}}\,(a\sqrt{by}+b\sqrt{ax}),$$

and $\therefore$ by Ex. 18, $$\rho=\left(1-2\cos\alpha\,.\sqrt{\frac{by}{ax}}+\frac{by}{ax}\right)^{\frac{3}{2}}\,.\,\frac{2\,(ax)^{\frac{3}{2}}}{(a\sqrt{by}+b\sqrt{ax})\sin\alpha}$$

$$=2\,(ax-2\cos\alpha\sqrt{abxy}+by)^{\frac{3}{2}}\,.\,\frac{1}{ab\sin\alpha}.$$

For the vertex $ax-2\cos\alpha\sqrt{abxy}+by$ is a minimum subject to

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1,\ \text{and}\ \therefore\ \frac{dx}{\sqrt{ax}}+\frac{dy}{\sqrt{by}}=0,$$

and $$dx\left(a-\cos\alpha\sqrt{\frac{aby}{x}}\right)+dy\left(b-\cos\alpha\sqrt{\frac{abx}{y}}\right)=0,$$

$$\therefore\ \sqrt{ax}\left(a-\cos\alpha\sqrt{\frac{aby}{x}}\right)=\sqrt{by}\left(b-\cos\alpha\sqrt{\frac{abx}{y}}\right),$$

or $$a\,(\sqrt{ax}-\cos\alpha\,.\sqrt{by})=b\,(\sqrt{by}-\cos\alpha\sqrt{ax});$$

$$\therefore\ \frac{\sqrt{ax}}{a\cos\alpha+b}=\frac{\sqrt{by}}{a+b\cos\alpha}=\frac{1}{\cos\alpha+\frac{b}{a}+\frac{a}{b}+\cos\alpha}=\frac{ab}{a^2+2ab\cos\alpha+b^2}.$$

These values of x and y are finite, a, b being so, and $\therefore$ apply to the vertex.

20. In the first case the circle of curvature of the curve at the origin touches the axis of y, and if it meet the curve in another adjacent point the circle is determinate, and then (x, y) being such adjacent point

$$x\,(2\rho-x)=y^2,$$

and $\therefore$ ultimately when x and y vanish, $2\rho=$ limit of $\frac{y^2}{x}$. Similarly in the 2nd case.

21. If the axes be turned through an angle a so that the axis of x' coincides with the tangent, $2\rho=$ limit of $\frac{x'^2}{y'}+y'$ where

$$x'=x\cos a+y\sin a,\ y'=y\cos a-x\sin a,$$

and $\therefore$ $2\rho=$ limit of $\quad \frac{x^2+y^2}{y'}=\frac{x^2+y^2}{y\cos a-x\sin a}$.

In the curve $y^2+2ay-2ax=0$, the tangent at the origin is $y=x$;

$$\therefore\ a=\frac{\pi}{4} \text{ and } \pm\rho=\text{limit of } \frac{\frac{1}{2}\cdot 2y^2}{(y-x)\frac{1}{\sqrt{2}}}=\frac{y^2\sqrt{2}}{-\frac{y^2}{2a}}=-2a\sqrt{2},\ \therefore\ \&c.$$

22. If $p=r\sin\phi$, and $\tan\phi=r\frac{d\theta}{dr}$, $\frac{\rho}{r}=\frac{dr}{dp}=\frac{dr}{d\theta}\div\frac{dp}{d\theta}$

$$=r\cot\phi\div\left(\sin\phi\frac{dr}{d\theta}+r\cos\phi\cdot\frac{d\phi}{d\theta}\right)$$

$$=r\cot\phi\div r\cos\phi\left(1+\frac{d\phi}{d\theta}\right),\ \therefore\ \rho=r\operatorname{cosec}\phi\div\left(1+\frac{d\phi}{d\theta}\right).$$

23. Here $\frac{dx}{d\theta}=-2a(\sin\theta-\sin 2\theta)$, $\frac{dy}{d\theta}=2a(\cos\theta-\cos 2\theta)$;

$$\therefore\ \frac{dy}{dx}=\tan\psi=\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\div\cos\frac{3\theta}{2}\sin\frac{\theta}{2}=\tan\frac{3\theta}{2},$$

$$\therefore\ \psi=\frac{3\theta}{2};\ \text{and}\ \left(\frac{ds}{d\theta}\right)=4a\sin\frac{\theta}{2},\ \therefore\ \rho=\frac{ds}{d\psi}=\frac{8}{3}a\sin\frac{\theta}{2}.$$

For the evolute, if (x', y') be the corresponding point,

$$x'=x-\rho\sin\psi=a\left(2\cos\theta-\cos 2\theta-\frac{8}{3}a\sin\frac{\theta}{2}\cdot\sin\frac{3\theta}{2}\right)$$

$$=a\left(\frac{2}{3}\cos\theta+\frac{1}{3}\cos 2\theta\right),$$

and

$$y'=y+\rho\cos\psi=a\left(2\sin\theta-\sin 2\theta+\frac{8}{3}a\sin\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

$$=a\left(\frac{2}{3}\sin\theta+\frac{1}{3}\sin 2\theta\right),$$

and if for θ there be written $\pi+\theta$, and the signs of x', y' be changed, comparing with Art. 360, this represents an epicycloid in which the radius of each circle $=\frac{a}{3}$.

24. Here $\frac{d\phi}{dy}$ cannot vanish, being unity, $\therefore$ the curve has no multiple points. For points of inflexion,

$$\frac{dy}{dx}=4x^3-12x^2-36x,\ \frac{d^2y}{dx^2}=12x^2-24x-36=0;$$

$\therefore\ (x-3)(x+1)=0$, and then $\frac{d^3y}{dx^3}=24(x-1)$ which is not zero when $x=3$ or -1, which values of x, $\therefore$, correspond to points of inflexion. When

$$x=\frac{3}{2}(1\pm\sqrt{5}),\ \left(x-\frac{3}{2}\right)^2=5\,.\left(\frac{3}{2}\right)^2 \text{ or } x^2-3x-9=0, \text{ and } \therefore\ \frac{dy}{dx}=0,$$

and y is then a minimum, and the curve parallel to the axis of x.

25. Here $\frac{dy}{dx}=-e^{-x^2}.\,2x$, $\frac{d^2y}{dx^2}=-e^{-x^2}(2-4x^2)$,

and $$\frac{d^3y}{dx^3}=e^{-x^2}(8x-2x\,.\,\overline{4x^2-2}), \text{ and } \therefore \text{ when } x=\pm\frac{1}{\sqrt{2}},$$

$$\frac{d^2y}{dx^2}=0, \text{ and } \frac{d^3y}{dx^3} \text{ is finite, } \therefore \text{ \&c.}$$

26. At a point of inflexion the tangent is stationary, $\therefore$ if ψ be the angle it makes with the prime radius $\frac{d\psi}{d\theta}=0$; but $\psi=\phi+\theta$, $\therefore$ &c.

27. $\text{Tan}\,\phi=r\frac{d\theta}{dr}$, $\therefore$ as in Art. 298, if the equation of the curve be $f(r,\theta)=0$, $\frac{dr}{d\theta}$ must be of the type $\frac{0}{0}$ at a multiple point, *if* $f(r,\theta)$ be rational: but if the equation be irrational the argument no longer holds, as the different signs of radicals give rise to more than one value of $\frac{dr}{d\theta}$ for particular values of r and θ corresponding to $f(r,\theta)=0$, and $\therefore$ $\frac{dr}{d\theta}$ is *not* necessarily of the form $\frac{0}{0}$.

28. (1). Rationalising the equation, $\{4(x^2+y^2)-1\}^3=27y^2$, $\therefore$ for multiple points, $x\,.\,\{4(x^2+y^2)-1\}^2=0$ and $24\{4(x^2+y^2)-1\}^2y=54y$: if $4(x^2+y^2)=1$, then $y=0$ and $\therefore\ x=\pm\frac{1}{2}$, which values agree with the equation to the curve, which is $4\left(x^2-\frac{1}{4}\right)=3y^{\frac{2}{3}}-4y^2=3y^{\frac{2}{3}}$ nearly, thus $x^2>\frac{1}{4}$ and the values of y are equal and opposite, so that there is a cusp of the first kind at each of the points $\left(\pm\frac{1}{2},\,0\right)$, $y=0$ being the tangent. But if $x=0$ either $y=0$, which does not satisfy the equation to the curve, or

$$24(4y^2-1)^2=54, \text{ i.e. } y=\pm\frac{1}{2}\left\{1\pm\frac{3}{2}\right\}=\pm\frac{5}{4} \text{ or } \pm\frac{1}{4},$$

which do not agree with the equation to the curve from which

$$y^2-1+3(y^2-y^{\frac{2}{3}})=0, \text{ and } \therefore\ y=\pm 1,$$

or is imaginary, when $x=0$.

For points of inflexion $8x\frac{dx}{dy}+8y-2y^{-\frac{1}{3}}=0$, and

$$8\left(\frac{dx}{dy}\right)^2+8+\frac{2}{3}y^{-\frac{4}{3}}=0,\ \therefore\ 8+\frac{2}{3}y^{-\frac{4}{3}}+(8y-2y^{-\frac{1}{3}})^2\cdot\frac{1}{8x^2}=0;$$

$$\therefore\ (8y^2-2-6y^{\frac{2}{3}})\left(8+\frac{2}{3}y^{-\frac{4}{3}}\right)=(8y-2y^{-\frac{1}{3}})^2,$$

whence $(2y^{\frac{2}{3}}+1)^3=0$, and $\therefore$ there are no real points of inflexion. This is an epicycloid, cf. p. 391, Ex. 21.

(2) When $y^2-2xy+2x^2-x^3=0$, for multiple points $2(y-x)=0$ and $4x-3x^2=2y$, $\therefore\ 3x^2=2x$, i. e. $x=0$ or $\frac{2}{3}$; but the given equation may be written $(y-x)^2+x^2(1-x)=0$, $\therefore\ \left(\frac{2}{3},\ \frac{2}{3}\right)$ is not on the curve, but the origin is, while if $y-x$ be not zero, x must be >1, $\therefore$ the origin is a conjugate point.

For points of inflexion $y=x\pm x\sqrt{x-1}$,

$$\therefore\ \frac{dy}{dx}=1\pm\frac{\left(\overline{x-1}+\frac{1}{2}x\right)}{\sqrt{x-1}},\ \therefore\ 0=\frac{3}{2}(x-1)-\frac{1}{2}\left(\frac{3}{2}x-1\right);$$

$\therefore\ x=\frac{4}{3}$ and y is then possible and $=\frac{4}{3}\pm\frac{4}{3\sqrt{3}}$, $\therefore$ there are 2 real points of inflexion, as $\frac{d^3y}{dx^3}$ clearly vanishes for a different value of x.

29. When $x=2$, $y=-1$, and changing the origin to $(2,-1)$ the equation becomes $y=(x+2)(-x)\pm(-x)^{\frac{5}{2}}$, $\therefore$ x is negative and the tangent at the new origin is $y+2x=0$, and $y+2x=-x^2$ nearly on the curve, $\therefore$ both branches lie on the same side of the tangent, i. e. there is a cusp of the 2nd kind, y on the curve being $<-2x$.

30. $\frac{d^2y}{dx^2}=6x-18$, $\frac{d^3y}{dx^3}=6$, $\therefore\ x=3$ and $y=2$, for the point of inflexion.

31. The angle of the spiral is α, and (r,θ) being a point on it, and (r',θ') any point on the corresponding parallel line, the perpendicular on the latter from the pole $=r\sin\alpha\cos\alpha=r'\sin(\theta-\theta')$,

$$\therefore\ r'\sin(\theta-\theta')=\sin\alpha\cos\alpha\,.\,Aa^{\theta},$$

and $\therefore$ for the envelope $r'\cos(\theta-\theta')=\sin\alpha\cos\alpha\,.\,Aa^{\theta}\,.\cot\alpha$, whence

$$\tan(\theta-\theta')=\tan\alpha,\ \text{and}\ \therefore\ \theta-\theta'=\alpha,$$

and $\therefore\ r'\sin\alpha=\sin\alpha\cos\alpha\,.\,Aa^{\theta'+\alpha}$, or the envelope is the curve

$$r=A\cos\alpha\,.\,a^{\theta+\alpha},$$

a similar equiangular spiral.

32. If P, P' be 2 points on the spiral, O the pole and Q the *other* point of intersection of the circles on the diameters OP, OP', it will be seen from a figure that, the angles OQP, OQP' being right angles, Q is the foot of the perpendicular from O on PP', $\therefore$ the locus in question is the locus of the foot of the perpendicular from the pole on the tangent. Thus when P, P' coincide, if P, Q be (r, θ), (r', θ'), and α be the angle of the spiral, and

$$r = Aa^{\theta}, \quad r' = r \sin \alpha, \text{ and } \theta' = \theta - \left(\frac{\pi}{2} - \alpha\right),$$

and $\therefore$ $r' = \sin \alpha \, . \, A \, . \, a^{\theta' + \frac{\pi}{2} - \alpha} = Aa^{\frac{\pi}{2} - \alpha} \, . \sin \alpha \, . \, a^{\theta'}$, a similar spiral.

CHAPTER XXVIII.

1. If the radius $= a$, and the straight line be axis of x, and the origin the point of contact when the base is $y = a$, then when the semi-circle has turned through an angle θ, the centre is at $(a\theta, a)$ and the equation of the base is $y - a + (x - a\theta) \tan \theta = 0$; $\therefore$ for its envelope

$$a \tan \theta = (x - a\theta) \sec^2 \theta, \text{ i.e. } x = a\theta + \frac{a}{2} \sin 2\theta = \frac{a}{2}(2\theta + \sin 2\theta),$$

and $y = a - a \sin^2 \theta = \frac{a}{2}(1 + \cos 2\theta)$, a cycloid (Art. 358), the vertex being at $(0, a)$ and the radius of the generating circle $= \frac{a}{2}$.

2. With the figure and notation of Art. 358, if the cycloid roll along $x = 0$, and has turned through an angle ϕ when P is in contact with $x = 0$, MD, which is parallel to the normal at P, is then perpendicular to $x = 0$, and $\phi =$ the angle through which AD has turned $= \frac{\theta}{2}$, $\therefore$ the equation to the base is $y - k + (x - h) \cot \frac{\theta}{2} = 0$, (h, k) being D.

Now AM is in this case parallel to the axis of y, and thus

$$h = DM + PM \sin \frac{\theta}{2} = 2a \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \left\{ a\theta + a \sin \theta - 2a \cos \frac{\theta}{2} \, . \, \sin \frac{\theta}{2} \right\}$$

$$= 2a \cos \frac{\theta}{2} + a\theta \sin \frac{\theta}{2};$$

and $k = \text{arc } AP - PM \cos \frac{\theta}{2} = 2AM - PM \cos \frac{\theta}{2} = 4a \sin \frac{\theta}{2} - a\theta \cos \frac{\theta}{2}$,

and $\therefore$ the equation to the base is

$$y \sin \phi + x \cos \phi = 4a \sin^2 \phi - 2a\phi \sin \phi \cos \phi + 2a \cos^2 \phi + 2a\phi \sin \phi \cos \phi$$
$$= 2a(1 + \sin^2 \phi);$$

$\therefore$ for the envelope $y \cos \phi - x \sin \phi = 4a \sin \phi \, . \cos \phi$, and eliminating x and y in succession, $\frac{y}{2a} = \sin \phi + \sin^3 \phi + 2 \sin \phi \cos^2 \phi = \sin \phi (2 + \cos^2 \phi)$, and $$\frac{x}{2a} = \cos\phi + \cos\phi \sin^2\phi - 2\sin^2\phi\cos\phi = \cos^3\phi,$$

and $$\therefore \frac{y}{2a} = \left\{1 - \left(\frac{x}{2a}\right)^{\frac{2}{3}}\right\}^{\frac{1}{2}} \cdot \left\{2 + \left(\frac{x}{2a}\right)^{\frac{2}{3}}\right\},$$

which is the same curve as that given in the text, the axes being interchanged.

3. If P, P' be 2 adjacent points on the hyperbola $x^2 - y^2 = a^2$, C being its centre, the radical axis of the 2 corresponding circles is the perpendicular from C on PP', and the 2 circles ultimately intersect in a *second* point on the perpendicular CY on the tangent at P, but twice as far from C as Y. Now if P be (x', y') the tangent at P is $xx' - yy' = a^2$, and CY is $\frac{x'}{x} + \frac{y'}{y} = 0$; hence at Y, $\frac{x'}{x} = -\frac{y'}{y} = \frac{a}{\sqrt{x^2 - y^2}} = \frac{a^2}{x^2 + y^2}$, and $\therefore$ the locus of Y is

$$(x^2 + y^2)^2 = a^2(x^2 - y^2),$$

a lemniscate, and the locus required is the similar lemniscate

$$(x^2 + y^2)^2 = 4a^2(x^2 - y^2).$$

4. (1) The lowest terms give $x(y^2 - x^2) = 0$, $\therefore$ there are 3 branches, touching $x = 0$ and $x = \pm y$. For the first of these, x being smaller than y, the equation is approximately $y^4 + 2ay^2x = 0$, i. e. $y^2 + 2ax = 0$, and this branch is $\therefore$ approximately parabolic to the left of $x = 0$. For the other 2 branches

$$y^2 = -ax \pm x\sqrt{(a^2 + 2ax - x^2)}$$

(the lower sign corresponding to the branch above given),

$$\therefore y^2 = -ax + ax\left\{1 + \frac{2ax - x^2}{2 \cdot a^2} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\left(\frac{2ax - x^2}{a^2}\right)^2\right\}$$

$$= ax\left\{\frac{x}{a} - \frac{x^2}{2a^2} - \frac{1}{2}\cdot\frac{x^2}{a^2}\right\} \text{ approximately,}$$

$\therefore y^2 \lessgtr x^2$, as x is positive or negative (supposing a positive), and $\therefore$ the branch touching $y = x$ lies on the same side of $y = x$ as the positive part of $y = 0$, and similarly for the branch touching $x + y = 0$; the results being reversed if a be negative, so that in neither case is there any inflexion near the origin.

(2) The lowest terms in x and y give

$$y^4 + x^4 + 3x^2y^2 = 0, \text{ i. e. } (y^2 + x^2)^2 + x^2y^2 = 0,$$

which give impossible values, except at the origin, which is $\therefore$ a conjugate point.

(3) The tangents at the origin are given by $xy(ay - bx) = 0$, $\therefore$ there are 3 branches. For that touching $y = 0$, when y is much smaller than x the equation to the curve gives $x^4 = 4bx^2y$ or $x^2 = 4by$, which is parabolic.

Similarly for the branch touching $x=0$. For the remaining branch, if $y=\frac{bx}{a}+z$ where z is small compared with either x or y,

$$\left(\frac{bx}{a}\right)^4+4\left(\frac{bx}{a}\right)^3 z-4azx\left(\frac{bx}{a}+z\right)-x^4=0 \text{ nearly, or}$$

$$\left\{\left(\frac{b}{a}\right)^4-1\right\}x^4=4bzx^2, \text{ and } \therefore\ x^2\left\{\left(\frac{b}{a}\right)^4-1\right\}=4bz \text{ nearly;}$$

$\therefore$ if b be positive and $>a$, z is positive and the curve lies on the same side of $ay=bx$ as the positive part of $x=0$; and so on for other cases: thus there is no inflexion at the origin.

(4) The tangents are $x^2y=0$, $\therefore$ there are a double branch touching $x=0$, and a single branch touching $y=0$. For the latter, y being much smaller than x, near the origin $x^5=2a^2x^2y$ or $x^3=2a^2y$, $\therefore$ x changes sign with y, and $\therefore$ there is a point of inflexion at the origin on this branch. For the double branch near the origin $y^4=2a^2x^2$, $\therefore$ $y^2=\pm ax\sqrt{2}$, i.e. 2 parabolas.

5. By Art. 277, $x=0$ would be an asymptote if $x=0$ and $y=\infty$ satisfied the equation to the curve, but they do not: for real values x^2 must lie between a^2 and b^2, and y being always finite, or zero (where $x^2=a^2$ or b^2), there are 2 loops, symmetrical with respect to the axes, their breadth in direction of the axis of x being $a\sim b$: for their breadth parallel to $x=0$, y^2 is to be a maximum, and taking a, b as positive,

$$y^2=a^2+b^2-x^2-\frac{a^2b^2}{x^2}=(a-b)^2-\left(x-\frac{ab}{x}\right)^2;$$

$\therefore$ the greatest value of y is $a\sim b$, and the corresponding breadth is

$$2(a\sim b),\ \therefore \text{ \&c.}$$

Changing the origin to $(\pm a,\ 0)$, when $b=a$, the equation is

$$(x\pm a)^2y^2+(x^2\pm 2ax)^2=0,$$

or approximately $a^2y^2+4a^2x^2=0$, $\because$ x is small, and this is elliptical in form.

6. The curve is clearly symmetrical as to both axes, and $\therefore$ it is sufficient to consider x and y as both positive, and then $y=\{b^2\pm x\sqrt{2a^2-x^2}\}^{\frac{1}{2}}$.

(a) With the upper sign, x may have any value from 0 to $a\sqrt{2}$, but no greater value for real values of y; for each of these limiting values of x, $y=b$, and for intermediate values of x, $y>b$, and there is only one positive value of y (x being positive). Thus there is a curved line connecting $(0,\ b)$ and $(a\sqrt{2},\ b)$ concave to the axis of x.

(β) With the lower sign of the inner radical, x must not only lie between 0 and $a\sqrt{2}$, but also b^2 be $>x\sqrt{2a^2-x^2}$, i.e. $x^4-2a^2x^2+a^4+b^4-a^4>0$.

(I) If $b^2>a^2$ the 2nd condition is at once fulfilled, and there is a corresponding curve line joining $(0,\ b)$ and $(a\sqrt{2},\ b)$ convex to the axis of x.

Also generally at $(0, b)$, $\frac{dy}{dx} = \pm\frac{1}{2} \cdot \frac{a\sqrt{2}}{b} = \pm\frac{a}{b\sqrt{2}}$, and thus the equation, when $b^2 > a^2$, represents 2 curves resembling figures of eight with $y = \pm b$ for their axes. Changing the origin to $(a\sqrt{2}, b)$, the equation becomes

$$\{(x + a\sqrt{2})^2 - a^2\}^2 + (y^2 + 2by)^2 = a^4,$$

or

$$(x^2 + 2ax\sqrt{2} + a^2)^2 + (y^2 + 2by)^2 = a^4,$$

$\therefore$ $x = 0$ is the tangent at the new origin.

(2) If $a^2 > b^2$, let $a^4 = b^4 + c^4$, so that $c^2 < a^2$. Then the 2nd condition becomes $(x^2 - a^2)^2 > c^4$, or $x^2 - a^2 - c^2$ and $x^2 - a^2 + c^2$ are of the same sign, $\therefore$ either $x^2 > a^2 + c^2$ or $< a^2 - c^2$, and $\therefore$ from the two conditions x^2 either lies between 0 and $a^2 - c^2$, or between $a^2 + c^2$ and $2a^2$ for real values of y. Now for (β), when $x^2 = a^2 \pm c^2$, $y^2 = b^2 - \sqrt{a^4 - c^4} = b^2 - b^2 = 0$, and changing the origin to $(\sqrt{a^2 \pm c^2}, 0)$, the equation becomes

$$\{(x + \sqrt{a^2 \pm c^2})^2 - a^2\}^2 + (y^2 - b^2)^2 = a^4,$$

or

$$\{x^2 + 2x\sqrt{a^2 \pm c^2} \pm c^2\}^2 + (y^2 - b^2)^2 = a^4,$$

and $\therefore$ $x = 0$ is the corresponding tangent.

Thus on the whole, when $a^2 > b^2$, the equation will be found, on drawing a figure, to represent 2 figures, each resembling a heart (without cusp), turned opposite ways, and with the axis of x for their common axis.

If $a = b$, the equation may be written

$$(x^2 + y^2 - a^2)^2 = 2x^2y^2,$$

$\therefore$ $x^2 \pm xy\sqrt{2} + y^2 = a^2$, which represents 2 ellipses, which both the previous pairs of curves turn into.

7. If the conic be $lu' = 1 + e\cos(\theta' - \alpha)$, then $lu = 1 + e\cos(\theta - \alpha)$, and (cf. Art. 325) at (u, θ), $\frac{du'}{d\theta'} = \frac{du}{d\theta}$, and $\frac{d^2u'}{d\theta'^2} = \frac{d^2u}{d\theta^2}$,

$$\therefore -l\frac{du}{d\theta} = e\sin(\theta - \alpha) \qquad (1),$$

$$-l\frac{d^2u}{d\theta^2} = e\cos(\theta - \alpha) \qquad (2).$$

Hence $l\left(u + \frac{d^2u}{d\theta^2}\right) = 1$, and by (1) and (2),

$$e\cos\alpha = -l\left(\sin\theta\frac{du}{d\theta} + \cos\theta\frac{d^2u}{d\theta^2}\right),$$

$$e\sin\alpha = l\left(\cos\theta\frac{du}{d\theta} - \sin\theta\frac{d^2u}{d\theta^2}\right);$$

$$\therefore lu' = 1 - \cos\theta' \cdot l\left(\sin\theta\frac{du}{d\theta} + \cos\theta\frac{d^2u}{d\theta^2}\right)$$

$$+ \sin\theta' \cdot l\left(\cos\theta\frac{du}{d\theta} - \sin\theta\frac{d^2u}{d\theta^2}\right);$$

$$\therefore\ u' = u + \frac{d^2u}{d\theta^2} + \frac{du}{d\theta} \cdot \sin(\theta' - \theta) - \frac{d^2u}{d\theta^2} \cdot \cos(\theta' - \theta)$$

$$= u + \frac{d^2u}{d\theta^2} - \frac{d}{d\theta}\left\{\frac{du}{d\theta}\sec(\theta' - \theta)\right\}\cos^2(\theta' - \theta).$$

8. If the straight line be axis of x, (x, y) the centre of curvature of the rolling curve at the point of contact in any position, ρ the corresponding radius of curvature and s the arc of the curve measured from some point fixed on it, clearly $x = s + \text{a constant}$, $y = \rho = \frac{ds}{d\psi}$ say; and eliminating s and ψ from these equations and the intrinsic equation to the curve, the equation to the required locus is obtained.

In the equiangular spiral $\frac{dr}{ds} = \cos\alpha$;

$$\therefore\ s \cdot \cos\alpha + \text{constant} = r = ae^{\theta\cot\alpha} = a \cdot e^{(\psi - \alpha)\cot\alpha},$$

which is of the form $s = ae^{\psi\cot\alpha} + \text{constant}$,

$$\therefore\ \frac{ds}{d\psi} = a\cot\alpha \cdot e^{\psi\cot\alpha} = \cot\alpha\,(s - c) \text{ say,}$$

and $\therefore\ x - b = s - c$ suppose $= y\tan\alpha$, a straight line.

In the cycloid (cf. Parkinson's *Mechanics*), $s = 4a\sin\psi$, $\therefore\ \rho = 4a\cos\psi$,

and $\therefore\ (x - b) = 4a\sin\psi,\ y = 4a\cos\phi,$

and $\therefore\ (x - b)^2 + y^2 = (4a)^2$, a circle.

In the catenary (cf. Todhunter's *Analytical Statics*),

$$\frac{dy}{dx} = \frac{s}{c} = \tan\psi \text{ or } s = c\tan\psi,\ \therefore\ y = c\sec^2\psi,$$

and $x = c\tan\psi + a$. Hence $y = c + \frac{1}{c}(x - a)^2$, a parabola.

9. If (h, k) be any point on the circle $(x - a)^2 + (y - b)^2 = c^2$, the 2 sides containing the right angle are $\frac{y}{k} = \frac{x}{h}$ and $(y - k)\,k + (x - h)\,h = 0$, also

$$(h - a)^2 + (k - b)^2 = c^2;$$

$\therefore$ the 2nd side is $hx + ky = 2ah + 2bk + c^2 - a^2 - b^2$; $\therefore$ for its envelope

$$dh\,(h - a) + dk\,(k - b) = 0, \text{ and } dh\,(x - 2a) + dk\,(y - 2b) = 0;$$

$$\therefore\ h = a + \lambda(x - 2a),\ k = b + \lambda(y - 2b), \text{ and}$$

$$\therefore\ (x - 2a)\{a + \lambda(x - 2a)\} + (y - 2b)\{b + \lambda \cdot (y - 2b)\} = c^2 - a^2 - b^2;$$

$$\therefore\ \lambda = (a^2 + b^2 + c^2 - ax - by) \div \{(x - 2a)^2 + (y - 2b)^2\},$$

and $$\lambda^2\{(x - 2a)^2 + (y - 2b)^2\} = c^2;$$

$$\therefore\ (a^2 + b^2 + c^2 - ax - by)^2 = c^2\{(x - 2a)^2 + (y - 2b)^2\} \text{ or \&c.}$$

10. With the centre for origin the equation of any such hyperbola is

$$c(x^2-y^2)-2bxy=1 \quad\ldots\ldots\ldots\ldots (1),$$

and when $x=a$, $\dfrac{dy}{dx}=0$, $\therefore\ cx=by$ or $y=\dfrac{ca}{b}$, and $\therefore\ c\left(a^2-\dfrac{c^2a^2}{b^2}\right)-2a^2c=1$,

or $$b^2(1+ca^2)=-c^3a^2 \quad\ldots\ldots\ldots\ldots (2);$$

$\therefore$, for the envelope, $\quad dc\,.\,(x^2-y^2)=2db\,.\,xy$,

and $\quad dc\,(3c^2a^2+a^2b^2)+2bdb\,(1+ca^2)=0$;

$$\therefore\ \lambda(x^2-y^2)=3c^2a^2+a^2b^2 \text{ and } 2\lambda xy=-2b(1+ca^2);$$

$$\therefore\ \lambda=3c^3a^2+ca^2b^2+2b^2(1+ca^2)=-b^2; \text{ by (2)}$$

$$\therefore\ b^2(a^2+x^2-y^2)=-3c^2a^2 \quad\ldots\ldots\ldots\ldots (3),$$

and $$bxy=ca^2+1 \quad\ldots\ldots\ldots\ldots (4).$$

From (1) and (4), $\quad c(a^2+x^2-y^2)=3bxy$,

$$\therefore\ \text{by (3) } (a^2+x^2-y^2)^3=-27a^2x^2y^2,$$

or $$a^2+x^2-y^2+3(axy)^{\frac{2}{3}}=0.$$

11. The axes of co-ordinates being rectangular, let the ends of the chord be $(0, \pm c)$, the focus of any one of the parabolas be $(0, a)$ and its directrix $x\cos\alpha+y\sin\alpha=p$; then its axis is

$$x\sin\alpha=(y-a)\cos\alpha \quad\ldots\ldots\ldots\ldots (1),$$

and the parabola is $\quad (x\cos\alpha+y\sin\alpha-p)^2=x^2+(y-a)^2$,

$$\therefore\ c\sin\alpha-p=\pm(c-a), \text{ and } c\sin\alpha+p=\pm(c+a),$$

$\therefore$ either $c\sin\alpha=\pm c$ *or* $\pm a$; the former gives $x=0$ for the axis, which is impossible; with the latter result, $x\sin\alpha-y\cos\alpha=\pm c\sin\alpha\,.\cos\alpha$, $\therefore$ for the envelope $\quad x\cos\alpha+y\sin\alpha=\pm c\cos 2\alpha$;

$$\therefore\ x=\pm c\cos\alpha(\sin^2\alpha+\cos 2\alpha)=\pm c\cos^3\alpha,$$

and $$y=\pm c\sin\alpha(\cos 2\alpha-\cos^2\alpha)=\mp c\sin^3\alpha,$$

and $$\therefore\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}.$$

12. Taking the axes touched as axes of co-ordinates, any such ellipse is given by $a^2(y-b)^2+b^2(x-a)^2=a^2b^2$, and $ab=c^2$ say;

$$\therefore\ a^2y^2+b^2x^2-2c^2(ay+bx)+c^4=0;$$

and for the envelope $\quad bda+adb=0$,

and $\quad da\,(ay^2-c^2y)+db\,(bx^2-c^2x)=0$;

$$\therefore\ ay(ay-c^2)=bx(bx-c^2),$$

or $\quad a^2y^2-b^2x^2=c^2(ay-bx)$, and $\therefore$ (1) $ay=bx$, or (2) $ay+bx=c^2$.

$$(1)\quad ay=bx=\frac{c^2x}{a},\ \therefore\ a=\pm c\sqrt{\frac{x}{y}}, \text{ and } b=\pm c\sqrt{\frac{y}{x}},$$

$$\therefore\ 2c^2xy\mp 4c^3\sqrt{xy}+c^4=0,$$

$$\therefore\ 2\sqrt{xy}=\pm 2c\pm c\sqrt{2}, \text{ or } 2xy=c^2(\sqrt{2}\pm 1)^2,$$

which represents 2 rectangular hyperbolæ, the product of the transverse axes of which $=4c^2$, and $\therefore$ &c.

(2) $ay+bx=c^2$ gives $c^4-2c^2xy-2c^4+c^4=0$, and $\therefore\ xy=0$, which is true by hypothesis.

13. If the circles be $x^2+y^2=a^2$, and $(x-c)^2+y^2=a^2$, and P be

$$(a\cos\theta,\ a\sin\theta),\ \text{then } Q \text{ is } (c\mp a\sin\theta,\ \pm a\cos\theta),$$

the upper or lower signs being taken together; $\therefore$ PQ is

$$(x-a\cos\theta)(a\sin\theta\mp a\cos\theta)=(y-a\sin\theta)(a\cos\theta-c\pm a\sin\theta),$$

or $$ax(\sin\theta\mp\cos\theta)+y(c-a\cos\theta\mp a\sin\theta)=ac\sin\theta\mp a^2;$$

$$\therefore\ \sin\theta(ax\mp ay-ac)-\cos\theta(\pm ax+ay)+cy\pm a^2=0,$$

and $\therefore$ for the envelope, $\cos\theta(ax\mp ay-ac)+\sin\theta(\pm ax+ay)=0$;

$\therefore$, eliminating θ, $(ax\mp ay-ac)^2+(\pm ax+ay)^2=(cy\pm a^2)^2$,

or $$2a^2x^2+2a^2y^2-2a^2cx+a^2(c^2-a^2)=c^2y^2,$$

i.e. $2a^2\left(x-\frac{c}{2}\right)^2+(2a^2-c^2)y^2=a^2\left(a^2-\frac{c^2}{2}\right)$, which represents an ellipse or hyperbola as $2a^2>$ or $<c^2$.

If $2AP^2=AB^2$, i.e. $2a^2=c^2$, then $x=\frac{c}{2}$, and

$$a\sin\theta\left(\mp y-\frac{c}{2}\right)-a\cos\theta\left(y\pm\frac{c}{2}\right)+c\left(y\pm\frac{c}{2}\right)=0,$$

$\therefore$ $y=\mp\frac{c}{2}$ for all values of θ, and $\therefore$ PQ has really no envelope but passes through $\left(\frac{c}{2},\ \mp\frac{c}{2}\right)$, i.e. through one or other of the points of intersection of the 2 circles, through the lower point corresponding to the upper sign of the ambiguity, and *vice versâ*.

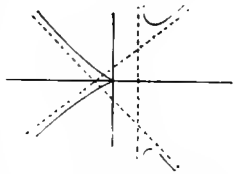
14. (1) $x^3-xy^2+ay^2=0$: if $y=mx+n$ be an asymptote,

$$x^3=(x-a)(mx+n)^2$$

for 2 ∞ values of x, $\therefore$ $1=m^2$, and $-am^2+2mn=0$, $\therefore$ $n=\frac{a}{2m}$, and the asymptotes are $y=\pm\left(x+\frac{a}{2}\right)$, and also $x=a$. Also $y^2=\frac{x^3}{x-a}$, $\therefore$ there are 2 values of y for each value of x and the curve is symmetrical as to $y=0$; and when x is positive, $x>a$; and then if y be positive

$$y^2=\frac{x^3}{x-a}>\text{ or }<\left(x+\frac{a}{2}\right)^2,$$

as $x^3>$ or $<x^3+\frac{a^2}{4}x-a^2x-\frac{a^3}{4}$, $\therefore$ for positive values of x and y, the ordinate of the curve $>$ than that of $y=x+\frac{a}{2}$ for the same abscissa, $\therefore$ the curve when x and y are both positive lies above $y=x+\frac{a}{2}$. Thus there are 2 branches extending to ∞ as in fig. p. 372 (but to the right of $x=0$). When x is negative, y is possible for all values of x; and the curve passes through the origin where the tangent is $y=0$ and there is a cusp of the 1st kind, $\because x^3+ay^2=0$ nearly, and y increasing numerically with x, there are 2 branches going off to $-\infty$, the upper one being above $y+x+\dfrac{a}{2}=0$, and ultimately touching it; and similarly for the lower branch, as the curve does not cut $y=0$ again, but must cut the asymptotes where $x=-\dfrac{a}{3}$. Here a has been supposed positive; if it be negative, and the sign of x be changed, there will be the same curve, $\therefore$ it will be obtained by turning the first curve round $x=0$ on to its own plane again.



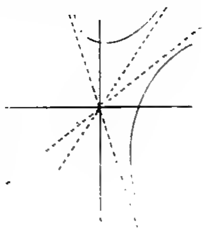
(2) $y^3-7yx^2+6x^3-a^3=0$: by the latter part of Art. 274 the asymptotes are $y^3-7yx^2+6x^3=0$, i.e. $(y-x)(y+3x)(y-2x)=0$. When $x=0$, $y=a$, and changing the origin to $(0, a)$, $(y+a)^3-7(y+a)x^2+6x^3=a^3$, $\therefore$ $y=0$ is the tangent and the curve is nearly $3ay=7x^2$, which is parabolic and convex to $y=0$, $\therefore$ there must be a branch here bounded by the asymptotes $y+3x=0$ and $y=2x$. When $y=0$, $x=\dfrac{a}{\sqrt[3]{6}}=c$ say, and changing the origin to $(c, 0)$ $y^3-7y(x+c)^2+6(x+c)^3-a^3=0$, and the curve is nearly

$$-14xy-7cy+18cx+18x^2=0,$$

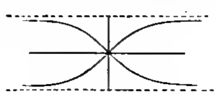
which is hyperbolic, and the tangent is $7y=18x$; thus there must be a branch bounded by $y=x$ and $y+3x=0$, cutting the positive part of $y=0$. For a branch between $y=x$ and $y=2x$, a test is, $y=\dfrac{3}{2}x$, say, meets the curve, if possible, where

$$27x^3-84x^3+48x^3=8a^3=-9x^3,$$

and $\therefore$ where x is negative, $\therefore$ there is such a branch. Also if $y=mx$, $\dfrac{a}{x}=(m^3-7m+6)^{\frac{1}{3}}$, $\therefore$ for each value of m there is only one point on the curve, and $\therefore$ there are no other branches. If a were negative, the equation would become the same by changing the signs of x and y, $\therefore$ the curve would be found by turning the above in its plane round the origin through an angle π.



(3) $y^4+x^2y^2-a^2x^2=0$: $y=\pm a$ are the asymptotes, the curve is symmetrical as to both axes, passes through the origin near which the curve is nearly $y^4=a^2x^2$, or $y^2=\pm ax$ (2 parabolas); also $y^4=x^2(a^2-y^2)$, $\therefore$ y lies between a and $-a$, and thus the curve consists of 2 branches, one extending from the origin to touch the asymptotes at ∞ to the right of $x=0$, and the other to the left. N.B. The branches should *touch* $x=0$.



(4) $a(x^3+7x^2y+7xy^2+y^3)-x^2y^2=0$: the origin is a triple point, the tangents there being given by $(x+y)(x^2+y^2+6xy)=0$.

The curve is symmetrical as to $x=y$, $\because$ x and y can be interchanged without altering the equation. If $y=mx+n$ be an asymptote, $m^2=0$, and $2mn=a(1+7m+7m^2+m^3)$, $\therefore$ $n=\infty$, and there are no finite asymptotes. Turning the axes through an angle $\frac{\pi}{4}$, the equation is

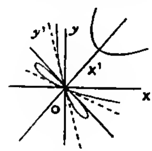
$$\left(\frac{x^2-y^2}{2}\right)^2=a\left\{\left(\frac{x-y}{\sqrt{2}}\right)^3+\left(\frac{x+y}{\sqrt{2}}\right)^3+7\cdot\frac{x^2-y^2}{2}\cdot x\sqrt{2}\right\},$$

or $(x^2-y^2)^2=8a\sqrt{2}(2x^3-xy^2)=c(2x^3-xy^2)$ say.

If then $y=mx$, $x(1-m^2)^2=c(2-m^2)$,

or $$x=\frac{c}{1-m^2}+\frac{c}{(1-m^2)^2}.$$

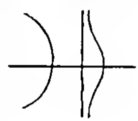
Hence, taking a and $\therefore$ c as positive, as m $\propto$ from 0 to 1, x increases from $2c$ to ∞, and for each value of m there is but one value of x (and $\therefore$ of y) other than $x=0$, unless $m^2=2$, when $x=0$ only. As m $\propto$ from 1 to $\sqrt{2}$, x is positive, but diminishes from ∞ to 0; as m $\propto$ from $\sqrt{2}$ to ∞, x is negative but always finite, and is zero at the limits and y is negative, so that there is a loop touching $x=0$ and $y=x\sqrt{2}$ below $y=0$. As m $\propto$ from ∞ to $-\sqrt{2}$, m and x are negative, and $\therefore$ y positive, and $\therefore$ there is a loop similar to the former above $y=0$. As m $\propto$ from $-\sqrt{2}$ to -1, x is positive and $\therefore$ y negative, and x $\propto$ from 0 to ∞ again: lastly, as m $\propto$ from -1 to 0, x is positive and y negative and x $\propto$ from ∞ to $2c$, thus completing the curve as annexed. If a be negative the curve is only turned round through an angle π.



(5) $xy^2+ax^2-a^3=0$; the curve is symmetrical as to $y=0$, $x=0$ is an asymptote; when $y=0$, $x=\pm a$, and as x is positive or negative, $x<a$ or $-a$; and changing the origin to $(\pm a, 0)$ the equation becomes

$$(x\pm a)y^2+a(x^2\pm 2ax)=0,$$

$\therefore$ at $(\pm a, 0)$ the curve is nearly parabolic, with its concavity towards or away from the original origin, as the upper or lower sign is taken. Thus, there being 2 real values of x for every value of y from the given equation, the curve consists of 2 branches, one through $(a, 0)$ ultimately touching $x=0$ at its 2 ends, the other through $(-a, 0)$ stretching to $-\infty$ at each end. The direction of the concavities will be inverted if a be negative.

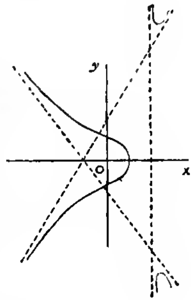


(6) $y^2(x-2a)=x^3-a^3$: if $y=mx+n$ be an asymptote,

$$(mx+n)^2(x-2a)=x^3-a^3$$

gives $m^2=1$, $2mn=2am^2$, $\therefore$ $y=\pm(x+a)$, also $x=2a$. The curve is symmetrical as to $y=0$, and cuts the axes at $(a, 0)$, $\left(0, \pm\frac{a}{\sqrt{2}}\right)$, and x cannot lie between a and $2a$ for real values of y. When

$$x>2a, \ y^2=\frac{x^3-a^3}{x-2a}> \text{ or } <(x+a)^2,$$



as $x^3-a^3>$ or $<x^3+a^2x-4a^2x-2a^3$, $\therefore$ the positive ordinate of the curve $>$ than that of the oblique asymptote, and $\therefore$ there are 2 branches stretching to ∞, similar to those in fig. p. 372 but to the right of $x=0$. So long as $x<a$, y is possible for all values of x, and $\therefore$ there is a branch through $(a, 0)$ stretching to $-\infty$ at both ends. Changing the origin to $(a, 0)$ the equation becomes $y^2(x-a)=x^3+3ax^2+3a^2x$, $\therefore$ the curve there is nearly the parabola $y^2+3ax=0$. The curve is the same in shape, if a be negative, but turned the reverse way towards $x=0$, as appears by changing the sign of x as well as of a.

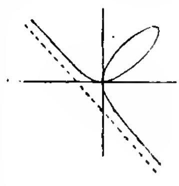
(7) $y^5-ax^3y-bxy^3+x^5=0$: for the asymptote

$$(mx+n)^5-ax^3(mx+n)-bx(mx+n)^3+x^5=0$$

has 2 ∞ roots, $\therefore$ $m^5+1=0$ or $m=-1$, and $5m^4n=am+bm^3$, and

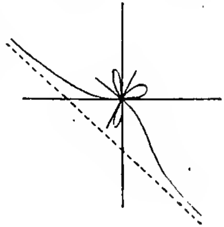
$$\therefore\ n=-\frac{a+b}{5}, \ \therefore\ x+y=-\frac{a+b}{5}$$

is the asymptote. If a, b be supposed both positive there is a double point at the origin, and approximately, neglecting x^5 when x is much smaller than y, $y^5=ax^3y+bxy^3$, but ax^3y is also much smaller than bxy^3, and $\therefore$ ax^3y must be neglected, and thus one branch at the origin is $y^2=bx$; similarly the other branch is $x^2=ay$. Also if $y=mx$, $x(1+m^5)=m(a+bm^2)$, $\therefore$ there is in general one value of x besides zero for each value of m, and when m is positive x is always positive and finite or zero, and so also y, thus the positive arms of the 2 branches unite to form an oval. As m $\propto$ from ∞ to -1, x is positive and $\therefore$ y is negative, and x $\propto$ from 0 to ∞, corresponding to the lower arm of $y^2=bx$ which goes off to ∞ to touch the asymptote below $y=0$. Similarly for the branch corresponding to the 2nd arm of $x^2=ay$. The curve is not symmetrical as to $x=y$ unless $a=b$. Similarly if a, b be both negative.



If $a>-b$ and b negative $=-c$ say, then $x(1+m^5)=m(a-cm^2)$, and there are 4 branches at the origin, viz. $x^2=ay$, $y^2+cx=0$, and two which touch $y\sqrt{c}=\pm x\sqrt{a}$. As m $\propto$ from 0 to $\sqrt{\frac{a}{c}}$, x is positive (and $\therefore$ also y) and $\propto$ from 0 to 0 again, being finite for intermediate values, $\therefore$ there is a loop above $y=0$ touching $y=0$ and $y=x\sqrt{\frac{a}{c}}$. As m $\propto$ from $\sqrt{\frac{a}{c}}$ to ∞, x is negative, and $\therefore$ also y, and x $\propto$ from 0 to 0 again, and ..

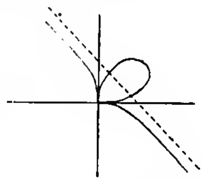
there is another loop below $y=0$ and touching $y=x\sqrt{\frac{a}{c}}$ and $x=0$. As $m \propto$ from ∞ to $\sqrt{\frac{a}{c}}$, x is negative and $\therefore y$ is positive (m being negative) and $x \propto$ from 0 to 0 again, and there is a 3rd loop above $y=0$ touching $x=0$ and $y=-x\sqrt{\frac{a}{c}}$. As $m \propto$ from $-\sqrt{\frac{a}{c}}$ to -1, x is positive and $\therefore y$ negative, and $x \propto$ from 0 to ∞, and there is a branch below $y=0$, touching $y=-x\sqrt{\frac{a}{c}}$ and the asymptote at ∞. Lastly, when $m \propto$ from -1 to 0, being negative, x is negative and $\therefore y$ is positive, and $x \propto$ from ∞ to 0, there being a branch above $y=0$, touching $y=0$ at the origin and the asymptote at ∞. Similarly for other cases (if $a<b$ change the signs of x and y &c.).



(8) $y^5-5ax^2y^2+x^5=0$: if $x+y=c$ be the asymptote, then when x is ∞, $(c-x)^5+x^5-5ax^2(x-c)^2=0$, $\therefore c=a$; the curve is symmetrical as to $x=y$ and meets it where $x=\frac{5a}{2}$: near the origin, neglecting x^5, i.e. when x is smaller than y, and $\therefore$ near $x=0$, $y^3=5ax^2$, thus there is a cusp where $x=0$ is the tangent; so $x^3=5ay^2$ gives another cusp, the tangent being $y=0$; the positive arms of the 2 branches meet forming a loop, and the other arms stretch to ∞ towards the asymptote $x+y=a$. That these are the only branches will appear thus: if $y=mx$, then

$$x(1+m^5)=5a(1+m^2),$$

$\therefore$ for each value of m there is but one value at most of x, besides zero, and x and y cannot be both negative for then m would be positive, and $\therefore$ by the last equation x positive also. Further x is ∞ only when $m=-1$.



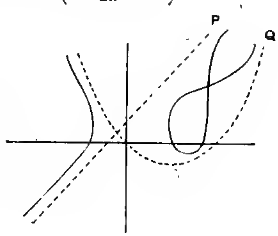
If a were negative, changing the signs of x and y would make the equation the same in form, so that the curve would merely be turned in its plane through 2 right angles.

(9) $y=\frac{x^2}{a}\pm(x-a)\frac{\sqrt{(x^2-b^2)}}{a}$: x cannot lie between b and $-b$, unless a lies between b and $-b$, when (a, a) is a conjugate point. The 2 curves corresponding to the double sign, meet at $\left(\pm b, \frac{b^2}{a}\right)$, and if $a>b$ or $<-b$, they meet again at (a, a).

Also $a\frac{dy}{dx}=2x\pm\left\{\sqrt{(x^2-b^2)}+\frac{x(x-a)}{\sqrt{x^2-b^2}}\right\}$, $\therefore$ when $x=\pm b$, $\frac{dy}{dx}=\mp\infty$, when

$x = -b$, $\frac{dy}{dx} = \pm\infty$, and when $x = a$, $a\frac{dy}{dx} = 2a \pm \sqrt{(a^2 - b^2)}$, and $\therefore$ there is a double point. When x is very great

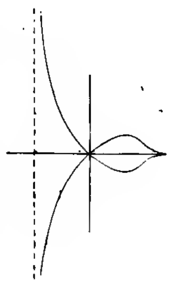
$$ay = x^2 \pm (x-a)\,x\left(1 - \frac{b^2}{x^2}\right)^{\frac{1}{2}} = x^2 \pm (x^2 - ax)\left(1 - \frac{b^2}{2x^2} - \dots\right)$$
$$= x^2 \pm \left(x^2 - ax - \frac{b^2}{2} + \frac{ab^2}{2x}\right)$$



nearly; $\therefore$ for the part of the curve corresponding to the upper sign the asymptote is $ay = 2x^2 - ax - \frac{b^2}{2}$, a parabola, and the curve is above or below it as x is + or −. For the other part of the curve the asymptote is $ay = ax + \frac{b^2}{2}$, which is above or below the corresponding curve, as x is + or −.

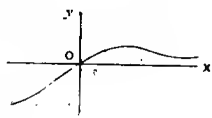
N.B. In the figure a being supposed $> b$ and b positive, the branch P should cut the rectilinear asymptote and touch the parabola at ∞: the branch Q should be convex to the rectilinear asymptote and ultimately touch it; and the parabola should *not* pass through the origin. If $a < b$ there is no loop.

(10) $y^2(a+x) = x^2(a-x)$: the curve is symmetrical as to the axis of x; $x + a = 0$ is the real asymptote, the directions of the other two being given by $x^2 + y^2 = 0$. When $x = 0$, $y = 0$; when $y = 0$, $x = 0$ or a; and for real values x must lie between a and $-a$; and at the origin the tangents are $x^2 = y^2$, and on the curve $\frac{x^2}{y^2} = \frac{a+x}{a-x}$, $\therefore$ when x is positive $x > y$, thus there is a loop from $x = 0$ to $x = a$ lying between the tangents $x = \pm y$; changing the origin to $(a, 0)$ the equation to the curve becomes $y^2 . (2a + x) = -x(x+a)^2$, $\therefore$ $x = 0$ is the tangent at the new origin. Also for values of x from 0 to $-a$, the curve extends to ∞ to touch the asymptote. If a be negative, changing the sign of x, the same curve is obtained turned round through 2 right angles.



N.B. The loop should be an oval.

(11) $y = x . e^{-x}$: when $x = 0$, $y = \frac{x}{e^x} = 0$, and $\frac{dy}{dx} = e^{-x}(1 - x)$, $\therefore$ at the origin $\frac{dy}{dx} = 1$, and e^{-x} is positive whether x be positive or negative, $\therefore$ y has the same sign as x for any value of x; and when $x = \infty$, $y =$ limit of $\frac{x}{e^x} = \frac{1}{e^x} = 0$ and $\frac{dy}{dx} = 0$; thus $y = 0$ is an asymptote to the right of $x = 0$. When $x = -\infty$, $y = -\infty . e^\infty = -\infty$; hence the curve goes to infinity without any finite asymptote to the left of $x = 0$.

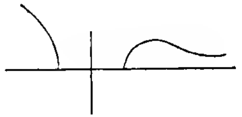


N.B. The curve to the left should turn *downwards*.

(12) $y=e^{-x}.\sqrt{x^2-1}$: x cannot lie between 1 and -1, and $y=0$ when $x=\pm 1$, and y is always positive (unless the $-$ sign of the radical be taken as well, when the curve is symmetrical as to $y=0$). Also

$$\frac{dy}{dx}=e^{-x}.\left\{\frac{x}{\sqrt{x^2-1}}-\sqrt{x^2-1}\right\}=e^{-x}\frac{(1+x-x^2)}{\sqrt{(x^2-1)}}=\pm\infty \text{ when } x=\pm 1,$$

$\therefore$ the 2 branches are perpendicular to $y=0$ where they meet it and stop. When $x=\infty$, $y^2=\frac{x^2-1}{e^{2x}}=$ limit of $\frac{x}{e^{2x}}=0$ and $\frac{dy}{dx}=0$, thus $y=0$ is an asymptote at its $+$ end. When

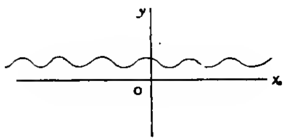


$$x=-\infty,\ y=e^{\infty}.\infty=\infty,$$

and there is no finite asymptote for $-$ values of x.

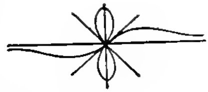
(13) $e^{\left(\frac{y}{a}\right)^2}=\sin\frac{x}{a}$: $\sin\frac{x}{a}$ must be always positive; and the least value of $e^{\left(\frac{y}{a}\right)^2}$ is when $y=0$, and is then 1. Hence the equation represents an ∞ series of conjugate points on $y=0$, which are given by $x=\frac{a\pi}{2}+2ma\pi$, where m is any integer.

(14) $y=e^{\cos x}$: y is always positive, and $\because \cos x=\cos(-x)$ the curve is symmetrical as to $x=0$. As $x \propto$ from 0 to $\frac{\pi}{2}$, $y \propto$ from e to 1; and as $x \propto$ from $\frac{\pi}{2}$ to π, $y \propto$ from 1 to $\frac{1}{e}$, and so on, and the curve undulates regularly ad infinitum both ways, being enclosed within the lines



$$y=e \text{ and } y=\frac{1}{e}.$$

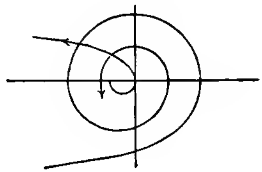
(15) $r^2\sin\theta=a^2\cos 2\theta$: when $\theta=0$ or π, $r=\infty$, and $r\sin\theta=\frac{a^2}{r}=0$, $\therefore$ $\theta=0$ is an asymptote touching the curve at both ends: $\sin\theta$ and $\cos 2\theta$ must have the same signs, whence all cases are included within values of θ from 0 to $\frac{\pi}{4}$, from $\frac{3\pi}{4}$ to π, and from $-\frac{\pi}{4}$ to $-\frac{3\pi}{4}$, and corresponding to each value of θ, there are 2 equal and opposite values of r. When $\theta=\pm\frac{\pi}{4}$, $r=0$, and also when $\theta=\frac{3\pi}{4}$; thus the positive values of θ give two



curves undulating through the pole and touching the asymptote at each end but on opposite sides; and when $\theta=-\frac{\pi}{2}$, $r^2=a^2$, and when $\theta=-\frac{3\pi}{4}$, $r=0$, thus the negative values of θ give 2 equal and similar loops through the pole, and symmetrical as to $\theta=\frac{\pi}{2}$.

N.B. There should be a 2nd undulating curve in the figure, similar but reversed symmetrically.

(16) $r(\theta-\pi)^2=a\left(\theta^2-\frac{\pi^2}{4}\right)$: considering first positive values of θ, when $\theta=0$, $r=-\frac{a}{4}$; when $\theta=\frac{\pi}{2}$, $r=0$; when $\theta=\pi$, $r=\infty$, and $r\sin\theta=$ the limit of



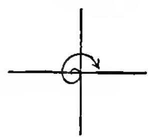
$$\frac{3a\pi^2}{4}\cdot\frac{\sin\theta}{(\theta-\pi)^2}=\frac{3a\pi^2}{8}\cdot\frac{\cos\theta}{(\theta-\pi)}=\infty,$$

$\therefore$ there is no finite rectilinear asymptote. When $\theta>\pi$, r is always positive, and

$$\frac{1}{a}\frac{dr}{d\theta}=\frac{d}{d\theta}\left\{\frac{\theta^2-\frac{\pi^2}{4}}{(\theta-\pi)^2}\right\}$$

$$=\frac{2\theta(\theta-\pi)-2\left(\theta^2-\frac{\pi^2}{4}\right)}{(\theta-\pi)^3}=-\frac{2\pi\left(\theta-\frac{\pi}{4}\right)}{(\theta-\pi)^3},$$

$\therefore$ r diminishes as θ increases from π, until when θ is ∞, r approximates to a, so that $r=a$ is a circular asymptote.



For negative values of θ, from $\theta=0$ to $-\frac{\pi}{2}$, r is negative, and $r=0$ when $\theta=-\frac{\pi}{2}$; and $\frac{dr}{d\theta}$ is always negative, $\therefore$ r continually increases algebraically, approaching the limit $r=a$.

15. If SP, HP make angles ϕ, ϕ' with the tangent at P, and that meet the perpendiculars from S and H in Q and R, and if $SP=r$, $HP=r'$ and $rr'=c^2$, then $PQ=r\sec\phi$, and $\cos\phi=\frac{dr}{ds}$, $\cos\phi'=-\frac{dr'}{ds}$, if r increase and $\therefore$ r' diminish as s increases, and $PR=r'\sec\phi'$;

$$\therefore\ \frac{PQ}{PR}=-\frac{r}{r'}\cdot\frac{dr'}{dr},\ \text{but}\ r'+r\frac{dr'}{dr}=0,\ \therefore\ PQ=PR.$$

16. $f\left(\frac{x}{a}, \frac{y}{b}\right)=1$ and $ab=c^2$ say, $\therefore$ for the envelope,

$$\frac{df}{d\left(\frac{x}{a}\right)} \cdot \frac{x}{a^2} \cdot da + \frac{df}{d\left(\frac{y}{b}\right)} \cdot \frac{y}{b^2} \cdot db = 0,$$

and
$$\frac{1}{a} da + \frac{1}{b} db = 0; \ \therefore \frac{x}{a} \cdot \frac{df}{d\left(\frac{x}{a}\right)} = \frac{y}{b} \frac{df}{d\left(\frac{y}{b}\right)} \ \ldots\ldots\ldots\ldots (1),$$

which is a homogeneous equation of n dimensions in $\frac{x}{a}$ and $\frac{y}{b}$ (cf. p. 109, Ex. 3), and solving it for $\frac{x}{y} \cdot \frac{b}{a}$ there are n values of $\frac{x}{a} : \frac{y}{b}$, and substituting any *one* of these values for $\frac{x}{a}$ in $f\left(\frac{x}{a}, \frac{y}{b}\right)=1$, there is an equation of the form $A\left(\frac{y}{b}\right)^n=1$, or $\frac{y}{b}=\frac{1}{A^{\frac{1}{n}}}$ for the real solution, and corresponding to this $\frac{x}{a}=\frac{1}{B^{\frac{1}{n}}}$ say, and $\therefore$ $\frac{xy}{c^2}=\frac{1}{\sqrt[n]{AB}}$, a rectangular hyperbola; thus there are in general n rectangular hyperbolæ. There may be equal or imaginary roots in (1) &c. In each case the real hyperbola has the axes for asymptotes.

17. If $AB=f$, $BC=g$, $CD=h$, and $DA=k$, then

$$BD^2=f^2+k^2-2fk\cos A=g^2+h^2-2gh\cos C.$$

$$\therefore fk\sin A=gh\sin C \cdot \frac{dC}{dA} \text{ or } \frac{dC}{dA}=\frac{DAB}{BCD};$$

so
$$\frac{dD}{dB}=\frac{ABC}{CDA}; \text{ and } A+B+C+D=2\pi,$$

$$\therefore \text{ ultimately } \frac{\Delta A}{BCD}=\frac{\Delta C}{DAB}=\lambda,\ \frac{\Delta B}{CDA}=\frac{\Delta D}{ABC}=\mu \text{ say},$$

and
$$\Delta A+\Delta B+\Delta C+\Delta D=0,$$

$$\therefore \lambda+\mu=0, \text{ and } \therefore \frac{\Delta A}{BCD}=\frac{-\Delta B}{CDA}=\frac{\Delta C}{DAB}=\frac{-\Delta D}{ABC}.$$

so that if A and C increase, B and D diminish, and *vice versâ*.

18. Suppose $\frac{y}{x}=\mu_1+\frac{b}{x}+\frac{c}{x^2}+\ldots$; then

$$\phi\left(\mu_1+\frac{b}{x}+\frac{c}{x^2}+.\right)+\frac{1}{x}\psi\left(\mu_1+\frac{b}{x}+\frac{c}{x^2}+.\right)+\frac{1}{x^2}\cdot\chi\left(\mu_1+\frac{b}{x}+\frac{c}{x^2}\right)+\ldots=0,$$

$$\text{and } \phi(\mu_1)=0, \text{ and } b=-\frac{\psi(\mu_1)}{\phi'(\mu_1)};$$

thus to the 2[nd] order of small quantities

$$\phi(\mu_1)+\left(\frac{b}{x}+\frac{c}{x^2}\right)\phi'(\mu_1)+\frac{1}{2}\cdot\left(\frac{b}{x}\right)^2\phi''(\mu_1)+\frac{1}{x}\psi(\mu_1)+\frac{b}{x^2}\cdot\psi'(\mu_1)+\frac{1}{x^2}\chi(\mu_1)=0;$$

$$\therefore\ c\phi'(\mu_1)+\frac{b^2}{2}\phi''(\mu_1)+b\psi'(\mu_1)+\chi(\mu_1)=0;$$

$$\therefore\ c=-\frac{2\chi(\mu_1)+2b\psi'(\mu_1)+b^2\phi''(\mu_1)}{2\phi'(\mu_1)},\ \text{and}\ \therefore\ \&c.$$

Hence $y=x\mu_1+b-\frac{f(\mu_1)}{x}$ say, approximately, when x and y are very large; thus when x is positive the rectilinear asymptote $y=\mu_1 x+b$ is above or below the curve as $f(\mu_1)$ is positive or negative; and when x is negative the asymptote is below or above the curve, as $f(\mu_1)$ is positive or negative, $\therefore$ the ends of the asymptote are on opposite sides of the curve, *if* $f(\mu_1)$ be not zero.

19. Here $\phi'(\mu_1)=0$, and if $\frac{y}{x}=\mu_1+\left(\frac{A}{x}\right)^{\frac{1}{2}}+\frac{B}{x}+\frac{C}{x^{\frac{3}{2}}}+\ldots,$

and
$$\phi\left(\frac{y}{x}\right)+\frac{1}{x}\psi\left(\frac{y}{x}\right)+\frac{1}{x^2}\chi\left(\frac{y}{x}\right)+\ldots=0,$$

$$\therefore\ \left\{\left(\frac{A}{x}\right)^{\frac{1}{2}}+\frac{B}{x}+\frac{C}{x^{\frac{3}{2}}}\right\}^2\cdot\frac{\phi''(\mu_1)}{2}+\left\{\left(\frac{A}{x}\right)^{\frac{1}{2}}+\frac{B}{x}+\frac{C}{x^2}\right\}^3\frac{\phi'''(\mu_1)}{6}$$

$$+\frac{1}{x}\left[\psi(\mu_1)+\left\{\left(\frac{A}{x}\right)^{\frac{1}{2}}+\frac{B}{x}+\frac{C}{x^{\frac{3}{2}}}\right\}\psi'(\mu_1)+\left\{\left(\frac{A}{x}\right)^{\frac{1}{2}}+\ldots\right\}^2\frac{\psi''(\mu_1)}{2}\right]$$

$$+\frac{1}{x^2}\{\chi(\mu_1)+\ldots\}=0;$$

thus to the 1[st] approximation $\frac{A}{2}\phi''(\mu_1)+\psi(\mu_1)=0$; .. to the 2[nd] approximation,

$$B\,.\,A^{\frac{1}{2}}\phi''(\mu_1)+A^{\frac{3}{2}}\,.\,\frac{\phi'''(\mu_1)}{6}+A^{\frac{1}{2}}\psi'(\mu_1)=0,$$

or
$$-B=\frac{\psi'(\mu_1)}{\phi''(\mu_1)}+\phi'''(\mu_1)\cdot\frac{A}{6\phi''(\mu_1)};$$

and $\therefore$ to the 3[rd] approximation or coefficient of $\frac{1}{x^2}$, $2A^{\frac{1}{2}}\,.\,\frac{C}{2}\,.\,\phi''(\mu_1)$

$$+\frac{B^2}{2}\cdot\phi''(\mu_1)+3AB\,.\,\frac{\phi'''(\mu_1)}{6}+B\,.\,\psi'(\mu_1)+\frac{A}{2}\cdot\psi''(\mu_1)+\chi(\mu_1)=0,$$

or $-C\,.\,A^{\frac{1}{2}}\phi''(\mu_1)=\chi(\mu_1)+\frac{A}{2}\psi''(\mu_1)+B\left\{\psi'(\mu_1)+\frac{A}{2}\phi'''(\mu_1)+\frac{B}{2}\cdot\phi''(\mu_1)\right\}.$

20. Moving the origin to (a, β) the equation to the curve is

$$\phi(x+a,\ y+\beta)=0,$$

and if there be n tangents at the new origin, they are given by the terms of the n^{th} degree in x and y (the equation being rational), and there are no terms of a degree lower than n; and, expanding, as in Art. 226,

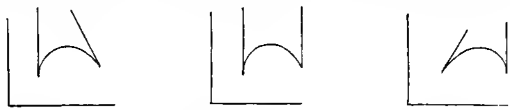
$$\phi(x+a,\ y+\beta)=\left(x\frac{d}{da}+y\frac{d}{d\beta}\right)^n\phi(a,\beta)+\text{higher powers of } x \text{ and } y.$$

Hence, with the original origin the n tangents are given by

$$\left\{(x-a)\frac{d}{da}+(y-\beta)\frac{d}{d\beta}\right\}^n\phi(a,\beta)=0.$$

21. The argument and $\therefore$ the conclusion is not altered if $\phi'(x)$ be very great either when $x=a$ or $=b$, .. &c.

This may be geometrically illustrated (cf. Art. 105) as follows, y being $\phi(x)$ and being supposed to increase and then diminish between the limits.



22. If $F'(x)$ or $f'(x)$ be very great, when $x=a$ or $a+h$, the theorem of course holds, and $\therefore$ when $F'(x)$ is ∞ at either of the limiting values of x, or if $f'(x)$ is ∞ at one of these points. This corresponds in the fig. to Art. 105 to the cases of the tangents at P or Q or both being parallel to the axis of y, or to the tangent at either p or q being in that direction.

23. Referring to Art. 373 (3), so long as $p+1$ is not $< q$ none of the differential coefficients of $(x-a)^{p+1}$ up to the $(q+1)^{\text{th}}$ become ∞, and R is finite so long as $p-q+1$ is positive, i.e. $p+1>q$.

24. If $p=q-\frac{1}{2}$ in Art. 373 (3), $f(x)$ being arbitrary,

$$R=\frac{\underline{|q}}{\underline{|n}}\cdot\frac{(1-\theta)^{n-q}\cdot\theta^{\frac{1}{2}}\cdot h^{n+1}\cdot F^{n+1}(a+h\theta)}{\left(q+\frac{1}{2}\right)\left(q-\frac{1}{2}\right)\cdots\cdots\left(\frac{1}{2}\right)}$$

$$=\frac{\underline{|q}}{\underline{|n}}\cdot\frac{(1-\theta)^{n-q}\cdot\theta^{\frac{1}{2}}\cdot h^{n+1}\cdot F^{n+1}(a+h\theta)\cdot 2^{q+1}}{(2q+1)(2q-1)\cdots\cdots 5\cdot 3\cdot 1}.$$

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