KEY TO

TODHUNTER'S INTEGRAL CALCULUS.

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PREFACE.

THE solutions to the questions, particularly in the earlier chapters, are exhibited in some detail, in the hope that they may thereby be better adapted to the purposes of a Key. In all I have aimed especially at clearness and accuracy, and trust that no errors will be found.

In one or two of the examples to Chap. XV., the complete solutions are so lengthy that I have limited myself to proving the character of the result, and indicating the subsequent steps, referring the reader for further information, if required, to the *History of the Calculus of Variations,* or to the *Researches* . . . .

The working of examples 13 and 14 of Chap. V. (based on the principle of Art. 66) which is given in the *History of Probability,* is shorter than that in the Key; but the latter may appear clearer, and supplies a different method.

All the references, not otherwise specified, are to the *Integral Calculus* (edition of 1886).

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CHAPTER I.

1. $$1-3x-x^2=-\left(x+\frac{3}{2}\right)^2+\frac{13}{4}, \qquad \therefore \text{ if } x+\frac{3}{2}=z,$$

$$\text{the integral}=\int\frac{dz}{\sqrt{\frac{13}{4}-z^2}}=\sin^{-1}\frac{2z}{\sqrt{13}} \text{ (Cf. Art. 9)}$$

$$=\sin^{-1}\frac{2x+3}{\sqrt{13}}.$$

2. Integrating by 'parts' (Art. 12),

$$\int\log x dx=x\log x-\int dx\,.\,x\,.\,\frac{1}{x}=x\log x-\int dx=x\log x-x.$$

3. So $$\int x^n\,.\log x dx=\frac{x^{n+1}}{n+1}\,.\log x-\int dx\,.\,\frac{x^{n+1}}{n+1}\,.\,\frac{1}{x}$$

$$=\frac{x^{n+1}}{n+1}\log x-\int dx\,.\,\frac{x^n}{n+1}=\frac{x^{n+1}}{n+1}\log x-\frac{x^{n+1}}{(n+1)^2}.$$

4. So $$\int\theta\sin\theta d\theta=-\theta\cos\theta+\int d\theta\,.\cos\theta\,.\,1=-\theta\cos\theta+\sin\theta.$$

5. $$\int\frac{dx}{e^x+e^{-x}}=\int\frac{e^x dx}{1+e^{2x}}, \quad \therefore \text{ if } e^x=z, \text{ the integral is, } \left(\because\ e^x\,.\,\frac{dx}{dz}=1\right),$$

$$\int\frac{dz}{1+z^2}=\tan^{-1}z=\tan^{-1}(e^x).$$

6. If $x=z^2$, $\qquad dx=2zdz$, and the integral becomes

$$\int\frac{\sqrt{m+z^2}}{z}\,.\,2zdz=2\int\sqrt{m+z^2}\,.\,dz, \text{ and } \therefore \text{ by Art. 14, Ex. 4,}$$

$$\text{the integral}=z\sqrt{m+z^2}+m\log\{z+\sqrt{m+z^2}\}$$

$$=\sqrt{x(m+x)}+m\log\{\sqrt{x}+\sqrt{m+x}\}.$$

7. Integrating by 'parts,'

$$\int x\tan^{-1}x\,dx=\frac{x^2}{2}\tan^{-1}x-\int dx\,.\,\frac{x^2}{2}\cdot\frac{1}{1+x^2}=\frac{x^2}{2}\tan^{-1}x-\tfrac{1}{2}\int dx\left(1-\frac{1}{1+x^2}\right)$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\tfrac{1}{2}\tan^{-1}x.$$

8. $(1-\cos x)^2=1-2\cos x+\tfrac{1}{2}(1+\cos 2x)=\tfrac{3}{2}-2\cos x+\tfrac{1}{2}\cos 2x,$

$$\therefore\ \text{the integral}=\frac{3x}{2}-2\sin x+\frac{\sin 2x}{4}.$$

9. $$\int\frac{x\,dx}{(1-x)^3}=\int dx\left\{\frac{x-1+1}{(1-x)^3}\right\}=\int dx\left(\frac{1}{(1-x)^3}-\frac{1}{(1-x)^2}\right)=\frac{1}{2(1-x)^2}-\frac{1}{1-x}.$$

10. $$\int\frac{x^2dx}{a^6-x^6}=\int\frac{x^2dx}{2a^3}\left(\frac{1}{a^3-x^3}+\frac{1}{a^3+x^3}\right)$$

$$=\frac{1}{6a^3}\int dx^3\left(\frac{1}{a^3-x^3}+\frac{1}{a^3+x^3}\right)=\frac{1}{6a^3}\cdot\log\frac{a^3+x^3}{a^3-x^3}.$$

11. $\int\sqrt{2ax-x^2}\,dx=\int dx\,.\,\sqrt{(a^2-\overline{x-a}|^2)}=$ etc. (Cf. Art. 14, Ex. 1.)

12. $$\int\frac{x\,dx}{\sqrt{2ax-x^2}}=\int\frac{dx}{\sqrt{2ax-x^2}}(\overline{x-a}+a)=-\sqrt{2ax-x^2}+a\int\frac{dx}{\sqrt{2ax-x^2}},\text{ and}$$

the latter integral $=a\int\frac{dx}{\sqrt{a^2-(x-a)^2}}=a\sin^{-1}\frac{x-a}{a}=a\left\{\frac{\pi}{2}-\cos^{-1}\frac{x-a}{a}\right\}$

$$=a\left\{\cos^{-1}\frac{a-x}{a}-\frac{\pi}{2}\right\}=a\cos^{-1}\left(1-\frac{x}{a}\right)=a\,\text{vers}^{-1}\frac{x}{a};$$

$\therefore$ etc., omitting the constant.

13. The given integral $=\int\frac{dx}{\{(x-3)^2+4\}^{\frac{1}{2}}}$,

and $\therefore$ by Art. 14, Ex. 2 it $=\log\{x-3+\sqrt{x^2-6x+13}\}$.

14. If $x+\sin x=z,\ \frac{dz}{dx}=1+\cos x,$

$$\therefore\int\frac{1+\cos x}{x+\sin x}dx=\int\frac{dz}{z}=\log z=\log(x+\sin x).$$

15. $$\frac{x+\sin x}{1+\cos x}=\frac{x}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2},$$

$$\therefore\ \text{the integral}=x\tan\frac{x}{2}-\int dx\,.\,\tan\frac{x}{2}+\int dx\,.\,\tan\frac{x}{2}=x\tan\frac{x}{2}.$$

16. $$\int\frac{dx}{x(\log x)^n}=\int\frac{d(\log x)}{(\log x)^n}=-\frac{1}{n-1}\cdot\frac{1}{(\log x)^{n-1}}.$$

17. $$\int \log(\log x)\frac{dx}{x} = \int \log z \,.\, dz \quad \text{if } \log x = z$$
$$= z\log z - \int dz \,.\, \frac{z}{z} \quad \text{integrating by parts}$$
$$= z(\log z - 1) = \log x \,.\, \log\log x - \log x.$$

18. $$\int \frac{dx}{x+\sqrt{x^2-1}} = \int dx(x-\sqrt{x^2-1}) = \frac{x^2}{2} - \frac{x}{2}\sqrt{x^2-1} + \tfrac{1}{2}\log\{x+\sqrt{x^2-1}\}.$$
Cf. Art. 14, Ex. 4.

19. If $x - 1 = z^2$,
$$\int \frac{x^3 dx}{(x-1)^{\frac{1}{2}}} = \int \frac{2zdz}{z}(z^2+1)^3 = 2\int dz(z^6+3z^4+3z^2+1)$$
$$= 2\left(\frac{z^7}{7} + \frac{3}{5}z^5 + z^3 + z\right) = 2\sqrt{x-1}\left\{\frac{(x-1)^3}{7} + \frac{3}{5}(x-1)^2 + x - 1 + 1\right\}, \text{etc.}$$

20. Since $\sin mx \cos nx = \frac{1}{2}\{\sin(m+n)x + \sin(m-n)x\}$, as in Art. 12, $\int e^{ax} . \sin m \cos nx dx = \frac{e^{ax}}{2}\left\{\frac{a\sin(m+n)x - (m+n)\cos(m+n)x}{a^2+(m+n)^2}\right\}$ + a similar expression changing the sign of n.

21. So by the method of Art. 12, $\frac{1}{4}\int e^{-x}\cos 3x dx + \frac{3}{4}\int e^{-x}\cos x dx$
$$= \frac{e^{-x}}{4}(-\cos 3x + 3\sin 3x) \div (1+9) + \frac{3e^{-x}}{4}\,\frac{(-\cos x + \sin x)}{2}, \text{ etc.}$$
Aliter: apply the result of Ex. 16, Chap. IV.

22. $\int_0^a \sqrt{a^2-x^2} \,.\, dx$: if $x = a\sin\theta$, the lts. of θ are 0 and $\frac{\pi}{2}$ (cf. Art. 46), and the integral becomes
$$\int_0^{\frac{\pi}{2}} a\cos\theta \,.\, a\cos\theta d\theta = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} d\theta(1+\cos 2\theta) = \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big/_0^{\frac{\pi}{2}}$$
$$= \frac{a^2}{2}\left(\frac{\pi}{2} + 0 - 0 - 0\right) = \frac{\pi a^2}{4}.$$
Cf. also Art. 35. The result may also be simply obtained by regarding the integral as giving the area of a quadrant of a circle of radius a (Art. 129), or be deduced from Art. 14, Ex. 1.

23. If $x - a = a\sin\theta$,
$$\int_0^{2a} \sqrt{2ax-x^2} \,.\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos^2\theta \,.\, d\theta = a^2\int_0^{\frac{\pi}{2}} d\theta\,(1+\cos 2\theta) = \frac{\pi a^2}{2}, \quad \text{as in Ex. 22;}$$
or, again, the integral is the area of a semicircle of radius a.

Aliter: as in Art. 14, Ex. 4, $\because\ 2ax - x^2 = a^2 - (x-a)^2$,

$$\int_0^{2a} \sqrt{2ax - x^2} \,.\, dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}\Big|_0^{2a}$$

$$= \frac{a^2}{2}\{\sin^{-1}(1) - \sin^{-1}(-1)\} = \frac{a^2}{2}\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = \frac{\pi a^2}{2}.$$

24. As in Ex. 4, $\int_0^\pi a\theta \sin\theta d\theta = a(-\theta\cos\theta + \sin\theta)\Big|_0^\pi = a\,.\,\pi.$

25. If $\text{vers}^{-1}\frac{x}{a} = \theta,\ x = a(1-\cos\theta),\quad \therefore\ dx = a\sin\theta d\theta$, and

$$\int_0^{2a} x\,\text{vers}^{-1}\frac{x}{a}dx = \int_0^\pi a^2 \sin\theta d\theta \,.\, \theta(1-\cos\theta) = a^2\int_0^\pi d\theta\left(\theta\sin\theta - \frac{\theta}{2}\sin 2\theta\right)$$

$$= a^2\left\{-\theta\cos\theta + \sin\theta + \frac{\theta}{4}\cos 2\theta - \int\frac{\cos 2\theta}{4}d\theta\right\}\Big|_0^\pi$$

$$= a^2\left\{-\theta\cos\theta + \sin\theta + \frac{\theta}{4}\cos 2\theta - \frac{\sin 2\theta}{8}\right\}\Big|_0^\pi$$

$$= a^2\left\{\pi + \frac{\pi}{4}\right\} = \frac{5\pi a^2}{4}.$$

26. Putting $x = a(1-\cos\theta)$,

$$\int_0^{2a} x^2\text{vers}^{-1}\frac{x}{a}dx \text{ becomes } \int_0^\pi a^3\sin\theta d\theta \,.\, \theta(1-\cos\theta)^2$$

$$= a^3\int_0^\pi d\theta(\theta\sin\theta - \theta\sin 2\theta + \theta\sin\theta\cos^2\theta)$$

$$= a^3\left\{-\theta\cos\theta + \int\cos\theta d\theta + \frac{\theta}{2}\cos 2\theta - \int\frac{d\theta}{2}\cos 2\theta - \frac{\theta\cos^3\theta}{3} + \int\frac{d\theta}{3}\left(\frac{\cos 3\theta + 3\cos\theta}{4}\right)\right\}\Big|_0^\pi$$

$$= a^3\left\{-\theta\cos\theta + \sin\theta + \frac{\theta}{2}\cos 2\theta - \frac{\sin 2\theta}{4} - \frac{\theta\cos^3\theta}{3} + \frac{\sin 3\theta}{36} + \frac{\sin\theta}{4}\right\}\Big|_0^\pi$$

$$= a^3\left(\pi + \frac{\pi}{2} + \frac{\pi}{3}\right) = \frac{11\pi a^3}{6}.$$

27. $\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^3\theta d\theta = \int_0^{\frac{\pi}{2}}\cos\theta d\theta(\sin^2\theta - \sin^4\theta) = \frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5}\Big|_0^{\frac{\pi}{2}}$

$$= \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.$$

28. $$\int\frac{dx}{\sin x+\cos x}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sin\left(x+\frac{\pi}{4}\right)}=\frac{1}{\sqrt{2}}\int\frac{dx\left\{\sin^2\left(\frac{x}{2}+\frac{\pi}{8}\right)+\cos^2\left(\frac{x}{2}+\frac{\pi}{8}\right)\right\}}{2\sin\left(\frac{x}{2}+\frac{\pi}{8}\right)\cos\left(\frac{x}{2}+\frac{\pi}{8}\right)}$$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{2}\left\{\frac{\sin\overline{\frac{x}{2}+\frac{\pi}{8}}}{\cos\left(\frac{x}{2}+\frac{\pi}{8}\right)}+\frac{\cos\overline{\frac{x}{2}+\frac{\pi}{8}}}{\sin\left(\frac{x}{2}+\frac{\pi}{8}\right)}\right\}$$

$$=\frac{1}{\sqrt{2}}\left\{\log\sin\left(\frac{x}{2}+\frac{\pi}{8}\right)-\log\cos\left(\frac{x}{2}+\frac{\pi}{8}\right)\right\}$$

$$=\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right).$$

29. If $x=\frac{1}{y}$,

$$\int\frac{dx}{x\sqrt{(a+bx+cx^2)}}=-\int\frac{1}{y^2}.dy.\frac{y^2}{\sqrt{(ay^2+by+c)}}=-\int\frac{dy}{\sqrt{(ay^2+by+c)}}$$

$$=-\frac{1}{\sqrt{a}}\log\{2ay+b+2\sqrt{a}\sqrt{(ay^2+by+c)}\}$$ (as in Art. 14, Ex. 5) and ∴

$$=-\frac{1}{\sqrt{a}}\log\{2a+bx+2\sqrt{a}\sqrt{(a+bx+cx^2)}\}+\frac{1}{\sqrt{a}}\log x.$$

Here a is positive; if a be negative the integral will involve not a log. but the inverse of a sine.

30. If $\sin^{-1}x=\theta$, $dx=\cos\theta d\theta$,

and ∴ $$\int\frac{\sqrt{1-x^2}}{x^4}\sin^{-1}x dx=\int\frac{\cos^2\theta d\theta\,.\,\theta}{\sin^4\theta}=\int\theta d\theta\,.\cot^2\theta\,\text{cosec}^2\theta$$

$$=-\frac{\theta}{3}\cot^3\theta+\int\frac{d\theta}{3}\cot^3\theta$$

$$=-\frac{\theta}{3}\cot^3\theta+\frac{1}{3}\int\cos\theta d\theta\left(\frac{1}{\sin^3\theta}-\frac{1}{\sin\theta}\right)$$

$$=-\frac{\theta}{3}\cot^3\theta-\frac{1}{6\sin^2\theta}-\frac{1}{3}\log\sin\theta$$

$$=-\frac{\sin^{-1}x(1-x^2)^{\frac{3}{2}}}{3x^3}-\frac{1}{6x^2}-\frac{1}{3}\log x.$$

31. If $\sin\theta=x$,

$$\int\frac{\sin^{-1}x dx}{(1-x^2)^{\frac{3}{2}}}=\int\frac{\theta\,.\cos\theta d\theta}{\cos^3\theta}=\int\theta d\theta\sec^2\theta=\theta\tan\theta-\int d\theta\tan\theta,$$

(integrating by parts) $$=\theta\tan\theta+\log\cos\theta,\quad\because\tan\theta=\frac{\sin\theta}{\cos\theta}.$$

32. If $x = a\cos\theta$,

$$\int \frac{dx}{(a^2-x^2)^{\frac{5}{2}}} = -\int \frac{a\sin\theta d\theta}{a^5\sin^5\theta} = -\frac{1}{a^4}\int \frac{d\theta}{\sin^4\theta} = -\frac{1}{a^4}\int d\theta \operatorname{cosec}^2\theta(1+\cot^2\theta)$$

$$= \frac{1}{a^4}\left(\cot\theta + \frac{\cot^3\theta}{3}\right).$$

33. $$\int \frac{\sin^2 x dx}{a+b\cos^2 x} = \int dx\left(-\frac{1}{b} + \frac{1+\frac{a}{b}}{a+b\cos^2 x}\right) = -\frac{x}{b} + \frac{a+b}{b}\int \frac{dx\sec^2 x}{a+b+a\tan^2 x}$$

$$= -\frac{x}{b} + \frac{a+b}{ab}\int \frac{dx\sec^2 x}{\frac{a+b}{a}+\tan^2 x}$$

$$= -\frac{x}{b} + \frac{a+b}{ab}\left(\frac{a}{a+b}\right)^{\frac{1}{2}}\tan^{-1}.\frac{\tan x . a^{\frac{1}{2}}}{\sqrt{a+b}}$$

$$= -\frac{x}{b} + \left(\frac{a+b}{ab^2}\right)^{\frac{1}{2}}\tan^{-1}\frac{\sqrt{a}\tan x}{\sqrt{a+b}}.$$

34. Integrating by parts,

$$\int x^3(a+bx^2)^{\frac{1}{2}}dx = x^2\frac{1}{3b}(a+bx^2)^{\frac{3}{2}} - \frac{1}{3b}\int 2x(a+bx^2)^{\frac{3}{2}}dx$$

$$= \frac{x^2}{3b}(a+bx^2)^{\frac{3}{2}} - \frac{2}{15b^2}(a+bx^2)^{\frac{5}{2}}$$

$$= (a+bx^2)^{\frac{5}{2}}\left(\frac{1}{3b^2} - \frac{2}{15b^2}\right) - \frac{a}{3b^2}(a+bx^2)^{\frac{3}{2}}$$

$$= (a+bx^2)^{\frac{3}{2}}\left(\frac{a+bx^2}{5b^2} - \frac{a}{3b^2}\right).$$

35. If $x = \tan\theta$,

$$\int \frac{dx}{x^4\sqrt{1+x^2}} = \int \frac{\sec^2\theta d\theta}{\tan^4\theta\sec\theta} = \int \frac{d\theta\cos^3\theta}{\sin^4\theta} = \int \cos\theta d\theta\left(\frac{1}{\sin^4\theta} - \frac{1}{\sin^2\theta}\right)$$

$$= -\frac{1}{3\sin^3\theta} + \frac{1}{\sin\theta},$$ or, $\because$ $\sin\theta = \frac{x}{\sqrt{1+x^2}}$,

the integral $= \frac{(1+x^2)^{\frac{3}{2}}}{3x^3}\left\{\frac{3x^2}{1+x^2} - 1\right\} = \frac{(2x^2-1)}{3x^3}\sqrt{1+x^2}.$

36. If $x = \tan\theta$, $\frac{d\theta}{dx} = \frac{1}{\sec^2\theta} = \frac{1}{1+x^2}$, and $\therefore$

$$\int \tan^{2n}\theta d\theta = \int \tan^{2n-2}\theta d\theta(\sec^2\theta - 1) = \frac{\tan^{2n-1}\theta}{2n-1} - \int \tan^{2n-2}\theta d\theta \text{ and } = \int \frac{x^{2n}dx}{1+x^2}$$

$$= \int \frac{dx}{1+x^2}\{(x^{2n}+x^{2n-2}) - (x^{2n-2}+x^{2n-4}) + \ldots - (-1)^n(x^2+1) + (-1)^n 1\}$$

$$= \frac{x^{2n-1}}{2n-1} - \frac{x^{2n-3}}{2n-3} + \ldots - (-1)^n x + (-1)^n\tan^{-1}x.$$

37. $\int_0^\pi \sin mx \sin nx dx = \frac{1}{2}\int_0^\pi dx\{\cos \overline{n-m}|x - \cos\overline{n+m}|x\} = u$, say; $\therefore$ if m, n are unequal integers, $u = \frac{\sin(n-m)x}{2(n-m)} - \frac{\sin(n+m)x}{2(n+m)}\Big|_0^\pi = 0$;

and so $\int_0^\pi \cos mx \cos nx dx = \frac{1}{2}\int_0^\pi dx\{\cos\overline{n-m}|x + \cos\overline{n+m}|x\} = 0$:

but if $n = m$ and they are integral,

$$\int_0^\pi \sin mx \sin nx dx = \frac{1}{2}\int_0^\pi dx(1-\cos 2mx) = \frac{1}{2}\left\{x - \frac{\sin 2mx}{2m}\right\}\Big|_0^\pi = \frac{\pi}{2}:$$

and so $\int_0^\pi \cos mx \cos nx dx = \frac{1}{2}\int_0^\pi dx(1+\cos 2mx) = \frac{\pi}{2}$.

38. Integrating by parts,

$$\int\left(\log\frac{x}{a}\right)^3 dx = x\left(\log\frac{x}{a}\right)^3 - \int dx\,.\,x\,.\,3\left(\log\frac{x}{a}\right)^2.\,\frac{1}{x} = x\left(\log\frac{x}{a}\right)^3 - 3\int dx\,.\log\left(\frac{x}{a}\right)^2$$

$$= x\left(\log\frac{x}{a}\right)^3 - 3x\left(\log\frac{x}{a}\right)^2 + 3\int xdx\,.\,2\left(\log\frac{x}{a}\right).\,\frac{1}{x}$$

$$= x\left(\log\frac{x}{a}\right)^3 - 3x\left(\log\frac{x}{a}\right)^2 + 6x\left(\log\frac{x}{a}\right) - 6\int xdx\,.\,\frac{1}{x}, \text{ etc.}$$

39. When $\cot\theta = x$,

$$\int\frac{\cot^{-1}xdx}{x^2(1+x^2)} = -\int\frac{\theta\operatorname{cosec}^2\theta d\theta}{\cot^2\theta\operatorname{cosec}^2\theta} = -\int\theta d\theta\tan^2\theta = -\int\theta d\theta(\sec^2\theta - 1)$$

$$= -\theta(\tan\theta - \theta) + \int d\theta(\tan\theta - \theta) = -\theta(\tan\theta - \theta) - \frac{\theta^2}{2} - \log\cos\theta$$

$$= \frac{\theta^2}{2} - \theta\tan\theta - \log\cos\theta.$$

40. $$\int dx\frac{2a+x}{a+x}\sqrt{\frac{a-x}{a+x}} = \int\frac{dx}{\sqrt{a^2-x^2}}\left(1+\frac{a}{a+x}\right)(a-x)$$

$$= \int\frac{dx}{\sqrt{a^2-x^2}}\left(a-x-a+\frac{2a^2}{a+x}\right)$$

$$= \sqrt{a^2-x^2} + \int\frac{2a^2dx}{(a+x)^{\frac{3}{2}}(a-x)^{\frac{1}{2}}},$$

and the latter integral $= a\int\frac{dx(a+x+a-x)}{(a+x)^{\frac{3}{2}}(a-x)^{\frac{1}{2}}}$

$$= \int\frac{adx}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{1}{2}}} + \int\frac{adx(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{3}{2}}}$$

$$= \int\frac{adx(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{3}{2}}} - 2a\sqrt{\frac{a-x}{a+x}} - a\int\frac{dx(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{3}{2}}}$$

$$= -2a\sqrt{\frac{a-x}{a+x}}, \text{ and } \therefore \text{ etc.}$$

41. Here $\dfrac{d}{dx}\left(\text{vers}^{-1}\dfrac{x}{a}\right) = \dfrac{d}{dx}\cos^{-1}\left(1-\dfrac{x}{a}\right) = -\dfrac{1}{a}\cdot\dfrac{-1}{\sqrt{1-\left(1-\dfrac{x}{a}\right)^2}}$

$$=\frac{1}{\sqrt{2ax-x^2}},$$

and $\therefore$ the given integral $=\displaystyle\int \text{vers}^{-1}\frac{x}{a}\cdot\frac{d}{dx}\left(\text{vers}^{-1}\frac{x}{a}\right)\cdot dx = \frac{1}{2}\left(\text{vers}^{-1}\frac{x}{a}\right)^2.$

Aliter: if $\text{vers}^{-1}\dfrac{x}{a}=\theta,\ x=a(1-\cos\theta)$,

and $$\int\frac{\text{vers}^{-1}\frac{x}{a}dx}{\sqrt{2ax-x^2}}=\int\frac{a\theta\sin\theta d\theta}{a\{(1-\cos\theta)(1+\cos\theta)\}^{\frac{1}{2}}}=\int\theta d\theta=\frac{\theta^2}{2},\text{ etc.}$$

42. $$\int\frac{dx}{1+c\cos x}=\int\frac{dx}{(1+c)\cos^2\frac{x}{2}+(1-c)\sin^2\frac{x}{2}}=\int\frac{2\frac{dx}{2}\sec^2\frac{x}{2}}{1+c+(1-c)\tan^2\frac{x}{2}}$$

$$=\frac{2}{1-c}\int\frac{\frac{dx}{2}\sec^2\frac{x}{2}}{\frac{1+c}{1-c}+\tan^2\frac{x}{2}}=\frac{2}{\sqrt{1-c^2}}\tan^{-1}\left\{\sqrt{\frac{1-c}{1+c}}\tan\frac{x}{2}\right\},\text{ if}$$

$c<1$; $\therefore \displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{1+c\cos x}=\frac{2}{\sqrt{1-c^2}}\tan^{-1}\sqrt{\frac{1-c}{1+c}}=\frac{1}{\sqrt{1-c^2}}\tan^{-1}\frac{2\sqrt{1-c^2}}{1+c-(1-c)}$

$$=\frac{1}{\sqrt{1-c^2}}\tan^{-1}\frac{\sqrt{1-c^2}}{c}=\frac{1}{\sqrt{1-c^2}}\cos^{-1}c\text{ (cf. Art. 14, Ex. 14).}$$

43. $\displaystyle\int e^{-\theta}\cos^3\theta d\theta$ (as in Ex. 21)

$$=\frac{e^{-\theta}}{40}(3\sin 3\theta-\cos 3\theta)+\frac{3e^{-\theta}}{8}(\sin\theta-\cos\theta),$$

and $\therefore \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-\theta}\cos^3\theta d\theta=\frac{e^{-\frac{\pi}{2}}}{40}(-3)-\frac{e^{\frac{\pi}{2}}}{40}(3)+\frac{3}{8}e^{-\frac{\pi}{2}}(1)-\frac{3}{8}e^{\frac{\pi}{2}}(-1)$

$$=\left(e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}\right)\left(\frac{3}{8}-\frac{3}{40}\right)=\frac{3}{10}\left(e^{\frac{\pi}{2}}+e^{-\frac{\pi}{2}}\right).$$

44. If $z=x+\dfrac{1}{x},\quad dz=dx\left(1-\dfrac{1}{x^2}\right)=\dfrac{dx}{x}\left(x-\dfrac{1}{x}\right),$

and $$\int\frac{(x^2-1)dx}{x\sqrt{(1+3x^2+x^4)}}=\int\frac{dx\left(x-\frac{1}{x}\right)}{x\sqrt{\left(x^2+\frac{1}{x^2}+3\right)}}=\int\frac{dz}{\sqrt{(z^2+1)}}$$

$$=\log\{z+\sqrt{z^2+1}\}=\log\frac{x^2+1+\sqrt{(x^4+3x^2+1)}}{x}.$$

45. If $a+bx^n=z^4$, $nbx^{n-1}dx=4z^3dz$,

$$\text{and} \int\frac{(a+bx^n)^{\frac{3}{4}}dx}{x}=\int\frac{z^3.\,4z^3dz}{nbx^n}=4\int\frac{z^6dz}{n(z^4-a)}=\frac{4}{n}\int dz\frac{(z^6-az^2+az^2)}{z^4-a}$$

$$=\frac{4}{n}\int dz\,.\,z^2+\frac{2a}{n}\int dz\left(\frac{1}{z^2-\sqrt{a}}+\frac{1}{z^2+\sqrt{a}}\right.$$

$$=\frac{4z^3}{3n}+\frac{a^{\frac{3}{4}}}{n}\int dz\left(\frac{1}{z-a^{\frac{1}{4}}}-\frac{1}{z+a^{\frac{1}{4}}}\right)+\frac{2a}{n}\cdot\frac{1}{a^{\frac{1}{4}}}\tan^{-1}\frac{z}{a^{\frac{1}{4}}}$$

$$=\frac{4z^3}{3n}+\frac{a^{\frac{3}{4}}}{n}\log\frac{z-a^{\frac{1}{4}}}{z+a^{\frac{1}{4}}}+\frac{2a^{\frac{3}{4}}}{n}\tan^{-1}\frac{z}{a^{\frac{1}{4}}}$$

$$=\frac{4}{3n}(a+bx^n)^{\frac{3}{4}}+\frac{a^{\frac{3}{4}}}{n}\log\frac{(a+bx^n)^{\frac{1}{4}}-a^{\frac{1}{4}}}{(a+bx^n)^{\frac{1}{4}}+a^{\frac{1}{4}}}+\frac{2a^{\frac{3}{4}}}{n}\tan^{-1}\left(\frac{a+bx^n}{a}\right)^{\frac{1}{4}}.$$

CHAPTER II.

1. Let $\dfrac{1}{x^3-1}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+x+1}$,

then $1\equiv A(x^2+x+1)+Bx(x-1)+C(x-1)$, $\therefore$ putting $x=1$ and 0 successively, $1=3A$, and $-C=1-A=\frac{2}{3}$, and the coefficient of x^2 gives $B=-A=-\frac{1}{3}$,

$$\therefore \quad \int\frac{dx}{x^3-1}=\int\frac{dx}{3(x-1)}-\frac{1}{3}\int\frac{(x+2)dx}{x^2+x+1}=\int\frac{dx}{3(x-1)}-\frac{1}{6}\int\frac{dx(2x+1+3)}{x^2+x+1}$$

$$=\frac{1}{3}\log(x-1)-\frac{1}{6}\log(x^2+x+1)-\frac{1}{2}\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$=\frac{1}{6}\log\frac{(x-1)^2}{x^2+x+1}-\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

2. $\displaystyle\int\frac{x^2-1}{x^2-4}dx=\int dx\left\{1+\frac{3}{4}\left(\frac{1}{x-2}-\frac{1}{x+2}\right)\right\}=x+\frac{3}{4}\log\frac{x-2}{x+2}$, etc.

3. $$\frac{x^3}{x^2+7x+12}=x-\frac{7x^2+12x}{x^2+7x+12}=x-7+\frac{A}{x+3}+\frac{B}{x+4},\ \text{say},$$

$$\therefore \quad x^3\equiv x^3-37x-84+A(x+4)+B(x+3),$$

and putting $x=-3$ and -4 successively,

$$A=84-111=-27,\quad B=148-84=64,$$

$$\text{and }\therefore\quad \int\frac{x^3dx}{x^2+7x+12}=\frac{x^2}{2}-7x-27\log(x+3)+64\log(x+4).$$

4. $$\int \frac{dx}{a^4 - x^4} = \frac{1}{2a^2}\int dx\left(\frac{1}{a^2 - x^2} + \frac{1}{a^2 + x^2}\right)$$

$$= \frac{1}{2a^2}\int dx\left\{\frac{1}{a^2 + x^2} - \frac{1}{2a}\left(\frac{1}{x - a} - \frac{1}{x + a}\right)\right\}$$

$$= \frac{1}{2a^3}\tan^{-1}\frac{x}{a} - \frac{1}{4a^3}\log\frac{x-a}{x+a}, \quad \text{if } x > a,$$

and $$= \frac{1}{2a^3}\tan^{-1}\frac{x}{a} - \frac{1}{4a^3}\log\frac{a-x}{a+x}, \quad \text{if } x < a,$$

5. $$\int \frac{2x^2 - 3a^2}{x^4 - a^4}dx = 2\int\frac{dx}{x^2 + a^2} - \frac{1}{2}\int dx\left(\frac{1}{x^2 - a^2} - \frac{1}{x^2 + a^2}\right)$$

$$= \frac{5}{2a}\tan^{-1}\frac{x}{a} - \frac{1}{4a}\int dx\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$= \frac{5}{2a}\tan^{-1}\frac{x}{a} - \frac{1}{4a}\log\frac{x-a}{x+a}.$$

6. $$\int \frac{dx}{(x^2+1)(x^2+x+1)} = \int dx\left(\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+x+1}\right), \text{ say},$$

and ∴ $$1 \equiv (Ax+B)(x^2+x+1) + (Cx+D)(x^2+1);$$

the coefficients of x^3, x^2, x, and 1 give $A + C = 0$,

$$A + B + D = 0, \quad A + B + C = 0, \text{ and } 1 = B + D.$$

Hence $$C = D = -A, \quad \therefore B = 0, \quad \therefore D = 1 = C = -A,$$

and $$\text{the integral} = \int dx\left(\frac{-x}{x^2+1} + \frac{x+1}{x^2+x+1}\right)$$

$$= \frac{1}{2}\int dx\left(\frac{-2x}{x^2+1} + \frac{2x+1+1}{x^2+x+1}\right)$$

$$= -\frac{1}{2}\log\frac{x^2+1}{x^2+x+1} + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

7. $$\frac{x^2}{x^4+x^2-2} = \frac{1}{2}\cdot\frac{2x^2+1-1}{x^4+x^2-2} = \frac{1}{2}\left\{\frac{1}{x^2-1} + \frac{1}{x^2+2} - \frac{1}{3}\left(\frac{1}{x^2-1} - \frac{1}{x^2+2}\right)\right\},$$

∴ $$\int\frac{dx \, . \, x^2}{x^4+x^2-2} = \int dx\left\{\frac{1}{3}\cdot\frac{1}{x^2-1} + \frac{2}{3}\cdot\frac{1}{2+x^2}\right\}$$

$$= \int dx\left\{\frac{1}{6}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) + \frac{2}{3}\cdot\frac{1}{2+x^2}\right\}$$

$$= \frac{1}{6}\log\frac{x-1}{x+1} + \frac{2}{3}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}.$$

8. If $\dfrac{x^2-1}{x^4+x^2+1}=\dfrac{Ax+B}{x^2+x+1}+\dfrac{Cx+D}{x^2-x+1}$,

$$x^2-1\equiv(Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1);$$

and the coefficients of x^3, x^2, x, and 1 give $A+C=0$,

$$-A+B+C+D=1, \quad A-B+C+D=0, \quad \text{and } -1=B+D.$$

Hence $B=D=-\frac{1}{2}$, and $A+C=0$, $-A+C=2$,

$\therefore$ $C=1$, and $A=-1$,

and $$\int\frac{(x^2-1)dx}{x^4+x^2+1}=\int\frac{-(x+\frac{1}{2})dx}{x^2+x+1}+\int\frac{(x-\frac{1}{2})dx}{x^2-x+1}=\frac{1}{2}\log\frac{x^2-x+1}{x^2+x+1}.$$

Aliter, more simply, if $z=x+\dfrac{1}{x}$,

$$\int\frac{(x^2-1)dx}{x^4+x^2+1}=\int\frac{x^2dz}{x^4+x^2+1}=\int\frac{dz}{\left(x+\frac{1}{x}\right)^2-1}=\frac{1}{2}\log\frac{z-1}{z+1}, \text{ etc.}$$

9. $$\frac{x^2-3x+3}{(x-1)(x-2)}=1+\frac{1}{(x-1)(x-2)}=1+\frac{1}{x-2}-\frac{1}{x-1},$$

$\therefore$ $$\int\frac{x^2-3x+3}{(x-1)(x-2)}dx=x+\log\frac{x-2}{x-1}.$$

10. If $\dfrac{3x-1}{x(x-2)(x+1)}=\dfrac{A}{x}+\dfrac{B}{x-2}+\dfrac{C}{x+1}$,

then $3x-1\equiv A(x-2)(x+1)+Bx(x+1)+Cx(x-2)$,

$\therefore$ putting $x=0$, 2, and -1 in succession,

$$-1=-2A \text{ or } A=\tfrac{1}{2}, \quad 5=6B \text{ or } B=\tfrac{5}{6},$$

and $-4=-C(-3)$ or $C=-\frac{4}{3}$;

$\therefore$ $$\int\frac{(3x-1)dx}{x^3-x^2-2x}=\frac{1}{2}\log x+\frac{5}{6}\log(x-2)-\frac{4}{3}\log(x+1).$$

11. If $\dfrac{1}{(x^2+a^2)(x+b)}=\dfrac{Ax+B}{x^2+a^2}+\dfrac{C}{x+b}$,

$$1\equiv(Ax+B)(x+b)+C(x^2+a^2),$$

$\therefore$ if $x=-b$, $1=C(a^2+b^2)$, and the coefficients of x^2 and x give

$$A+C=0 \text{ or } A=-C=-\frac{1}{a^2+b^2}; \text{ and } Ab+B=0 \text{ or } B=\frac{b}{a^2+b^2};$$

$\therefore$ $$\int\frac{dx}{(x^2+a^2)(x+b)}=\int dx\left\{-\frac{x-b}{x^2+a^2}+\frac{1}{x+b}\right\}\frac{1}{a^2+b^2}$$

$$=\frac{1}{a^2+b^2}\int dx\left\{\frac{1}{x+b}-\frac{1}{2}\left(\frac{2x}{x^2+a^2}\right)+\frac{b}{x^2+a^2}\right\}$$

$$=\frac{1}{a^2+b^2}\left\{\log(x+b)-\log\sqrt{x^2+a^2}+\frac{b}{a}\tan^{-1}\frac{x}{a}\right\}, \text{ etc.}$$

12. If $\frac{1}{x(1+x+x^2+x^3)}=\frac{1}{x(1+x)(1+x^2)}=\frac{A}{x}+\frac{B}{1+x}+\frac{Cx+D}{1+x^2}$,

then $$1\equiv A(1+x)(1+x^2)+Bx(1+x^2)+(Cx+D)x(1+x),$$

and putting $x=0$ and -1 in succession,

$A=1$, and $1=-2B$ or $B=-\frac{1}{2}$; also the coefficients of x^2 and x give

$$A+C+D=0 \text{ or } C+D=-1,$$

and $$A+B+D=0, \quad \therefore\ D=-\tfrac{1}{2} \text{ and } C=-\tfrac{1}{2}.$$

Hence $$\int\frac{dx}{x(1+x+x^2+x^3)}=\log x-\tfrac{1}{2}\log(1+x)-\tfrac{1}{2}\int\frac{(x+1)dx}{x^2+1},$$

and $$\text{the last integral}=-\tfrac{1}{4}\int dx\left(\frac{2x}{x^2+1}+\frac{2}{x^2+1}\right)$$

$$=-\tfrac{1}{4}\log(x^2+1)-\tfrac{1}{2}\tan^{-1}x, \quad \therefore \text{ etc.}$$

13. If $\frac{1}{(x-1)^2(x^2+1)^2}=\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{Cx+D}{(x^2+1)^2}+\frac{Ex+F}{x^2+1}$,

$$1\equiv A(x^2+1)^2+B(x-1)(x^2+1)^2+(Cx+D)(x-1)^2$$
$$+(Ex+F)(x-1)^2(x^2+1);\ \ldots\ldots\ldots\ldots(1)$$

and putting $x=1$, $\quad 1=4A$;

$\therefore$ $$0=A(x^2+3)(x^2-1)+B(x-1)(x^2+1)^2+(Cx+D)\overline{x-1}|^2$$
$$+(Ex+F)(x-1)^2(x^2+1),$$

$\therefore$ $$0=A(x^2+3)(x+1)+B(x^2+1)^2+(Cx+D)(x-1)$$
$$+(Ex+F)(x-1)(x^2+1),$$

and putting $x=1$, $\quad 0=A\,.\,8+B\,.\,4;\ \therefore\ B=-2A=-\frac{1}{2}$;

also the coefficients of x^5, x^4, x^3, and x^0 in (1) give

$$0=B+E \qquad \text{or } E=\tfrac{1}{2},$$
$$0=A-B-2E+F. \qquad \text{or } F=\tfrac{1}{4},$$
$$0=2B+C+E\,.\,2-2F \text{ or } C=1-1+\tfrac{1}{2}=\tfrac{1}{2},$$

and $$1=A-B+D+F \qquad \text{or } D=1-\tfrac{1}{4}-\tfrac{1}{2}-\tfrac{1}{4}=0;$$

and $\therefore$ $$\int\frac{dx}{(x-1)^2(x^2+1)^2}=-\frac{1}{4(x-1)}-\frac{1}{2}\log(x-1)+\int\frac{dx\cdot\frac{x}{2}}{(x^2+1)^2}+\frac{1}{2}\int\frac{dx\left(x+\frac{1}{2}\right)}{x^2+1}$$

$$=-\frac{1}{4(x-1)}-\frac{1}{2}\log(x-1)-\frac{1}{4}\cdot\frac{1}{x^2+1}+\frac{1}{4}\log(x^2+1)$$
$$+\frac{1}{4}\tan^{-1}x.$$

14. If $\frac{x}{(1+x)(1+2x)^2(1+x^2)}=\frac{A}{1+x}+\frac{B}{(1+2x)^2}+\frac{C}{1+2x}+\frac{Dx+E}{1+x^2}$,

then $$x\equiv A(1+2x)^2(1+x^2)+B(1+x)(1+x^2)$$
$$+(Dx+E)(1+x)(1+2x)^2+C(1+x)(1+2x)(1+x^2),$$

and putting $x=-1$, $-1=2A$,

and putting $x=-\frac{1}{2}$, $-\frac{1}{2}=\frac{B}{2}\cdot\frac{5}{4}$ or $B=-\frac{4}{5}$; also the coefficients of x^4, x^3, and x^0 give

$$0=4A+4D+2C \text{ or } 2D+C=1,$$

$$0=4A+B+8D+4E+3C \text{ or } 8D+4E+3C=2\tfrac{4}{5},$$

and $\quad A+B+E+C=0$ or $E+C=\frac{1}{2}+\frac{4}{5}=\frac{13}{10}$, $\therefore 4E+4C=\frac{26}{5}$,

and $\therefore \quad 8D-C=-\frac{12}{5}$, $\qquad \therefore \quad 10D=-\frac{7}{5}$ or $D=-\frac{7}{50}$,

and $\therefore \quad C=1-2D=1+\frac{7}{25}=\frac{32}{25}$, $\quad$ and $\quad E=\frac{13}{10}-\frac{32}{25}=\frac{1}{50}$,

and $\therefore$ the given integral $=\int dx\left\{-\frac{1}{2(1+x)}-\frac{4}{5(1+2x)^2}+\frac{32}{25}\cdot\frac{1}{1+2x}-\frac{\frac{7}{50}x-\frac{1}{50}}{1+x^2}\right\}$

$$=-\tfrac{1}{2}\log(1+x)+\tfrac{2}{5}\cdot\frac{1}{1+2x}+\tfrac{16}{25}\log(1+2x)$$

$$-\tfrac{7}{100}\log(1+x^2)+\tfrac{1}{50}\tan^{-1}x.$$

15. If $\frac{x^2}{x^4+1}\equiv\frac{Ax+B}{x^2-x\sqrt{2}+1}+\frac{Cx+D}{x^2+x\sqrt{2}+1}$, then equating coefficients of like powers of x in $x^2=(Ax+B)(x^2+x\sqrt{2}+1)+(Cx+D)(x^2-x\sqrt{2}+1)$,

$$0=A+C, \qquad 1=A\sqrt{2}+B-C\sqrt{2}+D,$$

$$0=A+B\sqrt{2}+C-D\sqrt{2}, \qquad 0=B+D.$$

Hence $\quad B+D=0=B-D, \qquad \therefore B=0=D,$

and $\quad A-C=\frac{1}{\sqrt{2}}, \qquad \therefore A=\frac{1}{2\sqrt{2}}$ and $C=-\frac{1}{2\sqrt{2}}$.

Thus $\quad \int\frac{x^2dx}{x^4+1}=\frac{1}{2\sqrt{2}}\int dx\left(\frac{x}{x^2-x\sqrt{2}+1}-\frac{x}{x^2+x\sqrt{2}+1}\right)$

$$=\frac{1}{4\sqrt{2}}\int dx\left(\frac{2x-\sqrt{2}+\sqrt{2}}{x^2-x\sqrt{2}+1}-\frac{2x+\sqrt{2}-\sqrt{2}}{x^2+x\sqrt{2}+1}\right)$$

$$=\frac{1}{4\sqrt{2}}\log\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}+\frac{1}{4}\int dx\left\{\frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}+\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\right\}$$

$$=\frac{1}{4\sqrt{2}}\log\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}+\frac{\sqrt{2}}{4}\{\tan^{-1}(x\sqrt{2}-1)+\tan^{-1}(x\sqrt{2}+1)\}.$$

16. If $\frac{x^3}{x^6+1}\equiv\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2-x\sqrt{3}+1}+\frac{Ex+F}{x^2+x\sqrt{3}+1}$,

equating coefficients of like powers of x in

$$x^3=(Ax+B)(x^4-x^2+1)+(Cx+D)(x^2+1)(x^2+x\sqrt{3}+1)$$

$$+(Ex+F)(x^2+1)(x^2-x\sqrt{3}+1),$$

$$0=A+C+E, \qquad 0=B+C\sqrt{3}+D-E\sqrt{3}+F,$$

$$1=-A+2C+D\sqrt{3}+2E-F\sqrt{3}, \quad 0=-B+C\sqrt{3}+2D-E\sqrt{3}+2F,$$

$$0 = A + C + D\sqrt{3} + E - F\sqrt{3}, \text{ and } 0 = B + D + F.$$

From the first and fifth of these equations

$$D = F = -\frac{B}{2} \text{ by the sixth equation,}$$

and from the second and sixth equations

$$C = E = -\frac{A}{2} \text{ by the first equation;}$$

$\therefore$ by the third and fourth equations

$$1 = -A - 2A \text{ or } A = -\tfrac{1}{3}, \text{ and } 0 = -B - 2B \text{ or } B = 0 = D = F.$$

Hence
$$\int \frac{x^3 dx}{x^6+1} = \int dx \left\{ -\frac{1}{3} \cdot \frac{x}{x^2+1} + \frac{1}{6} \cdot \frac{x}{x^2 - x\sqrt{3}+1} + \frac{1}{6} \cdot \frac{x}{x^2 + x\sqrt{3}+1} \right\}$$

$$= -\frac{1}{6} \log (x^2+1) + \frac{1}{12} \int dx \left(\frac{2x - \sqrt{3}}{x^2 - x\sqrt{3}+1} + \frac{2x+\sqrt{3}}{x^2+x\sqrt{3}+1} \right)$$

$$+ \frac{\sqrt{3}}{12} \int dx \left\{ \frac{1}{\left(x - \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4}} - \frac{1}{\left(x + \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4}} \right\}$$

$$= -\frac{1}{6} \log (x^2+1) + \frac{1}{12} \log (x^4 - x^2 + 1) + \frac{\sqrt{3}}{6} \{\tan^{-1}(2x - \sqrt{3})$$

$$- \tan^{-1}(2x + \sqrt{3})\}, \text{ etc.}$$

17. If $1 - y^3 = y^3 z^3$ or $\frac{1}{y^3} - 1 = z^3$, $-\frac{1}{y^4} \cdot dy = z^2 dz$, and

$$\int \frac{dy}{\sqrt[3]{(1-y^3)}} = -\int \frac{z^2 dz \cdot y^4}{yz} = -\int \frac{z dz}{1+z^3} = -\int dz \left(\frac{A}{1+z} + \frac{Bz+C}{1-z+z^2} \right), \text{ say,}$$

where $z \equiv A(1 - z + z^2) + (Bz + C)(1+z)$, and $\therefore$ putting $z = -1$, $A = -\frac{1}{3}$, and the coefficients of z^2 and z^0 give $A + B = 0$ or $B = \frac{1}{3}$, and $A + C = 0$ or $C = \frac{1}{3}$,

and $\therefore$
$$-\int \frac{z dz}{1+z^3} = \frac{1}{3} \int dz \left\{ \frac{1}{1+z} - \frac{z+1}{1-z+z^2} \right\}$$

$$= \frac{1}{3} \int dz \left\{ \frac{1}{1+z} - \frac{1}{2} \left(\frac{2z-1}{z^2 - z + 1} + \frac{3}{\overline{z - \frac{1}{2}}|^2 + \frac{3}{4}} \right) \right\}$$

$$= \frac{1}{3} \log (1+z) - \frac{1}{6} \log (z^2 - z + 1) - \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \frac{2z-1}{\sqrt{3}}$$

$$= \frac{1}{3} \log \left\{ 1 + \frac{\sqrt[3]{1-y^3}}{y} \right\} - \frac{1}{6} \log \left\{ \frac{1}{y^3 + y^2 . \sqrt[3]{1-y^3}} \right\}$$

$$- \frac{1}{\sqrt{3}} \tan^{-1} \frac{2\sqrt[3]{1-y^3} - y}{y\sqrt{3}}$$

$$= \frac{1}{2} \log \{ y + \sqrt[3]{1-y^3} \} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2\sqrt[3]{1-y^3} - y}{y\sqrt{3}}$$

18. If $y=\frac{x}{1+x}$, $x=\frac{y}{1-y}$, and the given integral becomes

$$\int\frac{ydy}{x}\cdot\frac{1}{(1-y)^2}\cdot\frac{1}{\sqrt[3]{(1+3x+3x^2)}}=\int\frac{ydy}{(1-y)^2x\sqrt[3]{\left(\frac{1}{1-y}\right)^3-\frac{y^3}{(1-y)^3}}}=\int\frac{dy}{\sqrt[3]{1-y^3}},$$

and $\therefore$ by Ex. 17 the proposed integral, replacing y by x,

$$=\frac{1}{2}\log\frac{x+\sqrt[3]{1+3x+3x^2}}{1+x}-\frac{1}{\sqrt{3}}\tan^{-1}\left\{\frac{2\sqrt[3]{1+3x+3x^2}-x}{x\sqrt{3}}\right\}.$$

CHAPTER III.

1. Integrating by parts

$$\int(a^2+x^2)^{\frac{n}{2}}.\,dx=x(a^2+x^2)^{\frac{n}{2}}-n\int dx\,.\,x^2(a^2+x^2)^{\frac{n}{2}-1}$$

$$=x(a^2+x^2)^{\frac{n}{2}}-n\int dx(a^2+x^2-a^2)(a^2+x^2)^{\frac{n}{2}-1}$$

$$=x(a^2+x^2)^{\frac{n}{2}}-n\int dx(a^2+x^2)^{\frac{n}{2}}+na^2\int dx(a^2+x^2)^{\frac{n}{2}-1};$$

$\therefore$ transposing and dividing by $n+1$

$$\int(a^2+x^2)^{\frac{n}{2}}dx=\frac{x}{n+1}(a^2+x^2)^{\frac{n}{2}}+\frac{na^2}{n+1}\int dx(a^2+x^2)^{\frac{n}{2}-1}.$$

2. $$\int x^m(2ax-x^2)^{\frac{1}{2}}dx=\int x^{m-1}(\overline{x-a}+a)(2ax-x^2)^{\frac{1}{2}}dx$$

$$=-\frac{x^{m-1}}{3}(2ax-x^2)^{\frac{3}{2}}+\frac{m-1}{3}\int dx\,.\,x^{m-2}(2ax-x^2)^{\frac{3}{2}}+a\int dx\,.\,x^{m-1}(2ax-x^2)^{\frac{1}{2}}$$

$$=-\frac{x^{m-1}}{3}(2ax-x^2)^{\frac{3}{2}}+\frac{m-1}{3}.2a\int dx.x^{m-1}(2ax\ \ x^2)^{\frac{1}{2}}-\frac{m-1}{3}\int dx.x^m(2ax-x^2)^{\frac{1}{2}}$$
$$+a\int dx\,.\,x^{m-1}(2ax-x^2)^{\frac{1}{2}},$$

$$\therefore\ \left(1+\frac{m-1}{3}\right)\int dx\,.\,x^m(2ax-x^2)^{\frac{1}{2}}$$

$$=-\frac{x^{m-1}}{3}(2ax-x^2)^{\frac{3}{2}}+\frac{a}{3}(2m-2+3)\int dx\,.\,x^{m-1}(2ax-x^2)^{\frac{1}{2}},$$

or $$\int x^m(2ax-x^2)^{\frac{1}{2}}dx=-\frac{x^{m-1}(2ax-x^2)^{\frac{3}{2}}}{m+2}+\frac{a(2m+1)}{m+2}\int x^{m-1}(2ax-x^2)^{\frac{1}{2}}dx.$$

3. $$\int x\sqrt{2ax-x^2}dx=\int(\overline{x-a}+a)\sqrt{2ax-x^2}dx$$

$$=-\tfrac{1}{3}(2ax-x^2)^{\frac{3}{2}}+a\int\sqrt{2ax-x^2}dx:$$

a simple case of Ex. 2.

4. By Ex. 3, $\int_0^{2a} x\sqrt{2ax-x^2}\,dx = -\frac{1}{3}(2ax-x^2)^{\frac{3}{2}}\Big|_0^{2a} + a\int_0^{2a}\sqrt{2ax-x^2}\,dx$

$$= a\int_0^{2a}\sqrt{2ax-x^2}\,dx = a\int_0^{2a}\{a^2-(x-a)^2\}^{\frac{1}{2}}dx,$$

$\therefore$ as in Art. 14, Ex. 1, the integral $= a\frac{x-a}{2}\sqrt{2ax-x^2}\Big|_0^{2a} + \frac{a^3}{2}\sin^{-1}\frac{x-a}{a}\Big|_0^{2a}$

$$= \frac{a^3}{2}\sin^{-1}\frac{x-a}{a}\Big|_0^{2a} = \frac{a^3}{2}\left(\frac{\pi}{2} - \overline{-\frac{\pi}{2}}\right) = \frac{\pi a^3}{2}.$$

5. $\int x^2\sqrt{2ax-x^2}\,dx = \int x(x-a+a)\sqrt{2ax-x^2}\,dx = -\frac{x}{3}(2ax-x^2)^{\frac{3}{2}}$

$$+\frac{1}{3}\int dx(2ax-x^2)^{\frac{3}{2}} + a\int x\sqrt{2ax-x^2}\,dx$$

$$= -\frac{x}{3}(2ax-x^2)^{\frac{3}{2}} + \frac{2a}{3}\int dx\,.\,x(2ax-x^2)^{\frac{1}{2}}$$

$$-\frac{1}{3}\int x^2(2ax-x^2)^{\frac{1}{2}}dx + a\int x\sqrt{2ax-x^2}\,.\,dx$$

$\therefore$ $\frac{4}{3}\int x^2\sqrt{2ax-x^2}\,dx = -\frac{x}{3}(2ax-x^2)^{\frac{3}{2}} + \frac{5}{3}a\int x\sqrt{2ax-x^2}\,dx$, etc.

6. By Ex. 5, $\int_0^{2a} x^2\sqrt{2ax-x^2}\,dx = -\frac{x}{4}(2ax-x^2)^{\frac{3}{2}}\Big|_0^{2a} + \frac{5a}{4}\int_0^{2a} x\sqrt{2ax-x^2}\,dx$

$$= \frac{5a}{4}\int_0^{2a} x\sqrt{2ax-x^2}\,dx,$$

and $\therefore$, as in Ex. 4, $= \frac{5a}{4}\cdot\frac{\pi a^3}{2} = \frac{5\pi a^4}{8}.$

Aliter: putting $x = a(1+\sin\theta)$, $dx = a\cos\theta\, d\theta$,

and $\int_0^{2a} x^2\sqrt{2ax-x^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^4(1+\sin\theta)^2\cos^2\theta\, d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^4 d\theta\{2\cos^2\theta - \cos^4\theta + 2\cos^2\theta\sin\theta\}$$

$$= -\frac{2a^4}{3}\cos^3\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 2a^4\int_0^{\frac{\pi}{2}} d\theta(2\cos^2\theta - \cos^4\theta),$$

and $\therefore$, by Art. 35, $= 2a^4\left\{2\,.\,\frac{\pi}{4} - \frac{3}{4\,.\,2}\,.\,\frac{\pi}{2}\right\} = \pi a^4\left(1-\frac{3}{8}\right) = \frac{5}{8}\pi a^4.$

7. In Ex. 2, putting $m=3$, 2, and 1 in succession, the result can be obtained, but practically the simplest way is to put $x = a(1+\sin\theta)$ when

the integral becomes

$$a^5\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta(1+\sin\theta)^3\cos^2\theta d\theta = 2a^5\int_0^{\frac{\pi}{2}} d\theta(\cos^2\theta + 3\sin^2\theta\cos^2\theta)$$

$$= 2a^5\int_0^{\frac{\pi}{2}} d\theta(4\cos^2\theta - 3\cos^4\theta)$$

$$= 2a^5 . \frac{\pi}{2}\left(4 . \frac{1}{2} - 3 . \frac{3}{4 . 2}\right) = \pi a^5 . \frac{7}{8},$$

assuming the results of Art. 35. Cf. also Arts. 41 and 42.

8. $$\int x^n(\log x)^m dx = \frac{x^{n+1}}{n+1}(\log x)^m - \int\frac{dx}{n+1} . x^{n+1} . m(\log x)^{m-1} . \frac{1}{x}$$

$$= \frac{x^{n+1}}{n+1}(\log x)^m - \frac{m}{n+1}\int x^n(\log x)^{m-1}dx.$$

9. $$\int x^n(\log x)^2 dx = \frac{x^{n+1}}{n+1}(\log x)^2 - \frac{2}{n+1}\int x^{n+1} . (\log x)\frac{1}{x} . dx$$

$$= \frac{x^{n+1}}{n+1}(\log x)^2 - \frac{2}{n+1}\int x^n(\log x)dx$$

$$= \frac{x^{n+1}}{n+1}(\log x)^2 - \frac{2}{(n+1)^2}x^{n+1}\log x + \frac{2}{(n+1)^2}\int x^{n+1}\frac{1}{x}dx$$

$$= \frac{x^{n+1}}{n+1}\left\{(\log x)^2 - 2\frac{\log x}{n+1} + \frac{2}{(n+1)^2}\right\}.$$

10. $$\int_0^{\frac{\pi}{4}}\sec^4\theta d\theta = \int_0^{\frac{\pi}{4}}\sec^2\theta(1+\tan^2\theta)d\theta = \tan\theta\Big/_0^{\frac{\pi}{4}} + \frac{\tan^3\theta}{3}\Big/_0^{\frac{\pi}{4}} = 1 + \frac{1}{3} = \frac{4}{3}.$$

11. $$\int_0^a \frac{x^2dx\sqrt{a-x}}{\sqrt{a+x}} = \int_0^a\frac{xdx(ax-x^2)}{\sqrt{a^2-x^2}}$$

$$= -(ax-x^2)\sqrt{a^2-x^2}\Big/_0^a + \int_0^a dx\sqrt{a^2-x^2}(a-2x)$$

$$= a\int_0^a\sqrt{a^2-x^2} . dx + \frac{2}{3}(a^2-x^2)^{\frac{3}{2}}\Big/_0^a$$

$$= \frac{a}{2}\left\{x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right\}\Big/_0^a - \frac{2}{3}a^3 = a^3\left(\frac{\pi}{4} - \frac{2}{3}\right).$$

12. $$\int\sin^3\theta\cos^3\theta d\theta = \int\sin\theta d\theta(\cos^3\theta - \cos^5\theta) = -\tfrac{1}{4}\cos^4\theta + \tfrac{1}{6}\cos^6\theta.$$

13. $$\int\frac{d\theta}{\sin^4\theta\cos^4\theta} = \int d\theta\left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}\right)^2$$

$$= \int d\theta\{\operatorname{cosec}^4\theta + \sec^4\theta + 2(1+\cot^2\theta)(1+\tan^2\theta)\}$$

$$= \int d\theta\{\operatorname{cosec}^4\theta + \sec^4\theta + 2(\operatorname{cosec}^2\theta + \sec^2\theta)\}$$

$$=\int d\theta\{\text{cosec}^2\theta(3+\cot^2\theta)+\sec^2\theta(3+\tan^2\theta)\}$$

$$=-3\cot\theta-\frac{\cot^3\theta}{3}+3\tan\theta+\frac{\tan^3\theta}{3},\text{ etc.}$$

14. $$\int\frac{\sin^2\theta d\theta}{\cos^3\theta}=\int\frac{\sin\theta}{\cos^3\theta}.\,d\theta.\sin\theta=\frac{\sin\theta}{2\cos^2\theta}-\frac{1}{2}\int d\theta.\frac{\cos\theta}{\cos^2\theta}$$

$$=\frac{\sin\theta}{2\cos^2\theta}-\frac{1}{2}\int\frac{d\theta.\cos\theta}{1-\sin^2\theta}$$

$$=\frac{\sin\theta}{2\cos^2\theta}-\frac{1}{4}\int d\theta\cos\theta\left(\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}\right)$$

$$=\frac{\sin\theta}{2\cos^2\theta}+\frac{1}{4}\log\frac{1-\sin\theta}{1+\sin\theta}.$$

15. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos 2\theta)^{\frac{3}{2}}\cos\theta d\theta$ becomes (putting $\sqrt{2}\sin\theta=\sin\phi$, and $\therefore$ $\sqrt{2}\cos\theta d\theta=\cos\phi d\phi$)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3\phi.\frac{\cos\phi d\phi}{\sqrt{2}}=\sqrt{2}\int_0^{\frac{\pi}{2}}\cos^4\phi d\phi=\sqrt{2}.\frac{\pi}{2}\,\frac{3.1}{4.2}=\frac{3\pi\sqrt{2}}{16}.$$

16. If $\cos^{-1}\frac{x}{a}=\theta$, $x=a\cos\theta$, and

$$\int_0^a\sqrt{a^2-x^2}.\cos^{-1}\frac{x}{a}dx=-\int_{\frac{\pi}{2}}^0 a\sin\theta.\theta.a\sin\theta d\theta=\frac{a^2}{2}\int_0^{\frac{\pi}{2}}d\theta.\theta(1-\cos 2\theta),$$

$\therefore$ integrating by parts, $$=\frac{a^2\theta}{2}\left(\theta-\frac{\sin 2\theta}{2}\right)\Big|_0^{\frac{\pi}{2}}-\frac{a^2}{2}\int_0^{\frac{\pi}{2}}d\theta\left(\theta-\frac{\sin 2\theta}{2}\right)$$

$$=\frac{a^2}{2}.\left(\frac{\pi}{2}\right)^2-\frac{a^2}{2}\left(\frac{\theta^2}{2}+\frac{\cos 2\theta}{4}\right)\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{a^2}{2}.\frac{\pi^2}{4}-\frac{a^2}{2}\left\{\frac{\pi^2}{8}+\frac{1}{4}(-1-1)\right\}=\frac{a^2}{4}\left(\frac{\pi^2}{4}+1\right).$$

17. If $\text{vers}^{-1}\frac{x}{a}=\theta$, $x=a(1-\cos\theta)$, and

$$\int_0^{2a}\left(\text{vers}^{-1}\frac{x}{a}\right)^2dx=\int_0^{\pi}\theta^2.\,a\sin\theta d\theta=-a\theta^2\cos\theta\Big|_0^{\pi}+2a\int_0^{\pi}\theta d\theta.\cos\theta$$

$$=\pi^2a+2a\theta\sin\theta\Big|_0^{\pi}-2a\int_0^{\pi}d\theta\sin\theta$$

$$=\pi^2a+2a\cos\theta\Big|_0^{\pi}=\pi^2a-4a,\text{ etc.}$$

18. Here $\dfrac{\sin^3 x}{1+c\cos x}=\dfrac{\sin x}{c}\cdot\dfrac{c-c\cos^2 x}{1+c\cos x}=\dfrac{\sin x}{c}\left\{\dfrac{c+\cos x}{1+c\cos x}-\cos x\right\}$

$$=\frac{\sin x}{c^2}\left\{-c\cos x+\frac{c^2-1}{1+c\cos x}+1\right\},$$

$$\int_0^{\frac{\pi}{2}}\frac{\sin^3 x\,dx}{1+c\cos x}=\int_0^{\frac{\pi}{2}}\frac{dx}{c^2}\left(\sin x-c\sin x\cos x+\overline{c^2-1}\,\frac{\sin x}{1+c\cos x}\right)$$

$$=\frac{1}{c^2}\left\{-\cos x-\frac{c}{2}\sin^2 x-\frac{c^2-1}{c}\log(1+c\cos x)\right\}\Bigg/_0^{\frac{\pi}{2}}.$$

$$=\frac{1}{c^2}\left\{1-\frac{c}{2}-\frac{c^2-1}{c}\log\frac{1}{1+c}\right\}=\frac{c^2-1}{c^3}\log(1+c)+\frac{2-c}{2c^2}.$$

19. $\phi(n)=\displaystyle\int\frac{dx}{(1+c\cos x)^n}=\int dx\,.\,\frac{\sin^2 x+\cos^2 x}{(1+c\cos x)^n}$

$$=\frac{1}{c(n-1)}\,\frac{\sin x}{(1+c\cos x)^{n-1}}-\frac{1}{c(n-1)}\int\frac{dx\,.\cos x}{(1+c\cos x)^{n-1}}+\int\frac{dx\,.\cos^2 x}{(1+c\cos x)^n}$$

$$-\frac{\sin x(1+c\cos x)^{-n+1}}{(n-1)c}+\phi(n)$$

$$=-\frac{1}{c^2(n-1)}\int\frac{dx(c\cos x+1-1)}{(1+c\cos x)^{n-1}}+\frac{1}{c^2}\int\frac{dx(c^2\cos^2 x+2c\cos x+1)}{(1+c\cos x)^n}$$

$$-\frac{1}{c^2}\int\frac{dx(2c\cos x+2-1)}{(1+c\cos x)^n}$$

$$=-\frac{1}{(n-1)c^2}\{\phi(n-2)-\phi(n-1)\}+\frac{1}{c^2}\phi(n-2)-\frac{2}{c^2}\phi(n-1)+\frac{1}{c^2}\phi(n),$$

and $\therefore\ \phi(n)(c^2-1)(n-1)$

$$=c\sin x(1+c\cos x)^{-n+1}+(n-2)\phi(n-2)-\phi(n-1)(2n-3).$$

20. If $\operatorname{vers}^{-1}\dfrac{x}{a}=\theta$, $\quad x=a(1-\cos\theta)$, and

$$\int_0^{2a}\sqrt{2ax-x^2}\,.\left(\operatorname{vers}^{-1}\frac{x}{a}\right)dx=\int_0^{\pi}a\sin\theta\,.\,\theta\,.\,a\sin\theta\,d\theta=a^2\int_0^{\pi}\theta\,.\,d\theta\frac{1-\cos 2\theta}{2}$$

$$=\frac{a^2\theta}{2}\left(\theta-\frac{\sin 2\theta}{2}\right)\Bigg/_0^{\pi}-\frac{a^2}{2}\int_0^{\pi}d\theta\left(\theta-\frac{\sin 2\theta}{2}\right)$$

$$=\frac{a^2\pi^2}{2}-\frac{a^2}{2}\left(\frac{\theta^2}{2}+\frac{\cos 2\theta}{4}\right)\Bigg/_0^{\pi}$$

$$=\frac{a^2\pi^2}{2}-\frac{a^2}{2}\left(\frac{\pi^2}{2}+\frac{1}{4}-\frac{1}{4}\right)=\frac{a^2\pi^2}{4}.$$

Aliter: by Art. 41 (3),

$$\int_0^{2a}\sqrt{2ax-x^2}\,\text{vers}^{-1}\frac{x}{a}dx=\int_0^{2a}\sqrt{2ax-x^2}\,\text{vers}^{-1}\left(2-\frac{x}{a}\right)dx$$

$$=\int_0^{2a}\sqrt{2ax-x^2}\left(\pi-\text{vers}^{-1}\frac{x}{a}\right)dx$$

$$\therefore\quad =\frac{1}{2}\int_0^{2a}\sqrt{2ax-x^2}\,\pi dx=\frac{\pi}{2}\int_0^{2a}dx\,.\sqrt{a^2-(x-a)^2}$$

$$=\frac{\pi}{2}\,.\,\text{area of a semicircle of radius } a=\frac{\pi^2a^2}{4}.$$

21. If $\text{vers}^{-1}\frac{x}{a}=\theta,\ \ x=a(1-\cos\theta)$, and

$$\int_0^{2a}x\sqrt{2ax-x^2}\,\text{vers}^{-1}\frac{x}{a}dx=\int_0^{\pi}a^3(1-\cos\theta)\sin\theta\,.\,\theta\,.\sin\theta d\theta$$

$$=a^3\int_0^{\pi}\theta d\theta\left(\frac{1-\cos 2\theta}{2}-\sin^2\theta\cos\theta\right)$$

$$=\frac{a^3\theta}{2}\left(\theta-\frac{\sin 2\theta}{2}\right)\Big/_0^{\pi}-\frac{a^3}{2}\int_0^{\pi}d\theta\left(\theta-\frac{\sin 2\theta}{2}\right)$$

$$-\theta\frac{a^3\sin^3\theta}{3}\Big/_0^{\pi}+a^3\int_0^{\pi}\frac{d\theta\sin^3\theta}{3}$$

$$=\frac{a^3\pi^2}{2}-\frac{a^3}{2}\left(\frac{\theta^2}{2}+\frac{\cos 2\theta}{4}\right)\Big/_0^{\pi}+\frac{a^3}{3}\int_0^{\pi}d\theta\sin\theta(1-\cos^2\theta)$$

$$=\frac{a^3\pi^2}{2}-\frac{a^3\pi^2}{4}+\frac{a^3}{3}\left(-\cos\theta+\frac{\cos^3\theta}{3}\right)\Big/_0^{\pi}$$

$$=\frac{\pi^2a^3}{4}+\frac{a^3}{3}\left(1+1-\frac{1}{3}-\frac{1}{3}\right)=\frac{\pi^2a^3}{4}+\frac{4}{9}a^3.$$

22. $$\int_0^{\frac{\pi}{4}}dx\tan^7x=\int_0^{\frac{\pi}{4}}dx(\tan^5x\sec^2x-\tan^5x)$$

$$=\int_0^{\frac{\pi}{4}}dx(\tan^5x\sec^2x-\tan^3x\sec^2x+\tan x\sec^2x-\tan x)$$

$$=\left(\frac{1}{6}\tan^6x-\frac{1}{4}\tan^4x+\frac{1}{2}\tan^2x+\log\cos x\right)\Big/_0^{\frac{\pi}{4}}$$

$$=\frac{1}{6}-\frac{1}{4}+\frac{1}{2}+\log\frac{1}{\sqrt{2}}=\frac{5}{12}-\frac{1}{2}\log 2.$$

23. Here x may $=\frac{\pi}{2}$, and $\therefore\ \sin x=1,\ \therefore\ c$ must be not >1 for real values of $\sqrt{1-c^2\sin^2x}$, and

$$\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-c^2\sin^2x}}=\int_0^{\frac{\pi}{2}}dx\left(1+\frac{c^2\sin^2x}{2}+\frac{1.3}{2.4}c^4\sin^4x+\frac{1.3.5}{2.4.6}c^6\sin^6x+\ldots\right)$$

by the Binomial Theorem, and $\therefore$ by Art. 35,

$$=\frac{\pi}{2}\left\{1+c^2.\left(\frac{1}{2}\right)^2+\frac{1.3}{2.4}c^4.\frac{3.1}{4.2}+\frac{1.3.5}{2.4.6}.c^6.\frac{5.3.1}{6.4.2}+\ldots\right\}$$

$$=\frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2.c^2+\left(\frac{1.3}{2.4}\right)^2.c^4+\left(\frac{1.3.5}{2.4.6}\right)^2.c^6+\ldots\right\}.$$

24. $\int x^m P^n dx=\int x^m.P^{n-1}.dx.P=\int x^m P^{n-1}dx(Ax^a++\ldots)$

$$=\Sigma\int Adx.x^{m+a}.P^{n-1}=\Sigma\{A\,V_{m+a,\,n-1}\}.\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Also, integrating by parts,

$$\int x^m P^n dx\equiv V_{m,\,n}=\frac{x^{m+1}.P^n}{m+1}-\frac{n}{m+1}\int x^{m+1}dx.P^{n-1}.\frac{dP}{dx},$$

or $\quad x^{m+1}.P^n=(m+1)V_{m,\,n}+n\int dx.P^{n-1}x^{m+1}(Aax^{a-1}+Bbx^{b-1}+\ldots)$

$$=(m+1)V_{m,\,n}+n\Sigma\int dx.P^{n-1}.Aax^{m+a}$$

$$=(m+1)V_{m,\,n}+n\Sigma\{Aa\,V_{m+a,\,n-1}\},$$

$\therefore$ by (1) $\quad =\Sigma\{A(\overline{m+1}+na)V_{m+a,\,n-1}\}=\Sigma aA\,V_{m+a,\,n-1}.$

CHAPTER IV.

1. If $x=a\sin^2\theta$,

$$\int_0^a\frac{x^{\frac{5}{2}}dx}{\sqrt{a-x}}=\int_0^{\frac{\pi}{2}}\frac{a^{\frac{5}{2}}\sin^5\theta.2a\sin\theta\cos\theta d\theta}{a^{\frac{1}{2}}\cos\theta}=2a^3\int_0^{\frac{\pi}{2}}d\theta.\sin^6\theta$$

$$=2a^3\frac{\pi}{2}.\frac{5.3.1}{6.4.2}=\frac{5}{16}\pi a^3.$$

2. $\int_0^{2a}\frac{xdx}{\sqrt{2ax-x^2}}=\int_0^{2a}\frac{(2a-x)dx}{\sqrt{2ax-x^2}}=\frac{1}{2}\int_0^{2a}\frac{dx.2a}{\sqrt{2ax-x^2}}=a\int_0^{2a}\frac{dx}{\sqrt{a^2-(x-a)^2}}$

$$=a\sin^{-1}\frac{x-a}{a}\Big/_0^{2a}=a\left\{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right\}=\pi a.$$

3. If $x=a\sin\theta$,

$$\int_0^a\frac{(a^2-e^2x^2)dx}{\sqrt{a^2-x^2}}=\int_0^{\frac{\pi}{2}}a^2(1-e^2\sin^2\theta)d\theta=a^2\left\{\frac{\pi}{2}-e^2.\frac{\pi}{2}.\frac{1}{2}\right\}=\frac{\pi a^2}{2}\left(1-\frac{e^2}{2}\right).$$

4. Here $\frac{1}{(x^2+x^2)(b^2+x^2)}=\frac{1}{a^2-b^2}\left(\frac{1}{b^2+x^2}-\frac{1}{a^2+x^2}\right),$

$$\therefore\quad\int_0^\infty\frac{dx}{(a^2+x^2)(b^2+x^2)}=\frac{1}{a^2-b^2}\left\{\frac{1}{b}\tan^{-1}\frac{x}{b}-\frac{1}{a}\tan^{-1}\frac{x}{a}\right\}\Big/_0^\infty$$

$$=\frac{1}{a^2-b^2}.\frac{\pi}{2}\left(\frac{1}{b}-\frac{1}{a}\right)=\frac{\pi}{2ab(a+b)}.$$

5. Hence $\int_0^a \phi(a+x)dx = \int_0^a \phi(x)dx$ and $= \int_a^{2a} \phi(x)dx$

$$= \int_a^{2a} \phi(a+x)dx = \int_{2a}^{3a} \phi(x)dx, \text{ and so on.}$$

Thus
$$\int_0^{na} \phi(x)dx = \int_0^a \phi(x)dx + \int_a^{2a} \phi(x)dx + \dots + \int_{(n-1)a}^{na} \phi(x)dx$$
$$= n\int_0^a \phi(x)dx.$$

6. If $\frac{b+a}{2} + \frac{b-a}{2c}x = y$, the lts. of y, corresponding to $-c$ and c for x, are $\frac{b+a}{2} - \frac{b-a}{2}$ or a, and $\frac{b+a}{2} + \frac{b-a}{2}$ or b,

$$\therefore \qquad \int_{-c}^{c} \phi(y)dx = \int_a^b \phi(y)dy \cdot \frac{2c}{b-a},$$

$$\therefore \qquad \int_a^b \phi(y)dy = \int_a^b \phi(x)dx = \frac{b-a}{2c}\int_{-c}^{c} \phi\left(\frac{b+a}{2} + \frac{b-a}{2c}x\right)dx.$$

7. $$\int_0^\pi \frac{x \sin x dx}{1+\cos^2 x} = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)dx}{1+\cos^2(\pi - x)} = \int_0^\pi \frac{(\pi - x)\sin x dx}{1+\cos^2 x}$$

by addition, $$= \frac{1}{2}\int_0^\pi \frac{\pi \sin x dx}{1+\cos^2 x} = -\frac{\pi}{2}\tan^{-1}(\cos x)\Big|_0^\pi = -\frac{\pi}{2}\left\{-\frac{\pi}{4} - \frac{\pi}{4}\right\} = \frac{\pi^2}{4}.$$

8. $$\int_0^{2a} (2ax - x^2)^{\frac{3}{2}} \text{vers}^{-1}\frac{x}{a} dx = \int_0^{2a} (2ax - x^2)^{\frac{3}{2}}\cos^{-1}\left(1 - \frac{x}{a}\right)dx = u, \text{ say,}$$

$$= \int_0^{2a} (2ax - x^2)^{\frac{3}{2}}\cos^{-1}\left(\frac{x}{a} - 1\right)dx, \text{ changing } x$$

into $2a - x$; $\therefore$ $$u = \frac{\pi}{2}\int_0^{2a} (2ax - x^2)^{\frac{3}{2}}dx, \text{ by addition; if now}$$

$x = a(1 + \sin\theta)$, $$u = \frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^4\cos^3\theta \cdot \cos\theta d\theta$$

$$= \pi a^4 \int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = \frac{\pi^2 a^4}{2} \cdot \frac{3 \cdot 1}{4 \cdot 2} = \frac{3}{16}\pi^2 a^4.$$

9. If $\frac{1}{n} = h$, the series $= h\left\{1 + \frac{1}{\sqrt{1-h^2}} + \frac{1}{\sqrt{1-(2h)^2}} + \dots + \frac{1}{\sqrt{1-\overline{n-1}|^2 h^2}}\right\}$,

and $\therefore$ (Cf. Art. 4) $= \int_0^1 \frac{dx}{\sqrt{1-x^2}}$ in the limit when $h = 0$, $\because$ the terms h, $2h$ .. $\overline{n-1}|h$ $\propto$ from 0 to $(n-1)h$ or 1, and differ successively by h: $\therefore$ the required limit is $\sin^{-1}x\Big|_0^1 = \frac{\pi}{2}$.

10. If $\frac{1}{n}=h$, the proposed fraction $=\frac{h^p+(2h)^p+(3h)^p+\ldots+(2nh)^p}{(1+h)^p+(1+2h)^p+\ldots+(1+nh)^p}$, and $\therefore$ multiplying numerator and denominator by h, the fraction in the limit, when $h=0$, $\;=\int_0^2 dx\,.\,x^p\div\int_0^1 dx(1+x)^p=\frac{x^{p+1}}{p+1}\Big/_0^2\div\frac{(1+x)^{p+1}}{p+1}\Big/_0^1$

$$=2^{p+1}\div(2^{p+1}-1)=\frac{1}{1-(\frac{1}{2})^{p+1}}.$$

11. If $u=\left\{\frac{\lfloor n}{n^n}\right\}^{\frac{1}{n}}$, and $h=\frac{1}{n}$,

$$\log u=\frac{1}{n}\{\log 1+\log 2+\ldots+\log n-n\log n\}$$

$$=\frac{1}{n}\left\{\log\frac{1}{n}+\log\frac{2}{n}+\ldots+\log\frac{n}{n}\right\}$$

$$=h\{\log h+\log 2h+\ldots+\log nh\},$$

$\therefore$ in the limit $\quad\log u=\int_0^1 dx\log x=x\log x\Big/_0^1-\int_0^1 x\,dx\,.\,\frac{1}{x}$,

and, $\because$ the limit of $x\log x$, when $x=0$, is 0,

$$\log u=-1,\text{ and }\therefore\ u=\frac{1}{e}.$$

12. $\int_0^{\frac{\pi}{2}}\log(\tan x)dx=\int_0^{\frac{\pi}{2}}\log\tan\overline{\frac{\pi}{2}-x}\,dx=\int_0^{\frac{\pi}{2}}\log\cot x\,dx$,

and $\therefore$ by addition $\quad 2\int=\int_0^{\frac{\pi}{2}}\log(\tan x\,.\cot x)dx=0$,

$\because$ every term of the last integral series is zero.

13. Integrating by parts,

$$\int_0^{\frac{\pi}{2}}\sin x\log\sin x\,dx=-\cos x\log\sin x\Big/_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}dx\,.\,\frac{\cos^2 x}{\sin x}$$

$$=-\cos x\log\sin x\Big/_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}dx\left(\frac{1}{\sin x}-\sin x\right)$$

$$=-\cos x\log\sin x\Big/_0^{\frac{\pi}{2}}+\left(\log\tan\frac{x}{2}+\cos x\right)\Big/_0^{\frac{\pi}{2}};$$

and when $x=\frac{\pi}{2}$, $-\cos x\log\sin x=0$, $\cos x=0$, and $\log\tan\frac{x}{2}=\log\tan\frac{\pi}{4}$ $=\log 1=0$, $\therefore$ the required integral ($\because\cos x=1$ when $x=0$) ultimately

$$=\left(\log\sin x-\log\tan\frac{x}{2}\right)_{x=0}-1$$

$$=\log 2\cos^2\frac{x}{2}\Big/_{x=0}-1=\log 2-1.$$

14. If $P=\left\{f(a)f\left(a+\frac{c}{n}\right)\ldots f\left(a+\frac{n-1}{n}c\right)\right\}^{\frac{1}{n}}$,

$$\log P=\frac{1}{n}\left\{\log f(a)+\log f\left(a+\frac{c}{n}\right)+\ldots+\log f\left(a+\frac{n-1}{n}c\right)\right\}:$$

$\therefore$ as in Art. 39, $\log P=\frac{1}{c}\int_a^{a+c}dx\log f(x)$, $\because$ $\frac{c}{n}=dx$, or $h=\frac{dx}{c}$,

$$\therefore \qquad P=e^{\frac{1}{c}\int_a^{a+c}dx\log f(x)}.$$

Here $f(x)$ must be positive from $x=a$ to $x=a+c$, for $\log f(x)$ to be always real for such values of x, and $f(x)$ must also be finite by Art. 2.

Also, if $f(x)$ be not constant,

$$P<\left\{f(a)+f\left(a+\frac{c}{n}\right)+\ldots+f\left(a+\frac{n-1}{n}c\right)\right\}\cdot\frac{1}{n},\ \text{i.e., } P<\frac{1}{c}\int_a^{a+c}f(x)dx.$$

If $a=0$, $c=1$ and $u=\log f(x)$, and $\therefore f(x)=e^u$, it follows that

$$e^{\int_0^1 udx}<\int_0^1 e^u dx,$$

unless u be constant from $x=0$ to $x=1$, when the two expressions are equal.

15. $\int_0^{\frac{\pi}{2}}d\theta\log(1+n\cos^2\theta)=\int_0^{\frac{\pi}{2}}d\theta\log(1+n\sin^2\theta)$, by Art. 41 (3),

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}d\theta\log(1+n+n^2\cos^2\theta\sin^2\theta) \text{ by addition,}$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}d\theta\{\log(1+n)+\log(1+n_1\sin^2 2\theta)\}, \text{ and } \therefore$$

if 2θ be replaced by θ, $\quad =\frac{1}{4}\int_0^{\pi}d\theta\{\log(1+n)+\log(1+n_1\sin^2\theta)\}$

$$=\frac{\pi}{4}\log(1+n)+\frac{1}{2}\int_0^{\frac{\pi}{2}}d\theta\log(1+n_1\sin^2\theta);$$

similarly $\int_0^{\frac{\pi}{2}}d\theta\log(1+n_1\sin^2\theta)=\frac{\pi}{4}\log(1+n_1)+\frac{1}{2}\int_0^{\frac{\pi}{2}}d\theta\log(1+n_2\sin^2\theta)$,

and so on; thus

$$\int_0^{\frac{\pi}{2}}\log(1+n\cos^2\theta)d\theta=\frac{\pi}{4}\log(1+n)+\frac{\pi}{8}\log(1+n_1)+\frac{\pi}{16}\log(1+n_2)+\ldots$$
$$+\frac{1}{2^{r+1}}\int_0^{\frac{\pi}{2}}d\theta\log(1+n_{r+1}\sin^2\theta);$$

but $\quad \frac{n_1}{n}=\frac{n}{4(n+1)}<\frac{1}{4}$, $\frac{n_2}{n_1}=\frac{n_1}{4(n_1+1)}<\frac{1}{4}$, and so on, $\therefore \frac{n_{r+1}}{n}<\frac{1}{4^{r+1}}$,

$\therefore$ the term $\frac{1}{2^{r+1}}\int_0^{\frac{\pi}{2}}d\theta\log(1+n_{r+1}\sin^2\theta)$ approximates to $\frac{1}{2^{r+1}}\int_0^{\frac{\pi}{2}}d\theta\,.\log(1)$, or zero, as r increases. Hence

$$\int_0^{\frac{\pi}{2}} \log(1+n\cos^2\theta)d\theta = \frac{\pi}{4}\log\{(1+n)(1+n_1)^{\frac{1}{2}}(1+n_2)^{\frac{1}{4}}\ldots\},$$

when the number of terms is increased indefinitely.

16. Integrating by parts,

$$\int e^{cx}.\cos ax dx = \frac{e^{cx}}{c}.\cos ax + \int \frac{e^{cx}}{c}.a\sin ax dx + \text{a constant}$$

$$= \frac{e^{cx}\cos ax}{c} + \frac{e^{cx}}{c^2}.a\sin ax - \int \frac{e^{cx}}{c^2}a^2\cos ax dx + \text{a constant},$$

$$\therefore \int e^{cx}\cos ax dx\left(1+\frac{a^2}{c^2}\right) = \frac{e^{cx}\cos ax}{c} + \frac{e^{cx}a\sin ax}{c^2} + \text{a constant};$$

$$\therefore \text{ if } \frac{a}{\sin\phi} = \frac{c}{\cos\phi}\left(=\sqrt{a^2+c^2}\right),$$

$$\int e^{cx}\cos ax dx = \frac{e^{cx}}{a^2+c^2}(\cos ax\cos\phi + \sin ax\sin\phi)\sqrt{a^2+c^2} + \text{a constant}$$

$$= \frac{e^{cx}\cos(ax-\phi)}{(a^2+c^2)^{\frac{1}{2}}} + \text{a constant}.$$

Hence if $ax - \phi = ax'$, the second integral with regard to x of

$$e^{cx}\cos ax = \int dx\left\{\frac{e^{cx}\cos(ax-\phi)}{(a^2+c^2)^{\frac{1}{2}}} + A\right\},$$

$$e^{\frac{c\phi}{a}}\int dx' \frac{e^{cx'}\cos ax'}{(a^2+c^2)^{\frac{1}{2}}} + Ax' = e^{\frac{c\phi}{a}}\left\{\frac{e^{cx'}\cos(ax'-\phi)}{a^2+c^2} + \text{a constant}\right\} + Ax'$$

$$= e^{cx}\frac{\cos(ax-2\phi)}{a^2+c^2} + Ax + A_1, \text{ and so on.}$$

Thus, integrating n times successively, the result is

$$e^{cx}\frac{\cos(ax-n\phi)}{(a^2+c^2)^{\frac{n}{2}}} + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \ldots + C_1x + C.$$

17. Comparing Art. 49 (III.), $\chi(x) = a^{\frac{1}{x}} - 1$,

$$\therefore \qquad \frac{-x\chi'(x)}{\chi(x)} = \frac{x.a^{\frac{1}{x}}}{a^{\frac{1}{x}}}.\frac{1}{x^2}\log a = \frac{\log a}{x} = 0 \text{ when } x = \infty,$$

and the limit is $\cdot < 1$, and the series accordingly divergent.

18. Here, by Art. 49 (III.), $\because \psi(x) = x^{a+\frac{b}{x}}$,

$$\frac{x\psi'(x)}{\psi(x)} = \frac{x}{\psi(x)}\left\{x^{a+\frac{b}{x}}.\log x\left(-\frac{b}{x^2}\right) + \left(a+\frac{b}{x}\right)x^{a+\frac{b}{x}-1}\right\}$$

$$= -\frac{b}{x}\log x + a + \frac{b}{x},$$

and $\therefore$ when $x = \infty$ $\dfrac{x\psi'(x)}{\psi(x)} = a$, and $\therefore$ etc.

19. Here $\chi(x)$ in Art. 50 is $\dfrac{p(p+a)(p+2a)\ldots(p+xa)}{q(q+a)(q+2a)\ldots(q+xa)}$, and

$$P_0 = \text{lt. of } x\left\{\frac{\chi(x)}{\chi(x+1)} - 1\right\} \text{ when } n = \infty,$$

$$= \text{lt. of } x\left\{\frac{q+(x+1)a}{p+(x+1)a} - 1\right\} = \text{lt. of } \frac{x(q-p)}{xa} = \frac{q-p}{a} \text{ when } x = \infty,$$

and $\therefore$ the series is convergent if $q - p > a$, i.e., $q > p + a$, and divergent if q be not $> p + a$.

20. By Art. 50 the series is convergent or divergent as $P_0 > 1$ or not > 1; and here $P_0 = \text{lt. of } x\left\{\dfrac{x^p + ax^{p-1} + bx^{p-2} + \ldots}{x^p + Ax^{p-1} + Bx^{p-2} + \ldots} - 1\right\}$

$$= \text{lt. of } x\left\{\frac{(a-A)x^{p-1}}{x^p}\right\} \text{ approximately} = a - A, \text{ etc.}$$

21. Comparing Art. 4, $\int u^2 dx$ is the limit of a series containing n terms, which may be written

$$\frac{1}{n}(a_1^2 + a_2^2 + \ldots + a_n^2) = A,$$

and similarly, the limits of the three integrals being the same,

$$B = \text{lt. of } \frac{1}{n}(a_1c_1 + a_2c_2 + \ldots + a_nc_n)$$

and

$$C = \text{lt. of } \frac{1}{n}(c_1^2 + c_2^2 + \ldots + c_n^2),$$

and $\therefore$ by the Algebraical Theorem quoted AC cannot be $< B^2$.

CHAPTER V.

1. If $x^{\frac{3}{2}} = y$,

$$\int\frac{\sqrt{x}dx}{\sqrt{a^3 - x^3}} = \int\frac{y^{\frac{1}{3}}.\frac{2}{3}y^{-\frac{1}{3}}.dy}{\sqrt{a^3 - y^2}} = \frac{2}{3}\int dy\,.\,\frac{1}{\sqrt{a^3 - y^2}} = \frac{2}{3}\sin^{-1}\frac{y}{a^{\frac{3}{2}}} = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{\frac{3}{2}}.$$

2. If $\dfrac{x^3}{(x-a)(x-b)(x-c)}=1+\dfrac{(a+b+c)x^2-(bc+ca+ab)x+abc}{(x-a)(x-b)(x-c)}$

$$=1+\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c},$$

then $(a+b+c)x^2-(bc+ca+ab)x+abc$

$$\equiv A(x-b)(x-c)+B(x-c)(x-a)+C(x-a)(x-b)$$

and $\therefore$ putting $x=a$,

$$A(a-b)(a-c)=a^2(a+b+c)-a(bc+ca+ab)+abc=a^3,$$

or $$A=\frac{a^3}{(a-b)(a-c)},$$

and similarly for B and C by symmetry: thus

$$\int\frac{dx\,.\,x^3}{(x-a)(x-b)(x-c)}=x+\frac{a^3\log(x-a)}{(a-b)(a-c)}++.$$

3. $$\int\frac{\tan x dx}{1+m^2\tan^2 x}=\int\frac{dx\,.\sin x\cos x}{\cos^2 x+m^2\sin^2 x},$$

and $$\frac{d}{dx}(\cos^2 x+m^2\sin^2 x)=2\sin x\cos x(-1+m^2),$$

$\therefore$ $$\int\frac{\tan x dx}{1+m^2\tan^2 x}=\frac{1}{2(m^2-1)}\log(\cos^2 x+m^2\sin^2 x).$$

4. If $x=\dfrac{a}{y}$, $\displaystyle\int\frac{dx}{x(a^{2n}+x^{2n})^{\frac{1}{2}}}=-a\int\frac{dy}{y^2}\cdot\frac{y}{a}\cdot\frac{1}{a^n\left(1+\frac{1}{y^{2n}}\right)^{\frac{1}{2}}}=\int\frac{dy\,.\,y^{n-1}}{a^n(1+y^{2n})^{\frac{1}{2}}}$

$$=-\frac{1}{na^n}\int\frac{dz}{(1+z^2)^{\frac{1}{2}}},\text{ if } y^n=z;$$

and $\therefore$ $$\text{the integral}=-\frac{1}{na^n}\log\{z+(1+z^2)^{\frac{1}{2}}\}$$

$$=-\frac{1}{na^n}\log\left\{\left(\frac{a}{x}\right)^n+\sqrt{1+\left(\frac{a}{x}\right)^{2n}}\right\}$$

$$=\frac{1}{na^n}\log\frac{x^n}{a^n+(a^{2n}+x^{2n})^{\frac{1}{2}}}.$$

5. $$\int\sec x\sec 2x dx=\int\frac{\cos x dx}{(1-\sin^2 x)(1-2\sin^2 x)}$$

$$=\int\cos x dx\left\{\frac{2}{1-2\sin^2 x}-\frac{1}{1-\sin^2 x}\right\}$$

$$=\int \cos x dx\left\{\frac{1}{1-\sqrt{2}\sin x}+\frac{1}{1+\sqrt{2}\sin x}-\frac{1}{2}\left(\frac{1}{1-\sin x}+\frac{1}{1+\sin x}\right)\right\}$$

$$=\frac{1}{\sqrt{2}}\log\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}-\frac{1}{2}\log\frac{1+\sin x}{1-\sin x}.$$

6. $$\int dx\frac{\tan a-\tan x}{\tan a+\tan x}=-\int dx\frac{\sin(x-a)}{\sin(x+a)}=-\int dx\frac{\sin(x+a-2a)}{\sin(x+a)}$$

$$=-\int dx\left\{\cos 2a-\sin 2a\frac{\cos(x+a)}{\sin(x+a)}\right\}$$

$$=\sin 2a\log\sin(x+a)-x\cos 2a.$$

7. $$\frac{1}{x^4+a^2x^2+a^4}=\frac{1}{(x^2+a^2)^2-a^2x^2}=\frac{1}{(x^2+ax+a^2)(x^2-ax+a^2)}$$

$$=\frac{Ax+B}{x^2+ax+a^2}+\frac{Cx+D}{x^2-ax+a^2}\text{ suppose,}$$

$$\therefore\quad 1\equiv(Ax+B)(x^2-ax+a^2)+(Cx+D)(x^2+ax+a^2),$$

$\therefore$ equating coefficients of x^3, x^2, x, and x^0,

$$A+C=0,\qquad -aA+B+aC+D=0,$$

$$Aa^2-Ba+Ca^2+Da=0,\qquad\text{and}\quad 1=Ba^2+Da^2;$$

$$\therefore\quad A=-C=\frac{1}{2a}(B+D)=\frac{1}{2a^3},\text{ and }B=D=\frac{1}{2a^2},$$

and $\therefore$ $$\int\frac{1}{x^4+a^2x^2+a^4}=\frac{1}{2a^3}\int\frac{dx(x+a)}{x^2+ax+a^2}-\frac{1}{2a^3}\int\frac{dx(x-a)}{x^2-ax+a^2}$$

$$=\frac{1}{4a^3}\int dx\left\{\frac{2x+a}{x^2+ax+a^2}+\frac{a}{\left(x+\frac{a}{2}\right)^2+\frac{3a^2}{4}}-\frac{2x-a}{x^2-ax+a^2}+\frac{a}{\left(x-\frac{a}{2}\right)^2+\frac{3a^2}{4}}\right\}.$$

$$=\frac{1}{4a^3}\left\{\log\frac{x^2+ax+a^2}{x^2-ax+a^2}+\frac{2}{\sqrt{3}}\tan^{-1}\frac{\left(x+\frac{a}{2}\right)}{\frac{a\sqrt{3}}{2}}+\frac{2}{\sqrt{3}}\tan^{-1}\frac{\left(x-\frac{a}{2}\right)}{\frac{a\sqrt{3}}{3}}\right\},$$

and $$\tan^{-1}\frac{x+\frac{a}{2}}{\frac{a\sqrt{3}}{2}}+\tan^{-1}\frac{x-\frac{a}{2}}{\frac{a\sqrt{3}}{2}}$$

$$=\tan^{-1}\frac{2x+a}{a\sqrt{3}}+\tan^{-1}\frac{2x-a}{a\sqrt{3}}=\tan^{-1}\frac{4ax\sqrt{3}}{3a^2-(4x^2-a^2)}=\tan^{-1}\left(\frac{xa\sqrt{3}}{a^2-x^2}\right),$$

and $\therefore$ etc.

8. If $\frac{a}{x}+bx=y$, $dx\left(-\frac{a}{x^2}+b\right)=dy=-\frac{dx}{x^2}$

and $\therefore \int\frac{(a-bx^2)dx}{x\{cx^2-(a-bx^2)^2\}^{\frac{1}{2}}}=-\int\frac{xdy}{\{cx^2-(a-bx^2)^2\}^{\frac{1}{2}}}$

$$=-\int\frac{dy}{(c^2+4ab-y^2)^{\frac{1}{2}}}=\cos^{-1}\frac{y}{(c^2+4ab)^{\frac{1}{2}}}$$

$$=\cos^{-1}\left\{\frac{a+bx^2}{x\sqrt{c^2+4ab}}\right\}.$$

9. If $u=$ limit (when $x=\infty$) of $\left\{\sin\frac{\pi}{n}.\sin\frac{2\pi}{n}...\sin\frac{\overline{n-1}|\pi}{n}\right\}^{\frac{1}{n}}$,

$$\log u=\frac{1}{n}\left\{\log\sin\frac{\pi}{n}+\log\sin\frac{2\pi}{n}+...\log\sin\frac{\overline{n-1}|\pi}{n}\right\}$$

$$=\frac{1}{\pi}.\frac{\pi}{n}\left\{\log\sin\frac{\pi}{n}+\log\sin\frac{2\pi}{n}+...+\log\sin\frac{n-1}{n}.\pi\right\}$$

$=\frac{1}{\pi}\int_0^\pi dx\log\sin x$, $\because$ $\frac{n-1}{n}\pi=\pi$ when $n=\infty$, and $\therefore$ by Art. 51, $\log u=\frac{1}{\pi}.\pi\log\frac{1}{2}$, or $u=\frac{1}{2}$.

10. If $\tan^{-1}x=\theta$, $x=\tan\theta$, $dx=\sec^2\theta d\theta$, and the limits of θ are 0 and $\frac{\pi}{4}$,

$$\therefore \int_0^1 x(\tan^{-1}x)^2dx=\int_0^{\frac{\pi}{4}}\tan\theta.\theta^2.\sec^2\theta d\theta=\frac{1}{2}\tan^2\theta.\theta^2\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}d\theta.\theta.\tan^2\theta$$

$$=\frac{\pi^2}{32}-\int_0^{\frac{\pi}{4}}d\theta.\theta(\sec^2\theta-1)=\frac{\pi^2}{32}-\theta(\tan\theta-\theta)\Big|_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}d\theta(\tan\theta-\theta)$$

$$=\frac{\pi^2}{32}-\frac{\pi}{4}+\frac{\pi^2}{16}-\left(\log\cos\theta+\frac{\theta^2}{2}\right)\Big|_0^{\frac{\pi}{4}}=\frac{\pi^2}{16}-\frac{\pi}{4}+\log\sqrt{2}.$$

11. $\int_0^a\int_0^x\int_0^{x+y}e^{x+y+z}dxdydz=\int_0^a\int_0^x dxdye^{x+y+z}\Big|_{z=0}^{z=x+y}$

$$=\int_0^a\int_0^x dxdy\{e^{2(x+y)}-e^{x+y}\}$$

$$=\int_0^a dx\left\{\frac{1}{2}e^{2(x+y)}-e^{x+y}\right\}\Big|_{y=0}^{y=x}$$

$$=\int_0^a dx\left\{\frac{1}{2}e^{4x}-e^{2x}-\frac{1}{2}e^{2x}+e^x\right\}$$

$$=\frac{1}{8}e^{4a}-\frac{3}{4}e^{2a}+e^a-\frac{1}{8}e^0+\frac{3}{4}e^0-e^0$$

$$=\frac{1}{8}e^{4a}-\frac{3}{4}e^{2a}+e^a-\frac{3}{8}.$$

12. Supposing (for simplicity only) the variables to be *equi-crescent*, i.e., in Art. 60 all the h's to be the same, say Δx, and all the k's the same, say Δy, as this will not alter the values of A, B, or C; then the limits of the integrations being the same in the three integrals, A may be regarded as the limit of the series

$$\Delta x\Delta y\{a_1^2+a_2^2+\ldots+a_m^2\}$$
$$+\Delta x\Delta y\{a_1'^2+a_2'^2+\ldots+a_m'^2\}+(n-2)\text{ similar rows},$$

and so on, as in Ex. 21 of Chap. IV., so that AC is never $<B^2$, m and n being each the same quantity for B and C as well as for A.

13. As in Art. 4 (comparing the previous example), $\int_b^a \phi(z)dz$ may be regarded as the limit, when $h=0$, of $h\{\phi(b)+\phi(b+h)+\phi(b+2h)+\ldots+\phi(a-h)\}$ which, $\therefore$, $=1$. Then $\int_b^a dz\phi(z)\cos cz$ is the limit of

$$h\{\phi(b)\cos cb+\phi(b+h)\cos c(b+h)+\ldots+\phi(a-h)\cos c(a-h)\}$$

and $\int_b^a \phi(z)\sin cz dz$ is the limit of

$$h\{\phi(b)\sin cb+\phi(b+h)\sin c(b+h)+\ldots+\phi(a-h)\sin c(a-h)\},$$

$$\therefore \left(\int_b^a dz\phi(z)\cos cz\right)^2+\left(\int_b^a dz\phi(z)\sin cz\right)^2=\text{limit of}$$

$$h^2\{\overline{\phi(b)}|^2+\overline{\phi(b+h)}|^2+\ldots+\overline{\phi(a-h)}|^2+2\phi(b+rh)\phi(b+sh)\cos c(r-s)h+\ldots\},$$

which, $\because$ $\phi(z)$ is always positive, $<h^2\{\phi(b)+\phi(b+h)+\ldots+\phi(a-h)\}^2$, and $\therefore$ <1.

14. If $\int_b^a \phi(z)dz$ be the limit when $h=0$ and $n=\infty$, as in Art. 4, of

$h\{\phi(b)+\phi(b+h)+\ldots+\phi(b+\overline{n-1}|h)\}$, this series $=1$; and

$$\int_b^a z\phi(z)dz=\text{the lt. of } h\{b\phi(b)+(b+h)\phi(b+h)+\ldots+(b+\overline{n-1}|h)\phi(b+\overline{n-1}|h)\}$$
$$=b+h^2\{\phi(b+h)+2\phi(b+2h)+\ldots+(n-1)\phi(a-h)\};\text{ also}$$

$$\int_b^a z^2\phi(z)dz=\text{the lt. of}$$
$$h\{b^2\phi(b)+(b+h)^2\phi(b+h)+(b+2h)^2\phi(b+2h)+\ldots+(b+\overline{n-1}|h)^2\phi(a-h)\}$$
$$=b^2+2h^2b\{\phi(b+h)+2\phi(b+2h)+3\phi(b+3h)+\ldots+(n-1)\phi(a-h)\}$$
$$+h^3\{1^2.\ \phi(b+h)+2^2\phi(b+2h)+\ldots+(n-1)^2\phi(a-h)\},$$

$$\therefore \int_b^a z^2\phi(z)dz-\left(\int_b^a z\phi(z)dz\right)^2$$
$$=2bh^2\{\phi(b+h)+2\phi(b+2h)+\ldots+(n-1)\phi(a-h)\}$$
$$+h^3\{1^2.\ \phi(b+h)+2^2\phi(b+2h)+\ldots+(n-1)^2\phi(a-h)\}$$
$$-2bh^2\{\phi(b+h)+2\phi(b+2h)+\ldots+(n-1)\phi(a-h)\}$$
$$-h^4\{\phi(b+h)+2\phi(b+2h)+\ldots+(n-1)\phi(a-h)\}^2$$

$$= h^3\{1^2.\,\phi(b+h)+2^2\phi(b+2h)+\ldots+(n-1)^2\phi(a-h)\}$$
$$-h^4\{\phi(b+h)+2\phi(b+2h)+\ldots+(n-1)\phi(a-h)\}^2; \ldots\ldots\ldots\ldots(1)$$

and if the coefficient of h^3 in this be multiplied by

$$h\{\phi(b+h)+\phi(b+2h)+\ldots+\phi(a-h)\}$$

which ultimately $=1$, the square terms cancel with those in the coefficient of h^4 in (1) and the general terms formed by products in the two series in (1) are respectively $\qquad \phi(b+rh)\phi(b+sh)\{r^2+s^2\}$,

and $\qquad \phi(b+rh)\phi(b+sh)(2rs)$,

therefore, as $\phi(z)$ is always positive and cannot be always 0,

$$\int_b^a z^2\phi(z)dz > \left\{\int_b^a z\phi(z)dz\right\}^2.$$

CHAPTER VI.

1. Here $y = bx^{\frac{m+n}{n}}$ suppose,

$$\therefore \qquad 1+\left(\frac{dy}{dx}\right)^2 = 1+c^2x^{\frac{2m}{n}} \text{ say, and } s=\int dx\left\{1+c^2x^{\frac{2m}{n}}\right\}^{\frac{1}{2}},$$

$\therefore$ (cf. Art. 15) $p=1$ and $q=2$ here, and the conditions for integrability are

that $\qquad$ (1) $\dfrac{1}{\dfrac{2m}{n}} = \dfrac{n}{2m}$ should be integral,

and $\qquad$ (2) $-\dfrac{1}{2}-\dfrac{n}{2m}$ or $\dfrac{n}{2m}+\dfrac{1}{2}$ should be integral.

2. By Art. 100, the equation to a tractory may be written

$$x+\sqrt{c^2-y^2} = c\log\left\{\frac{c+\sqrt{c^2-y^2}}{y}\right\},$$

$$\therefore \qquad \frac{dx}{dy} = \frac{y}{\sqrt{c^2-y^2}} + \frac{c}{c+\sqrt{c^2-y^2}}\left(-\frac{y}{\sqrt{c^2-y^2}}\right) - \frac{c}{y}$$

$$= \frac{y}{c+\sqrt{c^2-y^2}} - \frac{c}{y} = \frac{c-\sqrt{c^2-y^2}}{y} - \frac{c}{y} = -\frac{\sqrt{c^2-y^2}}{y}$$

$$\therefore \qquad s = \pm\int dy\left\{1+\frac{c^2-y^2}{y^2}\right\}^{\frac{1}{2}} = \pm\int\frac{cdy}{y} = \pm c\log y + A.$$

If, when $x=0$ and $y=c$, $s=0$, then $0=\pm c\log c + A$, and therefore the length of the arc from the cusp to the point (x, y) is $\pm c\log y \mp c\log c = \pm c\log\frac{y}{c}$. Since $\dfrac{dx}{dy}$ is negative for positive values of y, then for such values y diminishes as x increases, $\therefore \dfrac{dy}{ds}$ is here negative, and $\therefore$ the negative sign of the radical in the integral is to be taken, and $s=\log\dfrac{c}{y}$. Also, $\because$ $y<c$ except when $x=0$, $\log\dfrac{c}{y}$ is positive.

N.B.—It is a property of the tractory that the length of the tangent intercepted by the asymptote is constant; whence $y\frac{ds}{dy}=c$.

3. The equation of the cissoid being $y=\frac{x^{\frac{3}{2}}}{\sqrt{2a-x}}$ (cf. Todhunter's *Diff. Cal.* Chap. XXVI., Ex. 7),

$$\frac{dy}{dx}=\frac{\frac{3}{2}x^{\frac{1}{2}}(2a-x)+\frac{1}{2}x^{\frac{3}{2}}}{(2a-x)^{\frac{3}{2}}}, \quad \therefore \left(\frac{ds}{dx}\right)^2=1+\frac{x(3a-x)^2}{(2a-x)^3}=\frac{a^2(8a-3x)}{(2a-x)^3},$$

$$\therefore \qquad s=a\int dx\,.\frac{(8a-3x)^{\frac{1}{2}}}{(2a-x)^{\frac{3}{2}}}, \quad \therefore \text{ if } 2a-x=z,$$

$$\frac{s}{a}=-\int dz(2a-3z)^{\frac{1}{2}}.z^{-\frac{3}{2}}=-\int dz(2az^{-1}+3)^{\frac{1}{2}}z^{-1},$$

$\therefore$ if $2az^{-1}+3=t^2$, and $\therefore\ -az^{-2}dz=tdt$,

$$\frac{s}{a}=\int\frac{tdtz^2}{a}.tz^{-1}=\int\frac{t^2dt}{a}\left(\frac{t^2-3}{2a}\right)^{-1}=2\int\frac{dt\,.t^2}{t^2-3}=2t+6\int\frac{dt}{t^2-3}$$

$$=2t+\sqrt{3}\log\frac{t-\sqrt{3}}{t+\sqrt{3}}; \text{ thus the curve is rectifiable.}$$

4. The curve is symmetrical as to both axes, and meets them in the points $\left(\pm\frac{a}{2}, 0\right)$ and $(0, \pm a)$; and if s be measured in the first quadrant from $\left(\frac{a}{2}, 0\right)$, y and s increase together. Hence

$$s=4\int_0^a dy\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}}; \text{ and here}$$

$$2x=\{a^2+3a^{\frac{4}{3}}y^{\frac{2}{3}}-4y^2\}^{\frac{1}{2}},$$

$$\therefore \qquad 2\frac{dx}{dy}=\left\{a^2+3a^{\frac{4}{3}}y^{\frac{2}{3}}-4y^2\right\}^{-\frac{1}{2}}\left\{a^{\frac{4}{3}}y^{-\frac{1}{3}}-4y\right\},$$

$$\therefore \quad 1+\left(\frac{dx}{dy}\right)^2=\frac{4(a^2+3a^{\frac{4}{3}}y^{\frac{2}{3}}-4y^2)y^{\frac{2}{3}}+(a^{\frac{4}{3}}-4y^{\frac{4}{3}})^2}{4y^{\frac{2}{3}}\{a^2+3a^{\frac{4}{3}}y^{\frac{2}{3}}-4y^2\}}$$

$$=\frac{a^{\frac{4}{3}}(4a^{\frac{2}{3}}y^{\frac{2}{3}}+4y^{\frac{4}{3}}+a^{\frac{4}{3}})}{4y^{\frac{2}{3}}(a^{\frac{2}{3}}-y^{\frac{2}{3}})(a^{\frac{4}{3}}+4a^{\frac{2}{3}}y^{\frac{2}{3}}+4y^{\frac{4}{3}})}=\frac{a^{\frac{4}{3}}}{4y^{\frac{2}{3}}(a^{\frac{2}{3}}-y^{\frac{2}{3}})};$$

$$\therefore \qquad s=4\int_0^a\frac{dy\,.a^{\frac{2}{3}}}{2y^{\frac{1}{3}}(a^{\frac{2}{3}}-y^{\frac{2}{3}})^{\frac{1}{2}}}; \text{ or, if } y=az^3,$$

$$s=2\int_0^1\frac{3az^2dz\,.a^{\frac{2}{3}}}{a^{\frac{1}{3}}z\,.a^{\frac{1}{3}}(1-z^2)^{\frac{1}{2}}}=6a\int_0^1\frac{dz\,.z}{(1-z^2)^{\frac{1}{2}}}=-6a\sqrt{1-z^2}\Big/_0^1=6a.$$

5. If $x+y=a\sec^3\theta$, then $x-y=a\tan^3\theta$,

$$\therefore \quad x=\frac{a}{2}(\sec^3\theta+\tan^3\theta), \quad y=\frac{a}{2}(\sec^3\theta-\tan^3\theta),$$

$$\therefore \quad \frac{dx}{d\theta}=\frac{3a}{2}\sec^2\theta(\sec\theta\tan\theta+\tan^2\theta)=\frac{3a}{2}\sec^2\theta\tan\theta(\sec\theta+\tan\theta),$$

and
$$\frac{dy}{d\theta}=\frac{3a}{2}\sec^2\theta\tan\theta(\sec\theta-\tan\theta),$$

$$\therefore \quad \left(\frac{ds}{d\theta}\right)^2=\left(\frac{3a}{2}\right)^2\sec^4\theta\tan^2\theta\,.\,2(\sec^2\theta+\tan^2\theta),$$

$$\therefore \quad s=\frac{3a}{\sqrt{2}}\int_{\theta_1}^{\theta}d\theta(\sec^2\theta+\tan^2\theta)^{\frac{1}{2}}\sec^2\theta\tan\theta,$$

if θ_1 and θ correspond to the limits x_1 and x;

$$\therefore \quad s=\frac{3a}{\sqrt{2}}\int_{\theta_1}^{\theta}d\theta(\sec^2\theta+\tan^2\theta)^{\frac{1}{2}}\,.\,\frac{1}{4}\,\frac{d}{d\theta}(\sec^2\theta+\tan^2\theta)$$

$$=\frac{a}{2\sqrt{2}}(\sec^2\theta+\tan^2\theta)^{\frac{3}{2}}\Big/_{\theta_1}^{\theta}$$

$$=\frac{1}{2\sqrt{2}}\left\{(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}\right\}^{\frac{3}{2}}-\frac{1}{2\sqrt{2}}\left\{(x_1+y_1)^{\frac{2}{3}}+(x_1-y_1)^{\frac{2}{3}}\right\}^{\frac{3}{2}}.$$

6. Here $\dfrac{ds}{dx}=\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}=\dfrac{a}{c}e^{\frac{x}{c}}$, $\therefore\ \dfrac{dy}{dx}=\dfrac{1}{c}\sqrt{a^2e^{\frac{2x}{c}}-c^2}$,

or if $a^2e^{\frac{2x}{c}}-c^2=z^2$, and $\therefore\ \dfrac{a^2}{c}e^{\frac{2x}{c}}dx=zdz$,

$$y=\int\frac{z}{c}dx=\int\frac{z^2dz}{z^2+c^2}=z-c\tan^{-1}\frac{z}{c}=\sqrt{a^2e^{\frac{2x}{c}}-c^2}-c\tan^{-1}\frac{\sqrt{a^2e^{\frac{2x}{c}}-c^2}}{c}.$$

7. As in Art. 104, if the parabola be $y=2\sqrt{ax}$,

$$\cot\phi=\frac{dy}{dx}=\sqrt{\frac{a}{x}}, \text{ or } x=a\tan^2\phi, \ \therefore\ \frac{dx}{d\phi}=2a\tan\phi\sec^2\phi,$$

and
$$\frac{ds}{dx}=\operatorname{cosec}\phi, \ \therefore\ \frac{ds}{d\phi}=2a\sec^3\phi;$$

and
$$s=2a\int\sec^3\phi d\phi=2a\int d\phi\sec^2\phi(1+\tan^2\phi)^{\frac{1}{2}};$$

or if $\tan\phi=z$,
$$s=2a\int dz(1+z^2)^{\frac{1}{2}}=az(1+z^2)^{\frac{1}{2}}+a\log(z+\sqrt{1+z^2})$$

$$=a\tan\phi\sec\phi+a\log(\tan\phi+\sec\phi)$$

$$=\frac{a\sin\phi}{\cos^2\phi}+a\log\frac{1+\sin\phi}{\cos\phi}, \ \therefore \text{ etc.}$$

8. Here $\cot\phi = \frac{dy}{dx} = \frac{d}{dx}(a^{\frac{1}{3}}x^{\frac{2}{3}}) = \frac{2}{3}\left(\frac{a}{x}\right)^{\frac{1}{3}}$,

$\therefore$ $$x = \frac{8}{27}a\tan^3\phi, \text{ and } \frac{dx}{d\phi} = \frac{8}{9}a\tan^2\phi\sec^2\phi,$$

also $$\frac{ds}{dx} = \operatorname{cosec}\phi, \quad \therefore \quad \frac{ds}{d\phi} = \frac{8}{9}a\tan\phi\sec^3\phi,$$

and $\therefore$ $$s = \frac{8}{27}a\sec^3\phi + c, \text{ and if } s = 0 \text{ when } \phi = 0,$$

$$\frac{8}{27}a + c = 0, \qquad \therefore \qquad s = \frac{8}{27}a(\sec^3\phi - 1).$$

Clearly the curve is the evolute of a parabola.

9. The intrinsic equation to the parabola being $\frac{ds}{d\phi} = 2a\sec^3\phi (= \rho)$, $\because$ ρ increases with ϕ, the first figure of Art. 114 will apply, A being the vertex and AB the axis of the parabola AP, and BQ the corresponding position of the evolute; for which $\therefore$

$$s' = \frac{ds}{d\phi} - \left(\frac{ds}{d\phi}\right)_0 = 2a(\sec^3\phi - 1).$$

Also, if (x, y) be any point on the evolute,

$$\frac{dx}{ds'} = \cos\phi, \quad \therefore \quad \frac{dx}{d\phi} = \cos\phi \,.\, 6a\sec^3\phi\tan\phi,$$

and $$x = 6a\int d\phi\sec^2\phi\tan\phi = 3a\sec^2\phi + c,$$

or if, when $\phi = 0$, $$x = \left(\frac{ds}{d\phi}\right)_0 = 2a,$$

$$x = 3a\sec^2\phi - a\,; \text{ also } \frac{dy}{ds'} = \sin\phi,$$

$\therefore$ $$\frac{dy}{d\phi} = \sin\phi \,.\, 6a\sec^3\phi\tan\phi,$$

$\therefore$ $$y = 6a\int d\phi\sec^2\phi\tan^2\phi = 2a\tan^3\phi + c',$$

or if $y = 0$ when $\phi = 0$, $c' = 0$, and $\therefore$ $y = 2a\tan^3\phi$.

Thus, where the parabola and evolute meet

$$4a^2\tan^6\phi = 4a^2(3\sec^2\phi - 1) \text{ or } \tan^6\phi = 3\tan^2\phi + 2,$$

$\therefore$ $\tan^6\phi - \tan^2\phi = 2(\tan^2\phi + 1)$,

$\therefore$ $\tan^2\phi(\tan^2\phi - 1) - 2 = 0$ or $\tan^2\phi = 2$,

and $\therefore$ $$\sec^2\phi = 3, \text{ and } s' = 2a(3\sqrt{3} - 1).$$

10. The epicycloid is (cf. Todhunter's *Diff. Cal.*) given here by

$$x = \frac{a(a+b)}{a+2b}\cos\theta - \frac{ab}{a+2b}\cos\frac{a+b}{b}\theta$$

and $$y = \frac{a(a+b)}{a+2b}\sin\theta - \frac{ab}{a+2b}\sin\frac{a+b}{b}\theta,$$

$$\therefore \quad \tan\phi = \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{\frac{a(a+b)}{a+2b}\cos\theta - \frac{a(a+b)}{a+2b}\cos\frac{a+b}{b}\theta}{\frac{a(a+b)}{a+2b}\sin\frac{a+b}{b}\theta - \frac{a(a+b)}{a+2b}\sin\theta}$$

$$= \frac{\cos\theta - \cos\frac{a+b}{b}\theta}{\sin\frac{a+b}{b}\theta - \sin\theta} = \tan\frac{a+2b}{2b}\theta$$

$$\therefore \quad \phi = \frac{a+2b}{2b}\theta; \text{ also } \left(\frac{ds}{d\theta}\right)^2 = \frac{a^2(a+b)^2}{(a+2b)^2}\left\{2 - 2\cos\frac{a}{b}\theta\right\},$$

$$\therefore \quad \frac{ds}{d\phi} = \frac{2ab(a+b)}{(a+2b)^2}.2\sin\frac{a\theta}{2b} = c\sin\frac{a\phi}{a+2b}, \text{ say; } \therefore \text{ for the evolute}$$

$$s' = \frac{ds}{d\phi} + A = c\sin\frac{a\phi}{a+2b} + A, \quad \therefore \text{ putting } \frac{\pi}{2} + \frac{a\phi'}{a+2b} \text{ for } \frac{a\phi}{a+2b},$$

$$\frac{ds'}{d\phi'} = c'.\sin\frac{a\phi'}{a+2b}, \text{ and } \therefore \text{ the evolute is also an epicycloid.}$$

11. Here $\frac{dx}{d\theta} = \sin\theta\psi''(\theta) + \cos\theta\psi'''(\theta) - \sin\theta\psi''(\theta) + \cos\theta\psi'(\theta)$

$$= \cos\theta\{\psi'(\theta) + \psi'''(\theta)\},$$

and $$\frac{dy}{d\theta} = \cos\theta\psi''(\theta) - \sin\theta\psi'''(\theta) - \sin\theta\psi'(\theta) - \cos\theta\psi''(\theta)$$

$$= -\sin\theta\{\psi'(\theta) + \psi'''(\theta)\},$$

$$\therefore \quad \frac{ds}{d\theta} = \psi'(\theta) + \psi'''(\theta),$$

and $\therefore$ $$s = \psi(\theta) + \psi''(\theta) + \text{constant.}$$

12. Here $8a^3\frac{dy}{dx} = 4x^3 + 12a^2x,$

$$\therefore \quad 2a^3\frac{ds}{dx} = \{4a^6 + (x^3 + 3a^2x)^2\}^{\frac{1}{2}};$$

but $$4a^6 + x^2(x^2+3a^2)^2 = x^6 + 6a^2x^4 + 9a^4x^2 + 4a^6$$

$$= x^6 + 4a^2x^4 + 2a^2x^4 + 8a^4x^2 + a^4x^2 + 4a^6$$

$$= (x^2 + 4a^2)(x^4 + 2a^2x^2 + a^4),$$

$$\therefore \quad 2a^3s = \int dx(x^2 + a^2)(x^2 + 4a^2)^{\frac{1}{2}}, \text{ or if } x = 2a\tan\phi$$

$$2a^3s = \int 4a^4 d\phi\,.\sec^3\phi(1 + 4\tan^2\phi),$$

$$\therefore \quad \frac{s}{2a} = \int d\phi\sec^3\phi + \frac{4}{3}\sec^3\phi\tan\phi - \frac{4}{3}\int d\phi\sec^5\phi,$$

and also $\dfrac{s}{2a} = \int d\phi(4\sec^5\phi - 3\sec^3\phi)$, $\therefore$, eliminating $\int d\phi \,.\, \sec^5\phi$,

$$\frac{4s}{2a} - \frac{2s}{a} = 4\sec^3\phi\tan\phi = \frac{4x}{2a}\left\{\left|\frac{x}{2a}\right|^2 + 1\right\}^{\frac{3}{2}},$$

$\therefore$ $s = \dfrac{x}{8a^3}(x^2+4a^2)^{\frac{3}{2}}$, measured up to the point (x, y).

CHAPTER VII.

1. The equation to the catenary being $y = \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}})$, and x_1 the abscissa corresponding to s,

$$A = \int_0^{x_1} ydx = \int_0^{x_1} dx \,.\, \frac{c}{2}(e^{\frac{x}{c}} + e^{-\frac{x}{c}}) = \frac{c^2}{2}(e^{\frac{x}{c}} - e^{-\frac{x}{c}})\Big/_0^{x_1}$$

$$= \frac{c^2}{2}(e^{\frac{x_1}{c}} - e^{-\frac{x_1}{c}}), \text{ and } s = \frac{c}{2}(e^{\frac{x_1}{c}} - e^{-\frac{x_1}{c}}), \therefore A = cs.$$

2. The curve is symmetrical as to both axes, and when $y=0$, $x = \pm a$,

$\therefore$ whole area $= 4\int_0^a ydx = 4\int_0^a dxb\left\{1 - \left(\dfrac{x}{a}\right)^{\frac{2}{3}}\right\}^{\frac{3}{2}}$, or if $x = a\sin^3\phi$,

$$\text{the area} = 4\int_0^{\frac{\pi}{2}} 3a\sin^2\phi\cos\phi d\phi \,.\, b\cos^3\phi = 12ab\int_0^{\frac{\pi}{2}} d\phi(\cos^4\phi - \cos^6\phi)$$

$$= 12ab \,.\, \frac{\pi}{2}\left\{\frac{3 \,.\, 1}{4 \,.\, 2} - \frac{5 \,.\, 3 \,.\, 1}{6 \,.\, 4 \,.\, 2}\right\} = 6\pi ab \,.\, \frac{1}{6} \,.\, \frac{3}{8} = \frac{3\pi ab}{8}.$$

3. The area, bounded by $y=0$,

$$= \int_0^a dx \,.\, \frac{c^2(a-x)}{a^2+x^2} = \frac{c^2a}{a}\tan^{-1}\frac{x}{a} - \frac{c^2}{2}\log(a^2+x^2)\Big/_0^a = c^2\frac{\pi}{4} - \frac{c^2}{2}\log 2.$$

4. The coefficient of y^2 gives $x=0$ for the asymptote, the other two asymptotes being imaginary; also, the curve is symmetrical as to the axis of x, and, when $y=0$, $x=2a$, and x cannot be greater for real values of y, nor can x be negative. Hence

$$\text{the area} = 2\int_0^\infty xdy = 2\int_0^\infty \frac{dy \,.\, 8a^3}{4a^2+y^2} = 8a^2\tan^{-1}\frac{y}{2a}\Big/_0^\infty = 8a^2 \,.\, \frac{\pi}{2} = 4\pi a^2.$$

The above is simpler than integrating with respect to x.

5. The coefficients of x^2 and y^2 give for asymptotes $y = \pm a$, and two imaginary asymptotes, and the curve is symmetrical as to both axes, which

it meets only at the origin, and there are but two values of y for any value of x, and those equal and opposite. Hence

$$\text{the area} = 4\int_0^a dy\,.\,\frac{ay}{\sqrt{a^2-y^2}} = -4a\sqrt{a^2-y^2}\Big/_0^a = 4a^2.$$

6. The curve is symmetrical as to the axes of x; has only one real asymptote $x=a$; and when $y=0$, $x=0$ or $-a$, and y is real for all values of x from 0 to $-a$ but not for larger negative values, nor for positive values $>a$. Hence the area of the loop is $-2\int_{-a}^0 dx\,.\,x\sqrt{\frac{a+x}{a-x}}$, because x is negative and y must be positive,

$$= -2\int_{-a}^0 dx\frac{ax+x^2}{\sqrt{a^2-x^2}} = -2\int_{-a}^0 dx\frac{ax-a^2+x^2+a^2}{\sqrt{a^2-x^2}}$$

$$= 2a\sqrt{a^2-x^2} + x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a} - 2a^2\sin^{-1}\frac{x}{a}\Big/_{-a}^0$$

$$= 2a^2 + a^2\left(-\frac{\pi}{2}\right) = 2a^2\left(1-\frac{\pi}{4}\right).\quad \text{Cf. Art. 136.}$$

7. Here from Ex. 6 the area $= 2\int_0^a dx\frac{ax+x^2}{\sqrt{a^2-x^2}}$,

and ∴

$$= -2a\sqrt{a^2-x^2} - x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\Big/_0^a$$

$$= 2a^2 + a^2\,.\,\frac{\pi}{2}.$$

8. The real asymptote is given by the coefficient of y^2, i.e., $x=2a$; the curve is symmetrical as to $y=0$; x is not $>2a$ and cannot be negative, and when $y=0$, $x=0$, and for each value of x from 0 to $2a$ there are two real values of y, equal and opposite. Hence

$$\text{the area} = 2\int_0^{2a} dx\,.\,\frac{x^{\frac{3}{2}}}{\sqrt{2a-x}} = -4x^{\frac{3}{2}}\sqrt{2a-x}\Big/_0^{2a} + 6\int_0^{2a} dx\sqrt{2ax-x^2}$$

$$= 6\int_0^{2a} dx\sqrt{a^2-(x-a)^2} = 3(x-a)\sqrt{2ax-x^2}\Big/_0^{2a} + 3a^2\sin^{-1}\frac{x-a}{a}\Big/_0^{2a}$$

$$= 3a^2\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = 3\pi a^2.$$

9. For a given value of x if y be possible, $y \propto$ from $x-\sqrt{a^2-x^2}$ to $x+\sqrt{a^2-x^2}$, and the extreme possible values of x are obtained when these limits of y *coincide*, and therefore $a^2-x^2=0$, and $x=\pm a$. Thus the

$$\text{area} = \int_{-a}^a dx\{x+\sqrt{a^2-x^2} - x + \sqrt{a^2-x^2}\} = 2\int_{-a}^a dx\sqrt{a^2-x^2} = 4\int_0^a dx\sqrt{a^2-x^2}$$

$= 4$ times the quadrant of a circle of radius a obviously, ∴ $=\pi a^2$.

10. These two parabolas meet where $4ax = \left(\frac{x^2}{4a}\right)^2$ and therefore $x = 0$ or $4a$, and the included area will be seen from a figure to involve only positive values of the co-ordinates. Thus the

$$\text{area} = \int_0^{4a} dx\left\{2\sqrt{ax} - \frac{x^2}{4a}\right\} = \frac{4}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{x^3}{12a}\Big|_0^{4a}$$
$$= \left(\frac{4}{3}\,.\,8 - \frac{4^3}{12}\right)a^2 = \frac{4^3}{12}a^2 = \frac{16}{3}a^2.$$

11. The curve is symmetrical as to both axes, and when $y = 0$, $x^2 = 0$ or a^2, $\therefore$ the $\quad \text{area} = 4\int_0^a dx\frac{bx\sqrt{a^2 - x^2}}{a^2} = -\frac{4}{3}\frac{b}{a^2}(a^2 - x^2)^{\frac{3}{2}}\Big|_0^a = \frac{4}{3}ab.$

12. The curve is symmetrical as to both axes; has no real asymptotes; x must be between a and $-a$, and for each value of x between those limits there are only two real values of y, equal and opposite; and when $y = 0$, $x = 0$ or $\pm a$. Thus there are two equal loops and the area of either

$$= 2\int_0^a dx\frac{x}{a^{\frac{1}{2}}}(a^2 - x^2)^{\frac{1}{4}} = -\frac{4}{5\sqrt{a}}(a^2 - x^2)^{\frac{5}{4}}\Big|_0^a = \frac{4}{5}a^2.$$

13. Here $x = -\sqrt{c^2 - y^2} + c\log\frac{c + \sqrt{c^2 - y^2}}{y}$, and the lts. of y are 0 and c,

$$\therefore \quad \text{area} = \int_0^c dy\left\{-\sqrt{c^2 - y^2} + c\log\left(\frac{c + \sqrt{c^2 - y^2}}{y}\right)\right\}$$
$$= -\frac{\pi c^2}{4} + cy\log\frac{c + \sqrt{c^2 - y^2}}{y}\Big|_0^c - c\int_0^c dy\,.\,y\left\{\frac{-y}{\sqrt{c^2 - y^2}\,(c + \sqrt{c^2 - y^2})} - \frac{1}{y}\right\}$$
$$= -\frac{\pi c^2}{4} + c\int_0^c dy\left\{\frac{y^2 + c\sqrt{c^2 - y^2} + c^2 - y^2}{\sqrt{c^2 - y^2}\,(c + \sqrt{c^2 - y^2})}\right\}$$
$$= -\frac{\pi c^2}{4} + c^2\int_0^c \frac{dy}{\sqrt{c^2 - y^2}} = -\frac{\pi c^2}{4} + \frac{\pi}{2}\,.\,c^2 = \frac{\pi c^2}{4}.$$

14. The curve is symmetrical as to both axes; has no real asymptotes; x must lie between a and $-a$, and for each value of x, y has but two values, equal and opposite; and when $y = 0$, $x^2 = 0$ or a^2. Thus the curve consists of two equal loops, and the area of each

$$= 2\int_0^a dx\,.\,x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}} = 2\int_0^a dx\frac{a^2x - x^3}{\sqrt{a^4 - x^4}} = \int_0^a dx\left\{\frac{a^2\,.\,2x}{\sqrt{a^4 - (x^2)^2}} - \frac{2x^3}{\sqrt{a^4 - x^4}}\right\}$$
$$= a^2\sin^{-1}\frac{x^2}{a^2}\Big|_0^a + \sqrt{a^4 - x^4}\Big|_0^a = \frac{\pi a^2}{2} - a^2 = \frac{a^2}{2}(\pi - 2).$$

15. The curve is symmetrical as to $y=0$; x cannot be $>\frac{a}{2}$ but may have any less value up to $-\infty$; for each of such values of x, y has two equal and opposite values, and when $y=0$, $x=0$ or $\frac{a}{2}$, so that there is a loop from $x=0$ to $x=\frac{a}{2}$. Hence the area of the loop is

$$2\int_0^{\frac{a}{2}} \frac{dx \,.\, bx\sqrt{a^2-2ax}}{4a^2} = -\frac{b}{2a^2}\cdot\frac{1}{3a}x(a^2-2ax)^{\frac{3}{2}}\Big/_0^{\frac{a}{2}} + \frac{b}{6a^3}\int_0^{\frac{a}{2}} dx(a^2-2ax)^{\frac{3}{2}}$$

$$= -\frac{b}{6a^3}\cdot\frac{1}{5a}(a^2-2ax)^{\frac{5}{2}}\Big/_0^{\frac{a}{2}} = \frac{b\,.\,a^5}{30a^4} = \frac{ab}{30}.$$

16. The curve is symmetrical as to both axes; when $x=0$, $y^2=\frac{a^2}{2}$, and when $y=0$, $x=\pm a$, and between these values of x, y is finite, and there are two real values of y (equal, and opposite in sign) for each value of x. Thus there is a loop from $x=-a$ to a. For values of $x^2>a^2$, y increases indefinitely in magnitude with x. Hence the area of the loop is

$$4\int_0^a \frac{dx(a^2-x^2)}{\sqrt{2}\sqrt{a^2+x^2}} = \frac{4a^2}{\sqrt{2}}\log\{x+\sqrt{x^2+a^2}\}\Big|_0^a - 2\sqrt{2}x\sqrt{a^2+x^2}\Big|_0^a + 2\sqrt{2}\int_0^a dx\sqrt{a^2+x^2}$$

$$= 2\sqrt{2}a^2\{\log x+\sqrt{x^2+a^2}\} - 2\sqrt{2}x\sqrt{a^2+x^2} + \sqrt{2}x\sqrt{a^2+x^2}$$
$$+\sqrt{2}a^2\log\{x+\sqrt{x^2+a^2}\},$$

from $x=0$ to a, $\quad = 3\sqrt{2}a^2\log(1+\sqrt{2}) - \sqrt{2}a^2\sqrt{2} = a^2\{3\sqrt{2}\log(1+\sqrt{2})-2\}.$

17. The curve is symmetrical as to $x=0$; when $y=0$, $x^2=a^2$, and solving for y, $\quad 2y(a^2+x^2)-2a(a^2-x^2) = \pm(a^2-x^2)\{4a^2-2(a^2+x^2)\}^{\frac{1}{2}}$

$$= \pm\sqrt{2}(a^2-x^2)^{\frac{3}{2}},$$

$\therefore$ x must lie between $-a$ and a, and for each value of x between these lts. there are two values of y; and there are no real asymptotes, and the curve is therefore closed. Thus the area of the curve is

$$2\int_0^a dx\left\{\frac{2a(a^2-x^2)+\sqrt{2}(a^2-x^2)^{\frac{3}{2}}}{2(a^2+x^2)} - \frac{2a(a^2-x^2)-\sqrt{2}(a^2-x^2)^{\frac{3}{2}}}{2(a^2+x^2)}\right\}$$

$$= 2\sqrt{2}\int_0^a dx\frac{(a^2-x^2)^{\frac{3}{2}}}{a^2+x^2}, \text{ or if } x = a\sin\phi,$$

$$\text{the area} = 2\sqrt{2}a^2\int_0^{\frac{\pi}{2}} \frac{d\phi\,\cos^4\phi}{1+\sin^2\phi},$$

but $\quad \dfrac{\cos^4\phi}{1+\sin^2\phi} = \dfrac{\cos^4\phi}{2-\cos^2\phi} = \dfrac{\cos^4\phi - 2\cos^2\phi + 2\cos^2\phi - 4 + 4}{2-\cos^2\phi},$

$$\therefore \quad \text{the area} = 2\sqrt{2}a^2\int_0^{\frac{\pi}{2}} d\phi\left\{-\cos^2\phi - 2 + \frac{4}{2\sin^2\phi + \cos^2\phi}\right\}$$

$$= -\frac{\pi}{2}\left\{\frac{1}{2}+2\right\}2\sqrt{2}a^2 + 8\sqrt{2}a^2\int_0^{\frac{\pi}{2}} d\phi\frac{\sec^2\phi}{1+2\tan^2\phi}$$

$$= -\frac{5}{\sqrt{2}}\pi a^2 + 4\sqrt{2}a^2\int_0^{\frac{\pi}{2}} d\phi\frac{\sec^2\phi}{\frac{1}{2}+\tan^2\phi}$$

$$= -\frac{5\pi a^2}{\sqrt{2}} + 4\sqrt{2}a^2\sqrt{2}\tan^{-1}(\tan\phi\sqrt{2})\Big/_0^{\frac{\pi}{2}}$$

$$= -\frac{5\pi a^2}{\sqrt{2}} + 8a^2 \cdot \frac{\pi}{2} = \pi a^2\left(4 - \frac{5}{\sqrt{2}}\right).$$

18. Replacing $\dfrac{x}{a}$ by x, the area contained between the curve $y = c\sin x\log\sin x$ and $y = 0$ is $\int aydx$, y being taken positively and x between the proper limits. Here as $\sin x$ cannot be > 1, $\log\sin x$ is negative, and therefore y negative; and when $x = 0$, $y = 0$ $\left(\text{for } y = \text{limit of } c\dfrac{\log\sin x}{\frac{1}{\sin x}} = -c\sin x\right)$; when $x = \dfrac{\pi}{2}$, $y = c\log 1 = 0$ again, and when $x = \pi$, $y = 0$ again.

$$\text{Thus the above area} = -a\int_0^{\pi} cdx \,.\, \sin x\log\sin x = -2ac\int_0^{\frac{\pi}{2}} dx\sin x\log\sin x$$

$$= -2ac(\log 2 - 1), \text{ by Ex. 13, Chap. IV.}$$

19. The area $= \displaystyle\int_\alpha^\beta dx \,.\, c\left(\frac{x}{a}\right)^n = \frac{c}{(n+1)} \,.\, \frac{\beta^{n+1} - \alpha^{n+1}}{a^n}$, the area being bounded by the axis of x, as there is but one value of y for each value of x, n being integral or a fraction in its lowest terms with an odd denominator, for if n be a fraction which in its lowest terms has an even denominator there are two values of y (equal and opposite) for each positive value of $\dfrac{x}{a}$, and the area would then be doubled. For $xy = a^2$, $n = -1$, and $a = c$, therefore the

$$\text{area} = \text{the lt. (when } n = -1) \text{ of } \frac{a^2}{n+1}(\beta^{n+1} - \alpha^{n+1})$$

$$= \text{the lt. (when } x = 0) \text{ of } a^2 \,.\, \frac{\beta^x - \alpha^x}{x} = a^2 \,.\, \frac{\beta^x\log\beta - \alpha^x\log\alpha}{1}$$

$$= a^2(\log\beta - \log\alpha).$$

20. Solving for y, $cy = -bx \pm (b^2x^2 - acx^2 + c)^{\frac{1}{2}}$, and the lts. of x are given when these values of y are equal, and therefore by $(ac - b^2)x^2 = c$, say $x = \pm x_1$; therefore

$$\text{the area} = \int_{-x_1}^{x_1} \frac{dx}{c}[-bx+\{c-(ac-b^2)x^2\}^{\frac{1}{2}}+bx+\{c-(ac-b^2)x^2\}^{\frac{1}{2}}]$$
$$=\frac{4}{c}\int_0^{x_1} dx\{c-(ac-b^2)x^2\}^{\frac{1}{2}} = \frac{4\sqrt{ac-b^2}}{c}\int_0^{x_1} dx\{x_1^2-x^2\}^{\frac{1}{2}}$$
$$=\frac{4\sqrt{ac-b^2}}{c}\cdot\frac{\pi x_1^2}{4} = \frac{\pi}{\sqrt{ac-b^2}}.$$

21. This curve is the lemniscate, and the area of either loop is

$$2\int_0^{\frac{\pi}{4}} \frac{r^2}{2} d\theta = \int_0^{\frac{\pi}{4}} d\theta \, . \, a^2 \cos 2\theta = \frac{a^2}{2}\sin 2\theta \Big|_0^{\frac{\pi}{4}} = \frac{a^2}{2}.$$

22. If n be a positive integer, when $\theta=0$, $r=0$; when $n\theta=\frac{\pi}{2}$, $r=a$, and when $n\theta=\pi$, $r=0$ again; thus there is a loop from $\theta=0$ to $\frac{\pi}{n}$: and turning the initial line round the pole through the angle $\frac{\pi}{n}$, the equation becomes $r=a\sin n\left(\theta+\frac{\pi}{n}\right)=-a\sin n\theta$, thus there is another equal loop bounded by $\theta=\frac{\pi}{n}$ and $\theta=\frac{2\pi}{n}$, both produced through the pole; and so on. Thus the angular space 2π round the pole is divided into $2n$ equal parts, and in each of the odd divisions counting from the initial line there is a loop, in the even divisions a loop produced backwards. Thus on the whole there are $2n$ or n loops according as the vacant even divisions from $\theta=0$ to π, are or are not filled up by the negative values of r from $\theta=\pi$ to 2π. To test this, examining the case of the second division, the corresponding division on the opposite side of the pole, is the $\left(2+\frac{\pi}{\frac{\pi}{n}}\right)$th or $(2+n)$th division, and there are therefore $2n$ loops if $2+n$ be even (*when* r is negative), and there are only n loops if $2+n$ and therefore n be odd.

Also the area of each loop $=\int_0^{\frac{\pi}{n}} \frac{a^2\sin^2 n\theta}{2} \, . \, d\theta$,

or putting $n\theta$ for θ, $\quad = \int_0^{\pi} \frac{a^2}{2n}\sin^2\theta d\theta = \frac{a^2}{n}\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta = \frac{\pi a^2}{4n}$.

Hence the whole area is $\frac{\pi a^2}{2}$ or $\frac{\pi a^2}{4}$ as n is even or odd. If n be a negative integer, considering negative values of θ, a similar result follows, the only difference being that when n is odd the gaps in the first case become loops, and *vice versa*. *Generally* when n is odd, there are really n pairs of coincident loops.

If $n=\frac{p}{q}$ a fraction in its lowest terms, (1) if $q=2$, p is odd, and there are $2n=p$ divisions in all, and p being odd, no two divisions are exactly opposite, and therefore some of the loops overlap others; (2) if q be not 2, the number of divisions is fractional, and therefore again there is overlapping and the problem of determining the area would be very difficult and involve special interpretation.

23. If the initial be turned round the pole through an angle $-\frac{\pi}{2n}$, the equation becomes $r = a\cos n\left(\theta - \frac{\pi}{2n}\right) = a\sin n\theta$, and the greatest value of r being a, the area between the circle $r = a$, and the loops $= \pi a^2 - \frac{\pi a^2}{2} = \frac{\pi a^2}{4}$ or $\frac{3\pi a^2}{4}$ as n is even or odd, by Ex. 22.

24. From $\theta = 0$ to $\frac{\pi}{3}$, $\pm r$ increases from 0 to $\frac{\sqrt{2}a}{\sqrt[4]{3}}$ and then decreases to 0, thus forming 2 equal loops; from $\theta = \frac{\pi}{3}$ to $\frac{\pi}{2}$, r is impossible, since $\cos\theta$ is then positive and $\sin 3\theta$ is negative; from $\theta = \frac{\pi}{2}$ to $\frac{2\pi}{3}$, r decreases from $\pm\infty$ to 0. For the remaining range of angular space, from $\theta = 0$ to $-\frac{\pi}{3}$, r is impossible. Thus the area of a loop is

$$\int_0^{\frac{\pi}{3}} \frac{r^2}{2} \,.\, d\theta = \int_0^{\frac{\pi}{3}} \frac{a^2}{2} \,.\, d\theta \frac{\sin 3\theta}{\cos\theta} = \int_0^{\frac{\pi}{3}} \frac{a^2}{2} \frac{d\theta}{\cos\theta}\{3\sin\theta - 4\sin\theta(1-\cos^2\theta)\}$$

$$= \frac{a^2}{2}\{\log\cos\theta - \cos 2\theta\}\Big/_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left\{\log\frac{1}{2} + \frac{1}{2} + 1\right\} = \frac{3a^2}{4} - \frac{a^2}{2}\log 2.$$

25. Here $r = a(\cos 2\theta + \sin 2\theta) = a\sqrt{2}\,.\cos\left(2\theta - \frac{\pi}{4}\right)$, and turning the initial line round the pole through the angle $\frac{\pi}{8}$, this becomes $r = a\sqrt{2}\cos 2\theta$, and therefore by the method of Ex. 22, as the area $= \int \frac{r^2}{2} d\theta$ between the proper limits, the whole area $= \frac{\pi a^2}{2}\,.\,(\sqrt{2})^2 = \pi a^2$.

26. Changing to polar co-ordinates the equation becomes $\pm r = a\sin 2\theta$, therefore r is limited in value, and from $\theta = 0$ to $\frac{\pi}{2}$, $\pm r \propto$ from 0 to 0 again, thus there are two equal loops; turning the initial line through the angle $\frac{\pi}{2}$, the equation becomes $\pm r = a\sin(2\theta + \pi) = -a\sin 2\theta$, which gives the same results, and so on. Thus there must be four equal loops, and the area of each is

$$\int_0^{\frac{\pi}{2}} d\theta \,.\, \frac{a^2}{2}\sin^2 2\theta = \frac{a^2}{4}\int_0^{\frac{\pi}{2}} d\theta(1 - \cos 4\theta) = \frac{\pi a^2}{8} + \frac{\sin 4\theta}{16}a^2\Big/_0^{\frac{\pi}{2}} = \frac{\pi a^2}{8}.$$

27. Changing to polar co-ordinates $r^2=4a^2\cos^2\theta+4b^2\sin^2\theta$, and therefore r is limited in value, and for any particular value, as α of θ, r has only two equal and opposite values, corresponding also to $\alpha+\pi$, thus the curve is a single closed curve, and it is symmetrical as to both axes. Hence

$$\text{the area}=4\int_0^{\frac{\pi}{2}}\frac{r^2d\theta}{2}=8\int_0^{\frac{\pi}{2}}d\theta(a^2\cos^2\theta+b^2\sin^2\theta)$$
$$=4\pi\left(\frac{a^2}{2}+\frac{b^2}{2}\right)=2\pi(a^2+b^2).$$

28. If $x=ar\cos\theta$ and $y=br\sin\theta$, the equation becomes

$$r^2=\frac{c^2}{a^2}\cos^2\theta+\frac{c^2}{b^2}\sin^2\theta,$$

and $$\text{the area}=4\int_0^{\frac{\pi}{2}}\frac{abr^2d\theta}{2},\text{ as in Ex. 27,}$$

and ∴ $$=ab\,.\,2\pi\left(\frac{c^2}{4a^2}+\frac{c^2}{4b^2}\right)=\frac{\pi c^2}{2ab}(a^2+b^2).$$

29. If x and y be of the same sign, both x and y are finite, but they can be ∞ if of different signs; and if x and y be both negative they are unreal, therefore for a loop they must be both positive, and the tangents at the origin are the axes. Hence, changing to polar co-ordinates,

$$r=\frac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta},$$

and the area of the loop is

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{2}\cdot\frac{9a^2\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}=\frac{9a^2}{2}\int_0^{\frac{\pi}{2}}\frac{d\theta\tan^2\theta\sec^2\theta}{(\tan^3\theta+1)^2}=-\frac{3a^2}{2}\cdot\frac{1}{\tan^3\theta+1}\bigg|_0^{\frac{\pi}{2}}=\frac{3a^2}{2}.$$

30. The curve is symmetrical as to the initial line; from $\theta=0$ to $\frac{\pi}{4}$, r decreases from a to 0; from $\theta=\frac{\pi}{4}$ to $\frac{\pi}{2}$, $r\propto$ from 0 to $-\infty$; from $\theta=\frac{\pi}{2}$ to $\frac{3\pi}{4}$, $r\propto$ from ∞ to 0; and from $\theta=\frac{3\pi}{4}$ to π, $r\propto$ from 0 to $-a$. Thus there is a loop from $\theta=-\frac{\pi}{4}$ to $\frac{\pi}{4}$, and its area is

$$\int_0^{\frac{\pi}{4}}d\theta\frac{a^2\cos^2 2\theta}{\cos^2\theta}=\frac{1}{2}\int_0^{\frac{\pi}{2}}d\theta\frac{a^2\cos^2\theta}{\cos^2\frac{\theta}{2}},\text{ (putting }\theta\text{ for }2\theta)$$

$$=a^2\int_0^{\frac{\pi}{2}}d\theta\frac{\cos^2\theta}{1+\cos\theta}=a^2\int_0^{\frac{\pi}{2}}d\theta\left(\frac{\cos^2\theta+\cos\theta-\cos\theta-1+1}{1+\cos\theta}\right)$$

$$= a^2\int_0^{\frac{\pi}{2}} d\theta\left\{\cos\theta - 1 + \frac{1}{2}\sec^2\frac{\theta}{2}\right\} = a^2\sin\theta\Big/_0^{\frac{\pi}{2}} - \frac{\pi}{2}\cdot a^2 + a^2\tan\frac{\theta}{2}\Big/_0^{\frac{\pi}{2}}$$

$$= a^2\left(1 - \frac{\pi}{2}\right) + a^2 = a^2\left(2 - \frac{\pi}{2}\right).$$

31. With the positive sign of the radical, for each value of θ there is but one positive value of r, since $\frac{a^2}{\sqrt{a^2 - b^2\cos^2\theta}} > a$ and therefore $> b\cos\theta$, and the area is $\frac{1}{2}\int_0^{\pi} d\theta\{r_1^2 + r_2^2\}$, where $r_1 = \frac{a^2}{\sqrt{a^2 - b^2\cos^2\theta}} + b\cos\theta$, and

$$r_2 = \frac{a^2}{\sqrt{a^2 - b^2\cos^2(\theta + \pi)}} + b\cos(\theta + \pi) = \frac{a^2}{\sqrt{a^2 - b^2\cos^2\theta}} - b\cos\theta,$$

$$\text{area} = \int_0^{\pi} d\theta\left(\frac{a^4}{a^2 - b^2\cos^2\theta} + b^2\cos^2\theta\right) = 2\int_0^{\frac{\pi}{2}} d\theta\left\{\frac{a^4\sec^2\theta}{a^2\tan^2\theta + a^2 - b^2} + b^2\cos^2\theta\right\}$$

$$= 2b^2\cdot\frac{\pi}{4} + 2a^2\int_0^{\frac{\pi}{2}}\frac{d\theta\,.\sec^2\theta}{\frac{a^2 - b^2}{a^2} + \tan^2\theta} = \frac{\pi b^2}{2} + \frac{2a^3}{\sqrt{a^2 - b^2}}\tan^{-1}\frac{a\tan\theta}{\sqrt{a^2 - b^2}}\Big/_0^{\frac{\pi}{2}}$$

$$= \frac{\pi b^2}{2} + \frac{\pi a^3}{\sqrt{a^2 - b^2}}.$$

Comparing the values of r_1 and r_2 it will be seen that if the radical be considered negative, the same curve is only obtained over again, as if the initial line had been turned through the angle π.

32. See Art. 145.

33. (1) If $a < b$, putting $\theta + \pi$ for θ, $r = a - b\operatorname{cosec}\theta$, which is negative, therefore, corresponding to the *direction* θ, r measured positively above the initial line is either $a + b\operatorname{cosec}\theta$ or $b\operatorname{cosec}\theta - a$, and accordingly the area between the limits θ_1, θ_2 corresponding to the two radii vectores (on the same branch) is

$$\int_{\theta_1}^{\theta_2}\frac{d\theta}{2}(b\operatorname{cosec}\theta \pm a)^2 = \frac{1}{2}\int_{\theta_1}^{\theta_2} d\theta(b^2\operatorname{cosec}^2\theta \pm 2ab\operatorname{cosec}\theta + a^2)$$

$$= \frac{b^2}{2}(\cot\theta_1 - \cot\theta_2) \pm ab\log\frac{\tan\frac{\theta_2}{2}}{\tan\frac{\theta_1}{2}} + \frac{a^2}{2}(\theta_2 - \theta_1).$$

(2) If $a > b$ the above holds so long as $\sin\theta < \frac{b}{a}$, or, if α be the least value of $\sin^{-1}\frac{b}{a}$, so long as $\theta < \alpha$ or $> \pi - \alpha$; but if θ lie between these limits, for the upper branch $r = a + b\operatorname{cosec}\theta$ as before, but for the lower

branch, putting $\theta+\pi$ for θ, $r=a-b\operatorname{cosec}\theta$, which is positive and corresponds to a point on the loop below the pole. The integration will still be the same unless $\theta_1<a$ and $\theta_2>a$, and so on, i.e., unless one radius vector be drawn above the initial line and the other below, or both above the initial line but on opposite sides of $r\cos\theta=0$, in which cases some convention would be required. Thus the limits of θ might be taken as from θ_1 to a, and from $a+\pi$ to θ_2; or from θ_1 to a and from $\frac{\pi}{2}+a$ to θ_2.

34. The equation to the ellipse being

$$a^2\sin^2\theta+b^2\cos^2\theta=\frac{a^2b^2}{r^2}, \quad \text{.............................(1)}$$

the area included between the curve and the radii vectores r_1 and r_2 corresponding to the values θ_1 and θ_2 of θ is

$$\frac{1}{2}\int_{\theta_1}^{\theta_2}d\theta\cdot\frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}=\frac{b^2}{2}\int_{\theta_1}^{\theta_2}\frac{d\theta\,.\sec^2\theta}{\frac{b^2}{a^2}+\tan^2\theta}=\frac{ab}{2}\tan^{-1}\frac{a\tan\theta}{b}\Big/_{\theta_1}^{\theta_2}$$

$$=\frac{ab}{2}\left\{\tan^{-1}\left(\frac{a}{b}\tan\theta_2\right)-\tan^{-1}\left(\frac{a}{b}\tan\theta_1\right)\right\}.$$

This can be expressed in terms of r_1 and r_2 from equation (1).

35. The parabola $y^2=4ax$ in polar co-ordinates is $r\sin^2\theta=4a\cos\theta$, and

$$\text{the area}=\frac{1}{2}\int_{\theta_1}^{\theta_2}d\theta\,.\frac{16a^2\cos^2\theta}{\sin^4\theta}=8a^2\int_{\theta_1}^{\theta_2}d\theta\cot^2\theta\operatorname{cosec}^2\theta=-\frac{8a^2}{3}(\cot^3\theta_2-\cot^3\theta_1).$$

36. The part of the curve bounded by the asymptote is given by $\theta=0$ to π, $\therefore$

$$\text{the area}=\frac{1}{2}\int_0^{\pi}d\theta\{4a^2\sec^2\theta-a^2(\sec\theta+\tan\theta)^2\}$$

$$=\frac{a^2}{2}\int_0^{\pi}d\theta\{3\sec^2\theta-2\sec\theta\tan\theta+1-\sec^2\theta\}$$

$$=a^2\int_0^{\frac{\pi}{2}}d\theta(2\sec^2\theta-2\sec\theta\tan\theta+1)$$

$$=a^2\,.\frac{\pi}{2}+a^2(2\tan\theta-2\sec\theta)\Big/_0^{\frac{\pi}{2}}$$

$$=\frac{\pi a^2}{2}+2a^2\,.\,\text{lt. of}\,\frac{\sin\theta-1}{\cos\theta}\Big/_{\theta=\frac{\pi}{2}}+2a^2$$

$$=\frac{\pi a^2}{2}+2a^2\frac{\cos\theta}{-\sin\theta}\Big/_{\theta=\frac{\pi}{2}}+2a^2=a^2\left(\frac{\pi}{2}+2\right).$$

37. From $\theta=0$ to $\frac{2}{3}\pi$, r decreases from $3a$ to 0; and so from $\theta=0$ to $-\frac{2}{3}\pi$; from $\theta=\frac{2}{3}\pi$ to π, r ∝ from 0 to $-a$, and so from $-\frac{2}{3}\pi$ to $-\pi$. Thus

the curve consists of two closed curves meeting in the pole, and for any value of $\theta < \frac{\pi}{3}$, for the first curve $r_1 = 2a\cos\theta + a$, and in the same *direction* on the second curve, $r_2 = 2a\cos\theta - a$, therefore measured in the same direction $r_1 > r_2$; therefore it follows that the second curve lies within the first. Also the area of the inner loop is

$$2a^2\int_{\frac{2\pi}{3}}^{\pi}\frac{d\theta}{2}\cdot(2\cos\theta+1)^2 = a^2\int_{\frac{2\pi}{3}}^{\pi}d\theta\{2(1+\cos 2\theta)+4\cos\theta+1\}$$

$$= a^2\left\{3\frac{\pi}{3}+\frac{\sqrt{3}}{2}-\frac{4\sqrt{3}}{2}\right\} = a^2\left(\pi - \frac{3\sqrt{3}}{2}\right);$$

and the area of both loops is

$$a^2\int_0^{\pi}d\theta\{4\cos^2\theta+4\cos\theta+1\} = a^2\left\{8\cdot\frac{\pi}{2}\cdot\frac{1}{2}+\pi\right\} = 3\pi a^2,$$

therefore the area of the outer curve, which is the whole area bounded externally by the curve, $= a^2\left(2\pi + \frac{3\sqrt{3}}{2}\right)$.

38. This is only a more general case of Ex. 37, where instead of $\frac{2\pi}{3}$, the critical angle is α, where $\cos\alpha = -\frac{b}{a}$. Thus the area of the inner loop is

$$\int_{\alpha}^{\pi}d\theta\left\{\frac{a^2}{2}(1+\cos 2\theta)+2ab\cos\theta+b^2\right\} = \left(\frac{a^2}{2}+b^2\right)(\pi-\alpha) - \frac{a^2}{4}\sin 2\alpha - 2ab\sin\alpha;$$

and the area of both curves is

$$\int_0^{\pi}d\theta\{a^2\cos^2\theta+2ab\cos\theta+b^2\} = \frac{a^2\pi}{2}+b^2\pi,$$

and $\therefore$ outer area is $\quad a\left(\frac{a^2}{2}+b^2\right)+\frac{a^2}{4}\sin 2\alpha + 2ab\sin\alpha.$

39. If x', y' be current co-ordinates, and the area u be divided into strips parallel to the axis of x',

$$u = \int_0^y dy'\left\{\sqrt{a^2+y'^2}-\frac{y'x}{y}\right\} = \frac{y'}{2}\sqrt{a^2+y'^2}+\frac{a^2}{2}\log\{y'+\sqrt{a^2+y'^2}\} - \frac{y'^2x}{2y}\Big|_0^y$$

$$= \frac{y}{2}\sqrt{a^2+y^2} - \frac{xy}{2} + \frac{a^2}{2}\log\frac{y+\sqrt{a^2+y^2}}{a} = \frac{a^2}{2}\log\frac{y+\sqrt{a^2+y^2}}{a},$$

$$e^{\frac{2u}{a^2}} = \frac{y+\sqrt{a^2+y^2}}{a}, \text{ and } e^{-\frac{2u}{a^2}} = \frac{\sqrt{a^2+y^2}-y}{a},$$

$$e^{\frac{2u}{a^2}} + e^{-\frac{2u}{a^2}} = \frac{2\sqrt{a^2+y^2}}{a} = \frac{2x}{a}, \text{ and } e^{\frac{2u}{a^2}} - e^{-\frac{2u}{a^2}} = \frac{2y}{a}, \ \therefore \text{ etc.}$$

40. The curve is symmetrical as to both axes, and any line $\theta = \alpha$ cuts it in two points equidistant from the pole, so that it is a single closed curve. Hence

$$\text{the area} = 4\int_0^{\frac{\pi}{2}} \frac{d\theta}{2} \cdot \frac{(a^2-b^2)^2}{a^2+b^2} \frac{(a^2\sin^2\theta - b^2\cos^2\theta)^2}{(a^2\sin^2\theta + b^2\cos^2\theta)^2} = A \text{ say,}$$

$$\therefore \frac{A(a^2+b^2)}{2(a^2-b^2)^2} = \int_0^{\frac{\pi}{2}} d\theta \left\{1 - \frac{4a^2b^2\sin^2\theta\cos^2\theta}{(a^2\sin^2\theta + b^2\cos^2\theta)^2}\right\}$$

$$= \frac{\pi}{2} + \frac{2a^2b^2}{a^2-b^2} \left. \frac{\sin\theta\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}\right/_0^{\frac{\pi}{2}} - \frac{2a^2b^2}{a^2-b^2}\int_0^{\frac{\pi}{2}} \frac{d\theta \,.\, \cos 2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{\pi}{2} + \frac{4a^2b^2}{a^2-b^2}\int_0^{\frac{\pi}{2}} d\theta \frac{\sin^2\theta + \frac{b^2}{a^2-b^2} - \frac{1}{2} - \frac{b^2}{a^2-b^2}}{(a^2-b^2)\sin^2\theta + b^2}$$

$$= \frac{\pi}{2}\left\{1 + \frac{4a^2b^2}{(a^2-b^2)^2}\right\} - \frac{2a^2b^2(a^2+b^2)}{(a^2-b^2)^2}\int_0^{\frac{\pi}{2}} \frac{d\theta \,.\, \sec^2\theta}{a^2\tan^2\theta + b^2}$$

$$= \frac{\pi}{2}\left(\frac{a^2+b^2}{a^2-b^2}\right)^2 - \frac{2b^2(a^2+b^2)}{(a^2-b^2)^2} \frac{a}{b} \tan^{-1}\frac{a\tan\theta}{b} \Big/_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\left(\frac{a^2+b^2}{a^2-b^2}\right)^2 - \frac{2ab(a^2+b^2)}{(a^2-b^2)^2} \frac{\pi}{2} = \frac{\pi}{2} \frac{(a^2+b^2)}{(a^2-b^2)^2}(a-b)^2,$$

$$\therefore \qquad A = \pi(a-b)^2.$$

41. Here the straight line joining the extremities of the arc is the part of the radius vector in the initial position produced to meet the next branch of the spiral. Thus the area in question

$$= \int_{\theta_1}^{\theta_1+2\pi} \frac{r^2}{2} d\theta,$$ and r is given in terms of θ by the equation to the spiral. Hence when $r = a\left(\frac{\theta}{2\pi}\right)^n$,

$$\text{the area} = \frac{a^2}{2}\int_{\theta_1}^{\theta_1+2\pi} d\theta \,.\, \left(\frac{\theta}{2\pi}\right)^{2n},$$ or if θ denote the superior limit,

$$= \pi a^2 \int_{\theta-2\pi}^{\theta} d\theta \,.\, \frac{\theta^{2n}}{(2\pi)^{2n+1}} = \frac{\pi a^2}{2n+1}\left\{\left(\frac{\theta}{2\pi}\right)^{2n+1} - \left(\frac{\theta-2\pi}{2\pi}\right)^{2n+1}\right\}.$$

To avoid negative values of θ, when r would be positive or negative as n was even or odd, θ must be taken $\succ 2\pi$.

42. If ϕ_1, ϕ_2 be the angles, measured in the same sense, which the two radii of curvature make with the axis, the area is $\frac{1}{2}\int_{\phi_1}^{\phi_2}\rho^2 d\phi$, and in the parabola $\rho = \frac{ds}{d\phi} = 2a\sec^3\phi$,

$$\therefore \quad \text{the area} = 2a^2\int_{\phi_1}^{\phi_2} d\phi \sec^2\phi(1+\tan^2\phi)^2$$

$$= 2a^2\left\{\frac{1}{5}(\tan^5\phi_2 - \tan^5\phi_1) + \frac{2}{3}(\tan^3\phi_2 - \tan^3\phi_1) + \tan\phi_2 - \tan\phi_1\right\}.$$

43. In the cycloid $s = 4a\sin\phi$,

$$\therefore \quad \rho = \frac{ds}{d\phi} = 4a\cos\phi,$$

and the area in question $= \frac{1}{2}\int_{\phi_1}^{\phi_2} d\phi\,.\,\rho^2$, where ϕ_1, ϕ_2 correspond to the two radii of curvature,

$$= 8a^2\int_{\phi_1}^{\phi_2} d\phi\,.\cos^2\phi$$

$$= 4a^2\int_{\phi_1}^{\phi_2} d\phi(1+\cos 2\phi)$$

$$= 4a^2\left\{(\phi_2 - \phi_1) + \frac{1}{2}(\sin 2\phi_2 - \sin 2\phi_1)\right\}.$$

44. It follows from the equation $xy = \kappa^2$, that, corresponding to any value of x, there is one and but one value of y.

The area between the values x_1 and x_2 of x is (Art. 161)

$$2\pi\int_{x_1}^{x_2} y\frac{ds}{dx}\,.\,dx: \text{ and } y = \frac{\kappa^2}{x}, \ \therefore \ \frac{ds}{dx} = \sqrt{1+\frac{\kappa^4}{x^4}}, \ \therefore \ \text{area} = 2\pi\int_{x_1}^{x_2} dx\,.\,\frac{\kappa^2}{x^3}\sqrt{\kappa^4+x^4};$$

but
$$\int\frac{dx}{x^3}\sqrt{\kappa^4+x^4} = -\frac{\sqrt{\kappa^4+x^4}}{2x^2} + \frac{1}{2}\int dx\,.\,\frac{2x}{\sqrt{\kappa^4+x^4}} = -\frac{\sqrt{\kappa^4+x^4}}{2x^2} + \frac{1}{2}\int\frac{dz}{\sqrt{\kappa^4+z^2}},$$

(if $z = x^2$), and $\therefore$
$$= -\frac{\sqrt{\kappa^4+x^4}}{2x^2} + \frac{1}{2}\log\{x^2 + \sqrt{\kappa^4+x^4}\},$$

$$\therefore \quad \text{the area} = \pi\kappa^2\left\{\log\frac{x_2^2 + \sqrt{\kappa^4+x_2^4}}{x_1^2 + \sqrt{\kappa^4+x_1^4}} - \frac{\sqrt{\kappa^4+x_2^4}}{x_2^2} + \frac{\sqrt{\kappa^4+x_1^4}}{x_1^2}\right\}.$$

45. Here $1 = \frac{y}{c}\,.\,\frac{dx}{dy}$, $\therefore \ \frac{ds}{dy} = \frac{\sqrt{c^2+y^2}}{y}$, and the area is

$$2\pi\int_{y_1}^{y_2} y\frac{ds}{dy}\,.\,dy = 2\pi\int_{y_1}^{y_2} dy\sqrt{c^2+y^2},$$

where y_1, y_2 correspond to the limits of x supposed; thus the area is

$$\pi y\sqrt{c^2 - y^2} + \pi c^2 \log\left(y + \sqrt{c^2 - y^2}\right),$$

taken between the proper limits.

46. As $x=0$ bisects the curve, by a complete revolution round $x=0$, the two portions of the curve only produce the same surface, and this

$$= 2\pi\int_{x_1}^{x_2} x\frac{ds}{dx}\,.\,dx, \text{ and } s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right), \quad \therefore \quad \frac{ds}{dx} = \frac{y}{c},$$

and $\quad$ the area $= 2\pi\int_{x_1}^{x_2} dx\,.\,\frac{x}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) = \pi cx\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)\Big/_{x_1}^{x_2} - \pi c\int_{x_1}^{x_2} dx\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$

$$= \pi cx\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) - \pi c^2\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) \text{ taken between proper limits.}$$

47. With the equation $a^2y^2 + b^2x^2 = a^2b^2$,

$$\frac{dx}{dy} = -\frac{a^2y}{b^2x}, \quad \therefore \quad \frac{ds}{dy} = \frac{1}{b^2x}\sqrt{a^4y^2 + a^2b^4 - a^2b^2y^2} = \frac{a}{b^2x}\sqrt{a^2e^2y^2 + b^4},$$

and $\therefore$ the whole surface is

$$2\int_0^b 2\pi x\frac{ds}{dy}\,.\,dy = 4\pi\int_0^b dy\,.\,\frac{a^2e}{b^2}\sqrt{y^2 + \frac{b^4}{a^2e^2}}$$

$$= 2\pi\frac{a^2e}{b^2}\left\{y\sqrt{y^2 + \frac{b^4}{a^2e^2}} + \frac{b^4}{a^2e^2}\log\left(y + \sqrt{y^2 + \frac{b^4}{a^2e^2}}\right)\right\}\Bigg|_0^b$$

$$= \frac{2\pi a^2e}{b^2}\left\{b^2\sqrt{1 + \frac{b^2}{a^2e^2}} + \frac{b^4}{a^2e^2}\log\frac{b + b\sqrt{1 + \frac{b^2}{a^2e^2}}}{\frac{b^2}{ae}}\right\}$$

$$= 2\pi a^2e\left\{\frac{1}{e} + \frac{(1-e^2)}{e^2}\log\frac{1 + \frac{1}{e}}{\frac{1}{e}\sqrt{(1-e^2)}}\right\}$$

$$= 2\pi a^2\left\{1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right\}.$$

48. Here (cf. Todhunter's *Diff. Cal.*, Art. 358) $x = a(1 - \cos\theta)$, also $s = 4a\sin\frac{\theta}{2}$, $\therefore$ the whole surface in question

$$= 2\int_0^\pi 2\pi x\,.\,\frac{ds}{d\theta}\,.\,d\theta = 4\pi a^2\int_0^\pi (1 - \cos\theta)2\cos\frac{\theta}{2}d\theta = 16\pi a^2\int_0^\pi d\theta\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 32\pi a^2\int_0^{\frac{\pi}{2}} d\theta\sin^2\theta\cos\theta = \frac{32}{3}\pi a^2.$$

49. Here the surface $= 2\int_0^\pi 2\pi(2a - x)\frac{ds}{d\theta} . d\theta = 4\pi a^2 \int_0^\pi d\theta(1 + \cos\theta)2\cos\frac{\theta}{2}$

$$= 16\pi a^2 \int_0^\pi d\theta . \cos^3\frac{\theta}{2} = 32\pi a^2 \int_0^{\frac{\pi}{2}} d\theta . \cos^3\theta$$

$$= 32\pi a^2 . \frac{2}{3.1} = \frac{64}{3}\pi a^2.$$

50. Here the surface is, since $y = a(\theta + \sin\theta)$,

$$\int_0^\pi 2\pi y \frac{ds}{d\theta} . d\theta = \int_0^\pi 2\pi a(\theta + \sin\theta) . 2a\cos\frac{\theta}{2} d\theta$$

$$= 4\pi a^2 \int_0^\pi d\theta\left(\theta\cos\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)$$

$$= 8\pi a^2 \int_0^{\frac{\pi}{2}} d\theta(2\theta\cos\theta + 2\sin\theta\cos^2\theta),$$

and $\int\theta\cos\theta d\theta = \theta\sin\theta + \cos\theta,$

$\therefore$ the surface is $8\pi a^2 \left\{2\theta\sin\theta + 2\cos\theta - \frac{2}{3}\cos^3\theta\right\}\Big/_0^{\frac{\pi}{2}}$

$$= 8\pi a^2\left\{\pi - 2 + \frac{2}{3}\right\} = 8\pi a^2\left(\pi - \frac{4}{3}\right).$$

51. The equation to the tractory being (Art. 100)

$$x + \sqrt{c^2 - y^2} = c\log\frac{c + \sqrt{c^2 - y^2}}{y}, \text{ as in Ex. 2, Chap. VI.}$$

$\frac{ds}{dy} = -\frac{c}{y}$, and $\therefore$ the whole surface is

$$2\int_c^0 2\pi y . \frac{ds}{dy} . dy = 4\pi c \int_0^c dy = 4\pi c^2.$$

52. If the equations of the sphere and one of the cylinders be $x^2 + y^2 + z^2 = a^2$, and $x^2 + y^2 = ax$, as in Art. 170 the *surface* of the sphere included within the cylinder is $2\iint dxdy\sec\gamma$, taken between the proper limits for x and y, which are determined by the equation $x^2 + y^2 = ax$, so that the limits of y are $-\sqrt{ax - x^2}$ and $\sqrt{ax - x^2}$ (say y_1, y_2), and those of x are 0 and a: also $\sec\gamma$ on the sphere $= a \div z$, therefore the above surface

$$= 2\int_0^a \int_{y_1}^{y_2} dxdy . \frac{a}{\sqrt{a^2 - x^2 - y^2}} = 4a\int_0^a dx . \sin^{-1}\frac{y}{\sqrt{a^2 - x^2}}\Big/_0^{y_2}$$

$$= 4a\int_0^a dx\left\{\sin^{-1}\sqrt{\frac{x}{a + x}}\right\};$$

and if $\sqrt{\frac{x}{a + x}} = z$, $x = \frac{az^2}{1 - z^2}$,

$$\therefore \qquad \frac{dx}{dz}=\frac{a}{(1-z^2)^2}\left\{2z(1-z^2)+2z^3\right\}=\frac{2az}{(1-z^2)^2}, \text{ and}$$

$$\therefore \int dx \sin^{-1}\sqrt{\frac{x}{a+x}}=\int\frac{2azdz\sin^{-1}z}{(1-z^2)^2}=\frac{a\sin^{-1}z}{1-z^2}-\int\frac{adz}{(1-z^2)^{\frac{3}{2}}}$$

$$=\frac{a\sin^{-1}z}{1-z^2}-\frac{az}{\sqrt{1-z^2}}, \text{ as above,}$$

$\therefore$ the above surface $=4a\left\{\frac{a\sin^{-1}z}{1-z^2}-\frac{az}{\sqrt{1-z^2}}\right\}_0^{\frac{1}{\sqrt{2}}}$, the lts. corresponding to 0 and a for x,

$$=\frac{4a^2\cdot\frac{\pi}{4}}{\frac{1}{2}}-4a^2=2\pi a^2-4a^2;$$

$\therefore$ the surface not included within the two cylinders

$$=4\pi a^2-2(2\pi a^2-4a^2)=8a^2.$$

53. Here $y=a$ bisects the curve, and the extreme value of x, i.e., ae, gives $y=a(1\pm\log e)=2a$ or 0, therefore the curve meets but is not cut by the axis of x from $x=a$ to ae. Hence (cf. Art. 167)

the surface $=2\pi a$ times the length of the curve between the given lts.:

and $$\frac{dy}{dx}=\pm\frac{a}{x}, \ \therefore \ \frac{ds}{dx}=\frac{\sqrt{a^2+x^2}}{x},$$

and $$s=\int_a^{ae}dx\frac{\sqrt{a^2+x^2}}{x} \text{ for each half of the curve:}$$

and putting $\quad x=a\tan\phi, \quad dx=a\sec^2\phi d\phi,$

and $$s=\int_{\phi_1}^{\phi_2}a\sec^2\phi d\phi\cdot\frac{\sec\phi}{\tan\phi}=\int_{\phi_1}^{\phi_2}ad\phi\frac{\sin^2\phi+\cos^2\phi}{\cos^2\phi\sin\phi}$$

$$=\int_{\phi_1}^{\phi_2}ad\phi\left(\frac{\sin\phi}{\cos^2\phi}+\frac{1}{\sin\phi}\right)=a\left\{\frac{1}{\cos\phi}+\log\tan\frac{\phi}{2}\right\}\Bigg|_{\phi_1}^{\phi_2};$$

also $$\sec\phi=\frac{1}{a}\sqrt{x^2+a^2},$$

$$\therefore \qquad \tan^2\frac{\phi}{2}=(\sec\phi-1)\div(\sec\phi+1)=(\sqrt{x^2+a^2}-a)^2\div x^2,$$

$\therefore$ (x and $\therefore$ $\tan\phi$ being positive)

$$s=a\left\{\frac{1}{a}\sqrt{x^2+a^2}+\log\frac{\sqrt{a^2+x^2}-a}{x}\right\}_a^{ae}$$

$$=a\left\{\sqrt{1+e^2}-\sqrt{2}\right\}+a\log\left\{\frac{\sqrt{1+e^2}-1}{e}\cdot\frac{1}{\sqrt{2}-1}\right\}$$

$$=a\left\{\sqrt{1+e^2}-\sqrt{2}\right\}+a\log\left[\frac{(1+e^2)-1}{e\{(1+e^2)^{\frac{1}{2}}+1\}}\cdot(\sqrt{2}+1)\right]$$

$$\left.\sqrt{2}\right\} + a\log\frac{\sqrt{2}+1}{\sqrt{1+e^2}+1}, \ldots$$

and the surface is $4\pi a$ times this quantity.

54. Comparing $lx'+my'+nz'=p$ with $\frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}=1$, $p=\frac{c^2n}{z}=\frac{c^2\cos\gamma}{z}$, and $dS=dxdy\sec\gamma$,

$$\therefore \qquad \int\frac{dS}{p}=\int\int dxdy\sec^2\gamma\,.\,\frac{z}{c^2}=\int\int dxdy\frac{z}{c^2}\left(1+\overline{\frac{dz}{dx}}\Big|^2+\overline{\frac{dz}{dy}}\Big|^2\right),$$

and here $\quad \frac{dz}{dx}=-\frac{c^2}{a^2}\cdot\frac{x}{z},\ \frac{dz}{dy}=-\frac{c^2}{b^2}\cdot\frac{y}{z},\ \therefore\ \sec^2\gamma=\frac{c^4}{z^2}\left\{\frac{z^2}{c^4}+\frac{x^2}{a^4}+\frac{y^2}{b^4}\right\}$,

and $\therefore \quad \int\frac{dS}{p}=\int\int dxdy\,.\,\frac{c^2}{z}\left\{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right\}$,

and the limits of x and y are determined by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\because$ for a given value of x the extreme lts. of y are $\pm b\sqrt{1-\frac{x^2}{a^2}}$, and thence the extreme lts. of x are (given by equating those of y) $x=\pm a$. Thus the whole integral $\frac{dS}{p}$ is

$$8\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\frac{dxdy\,.\,c}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\left\{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{1}{c^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right\},$$

or if $\frac{x}{a}=x'$ and $\frac{y}{b}=y'$,

$$\int\frac{dS}{p}=8abc\int_0^1\int_0^{\sqrt{1-x'^2}}\frac{dx'dy'}{\sqrt{1-x'^2-y'^2}}\left\{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{1}{c^2}(1-x'^2-y'^2)\right\},$$

or, dropping the dashes,

$$\int\frac{dS}{p}=\int_0^1\int_0^{\sqrt{1-x^2}}8\frac{abcdxdy}{\sqrt{1-x^2-y^2}}\left\{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)\left(1-x^2-y^2\right)+\frac{x^2}{a^2}-\frac{x^2}{b^2}+\frac{1}{b^2}\right\}$$

$$=\int_0^1 8abcdx\left\{\frac{b^2-c^2}{b^2c^2}\left(\frac{y}{2}\sqrt{1-x^2-y^2}+\frac{1-x^2}{2}\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right)\right.$$

$$\left.+\frac{x^2(b^2-a^2)+a^2}{a^2b^2}\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right\}\Bigg/_0^{\sqrt{1-x^2}}$$

$$=\int_0^1 8abcdx\left\{\frac{b^2-c^2}{b^2c^2}\cdot\frac{\pi}{4}(1-x^2)+\frac{\pi}{2}\ \frac{x^2(b^2-a^2)+a^2}{a^2b^2}\right\}$$

$$= 8abc\pi\left\{\frac{b^2-c^2}{b^2c^2}\cdot\frac{1-\frac{1}{3}}{4}+\frac{1}{2}\frac{\frac{1}{3}(b^2-a^2)+a^2}{a^2b^2}\right\}$$

$$= \frac{8\pi}{abc}\left\{\frac{a^2}{6}(b^2-c^2)+\tfrac{1}{2}c^2\left(\frac{b^2}{3}+\frac{2a^2}{3}\right)\right\}$$

$$= \frac{4}{3}\cdot\frac{\pi}{abc}\{a^2(b^2-c^2)+b^2c^2+2c^2a^2\} = \frac{4\pi}{3abc}(a^2b^2+b^2c^2+c^2a^2).$$

CHAPTER VIII.

1. The volume $= \int_0^{3a}\pi y^2 dx = \int_0^{3a}\pi dx\,.\,\frac{ax(x-3a)}{x-4a}$

$$= \pi a\int_0^{3a}\frac{dx}{x-4a}\{x(x-4a)+a(x-4a)+4a^2\}$$

$$= \pi a\left\{\frac{x^2}{2}+ax+4a^2\log(4a-x)\right\}\Big/_0^{3a} = \pi a^3\left\{\frac{9}{2}+3+4\log\frac{1}{4}\right\}$$

$$= \frac{\pi a^3}{2}\left\{15-8\log 4\right\} = \frac{\pi a^3}{2}\left\{15-16\log 2\right\}.$$

2. With the notation of Todhunter's *Differential Calculus*, Art. 358, the volume is

$$2\int_0^{\pi}\pi x^2\frac{dy}{d\theta}\,.\,d\theta = 2\pi a^3\int_0^{\pi}d\theta(1-\cos\theta)^2(1+\cos\theta)$$

$$= 2\pi a^3\int_0^{\pi}d\theta\{1-2\cos\theta+\cos^2\theta+\cos\theta-2\cos^2\theta+\cos^3\theta\}$$

$$= 2\pi a^3\int_0^{\pi}d\theta\{1-\cos^2\theta\} = 4\pi a^3\int_0^{\frac{\pi}{2}}d\theta\sin^2\theta = \pi^2a^3.$$

3. Here the volume $= 2\int_0^{\pi}\pi(2a-x)^2\frac{dy}{d\theta}\,.\,d\theta = 2\pi a^3\int_0^{\pi}d\theta(1+\cos\theta)^3$

$$= 2\pi a^3\int_0^{\pi}d\theta(1+3\cos^2\theta),$$

the other terms vanishing between the limits,

$$= 2\pi a^3\left\{\pi+6\,.\,\frac{\pi}{2}\cdot\frac{1}{2}\right\} = 5\pi^2a^3.$$

4. The asymptote is $x = 2a$, therefore the volume in question is $\int_0^{\infty}dy\,.\,\pi(2a-x)^2$, as x must lie between 0 and $2a$; and corresponding to the limits 0 and ∞ of y those of x are 0 and $2a$; also, $y = \frac{x^{\frac{3}{2}}}{(2a-x)^{\frac{1}{2}}}$, therefore if $x = 2a\sin^2\theta$, $y = 2a\frac{\sin^3\theta}{\cos\theta}$, and the limits of θ are 0 and $\frac{\pi}{2}$, and $\frac{dy}{d\theta} = 2a\left(3\sin^2\theta+\frac{\sin^4\theta}{\cos^2\theta}\right)$. Thus the volume is

$$2\pi\int_0^{\frac{\pi}{2}} d\theta \,.\, 2a\left(3\sin^2\theta+\frac{\sin^4\theta}{\cos^2\theta}\right).(2a)^2\cos^4\theta$$

$$=16\pi a^3\int_0^{\frac{\pi}{2}} d\theta\{3\sin^2\theta\cos^4\theta+\sin^4\theta\cos^2\theta\}$$

$$=16\pi a^3\int_0^{\frac{\pi}{2}} d\theta\{3\cos^4\theta-3\cos^6\theta+\sin^4\theta-\sin^6\theta\}$$

$$=8\pi^2a^3\left\{\frac{3.3}{4.2}-\frac{3.5.3}{6.4.2}+\frac{3.1}{4.2}-\frac{5.3.1}{6.4.2}\right\}$$

$$=8\pi^2a^3\left\{\frac{3}{2}-\frac{15}{12}\right\}=2\pi^2a^3.$$

5. The asymptote is $x=0$, and y^2 must lie between 0 and ∞, and therefore the volume in question is

$$2\int_0^\infty \pi x^2 dy = 2\pi\int_0^\infty \frac{dy(8a^3)^2}{(y^2+4a^2)^2};$$

or if $y=2a\tan\phi$, the limits of ϕ are 0 and $\frac{\pi}{2}$, and the volume is

$$2\pi\int_0^{\frac{\pi}{2}}\frac{(8a^3)^2.\,2a\sec^2\phi d\phi}{16a^4\sec^4\phi}=16\pi a^3\int_0^{\frac{\pi}{2}} d\phi\cos^2\phi=16\pi a^3.\frac{\pi}{4}=4\pi^2a^3.$$

6. The limits of y are $\pm b$, and therefore the volume is

$$2\int_0^b \pi x^2 dy = 2\pi\int_0^b dy\frac{(y^2-b^2)^4}{a^6}=\frac{2\pi}{a^6}\int_0^b dy\{y^8-4y^6b^2+6y^4b^4-4y^2b^6+b^8\}$$

$$=\frac{2b^9\pi}{a^6}\left\{\frac{1}{9}-\frac{4}{7}+\frac{6}{5}-\frac{4}{3}+1\right\}=\frac{2\pi b^9}{a^6}\left(\frac{35-180+378-420+315}{315}\right)$$

$$=\frac{256}{315}\cdot\frac{\pi b^9}{a^6}.$$

7. If the sphere be $x^2+y^2+z^2=a^2$, the volume of the frustum from $x=x_1$ to $x=x_2$ is

$$\int_{x_1}^{x_2}\pi(a^2-x^2)dx=\pi a^2(x_2-x_1)-\frac{\pi}{3}(x_2^3-x_1^3),$$

and here $x_2-x_1=h$; $r_1^2=a^2-x_1^2$, $r_2^2=a^2-x_2^2$,

$$\therefore \text{ the volume}=\frac{\pi h}{3}\{3a^2-(x_2^2+x_1x_2+x_1^2)\}$$

$$=\frac{\pi h}{6}\{6a^2+(x_2-x_1)^2-3x_2^2-3x_1^2\}$$

$$=\frac{\pi h}{6}\{h^2+3(r_1^2+r_2^2)\}.$$

8. The volume of the frustum from $x = x_1$ to x_2 is

$$\int_{x_1}^{x_2} \pi y^2 dx = \int_{x_1}^{x_2} \pi(2mx + nx^2)dx = \pi m(x_2^2 - x_1^2) + \frac{\pi n}{3}(x_2^3 - x_1^3).$$

If then $x_2 - x_1 = h$, and $b^2 = 2mx_1 + nx_1^2$, and $c^2 = 2mx_2 + nx_2^2$,

$$\text{the volume} = \pi h\left\{m(x_2 + x_1) + \frac{n}{3}(x_2^2 + x_2x_1 + x_1^2)\right\}$$

$$= \pi h\left\{\frac{c^2 + b^2}{2} - \frac{n}{2}(x_2^2 + x_1^2) + \frac{n}{3}(x_2^2 + x_1^2 + x_2x_1)\right\}$$

$$= \frac{\pi h}{2}\left\{(c^2 + b^2) - \frac{n}{3}(x_2^2 + x_1^2 - 2x_2x_1)\right\} = \frac{\pi h}{2}\left\{(c^2 + b^2) - \frac{n}{3}h^2\right\}.$$

Also
$$r^2 = 2m\frac{x_2 + x_1}{2} + n\left(\frac{x_2 + x_1}{2}\right)^2$$

$$= \frac{b^2 + c^2}{2} + \frac{n}{4}(x_1^2 + x_2^2 + 2x_2x_1) - \frac{n}{2}(x_1^2 + x_2^2) = \frac{b^2 + c^2}{2} - \frac{n}{4}h^2,$$

$$\therefore \text{ the volume} = \pi h\left\{\frac{b^2 + c^2}{2} - \frac{n}{6}h^2\right\} = \pi h\left\{r^2 + \frac{n}{4}h^2 - \frac{n}{6}h^2\right\} = \pi h\left\{r^2 + \frac{nh^2}{12}\right\}.$$

For a cone, m must vanish, and $n = \tan^2\alpha$, where 2α is the vertical angle of the cone. Accordingly the frustum of the cone is given either by $\frac{\pi h}{2}\left(b^2 + c^2 - \frac{h^2}{3}\tan^2\alpha\right)$ or by $\pi h\left(r^2 + \frac{h^2}{12}\tan^2\alpha\right)$. If $b = 0$, $c = h\tan\alpha$, and the volume is $\frac{\pi}{3}h^3\tan^2\alpha$.

For a spheroid n is negative, and for the whole volume $b = 0 = c$, and therefore the whole volume is $-\frac{\pi h^3 n}{6}$; thus if the generating ellipse be $\alpha^2y^2 + \beta^2x^2 = 2mx$, the volume is $\frac{\pi}{6}\frac{\beta^2}{\alpha^2}\left(\frac{2m}{\beta^2}\right)^3 = \frac{4\pi . m^3}{3\alpha^2\beta^4}$.

9. If the cone be generated by the revolution, round the axis of x, of $y = \frac{x}{\sqrt{3}}$, the sphere of radius r is generated by the revolution of $(x-c)^2 + y^2 = r^2$ $\left(\text{where } r = c\sin\frac{\pi}{6}\right)$, i.e., of $(x - 2r)^2 + y^2 = r^2$. Hence the volume required

$$= \int_0^{\frac{r\sqrt{3}}{2}}\int_{y\sqrt{3}}^{2r - \sqrt{r^2 - y^2}} 2\pi y\,dy\,dx = 2\pi\int_0^{\frac{r\sqrt{3}}{2}} dy\{2ry - y\sqrt{r^2 - y^2} - y^2\sqrt{3}\}$$

$$= 2\pi\left\{ry^2 + \frac{1}{3}(r^2 - y^2)^{\frac{3}{2}} - \frac{y^3}{\sqrt{3}}\right\}\Bigg/_0^{\frac{r\sqrt{3}}{2}}$$

$$= 2\pi r^3\left\{\frac{3}{4} + \frac{1}{3}\cdot\frac{1}{8} - \frac{1}{3} - \frac{3}{8}\right\} = \frac{\pi r^3}{12}(18 + 1 - 8 - 9) = \frac{\pi r^3}{6}.$$

10. If the vertex of the paraboloid (of revolution) be at the origin, its axis, the axis of x, and latus rectum $= 2a$, its equation is $y^2 + z^2 = 2ax$; and

if the cylinder touch the axis of y its equation is $y^2+z^2-2az=0$, and it is supposed to extend from $x=0$ to $x=2a$. The whole volume of the cylinder is then $2a.\pi a^2=2\pi a^3$: for the volume (V) bounded by the cylinder, the paraboloid and the plane $x=0$, for given possible values of y and z, the element of volume is a small prism on the base $\Delta z\Delta y$ extending from the plane of yz to the paraboloid, therefore $x \propto$ from 0 to $\frac{y^2+z^2}{2a}$; and for a given possible value of z, y is limited by the cylinder, i.e., the limits of y are $-\sqrt{2az-z^2}$ and $\sqrt{2az-z^2}$, and lastly z can $\propto$ from 0 to $2a$; therefore

$$V=\int_0^{2a}\int_{-\sqrt{2az-z^2}}^{\sqrt{2az-z^2}} dzdy.\frac{y^2+z^2}{2a}=\frac{1}{a}\int_0^{2a} dz\left(\frac{y^3}{3}+yz^2\right)\Big/_0^{\sqrt{2az-z^2}}$$

$$=\frac{1}{a}\int_0^{2a} dz\sqrt{2az-z^2}\left(\frac{2az}{3}+\frac{2z^2}{3}\right) \quad \text{or if } z=a(1+\sin\theta),$$

$$V=\frac{2a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\cos^2\theta(1+\sin\theta+\overline{1+\sin\theta}|^2)$$

$$=\frac{2a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta(2\cos^2\theta+3\cos^2\theta\sin\theta+\cos^2\theta-\cos^4\theta)$$

$$=\frac{4a^3}{3}\int_0^{\frac{\pi}{2}} d\theta(3\cos^2\theta-\cos^4\theta)=\frac{4a^3}{3}\cdot\frac{\pi}{2}\left(\frac{3}{2}-\frac{3}{8}\right)$$

$$=\frac{4}{6}\pi a^3.\frac{9}{8}=\frac{3}{4}\pi a^3, \text{ and } \therefore\ V:2\pi a^3-V::3:5.$$

11. If the generating parabola be $y^2=lx$, and the axis $=a$, the radius of the base $\sqrt{la}$, and the volume (V_1) between the paroboloid and cone is

$$\int_0^a dx\left(\pi y^2-\frac{\pi x^2}{a^2}.al\right)=\pi\int_0^a dx\left(lx-\frac{lx^2}{a}\right)=\pi l\left(\frac{a^2}{2}-\frac{a^2}{3}\right)=\frac{\pi la^2}{6}; \text{ and}$$

the volume V_2 of the sphere $=\frac{4}{3}\pi\left(\frac{a}{2}\right)^3$; $\therefore\ V_1:V_2::l:a$.

12. If for x, y, z there be written ax, by, cz, the volume (V) is $abc\iiint dzdxdy$, subject to $x^2+y^2+z^4=1$; or $V=\int dz.A$ where A is the area of $x^2+y^2=1-z^4$ (z being constant), therefore $A=\pi(1-z^4)$, and the extreme values of z are given when $A=0$, which leads to the two, viz., $z=\pm 1$; thus

$$V=abc\int_{-1}^1 \pi dz(1-z^4)=2abc\pi\left(z-\frac{z^5}{5}\right)_0^1=\frac{8\pi}{5}al$$

13. For real values x, y, z are all positive, or else one of them is positive and the other two are negative, and by changing the signs of these negative co-ordinates the form of the equation is unaltered, and therefore

the surface is composed of four equal portions, which meet only at the origin. Changing to polar co-ordinates, where $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, and $y = r\sin\theta\sin\phi$, the equation to the surface is

$$r^3 = 27\,a^3\cos\theta\sin^2\theta\cos\phi\sin\phi, \quad\ldots\ldots\ldots\ldots(1)$$

so that r is limited in value, and for given (permissible) values of θ and ϕ, r has but one real value; and therefore the surface is composed of four equal single closed surfaces meeting at (0, 0, 0). The limits of θ and ϕ are therefore obtained by putting $r = 0$ in (1), and are therefore 0 and $\frac{\pi}{2}$ for each. Hence the volume required is

$$4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} 9a^3\cos\theta\sin^3\theta\sin\phi\cos\phi\,d\theta\,d\phi = 18a^3\int_0^{\frac{\pi}{2}} d\theta\cos\theta\sin^3\theta\sin^2\phi\Big/_0^{\frac{\pi}{2}}$$

$$= \frac{9}{2}a^3\sin^4\theta\Big/_0^{\frac{\pi}{2}} = \frac{9a^3}{2}.$$

Aliter: for the limits of θ, ϕ, when x, y, z are small the equation to the surface approximates to $xyz = 0$, and therefore the co-ordinate planes touch it, etc.

14. The curve is symmetrical as to both axes and the tangents at the origin are given by $a^2x^2 + b^2y^2 = 0$, therefore (0, 0) is a conjugate point. Since the asymptotes are imaginary, the curve is closed, and any straight line $y = mx$ meets it where $x = 0$ or $x^2(1 + m^2)^2 = a^2 + b^2m^2$, i.e., in only two points besides (0, 0). Hence the *curve* is a single closed curve around (0, 0), and solving for y^2 from the equation

$$y^4 - y^2(b^2 - 2x^2) + x^4 - a^2x^2 = 0,$$

$$2y^2 = b^2 - 2x^2 \pm \{(b^2 - 2x^2)^2 - 4(x^4 - a^2x^2)\}^{\frac{1}{2}},$$

therefore for real values of y, so long as x^2 is not $> a^2$, the upper sign of the radical must be taken; and if $x > a$, the extreme values of x are given by equating the two values of $2y^2$, which gives $b^4 + 4(a^2 - b^2)x^2 = 0$, i.e., imaginary values of x, therefore the limits of x are $\pm a$. Thus the required volume is

$$2\int_0^a \pi y^2dx = \pi\int_0^a dx\{b^2 - 2x^2 + \sqrt{b^4 + 4(a^2 - b^2)x^2}\}$$

$$= \pi ab^2 - \frac{2}{3}\pi a^3 + \frac{\pi}{2\sqrt{a^2 - b^2}}\int_0^{2a\sqrt{a^2-b^2}} dz\sqrt{z^2 + b^4}$$

$$= \pi ab^2 - \frac{2}{3}\pi a^3 + \frac{\pi}{4\sqrt{a^2 - b^2}}\{z\sqrt{z^2 + b^4} + b^4\log(z + \sqrt{z^2 + b^4})\}\Big/_0^{2a\sqrt{a^2-b^2}}$$

$$= \pi ab^2 - \frac{2}{3}\pi a^3 + \frac{\pi}{4\sqrt{a^2 - b^2}}\left\{2a\sqrt{a^2 - b^2}(2a^2 - b^2) + b^4\log\frac{2a\sqrt{a^2 - b^2} + 2a^2 - b^2}{b^2}\right\}$$

$$=\pi ab^2-\frac{2}{3}\pi a^3+\frac{\pi a}{2}(2a^2-b^2)+\frac{\pi b^4}{4\sqrt{a^2-b^2}}\log\left\{\frac{a+(a^2-b^2)^{\frac{1}{2}}}{b}\right\}^2, \text{ etc.}$$

When $a=b$, the curve becomes the circle $x^2+y^2=b^2$, and therefore the volume is $\frac{4}{3}\pi b^3$: also the above result becomes

$$\frac{5}{6}\pi b^3+\text{the lt. of } \frac{\pi b^4}{2\sqrt{a^2-b^2}}\log\left\{1+\frac{\sqrt{a^2-b^2}}{b}\right\}$$

$$=\frac{5}{6}\pi b^3+\frac{\pi b^4}{2\sqrt{a^2-b^2}}\,\frac{\sqrt{a^2-b^2}}{b}+\text{vanishing terms (expanding the}$$

log) $\qquad =\frac{5}{6}\pi b^3+\frac{\pi b^3}{2}=\frac{4}{3}\pi b^3$, as before.

15. The volume required $=\int\pi x^2dy$ taken between proper limits, and, as in Ex. 14, $\qquad 2x^2=a^2-2y^2\pm\{(a^2-2y^2)^2-4(y^4-b^2y^2)\}^{\frac{1}{2}}$,
and for real values of x, so long as y^2 is not $>b^2$, the upper sign of the radical must be taken; and if $y>b$, the extreme values of y are given by equating the two values of $2x^2$, which gives $a^4=4(a^2-b^2)y^2$, and then

$$2x^2=a^2-\frac{a^4}{2(a^2-b^2)}=\frac{a^2(a^2-2b^2)}{2(a^2-b^2)},$$

$\therefore$ these values of x^2 are real or imaginary as a^2 is $>$ or $<2b^2$. Thus the required volume (1), if $a^2<2b^2$, is

$$\pi\int_0^b dy\{a^2-2y^2+\sqrt{a^4-4(a^2-b^2)y^2}\}$$

$$=\pi a^2b-\frac{2}{3}\pi b^3+\frac{\pi}{2(a^2-b^2)^{\frac{1}{2}}}\int_0^{2b\sqrt{a^2-b^2}} dz\sqrt{a^4-z^2}$$

$$=\pi a^2b-\frac{2}{3}\pi b^3+\frac{\pi}{4\sqrt{a^2-b^2}}\left\{z\sqrt{a^4-z^2}+a^4\sin^{-1}\frac{z}{a^2}\right\}\Bigg|_0^{2b\sqrt{a^2-b^2}}$$

$$=\pi a^2b-\frac{2}{3}\pi b^3+\frac{\pi}{4\sqrt{a^2-b^2}}\left\{2b\sqrt{a^2-b^2}(2b^2-a^2)+a^4\sin^{-1}\frac{2b\sqrt{a^2-b^2}}{a^2}\right\};$$

also as $z \propto$ from 0 to $2b\sqrt{a^2-b^2}$, $\sin^{-1}\frac{z}{a^2}$ will only pass through the value $\frac{\pi}{2}$ for some value of z such as $2c\sqrt{a^2-c^2}$, if $c<b$, and therefore if $\frac{2b\sqrt{a^2-b^2}}{a^2}>1$, which is impossible, $\therefore \cos\left(\sin^{-1}\frac{z}{a^2}\right)$ is positive throughout, therefore, if $\frac{2b\sqrt{a^2-b^2}}{a^2}=\sin 2\theta$, $1-2\sin^2\theta$ is positive and $=\frac{a^2\sim 2b^2}{a^2}$, therefore

$\sin^2\theta=\frac{1}{2}\left(1-\frac{2b^2-a^2}{a^2}\right)$, and $\theta=\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}$; and the volume required is

$$\pi a^2 b - \frac{2}{3}\pi b^3 + \frac{\pi b}{2}(2b^2 - a^2) + \frac{\pi a^4}{2\sqrt{a^2-b^2}} \sin^{-1}\frac{\sqrt{a^2-b^2}}{a}.$$

When $a = b$ this becomes $\pi a^3 . \frac{5}{6} + \frac{\pi a^3}{2} = \frac{4}{3}\pi a^3$.

But (2) if $a^2 > 2b^2$, the volume may be divided into two parts V_1, V_2, where V_1 is the same as before on changing the sign of $2b^2 - a^2$, so that

$$\sin^2\theta = \frac{b^2}{a^2},$$

and $$V_1 = \pi a^2 b - \frac{2}{3}\pi b^3 + \frac{\pi b}{2}(a^2 - 2b^2) + \frac{\pi a^4}{2\sqrt{a^2-b^2}} \sin^{-1}\frac{b}{a};$$

and $$V_2 = 2\pi \int_b^{\frac{a^2}{2\sqrt{a^2-b^2}}} dy(x_1^2 - x_2^2),$$

where x_1, x_2 correspond to the upper and lower signs of the radical in the equation for $2x^2$,

$$\therefore \quad V_2 = 2\pi \int_b^{\frac{a^2}{2\sqrt{a^2-b^2}}} dy\{a^4 - 4(a^2-b^2)y^2\}^{\frac{1}{2}}$$

$$= \frac{\pi}{2\sqrt{a^2-b^2}}\left\{z\sqrt{a^4-z^2} + a^4\sin^{-1}\frac{z}{a^2}\right\}\Bigg/_{2b\sqrt{a^2-b^2}}^{a^2}. \quad \text{Thus}$$

$$V_1 + V_2 = \pi a^2 b - \frac{2}{3}\pi b^3 - \frac{\pi b}{2}(a^2 - 2b^2) - \frac{\pi a^4}{2\sqrt{a^2-b^2}}\sin^{-1}\frac{b}{a} + \frac{\pi a^4}{2\sqrt{a^2-b^2}} \cdot \frac{\pi}{2}$$

$$= \frac{1}{3}\pi b^3 + \frac{1}{2}\pi a^2 b + \frac{\pi a^4}{2\sqrt{a^2-b^2}}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a},$$

which is the same as when $2b^2 > a^2$. This is explained by the fact that the volume might have been integrated with regard to x, by transformation, the limits being then the same in both cases. The transformed integral $\left(\because \int \pi x^2 dy = \int\int 2\pi x dx dy\right)$ is $4\pi\int_0^a xy dx$, and the value of y would have to be substituted, which does not involve any ambiguity, so that the second part might have been assumed.

16. The curve is the *lemniscate*, and solving for y^2, the volume

$$= 2\int_0^a dx \cdot \frac{\pi}{2}\{-(a^2+2x^2) + \sqrt{a^4 + 8a^2x^2}\}$$

$$= -\pi\left(a^3 + \frac{2}{3}a^3\right) + 2\pi a\sqrt{2}\int_0^a dx\sqrt{x^2 + \frac{a^2}{8}}$$

$$= -\frac{5}{3}\pi a^3 + \pi a\sqrt{2}\left\{x\sqrt{x^2 + \frac{a^2}{8}} + \frac{a^2}{8}\log\left(x + \sqrt{x^2 + \frac{a^2}{8}}\right)\right\}\Bigg/_0^a$$

$$=-\frac{5}{3}\pi a^3+\pi a\sqrt{2}\left\{\frac{a^2 . 3}{2\sqrt{2}}+\frac{a^2}{8}\log\left(\frac{1+\frac{3}{2\sqrt{2}}}{\frac{1}{2\sqrt{2}}}\right)\right\}$$

$$=-\frac{\pi a^3}{6}+\frac{\pi a^3}{4\sqrt{2}}\log(1+\sqrt{2})^2=\frac{\pi a^3}{2}\left\{\frac{1}{\sqrt{2}}\log(1+\sqrt{2})-\frac{1}{3}\right\}.$$

17. If the paraboloid be $y^2=mx$, and the sphere be $(x-c)^2+y^2=r^2$, and if they meet where $x=x_1$ and $x=x_2$, the volume included between them is

$$\int_{x_1}^{x_2}\pi y^2dx=\pi\int_{x_1}^{x_2}dx\{r^2-\overline{x-c}|^2-mx\};$$

also $x_2-x_1=h$, and x_1, x_2 are given by the equation

$$(x-c)^2+mx=r^2; \qquad\text{.........}(1)$$

thus the volume $=\pi\left\{(r^2-c^2)h+(2c-m)\frac{x_2^2-x_1^2}{2}-\frac{1}{3}(x_2^3-x_1^3)\right\}$

$$=\pi . h\left\{r^2-c^2+\frac{2c-m}{2}(2c-m)-\frac{1}{3}(x_2^2+x_2x_1+x_1^2\right\}$$

$$=\pi . h\{r^2-c^2+\tfrac{1}{2}(2c-m)^2-\tfrac{1}{3}h^2-(c^2-r^2)\}$$

$$=\pi . h\{2(r^2-c^2)+\tfrac{1}{2}(2c-m)^2-\tfrac{1}{3}h^2\},$$

but $\qquad h^2=(2c-m)^2-4(c^2-r^2),$

$\therefore\qquad$ the volume $=\pi . h\left\{-\frac{h^2}{3}+\frac{h^2}{2}\right\}=\frac{\pi h^3}{6}.$

18. The surface passes through the origin, and for a given value of x the section is an ellipse, if b, c be of the same sign. Thus the volume required $=\int_0^a dx . \pi\sqrt{bc} . 2x=2\pi\sqrt{bc} . \frac{a^2}{2}=\pi a^2\sqrt{bc}.$

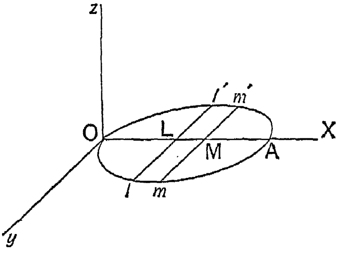
19. The ellipse being (referred to this tangent and the minor axis as axes) $a^2(y+b)^2+b^2x^2=a^2b^2$, or $y=-b\pm\frac{b}{a}\sqrt{a^2-x^2}$, taking the upper quadrant and x as positive, the volume required

$$=\int_0^a dx . \pi\left\{-b+\frac{b}{a}\sqrt{a^2-x^2}\right\}^2=\frac{\pi b^2}{a^2}\int_0^a dx\{2a^2-x^2-2a\sqrt{a^2-x^2}\}$$

$$=\frac{\pi b^2}{a^2}\left\{2a^3-\frac{a^3}{3}-2a . \frac{\pi}{4}a^2\right\}=\frac{\pi ab^2}{6}(10-3\pi).$$

20. Referring to the Fig. on p. 177 of Todhunter's *Integral Calculus*, if P, Q be adjacent points within the circle $\{x^2+y^2=ax,\ z=0\}$, the integration with regard to z in $\iiint dxdydz$, is the summation of all the rectangular parallelepipeds such as st from the plane $z=0$, to the corresponding point

(for the same values of x and y) of the paraboloid $x^2+y^2=cz$, so that the limits of z are 0 and $\frac{x^2+y^2}{c}$: secondly, the integration with respect to y, is the summation of all such columns as $pqQP$, as P and Q move along lLl' and mMm', from one side of the circle $OlmAm'l'$, to the other side, so that the limits of y are the values of y in the equation $x^2+y^2=ax$. Lastly, the integration with respect to x is the summation of the slices $lpl'm'qm$, as L is moved from O to A, so that the limits of x are 0 and a. Thus the required volume is



$$2\int_0^a\int_0^{\sqrt{ax-x^2}} dxdy\,.\,\frac{x^2+y^2}{c}=\frac{2}{c}\int_0^a dx\left\{x^2\sqrt{ax-x^2}+\frac{1}{3}(ax-x^2)^{\frac{3}{2}}\right\},$$

or if $x=\frac{a}{2}(1+\sin\theta)$,

$$\text{the volume}=\frac{2}{c}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a}{2}\cos\theta d\theta\left\{\left(\frac{a}{2}\right)^3(1+\sin\theta)^2\cos\theta+\frac{\cos^3\theta}{3}\,.\left(\frac{a}{2}\right)^3\right\}$$

$$=\frac{a^4}{8c}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\left\{\cos^2\theta+2\cos^2\theta\sin\theta+\cos^2\theta(1-\cos^2\theta)+\frac{\cos^4\theta}{3}\right\}$$

$$=\frac{a^4}{4c}\int_0^{\frac{\pi}{2}}d\theta\left\{2\cos^2\theta-\frac{2}{3}\cos^4\theta\right\}=\frac{\pi a^4}{4c}\left(\frac{1}{2}-\frac{1}{3}\,.\,\frac{3}{8}\right)=\frac{3\pi a^4}{32c}.$$

21. If straight lines be drawn from O (supposed within S) to all points in the boundary of dS, these straight lines enclose a thin cone of height $r\cos\phi$, and base ultimately dS, and the volume of this cone is therefore $\frac{1}{3}r\cos\phi dS$, and summing all such elements as dS of the surface S, the volume contained by $S=\frac{1}{3}\int r\cos\phi dS$, taken over the whole surface. In the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, if the tangent plane at P be $lx+my+nz=p$, then $p=r\cos\phi$, and if P be (x, y, z), $\frac{1}{p^2}=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$, and if $dS=dxdy\sec\gamma$,

$$\sec^2\gamma=1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2=\frac{c^4}{z^2p^2},\quad\therefore\ pdS=\frac{c^2}{z}dxdy,$$ and the volume of the ellipsoid $=\frac{c^2}{3}\int\int\frac{dxdy}{z}$ extended over the surface

$$=\frac{8c}{3}\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\frac{dxdy}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}=\frac{8abc}{3}\int_0^1\int_0^{\sqrt{1-x^2}}\frac{dxdy}{\sqrt{1-x^2-y^2}}$$

$$=\frac{8abc}{3}\int_0^1 dx \sin^{-1}\frac{y}{\sqrt{1-x^2}}\Big|_0^{\sqrt{1-x^2}}=\frac{4\pi abc}{3}.$$

Here, in integrating with respect to y, the elementary cones are summed which correspond to a zone of surface between the planes $x'=x$ and $x'=x+dx$, and the integration with respect to x sums all such elementary conical volumes.

22. For a given value of x, the $\iint dydz$ extended over the volume of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, contained between $x'=x$ and $x'=x+dx$, is $dx\,.\,\pi bc\left(1-\frac{x^2}{a^2}\right)$,

$$\therefore\quad \iiint x^2dxdydz \text{ here } =2\pi bc\int_0^a dx\left(x^2-\frac{x^4}{a^2}\right)=2\pi a^3bc\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{4}{15}\pi a^3bc.$$

23. If the volume be $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}dxdydz$, the limits of z are 0 and the value of z from the equation to the surface, therefore

$$z_1=0 \text{ and } z_2=(Ax^2+Bxy+Cy^2-F)\frac{1}{D};$$

and for a given value of x, y_1, y_2 are the values of y when $z_1=z_2$, therefore y_1, y_2 are the roots of

$$Ax^2+Bxy+Cy^2=F,$$

and x_1, x_2 are the values of x when $y_1=y_2$, and are therefore given by

$$B^2x^2=4C(Ax^2-F), \text{ or } x^2=\frac{4CF}{4AC-B^2}.$$

24. If the volume be $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}dxdydz$, z_1, z_2 are the roots of

$$ax^2+by^2+cz^2+a'yz+b'zx+c'xy=1,$$

and y_1, y_2 are the values of y when $z_1=z_2$, and are therefore given by

$$(a'y+b'x)^2=4c(ax^2+by^2+c'xy-1), \text{ or}$$

$$y^2(a'^2-4bc)+2yx(a'b'-2cc')+x^2(b'^2-4ca)+4c=0\,;$$

and so x_1, x_2 are given when $y_1=y_2$, and therefore by

$$(a'b'-2cc')^2x^2=(a'^2-4bc)\{x^2(b'^2-4ca)+4c\},$$

or $\quad 4x^2(c^2c'^2+bc\,.\,b'^2+caa'^2-a'b'c'\,.\,c-4abc^2)=4c(a'^2-4bc),$

or $\quad x^2(aa'^2+bb'^2+cc'^2-a'b'c'-4abc)=a'^2-4bc.$

25. The trace of the cone on the plane $y=0$ shows that with the negative sign of the radical the cone extends from the vertex $(0, 0, a)$ towards the plane $z=0$; $\therefore$ if the volume be $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2}dxdydz$, then $z_1=x$, $z_2=a-\sqrt{x^2+y^2}$;

for y_1, y_2, $x = a - \sqrt{x^2+y^2}$, or $y^2 = a^2 - 2ax$; and $x_1 = 0$ and x_2 is given by $y_1 = y_2$ or $x = \frac{a}{2}$. Hence the volume is $\int_0^{\frac{a}{2}}\int_{y_1}^{y_2} dxdy(a - x - \sqrt{x^2+y^2})$

$$= \int_0^{\frac{a}{2}} dx\left\{(a-x)(y_2-y_1) - \frac{y}{2}\sqrt{x^2+y^2}\Big|_{y_1}^{y_2} - \frac{x^2}{2}\log(y+\sqrt{x^2+y^2})\Big|_{y_1}^{y_2}\right\}$$

$$= \int_0^{\frac{a}{2}} dx\left\{2(a-x)\sqrt{a^2-2ax} - \sqrt{a^2-2ax}(a-x) - \frac{x^2}{2}\log\frac{a-x+\sqrt{a^2-2ax}}{a-x-\sqrt{a^2-2ax}}\right\}.$$

If $x = \frac{a}{2}\sin^2\theta$, this becomes

$$\int_0^{\frac{\pi}{2}} a\sin\theta\cos\theta d\theta\left\{\frac{a}{2}(1+\cos^2\theta)a\cos\theta - \frac{a^2}{8}\sin^4\theta\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^2\right\}$$

$$= \frac{a^3}{2}\int_0^{\frac{\pi}{2}} d\theta\left\{\sin\theta\cos^2\theta + \sin\theta\cos^4\theta - \sin^5\theta\cos\theta\log\frac{1+\cos\theta}{\sin\theta}\right\}$$

$$= \frac{a^3}{2}\left(\frac{1}{3}+\frac{1}{5}\right) - \frac{a^3\sin^6\theta}{12}\log\frac{1+\cos\theta}{\sin\theta}\Big|_0^{\frac{\pi}{2}} - \frac{a^3}{12}\int_0^{\frac{\pi}{2}} d\theta \,.\, \sin^6\theta\left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \frac{4}{15}a^3 - \frac{a^3}{12}\int_0^{\frac{\pi}{2}} d\theta(\sin^5\theta\cos\theta + \sin^5\theta\,\overline{1-\cos\theta})$$

$$= \frac{4}{15}a^3 - \frac{a^3}{12}\cdot\frac{4\,.\,2}{5\,.\,3} = \frac{4}{15}a^3\left(1-\frac{1}{6}\right) = \frac{20}{90}a^3.$$

26. If the volume be $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} dxdydz$, $z_1 = \frac{1}{c}(mx^2+ny^2)$, $z_2 = ax + by$; y_1, y_2 are the roots of $z_1 = z_2$, i.e., of $mx^2 + ny^2 = c(ax+by)$; and x_1, x_2 are given by $y_1 = y_2$, i.e., $b^2c^2 = 4n(mx^2 - cax)$.

If $m = n = a = b = 1$, $z_1 = \frac{x^2+y^2}{c}$, $z_2 = x+y$, $y_1 + y_2 = c$, $y_1y_2 = x^2 - cx$, and the

$$\text{volume} = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\frac{1}{c}dxdy(cx+cy-x^2-y^2)$$

$$= \frac{1}{c}\int_{x_1}^{x_2}\int_{y_1}^{y_2} dxdy(-y^2 - y_1y_2 + y\,\overline{y_1+y_2})$$

$$= \frac{1}{c}\int_{x_1}^{x_2}\int_{y_1}^{y_2} dxdy(y-y_1)(y_2-y)$$

$$= \frac{1}{c}\int_{x_1}^{x_2}\int_{y_1}^{y_2} dxdy\{(y-y_1)(y_2-y_1) - (y-y_1)^2\} = \frac{1}{c}\int_{x_1}^{x_2} dx\frac{(y_2-y_1)^3}{6},$$

but $\quad (y_2-y_1)^2 = c^2 - 4(x^2-cx)$,

$\therefore$ the volume $= \frac{4}{3c}\int_{x_1}^{x_2} dx\left\{\frac{c^2}{2} - \left(x-\frac{c}{2}\right)^2\right\}^{\frac{3}{2}}$, or if $x - \frac{c}{2} = \frac{c}{\sqrt{2}}\sin\theta$, the volume $=\dfrac{2\sqrt{2}}{3}\displaystyle\int_{\theta_1}^{\theta_2}\cos\theta d\theta\,.\,\frac{c^3}{2\sqrt{2}}\,.\cos^3\theta$, where θ_1, θ_2 may be shown to be $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$, since the equation for x_1, x_2 is $\dfrac{c^2}{2}=\left(x-\dfrac{c}{2}\right)^2$; thus

$$\text{the volume}=\frac{2c^3}{3}\,.\,\frac{\pi}{2}\,.\,\frac{3}{8}=\frac{\pi c^3}{8}.$$

27. If the centre describe the circle $\{x^2+y^2=c^2,\ z=0\}$, and the fixed plane be that of yz, the volume of the cavity $=\displaystyle\int_{-c}^{c}dx\,.\,S$, where, corresponding to a given value of x between the limits, S is the extreme area contained by the circle in the two positions corresponding to y and $-y$, where $x^2+y^2=c^2$. Thus $S=2\left\{\dfrac{\pi c^2}{2}+2\displaystyle\int_0^y y^2dx\right\}$, since the common chord of these two circles is $\{y'=0,\ x'=x\}$, therefore

$$S=\pi c^2+4\int_0^y dx\sqrt{c^2-x^2}=\pi c^2+2x\sqrt{c^2-x^2}\Big/_0^y+2c^2\sin^{-1}\frac{x}{c}\Big/_0^y$$

$$=\pi c^2+2yx+2c^2\sin^{-1}\frac{y}{c},$$

$$\therefore\quad \text{the volume}=2\int_0^c dx\left\{\pi c^2+2x\sqrt{c^2-x^2}+2c^2\sin^{-1}\frac{\sqrt{c^2-x^2}}{c}\right\}$$

$$=2\pi c^3-(c^2-x^2)^{\frac{3}{2}}\frac{4}{3}\Big/_0^c+4c^2\int_0^c dx\cos^{-1}\frac{x}{c}$$

$$=2\pi c^3+\frac{4}{3}c^3+4c^2x\cos^{-1}\frac{x}{c}\Big/_0^c+4c^2\int_0^c\frac{dx\,.\,x}{\sqrt{c^2-x^2}}$$

$$=2\pi c^3+\frac{4}{3}c^3-4c^2\sqrt{c^2-x^2}\Big/_0^c=\frac{2c^3}{3}(3\pi+2+6)=\frac{2c^3}{3}(3\pi+8).$$

28. $a^2y^2=x^2(c^2-z^2)$, therefore for the limits of z in $\iiint dxdzdy$, $z=\pm c$, therefore the volume is

$$4\int_0^a\int_0^c dxdz\,.\,\frac{x}{a}\sqrt{c^2-z^2}=\int_0^a dx\,.\,\frac{x}{a}\,.\,\pi c^2=\frac{\pi c^2a}{2}.$$

29. If the base of the cone be on the plane of xy, its axis along the axis of z, and the altitude $=2a$, its equation is $2a-z=2\sqrt{x^2+y^2}$, or, in cylindrical co-ordinates, $z=2(a-r)$, for the finite cone in question: also, if the cylinder touch the axis of y, its equation in cylindrical co-ordinates is $r=2a\cos\theta$; and the circular sections by the plane $z=0$ meet where $\cos\theta=\dfrac{1}{2}$, and therefore $\theta=\pm\dfrac{\pi}{3}$. Thus the volume V_1 of the cone included within the cylinder is $\iint rd\theta dr\,.\,z$, integrated over the area included between

the two above-mentioned circles, or

$$V_1 = 2\int_0^{\frac{\pi}{3}}\int_0^a r d\theta dr \,.\, 2(a-r) + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{2a\cos\theta} r d\theta dr \,.\, 2(a-r)$$

$$= 4\frac{\pi}{3}a^3\left(\frac{1}{2}-\frac{1}{3}\right) + 4a^3\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta\left(\frac{4\cos^2\theta}{2} - \frac{8\cos^3\theta}{3}\right)$$

$$= \frac{2}{9}\pi a^3 + 4a^3\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta\left\{1+\cos 2\theta - \frac{2}{3}(\cos 3\theta + 3\cos\theta)\right\}$$

$$= \frac{2}{9}\pi a^2 + 4a^3\left\{\theta + \frac{\sin 2\theta}{2} - \frac{2}{9}\sin 3\theta - 2\sin\theta\right\}\Bigg/_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \frac{2}{9}\pi a^3 + 4a^3\left\{\frac{\pi}{6} - \frac{\sqrt{3}}{4} + \frac{2}{9} - 2 + \sqrt{3}\right\}$$

$$= a^3\left\{\frac{8}{9}\pi + 3\sqrt{3} - \frac{64}{9}\right\} = \frac{a^3}{9}\{8\pi + 27\sqrt{3} - 64\};$$

and the whole volume of the cone is $\frac{2}{3}\pi a^3$, therefore volume of the remainder is $\frac{a^3}{9}\{64 - 27\sqrt{3} - 2\pi\}$.

If 2α be the vertical angle of the cone, $\cos\alpha = \frac{2}{\sqrt{5}}$, and for any given value of z between 0 and $2a$, the element of the surface S_1 of the cone included within the cylinder from z to $z + dz = dz\sec\alpha \,.\, 2r\theta$, where $2r = 2a - z$, and θ is given by $r = 2a\cos\theta$; therefore

$$S_1 = 2\int_0^{2a} dz\sec\alpha \,.\, \frac{2a-z}{2}\cos^{-1}\frac{2a-z}{4a} = \int_0^{2a} dz\sec\alpha \,.\, z\cos^{-1}\frac{z}{4a},$$

or if $z = 4a\cos\phi$,

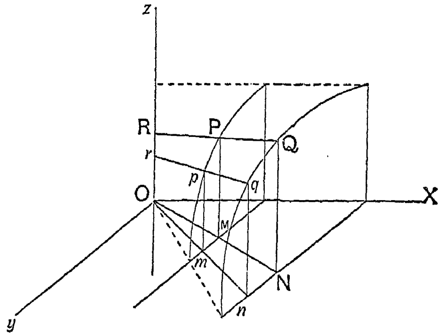
$$S_1\cos\alpha = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 4a\sin\phi d\phi \,.\, 4a\cos\phi \,.\, \phi = -4a^2\cos 2\phi \,.\, \phi\Bigg/_{\frac{\pi}{3}}^{\frac{\pi}{2}} + 4a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\phi \,.\, \cos 2\phi$$

$$= 4a^2\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + 2a^2\left(\sin\pi - \sin\frac{2\pi}{3}\right) = \frac{4}{3}\pi a^2 - a^2\sqrt{3},$$

and $\therefore$ $S_1 = \frac{a^2}{6}(4\pi\sqrt{5} - 3\sqrt{15})$; and the whole surface of the cone $= \pi a^2\operatorname{cosec}\alpha = \pi a^2\sqrt{5}$, therefore the other portion of the surface of the cone $= \frac{a^2}{6}(2\pi\sqrt{5} + 3\sqrt{15})$.

30. The parallel planes $PpMm$ and $QqNn$ may be taken to be $x' = x_1$ and $x' = x_2$, and if RPQ, rpq be two adjacent positions of the generator, the corresponding elements of area between the conoid and $z = 0$ on $x' = x_1$ and $x' = x_2$, are $PpmM$ and $QqnN$, which (since $PM = QN$) are ultimately as $Mm : Nn$, or by similar triangles as $x_1 : x_2$. Thus when RPQ has made a

complete revolution round the axis of z, if A_x denote the corresponding



section by $x' = x$, then $A_x \propto x$, and the volume between the planes $x' = x_1$ and $x' = x_2$ $= \int_{x_1}^{x_2} dx \,.\, A_x = \frac{\kappa}{2}(x_2^2 - x_1^2)$, where κ is a constant and $A_{x_1} = \kappa x_1$, $A_{x_2} = \kappa x_2$, therefore the volume $= (x_2 - x_1)\frac{A_{x_2} + A_{x_1}}{2}$, and similarly for any number of revolutions of the generator.

From the above, the same result holds if the volume and areas be bounded by the conoid and the plane $z = 0$.

CHAPTER IX.

1. If, at P_1, $\theta = \alpha$, the area $= \int_\alpha^{2\pi+\alpha} \frac{d\theta}{2}(r'^2 - r^2)$, where $r = a^\theta$, and $r' = a^{\theta+2\pi}$,

$$\therefore \text{ the area is } \int_\alpha^{\alpha+2\pi} \frac{d\theta}{2}(a^{2\theta+4\pi} - a^{2\theta}) = \frac{1}{4\log_e a}(a^{2\theta+4\pi} - a^{2\theta})\Big|_\alpha^{\alpha+2\pi}$$

$$= \frac{1}{4\log_e a}(a^{2\alpha+8\pi} - 2a^{2\alpha+4\pi} + a^{2\alpha})$$

$$= \frac{1}{4\log_e a}(SP_3 - SP_1)^2, \ S \text{ being the pole;}$$

i.e., the area $= (P_3P_1)^2 \div 4\log_e a$.

2. The curve is symmetrical as to $y = 0$, and, solving for y, $2y^2 = ax \pm x\sqrt{a^2 - 4x^2}$, therefore x cannot be negative, and must lie between 0 and $\frac{a}{2}$; and if $2y_2^2 = ax + x\sqrt{a^2 - 4x^2}$, and $2y_1^2 = ax - x\sqrt{a^2 - 4x^2}$, the area is $2\int_0^{\frac{a}{2}} dx(y_2 - y_1)$; but

$$(y_2 - y_1)\sqrt{2} = \{2(y_2^2 + y_1^2 - 2y_1y_2)\}^{\frac{1}{2}} = \sqrt{2ax - 2 \, . \, 2x^2},$$

$$\therefore \quad \text{the area} = 2\sqrt{2}\int_0^{\frac{a}{2}} dx \sqrt{\frac{a^2}{16} - \left(x - \frac{a}{4}\right)^2} = 2\sqrt{2} \, . \, \frac{\pi}{2} \, . \, \frac{a^2}{16} = \frac{\pi a^2\sqrt{2}}{16}.$$

3. If n be even, x and y must be of the same sign; if n be odd, x and y may be of the same or opposite signs; also the axes are tangents at the origin, and the equation in polar co-ordinates being $r^2(\cos^{2n}\theta + \sin^{2n}\theta) = a^2\cos^{n-1}\theta \sin^{n-1}\theta$, for any value of θ there are 2 or 0 real values of r; thus there are four loops or two as n is odd or even, and the loops are clearly all equal. Hence, if n be even, the area is

$$2a^2\int_0^{\frac{\pi}{2}} \frac{d\theta}{2} \, . \, \frac{\cos^{n-1}\theta \sin^{n-1}\theta}{\cos^{2n}\theta + \sin^{2n}\theta} = a^2\int_0^{\frac{\pi}{2}} d\theta \frac{\sec^2\theta \tan^{n-1}\theta}{1 + \tan^{2n}\theta} = \frac{a^2}{n}\tan^{-1}(\tan^n\theta)\Big/_0^{\frac{\pi}{2}} = \frac{a^2\pi}{2n}.$$

Hence, if n be odd, the area $= \dfrac{a^2\pi}{n}$.

4. If O be a fixed point on the oval, P the point from which the unwinding begins, QP' the tangent to the oval at Q (any further point on it); and $OP = c$, $OQ = s$, $QP' = \text{arc } QP$, then P' is on the involute; and if $\sigma = \text{arc } PP'$ of the involute, and ψ_Q be the angle between the tangents at Q and O to the oval, and $\psi_P = \alpha$, then $d\sigma = PQd\psi_Q$ approximately, and therefore

$$\sigma = \int_\alpha^{\alpha+2\pi} d\psi(s - c), \text{ or if } s = f(\psi),$$

$$\sigma = \int_\alpha^{\alpha+2\pi} d\psi\{f(\psi) - f(\alpha)\}, \text{ therefore by Art. 216}$$

$$\frac{d\sigma}{d\alpha} = \int_\alpha^{\alpha+2\pi} d\psi\{-f'(\alpha)\} + f(\alpha + 2\pi) - f(\alpha);$$

therefore, if l be the length of the string, and ρ_P the radius of curvature at P, when the length of the involute is a maximum or minimum, since $\dfrac{d\sigma}{d\alpha}$ then vanishes,

$$\int_\alpha^{\alpha+2\pi} d\psi \, . \, f'(\alpha) = l = 2\pi\frac{d}{d\alpha}\{f(\alpha)\} = 2\pi\frac{dc}{d\alpha} = 2\pi\rho_P.$$

5. Here z is limited by the two planes, and if the cylinder be given in cylindrical co-ordinates ρ and ϕ, its equation is $\rho^2 = \rho r\cos\phi$, and the limits of ρ are 0 and $r\cos\phi$, and of ϕ are $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$. Thus the volume required is

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{r\cos\phi} \rho d\phi d\rho(a' - a)\rho\frac{\cos\phi}{c} = (a' - a)\frac{r^3}{3c}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \, . \cos^4\phi$$

$$= (a' - a)\frac{r^3}{3c} \, . \, \pi \, . \, \frac{3}{8} = \frac{\pi(a' - a)r^3}{8c}.$$

6. The surfaces meet where $x^2+4a(x+a)=c^2$, and therefore $x=-2a\pm c$, and therefore $x+a=\pm c-a$, therefore if $c>a$, the upper sign only of the ambiguity gives real values of y and z, and the volume bounded by the mutually concave parts of the two surfaces is

$$\int_{-a}^{c-2a} dx\,.\,\pi\,.\,4a(x+a)+\int_{c-2a}^{c} dx\,.\,\pi\,.\,(c^2-x^2)$$

$$=2\pi a(x+a)^2\Big/_{-a}^{c-2a}+\pi\left(c^2x-\frac{x^3}{3}\right)\Big/_{c-2a}^{c}$$

$$=2\pi a(c-a)^2+\pi\left\{\frac{2c^3}{3}-c^2(c-2a)+\frac{(c-2a)^3}{3}\right\}$$

$$=\pi a\left\{(2c^2-4ca+2a^2)+\left(2c^2-2c^2+4ca-\frac{8a^2}{3}\right)\right\}=2\pi a\left(c^2-\frac{a^2}{3}\right).$$

The other part of the volume is the difference between this and the volume of the sphere.

CHAPTER X.

1. If the volume be $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2}\int_{z_1}^{z_2} r\,d\theta\,dr\,dz$, z_1, z_2 are the values of z in the given equation, which in polar co-ordinates is $z^2=ar\sin 2\theta-r^2$; therefore the limits of r are 0 and $a\sin 2\theta$, and therefore those of θ are 0 and $\frac{\pi}{2}$, assuming x and y both positive. Hence

$$\text{the volume}=2\int_0^{\frac{\pi}{2}}\int_0^{a\sin 2\theta} r\,d\theta\,dr\sqrt{ar\sin 2\theta-r^2},$$

or if $r=\frac{a\sin 2\theta}{2}(1+\sin\phi)$, the limits of ϕ are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, therefore

$$\text{volume}=\int_0^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\,.\,a\sin 2\theta(1+\sin\phi)\frac{a}{2}\sin 2\theta\cos\phi\,d\phi\frac{a}{2}\sin 2\theta\cos\phi$$

$$=\frac{a^3}{4}\int_0^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\,d\phi\sin^3 2\theta\cos^2\phi(1+\sin\phi)=\frac{\pi a^3}{8}\int_0^{\frac{\pi}{2}} d\theta\,.\,\sin^3 2\theta$$

$$=\frac{\pi a^3}{16}\int_0^{\pi} d\theta\sin^3\theta=\frac{\pi a^3}{8}\,.\,\frac{2}{3}=\frac{\pi a^3}{12}.$$

Similarly when x and y are both negative, the form of the given equation being unaltered, the volume is the same, and thus the whole volume with the restriction specified is $\frac{\pi a^3}{6}$.

2. If the volume $= V$, then

$$\frac{V}{abc} = \text{volume of } x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}=1, \; = 2\int_{x_1}^{x_2}\int_{y_1}^{y_2} dxdy(1-x^{\frac{2}{3}}-y^{\frac{2}{3}})^{\frac{3}{2}},$$

and the lts. of y are given by $1-x^{\frac{2}{3}}-y^{\frac{2}{3}}=0$, and $\therefore$ the lts. of x are ± 1; hence

$$\frac{V}{abc} = 8\int_0^1\int_0^{y_2} dxdy(y_2^{\frac{2}{3}}-y^{\frac{2}{3}})^{\frac{3}{2}};$$

and if $y = y_2\sin^3\theta$, $dy = 3y_2\sin^2\theta\cos\theta d\theta$, and the limits of θ are 0 and $\frac{\pi}{2}$, and

$$\therefore \quad \frac{V}{8abc} = \int_0^1\int_0^{\frac{\pi}{2}} dxd\theta \,.\, 3y_2^2\sin^2\theta\cos^4\theta = 3\int_0^1 dx \,.\, y_2^2 \,.\, \frac{\pi}{2}\left(\frac{3}{8}-\frac{5}{6}\,.\,\frac{3}{8}\right)$$

$$= \frac{3\pi}{32}\int_0^1 dx(1-x^{\frac{2}{3}})^3, \text{ or putting } x = \sin^3\phi,$$

$$\frac{V}{8abc} = \frac{3\pi}{32}\int_0^{\frac{\pi}{2}} 3\sin^2\phi\cos^7\phi d\phi = \frac{9\pi}{32}\int_0^{\frac{\pi}{2}} d\phi(\cos^7\phi-\cos^9\phi),$$

$$\therefore \quad V = \frac{9\pi abc}{4}\left(\frac{6\,.\,4\,.\,2}{7\,.\,5\,.\,3} - \frac{8\,.\,6\,.\,4\,.\,2}{9\,.\,7\,.\,5\,.\,3}\right) = \frac{\pi abc}{4}\,.\,\frac{6\,.\,4\,.\,2}{7\,.\,5\,.\,3} = \frac{4\pi abc}{35}.$$

3. If the volume $V = \int_0^{z_2}\int_{x_1}^{x_2}\int_{y_1}^{y_2} dzdxdy$, y_1, y_2 are given by $ay^2 = z(a^2-z^2)-x^2z$, therefore x_1, x_2 are given by $x^2 = a^2-z^2$, and therefore $z_2 = a$. Hence

$$V = 4\int_0^a\int_0^{\sqrt{a^2-z^2}} \frac{dzdx}{a^{\frac{1}{2}}}\,.\,z^{\frac{1}{2}}(a^2-z^2-x^2)^{\frac{1}{2}} = \int_0^a dz\,.\sqrt{\frac{z}{a}}\,.\,\pi(a^2-z^2)$$

$$= \pi a^3\left(\frac{2}{3}-\frac{2}{7}\right) = \frac{8\pi a^3}{21}.$$

4. If O be the centre and C the fixed point, by revolution round OC, dS will generate a ring of surface the distance of each point of which from C is r; and if dS be at P on the sphere and the angle $POC = \theta$, $r^2 = c^2+a^2-2ca\cos\theta$, and therefore $\int\frac{dS}{r^n}$ over the whole of the surface

$$= \int_0^\pi \frac{ad\theta\,.\,2\pi a\sin\theta}{(a^2+c^2-2ac\cos\theta)^{\frac{n}{2}}} = \frac{-\pi a}{c\left(\frac{n}{2}-1\right)}\,.\,\frac{1}{(a^2+c^2-2ac\cos\theta)^{\frac{n}{2}-1}}\Bigg|_0^\pi$$

$$= \frac{2\pi a}{c(n-2)}\left\{\frac{1}{(c-a)^{n-2}} - \frac{1}{(a+c)^{n-2}}\right\}.$$

5. In cylindrical co-ordinates the equation to the sphere is $r^2+z^2 = a^2$, and $\therefore$ the volume

$$= 4\int_0^{\frac{\pi}{2n}}\int_0^{a\cos n\theta} rd\theta dr\,.\sqrt{a^2-r^2} = -\frac{4}{3}\int_0^{\frac{\pi}{2n}} d\theta\,.\,(a^2-r^2)^{\frac{3}{2}}\Bigg|_0^{a\cos n\theta}$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2n}} d\theta\{a^3-a^3\sin^3 n\theta\} = \frac{4a^3}{3n}\int_0^{\frac{\pi}{2}} d\theta(1-\sin^3\theta) = \frac{4a^3}{3n}\left(\frac{\pi}{2}-\frac{2}{3}\right).$$

For the surface S, $\sec\gamma$ on the sphere is $\frac{a}{z}$, and therefore

$$S = 4\int_0^{\frac{\pi}{2n}}\int_0^{a\cos n\theta} r\,d\theta\,dr \,.\, \frac{a}{\sqrt{a^2 - r^2}} = 4a\int_0^{\frac{\pi}{2n}} d\theta(a - a\sin n\theta)$$

$$= \frac{4a^2}{n}\int_0^{\frac{\pi}{2}} d\theta(1 - \sin\theta) = \frac{4a^2}{n}\left(\frac{\pi}{2} - 1\right).$$

6. In polar co-ordinates the curve is $r(\cos^3\theta + \sin^3\theta) = 3a\sin\theta\cos\theta$,

and the volume $V = \int_0^{\frac{\pi}{2}}\int_0^{r_1} r\,d\theta\,dr \,.\, 2\pi r\sin\left(\theta + \frac{\pi}{4}\right)$

$$= \pi\sqrt{2}\int_0^{\frac{\pi}{2}} d\theta(\sin\theta + \cos\theta)\frac{9a^3\sin^3\theta\cos^3\theta}{(\cos^3\theta + \sin^3\theta)^3}$$

$$= 9\pi a^3\sqrt{2}\int_0^{\frac{\pi}{2}} \frac{d\theta(1 + \tan\theta)\tan^3\theta\sec^2\theta}{(1 + \tan^3\theta)^3}, \text{ or if } \tan\theta = z,$$

$$V = 9\pi a^3\sqrt{2}\int_0^{\infty}\frac{dz(1 + z)z^3}{(1 + z^3)^3}, \text{ or if } 1 + z^3 = \frac{1}{x},$$

$$V = 9\pi a^3\sqrt{2}\int_0^1 \frac{dx}{3x^2z^2}(1 + z)z^3x^3$$

$$= 3\pi a^3\sqrt{2}\int_0^1 x\,dx\left\{\left(\frac{1-x}{x}\right)^{\frac{1}{3}} + \left(\frac{1-x}{x}\right)^{\frac{2}{3}}\right\}$$

$$= 3\pi a^3\sqrt{2}\int_0^1 dx\left\{x^{\frac{2}{3}}(1 - x)^{\frac{1}{3}} + x^{\frac{1}{3}}(1 - x)^{\frac{2}{3}}\right\}$$

$$= 6\pi a^3\sqrt{2}\,\Gamma(\tfrac{5}{3})\Gamma(\tfrac{4}{3}) \div \Gamma(3), \text{ by Art. 261,}$$

$$= 6\pi a^3\sqrt{2} \,.\, \tfrac{2}{3}\Gamma(\tfrac{2}{3}) \,.\, \tfrac{1}{3}\Gamma(\tfrac{1}{3}) \div \lfloor 2$$

$$= \frac{2}{3}\pi a^3\sqrt{2} \,.\, \frac{\pi}{\sin\frac{\pi}{3}}, \text{ by Art. 262} = \frac{4\pi^2a^3\sqrt{2}}{3\sqrt{3}} = \frac{8\pi^2a^3}{3\sqrt{6}}.$$

Aliter: to avoid the use of Gamma Functions the above integral in might be evaluated by the method of partial fractions, but V is more easily found by turning the initial line through an angle $\frac{\pi}{4}$, when the polar equation of the curve becomes

$$r\left\{\cos^3\left(\theta + \frac{\pi}{4}\right) + \sin^3\left(\theta + \frac{\pi}{4}\right)\right\} = \frac{3a}{2}\sin 2\left(\theta + \frac{\pi}{4}\right),$$

or
$$\frac{r}{\sqrt{2}}(\cos^3\theta + 3\cos\theta\sin^2\theta) = \frac{3a}{2}\cos 2\theta,$$

and $$V=2\int_0^{\frac{\pi}{4}}\int_0^{r_1} 2\pi r\cos\theta\,.\,rd\theta dr=\frac{4\pi}{3}\int_0^{\frac{\pi}{4}} d\theta\cos\theta\,.\,\frac{27a^3}{2\sqrt{2}}\,.\,\frac{\cos^3 2\theta}{\cos^3\theta(1+2\sin^2\theta)^3}$$

$$=9a^3\sqrt{2}\pi\int_0^{\frac{\pi}{4}} d\theta\sec^2\theta\left(\frac{1-\tan^2\theta}{1+3\tan^2\theta}\right)^3,\text{ or if }\sqrt{3}\tan\theta=\tan\phi,$$

$$V=3a^3\sqrt{6}\pi\int_0^{\frac{\pi}{3}} d\phi\sec^2\phi\left(1-\frac{\tan^2\phi}{3}\right)^3\frac{1}{\sec^6\phi}$$

$$=\frac{a^3}{9}\sqrt{6}\pi\int_0^{\frac{\pi}{3}}\frac{d\phi}{\sec^4\phi}(4-\sec^2\phi)^3=\frac{2\pi a^3}{3\sqrt{6}}\int_0^{\frac{\pi}{3}} d\phi(64\cos^4\phi-48\cos^2\phi+12-\sec^2\phi)$$

$$=\frac{2\pi a^3}{3\sqrt{6}}\int_0^{\frac{\pi}{3}} d\phi\{16(1+\cos 2\phi)^2-24(1+\cos 2\phi)+12-\sec^2\phi\}$$

$$=\frac{2\pi a^3}{3\sqrt{6}}\int_0^{\frac{\pi}{3}} d\phi\{4-\sec^2\phi+8\cos 2\phi+8(1+\cos 4\phi)\}$$

$$=\frac{2\pi a^3}{3\sqrt{6}}(12\phi-\tan\phi+4\sin 2\phi+2\sin 4\phi)\Big/_0^{\frac{\pi}{3}}$$

$$=\frac{2\pi a^3}{3\sqrt{6}}(4\pi-\sqrt{3}+2\sqrt{3}-\sqrt{3})=\frac{8\pi^2 a^3}{3\sqrt{6}}.$$

7. If the equation of a cylinder be $x^2+y^2=a^2$, and the axes of z and x be turned round the axis of y through an angle α, y is unaltered and x becomes $x\cos\alpha-z\sin\alpha$, and therefore the equation of the cylinder becomes

$$(x\cos\alpha-z\sin\alpha)^2+y^2=a^2.$$

Hence if $\beta=2a$, and the last equation represent one of the cylinders, the other may be represented by

$$(x\cos\alpha+z\sin\alpha)^2+y^2=a^2,$$

by changing the sign of α: and it follows that when x and z are both positive or both negative, the corresponding value of y on the second cylinder $<$ the value of y on the first cylinder, or the second cylinder lies within the first. Thus if V be the whole common volume

$$\frac{V}{2}=2\int_0^{x_2}\int_0^{z_2} dxdz\,.\,2\{a^2-(x\cos\alpha+z\sin\alpha)^2\}^{\frac{1}{2}},$$

where z_2 is the positive value of z when $y=0$, and therefore

$z_2\sin\alpha=a-x\cos\alpha$, and therefore $x_2=a\sec\alpha$; therefore

$$\frac{V}{4}=\int_0^{a\sec\alpha}\frac{dx}{\sin\alpha}\left\{(x\cos\alpha+z\sin\alpha)\sqrt{a^2-(x\cos\alpha+z\sin\alpha)^2}\right.$$
$$\left.+a^2\sin^{-1}\frac{x\cos\alpha+z\sin\alpha}{a}\right\}\Big/_0^{z_2}$$

$$=\int_0^{a\sec\alpha}\frac{dx}{\sin\alpha}\left\{a\,.\,0+a^2\frac{\pi}{2}-x\cos\alpha\sqrt{a^2-x^2\cos^2\alpha}-a^2\sin^{-1}\frac{x\cos\alpha}{a}\right\}$$

$$=\frac{\pi a^3}{2}\frac{\sec\alpha}{\sin\alpha}+\frac{(a^2-x^2\cos^2\alpha)^{\frac{3}{2}}}{3\cos\alpha\sin\alpha}\Bigg|_0^{a\sec\alpha}-\frac{a^2x}{\sin\alpha}\sin^{-1}\frac{x\cos\alpha}{a}\Bigg|_0^{a\sec\alpha}$$

$$+\frac{a^2}{\sin\alpha}\int_0^{a\sec\alpha}\frac{dx\,.\,xa}{\sqrt{a^2-x^2\cos^2\alpha}}\cdot\frac{\cos\alpha}{a}$$

$$=\frac{\pi a^3}{\sin\beta}-\frac{2a^3}{3\sin\beta}-\frac{a^3\pi}{\sin\beta}-\frac{2a^2}{\sin\beta}\sqrt{a^2-x^2\cos^2\alpha}\Bigg|_0^{a\sec\alpha}=-\frac{2a^3}{3\sin\beta}+\frac{2a^3}{\sin\beta},$$

$$\therefore\quad V=\frac{16a^3}{3\sin\beta}.\qquad\text{By Art. 170, } S=\iint dxdz\sqrt{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dz}\right)^2},$$

and for the second cylinder, the limits being the same as before,

$$\cos\alpha(x\cos\alpha+z\sin\alpha)+y\frac{dy}{dx}=0,$$

$$\sin\alpha(x\cos\alpha+z\sin\alpha)+y\frac{dy}{dz}=0,$$

$$\therefore\ \sec^2\gamma=1+\frac{(x\cos\alpha+z\sin\alpha)^2}{a^2-(x\cos\alpha+z\sin\alpha)^2},$$

$\therefore$, by symmetry, the surface of either cylinder intercepted by the other

$$=2\int_0^{x_2}\int_0^{z_2}\frac{dxdz\,.\,2a}{\{a^2-(x\cos\alpha+z\sin\alpha)^2\}^{\frac{1}{2}}}=\frac{4a}{\sin\alpha}\int_0^{x_2}dx\sin^{-1}\frac{x\cos\alpha+z\sin\alpha}{a}\Bigg|_0^{z_2}$$

$$=\frac{4a}{\sin\alpha}\int_0^{x_2}dx\left\{\frac{\pi}{2}-\sin^{-1}\frac{x\cos\alpha}{a}\right\}$$

$$=\frac{2a\pi}{\sin\alpha}\,.\,a\sec\alpha-\frac{4ax}{\sin\alpha}\,.\sin^{-1}\frac{x\cos\alpha}{a}\Bigg|_0^{a\sec\alpha}+\frac{4a}{\sin\alpha}\int_0^{a\sec\alpha}\frac{dx\,.\,xa}{\sqrt{a^2-x^2\cos^2\alpha}}\cdot\frac{\cos\alpha}{a}$$

$$=\frac{4\pi a^2}{\sin\beta}-\frac{8a^2}{\sin\beta}\cdot\frac{\pi}{2}-\frac{8a}{\sin\beta}\sqrt{a^2-x^2\cos^2\alpha}\Bigg|_0^{a\sec\alpha}=\frac{8a^2}{\sin\beta}.$$

8. The plane of the moving circle, being normal to the fixed circle, must pass through its centre; and if two adjacent positions of the moving centre, in polar co-ordinates, be $(a,\ \theta)$ and $(a,\ \theta+\Delta\theta)$, the element of area between the corresponding diameters $2r$ and $2(r+\Delta r)$ in the plane of the fixed circle, neglecting higher powers of $\Delta\theta$ than the first, is

$$\frac{\Delta\theta}{2}\{(a+r)^2-(a-r)^2\}=2ar\Delta\theta,$$

and $r=a\sin\theta$, if the initial line be the given fixed diameter. Similarly the element of area between the planes of the moving circle in the above adjacent positions at a distance z from the plane of the fixed circle is $2a\Delta\theta\sqrt{r^2 - z^2}$.

Then the whole volume of the tube surface, which $\propto$ from a point when $\theta = 0$ to a maximum normal section $= \pi a^2$ when $\theta = \frac{\pi}{2}$, is

$$8 \cdot 2a \int_0^{\frac{\pi}{2}}\int_0^{a\sin\theta} d\theta dz \sqrt{a^2\sin^2\theta - z^2} = 4a^3\pi \int_0^{\frac{\pi}{2}} d\theta \cdot \sin^2\theta = \pi^2 a^3.$$

The trace of the tube surface on the plane of the fixed circle is bounded by the two curves $r = a(1 \pm \sin\theta)$, and therefore the volume swept out by the tube in a complete revolution round the fixed diameter = the difference of the volumes described by these curves

$$= 2\int_0^{\frac{\pi}{2}}\int_{a(1-\sin\theta)}^{a(1+\sin\theta)} 2\pi r \sin\theta \cdot r d\theta dr = \frac{4}{3}\pi a^3 \int_0^{\frac{\pi}{2}} d\theta \cdot \sin\theta\{(1+\sin\theta)^3 - (1-\sin\theta)^3\}$$

$$= \frac{8}{3}\pi a^3 \int_0^{\frac{\pi}{2}} d\theta (3\sin^2\theta + \sin^4\theta) = \frac{4}{3}\pi^2 a^3\left(\frac{3}{2} + \frac{3}{8}\right) = \frac{\pi^2 a^3}{2} \cdot 5$$

$$= \frac{5}{2} \cdot \text{volume of the tube.}$$

9. If A be the area of a regular hexagon of diagonal $2a$, the volume required is, if $x^2 + y^2 = a^2$ be the circle,

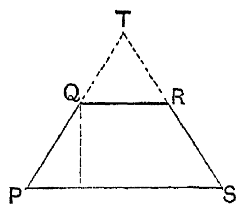
$$2\int_0^a dx \cdot \frac{Ay^2}{a^2} = \frac{2A}{a^2}\int_0^a dx(a^2 - x^2) = \frac{4}{3}Aa:$$

and the side of the hexagon $= a$,

and $$A = 6 \cdot \frac{a^2}{2}\sin\frac{\pi}{3} = 3a^2 \cdot \frac{\sqrt{3}}{2},$$

$\therefore$ the volume generated $= 2\sqrt{3}a^3$.

If $PQRS$ be half the hexagon on one side of the circle, when the side of the hexagon $= y$, the element of surface between QR and its adjacent position is to the first order of small quantities



$$PQ\sqrt{\Delta x^2 + \Delta y^2 \sin^2\frac{\pi}{6}} = PQ \cdot d \text{ say};$$

and if PQ, SR produced meet in T, by turning PQT in the plane PQR round T, till PQ is parallel to QR, it will be seen that the distance between PQ and its adjacent position also $= d$. Hence, as $PQ = y = QR$ and $\frac{dy}{dx} = -\frac{x}{y}$, $PQ \cdot d = \Delta x\sqrt{a^2 - x^2 + \frac{3}{4}x^2}$, and the whole surface is therefore

$$12\int_0^a dx\sqrt{a^2-\frac{x^2}{4}}=24\int_0^{\frac{a}{2}} dx\sqrt{a^2-x^2}=12\left\{x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right\}\Big/_0^{\frac{a}{2}}$$

$$=12a^2\left\{\frac{\sqrt{3}}{4}+\frac{\pi}{6}\right\}=a^2(3\sqrt{3}+2\pi).$$

10. If $x=a\sin^2\theta$,

$$\int_0^{\frac{a}{2}}\frac{dx}{\sqrt{2ax-x^2}\sqrt{a^2-x^2}}=\frac{1}{a}\int_0^{\frac{\pi}{4}}\frac{2\sin\theta\cos\theta d\theta}{\sin\theta\sqrt{2-\sin^2\theta}\sqrt{1-\sin^2\theta}}=\frac{1}{a}\int_0^{\frac{\pi}{4}}\frac{2d\theta}{\sqrt{(1+\sin^2\theta)(2-\sin^2\theta)}}$$

$$=\frac{1}{a}\int_0^{\frac{\pi}{4}}\frac{2d\theta}{\left\{\left(1+\frac{1-\cos 2\theta}{2}\right)\left(2-\frac{1-\cos 2\theta}{2}\right)\right\}^{\frac{1}{2}}}$$

$$=\frac{2}{a}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\{(3-\cos\theta)(3+\cos\theta)\}^{\frac{1}{2}}}=\frac{2}{3a}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{1}{9}\cos^2\theta}}$$

$$=\frac{2}{3a}F\left(c,\frac{\pi}{2}\right)\text{ when } c=\frac{1}{3}.$$

11. Here $s=\int d\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$, and $r\frac{dr}{d\theta}=-a^2\sin 2\theta$,

$$\therefore\quad s=\int d\theta\left\{a^2\cos 2\theta+\frac{a^2\sin^2 2\theta}{\cos 2\theta}\right\}^{\frac{1}{2}}=\int\frac{a d\theta}{\sqrt{\cos 2\theta}}=\int\frac{a d\theta}{\sqrt{1-2\sin^2\theta}},$$

or if $\sqrt{2}\sin\theta=\sin\phi$, and therefore $\sqrt{2}\cos\theta d\theta=\cos\phi d\phi$,

$$s=\int\frac{a\cos\phi d\phi}{\sqrt{2-\sin^2\phi}\,.\cos\phi}=\frac{a}{\sqrt{2}}\int\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}},$$

which is an elliptic integral of the first kind.

12. If the angles AOP, BOQ be θ, ϕ, since the arc $AP=QO$, the arc $AP+AQ=$ arc AO, therefore

$$\int_0^\theta\frac{d\theta}{\sqrt{1-2\sin^2\theta}}+\int_0^\phi\frac{d\phi}{\sqrt{1-2\sin^2\phi}}=\int_0^{\frac{\pi}{4}}\frac{dx}{\sqrt{1-2\sin^2 x}}=\text{a constant},$$

and the proof of the proposition in Art. 225 holds good for positive values of c^2, so long as $c^2\sin^2\mu$ is not >1, and here $\mu=\frac{\pi}{4}$ and $c^2=2$, therefore

$$\cos\theta\cos\phi-\sin\theta\sin\phi\sqrt{1-2\sin^2\frac{\pi}{4}}=\cos\frac{\pi}{4},\quad\text{or}\quad\cos\theta\cos\phi=\frac{1}{\sqrt{2}}.$$

CHAPTER XI.

1. By Art. 239 (6) $dxdy$ is changed into $d\phi d\theta\left(\frac{dx}{d\theta}\cdot\frac{dy}{d\phi}-\frac{dy}{d\theta}\cdot\frac{dx}{d\phi}\right)$ which here $=d\phi d\theta ab(\cos^2\theta\sin\phi\cos\phi+\sin\theta\sin\phi\sin\theta\cos\phi)$, therefore

$$\iint dxdy=\iint abd\phi d\theta\sin\phi\cos\phi.$$

2. Here $\frac{dx}{du}=\sin\alpha$, $\frac{dx}{dv}=\cos\alpha$, $\frac{dy}{du}=\cos\alpha$, $\frac{dy}{dv}=-\sin\alpha$, and $x^2+y^2=u^2+v^2$; thus

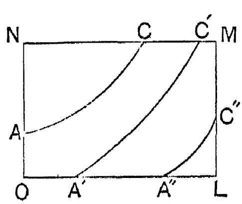
$$\frac{dx}{dv}\cdot\frac{dy}{du}-\frac{dx}{du}\cdot\frac{dy}{dv}=1,$$

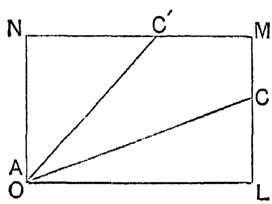
and

$$\iint f(x,y)\frac{dxdy}{\sqrt{1-x^2-y^2}}=\iint f_1(u,v)\frac{dudv}{\sqrt{1-u^2-v^2}},$$

f_1 denoting the function of u, v which $f(x,y)$ becomes on substitution.

3. If the limits of x and y are both constants the boundary $ABCD$ in the figure to Art. 246 is a rectangle with its sides parallel to the axes, suppose $OLMN$. Then the curve AC will in general, as u $\propto$, meet some two sides of the rectangle in three different sets of ways, for it must cut two sides, say ON and NM, and as A and C approach O and M, A and O will coincide before C and M or *vice versa*, or simultaneously. In the last case there may be only two integrals in the transformed integral, but otherwise three. If AC be a straight line parallel to either axis there will be only





one integral. The method assumes that AC actually cuts the rectangle. As explained in Art. 246, if the integration be with regard to v first, the limits of v are determined by the values of x and y corresponding to the extremities A, C of the curve AC within the boundary, for a given value of u. Similarly for the integrals corresponding to $A'C'$, $A''C''$. The limits of u are found from the values of x and y at (1) O and N, (2) at O and M, and (3) at L and M. For example in Art. 240, $OL=a$, $ON=b$, $u=x+y$, and

$v=\frac{y}{x+y}$, therefore AC is a straight line through the origin; and while C is between L and M, $u \propto$ from 0 at A to $\frac{y}{v}$ at C where $x=a$, and therefore $\frac{y}{v}=\frac{a}{1-v}$; and the corresponding limits of v are 0 at L where $y=0$, and $\frac{b}{a+b}$ at M; therefore the first part of the integral is

$$\int_0^{\frac{b}{a+b}}\int_0^{\frac{a}{1-v}} udvduV'.$$

For the second part of the integral, $u \propto$ from 0 at A to $\frac{y}{v}=\frac{b}{v}$ at C', and the corresponding limits of v are 1 at N and $\frac{b}{a+b}$ at M, and therefore the second part of the integral is

$$\int_{\frac{b}{a+b}}^1 \int_0^{\frac{b}{v}} V'udvdu.$$

4. If $ax=x'$, and $by=y'$,

$\int_0^\infty\int_0^\infty \phi(a^2x^2+b^2y^2)dxdy=\int_0^\infty\int_0^\infty \phi(x'^2+y'^2)\frac{dx'dy'}{ab}$, and transforming to polar

co-ordinates this $$=\frac{1}{ab}\int_0^\infty\int_0^{\frac{\pi}{2}} \phi(r^2)\,.\,rdrd\theta=\frac{\pi}{2ab}\int_0^\infty \phi(r^2)\,.\,\frac{dr^2}{2},$$

which is the same as $$\frac{\pi}{4ab}\int_0^\infty \phi(x)dx.$$

5. Here $x=\frac{v}{1+u}$, $y=\frac{uv}{1+u}=v-\frac{v}{1+u}$,

$$\therefore \quad \frac{dx}{dv}=\frac{1}{1+u},\ \frac{dy}{du}=\frac{v}{(1+u)^2},\ \frac{dx}{du}=-\frac{v}{(1+u)^2},\ \text{and}\ \frac{dy}{dv}=\frac{u}{1+u},$$

and $\therefore$ $dxdy$, which becomes $dudv\left(\frac{dx}{dv}\cdot\frac{dy}{du}-\frac{dx}{du}\cdot\frac{dy}{dv}\right)$,

$$=dudv\left\{\frac{v}{(1+u)^3}+\frac{vu}{(1+u)^3}\right\}=\frac{dudv\,.\,v}{(1+u)^2}.$$

For the limits, eliminating v, $y=ux$, therefore the limits of u corresponding to 0 and x of y are 0 and 1; and changing the order of integration between x and u their limits, being constant are unaltered, thus those of x are still 0 and a. Lastly, eliminating y, $x=\frac{v}{1+u}$, and therefore the limits of v are 0 and $a(1+u)$. Hence

$$\int_0^a\int_0^x Vdxdy=\int_0^1\int_0^{a(1+u)} V'.v(1+u)^{-2}dudv.$$

6. In polar co-ordinates

$$\int_0^\infty\int_0^\infty \frac{dxdy}{e^{(x^2+2xy\cos\alpha+y^2)}} = U\text{ (say)} = \int_0^{\frac{\pi}{2}}\int_0^\infty \frac{rd\theta dr}{e^{r^2(1+\cos\alpha\sin 2\theta)}}$$

and $\therefore$ $$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta \,.\, e^{-r^2(1+\cos\alpha\sin 2\theta)}}{1+\cos\alpha\sin 2\theta},$$ r being taken between the lts. 0 and ∞, $\therefore$

$$U = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\cos\alpha\sin 2\theta} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta \,.\, \sec^2\theta}{1+\tan^2\theta+2\cos\alpha\tan\theta}$$

$$= \frac{1}{2}\int_0^\infty \frac{dz}{(z+\cos\alpha)^2+\sin^2\alpha} = \frac{1}{2\sin\alpha}\,.\,\tan^{-1}\frac{z+\cos\alpha}{\sin\alpha},$$

when z is taken between the limits 0 and ∞, and therefore

$$U = \frac{1}{2\sin\alpha}\left\{\frac{\pi}{2} - \overline{\frac{\pi}{2}-\alpha}\right\} = \frac{\alpha}{2\sin\alpha}. \quad \text{Cf. Art. 46.}$$

7. Clearly $\int_0^a\int_0^b \phi(x, y)dxdy$ in polar co-ordinates

$$= \int_0^{\tan^{-1}\frac{b}{a}}\int_0^{a\sec\theta} \phi_1(x, y)rd\theta dr + \int_{\tan^{-1}\frac{b}{a}}^{\frac{\pi}{2}}\int_0^{b\,\mathrm{cosec}\,\theta} \phi_1(x, y)rd\theta dr,$$

supposing a and b both positive. If a be negative and b positive, the limits of θ will be successively $\pi-\tan^{-1}\frac{b}{-a}$ and π; and $\frac{\pi}{2}$ and $\pi-\tan^{-1}\frac{b}{-a}$: and similarly for the other possible cases. Thus

$$\int_0^a\int_0^b \frac{dxdy}{(c^2+x^2+y^2)^{\frac{3}{2}}} = U, \text{ suppose,}$$

$$= \int_0^{\tan^{-1}\frac{b}{a}}\int_0^{a\sec\theta} \frac{rd\theta dr}{(c^2+r^2)^{\frac{3}{2}}} + \int_{\tan^{-1}\frac{b}{a}}^{\frac{\pi}{2}}\int_0^{b\,\mathrm{cosec}\,\theta} \frac{rd\theta dr}{(c^2+r^2)^{\frac{3}{2}}}$$

$$= -\int_0^{\tan^{-1}\frac{b}{a}} \frac{d\theta}{(c^2+r^2)^{\frac{1}{2}}}\Bigg/_0^{a\sec\theta} - \int_{\tan^{-1}\frac{b}{a}}^{\frac{\pi}{2}} \frac{d\theta}{(c^2+r^2)^{\frac{1}{2}}}\Bigg/_0^{b\,\mathrm{cosec}\,\theta}$$

$$= -\int_0^{\tan^{-1}\frac{b}{a}} d\theta\left\{\frac{1}{(c^2+a^2\sec^2\theta)^{\frac{1}{2}}} - \frac{1}{c}\right\} - \int_{\tan^{-1}\frac{b}{a}}^{\frac{\pi}{2}} d\theta\left\{\frac{1}{(c^2+b^2\mathrm{cosec}^2\theta)^{\frac{1}{2}}} - \frac{1}{c}\right\}$$

$$= \frac{1}{c}\int_0^{\frac{\pi}{2}} d\theta - \int_0^{\tan^{-1}\frac{b}{a}} d\theta \frac{\cos\theta}{(c^2+a^2-c^2\sin^2\theta)^{\frac{1}{2}}} - \int_{\tan^{-1}\frac{b}{a}}^{\frac{\pi}{2}} d\theta \frac{\sin\theta}{(b^2+c^2-c^2\cos^2\theta)^{\frac{1}{2}}},$$

$$\therefore \quad cU = \frac{\pi}{2} - \sin^{-1}\frac{c\sin\theta}{\sqrt{c^2+a^2}}\Bigg|_0^{\tan^{-1}\frac{b}{a}} + \sin^{-1}\frac{c\cos\theta}{\sqrt{b^2+c^2}}\Bigg|_{\tan^{-1}\frac{b}{a}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - \sin^{-1}\frac{bc}{\sqrt{c^2+a^2}\sqrt{a^2+b^2}} - \sin^{-1}\frac{ca}{\sqrt{b^2+c^2}\sqrt{a^2+b^2}}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{bc}{a(a^2+b^2+c^2)^{\frac{1}{2}}} - \tan^{-1}\frac{ca}{b(a^2+b^2+c^2)^{\frac{1}{2}}}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{c(a^2+b^2)}{ab(a^2+b^2+c^2)^{\frac{1}{2}}} \div \left\{\frac{ab(a^2+b^2+c^2)-abc^2}{ab(a^2+b^2+c^2)}\right\}$$

$$= \cot^{-1}\left\{\frac{c}{ab}(a^2+b^2+c^2)^{\frac{1}{2}}\right\} = \tan^{-1}\frac{ab}{c(a^2+b^2+c^2)^{\frac{1}{2}}},$$

$$\therefore \quad U = \frac{1}{c}\tan^{-1}\frac{ab}{c\sqrt{a^2+b^2+c^2}}.$$

8. The integral $= \int_0^{2\pi}\int_0^{\infty}\frac{ar\,d\theta\,dr}{(r^2+a^2)^{\frac{3}{2}}(r^2+a'^2)^{\frac{1}{2}}} = \pi a \,.\int_0^{\infty}\frac{d(r^2)}{(r^2+a^2)^{\frac{3}{2}}(r^2+a'^2)^{\frac{1}{2}}};$

let $r^2+a'^2=u^2$, and therefore $d(r^2)=2udu$, and the integral becomes

$$2\pi a\int_{a'}^{\infty}\frac{udu}{(u^2+a^2-a'^2)^{\frac{3}{2}}u} = \frac{2\pi au}{(a^2-a'^2)(u^2+a^2-a'^2)^{\frac{1}{2}}}\Bigg|_{a'}^{\infty} \text{ (per Ex. 3, Art. 15)}$$

$$= \frac{2\pi a}{a^2-a'^2} - \frac{2\pi aa'}{(a^2-a'^2)\,.\,(a)} = \frac{2\pi}{a+a'}.$$

9. Here $\frac{dx}{dr}=\cos\theta$, $\frac{dy}{d\theta}=r\cos\theta - a\sin\theta$, $\frac{dx}{d\theta}=-r\sin\theta+a\cos\theta$,

$\frac{dy}{dr}=\sin\theta$; $\therefore$ $dxdy$ transforms into

$$(r\cos^2\theta - a\sin\theta\cos\theta + r\sin^2\theta - a\sin\theta\cos\theta)d\theta dr$$

or

$$(r - a\sin 2\theta)d\theta dr; \quad \therefore \text{ etc.}$$

10. Here $x = \frac{r}{(1+t^2)^{\frac{1}{2}}}$, $\quad y = \frac{rt}{(1+t^2)^{\frac{1}{2}}}$,

$$\therefore \quad \frac{dx}{dt} = -\frac{rt}{(1+t^2)^{\frac{3}{2}}},\ \frac{dy}{dr} = \frac{t}{(1+t^2)^{\frac{1}{2}}},\ \frac{dx}{dr} = \frac{1}{(1+t^2)^{\frac{1}{2}}},\ \text{and } \frac{dy}{dt} = \frac{r}{(1+t^2)^{\frac{3}{2}}};$$

and $\therefore$ $\frac{dxdy}{drdt} = -\frac{rt^2}{(1+t^2)^2} - \frac{r}{(1+t^2)^2} = -\frac{r}{1+t^2}.$

For the limits changing from y to r, $r^2=x^2+y^2$, therefore limits of r are x and ∞, and the integral U (say) is of the form $\int_0^{\infty}\int_x^{\infty}Vdxdr$: now changing the order of integration, the limits of r are 0 and ∞, and of x are 0 and r, therefore $U=\int_0^\infty\int_0^r Vdrdx$. Lastly, to change from x to t, eliminating y, $x=\frac{r}{(1+t^2)^{\frac{1}{2}}}$, and therefore the limits of t are ∞ and 0. Thus

$$U=\int_0^\infty\int_0^\infty\frac{e^{-r^2}rdrdt}{1+t^2}.$$

11. As in Art. 239, the integral becomes

$$\iiint drd\theta dz\left(\frac{dx}{d\theta}\cdot\frac{dy}{dr}-\frac{dx}{dr}\cdot\frac{dy}{d\theta}\right).$$

With cylindrical co-ordinates the integral is

$$\iiint rdrd\theta dz,$$

and a figure will readily show that the limits of r are 0 and a, of θ are 0 and $\frac{\pi}{2}$, and of z are 0 and $mr\cos\theta$. Hence

$$\text{the volume}=\int_0^a\int_0^{\frac{\pi}{2}}\int_0^{mr\cos\theta} rdrd\theta dz=\int_0^a\int_0^{\frac{\pi}{2}} mr^2drd\theta\,.\cos\theta=\int_0^a mr^2dr=\frac{ma^3}{3}.$$

12. Here $x=\sqrt{\beta\gamma}$, $y=\sqrt{\gamma\alpha}$, $z=\sqrt{\alpha\beta}$, $\therefore$ as in Art. 245, $dxdydz$ becomes

$$d\alpha d\beta d\gamma\left\{\frac{dx}{d\alpha}\left(\frac{dy}{d\beta}\cdot\frac{dz}{d\gamma}-\frac{dy}{d\gamma}\cdot\frac{dz}{d\beta}\right)+\frac{dy}{d\alpha}\left(\frac{dz}{d\beta}\cdot\frac{dx}{d\gamma}-\frac{dz}{d\gamma}\cdot\frac{dx}{d\beta}\right)+\frac{dz}{d\alpha}\left(\frac{dx}{d\beta}\cdot\frac{dy}{d\gamma}-\frac{dx}{d\gamma}\cdot\frac{dy}{d\beta}\right)\right\}$$

$$=\left\{\frac{1}{2}\sqrt{\frac{\gamma}{\alpha}}\left(\frac{1}{2}\sqrt{\frac{\alpha}{\beta}}\cdot\frac{1}{2}\sqrt{\frac{\beta}{\gamma}}\right)+\frac{1}{2}\sqrt{\frac{\beta}{\alpha}}\left(\frac{1}{2}\sqrt{\frac{\gamma}{\beta}}\cdot\frac{1}{2}\sqrt{\frac{\alpha}{\gamma}}\right)\right\}d\alpha d\beta d\gamma$$

$$=\frac{1}{4}d\alpha d\beta d\gamma,\quad\therefore\text{ etc.}$$

13. To change x_4 to ψ, eliminating r, θ, and ϕ, $x_4=x_3\tan\psi$, therefore $dx_4=x_3\sec^2\psi d\psi$, and the integral U becomes $\iiiint V_1dx_1dx_2dx_3\,.\,x_3\sec^2\psi d\psi$: next, to change x_3 to θ, eliminating r, ϕ, and x_4, $\sqrt{x_1^2+x_2^2}=r\sin\theta=x_3\tan\theta\sec\psi$, therefore $dx_3=-\cos\psi\,.\sqrt{x_1^2+x_2^2}\,.\operatorname{cosec}^2\theta d\theta=-r\cos\psi\operatorname{cosec}\theta d\theta$, therefore U becomes $-\iiiint V_2dx_1dx_2d\theta d\psi\,.\,r^2\cot\theta$: thirdly, to change x_2 to ϕ, eliminating x_3, x_4, and r, $x_2=x_1\tan\phi$, therefore $dx_2=x_1\sec^2\phi d\phi=r\sin\theta\sec\phi d\phi$, therefore U becomes $-\iiiint V_3dx_1d\phi d\theta d\psi\,.\,r^3\cos\theta\sec\phi$: lastly, to change x_1 to r, eliminating x_2, x_3, x_4, $dx_1=\sin\theta\cos\phi dr$, and therefore

$$U=-\iiiint V'.\,drd\phi d\theta d\psi\,.\,r^3\sin\theta\cos\theta.$$

14. By Art. 246 the elementary area $=\pm dudv\left(\frac{dx}{du}\cdot\frac{dy}{dv}-\frac{dx}{dv}\cdot\frac{dy}{du}\right)$. The

tangents being at right angles, let them be axes, and the fixed parabola $\sqrt{x}+\sqrt{y}=\sqrt{a}$, any variable parabola $\sqrt{x}+\sqrt{y}=\sqrt{u}$, so that $u \propto$ from 0 to a, and let one of the straight lines be $y=vx$, so that $v \propto$ from 0 to ∞. Then here $u=(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2$, $v=\frac{y}{x}$. $\therefore \frac{du}{dx}=x^{-\frac{1}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}})$, $\frac{dv}{dy}=\frac{1}{x}$,

$$\frac{du}{dy}=y^{-\frac{1}{2}}(x^{\frac{1}{2}}+y^{\frac{1}{2}}),\ \frac{dv}{dx}=-\frac{y}{x^2},$$

$$\therefore \qquad \frac{du}{dx}\cdot\frac{dv}{dy}-\frac{du}{dy}\cdot\frac{dv}{dx}=x^{-\frac{3}{2}}.u^{\frac{1}{2}}+x^{-2}y^{\frac{1}{2}}u^{\frac{1}{2}}=\frac{u}{x^2},$$

or since $x^{\frac{1}{2}}(1+v^{\frac{1}{2}})=u^{\frac{1}{2}}$, the area, by Art. 239 (8),

$$=\int\int dxdy=\int_0^\infty\int_0^a dvdu\,.\frac{u}{(1+v^{\frac{1}{2}})^4}$$

$$=\frac{a^2}{2}\int_0^\infty\frac{dv}{(1+v^{\frac{1}{2}})^4}\ :\ \text{let }1+v^{\frac{1}{2}}=z,$$

then $$\text{area}=a^2\int_1^\infty\frac{(z-1)dz}{z^4}=a^2\left(-\frac{1}{2z^2}+\frac{1}{3z^3}\right)\Big/_1^\infty=a^2\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{a^2}{6}.$$

If $4c$ be the latus rectum of the given parabola, $a^2=8c^2$, and therefore the area $=\frac{4}{3}c^2$; which may be verified geometrically.

15. (1) y and z being constants, $\frac{d\psi}{dx}\cdot\frac{dx}{dr}=-\frac{d\psi}{dr}$,

$$\therefore \qquad \int\int\int f(x,\,y,\,z)dxdydz=-\int\int\int f_1(r,\,y,\,z)\frac{drdydz}{\frac{d\psi}{dx}}\cdot\frac{d\psi}{dr}.$$

(2) In changing from x to r, y and z are constants,

$$\therefore \qquad \frac{d\psi}{dx}\cdot\frac{dx}{dr}+\frac{d\psi}{dr}=0;$$

in changing from y to θ, z and r are constants,

$$\therefore \qquad \frac{d\psi_1}{dy}\cdot\frac{dy}{d\theta}+\frac{d\psi_1}{d\theta}=0;$$

and in changing from z to ϕ, r and θ are constants,

$$\therefore \qquad \frac{d\psi_2}{dz}\cdot\frac{dz}{d\phi}+\frac{d\psi_2}{d\phi}=0.$$

Hence $$\int\int\int f(x,\,y,\,z)dxdydz=-\int\int\int f_1(r,\,\theta,\,\phi)drd\theta d\phi\,.\frac{\frac{d\psi}{dr}}{\frac{d\psi}{dx}}\cdot\frac{\frac{d\psi_1}{d\theta}}{\frac{d\psi_1}{dy}}\cdot\frac{\frac{d\psi_2}{d\phi}}{\frac{d\psi_2}{dz}}.$$

16. Since $x^2+y^2+z^2=1$, $\frac{dz}{dx}=-\frac{x}{z}$, $\frac{dz}{dy}=-\frac{y}{z}$,

and $\therefore$ $\left\{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}}=\frac{1}{z}=\frac{1}{\sin\theta(1-n^2\sin^2\phi)^{\frac{1}{2}}}$.

Also $\frac{dx}{d\phi}=\cos\phi(1-m^2\sin^2\theta)^{\frac{1}{2}}$, $\qquad \frac{dy}{d\theta}=-\sin\theta\cos\phi$,

$$\frac{dx}{d\theta}=-\frac{m^2\sin\theta\cos\theta\sin\phi}{(1-m^2\sin^2\theta)^{\frac{1}{2}}},\text{ and }\frac{dy}{d\phi}=-\cos\theta\sin\phi\,;$$

$\therefore$ $dxdy$ becomes $\frac{-d\theta d\phi\,.\sin\theta\{\cos^2\phi(1-m^2\sin^2\theta)+m^2\sin^2\phi\cos^2\theta\}}{(1-m^2\sin^2\theta)^{\frac{1}{2}}}$ and the double bracket in the numerator $=\cos^2\phi-m^2\cos^2\phi+m^2\cos^2\theta$

$$=n^2\cos^2\phi+m^2\cos^2\theta,$$

and $\therefore$ the integral transforms to $-\iint\frac{d\theta d\phi(n^2\cos^2\phi+m^2\cos^2\theta)}{(1-m^2\sin^2\theta)^{\frac{1}{2}}(1-n^2\sin^2\phi)^{\frac{1}{2}}}$. When this is taken between the limits 0 and $\frac{\pi}{2}$ for both θ and ϕ, to find the corresponding lts. of y and x in $\iint\frac{dxdy}{z}$, changing from ϕ to y, and therefore eliminating x, $y=\cos\theta\cos\phi$; therefore the limits of y so far, corresponding to 0 and $\frac{\pi}{2}$ for ϕ, are $\cos\theta$ and 0; now changing the order of $d\theta$ and dy, the limits of y are easily seen to be 0 and 1, and of θ (viewed as an abscissa) to be 0 and $\cos^{-1}y$; lastly, changing from θ to x, eliminating ϕ, $\frac{x^2}{1-m^2\sin^2\theta}+\frac{y^2}{\cos^2\theta}=1$, and $\therefore$ corresponding to the limits of 0 and $\cos^{-1}y$ of θ, those of x are $\sqrt{1-y^2}$ and 0. Hence the proposed integral in θ and ϕ

$$=-\int_0^1\int_{\sqrt{1-y^2}}^0\frac{dydx}{(1-x^2-y^2)^{\frac{1}{2}}}=\int_0^1 dy\,.\sin^{-1}\frac{x}{\sqrt{1-y^2}}\Big/_0^{\sqrt{1-y^2}}=\frac{\pi}{2}.$$

17. Let $(1-n^2\cos^2\theta)^{\frac{1}{2}}=p$, and $(\cos^2\phi+n^2\sin^2\phi)^{\frac{1}{2}}=q$; then changing from z to θ, and therefore eliminating r and ϕ,

$$z^2=\cos^2\theta\left\{\frac{y^2}{\sin^2\theta}+\frac{n^2x^2}{1-n^2\cos^2\theta}\right\}=y^2\cot^2\theta+x^2\left\{-1+\frac{1}{1-n^2\cos^2\theta}\right\},$$

$$\therefore\quad z\frac{dz}{d\theta}=-y^2\cot\theta\operatorname{cosec}^2\theta-\frac{n^2x^2\sin\theta\cos\theta}{(1-n^2\cos^2\theta)^2}=-r^2\cos^2\phi\cot\theta-\frac{n^2r^2\sin^2\phi\sin\theta\cos\theta}{1-n^2\cos^2\theta}$$

$$=-\frac{r^2}{p^2}\cot\theta\{\cos^2\phi-n^2\cos^2\phi\cos^2\theta+n^2\sin^2\phi\sin^2\theta\}$$

$$\therefore\quad \frac{dz}{d\theta}=-\frac{r\operatorname{cosec}\theta}{p^2q}\{\cos^2\phi(1-n^2)+n^2\sin^2\theta\}\,;$$

next, changing from y to ϕ, and therefore eliminating z and r,

$$y = \sin\theta \,.\, \frac{x\cot\phi}{p}, \;\therefore\; \frac{dy}{d\phi} = -\frac{x}{p}\sin\theta\,\text{cosec}^2\phi = -\frac{r\sin\theta}{\sin\phi};$$

and, lastly, changing x to r, and therefore eliminating y and z,

$$\frac{dx}{dr} = p\sin\phi. \quad \text{Hence } \iiint dxdydz \text{ transforms into}$$

$$\iiint dr d\theta d\phi \,.\, \frac{r^2}{pq}\text{cosec}\,\theta\{\cos^2\phi(1-n^2)+n^2\sin^2\theta\}\frac{\sin\theta}{\sin\phi}\,.\sin\phi$$

$$= \iiint dr d\theta d\phi \,.\, \frac{r^2\{\cos^2\phi(1-n^2)+n^2\sin^2\theta\}}{(1-n^2\cos^2\theta)^{\frac{1}{2}}.\,(\cos^2\phi+n^2\sin^2\phi)^{\frac{1}{2}}}.$$

18. If $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, and $y = r\sin\theta\sin\phi$, then eliminating r, $\tan\phi = \frac{y}{x}$, $\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$, and, as in Art. 244, $\sec^2\phi\frac{d\phi}{dy} = \frac{1}{x}$ or $\frac{d\phi}{dy} = \frac{x}{x^2+y^2}$;

$$\sec^2\theta\frac{d\theta}{dx} = \frac{x}{z\sqrt{x^2+y^2}} - \frac{\sqrt{x^2+y^2}}{z^2}\frac{dz}{dx} \text{ or } \frac{d\theta}{dx} = \frac{xz - p(x^2+y^2)}{r^2\sqrt{x^2+y^2}};$$

so $$\frac{d\theta}{dy} = \frac{yz - q(x^2+y^2)}{r^2\sqrt{x^2+y^2}}; \text{ and } \frac{d\phi}{dx} = -\frac{y}{x^2\sec^2\phi} = -\frac{y}{x^2+y^2},$$

$$\therefore \iint\frac{r^3}{3}\sin\theta d\theta d\phi = \iint\frac{r^3}{3}\sin\theta dxdy\left\{\frac{x^2z - xp(x^2+y^2) + y^2z - yq(x^2+y^2)}{r^2(x^2+y^2)^{\frac{3}{2}}}\right\}$$

$$= \frac{1}{3}\iint dxdy\frac{(z-px-qy)r\sin\theta}{\sqrt{x^2+y^2}} = \frac{1}{3}\iint dxdy(z-px-qy).$$

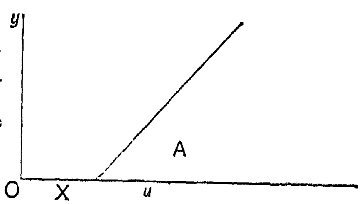
Now if l, m, n be the direction cosines of the normal to a surface at a given point, $p = -\frac{l}{n}$, $q = -\frac{m}{n}$, and

$$\therefore \qquad z - px - qy = \frac{1}{n}(lx + my + nz),$$

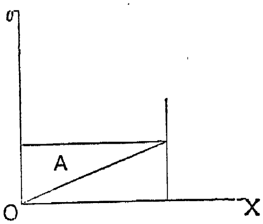
$$\therefore \qquad dxdy(z - px - qy) = dS(lx + my + nz),$$

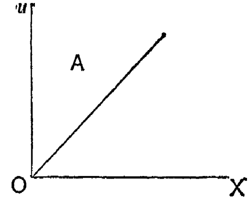
$\therefore$ the integration amounts to summing the product of an element of the surface by $\frac{1}{3}$ of its distance from the origin, so that the volume is divided into thin cones with a common vertex at the origin.

19. If the given integral be denoted by U, changing from z to u, and therefore eliminating v and w, $\frac{dz}{du} = 1$, and the limits of u are $x+y$ and ∞, there ore $U = \int_0^\infty\int_0^\infty\int_{x+y}^\infty V dxdydu$: now changing the *order* of y and u, the integration extending over A, U becomes $\int_0^\infty\int_x^\infty\int_0^{u-x} V dxdudy$; next, changing from y to v, and therefore eliminating z and w, $x+y = uv$, and therefore



$\frac{dy}{dv}=u$, and the lts. of v are $\frac{x}{u}$ and 1, therefore U becomes $\int_0^\infty\int_x^\infty\int_{\frac{x}{u}}^1 Vudxdudv$: now changing the order of x and u, U becomes $\int_0^\infty\int_0^u\int_{\frac{x}{u}}^1 Vududxdv$; again, changing the order of x and v, U becomes $\int_0^\infty\int_0^1\int_0^{uv} Vududvdx$; lastly, to change from x to w, eliminating y and z, $x+uvw=uv$, therefore $\frac{dx}{dw}=-uv$,





and the limits of w are 1 and 0, therefore, finally,

$$U=\int_0^\infty\int_0^1\int_0^1 Vu^2vdudvdw.$$

20. Changing from x_n to θ_{n-1}, and therefore eliminating $r, \theta_1, \ldots, \theta_{n-2}$, $x_n=x_{n-1}\tan\theta_{n-1}$, and $\therefore\ dx_n=x_{n-1}\sec^2\theta_{n-1}\,.\,d\theta_{n-1}$; and changing from x_{n-1} to θ_{n-2},

$$x_{n-1}=x_{n-2}\tan\theta_{n-2}\cos\theta_{n-1},$$

$\therefore$ $dx_{n-1}=x_{n-2}\cos\theta_{n-1}\sec^2\theta_{n-2}d\theta_{n-2}$, and so on up to $dx_2=x_1\cos\theta_2\sec^2\theta_1d\theta_1$, and then eliminating $x_2, x_3, \ldots, x_n$, $dx_1=dr\cos\theta_1$. Hence the given integral becomes $\iiint\ldots V'r^{n-1}.\,Hdrd\theta_1d\theta_2\ldots d\theta_{n-1}$, if

$$r^{n-1}.\,H=x_1x_2\ldots x_{n-1}\sec\theta_{n-1}\sec\theta_{n-2}\ldots\sec\theta_1$$
$$=r^{n-1}(\sin\theta_1)^{n-2}(\sin\theta_2)^{n-3}\ldots\sin\theta_{n-2},\ \therefore \text{ etc.}$$

CHAPTER XII.

1. If $\frac{x^2+a^2}{x^4+b^2x^2+b^4}=\frac{A_1x+B_1}{x^2-bx+b^2}+\frac{A_2x+B_2}{x^2+bx+b^2}$, then

$$x^2+a^2\equiv(A_1+A_2)x^3+x^2\{(A_1-A_2)b+B_1+B_2\}+x\{(A_1+A_2)b^2+(B_1-B_2)b\}+(B_1+B_2)b^2,$$

$\therefore$ $A_1+A_2=0,\quad 1=(A_1-A_2)b+B_1+B_2,$

$$0=(A_1+A_2)b^2+(B_1-B_2)b \text{ or } B_1=B_2,$$

and $\frac{a^2}{b^2}=B_1+B_2,\quad\therefore\ B_1=B_2=\frac{a^2}{2b^2},$

and $A_1=-A_2=\frac{1}{2b}\left(1-\frac{a^2}{b^2}\right)=\frac{b^2-a^2}{2b^3};$

$$\therefore \quad 2\int \frac{(x^2+a^2)dx}{x^4+b^2x^2+b^4} = \int dx\left\{\frac{2A_1x-A_1b+A_1b+2B_1}{x^2-bx+b^2}+\frac{2A_2x+A_2b-A_2b+2B_2}{x^2+bx+b^2}\right\}$$

$$= A_1\log\frac{x^2-bx+b^2}{x^2+bx+b^2}+\frac{A_1b+2B_1}{\frac{b\sqrt{3}}{2}}\left\{\tan^{-1}\frac{2x-b}{b\sqrt{3}}+\tan^{-1}\frac{2x+b}{b\sqrt{3}}\right\},$$

and when $x=0$ or ∞, the logarithmic part $=A_1\log 1=0$, therefore the

$$\text{given integral} = \frac{A_1b+2B_1}{b\sqrt{3}}\left\{\frac{\pi}{2}+\frac{\pi}{6}+\frac{\pi}{2}-\frac{\pi}{6}\right\} = \frac{\pi}{b\sqrt{3}}\left(\frac{b^2-a^2}{2b^2}+\frac{a^2}{b^2}\right) = \frac{a^2+b^2}{2b^3\sqrt{3}}\pi.$$

2. If $u=\int_0^{\frac{\pi}{2}}\cos(a\tan x)dx$, and $\tan x=\theta$,

$$u=\int_0^{\infty}\frac{\cos a\theta d\theta}{1+\theta^2}, \text{ and } \therefore \text{ by Art. 290, } =\frac{\pi}{2}e^{-a}.$$

3. If $u=\int_0^1 x^{2n-1}.e^{x^n}dx$, and $x^n=y$,

$$nu=\int_0^1 dy\,.\,ye^y = ye^y\Big|_0^1 - \int_0^1 e^y dy = e-e+1, \ \therefore\ u=\frac{1}{n}.$$

4. If $b\tan x=a\tan\theta$,

$$\int_0^{\frac{\pi}{2}}\frac{dx}{(a^2\cos^2x+b^2\sin^2x)^2} = \int_0^{\frac{\pi}{2}}dx\frac{(1+\tan^2x)\sec^2x}{(a^2+b^2\tan^2x)^2} = \int_0^{\frac{\pi}{2}}\frac{d\theta\,.\,a\sec^2\theta\left(1+\frac{a^2}{b^2}\tan^2\theta\right)}{ba^4\sec^4\theta}$$

$$= \frac{1}{a^3b^3}\int_0^{\frac{\pi}{2}}d\theta(a^2\sin^2\theta+b^2\cos^2\theta) = \frac{\pi}{2a^3b^3}\left(\frac{a^2}{2}+\frac{b^2}{2}\right)$$

$$= \frac{\pi}{4}\left(\frac{1}{ab^3}+\frac{1}{a^3b}\right).$$

5. If $\tan\phi=x^2$,

$\int_0^{\frac{\pi}{4}}\sqrt{\tan\phi}\,.\,d\phi = \int_0^1\frac{2x^2dx}{1+x^4}$, and therefore, as in Ex. 15 of Chapter II., the

$$\text{given integral} = \frac{1}{2\sqrt{2}}\log\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}+\frac{1}{\sqrt{2}}\left\{\tan^{-1}(x\sqrt{2}+1)+\tan^{-1}(x\sqrt{2}-1)\right\}$$

(taken between the limits 0 and 1)

$$= \frac{1}{2\sqrt{2}}\log\frac{2-\sqrt{2}}{2+\sqrt{2}}+\frac{1}{\sqrt{2}}\left\{\tan^{-1}(\sqrt{2}+1)+\tan^{-1}(\sqrt{2}-1)\right\}$$

$$= \frac{1}{\sqrt{2}}\log\frac{2-\sqrt{2}}{\sqrt{2}}+\frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{2\sqrt{2}}{1-1}\right\} = \frac{1}{\sqrt{2}}\left\{\log(\sqrt{2}-1)+\frac{\pi}{2}\right\}.$$

6. If $\cot\phi = x^2$, $\int_0^{\frac{\pi}{4}} \sqrt{\cot\phi}\,d\phi = \int_1^\infty \frac{2x^2dx}{1+x^4}$, and using the limits 1 and ∞ in Ex. 5, instead of 0 and 1, the result is

$$\frac{-1}{\sqrt{2}}\log(\sqrt{2}-1) + \frac{\pi - \frac{\pi}{2}}{\sqrt{2}}, \text{ and } \therefore \text{ etc.}$$

7. If $u = \int_0^x e^{x^2}dx \div \frac{1}{x}e^{x^2}$; when $x = \infty$, u is of the form $\frac{\infty}{\infty}$, and

$$\therefore \quad u = \text{lt. of } e^{x^2} \div e^{x^2}\left(2 - \frac{1}{x^2}\right) = \frac{1}{2} \text{ when } x = \infty.$$

8. By Art. 291 $\int_0^\infty \sin rx dx = \frac{1}{r}$, $\therefore$ $\int_a^b\int_0^\infty \sin rx dr dx = \int_a^b dr \cdot \frac{1}{r}$

i.e., $\int_0^\infty \frac{dx}{x}(\cos ax - \cos bx) = \log\frac{b}{a}$.

Aliter:
$$\int_{c'}^{c} \frac{\cos ax - \cos bx}{x}dx = \int_{ac'}^{ac}\frac{\cos x}{x}dx - \int_{bc'}^{bc}\frac{\cos x}{x}dx,$$
$$= \int_{bc}^{ac}\frac{\cos x}{x}dx - \int_{bc'}^{ac'}\frac{\cos x}{x}dx,$$

$\therefore$ when $c = \infty$ and $c' = 0$, $\int_0^\infty \frac{\cos ax - \cos bx}{x}dx$ approximates to

$$\int_{bc}^{ac}\frac{1}{\infty}dx - \int_{bc'}^{ac'}\frac{1}{x}dx = \log\frac{b}{a}.$$

9. $$\int_0^\infty \frac{dx}{xF\left(x, \frac{1}{x}\right)} = \int_0^1 \frac{dx}{xF\left(x, \frac{1}{x}\right)} + \int_1^\infty \frac{dx}{xF\left(x, \frac{1}{x}\right)}, \text{ and if } x = \frac{1}{z}$$

the latter integral $= \int_0^1 \frac{dz}{zF\left(\frac{1}{z}, z\right)} = \int_0^1 \frac{dx}{xF\left(x, \frac{1}{x}\right)}$, $\therefore$ etc.

10. By Art. 20, $\frac{F(x)}{(x-c)^n} = \frac{A_1}{(x-c)^n} + \frac{A_2}{(x-c)^{n-1}} + \dots + \frac{A_n}{x-c}$,

where $A_{r+1} = \frac{F^r(c)}{\lfloor r}$, and $A_1 = F(c)$,

$$\therefore \quad \frac{F(x)}{(x-c)^n} = \frac{F(c)}{(x-c)^n} + \frac{F'(c)}{(x-c)^{n-1}} + \frac{F''(c)}{\lfloor 2(x-c)^{n-2}} + \dots + \frac{F^{n-1}(c)}{\lfloor n-1(x-c)},$$

and by Leibnitz's Theorem,

$$\underline{|n-1}\frac{F(c)}{(x-c)^n}+\underline{|n-2}(n-1)\frac{F'(c)}{(x-c)^{n-1}}+\underline{|n-3}\frac{(n-1)(n-2)F''(c)}{\underline{|2}(x-c)^{n-2}}+\ldots+\frac{F^{n-1}(c)}{x-c}$$

$$=\frac{d^{n-1}}{dc^{n-1}}\left\{\frac{F(c)}{x-c}\right\},$$

$$\therefore \int_b^a\frac{F(x)dx}{(x-c)^n}=\frac{1}{\underline{|n-1}}\frac{d^{n-1}}{dc^{n-1}}\left\{F(c)\log\frac{a-c}{b-c}\right\}.$$

11. $\int_0^{2\pi}e^{\cos\theta}\cos(\sin\theta)d\theta$

$$=2\int_0^{\pi}e^{\cos\theta}\cos(\sin\theta)d\theta=2\int_0^{\frac{\pi}{2}}d\theta(e^{\cos\theta}+e^{-\cos\theta})\cos(\sin\theta)$$

$$=\int_0^{\frac{\pi}{2}}d\theta(e^{\cos\theta}+e^{-\cos\theta})(e^{\sin\theta\sqrt{-1}}+e^{-\sin\theta\sqrt{-1}})$$

$$=\int_0^{\frac{\pi}{2}}d\theta(e^{\cos\theta+\sqrt{-1}\sin\theta}+e^{\cos\theta-\sqrt{-1}\sin\theta}+e^{-\cos\theta+\sin\theta\sqrt{-1}}+e^{-\cos\theta-\sin\theta\sqrt{-1}})$$

$$=\int_0^{\frac{\pi}{2}}d\theta\Big\{1+\cos\theta+\sqrt{-1}\sin\theta+\frac{1}{\underline{|2}}(\cos 2\theta+\sqrt{-1}\sin 2\theta)+\ldots$$

$$+1+\cos\theta-\sqrt{-1}\sin\theta+\frac{1}{\underline{|2}}(\cos 2\theta-\sqrt{-1}\sin 2\theta)+\ldots$$

$$+1-\cos\theta+\sin\theta\sqrt{-1}+\frac{1}{\underline{|2}}(-\cos 2\theta+\sqrt{-1}\sin 2\theta)+\ldots$$

$$+1-\cos\theta-\sin\theta\sqrt{-1}+\frac{1}{\underline{|2}}(-\cos 2\theta-\sqrt{-1}\sin 2\theta)+\ldots\Big\}$$

$$=\int_0^{\frac{\pi}{2}}d\theta\,.\,4=2\pi.$$

12. The object here is (1) either to integrate, if possible, and then find the limit when $c=1$, or (2) to expand the function in the denominator of the integral in a series of powers of $1-c$; and as this is not immediately possible, the expression must be transformed by a suitable change of variable.

If then (1) n be a whole number, and $a^n=c$ as in Arts. 25 and 26, $\int\frac{d\theta}{1-a^n\cos^n\theta}$ may be separated into a series of integrals of the form $\int\frac{d\theta}{n(1+b\cos\theta)}$, which is integrable as in Art. 14, Ex. 14, and it will be found that the only integral which vanishes when $a=1$ is $\int\frac{d\theta}{1-a\cos\theta}$.

Hence the proposed integral reduces to

$$\sqrt{1-c}\int_0^{\frac{\pi}{2}}\frac{d\theta}{n(1-a\cos\theta)}=\frac{\sqrt{1-c}}{n}\cdot\frac{2}{\sqrt{1-a^2}}\tan^{-1}\left(\sqrt{\frac{1+a}{1-a}}\tan\frac{\theta}{2}\right)\Big/_0^{\frac{\pi}{2}}$$

$$=\frac{2}{n}\sqrt{\frac{1-c}{1-a^2}}\tan^{-1}(\infty)\text{ when } a=1,$$

$$=\frac{\pi}{n}\sqrt{\frac{1-c}{1-a^2}},\text{ and the lt. of }\frac{1-c}{1-a^2}=\frac{1-c}{1-c^{\frac{2}{n}}}=\frac{1}{\frac{2}{n}},$$

$\therefore$ the integral $=\dfrac{\pi}{\sqrt{2n}}$.

This only applies when n is integral, therefore (2) for a more general investigation, suppose $\cos^n\theta=x$; then

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{1-c\cos^n\theta}=\int_0^1 dx\,.\,\frac{x^{\frac{1}{n}-1}}{n}\cdot\frac{1}{(1-cx)(1-x^{\frac{2}{n}})^{\frac{1}{2}}}=u,\text{ say,}$$

or if $\dfrac{1-cx}{1-c}=\dfrac{x}{y}\left(=\dfrac{1}{1-c+cy}\right)$,

$$nu=\int_0^1\frac{dy(1-c)}{(1-c+cy)^2}\left(\frac{y}{1-c+cy}\right)^{\frac{1}{n}-1}\frac{1-c+cy}{1-c}\,\frac{(1-c+cy)^{\frac{1}{n}}}{\{(1-c+cy)^{\frac{2}{n}}-y^{\frac{2}{n}}\}^{\frac{1}{2}}}$$

$\therefore$ $$u=\int_0^1\frac{1}{n}dy\,.\,y^{\frac{1}{n}-1}\frac{1}{\{(1-c+cy)^{\frac{2}{n}}-y^{\frac{2}{n}}\}^{\frac{1}{2}}},$$

or, if $y=z^n$, $$u=\int_0^1\frac{dz}{\{(1-c+cz^n)^{\frac{2}{n}}-z^2\}^{\frac{1}{2}}},$$

which $=\infty$ when $c=1$, and expanding in powers of $1-c$,

$$u\sqrt{1-c}=\int_0^1\frac{dz}{\left\{\frac{(cz^n)^{\frac{2}{n}}-z^2}{1-c}+\frac{2}{n}(cz^n)^{\frac{2}{n}-1}\right\}^{\frac{1}{2}}}=\int_0^1\frac{dz}{\left\{\frac{2}{n}(-z^2+z^{2-n})\right\}^{\frac{1}{2}}}$$

$$=\int_0^1\frac{dz\,.\,z^{\frac{n}{2}-1}}{\sqrt{\frac{2}{n}(1-z^n)}}=\frac{2}{n}\int_0^1\frac{dx}{\sqrt{\frac{2}{n}(1-x^2)}}=\sqrt{\frac{2}{n}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{2n}}.$$

13. $$\int_0^{\pi}d\theta(a\cos\theta+b\sin\theta)\log(a\cos^2\theta+b\sin^2\theta)$$

$$=\int_0^{\pi}d\theta(-a\cos\theta+b\sin\theta)\log(a\cos^2\theta+b\sin^2\theta)$$

$\therefore$ $$=b\int_0^{\pi}d\theta\,.\sin\theta\log(a\cos^2\theta+b\sin^2\theta)$$

$$=-b\cos\theta\log(a\cos^2\theta+b\sin^2\theta)\Big/_0^{\pi}+b\int_0^{\pi}d\theta\,.\frac{\cos\theta(b-a)\sin2\theta}{a\cos^2\theta+b\sin^2\theta}$$

$$=2b\log a+2b(b-a)\int_0^\pi\frac{\sin\theta d\theta\left(\cos^2\theta+\frac{b}{a-b}-\frac{b}{a-b}\right)}{(a-b)\cos^2\theta+b}$$

$$=2b\log a+2b\cos\theta\Big/_0^\pi-\frac{2b^{\frac{3}{2}}}{(a-b)^{\frac{1}{2}}}\tan^{-1}\frac{\cos\theta\sqrt{a-b}}{\sqrt{b}}\Big/_0^\pi$$

$$=2b\log a-4b+\frac{4b^{\frac{3}{2}}}{(a-b)^{\frac{1}{2}}}\tan^{-1}\sqrt{\frac{a-b}{b}}$$

$$=2b\left(\log a-2+2\sqrt{\frac{b}{a-b}}\cos^{-1}\frac{\sqrt{b}}{\sqrt{a}}\right).$$

14. As in Art. 292, $\log\frac{1+2n\cos ax+n^2}{1+2n\cos bx+n^2}$

$$=2\left\{n(\cos ax-\cos bx)-\frac{n^2}{2}(\cos 2ax-\cos 2bx)+\frac{n^3}{3}(\cos 3ax-\cos 3bx)-+\dots\right\},$$

$\therefore$ the given integral, u say, by Ex. 8

$$=2\left\{n\log\frac{b}{a}-\frac{n^2}{2}\log\frac{b}{a}+\frac{n^3}{3}\log\frac{b}{a}-+\dots\right\}\text{ if } n<1,$$

$\therefore\ u=\log\frac{b^2}{a^2}\left(n-\frac{n^2}{2}+\frac{n^3}{3}-+\dots\right)=\log(1+n)\log\frac{b^2}{a^2}.$ Similarly if $n>1$,

$$u=\log\left(1+\frac{1}{n}\right)\log\frac{b^2}{a^2}+\int_0^\infty\frac{dx}{x}(\log n^2-\log n^2)=\log\left(1+\frac{1}{n}\right)\log\frac{b^2}{a^2}.$$

15. By Art. 288,

$$\int_0^\infty(e^{-ax-\alpha x\sqrt{-1}}-e^{-bx-\beta\sqrt{-1}})\frac{dx}{x}=\log\frac{b+\beta\sqrt{-1}}{a+\alpha\sqrt{-1}}=\log\frac{(a-\alpha\sqrt{-1})(b+\beta\sqrt{-1})}{a^2+\alpha^2}.$$

This is consequently real if $ab=\alpha\beta$.

In Art. 288, κ must be positive and therefore a and b must be both positive here, but the signs of α and β, the coefficients of imaginary quantities are necessarily immaterial. When the final expression is real, since $\frac{\alpha}{a}=\frac{\beta}{b}$, α, β are of the same sign, and therefore $\log\frac{ab+\alpha\beta}{a^2+\alpha^2}$ is real, whether that sign be positive or negative.

16. If $x-\frac{1}{2}=z$,

$$\int_0^1\cot^{-1}(1-x+x^2)=\int_{-\frac{1}{2}}^{\frac{1}{2}}dz\cot^{-1}\left(z^2+\frac{3}{4}\right)=z\cot^{-1}\left(z^2+\frac{3}{4}\right)\Big/_{-\frac{1}{2}}^{\frac{1}{2}}+\int_{-\frac{1}{2}}^{\frac{1}{2}}dz\frac{2z^2}{1+\left(z^2+\frac{3}{4}\right)^2}$$

$$=\frac{\pi}{4}+\int_{-\frac{1}{2}}^{\frac{1}{2}}dz\,.\,\frac{2z^2}{\left(z^2+\frac{5}{4}\right)^2-z^2}=\frac{\pi}{4}+\int_{-\frac{1}{2}}^{\frac{1}{2}}dz\left(\frac{z}{z^2-z+\frac{5}{4}}-\frac{z}{z^2+z+\frac{5}{4}}\right)$$

$$=\frac{\pi}{4}+\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}dz\left(\frac{2z-1+1}{z^2-z+\frac{5}{4}}-\frac{2z+1-1}{z^2+z+\frac{5}{4}}\right)$$

$$=\frac{\pi}{4}+\frac{1}{2}\left\{\log\frac{z^2-z+\frac{5}{4}}{z^2+z+\frac{5}{4}}+\tan^{-1}\left(z-\frac{1}{2}\right)+\tan^{-1}\left(z+\frac{1}{2}\right)\right\}\Bigg|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$=\frac{\pi}{4}+\frac{1}{2}\left\{\log\left(\frac{1}{2}\right)^2-\left(-\frac{\pi}{4}\right)+\frac{\pi}{4}\right\}=\frac{\pi}{2}-\log 2.$$

17. If $x=\tan y$,

$$\int_0^\infty\frac{dx}{1+x^2}\log\left(x+\frac{1}{x}\right)=\int_0^{\frac{\pi}{2}}dy\log(\tan y+\cot y)=-\int_0^{\frac{\pi}{2}}dy\log(\sin y\cos y)$$

$$=-\int_0^{\frac{\pi}{2}}dy(\log\sin y+\log\cos y)=-2\int_0^{\frac{\pi}{2}}dy\log\sin y$$

$$=-\pi\log\frac{1}{2}\text{ (by Art. 51) }=\pi\log 2.$$

18. By Art. 285 $\int_0^\infty\frac{\sin(rx)}{x}dx=\frac{\pi}{2}$, if r be positive,

$$\therefore\quad \frac{\pi}{2}=\int_0^\infty\frac{\sin 2x}{x}dx=\frac{\sin^2 x}{x}\Big|_0^\infty+\int_0^\infty\frac{\sin^2 x}{x^2}dx,$$

$$\therefore\quad \int_0^\infty\left(\frac{\sin x}{x}\right)^2dx=\frac{\pi}{2}=\int_0^\infty\frac{\sin x}{x}dx.$$

19. $\int_0^\infty\frac{dx}{x^2}(e^{-ax}-e^{-bx})^2=-\frac{1}{x}(e^{-ax}-e^{-bx})^2\Big|_0^\infty+2\int_0^\infty\frac{dx}{x}(e^{-ax}-e^{-bx})(-ae^{-ax}+be^{-bx})$,

and when $x=\infty$,

$$\frac{(e^{-ax}-e^{-bx})^2}{x}=\frac{0}{\infty}=0;$$

when $x=0$, $\frac{(e^{-ax}-e^{-bx})^2}{x}=\frac{2(e^{-ax}-e^{-bx})}{1}(-ae^{-ax}-be^{-bx})=0$,

$\therefore$ the given integral $=2\int_0^\infty\frac{dx}{x}\{-a(e^{-2ax}-e^{-(a+b)x})+b(e^{-(a+b)x}-e^{-2bx})\}$,

$\therefore$ (by Art. 288), $=-2a\log\frac{a+b}{2a}+2b\log\frac{2b}{a+b}=\log\left\{\frac{(2a)^{2a}.(2b)^{2b}}{(a+b)^{2(a+b)}}\right\}$.

20. $\int_0^\infty \frac{x^r \log x dx}{(1+x)^2} = \frac{d}{dr}\int_0^\infty \frac{x^r dx}{(1+x)^2} = \frac{d}{dr}\left\{\frac{\Gamma(1+r)\Gamma(1-r)}{\Gamma(2)}\right\}$ by Art. 261, when $r<1$, $\therefore$ by Art. 265 $=\frac{d}{dr}\left(\frac{r\pi}{\sin r\pi}\right) = \pi\left(\frac{1}{\sin r\pi} - \frac{r\pi\cos r\pi}{\sin^2 r\pi}\right)$, $\therefore$ putting $r=\frac{1}{2}$,

$$\int_0^\infty \frac{\sqrt{x}\,.\log x dx}{(1+x)^2} = \pi.$$

21. If $x = \frac{1}{\sqrt{y}}$, $\int_0^\infty (e^{-\frac{a^2}{x^2}} - e^{-\frac{b^2}{x^2}})dx = \frac{1}{2}\int_0^\infty \frac{dy}{y^{\frac{3}{2}}}(e^{-a^2y} - e^{-b^2y})$

$$= -\frac{e^{-a^2y} - e^{-b^2y}}{\sqrt{y}}\Big|_0^\infty - \int_0^\infty \frac{dy}{\sqrt{y}}(a^2e^{-a^2y} - b^2e^{-b^2y})$$

and $$\int_0^\infty \frac{dy}{\sqrt{y}} a^2 e^{-a^2y} = a\int_0^\infty dz\,.\,e^{-z}\,.\,z^{-\frac{1}{2}} = a\sqrt{\pi};$$

also when $y=\infty$, $$\frac{e^{-a^2y} - e^{-b^2y}}{\sqrt{y}} = \frac{0}{\infty} = 0,$$

and when $y=0$, $$\frac{e^{-a^2y} - e^{-b^2y}}{\sqrt{y}} = -\frac{1}{2}(a^2e^{-a^2y} - b^2e^{-b^2y})\sqrt{y} = 0,$$

$\therefore$ the given integral $=(b-a)\sqrt{\pi}$.

22. If $e^{-x}=y$, $dx = -\frac{dy}{y}$,

and $$\int_0^\infty \log\frac{e^x+1}{e^x-1}dx = \int_0^1 \log\frac{1+y}{1-y}\,.\,\frac{dy}{y}$$

$$= \int_0^1 \frac{dy}{y}\left\{y - \frac{y^2}{2} + \frac{y^3}{3} - + \ldots + y + \frac{y^2}{2} + \frac{y^3}{3} + + \ldots\right\}$$

$$= 2\int_0^1 dy\,.\,\Sigma\frac{y^{2r}}{2r+1} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) = \frac{\pi^2}{4}.$$

23. If $x=e^{-z}$, $$\int_0^1 \frac{x^m - x^n}{\log x}\,.\,\frac{dx}{x} = \int_0^\infty dz\left(\frac{e^{-mz} - e^{-nz}}{-z}\right),$$

and $\therefore$ (by Art. 288), $$= \log\left(\frac{m}{n}\right).$$

If $x^m = y$, $$\int_0^1 \frac{x^{m-1}}{\log x}dx = \int_0^1 \frac{dy}{\log y},$$

and $\therefore$ the first given integral would appear to vanish, but, as in Art. 288, $\int_0^1 \frac{x^{m-1}dx}{\log x}$ which equals $-\int_0^\infty dz\,.\,\frac{e^{-mz}}{z} = \infty$, and $\infty - \infty$ may be finite, and in this case is so, and not zero, unless $m=n$.

24. If $\sin^2\theta = x$, $\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta = \int_0^1 \frac{x^{\frac{n}{2}}dx}{2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} = \frac{1}{2}\int_0^1 dx \,.\, x^{\frac{n-1}{2}} \,.\, (1-x)^{-\frac{1}{2}}$

$$= \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right) \div \Gamma\left(\frac{n+2}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{n+1}{2}\right) \div \Gamma\left(\frac{n+2}{2}\right) \text{ so long as } n > -1.$$

25. If $\frac{x}{b+cx} = \frac{y}{b+c}$, $\frac{b}{b+cx} = 1 - \frac{cy}{b+c}$, therefore $\frac{bdx}{(b+cx)^2} = \frac{dy}{(b+c)}$, and $1 - y = 1 - \frac{(b+c)x}{b+cx} = \frac{b(1-x)}{b+cx}$, therefore

$$\int_0^1 \frac{x^{l-1}(1-x)^{m-1}dx}{(b+cx)^{l+m}} = \int_0^1 \frac{dy}{b}\,\frac{(b+cx)^2}{b+c}\cdot\left(\frac{y}{b+c}\right)^{l-1}\cdot\frac{1}{(b+cx)^{m+1}}\cdot(1-y)^{m-1}\left(\frac{b+cx}{b}\right)^{m-1}$$

$$= \int_0^1 dy \,.\, y^{l-1}(1-y)^{m-1}\frac{1}{b^m(b+c)^l} = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}\cdot\frac{1}{b^m(b+c)^l}.$$

26. If $\sin^2\theta = x$,

$$\int_0^{\frac{\pi}{2}} \frac{d\theta \cos^{2l-1}\theta \,.\, \sin^{2m-1}\theta}{(a\cos^2\theta + b\sin^2\theta)^{l+m}} = \frac{1}{2}\int_0^1 \frac{dx \,.\, \cos^{2l-2}\theta \sin^{2m-2}\theta}{(a\cos^2\theta + b\sin^2\theta)^{l+m}} = \frac{1}{2}\int_0^1 \frac{dx(1-x)^{l-1} \,.\, x^{m-1}}{\{a-(a-b)x\}^{l+m}},$$

and therefore by the method of the preceding example,

$$= \frac{1}{2}\cdot\frac{\Gamma(m)\Gamma(l)}{\Gamma(l+m)} \div a^l b^m.$$

27. If $\sin^2\theta = x$,

$$\int_0^{\frac{\pi}{2}} \frac{d\theta \,.\, \tan^n\theta}{a\cos^2\theta + b\sin^2\theta} = \int_0^{\frac{\pi}{2}} \frac{d\theta \sin^n\theta \cos^{-n}\theta}{a\cos^2\theta + b\sin^2\theta} = \int_0^1 \frac{dx}{2}\cdot\frac{x^{\frac{n-1}{2}}(1-x)^{-\frac{n+1}{2}}}{a+(b-a)x}, \text{and } \therefore$$

by the *method* of Ex. 25, $= \frac{1}{2}\,\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{\Gamma(1)}\cdot\frac{1}{a^{\frac{1-n}{2}}\,.\,b^{\frac{1+n}{2}}}$, $\therefore$ n must be <1,

and $\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right) = \frac{\pi}{\sin\frac{1-n}{2}\pi} = \frac{\pi}{\cos\frac{n\pi}{2}}$, by Art. 262, $\therefore$ etc.

28. The $\int_0^{\pi} \frac{\sin^{n-1}\theta d\theta}{(\alpha+\beta\cos\theta)^n} = \int_0^{\pi} \frac{2^{n-1}\sin^{n-1}\frac{\theta}{2}\cos^{n-1}\frac{\theta}{2}\,.\,d\theta}{\left(\alpha+\beta-2\beta\sin^2\frac{\theta}{2}\right)^n}$

$$= 2^n\int_0^{\frac{\pi}{2}} \frac{\sin^{n-1}\theta\cos^{n-1}\theta d\theta}{(\alpha+\beta-2\beta\sin^2\theta)^n},$$

$\therefore$ if $\sin^2\theta = x$, the integral $= 2^{n-1}\int_0^1 \frac{x^{\frac{n-2}{2}}(1-x)^{\frac{n-2}{2}}}{(\alpha+\beta-2\beta x)^n}$, and $\therefore$ (by the *method* of Ex. 25),

$$= \frac{2^{n-1}\left\{\Gamma\left(\frac{n}{2}\right)\right\}^2}{\Gamma(n)} \cdot \frac{1}{(\alpha+\beta)^{\frac{n}{2}}(\alpha-\beta)^{\frac{n}{2}}} = \text{etc.}$$

29. If $x^n = z$, $\frac{dx}{dz} = \frac{1}{nx^{n-1}}$,

and $\int_0^1 \frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}} = \int_0^1 \frac{dz}{n} x^{m-n}(1-z)^{-\frac{m}{n}} = \int_0^1 \frac{dz}{n} z^{\frac{m}{n}-1}(1-z)^{-\frac{m}{n}} = \frac{1}{n}\frac{\Gamma\left(\frac{m}{n}\right)\Gamma\left(1-\frac{m}{n}\right)}{\Gamma(1)}$,

if $m < n$, $= \frac{1}{n} \cdot \frac{\pi}{\sin\frac{m\pi}{n}}$, by Art. 262.

30. As in Ex. 25,

$$\int_0^1 \frac{x^{n-1}(1-x)^{-n}dx}{1+cx} = \frac{\Gamma(n)\Gamma(1-n)}{\Gamma(1)} \cdot \frac{1}{(1+c)^n} = \frac{\pi}{\sin n\pi} \cdot \frac{1}{(1+c)^n}, \text{ by Art. 262.}$$

31. $\operatorname{Sin} ax \sin^2 cx = \frac{1}{2}\sin ax(1-\cos 2cx)$

$$= \tfrac{1}{2}\sin ax - \tfrac{1}{4}\sin(a+2c)x - \tfrac{1}{4}\sin(a-2c)x,$$

and this is unaltered in value by changing the sign of c; c may, therefore, be taken as positive. Thus, by Art. 285, if $a > 2c$, numerically, as a is positive or negative the given integral u, say,

$= \pm\frac{\pi}{4} \mp \frac{\pi}{8} \mp \frac{\pi}{8} = 0$; but if $a < 2c$ numerically, as a is positive or negative, $u = \pm\frac{\pi}{4} - \frac{\pi}{8} + \frac{\pi}{8} = \pm\frac{\pi}{4}$; lastly, if $a = \pm 2c$ accordingly

$$u = \pm\frac{\pi}{4} \mp \frac{\pi}{8} = \pm\frac{\pi}{8}.$$

This is assuming that neither a nor $c = 0$, for then u would vanish altogether.

32. The equation $y = \int_0^\infty \frac{\sin\theta}{\theta}\cos(\theta x)d\theta$ is unaltered by changing the sign of x, therefore the locus is symmetrical as to $x = 0$; and

$$2y = \int_0^\infty \frac{\sin(1+x)\theta}{\theta}d\theta + \int_0^\infty \frac{\sin(1-x)\theta}{\theta}d\theta,$$

and $\therefore$ by Art. 285, $y = \frac{\pi}{2}$ if $(1+x)$ and $(1-x)$ be both positive,

$y = -\frac{\pi}{2}$ if $(1+x)$ and $(1-x)$ be both negative,

and $y = 0$ if $(1+x)$ and $(1-x)$ be of opposite signs;

also, if one of the two quantities $(1+x)$ and $(1-x)$ vanish, $y=\pm\frac{\pi}{4}$ according as the other of these two quantities is positive or negative. Hence for the locus (1) ultimately from $x=-1$ to $x=1$, $y=\frac{\pi}{2}$, a finite straight line parallel to $y=0$: (2) from $x=1$ to ∞, and from $x=-1$ to $-\infty$, $y=0$, i.e., two infinite parts of a straight line: (3) $y=-\frac{\pi}{2}$ is impossible, for the sum of two negative quantities cannot $=2$: (4) if $x=-1$, $y=\frac{\pi}{4}$, and if $x=1$, $y=\frac{\pi}{4}$, which values correspond to two conjugate points.

33. By Art. 292, $\frac{y}{b}=2\pi\log e^{-u}$ or 0, as $e^{-u}>$ or not >1, i.e., as u is negative or positive, and therefore for positive values of $\sin\frac{x}{a}$, $y=0$, and for negative values of $\sin\frac{x}{a}$, $y=-2\pi\sin\frac{x}{a}$ (and therefore y is positive). Hence the locus consists of an infinite series of segments of $y=0$, of length πa at intervals $=\pi a$, the intervals being bounded on the $+y$ side of the axis of x by equal concave arcs.

34. Here $y=+\sqrt{x^2+2x\sin\theta+1}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\pm(x+1)\mp(x-1)$,

the ambiguities being so taken that both $x+1$ and $x-1$ shall be positive; if then $x>1$, $y=2$, part of an ∞ straight line; so if $x<-1$, $y=-2$; and if x lie between 1 and -1, $y=2x$, a straight line connecting the other two.

35. If $\sin x\sin y=\sin\theta$,

$$u=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin x\sin^{-1}(\sin x\sin y)\,dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^{x}dx\,d\theta\frac{\cos\theta}{\cos y}\,.\,\theta$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{x}dx\,d\theta\frac{\cos\theta\sin x\,.\,\theta}{\sqrt{\sin^2 x-\sin^2\theta}},$$ or changing the order of integration,

$$u=\int_0^{\frac{\pi}{2}}\int_\theta^{\frac{\pi}{2}}d\theta\,dx\,.\,\frac{\theta\cos\theta\sin x}{\sqrt{\cos^2\theta-\cos^2 x}}=-\int_0^{\frac{\pi}{2}}d\theta\,.\,\theta\cos\theta\sin^{-1}\frac{\cos x}{\cos\theta}\Big|_\theta^{\frac{\pi}{2}}$$

$$=\int_0^{\frac{\pi}{2}}d\theta\,.\,\theta\cos\theta\,.\,\frac{\pi}{2}=\frac{\pi}{2}(\theta\sin\theta+\cos\theta)\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}\left(\frac{\pi}{2}-1\right).$$

36. Here (1) $\int_0^\infty e^{-xy}\sin ax\,dx=\frac{e^{-xy}}{a^2+y^2}(-y\sin ax-a\cos ax)\Big|_{x=0}^{x=\infty}$

as in Art. 12, and $\therefore$ $\qquad=\frac{a}{a^2+y^2}$,

$\therefore \qquad \int_0^\infty\int_0^\infty e^{-xy}\sin axdydx = \tan^{-1}\frac{y}{a}\Big/_0^\infty = \pm\frac{\pi}{2}$, as a is positive or negative: and (2) $\qquad \int_0^\infty e^{-xy}dy = -\frac{e^{-xy}}{x}\Big/_0^\infty = \frac{1}{x}$,

$\therefore \qquad \int_0^\infty\int_0^\infty e^{-xy}dxdy\sin ax = \int_0^\infty \frac{\sin ax}{x}dx = \pm\frac{\pi}{2}$, as a is positive or negative, by Art. 285, which is therefore corroborated by (1).

37. By Art. 286, $\int_0^\infty e^{-\left(x^2+\frac{c^2}{x^2}\right)}dx = \frac{\sqrt{\pi}}{2}e^{-2c}$, $\therefore$ putting $\frac{x}{a}$ for x,

$$\int_0^\infty e^{-\left(\frac{x^2}{a^2}+\frac{a^2c^2}{x^2}\right)}\frac{dx}{a} = \frac{\sqrt{\pi}}{2}e^{-2c}, \ \therefore \text{ if } ca = b,$$

$$\int_0^\infty e^{-\left(\frac{x^2}{a^2}+\frac{b^2}{x^2}\right)}dx = \frac{\sqrt{\pi}}{2}ae^{-\frac{2b}{a}}.$$

Hence $\qquad \int_0^\infty e^{-\kappa\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)}dx = \frac{\sqrt{\pi}}{2}\cdot\frac{a}{\sqrt{\kappa}}e^{-\kappa}$,

$\therefore$, differentiating with respect to κ,

$$-\int_0^\infty\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)e^{-\kappa\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)}dx = \frac{a\sqrt{\pi}}{2}e^{-2\kappa}\left(-\frac{2}{\sqrt{\kappa}}-\frac{1}{2\kappa^{\frac{3}{2}}}\right),$$

$\therefore$ when $\kappa=1$, $\int_0^\infty\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)e^{-\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)}dx = \frac{a\sqrt{\pi}}{2e^2}\cdot\frac{5}{2} = \frac{5a\sqrt{\pi}}{4e^2}$.

Also $\qquad \int_0^\infty dxe^{-\left(\frac{x^2}{a^2\kappa}+\frac{\kappa a^2}{x^2}\right)} = \frac{\sqrt{\pi}}{2}a\sqrt{\kappa}e^{-2}$,

$\therefore$ differentiating with respect to κ,

$$\int_0^\infty dxe^{-\left(\frac{x^2}{a^2\kappa}+\frac{\kappa a^2}{x^2}\right)}\left(\frac{x^2}{\kappa^2a^2}-\frac{a^2}{x^2}\right) = \frac{\sqrt{\pi}}{4e^2}a\kappa^{-\frac{1}{2}}, \text{ and } \therefore \text{ when } \kappa = 1,$$

$$\int_0^\infty dxe^{-\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)}\left(\frac{x^2}{a^2}-\frac{a^2}{x^2}\right) = \frac{a\sqrt{\pi}}{4e^2}.$$

38. Changing to polar co-ordinates, the given integral

$$= \int_0^{\frac{\pi}{2}}\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}.rd\theta dr = \frac{\pi}{4}\int_0^1 dz\sqrt{\frac{1-z}{1+z}}$$

$$= \frac{\pi}{4}\int_0^1 dz\left\{\frac{1}{\sqrt{1-z^2}}-\frac{z}{\sqrt{1-z^2}}\right\} = \frac{\pi}{4}\{\sin^{-1}z+\sqrt{1-z^2}\}\Big/_0^1 = \frac{\pi}{4}\left(\frac{\pi}{2}-1\right).$$

39. Putting $x, y, z, \dots$ for $x^2, y^2, z^2, \dots$, the given integral u becomes $\iiint \dots \frac{1}{2^n} \frac{x^{-\frac{1}{2}}y^{-\frac{1}{2}}z^{-\frac{1}{2}}\dots dxdydz\dots}{\sqrt{1-x-y-z-\dots}}$, and therefore, by Art. 277,

$$u = \frac{1}{2^n} \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^n}{\Gamma\left(\frac{n}{2}\right)} \cdot \int_0^1 \frac{h^{\frac{n}{2}-1}dh}{\sqrt{1-h}} = \frac{\pi^{\frac{n}{2}}}{2^n\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \frac{\pi^{\frac{n+1}{2}}}{2^n\Gamma\left(\frac{n+1}{2}\right)}.$$

40. $$F(u) = A_0 + A_1 xe^{\theta\sqrt{-1}} + A_2 x^2 e^{2\theta\sqrt{-1}} + \dots$$
$$F(v) = A_0 + A_1 xe^{-\theta\sqrt{-1}} + A_2 x^2 e^{-2\theta\sqrt{-1}} + \dots$$
$$\therefore \quad \frac{F(u)+F(v)}{2} = A_0 + A_1 x\cos\theta + A_2 x^2 \cos 2\theta + \dots,$$

and so $\frac{f(u)+f(v)}{2} = a_0 + a_1 x\cos\theta + a_2 x^2\cos 2\theta + \dots,$

$$\begin{aligned}\therefore \tfrac{1}{4}\{F(u)+F(v)\}\{f(u)+f(v)\} &= a_0A_0 + (a_0A_1 + a_1A_0)x\cos\theta \\ &+ x^2(a_1A_1\cos^2\theta + \overline{a_0A_2 + a_2A_0}\cos 2\theta) \\ &+ x^3\{(a_0A_3 + a_3A_0)\cos 3\theta + (a_1A_2 + a_2A_1)\cos\theta\cos 2\theta\} \\ &+ x^4\{(a_0A_4 + a_4A_0)\cos 4\theta + (a_1A_3 + a_3A_1)\cos\theta\cos 3\theta + a_2A_2\cos^2 2\theta\} + \dots,\end{aligned}$$

and this involves three types of functions of cosines, viz., (1) $\cos n\theta$, n being any integer; (2) $\cos r\theta \cos s\theta$, r and s being integers; and (3) $\cos^2 n\theta$;

and (1) $$\int_0^\pi \cos n\theta d\theta = \frac{\sin n\theta}{n}\Big/_0^\pi = 0,$$

(2) $$\int_0^\pi \cos r\theta\cos s\theta d\theta = \int_0^\pi \frac{d\theta}{2}(\cos\overline{r+s}\theta + \cos\overline{r-s}\theta),$$ and $\therefore$ as in (1), $=0$,

(3) $$\int_0^\pi \cos^2 n\theta d\theta = \frac{1}{2}\int_0^\pi (1+\cos 2n\theta)d\theta = \frac{\pi}{2}.$$

Hence $$\frac{1}{4}\int_0^\pi \{F(u)+F(v)\}\{f(u)+f(v)\}d\theta$$
$$= \pi a_0A_0 + \frac{\pi}{2}\{a_1A_1x^2 + a_2A_2x^4 + \dots\},$$

and $\therefore$ $a_0A_0 + a_1A_1x^2 + a_2A_2x^4 + \dots$
$$= \frac{1}{2\pi}\int_0^\pi \{F(u)+F(v)\}\{f(u)+f(v)\}d\theta - a_0A_0.$$

41. If $a_0 + a_1x + a_2x^2 + \dots = f(x)$, then by Ex. 40,

$$\begin{aligned}a_0^2 + a_1^2x^2 + a_2^2x^4 + \dots &= -a_0^2 + \frac{1}{2\pi}\int_0^\pi \{f(u)+f(v)\}^2 d\theta \\ &= -a_0^2 + \frac{1}{2\pi}\int_0^\pi d\theta\{f(xe^{\theta\sqrt{-1}}) + f(xe^{-\theta\sqrt{-1}})\}^2\end{aligned}$$

$\sim 2 , 2$

Hence if $(1+x)^n \equiv f(x)$, putting $x=1$, the sum of the squares of the coefficients of the terms of the expansion of $(1+x)^n$, when n is a positive integer,

$$=-1+\frac{1}{2\pi}\int_0^\pi d\theta\{(1+e^{\theta\sqrt{-1}})^n+(1+e^{-\theta\sqrt{-1}})^n\}^2$$

$$=-1+\frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta\{(1+e^{2\theta\sqrt{-1}})^n+(1+e^{-2\theta\sqrt{-1}})^n\}^2$$

$$=-1+\frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta\{e^{n\theta\sqrt{-1}}(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}})^n + e^{-n\theta\sqrt{-1}}(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}})^n\}^2$$

$$=-1+\frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\theta \cos^2 n\theta \cos^{2n}\theta \,.\, 2^{2(n+1)}.$$

It follows from Algebra that when n is a positive integer

$$\int_0^{\frac{\pi}{2}} \cos^{2n}\theta \cos^2 n\theta d\theta = \frac{\pi}{2^{2n+2}}\left\{1+\frac{\lfloor 2n}{(\lfloor n)^2}\right\}.$$

42. From Art. 290, the given integral $=\frac{\pi}{2}e^{-c}$, $\frac{\pi}{2}e^{c}$, or $\frac{\pi}{2}$, as c is positive, negative, or zero, and $\frac{\pi}{2}\left\{\frac{e^c}{1+0^{-c}}+\frac{e^{-c}}{1+0^c}\right\}$ reduces to one of these forms according to the value of c; for suppose c positive, then $0^{-c}=\frac{1}{0}=\infty$, therefore $\frac{e^c}{1+0^{-c}}=0$, and $\frac{e^{-c}}{1+0^c}=e^{-c}$; similarly if c be negative and if $c=0$,

$$\frac{e^c}{1+0^{-c}}+\frac{e^{-c}}{1+0^c}=\frac{1}{2}+\frac{1}{2}, \quad \therefore \text{ etc.}$$

43. If $u=\int_0^{\frac{\pi}{2}}\phi(\sin 2x)\cos x dx$, putting x for $2x$,

$$u=\frac{1}{2}\int_0^\pi \phi(\sin x)\cos\frac{x}{2}dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\phi(\sin x)\cos\frac{x}{2}dx+\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\phi(\sin x)\cos\frac{x}{2}dx$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\phi(\sin x)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)dx=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\phi(\sin x)\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\phi(\cos x)\cos\frac{x}{2}dx, \text{ and if } \cos x=\cos^2 z=1-2\sin^2\frac{x}{2},$$

$\sin\frac{x}{2}=\frac{1}{\sqrt{2}}\sin z$, therefore $\cos\frac{x}{2}dx=\sqrt{2}\cos z dz$, and

$$u=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\phi(\cos^2 z)\,.\,\sqrt{2}\cos z dz=\int_0^{\frac{\pi}{2}}\phi(\cos^2 x)\cos x dx.$$

44. $$\int_0^{\frac{\pi}{2}}\cos(x\sin y)dy=\int_0^{\frac{\pi}{2}}dy\left(1-\frac{x^2\sin^2 y}{\lfloor 2}+\frac{x^4\sin^4 y}{\lfloor 4}-+\ldots\right)$$
$$=\frac{\pi}{2}\left\{1-\frac{x^2}{\lfloor 2}\cdot\frac{1}{2}+\frac{x^4}{\lfloor 4}\cdot\frac{3.1}{4.2}-\frac{x^6}{\lfloor 6}\cdot\frac{5.3.1}{6.4.2}+-\ldots\right\}$$
$$=\frac{\pi}{2}\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+-\ldots\right), \text{ and } \therefore \text{ etc.}$$

45. If $y^n=z$,
$$\int_0^\infty e^{-y^n}.\,y^{n-m-1}dy=\int_0^\infty e^{-z}z^{\frac{n-m-1}{n}}\frac{dz}{nz^{\frac{n-1}{n}}}=\frac{1}{n}\int_0^\infty e^{-z}z^{-\frac{m}{n}}dz=\frac{1}{n}\Gamma\left(1-\frac{m}{n}\right), \text{ if } \frac{m}{n}<1.$$
Hence $\int_0^\infty x^{m-1}e^{-x^n}dx\int_0^\infty e^{-y^n}y^{n-m-1}dy$ in the same way
$$=\frac{1}{n}\Gamma\left(1-\frac{m}{n}\right)\cdot\frac{1}{n}\Gamma\left(1-\frac{n-m}{n}\right)=\frac{1}{n^2}\Gamma\left(1-\frac{m}{n}\right)\Gamma\left(\frac{m}{n}\right)$$
$$=\frac{1}{n^2}\cdot\frac{\pi}{\sin\frac{m\pi}{n}}, \text{ by Art. 262.}$$

46. By Art. 286,
$$\int_{-\infty}^\infty dx\,.\,e^{-\left(x^2+\frac{c^2}{x^2}\right)}=\sqrt{\pi}e^{-2c}, \;\therefore\; \int_{-\infty}^\infty dx e^{-\left(\kappa x^2+\frac{a^2\kappa'}{2x^2}\right)}=\sqrt{\frac{\pi}{\kappa}}e^{-2a\sqrt{\frac{\kappa\kappa'}{2}}},$$
and putting $\quad\kappa=\cos 2\theta+\sqrt{-1}\sin 2\theta$

and $\quad\kappa'=\sin 2\theta+\sqrt{-1}\cos 2\theta$,

as in Art. 303, the given integrals are given respectively by the real part and the coefficient of $-\sqrt{-1}$ in the imaginary part of $u=\frac{\sqrt{\pi}}{\sqrt{\kappa}}e^{-a\sqrt{2\kappa\kappa'}}$, but
$$\sqrt{\kappa}=\cos\theta+\sqrt{-1}\sin\theta \text{ and } \frac{1}{\sqrt{\kappa}}=\cos\theta-\sqrt{-1}\sin\theta,$$
$$\sqrt{\kappa'}=\left\{\cos\overline{\frac{\pi}{2}-2\theta}+\sqrt{-1}\sin\overline{\frac{\pi}{2}-2\theta}\right\}^{\frac{1}{2}}$$
$$=\cos\left(\frac{\pi}{4}-\theta\right)+\sqrt{-1}\sin\overline{\frac{\pi}{4}-\theta},$$
$$\sqrt{\kappa\kappa'}=\cos\theta\cos\overline{\frac{\pi}{4}-\theta}-\sin\theta\sin\overline{\frac{\pi}{4}-\theta}$$
$$+\sqrt{-1}\left(\sin\theta\cos\overline{\frac{\pi}{4}-\theta}+\cos\theta\sin\overline{\frac{\pi}{4}-\theta}\right)$$
$$=\frac{1}{\sqrt{2}}+\frac{\sqrt{-1}}{\sqrt{2}},$$

$$\therefore \quad u = \sqrt{\pi}(\cos\theta - \sqrt{-1}\sin\theta)e^{-a(1+\sqrt{-1})}$$
$$= \sqrt{\pi}(\cos\theta - \sqrt{-1}\sin\theta)e^{-a}(\cos a - \sqrt{-1}\sin a)$$
$$= \sqrt{\pi}e^{-a}\{\cos(\theta+a) - \sqrt{-1}\sin(\theta+a)\}, \quad \therefore \text{ etc.}$$

In the given integrals neither $\cos 2\theta$ nor $\sin 2\theta$ must be negative, and therefore the result holds if θ lies between 0 and $\frac{\pi}{4}$.

47. If $u = \Gamma(x)\Gamma\left(x+\frac{1}{n}\right)..\ \Gamma\left(x+\frac{n-1}{n}\right) = \Gamma(nx)(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}$

$$\log u = \sum_x^{x+1}\log\Gamma(x) \text{ (when } n=\infty\text{)} = \text{lt. of } \log\{\Gamma(nx)(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}\},$$

and $\therefore$ putting dx for $\frac{1}{n}$,

$$\int_x^{x+1}\log\Gamma(x)dx = \text{lt. of } \frac{1}{n}\log\{\Gamma(nx)(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}\}.$$

48. By Art. 282, when n is infinite,

$$\Gamma(n) = e^{-n}n^n(2n\pi)^{\frac{1}{2}}, \quad \therefore \Gamma(nx) = e^{-nx}(nx)^{nx}(2\pi)^{\frac{1}{2}}(nx)^{\frac{1}{2}},$$

$$\therefore \quad \int_x^{x+1}\log\Gamma(x)dx = \text{lt. of } \frac{1}{n}\log\{e^{-nx}(nx)^{nx}(2\pi)^{\frac{1}{2}}(nx)^{\frac{1}{2}}(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}\}$$
$$= \frac{1}{n}\left\{-nx + \log n + nx\log x + \frac{n}{2}\log 2\pi + \frac{1}{2}\log x\right\}$$
$$= -x + x\log x + \frac{1}{2}\log 2\pi.$$

CHAPTER XIII.

Examples in the Text.

Art. 314. In formula (4) of Art. 309, putting $l=\pi$, $\phi(x)\equiv e^{ax}$, and

$$\int_0^\pi \sin nv\,.\,e^{av}dv = e^{av}\frac{(a\sin nv - n\cos nv)}{a^2+n^2}\Big|_0^\pi \text{(Art. 12)} = \frac{n}{a^2+n^2}(1-e^{a\pi}\cos n\pi),$$

and $\therefore$ $$e^{ax} = \frac{2}{\pi}\sum_1^\infty \frac{n}{a^2+n^2}(1-e^{a\pi}\cos n\pi)\sin nx.$$

This holds from $x=0$ to π, but obviously not when $x=0$ or π, therefore the limits are excluded. The series would be unmanageable if the limit l were retained.

Art. 315. In formula (3) of Art. 309, putting $l=\pi$, $\phi(x)$

$$\int_0^\pi \phi(v)dv = \int_0^\pi e^{av}dv = \frac{e^{a\pi}-1}{a};$$

also $$\int_0^\pi \cos nv \,.\, e^{av}dv = \frac{e^{av}}{a^2+n^2}(a\cos nv + n\sin nv)\Big/_0^\pi = \frac{1}{a^2+n^2}(ae^{a\pi}\cos n\pi - a),$$

and ∴ $$e^{ax} = \frac{1}{\pi}\,\frac{e^{a\pi}-1}{a} + \frac{2}{\pi}\sum_1^\infty \frac{a}{a^2+n^2}(e^{a\pi}\cos n\pi - 1)\cos nx.$$

This holds from $x=0$ to $x=\pi$, both inclusive, by the text.

Art. 316. Here $\phi(x)$ is $\sin ax$, and in formula (4) of Art. 309, putting $l=\pi$,

$$\int_0^\pi \sin nv \sin av dv = \frac{1}{2}\int_0^\pi \{\cos(n-a)v - \cos(n+a)v\}dv$$

$$= \left\{\frac{\sin(n-a)v}{2(n-a)} - \frac{\sin(n+a)v}{2(n+a)}\right\}\Big/_0^\pi = \frac{1}{2}\left\{\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right\}$$

$$= -\frac{1}{2}\,\frac{\sin a\pi \cos n\pi \,.\, 2n}{n^2-a^2} = -\frac{n\sin a\pi}{n^2-a^2}\cos n\pi,$$ a being a fraction,

and ∴ $$\sin ax = \frac{-2}{\pi}\sum_1^\infty \sin nx \,.\, \frac{n\sin a\pi\cos n\pi}{n^2-a^2},$$

$$\therefore \frac{\pi}{2}\,.\,\frac{\sin ax}{\sin a\pi} = \frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - + \ldots \Big\}.$$

This formula in general holds from $x=0$ to $x=\pi$ both exclusive, but here both sides vanish when $x=0$, but only the right-hand side when $x=\pi$; therefore the limit $x=0$ is included.

Art. 317. Here $\phi(x)$ is $\cos ax$, and putting $l=\pi$ in formula (3) of Art. 309,

$$\int_0^\pi \phi(v)dv = \int_0^\pi \cos av dv = \frac{1}{a}\sin av\Big/_0^\pi = \frac{\sin a\pi}{a};$$

also $$\int_0^\pi \cos(nv)\cos av dv = \frac{1}{2}\int_0^\pi dv\{\cos(n+a)v + \cos(n-a)v\}$$

$$= \frac{\sin(n+a)\pi}{2(n+a)} + \frac{\sin(n-a)\pi}{2(n-a)} = -\frac{\sin a\pi\cos n\pi \,.\, a}{n^2-a^2},$$

and ∴ $$\cos ax = \frac{1}{\pi}\,.\,\frac{\sin a\pi}{a} - \frac{2}{\pi}\sum_1^\infty \cos nx \,.\, \frac{a\sin a\pi\cos n\pi}{n^2-a^2},$$

$$\therefore \quad \frac{\pi}{2}\,.\,\frac{\cos ax}{\sin a\pi} = \frac{1}{2a} + \frac{a\cos x}{1^2-a^2} - \frac{a\cos 2x}{2^2-a^2} + - \ldots$$

This may also be obtained from Art. 316 by integration.

Art. 325. In the second example, putting $l=\pi$ in Art. 309 (4), $\int_0^\pi \sin nv\phi(v)dv$ becomes $\int_0^\alpha \sin(nv).vdv+\int_\alpha^{\pi-\alpha}\sin(nv)\alpha dv+\int_{\pi-\alpha}^\pi \sin(nv)(\pi-v)dv$

$$=\left(-\frac{v}{n}\cos nv+\frac{1}{n^2}\sin nv\right)\Big/_0^\alpha-\frac{\alpha}{n}\cos nv\Big/_\alpha^{\pi-\alpha}-\frac{\pi}{n}\cos nv\Big/_{\pi-\alpha}^\pi$$
$$-\left(-\frac{v}{n}\cos nv+\frac{1}{n^2}\sin nv\right)\Big/_{\pi-\alpha}^\pi$$

$$=-\frac{\alpha}{n}\cos n\alpha+\frac{1}{n^2}\sin n\alpha-\frac{\alpha}{n}\{\cos n\overline{\pi-\alpha}-\cos n\alpha\}-\frac{\pi}{n}\cos n\pi$$
$$+\frac{\pi}{n}\cos n\overline{\pi-\alpha}+\frac{\pi}{n}\cos n\pi-\frac{\pi-\alpha}{?}\cos n\overline{\pi-\alpha}+\frac{1}{n^2}\sin n\overline{\pi-\alpha}$$

$$=\frac{1}{n^2}\sin n\alpha(1-\cos n\pi),$$

$$\therefore\quad \phi(x)=\frac{2}{\pi}\sum_1^\infty \sin nx.\frac{\sin n\alpha}{n^2}(1-\cos n\pi)$$
$$=\frac{2}{\pi}\left\{\frac{2}{1^2}\sin x\sin\alpha+\frac{2\sin 3x\sin 3\alpha}{3^2}+\dots\right\}.$$

In the third example, using Art. 309 (3), and putting $l=a$, when x is positive, $\int_0^a\phi(v)dv=\int_0^a dv(a-v)=\frac{a^2}{2}$, and

$$\int_0^a\phi(v)\cos\frac{n\pi v}{a}dv=\int_0^a(a-v)\cos\frac{n\pi v}{a}dv$$
$$=\frac{a}{n\pi}(a-v)\sin\frac{n\pi v}{a}\Big/_0^a-\left(\frac{a}{n\pi}\right)^2\cos\frac{n\pi v}{a}\Big/_0^a=-\left(\frac{a}{n\pi}\right)^2(\cos n\pi-1),$$

$$\therefore\quad a-x=\frac{1}{a}\cdot\frac{a^2}{2}-\frac{2}{a}\sum\left(\frac{a}{n\pi}\right)^2(\cos n\pi-1)\cos\frac{n\pi x}{a}$$
$$=\frac{a}{2}+\frac{4a}{\pi^2}\left(\frac{\cos\frac{\pi x}{a}}{1^2}+\frac{\cos\frac{3\pi x}{a}}{3^2}+\dots\right)$$
$$=\frac{a}{2}+\frac{8a}{\pi^2}\left(\frac{\cos^2\frac{\pi x}{2a}}{1^2}+\frac{\cos^2\frac{3\pi x}{2a}}{3^2}+\dots\right)-\frac{4a}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\dots\right)$$
$$=\frac{a}{2}+\frac{8a}{\pi^2}\sum_0^\infty\left\{\frac{\cos(2n+1)\frac{\pi x}{2a}}{2n+1}\right\}^2-\frac{4a}{\pi^2}\cdot\frac{\pi^2}{8},\ \therefore\text{ etc.}$$

If x be negative this clearly $=a+x$.

In the fourth example, for values of x between $-\pi$ and π, by Art. 310,

$$x = 2\{\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \ldots\},$$

$\therefore$ by integration $\dfrac{x^2}{4} = C - \cos x + \dfrac{\cos 2x}{2^2} - \dfrac{\cos 3x}{3^2} + \ldots,$

and when $x = 0$, $\quad C = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \ldots = \dfrac{\pi^2}{8} - \left(\dfrac{\pi^2}{6} - \dfrac{\pi^2}{8}\right) = \dfrac{\pi^2}{12}$, $\therefore$ etc.

In the fifth example, putting $l = \pi$ in Art. 309 (4),

$$\int_0^\pi \sin nv \,.\, \phi(v)dv \text{ becomes } \int_0^a \sin(nv)\sin\frac{\pi v}{a}dv + \int_a^\pi (0)dv$$

$$= \frac{1}{2}\int_0^a \left\{\cos\left(n - \frac{\pi}{a}\right)v - \cos\left(n + \frac{\pi}{a}\right)v\right\}dv$$

$$= \frac{1}{2}\left\{\frac{\sin\left(n - \frac{\pi}{a}\right)a}{n - \frac{\pi}{a}} - \frac{\sin\left(n + \frac{\pi}{a}\right)a}{n + \frac{\pi}{a}}\right\} = -\frac{1}{2}\left\{\frac{\sin na}{n - \frac{\pi}{a}} - \frac{\sin na}{n + \frac{\pi}{a}}\right\} = -\frac{a\pi \sin na}{n^2a^2 - \pi^2}$$

$$\therefore \quad \phi(x) = -\frac{2}{\pi}\sum a\pi\frac{\sin na \sin nx}{n^2a^2 - \pi^2} = 2a\left(\frac{\sin x \sin a}{\pi^2 - a^2} + \frac{\sin 2x \sin 2a}{\pi^2 - 2^2a^2} + \ldots\right).$$

In the sixth and last example, putting $l = \pi$ in Art. 309 (3),

$$\int_0^\pi \phi(v)dv = \int_0^{\frac{\pi}{2}}\left(\frac{\pi^2}{4} - v^2\right)dv = \frac{\pi^3}{8} - \frac{1}{3}\cdot\frac{\pi^3}{8} = \frac{\pi^3}{12}; \text{ and } \int_0^\pi \cos nv\phi(v)dv$$

$$= \int_0^{\frac{\pi}{2}} \cos(nv)\left(\frac{\pi^2}{4} - v^2\right)dv = \frac{\pi^2}{4n}\sin\frac{n\pi}{2} - \frac{v^2}{n}\sin nv\Big|_0^{\frac{\pi}{2}} + \frac{2}{n}\int_0^{\frac{\pi}{2}} v \sin nv dv$$

$$= -\frac{2}{n^2}v\cos nv\Big|_0^{\frac{\pi}{2}} + \frac{2}{n^3}\sin\frac{n\pi}{2} = -\frac{\pi}{n^2}\cos\frac{n\pi}{2} + \frac{2}{n^3}\sin\frac{n\pi}{2},$$

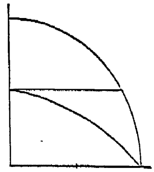
$$\therefore \quad \phi(x) = \frac{1}{\pi}\cdot\frac{\pi^3}{12} + \frac{4}{\pi}\left(\frac{\cos x \,.\, \sin\frac{\pi}{2}}{1^3} + \frac{\cos 3x \,.\, \sin\frac{3\pi}{2}}{3^3} + \ldots\right)$$

$$-2\left(\frac{\cos x}{1^2}\cos\frac{\pi}{2} + \frac{\cos 2x}{2^2}\cos\frac{2\pi}{2} + \ldots\right)$$

$$= \frac{\pi^2}{12} + \frac{4}{\pi}\left(\frac{\cos x}{1^3} - \frac{\cos 3x}{3^3} + \ldots\right) + 2\left(\frac{\cos 2x}{2^2} - \frac{\cos 4x}{4^2} + \ldots\right).$$

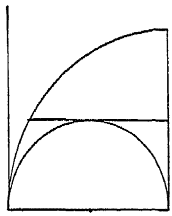
Miscellaneous Examples.

1. The limiting curves for y are the circle $x^2+y^2=a^2$, and the parabola $x^2=-2a\left(y-\frac{a}{2}\right)$, therefore the *area* of *integration*, which is bounded by the two curves and the axis of y, on changing the order must be separated into two parts, and when y lies between 0 and $\frac{a}{2}$, $x \propto$ from $\sqrt{a^2-2ay}$ to $\sqrt{a^2-y^2}$; and when y lies between $\frac{a}{2}$ and a, $x \propto$ from 0 to $\sqrt{a^2-y^2}$. Hence the giv



integral becomes $\int_0^{\frac{a}{2}}\int_{\sqrt{a^2-2ay}}^{\sqrt{a^2-y^2}}\phi(x, y)dydx+\int_{\frac{a}{2}}^{a}\int_0^{\sqrt{a^2-y^2}}\phi(x, y)dydx.$

2. The limiting curves for y are the circle $x^2+y^2=2ax$ and the parabola $y^2=4ax$, and to change the order of integration, the *area* of *integration*, which is bounded by the two curves and $x=2a$, must be separated into three parts, thus: from $y=0$ to $y=a$, (1) $x \propto$ from $\frac{y^2}{4a}$ to $a-\sqrt{a^2-y^2}$, and (2) $x \propto$ from $a \div \sqrt{a^2-y^2}$ to $2a$; and from $y=a$ to $y=2a\sqrt{2}$, $x \propto$ from $\frac{y^2}{4a}$ to $2a$. Hence the given integral becomes



$$\int_0^a\int_{\frac{y^2}{4a}}^{a-\sqrt{a^2-y^2}}\phi(x, y)dydx+\int_0^a\int_{a+\sqrt{a^2-y^2}}^{2a}\phi(x, y)dydx+\int_a^{2a\sqrt{2}}\int_{\frac{y^2}{4a}}^{2a}\phi(x, y)dydx.$$

3. Changing from y to v, and therefore eliminating u,

$$y=\frac{vx}{1-v}=\frac{x}{1-v}-x, \therefore \frac{dy}{dv}=\frac{x}{(1-v)^2}, \text{ and } v=\frac{y}{x+y},$$

therefore the limits of v, corresponding to those of y, are $\frac{a}{a+1}$ and $\frac{b}{b+1}$, so that the given integral U becomes $\int_0^c\int_{\frac{a}{a+1}}^{\frac{b}{b+1}}\phi'(x, v)dxdv\frac{x}{(1-v)^2}$: next, changing the *order* of the integration, since the limits are all constant,

$$U=\int_{\frac{a}{a+1}}^{\frac{b}{b+1}}\int_0^c\phi'(x, v)\frac{xdvdx}{(1-v)^2};$$

now changing from x to u, and therefore eliminating y,

$$x=u(1-v), \quad \therefore \frac{dx}{du}=1-v, \text{ and the limits of } u \text{ are } 0 \text{ and } \frac{c}{1-v},$$

$$\therefore \qquad U=\int_{\frac{a}{a+1}}^{\frac{b}{b+1}}\int_0^{\frac{c}{1-v}}\phi_1(u,\, v)u\,dv\,du,$$

where $\phi_1(u, v)$ is equivalent to $\phi\{u(1-v),\, uv\}$.

4. Changing from y to v, and therefore eliminating u,

$$y=\frac{cvx}{1-v}=\frac{cx}{1-v}-cx, \quad \therefore \frac{dy}{dv}=\frac{cx}{(1-v)^2}, \text{ and } v=\frac{y}{y+cx},$$

therefore the limits of v are 0 and $\frac{b}{b+cx}$, and the given integral

$$U=\int_0^a\int_0^{\frac{b}{b+cx}}\phi'(x,\, v)dx\,dv\frac{cx}{(1-v)^2};$$

therefore the *area of integration*, regarding v as an ordinate, is bounded by $v=0$, $x=0$, the hyperbola $v=\frac{b}{b+cx}$, and $x=a$, and the hyperbola meets $x=0$ and $x=a$ where $v=1$ and $\frac{b}{b+ca}$ respectively: thus to change the order of integration, the area must be divided into two parts, and from $v=0$ to $\frac{b}{b+ca}$, $x \propto$ from 0 to a; from $v=\frac{b}{b+ca}$ to 1, $x \propto$ from 0 to $\frac{b(1-v)}{cv}$.

Hence $$U=\int_0^{\frac{b}{b+ca}}\int_0^a cx\frac{\phi'(x,\, v)dv\,dx}{(1-v)^2}+\int_{\frac{b}{b+ca}}^1\int_0^{\frac{b}{cv}(1-v)}cx\frac{\phi'(x,\, v)dv\,dx}{(1-v)^2}:$$

now changing from x to u, and therefore eliminating y,

$$uv+cx=u, \quad \frac{dx}{du}=\frac{1-v}{c}, \text{ and the limits of } u \text{ are (1) } 0 \text{ and } \frac{ca}{1-v}, \text{ and}$$

(2) 0 and $\frac{b}{v}$; hence, since $\frac{(1-v)^2}{cx}=\frac{1-v}{cu}$,

$$U=\int_0^{\frac{b}{b+ca}}\int_0^{\frac{ca}{1-v}}\phi_1(u,\, v)u\,dv\,du+\int_{\frac{b}{b+ca}}^1\int_0^{\frac{b}{v}}\phi_1(u,\, v)u\,dv\,du.$$

If $c=1$, this agrees with the result of Art. 240.

5. To change from z to u, eliminating v and w,

$$\frac{dz}{du}=\frac{3u^2}{xy};$$

to change from y to v, eliminating z and w,

$$\frac{u^3}{v}=u^3\left(\frac{1}{x}+\frac{1}{y}\right)+xy,$$

$$\therefore \qquad \frac{dy}{dv} = \frac{u^3}{v^2} \div \left(\frac{u^3}{y^2} - x\right) = \frac{u^3 y^2}{v^2(xyz - xy^2)} = \frac{u^3 y}{v^2 x(z-y)};$$

and to change from x to w, y and z must be eliminated. Now the first and second given equations give

$(y+z)x = u^3\left(\frac{1}{v} - \frac{1}{x}\right)$, and the first and third given equations give $(y+z)^2 = w^2 - x^2 + \frac{2u^3}{x} = \frac{u^6}{v^2x^4}(v-x)^2$, consequently ; therefore,

$$w\frac{dw}{dx} = x + \frac{u^3}{x^2} - \frac{u^6}{v^2x^4}(v-x) - \frac{2u^6(v-x)^2}{v^2x^5},$$

and
$$v - x = x\left(\frac{yz}{yz+zx+xy} - 1\right) = -\frac{x^2(y+z)}{yz+zx+xy},$$

$$\therefore \qquad w\frac{dw}{dx} = x + \frac{u^3}{x^2} + \frac{u^6}{v^2x^2}\,\frac{y+z}{yz+zx+xy} - \frac{2u^6}{x^3}\,\frac{(y+z)^2}{y^2z^2}$$

$$= x + \frac{yz}{x} + \frac{(yz+zx+xy)(y+z)}{x^2} - \frac{2(y+z)^2}{x}$$

$$= \frac{1}{x^2}\{x^3 + xyz + (y+z)(zx+xy+yz-2xy-2zx)\}$$

$$= \frac{1}{x^2}\{x^3 - xy^2 - xyz + y^2z - xz^2 + yz^2\}$$

$$= \frac{x-y}{x^2}\{(x+y)x - yz - z^2\} = \frac{(x-y)(x-z)(x+y+z)}{x^2},$$

$$\therefore \iiint (x-y)(y-z)(z-x)\,dx\,dy\,dz$$

$$= U = -\iiint (x-y)(y-z)(z-x)\,.\,\frac{3u^2}{xy}du\,.\,\frac{u^3y}{v^2x(z-y)}dv\,.\,\frac{wx^2dw}{(x-y)(z-x)(x+y+z)}$$

$$= -\iiint \frac{3u^5w\,du\,dv\,dw}{v^2(x+y+z)}, \text{ and } (x+y+z)^2 = w^2 + \frac{2u^3}{v},$$

$$\therefore \quad U = -\iiint \frac{3u^5w\,du\,dv\,dw}{v^2\left(w^2 + \frac{2u^3}{v}\right)^{\frac{1}{2}}}.$$

6. Here $\tau = t^2 = x^{2n}$, therefore

$$\frac{d\tau}{dx} = 2nx^{2n-1}, \text{ and } \int_0^\infty e^{-\tau}dx = \int_0^\infty \frac{e^{-\tau}d\tau}{2n}\,.\,\tau^{-\frac{2n-1}{2n}} = \frac{1}{2n}\Gamma\left(\frac{1}{2n}\right).$$

Similarly $\int_0^\infty e^{-x^n}dx = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)$;

and
$$\int_0^\infty \frac{dx}{(1+x^{2n})^{\frac{1}{n}}} = \int_0^\infty \frac{d\tau}{(1+\tau)^{\frac{1}{n}}}\cdot\frac{1}{2n\tau^{\frac{2n-1}{2n}}} = \int_0^\infty \frac{d\tau\,.\,\tau^{\frac{1}{2n}-1}}{2n(1+\tau)^{\frac{1}{n}}},$$

. if $\frac{\tau}{1+\tau}=y$, and $\therefore \frac{1}{1+\tau}=1-y$ and $\tau=\frac{y}{1-y}$,

$$\int_0^\infty \frac{dx}{(1+x^{2n})^{\frac{1}{n}}}=\int_0^1 \frac{dy}{2n}(1-y)^{-2}\cdot\left(\frac{y}{1-y}\right)^{\frac{1}{2n}-1}(1-y)^{\frac{1}{n}}$$

$$=\frac{1}{2n}\int_0^1 dy\,.\,y^{\frac{1}{2n}-1}(1-y)^{\frac{1}{2n}-1}=\frac{1}{2n}\frac{\left\{\Gamma\left(\frac{1}{2n}\right)\right\}^2}{\Gamma\left(\frac{1}{n}\right)},$$

$\therefore$ the required result follows.

7. If $\tan\theta=x$, $\int_0^{\frac{\pi}{4}}\tan\theta\log\cot\theta d\theta$

$$=-\int_0^1\frac{x\log x dx}{1+x^2}=-\int_0^1 dx\,.\,x\log x(1-x^2+x^4-+\dots)$$

$$=-\int_0^1 dx\log x(x-x^3+x^5-+\dots)$$

$$=-\log x\left(\frac{x^2}{2}-\frac{x^4}{4}+\frac{x^6}{6}-+\dots\right)\Big/_0^1+\left(\frac{x^2}{2^2}-\frac{x^4}{4^2}+\frac{x^6}{6^2}-+\dots\right)\Big/_0^1$$

$$=0+\left(\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{6^2}-+\dots\right)=\frac{1}{4}\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\dots\right)=\frac{1}{4}\left(2\,.\,\frac{\pi^2}{8}-\frac{\pi^2}{6}\right)=\frac{\pi^2}{48}.$$

8. Transforming to polar co-ordinates the given integral

$$U=\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^4(\cos^4\theta+2\cos\alpha\cos^2\theta\sin^2\theta+\sin^4\theta)}r d\theta dr,$$

$$\therefore\qquad U=\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^4P^2}.\,r d\theta dr=\int_0^{\frac{\pi}{2}}\int_0^\infty\frac{e^{-u^2}d\theta du}{2P}=\int_0^{\frac{\pi}{2}}d\theta\,.\,\frac{\sqrt{\pi}}{4P},$$

where $P^2=\cos^4\theta+\sin^4\theta+2\cos^2\theta\sin^2\theta\cos\alpha$

$$=1-4\cos^2\theta\sin^2\theta\,.\,\sin^2\frac{\alpha}{2}=1-\sin^2\frac{\alpha}{2}\sin^2 2\theta,$$

$$\therefore\qquad U=\int_0^{\frac{\pi}{2}}\frac{d\theta\,.\,\sqrt{\pi}}{4\sqrt{\left(1-\sin^2\frac{\alpha}{2}\sin^2 2\theta\right)}}=\frac{\sqrt{\pi}}{8}\int_0^\pi\frac{d\theta}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\theta}}$$

$$=\frac{\sqrt{\pi}}{4}F\left(\sin\frac{\alpha}{2}\right).$$

9. $\int_0^\infty dx e^{-nx^2\cot 2\beta}\sin(nx^2+\alpha)$

$$=\int_0^\infty\frac{dx}{2\sqrt{-1}}\{e^{-nx^2(\cot 2\beta-\sqrt{-1})+\alpha\sqrt{-1}}-e^{-nx^2(\cot 2\beta+\sqrt{-1})-\alpha\sqrt{-1}}\},$$

and, therefore, by Art. 272,

$$=\frac{\sqrt{\pi}}{4\sqrt{-n}}\left\{\frac{e^{\alpha\sqrt{-1}}}{\sqrt{\cot 2\beta-\sqrt{-1}}}-\frac{e^{-\alpha\sqrt{-1}}}{\sqrt{\cot 2\beta+\sqrt{-1}}}\right\}$$

$$=\frac{1}{4}\sqrt{\frac{\pi\sin 2\beta}{-n}}\{(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)$$
$$-(\cos\alpha-\sqrt{-1}\sin\alpha)(\cos\beta-\sqrt{-1}\sin\beta)\}$$

$$=\frac{1}{2}\sqrt{\frac{\pi\sin 2\beta}{n}}\,.\sin(\alpha+\beta).$$

10. If the given integral be denoted by u,

$$\frac{du}{dn}=\int_0^{\frac{\pi}{4}}\frac{dx\sqrt{1-\tan^2x}}{1+n^2(1-\tan^2x)}=\int_0^{\frac{\pi}{4}}\frac{dx\cos x\sqrt{\cos 2x}}{1+n^2-(1+2n^2)\sin^2x},$$

therefore if $\sqrt{2}\sin x=\sin z$, and therefore $\sqrt{2}\cos x\frac{dx}{dz}=\cos z$,

$$\frac{du}{dn}=\int_0^{\frac{\pi}{2}}\frac{\cos z dz}{\sqrt{2}}\cdot\frac{\cos z}{1+n^2-\frac{1+2n^2}{2}\sin^2z}=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{\cos^2z dz}{(1+2n^2)\cos^2z+1}$$

$$=\frac{\sqrt{2}}{1+2n^2}\int_0^{\frac{\pi}{2}}dz\left(1-\frac{1}{1+(1+2n^2)\cos^2z}\right)$$

$$=\frac{\pi}{2}\cdot\frac{\sqrt{2}}{1+2n^2}-\frac{2\sqrt{2}}{1+2n^2}\int_0^{\frac{\pi}{2}}\frac{dz}{3+2n^2+(1+2n^2)\cos 2z}$$

$$=\frac{\pi}{2}\cdot\frac{\sqrt{2}}{1+2n^2}-\frac{\sqrt{2}}{1+2n^2}\int_0^{\pi}\frac{dz}{3+2n^2+(1+2n^2)\cos z},$$ and ∴ by Art. 14,

Ex. 14, $$\frac{du}{dn}=\frac{\pi}{2}\cdot\frac{\sqrt{2}}{1+2n^2}-\frac{\sqrt{2}}{1+2n^2}\cdot\frac{2\cdot\frac{\pi}{2}}{\sqrt{(4+4n^2)2}}=\frac{\pi}{2}(P-Q),$$

where $P=\frac{\sqrt{2}}{1+2n^2}$ and $Q=\frac{1}{(1+2n^2)\sqrt{1+n^2}}$; and

$$\int Pdn=\int\frac{d(n\sqrt{2})}{1+2n^2}=\tan^{-1}(n\sqrt{2})+\text{ a constant, and if } n=\cot\phi,$$

$$\int Qdn=-\int\frac{\operatorname{cosec}^2\phi d\phi}{(1+2\cot^2\phi)\operatorname{cosec}\phi}=-\int\frac{d\phi\sin\phi}{1+\cos^2\phi}=\tan^{-1}(\cos\phi)+\text{a constant},$$

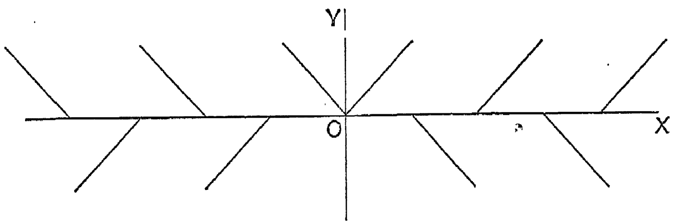
and ∴ $u=\frac{\pi}{2}\tan^{-1}(n\sqrt{2})-\frac{\pi}{2}\tan^{-1}\frac{n}{\sqrt{1+n^2}}+c,$

and when $n=0$, $u=0$, therefore $c=0$.

11. If ξ be a function of θ, the definite integral gives the functional equation $f(\xi)=$ some constant independent of θ and therefore of ξ, $=c$ say, therefore $y=cx$, a straight line, and $f(\xi)$ does not really involve ξ. If ξ be independent of θ, the evaluation of the definite integral gives $f(\xi)=$ some quantity involving ξ in general, and therefore the form or value of the function is determined. In this case, putting $\tan\frac{\theta}{2}=z$,

$$f(\xi)=\int_0^\infty \frac{\sin(\xi z)2dz}{\sin\theta\sec^2\frac{\theta}{2}}=\int_0^\infty \frac{\sin(\xi z)dz}{z},$$

and therefore by Art. 285, $f(\xi)=\frac{\pi}{2}$, 0, or $-\frac{\pi}{2}$, as ξ is positive, zero, or negative. Hence $y=\frac{\pi}{2}x$, 0, or $-\frac{\pi}{2}x$, as $\sin x$ is positive, zero, or negative; thus from $x=0$ to π, $y=\frac{\pi}{2}x$, and so on, and therefore the equation $y=xf(\sin x)$



geometrically represents an ∞ series of equal finite straight lines terminated at one end by the axis of x, and alternately on either side of that axis.

12. If an element Δs of the curve connect the points $P(x, y, z)$ and $Q(x+\Delta x, y+\Delta y, z+\Delta z)$, and $y^2+z^2=r^2$; and if the curve revolve round the axis of x through a small angle $\Delta\phi$, so that P and Q take the positions P', Q', then $PP'=r\Delta\phi$, $QQ'=(r+\Delta r)\Delta\phi$. The corresponding element of surface generated is the area of the skew quadrilateral $PQQ'P'$, which can be divided into the two plane triangles PQP', $P'Q'Q$. Since $PQ=\Delta s=P'Q'$, and $PP'=QQ'$ ultimately, these triangles are equal. Let A be the area of each, and A_x, A_y, A_z be the projections of A on the co-ordinate planes; then the co-ordinates of P' are respectively x, $y+r\Delta\phi\,.\,\frac{z}{r}$ or $y+z\Delta\phi$, and $z-r\Delta\phi\,.\,\frac{y}{r}$ or $z-y\Delta\phi$; therefore

$$\begin{aligned}\pm 2A_x &= \text{the projection, on the plane of } yz, \text{ of the area } PP'Q\\ &= z\Delta\phi(2z-y\Delta\phi)+(\Delta y-z\Delta\phi)(2z+\Delta z-y\Delta\phi)-\Delta y(2z+\Delta z)\\ &= -yz(\Delta\phi)^2+\Delta y\Delta z-y\Delta y\Delta\phi-z\Delta z\Delta\phi+yz(\Delta\phi)^2-\Delta y\Delta z\\ &= -(y\Delta y+z\Delta z)\Delta\phi;\end{aligned}$$

$$\pm 2A_y=-y\Delta\phi(2x)+(\Delta z+y\Delta\phi)(2x+\Delta x)-\Delta z(2x+\Delta x)=y\Delta x\Delta\phi,$$

and ± 2

Hence the area of the skew quadrilateral

$$=\{(y\Delta y+z\Delta z)^2+y^2(\Delta x)^2+z^2(\Delta x)^2\}^{\frac{1}{2}}.\,\Delta\phi,$$

and therefore the surface generated in a complete revolution

$$=\int_0^{2\pi}d\phi\int\sqrt{\{(ydy+zdz)^2+(y^2+z^2)(dx)^2\}}$$

$$=2\pi\int\sqrt{\{(ydy+zdz)^2+(y^2+z^2)(dx)^2\}}.$$

13. If the roots of $x^4+bx^2+a^2=0$ be all imaginary, the two values of x^2 in $(x^2)^2+bx^2+a^2=0$ are either both imaginary or both negative, therefore either $b^2<4a^2$, or $b^2>4a^2$, and b is positive. In the latter case, if $b^2-4a^2=4c^2$, $x^4+bx^2+a^2=\left(x^2+\frac{b}{2}\right)^2-c^2$,

and $$\int_0^\infty\frac{dx}{a^2+bx^2+x^4}=\int_0^\infty dx\left(\frac{1}{x^2+\frac{b}{2}-c}-\frac{1}{x^2+\frac{b}{2}+c}\right)\frac{1}{2c}$$

$$=\frac{1}{2c}\cdot\frac{\pi}{2}\left(\frac{1}{\sqrt{\frac{b}{2}-c}}-\frac{1}{\sqrt{\frac{b}{2}+c}}\right)=\frac{\pi}{4c}\left(\frac{1}{\frac{b}{2}-c}+\frac{1}{\frac{b}{2}+c}-\frac{2}{a}\right)^{\frac{1}{2}}$$

$$=\frac{\pi}{4c}\sqrt{\frac{b}{a^2}-\frac{2}{a}}=\frac{\pi}{2a}\frac{\sqrt{b-2a}}{\sqrt{b^2-4a^2}}=\frac{\pi}{2a\sqrt{b+2a}}.$$

This still holds if c be imaginary, if $b+2a$ be positive, and therefore a be taken as the positive root of a^2. If $x=\frac{1}{z}$,

$$\int_0^\infty\frac{x^2dx}{a^2+bx^2+x^4}=\int_0^\infty\frac{dz}{1+bz^2+a^2z^4}=\frac{1}{a^2}\int_0^\infty\frac{dx}{\frac{1}{a^2}+\frac{bx^2}{a^2}+x^4},$$

and therefore by the preceding, or in the same way, the result is

$$\frac{1}{a^2}\cdot\frac{\pi}{\frac{2}{a}\sqrt{\frac{b}{a^2}+\frac{2}{a}}}=\frac{\pi}{2\sqrt{b+2a}}.$$

Aliter :

$$\int_0^\infty\frac{x^2dx}{a^2+bx^2+x^4}=\int_0^\infty\frac{dx}{2}\left(\frac{1}{x^2+\frac{b}{2}-c}+\frac{1}{x^2+\frac{b}{2}+c}-\frac{b}{2c}\cdot\frac{1}{x^2+\frac{b}{2}-c}+\frac{b}{2c}\cdot\frac{1}{x^2+\frac{b}{2}+c}\right)$$

and so on as in the first case. The integrals are both clearly essentially positive if $x^4+bx^2+a^2=0$ have no real roots ; therefore even if $b^2>4a^2$ and b positive a must be positive.

14. In Art. 309 (4) putting $l=\pi$, as in Art. 324, $\int_0^\pi \sin(nv)\phi(v)dv$ becomes

$$\int_0^{\frac{\pi}{2}} \sin(nv)\,.\,vdv = -\frac{v}{n}\cos nv + \frac{1}{n^2}\sin nv \Big/_0^{\frac{\pi}{2}} = -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2},$$

and $\therefore$

$$\phi(x) = \frac{2}{\pi}\sum_1^\infty \sin nx\left(\frac{\sin\frac{n\pi}{2}}{n^2} - \frac{\pi\cos\frac{n\pi}{2}}{2n}\right)$$

$$= \frac{2}{\pi}\left\{\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right\}$$

$$+\left\{\frac{\sin 2x}{2} - \frac{\sin 4x}{4} + \frac{\sin 6x}{6} - \ldots\right\}, \therefore \text{ etc.}$$

CHAPTER XIV.

1. $A = \int_{\theta_1}^{\theta_2} \frac{r^2 d\theta}{2}$, and $M = \int_{a\theta_1}^{a\theta_2} \pi y^2 dx \div \int_{a\theta_1}^{a\theta_2} dx$ (Art. 333), therefore if $x = a\theta$,

$$M = \pi\int_{\theta_1}^{\theta_2} a d\theta \{f(\theta)\}^2 \div a(\theta_2 - \theta_1) = \frac{2\pi A}{\theta_2 - \theta}.$$

2. If c be the greatest range, then when θ is the angle of elevation the range is $a\sin 2\theta$, therefore if the ball reach more than $\frac{c}{m}$, θ must lie between $\frac{1}{2}\sin^{-1}\left(\frac{1}{m}\right)$ and $\frac{\pi}{2} - \frac{1}{2}\sin^{-1}\left(\frac{1}{m}\right)$. Hence by the method of the latter part of Art. 335 the chance required

$$= \int_0^{2\pi}\int_{\frac{1}{2}\sin^{-1}\left(\frac{1}{m}\right)}^{\frac{\pi}{2}-\frac{1}{2}\sin^{-1}\left(\frac{1}{m}\right)} a\cos\theta d\phi\,.\,a d\theta \div 2\pi a^2 = \sin\theta \Big/_{\frac{1}{2}\sin^{-1}\left(\frac{1}{m}\right)}^{\frac{\pi}{2}-\frac{1}{2}\sin^{-1}\left(\frac{1}{m}\right)}$$

$$= \cos\beta - \sin\beta, \text{ if } \sin 2\beta = \frac{1}{m}, \text{ and therefore}$$

$2\sin\beta\cos\beta = \frac{1}{m}$ and $(\cos\beta - \sin\beta)^2 = 1 - \frac{1}{m}$, therefore

the chance $= \sqrt{1 - \frac{1}{m}}$.

3. If the diameter $= a$, and with the point of projection as pole the circle be $r = a\cos\phi$, then with a given value of ϕ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, for favourable cases $a\sin 2\theta$ must not $> a\cos\phi$, θ being the angle of elevation. Thus the limits of θ are (1) 0 and $\frac{1}{2}\left(\frac{\pi}{2} - \phi\right)$, and (2) $\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \phi\right)$ and $\frac{\pi}{2}$, and

by the method of Art. 335, the required chance

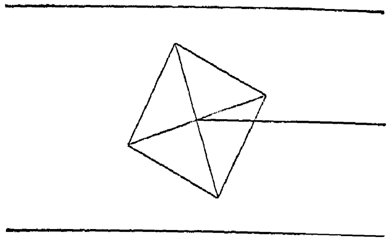
$$=2\int_0^{\frac{\pi}{2}} d\phi \left\{ \int_0^{\frac{\pi}{4}-\frac{\phi}{2}} \cos\theta d\theta + \int_{\frac{\pi}{4}+\frac{\phi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta \right\} \div 2\pi$$

$$=\frac{1}{\pi}\int_0^{\frac{\pi}{2}} d\phi \left\{ \sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)+1-\sin\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right\} = \frac{1}{2}-\frac{2}{\pi\sqrt{2}}\int_0^{\frac{\pi}{2}} d\phi \sin\frac{\phi}{2}$$

$$=\frac{1}{2}+\frac{2}{\pi}\sqrt{2}\left(\frac{1}{\sqrt{2}}-1\right)=\frac{1}{2}+\frac{2}{\pi}(1-\sqrt{2}).$$

Roughly this may be taken as equal to $\frac{1}{6}$.

4. The centre of the face of the cube on the table must lie on some line joining two consecutive lines of the system and perpendicular to both. If x be the distance of this centre from one of the two lines, and θ the angle between one of the diagonals of the face and the system of straight lines, then if the cube do not rest on a line of the system for a given value of θ,



x must lie between $\frac{a}{\sqrt{2}}\cos\theta$ and $c-\frac{a}{\sqrt{2}}\cos\theta$, and all varieties of cases will occur by varying θ from 0 to $\frac{\pi}{4}$. Hence the required chance

$$=\int_0^{\frac{\pi}{4}}\left(c-\frac{a}{\sqrt{2}}\cos\theta-\frac{a}{\sqrt{2}}\cos\theta\right)\frac{d\theta}{c}\div\int_0^{\frac{\pi}{4}} d\theta=\frac{4}{\pi c}\int_0^{\frac{\pi}{4}} d\theta(c-a\sqrt{2}\cos\theta)$$

$$=\frac{4}{\pi c}\left(\frac{\pi c}{4}-a\right)=1-\frac{4a}{\pi c}.$$

In the limit 0 of θ, it is assumed that $a\sqrt{2}<c$.

5. If the ellipse be given by $r=\frac{l}{1+e\cos\theta}$, the mean value is

$$\int_0^{2\pi}\frac{l d\theta}{1+e\cos\theta}\div 2\pi=\frac{l}{\pi}\int_0^{\pi}\frac{d\theta}{1+e\cos\theta}=\frac{l}{\pi}\cdot\frac{\pi}{\sqrt{1-e^2}}=b, \text{ (cf. Art. 14, Ex. 14).}$$

If the abscissa, say x, of the extremity increase uniformly, the factor by which the numerator and denominator of the fraction $\frac{r}{n}$ must be multiplied is not $\Delta\theta$ but $\Delta x=\Delta(r\cos\theta)=l\Delta\left(\frac{\cos\theta}{1+e\cos\theta}\right)=-\frac{l\sin\theta\Delta\theta}{(1+e\cos\theta)^2}$, and therefore

$$\text{the mean value} = \int_0^\pi \frac{l^2 \sin\theta d\theta}{(1+e\cos\theta)^3} \div \int_0^\pi \frac{l\sin\theta d\theta}{(1+e\cos\theta)^2}$$
$$= \frac{l}{2}(1+e\cos\theta)^{-2}\Big/_0^\pi \div (1+e\cos\theta)^{-1}\Big/_0^\pi$$
$$= \frac{l}{2}\left\{\frac{1}{(1-e)^2} - \frac{1}{(1+e)^2}\right\} \div \left(\frac{1}{1-e} - \frac{1}{1+e}\right) = \frac{l}{1-e^2} = a.$$

6. The rod (PQ) can fall at most on m of the straight lines where m is the greatest integer in r, or is $r-1$ if r be integral, therefore $n < r$: and one end P of the rod may be supposed to lie on a fixed perpendicular to two fixed consecutive lines A, B, as it must fall between some such pair. If a be the distance between A and B, x the distance of P from A, then P is at a distance from the nth line on the same side of B as $A = x + (n-1)a$. Hence the chance that the rod will fall on n lines at least is u, where

$$u = \int_0^a \frac{dx}{a} \cdot \frac{4\phi}{2\pi}, \text{ and } x + (n-1)a = ra\cos\phi,$$

$$\text{and } \therefore 2u = \frac{2}{\pi a}\int_{\cos^{-1}\frac{n}{r}}^{\cos^{-1}\frac{n-1}{r}} \phi \cdot ra\sin\phi d\phi = \frac{2r}{\pi}(-\phi\cos\phi + \sin\phi)\Big/_{\cos^{-1}\frac{n}{r}}^{\cos^{-1}\frac{n-1}{r}}$$
$$= \frac{2}{\pi}\left\{n\cos^{-1}\frac{n}{r} - (n-1)\cos^{-1}\frac{n-1}{r} + \sqrt{r^2-(n-1)^2} - \sqrt{r^2-n^2}\right\}.$$

If $n = r = 1$, this reduces to $\frac{2}{\pi}$, which agrees with Art. 336, when $a = c$.

If the chance be required that the rod should fall on exactly n of the lines, denoting such chance by u_n, and the result above by $u_{n,r}$, clearly

$$u_{n,r} = u_n + u_{n+1} + \ldots + u_m,$$
$$\therefore\ u_{n+1,r} = u_{n+1} + \ldots + u_m,$$
$$\text{and}\quad u_n = u_{n,r} - u_{n+1,r} = \frac{2}{\pi}\left\{2n\cos^{-1}\frac{n}{r} - (n-1)\cos^{-1}\frac{n-1}{r} - (n+1)\cos^{-1}\frac{n+1}{r}\right.$$
$$\left. + \sqrt{r^2-(n-1)^2} + \sqrt{r^2-(n+1)^2} - 2\sqrt{r^2-n^2}\right\},$$

so long as $n < m$, but if $n = m$, $u_{n,r} = u_n$.

7. If the arrows are at the points A, B on the target (of radius a), and A be at a distance x from the centre, the number of cases in which A lies between the distances x and $x + \Delta x$ on a fixed diameter may be measured by $x\Delta x$, which is equivalent to dividing the circle into a large number of very thin equal sectors, on any fixed one of which A may be supposed to lie. Also, when A is at the distance x from the centre, the number of cases in which $AB < a$, may be measured by the area X included between the given circle and an equal circle with centre A; and the whole

number of cases may be measured by πa^2. Hence, if u be the required chance,

$$1-u=\int_0^a xdxX \div \int_0^a xdx\,.\,\pi a^2, \text{ and by geometry}$$

$$\frac{1}{2}X = 2\frac{a^2\theta}{2} - \frac{x}{2}a\sin\theta, \text{ where } \frac{x}{2}=a\cos\theta, \text{ and therefore}$$

$$(1-u)\frac{\pi}{2} = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\sin 2\theta d\theta(2\theta-\sin 2\theta),$$

$$\therefore \quad (1-u)\frac{\pi}{4} = \left(-\theta\cos 2\theta+\frac{1}{2}\sin 2\theta\right)\Big/_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1-\cos 4\theta}{2}d\theta$$

$$=\frac{\pi}{2}-\frac{\pi}{6}-\frac{\sqrt{3}}{4}-\frac{\pi}{12}+\frac{1}{8}\left(\sin 2\pi-\sin\frac{4\pi}{3}\right),$$

$$\therefore \quad u=1-\frac{4}{\pi}\left(\frac{\pi}{4}-\frac{3\sqrt{3}}{16}\right)=\frac{3\sqrt{3}}{4\pi}.$$

8. With the notation of Art. 335, the number of orbits making an angle θ to $\theta+\Delta\theta$ with the ecliptic, may be measured by $\sin\theta\Delta\phi\Delta\theta$, therefore the mean inclination in the first octant

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\theta\sin\theta d\phi d\theta \div \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin\theta d\phi d\theta = (-\theta\cos\theta+\sin\theta)\Big/_0^{\frac{\pi}{2}}=1$$

= the angle subtended by an arc of a circle equal to the radius. Similarly for any other octant, the inclination being always taken as a positive angle $< \frac{\pi}{2}$. If a normal to the plane of the ecliptic meet a celestial sphere in the zenith, and a normal to the orbit of a comet through the centre of the sphere meet it in C, then θ and ϕ are C's zenith distance and azimuth.

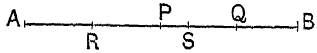
9. The parallels of latitude being circles parallel to the equator, when the territory lies between latitude $\theta°$ and $(\theta+1)°$, its area, if R be the radius of the earth (supposed spherical), is $\frac{1}{360}\,.\,2\pi R\,.\,R\{\sin(\theta+1)-\sin\theta\}$, and therefore if λ_1, λ_2 be the limits of the latitude, the mean area

$$=\frac{\pi R^2}{180}\int_{\lambda_1}^{\lambda_2-1}\{\sin(\theta+1)-\sin\theta\}d\theta \div \int_{\lambda_1}^{\lambda_2-1}d\theta$$

$$=\frac{\pi R^2}{180}\{\cos(\lambda_2-1)-\cos\lambda_1-\cos\lambda_2+\cos(\lambda_1+1)\}\div(\lambda_2-\lambda_1-1),$$

the angles being measured in degrees.

10. If AB, PQ, RS be the lines a, b, b', and $AP=x$, the whole number of different positions of PQ may be measured by $a-b$, and so the number for $RS=a-b'$, therefore the whole number of different and equally likely



positions of b and b' concurrently $=(a-b)(a-b')$: also, if x be great enough, and not too great, RS may take the position in the figure where $PS=c$, and the conditions for this are $x+c>b'$, and $x<a-b$, and therefore $a-b>b'-c$, which latter and $c<b<b'$ limit c. Now R may take any position up to A, and therefore the number of different equally likely positions of RS, subject to PQ and RS not having a common part $>c$, when R lies between A and P, is measured by $AR=x+c-b'$, and therefore as the whole number of different positions of PQ and RS must be the same when S is between B and Q, the required probability

$$=\frac{2}{(a-b)(a-b')}\int_{b'-c}^{a-b}dx(x+c-b')=\frac{(a-b+c-b')^2}{(a-b)(a-b')}.$$

11. If the point be taken as pole, and $r=$ the length of any one of the straight lines, then the area of the curve $=\int_0^{2\pi}\frac{r^2d\theta}{2}$, and therefore the mean value of the squares, which $=\int_0^{2\pi}r^2d\theta\div 2\pi=\frac{1}{\pi}$. area.

12. Here a is measured in miles, and if M takes t hours he gets $\frac{n}{t}$ shillings, and the shower begins not more than $\frac{a}{v}$ hours after M starts. Then (1) if he is not caught in the shower he takes $\frac{a}{v}$ hours and gets $\frac{nv}{a}$ shillings, and for the probability of this hypothesis, if the shower begins after a time t hours, t may $\propto$ from 0 to $\frac{a}{v}$, and z from 0 to vt, therefore

$$\text{the chance}=\int_0^{\frac{a}{v}}\int_0^{vt}dtdz\div\int_0^{\frac{a}{v}}\int_0^{a}dtdz=\frac{v}{a^2}\int_0^{\frac{a}{v}}vtdt=\frac{1}{2},$$

therefore the value of M's expectation so far $=\frac{nv}{2a}$ shillings.

But (2) if the shower begins t hours after he started, t as before may $\propto$ from 0 to $\frac{a}{v}$, but z must $\propto$ from vt to a, and M stops for $\frac{z-vt}{u}$ hours, and therefore gets $\dfrac{n}{\frac{a}{v}+\frac{z-vt}{u}}$ shillings. Thus the value of his expectation is then

$$\frac{v}{a^2}\int_0^{\frac{a}{v}}\int_{v}^{a}\frac{dtdz\,.\,n}{\frac{a}{v}+\frac{z-vt}{u}}=\frac{nuv}{a^2}\int_0^{\frac{a}{v}}dt\log\left(\frac{a}{v}+\frac{z-vt}{u}\right)\Big/_{vt}^{a}$$

$$=\frac{nuv}{a^2}\int_0^{\frac{a}{v}}dt\log\frac{a\left(\frac{1}{u}+\frac{1}{v}\right)-\frac{vt}{u}}{\frac{a}{v}}$$

$$=\frac{nuv}{a^2}.t\log\frac{a\left(\frac{1}{u}+\frac{1}{v}\right)-\frac{vt}{u}}{\frac{a}{v}}\Bigg|_0^{\frac{a}{v}}+\frac{nuv}{a^2}\cdot\frac{v}{u}\int_0^{\frac{a}{v}}\frac{dt.t}{a\left(\frac{1}{u}+\frac{1}{v}\right)-\frac{vt}{u}}$$

$$=\frac{nuv}{a^2}\int_0^{\frac{a}{v}}dt\left\{-1+\frac{a\left(\frac{1}{u}+\frac{1}{v}\right)}{a\left(\frac{1}{u}+\frac{1}{v}\right)-\frac{vt}{u}}\right\}$$

$$=\frac{nuv}{a^2}\left\{-t-a\left(\frac{1}{u}+\frac{1}{v}\right)\frac{u}{v}\log\left[a\left(\frac{1}{u}+\frac{1}{v}\right)-\frac{vt}{u}\right]\right\}\Bigg/_0^{\frac{a}{v}}$$

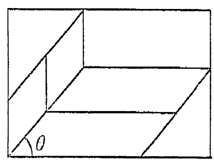
$$=\frac{nuv}{a^2}\left\{-\frac{a}{v}-\frac{a(u+v)}{v^2}\log\left(\frac{a}{v}\div a\overline{\frac{1}{u}+\frac{1}{v}}\right)\right\}$$

$$=\frac{nv}{a}\left\{-\frac{u}{v}+\frac{u(u+v)}{v^2}\log\frac{u+v}{u}\right\},$$

and therefore the value of the whole expectation is

$$\frac{nv}{a}\left\{\frac{1}{2}-\frac{u}{v}+\frac{u(u+v)}{v^2}\log\frac{u+v}{u}\right\}\text{ shillings.}$$

13. The two systems of lines form a series of equal rectangles, and the centre of the rod must fall within one of these rectangles. If a be the distance of two consecutive lines of one system, b the corresponding distance for the other system, θ the inclination of the rod to the lines of the first system, c the length of the rod, then it will be seen that in order that the rod should fall on a line, its centre may rest anywhere within the rectangle, except in an area $=(a-c\cos\theta)(b-c\sin\theta)$: and all varieties of cases will be included by supposing θ to vary from 0 to $\frac{\pi}{2}$. Hence the chance of crossing a line is



$$\int_0^{\frac{\pi}{2}}\{ab-(a-c\cos\theta)(b-c\sin\theta)\}d\theta\div\int_0^{\frac{\pi}{2}}ab\,d\theta$$

$$=\frac{2}{\pi ab}\left\{c(a+b)+\frac{c^2}{4}(\cos\pi-\cos 0)\right\}=\frac{2c(a+b)-c^2}{\pi ab}.$$

The result of Art. 336 may be deduced by putting $a=\infty$.

This assumes that $c<a$ and $<b$. If $a>b$, and $c<a>b$, the limits of θ are 0 and $\sin^{-1}\frac{b}{c}$ in the expression for the area above, while from $\theta=\sin^{-1}\frac{b}{c}$ to $\frac{\pi}{2}$, the chance is unity. Thus in this case the probability is

$$\frac{\int_0^{\sin^{-1}\frac{b}{c}}\left\{c(a\sin\theta+b\cos\theta)-\frac{c^2}{2}\sin 2\theta\right\}d\theta+\int_{\sin^{-1}\frac{b}{c}}^{\frac{\pi}{2}}ab\,d\theta}{\int_0^{\frac{\pi}{2}}ab\,d\theta}$$

$$=\frac{2}{\pi ab}\left\{ca\left(1-\frac{\sqrt{c^2-b^2}}{c}\right)+bc\,.\,\frac{b}{c}+\frac{c^2}{4}\left(1-\frac{2b^2}{c^2}-1\right)+ab\left(\frac{\pi}{2}-\sin^{-1}\frac{b}{c}\right)\right\}$$

$$=\frac{1}{\pi ab}\left\{2ca-2a\sqrt{c^2-b^2}+b^2+\pi ab-2ab\sin^{-1}\frac{b}{c}\right\}.$$

If however $c>a$, the limits of θ in the original expression for the area are $\cos^{-1}\frac{a}{c}$ and $\sin^{-1}\frac{b}{c}$, which holds so long as $c^2-a^2<b^2$. Of course, if c be not $<\sqrt{a^2+b^2}$, the chance is a certainty, but if $c>a$ and $<\sqrt{a^2+b^2}$, the chance is $\int_{\cos^{-1}\frac{a}{c}}^{\sin^{-1}\frac{b}{c}}\left\{c(a\sin\theta+b\cos\theta)-\frac{c^2}{2}\sin 2\theta\right\}\frac{2d\theta}{\pi ab}+\int_0^{\cos^{-1}\frac{a}{c}}\frac{2d\theta}{\pi}+\int_{\sin^{-1}\frac{b}{c}}^{\frac{\pi}{2}}\frac{2d\theta}{\pi}$, etc.

14. If a and b have the same meanings as in Ex. 13, and c be the edge of the cube, then as in Exs. 4 and 13 the centre of the face of the cube on the table must lie within an area $ab-(a-c\sqrt{2}\cos\theta)(b-c\sqrt{2}\cos\theta)$ in order to fall across a line, and if $c\sqrt{2}<b$, the limits of θ are 0 and $\frac{\pi}{4}$, and therefore the chance required is

$$\int_0^{\frac{\pi}{4}}\{c(a+b)\sqrt{2}\cos\theta-c^2(1+\cos 2\theta)\}d\theta\div\int_0^{\frac{\pi}{4}}d\theta\,.\,ab$$

$$=\frac{4}{\pi ab}\left\{c(a+b)-c^2\left(\frac{\pi}{4}+\frac{1}{2}\right)\right\}=\frac{4c(a+b)-c^2(\pi+2)}{\pi ab}.$$

If $c=b$ or $>b$, the cube must fall across a line, but if $c<b$ but $c\sqrt{2}>b$, the inferior limit of θ in the above integral becomes $\cos^{-1}\frac{b}{c\sqrt{2}}$, and the chance required is

$$\frac{4}{\pi ab}\int_{\cos^{-1}\frac{b}{c\sqrt{2}}}^{\frac{\pi}{4}}d\theta\{c(a+b)\sqrt{2}\cos\theta-c^2(1+\cos 2\theta)\}+\frac{4}{\pi}\int_0^{\cos^{-1}\frac{b}{c\sqrt{2}}}d\theta$$

$$=\frac{4}{\pi ab}\left\{c(a+b)-(a+b)\sqrt{2c^2-b^2}-c^2\left(\frac{\pi}{4}-\cos^{-1}\frac{b}{c\sqrt{2}}\right)-\frac{c^2}{2}\left(1-\frac{b}{c^2}\sqrt{2c^2-b^2}\right)\right.$$
$$\left.+ab\cos^{-1}\frac{b}{c\sqrt{2}}\right\}$$

$$=\frac{1}{\pi ab}\left\{4c(a+b)-2(2a+b)\sqrt{2c^2-b^2}-c^2(\pi+2)+4(ab+c^2)\cos^{-1}\frac{b}{c\sqrt{2}}\right\}.$$

This includes the case when $c\sqrt{2}>a$ but $<b\sqrt{2}$. By putting $a=\infty$ in the

above expression, the solution of Ex. 4 can be extended to the case in which the diagonal of a face of the cube is greater than the distance between consecutive straight lines.

15. In order that the inferior limits in the integrations shall be all zero, the ordinates may be represented—the first by z_1, the second by z_1+z_2, the third by $z_1+z_2+z_3$, and so on, and therefore

$$s = \text{their sum} = nz_1 + (n-1)z_2 + (n-2)z_3 + \ldots + z_n;$$

and subject to all positive values of the variables consistent with this equation, the mean value of the rth ordinate is

$$\frac{\iiint \ldots (z_1+z_2+\ldots+z_r)dz_1dz_2\ldots dz_{n-1}}{\iiint \ldots dz_1dz_2\ldots dz_{n-1}},$$

since z_n is invariable when $z_1z_2\ldots z_{n-1}$ are given. If now x_1 be put for nz_1, x_2 for $(n-1)z_2$, and so on, the mean value

$$=\frac{\iiint \ldots\left(\frac{x_1}{n}+\frac{x_2}{n-1}+\frac{x_3}{n-2}+\ldots+\frac{x_r}{n-r+1}\right)dx_1dx_2\ldots dx_{n-1}}{\iiint \ldots dx_1dx_2\ldots dx_{n-1}},$$

subject to $x_1+x_2+\ldots+x_n=s$: but by Dirichlet's Theorem (Art. 276) $\iiint y_1^{\alpha-1}y_2^{\beta-1}y_3^{\gamma-1}\ldots dy_1dy_2dy_3\ldots$, subject to $y_1+y_2+y_3+\ldots<h$, is

$$h^{\alpha+\beta+\gamma+\ldots}\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\ldots}{\Gamma(\alpha+\beta+\gamma+\ldots+1)},$$

therefore putting any one of the quantities α, β, γ, ... equal to 2, and each of the others equal to unity, if p be a whole number between 0 and n, $\iiint \ldots x_pdx_1dx_2\ldots dx_{n-1}$ (subject to $x_1+x_2+\ldots+x_{n-1}<s$, and the quantities all positive) $=s^n\div\Gamma(n+1)$, and putting $\alpha=1=\beta=\gamma=\ldots$,

$$\iiint \ldots dx_1dx_2\ldots dx_{n-1}=s^{n-1}\div\Gamma(n),$$

and $\therefore\ \iiint \ldots x_pdx_1dx_2\ldots dx_{n-1}\div\iiint \ldots dx_1dx_2\ldots dx_{n-1}=\frac{s}{n}$.

Hence putting $p=1, 2, \ldots, r$, successively, the mean value of the rth ordinate, if $r<n$, is $\quad \frac{s}{n}\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\ldots+\frac{1}{n-r+1}\right)$.

The mean value of the nth ordinate $=s-$ sum of mean values of the other ordinates, and it is easily seen that if this follow the law for the other ordinates, the sum of all the mean values $=\frac{s}{n}(1+1+\ldots+1$ to n terms$)=s$, therefore the mean value of the nth ordinate is

$$\frac{s}{n}\left(\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{2}+1\right),$$

16. To prove $\int dw(\psi - \sin\psi) = \frac{l^2}{2} - \pi\Omega$, where, if the circle be $r=a$, its centre being the pole, $l = 2\pi a$, $\Omega = \pi a^2$, ψ = the angle between the tangents from any point P, and dw = areal element at P, so that the integral extends over the whole area of the plane of the circle external to it. If P be (r, θ) the external area may be divided into elementary circular rings concentric with $r = a$, and such an element of area corresponding to $P = 2\pi r dr$; also $\sin\frac{\psi}{2} = \frac{a}{r}$, therefore

$$\int dw(\psi - \sin\psi) = 2\pi\int_a^\infty r dr\{\psi - \sin\psi\},$$

and $$\frac{1}{2}\cos\frac{\psi}{2}d\psi = -\frac{a}{r^2}dr,$$ and the corresponding lts. of ψ are π and 0, $\therefore$

$$\text{the proposed integral} = 2\pi\int_0^\pi \cos\frac{\psi}{2}\,\frac{d\psi}{2a}\cdot\frac{a^3}{\sin^3\frac{\psi}{2}}(\psi - \sin\psi)$$

$$= \pi a^2\int_0^\pi d\psi\left(\psi\cos\frac{\psi}{2}\operatorname{cosec}^3\frac{\psi}{2} - 2\cot^2\frac{\psi}{2}\right)$$

$$= 4\pi a^2\int_0^{\frac{\pi}{2}} d\psi\left(\frac{\psi\cos\psi}{\sin^3\psi} - \operatorname{cosec}^2\psi + 1\right)$$

$$= 4\pi a^2\left\{\frac{-\psi}{2\sin^2\psi} - \frac{1}{2}\cot\psi + \cot\psi + \psi\right\}_0^{\frac{\pi}{2}}$$

$$= 4\pi a^2\left\{-\frac{\pi}{4} + \frac{\pi}{2}\right\} + 2\pi a^2\left\{\frac{\psi}{\sin^2\psi} - \cot\psi\right\}[\psi = 0]$$

$$= \pi^2 a^2 + 2\pi a^2\left(\frac{1}{\psi} - \frac{1}{\psi}\right) = \pi^2 a^2 = \frac{1}{2}(2\pi a)^2 - \pi^2 a^2 = \frac{l^2}{2} - \pi\Omega.$$

CHAPTER XV.

1. If the given straight lines be $y=0$ and $y=mx$, the chord is $y=m'x+c$, where m' and c are functions of the co-ordinates of the extremities of the curve. Hence $V = a\sqrt{1+p^2} - y - (m'x+c)$, where a is some constant (Art. 378), therefore $K = 0 = \frac{dV}{dy} - \frac{d}{dx}\cdot\frac{dV}{dp} = 1 - \frac{d}{dx}\frac{ap}{\sqrt{1+p^2}}$,

$$\therefore \qquad \frac{ap}{\sqrt{1+p^2}} = x + b, \quad \therefore\ p = \frac{dy}{dx} = \pm\frac{x+b}{\sqrt{a^2-(x+b)^2}},$$

$$\therefore \qquad y + c' = \mp\sqrt{a^2-(x+b)^2},$$

and therefore the curve is a circular arc. Now it may be easily shown by the *Differential Calculus* that the area contained by a circular arc of *given* length and its chord is a maximum when the arc is a semicircle. Thus the length of the chord is given, and any position of it connecting the two given straight lines will suit the question so long as neither of the angles between the chord and the given lines on the side of the chord opposite to the origin (on which side the arc must clearly lie) is less than $\frac{\pi}{2}$, for otherwise the semicircle would cut the corresponding given line. If the area were a minimum the curve would be a straight line.

2. Taking the pole inside the curve,

$$\text{the perimeter} = \int_0^{2\pi} d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}, \text{ and the area} = \int_0^{2\pi} \frac{r^2}{2} d\theta,$$

$$\therefore \quad V = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} + ar^2, \text{ which involves only } r \text{ and } \frac{dr}{d\theta},$$

$$\therefore \text{ by Art. 357} \quad V = ar^2 + \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \frac{dr}{d\theta} \cdot \frac{\frac{dr}{d\theta}}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} + c,$$

$\therefore \quad ar^2 + \dfrac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}} = c$, or, if ϕ be the angle between the tangent and radius vector, and p the perpendicular on the tangent from the pole,

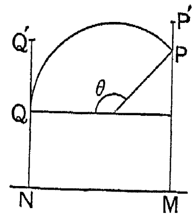
$$ar^2 = c - r \sin \phi = c - p,$$

$$\therefore \quad 2ar\frac{dr}{dp} = -1, \text{ or } \rho = \text{radius of curvature} = \text{a constant},$$

and therefore the curve is a circle.

This must correspond to a maximum area, for if the curve were an ellipse for instance, of eccentricity indefinitely near to unity, its area would vanish. The same result follows, by considering the curve as the limit of a polygon with indefinitely small sides, from Todhunter's *Diff. Cal.*, Chap. XVI., Ex. 2.

3. If P, Q be the two fixed points, $PM = h$, $QN = k$, the ordinates, when the arc is the greatest possible, and $h > k$, NQ is a tangent to the arc at Q, and if r be the radius, θ the angle of the arc, and l its length,



$$r\theta = l,$$

$$r(1 - \cos\theta) = x_1 - x_0,$$

$$r \sin\theta = h - k,$$

and eliminating r and θ from these equations, l is given in terms of x_1, x_0, h and k, and the difficulty arises when l exceeds the value thus given. It is clear that if P, Q were moved to some points P',

Q' in MP, NQ produced, the area would be increased, the arc would still lie between the ordinates, and the given length l of the curve could be made up of the arc $P'Q'$ and the straight lines PP' and QQ'. Adopting this form of solution, a maximum is now required of

$$\int_{x_0}^{x_1} y\,dx + a\left\{\int_{x_0}^{x_1}\sqrt{1+p^2}\,dx + y_1 - h + y_0 - \kappa\right\},$$

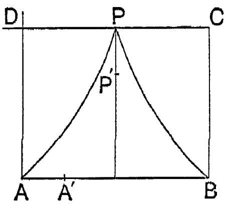
y_1, y_0 being the ordinates of P', Q'. Thus $K=0$ gives a circle as before for the arc $P'Q'$, and δU then reduces to

$$\left(\frac{ap}{\sqrt{1+p^2}}\right)_1 \delta y_1 - \left(\frac{ap}{\sqrt{1+p^2}}\right)_0 \delta y_0 + a\delta y_1 + a\delta y_0,$$

and therefore equating to zero the coefficients of δy_1 and δy_0 respectively $p_1 = -\infty$, $p_0 = \infty$, i.e., the arc $P'Q'$ touches the ordinates at P', Q', so that $P'Q'$ is parallel to MN, and $l = y_1 - h + y_1 - \kappa + \pi(x_1 - x_0)$, which gives the positions of P', Q'. If however $l >$ arc PQ but $< h - \kappa + \pi(x_1 - x_0)$, then P' will be at P, and therefore $\delta y_1 = 0$, and $p_0 = \infty$ as before, so that Q' is above Q but $Q'N < PM$, and the arc PQ' touches $Q'N$ at Q'.

N.B.—In the figure, P should be one extremity of the circular arc.

4. If $2a$, $2b$ be the lengths of the sides of the dish, c the given height, APB a vertical section in the plane $ABCD$ of the roof of the cover (supposed cylindrical) parallel to the ends; AD, BC vertical and each $=c$, and therefore P on CD; and AB, AD be taken as axes,



the area of the roof $= 2a\int_0^{2b}\sqrt{1+p^2}\,dx$,

and the area of the ends $= 2\int_0^{2b} y\,dx$.

Hence V may be written $\quad a\sqrt{1+p^2}+y$,

$$\therefore \qquad \frac{dV}{dy} = 1 = \frac{d}{dx}\frac{ap}{\sqrt{1+p^2}}, \quad \therefore \frac{ap}{\sqrt{1+p^2}} = x - h,$$

and since y is positive and must increase with x for some range of values from $x=0$, p is positive for such values, and therefore h must be negative;

$$\text{also} \qquad p = \frac{x-h}{\sqrt{a^2-(x-h)^2}}, \quad \therefore y - \kappa = -\sqrt{a^2-(x-h)^2},$$

and the curve is circular but convex to AB. Thus the solution must be discontinuous, and may consist of two circular arcs as AP, BP. For both arcs $K=0$, and therefore if the subscripts 1, 2 apply to the abscissa of P considered as on the arcs AP, PB respectively, as A, B are fixed and y is invariable along DC, the terms at the limits reduce (Art. 368) to

$V_1dx_1 - V_2dx_2$, but $dx_1 = dx_2$, therefore $V_1 = V_2$ as dx_1 may change sign, therefore $\sqrt{1+p_1^2} = \sqrt{1+p_2^2}$, therefore it follows that $p_1 = -p_2$, so that the arcs are equally inclined at P to CD, and therefore their radii being each $= a$, P is the centre of CD, which determines the arcs completely, subject to each arc being throughout convex to AB. Considering AP this will be the case if h be negative and $\kappa > c$. From the equation to AP, $\kappa = \sqrt{a^2 - h^2}$, and $c - \kappa = \sqrt{a^2-(b-h)^2}$, therefore

$$c - \sqrt{a^2 - h^2} = \sqrt{a^2-(b-h)^2},$$

$$\therefore \qquad c^2 + b^2 - 2bh = 2c\sqrt{a^2-h^2},$$

$$\therefore \quad 4h^2(b^2+c^2) - 4bh(b^2+c^2) + b^2(b^2+c^2) = 4a^2c^2 + b^2(b^2+c^2) - (b^2+c^2)^2,$$

and h is negative if $2ac > b^2+c^2$; the other condition follows symmetrically; thus $a > \frac{b^2+c^2}{2c}$ and $> \frac{b^2+c^2}{2b}$, therefore if $b > c$, $a > \frac{b^2+c^2}{2c}$, which $> b$, therefore $a > b$; if $b < c$, $a > \frac{b^2+c^2}{2b}$ which then $> b$, therefore $a > b$ in any case, and therefore the ends of the cover fit the shorter sides of the dish. Supposing now that $b > c$, $\frac{b^2+c^2}{2c}$ is diminished by diminishing b, but increased by diminishing c, therefore if $a < \frac{b^2+c^2}{2c}$ but $> \frac{b^2+c^2}{2b}$, a discontinuous solution may be obtained by taking A at some point A' in AB and combining the circular arc $A'P$ (touching AB at A') with the straight line AA'. Similarly if $c > b$ and $a < \frac{b^2+c^2}{2b}$ but $> \frac{b^2+c^2}{2c}$, if P be moved vertically down to some point P', a solution may be obtained by combining the straight line PP' with the circular arc AP' (touching PP' at P'). There is no other case if $b > c$, because $a > b$, and therefore $a > \frac{b^2+c^2}{2b}$.

If $c > b$ and $a < \frac{b^2+c^2}{2c}$, a solution may be obtained by combining an arc $A'P'$ with straight lines PP' and AA'.

Changing the independent variable from x to y, if $\frac{dx}{dy} = q$, V becomes $V' = a\sqrt{1+q^2} + yq$, and as y does not vary, the terms of the second order in the variation of $V' = \frac{a(\delta q)^2}{2(1+q^2)^{\frac{3}{2}}}$, which is positive, and therefore the result is a minimum.

5. If the centre of the sphere be taken as pole, and θ be measured from that vertical radius which is above the horizontal great circle, the time varies as $\int_{\theta_0}^{\theta_1} \frac{d\theta}{\cos\theta}\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}$, and taking θ and ϕ as the independent and dependent variables, V is a function of θ and $\frac{d\phi}{d\theta}$ or p, therefore

$$N=0,\ \ P=\frac{dV}{dp},\ \text{ and } K=0=-\frac{dP}{d\theta},$$

$$\therefore \qquad P=c=\frac{\sin^2\theta\frac{d\phi}{d\theta}}{\cos\theta\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}},\ \therefore\ \frac{d\phi}{d\theta}=\frac{c\cos\theta}{\sin\theta\sqrt{\sin^2\theta-c^2\cos^2\theta}};$$

therefore if $\sin\theta=n\operatorname{cosec}\psi$,

$$\frac{d\phi}{d\psi}=\frac{-cn\operatorname{cosec}\psi\cot\psi}{n\operatorname{cosec}\psi\sqrt{(1+c^2)n^2\operatorname{cosec}^2\psi-c^2}}=\frac{-\cos\psi}{\sqrt{\frac{n^2}{c^2}(1+c^2)-\sin^2\psi}},$$

$$\therefore \qquad \phi+\alpha=\cos^{-1}\left(\frac{m}{n}\sin\psi\right)\text{ say, } \therefore\ m\operatorname{cosec}\theta=\cos(\phi+\alpha)$$

or $$m=\sin\theta\cos\overline{\phi+\alpha}=\cos\alpha\sin\theta\cos\phi-\sin\alpha\sin\theta\sin\phi,$$

or in rectangular co-ordinates if the axis of x be vertical $y\cos\alpha-z\sin\alpha=m$, or every point of the path lies in this plane, which is vertical. Of course m and α will be determined by the co-ordinates of the limiting points. As in Art. 386, approximating to the second powers of small variations,

$$\delta U=\pm\int_{\theta_0}^{\theta_1}\frac{1}{2}\,\frac{(\delta p)^2\sin^2\theta\,.\,d\theta}{\cos\theta\left\{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right\}^{\frac{3}{2}}};$$

if the man walk upwards $\theta_1<\theta_0$, but the time must be positive, therefore the lower sign of the ambiguity must be taken, and *vice versa*. Hence δU is essentially positive, and therefore the result gives the least time.

6. Taking the plane to be that of yz, the surface may be supposed given by $x^2=f(y, z)$; and, as in Art. 376, the condition for a line of minimum length is

$$\frac{d^2y}{ds^2}\div\frac{df}{dy}=\frac{d^2z}{ds^2}\div\frac{df}{dz}.$$

Here $$\frac{df}{dy}\,.\,\frac{dy}{ds}+\frac{df}{dz}\,.\,\frac{dz}{ds}=2x\,.\,\frac{dx}{ds}=0\text{ when } x=0;$$

also $\frac{dx}{ds}\,.\,\frac{d^2x}{ds^2}+\frac{dy}{ds}\,.\,\frac{d^2y}{ds^2}+\frac{dz}{ds}\,.\,\frac{d^2z}{ds^2}=0$, and when $x=0$, along the curve of intersection,

$$\frac{dx}{ds}=0,$$

$$\therefore \qquad \frac{dy}{ds}\,.\,\frac{d^2y}{ds^2}+\frac{dz}{ds}\,.\,\frac{d^2z}{ds^2}=0,$$

$\therefore$ in general $$\frac{d^2y}{ds^2}\div\frac{d^2z}{ds^2}=-\frac{dz}{ds}\div\frac{dy}{ds}=\frac{df}{dy}\div\frac{df}{dz},\ \therefore\text{ etc.}$$

7. $V=p^2\sin x+\dfrac{(y+x-\sin x)^2}{\sin x}$, and by choosing for the upper limit of x a value x_1 which makes neither p nor $y=\infty$, nor $\sin x=0$, it is clear that

$V=0$ if the lower limit approach in value to x_1, therefore it must be understood that the limits of x are fixed. Then the terms of the second order in the variation of V are $(\delta p)^2 \sin x + \dfrac{(\delta y)^2}{\sin x}$, and therefore to ensure a minimum $\sin x$ must be positive throughout the given integral U, and

$$\delta U = \delta y_1(2p\sin x)_1 - \delta y_0(2p\sin x)_0 + \int_{x_0}^{x_1}\left(\frac{dV}{dy} - \frac{d}{dx}\frac{dV}{dp}\right)\delta y\,dx.$$

Thus $$\frac{dV}{dy} = \frac{d}{dx}\left(\frac{dV}{dp}\right), \text{ which gives}$$

$$\frac{x+y-\sin x}{\sin x} = \frac{d}{dx}(p\sin x) = \frac{d^2y}{dx^2}\sin x + \cos x\frac{dy}{dx}, \text{ therefore}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx}\cot x - y\,\text{cosec}^2 x = x\,\text{cosec}^2 x - \frac{1}{\sin x}, \text{ therefore}$$

$$\frac{dy}{dx} + y\cot x = -x\cot x - \int dx\frac{1-\cos x}{\sin x} = -x\cot x - \int \frac{2dx}{2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$$

$$= -x\cot x + \log(1+\cos x) - c', \text{ therefore}$$

$$\frac{dy}{dx}\sin x + y\cos x = -x\cos x + \sin x\log(1+\cos x) - c'\sin x, \text{ therefore}$$

$$y\sin x = c - x\sin x - (1-c')\cos x - (1+\cos x)\log(1+\cos x) - \int \sin x\,dx$$
$$= c - x\sin x + c'\cos x - (1+\cos x)\log(1+\cos x),$$

$$\therefore \quad y = c\,\text{cosec}\,x - x + c'\cot x - (\text{cosec}\,x + \cot x)\log(1+\cos x),$$

$$\therefore \quad p = -c\,\text{cosec}\,x\cot x - 1 - c'\,\text{cosec}^2 x + 1$$
$$+ (\text{cosec}\,x\cot x + \text{cosec}^2 x)\log(1+\cos x)$$
$$= \text{cosec}^2 x\{-c\cos x - c' + (1+\cos x)\log(1+\cos x)\};$$

and the terms at the limits give $p_1 = 0 = p_0$, therefore c, c' are given by putting $x = x_0$ and x_1 successively in the equation

$$c' + c\cos x = (1+\cos x)\log(1+\cos x).$$

This is assuming, which seems most reasonable, that the limits of y are not given. Thus

$$x + y - \sin x = \text{cosec}\,x\{c + c'\cos x - \sin^2 x - (1+\cos x)\log(1+\cos x)\},$$

and $$V = \text{cosec}^3 x\{c' + c\cos x - (1+\cos x)\log(1+\cos x)\}^2$$
$$+ \text{cosec}^3 x\{c + c'\cos x - \sin^2 x - (1+\cos x)\log(1+\cos x)\}^2$$

$$= \text{cosec}^3 x\{(c^2+c'^2)(1+\cos^2 x) + 4cc'\cos x - 2\sin^2 x(c + c'\cos x)$$
$$+ \sin^4 x + 2\sin^2 x(1+\cos x)\log(1+\cos x)$$
$$- 2(c+c')(1+\cos x)^2\log(1+\cos x) + 2(1+\cos x)^2\log^2(1+\cos x)\},$$

$$\therefore \qquad U=\int_{x_0}^{x_1} dx\left\{(c^2+c'^2)\left(\frac{2\operatorname{cosec}^2 x}{\sin x}-\frac{1}{\sin x}\right)+\frac{4cc'\cos x}{\sin^3 x}-\frac{2c}{\sin x}-\frac{2c'\cos x}{\sin x}\right.$$

$$+\sin x+2\left(\frac{1}{\sin x}+\frac{\cos x}{\sin x}\right)\log(1+\cos x)-\frac{4(c+c')}{\sin^3 x}\log(1+\cos x)$$

$$-4(c+c')\frac{\cos x}{\sin^3 x}\log(1+\cos x)+\frac{2(c+c')}{\sin x}\log(1+\cos x)$$

$$\left.+2\left(\frac{2}{\sin^3 x}+\frac{2\cos x}{\sin^3 x}-\frac{1}{\sin x}\right)\log^2(1+\cos x)\right\},\text{ now}$$

$$\int\frac{\operatorname{cosec}^2 x\,dx}{\sin x}=-\cot x\operatorname{cosec} x-\int dx\cot x\,.\operatorname{cosec} x\cot x$$

$$=-\cot x\operatorname{cosec} x-\int dx(\operatorname{cosec}^3 x-\operatorname{cosec} x),$$

therefore $2\int\operatorname{cosec}^3 x\,dx-\int\frac{dx}{\sin x}=-\cot x\operatorname{cosec} x$, therefore the indefinite integral

$$U=-(c^2+c'^2)\cot x\operatorname{cosec} x-2cc'\operatorname{cosec}^2 x-2c\log\tan\frac{x}{2}-2c'\log\sin x$$

$$-\cos x+P+Q+R\text{ say},$$

where $\quad P=2\int dx\left(\frac{1}{\sin x}+\frac{\cos x}{\sin x}\right)\log(1+\cos x)$, or if $\cos x=z$

$$P=-2\int dz\frac{1+z}{1-z^2}\log(1+z),$$

and

$$Q=-2(c+c')\int dx\log(1+\cos x)\left(\frac{2}{\sin^3 x}+\frac{2\cos x}{\sin^3 x}-\frac{1}{\sin x}\right)$$

$$=2(c+c')\log(1+\cos x)\,.\,(\cot x\operatorname{cosec} x+\operatorname{cosec}^2 x)+2(c+c')\int\frac{dx}{\sin x}$$

$$=\frac{2(c+c')}{\sin^2 x}\log(1+\cos x)\,.\,(1+\cos x)+2(c+c')\log\tan\frac{x}{2};$$

and

$$R=2\int dx\log^2(1+\cos x)\left(\frac{2}{\sin^3 x}+\frac{2\cos x}{\sin^3 x}-\frac{1}{\sin x}\right)$$

$$=-2\log^2(1+\cos x)\,.\frac{1+\cos x}{\sin^2 x}-4\int\frac{dx\log(1+\cos x)}{\sin x}$$

$$=-2\frac{(1+\cos x)}{\sin^2 x}\log^2(1+\cos x)+4\int\frac{dz\log(1+z)}{1-z^2},\ \therefore\text{ the indefinite}$$

integral

$$U=-(c^2+c'^2)\frac{\cos x}{\sin^2 x}-\frac{2cc'}{\sin^2 x}-2c\log\tan\frac{x}{2}-2c'\log\sin x-\cos x$$

$$-2\int\frac{dz}{1-z}\log(1+z)+2(c+c')\frac{1+\cos x}{\sin^2 x}\log(1+\cos x)$$

$$+2(c+c')\log\tan\frac{x}{2}-\frac{2(1+\cos x)}{\sin^2 x}\log^2(1+\cos x)+4\int\frac{dz\log(1+z)}{1-z^2},$$

and the two integrals in this expression reduce to $2\int\frac{dz\log(1+z)}{1+z}$ which $=\log^2(1+z)$.

Hence putting $\cos x = z$, if z_1, z_2 be the corresponding limits of z, the minimum value of $U = \phi(z_1) - \phi(z_0)$, where

$$\phi(z) = -(c^2 + c'^2)\frac{z}{1-z^2} - \frac{2cc'}{1-z^2} - 2c'\log(1+z) - z + \log^2(1+z)$$

$$+ 2(c+c')\frac{1}{1-z}\log(1+z) - \frac{2}{1-z}\log^2(1+z)\ ;\ \text{where}$$

$c' + cz = (1+z)\log(1+z)$ when for z are put z_0 and z_1 in succession.

8. Here $V = p^2 + \frac{ay}{y_1}$, where a and y_1 are undetermined constants, and since V contains y_1, by Art. 372,

the coefficient of $\delta y_1 = 2p_1 + \int_0^1 dx\left(-\frac{ay}{y_1^2}\right) = 0$,

$$\therefore \quad 2p_1 = -\frac{a}{y_1}\ ;\quad \text{also } K = 0 = \frac{a}{y_1} - \frac{2dp}{dx},$$

$$\therefore \quad 2p = \frac{ax}{y_1} + b,\quad \therefore\quad -\frac{a}{y_1} = \frac{a}{y_1} + b \quad \text{or} \quad b = -\frac{2a}{y_1},$$

and $\therefore$ $\quad 2y = \frac{ax^2}{2y_1} - \frac{2ax}{y_1} + c,\quad \therefore\quad c = 2y_0 = 2,$ and therefore

$$\int_0^1 dx\left(\frac{ax^2}{4y_1} - \frac{ax}{y_1} + 1\right) = -y_1 = \frac{a}{12y_1} - \frac{a}{2y_1} + 1,\ \text{and}\ 2y_1 = \frac{a}{2y_1} - \frac{2a}{y_1} + 2,$$

$$\therefore \quad y_1 + 1 = \frac{5a}{12y_1} \quad\text{and}\quad y_1 - 1 = -\frac{3a}{4y_1},\quad \therefore\quad \frac{y_1+1}{y_1-1} = -\frac{5}{9},$$

and $\therefore$ $\quad y_1 = -\frac{2}{7}$ and $\frac{a}{y_1} = \frac{12}{7}$. Hence $p = \frac{6x}{7} - \frac{12}{7}$,

$\therefore$ the required value $= \frac{6^2}{7^2}\int_0^1 (x-2)^2 dx = \left(\frac{6}{7}\right)^2 \frac{2^3 - 1}{3} = \frac{12}{7}$.

Now to the second order of variations,

$$\delta V = \frac{1}{2}\left\{2(\delta p)^2 + \frac{2ay}{y_1^3}(\delta y_1)^2 - \frac{2a}{y_1^2}\delta y_1 \delta y\right\},$$

$$\therefore \quad \delta U = \int_0^1 \left\{(\delta p)^2 + \frac{ay}{y_1^3}(\delta y_1)^2 - \frac{a}{y_1^2}\delta y_1 \delta y\right\} dx,$$

and $\quad \int_0^1 \delta y\, dx = x\delta y\Big/_0^1 - \int_0^1 x\delta p\, dx = \delta y_1 - \int_0^1 x\delta p\, dx,$

$$\therefore \quad \delta U = \int_0^1 dx\left\{(\delta p)^2 + \frac{ay}{y_1^3}(\delta y_1)^2 + \frac{ax}{y_1^2}\delta p \delta y_1\right\} - \frac{a}{y_1^2}(\delta y_1)^2$$

$$= \int_0^1 dx\left\{\delta p + \frac{ax\delta y_1}{2y_1^2}\right\}^2 - \frac{a}{y_1^2}(\delta y_1)^2 - \frac{a^2 \delta y_1^2}{4y_1^4}\int_0^1 x^2 dx$$

$$= \int_0^1 dx\left\{\delta p + \frac{ax\delta y_1}{2y_1^2}\right\}^2 - \frac{a}{y_1^2}(\delta y_1)^2\left\{1 + \frac{a}{12y_1^2}\right\},$$

and $\quad -a\left(1 + \frac{a}{12y_1^2}\right) = \frac{24}{49}\left(1 - \frac{12}{7}\cdot\frac{1}{12}\cdot\frac{7}{2}\right),$

therefore δU is positive to the second order, and therefore the result obtained is a minimum.

9. With the notation of Arts. 352 and 353, no limits being specified, the variation of $\int V dx$ may be divided into two parts δU_1 and δU_2, the former arising supposing v constant, and the latter from the variation of v, and thus $\delta U_1 = H + \int K\delta y dx$, $\delta U_2 = \int \frac{dV}{dv}\delta v dx$,

and if N' represents for V' what N represents for V, and so on,

$$\delta v = \int (N'\delta y + P'\delta p + \ldots)dx, \text{ therefore if } L = \frac{dV}{dv}, \text{ and } I = \int L dx,$$

$$\begin{aligned}\delta U_2 &= \int L\delta v dx = I\delta v - \int I\frac{d\delta v}{dx}dx \\ &= I\left(H' + \int K'\delta y dx\right) - \int I(N'\delta y + P'\delta p + \ldots)dx \\ &= I\left(H' + \int K'\delta y dx\right) - \left(H_i' + \int K_i'\delta y dx\right),\end{aligned}$$

where H_i', K_i' denote what H', K' become when IN', IP', ... are substituted for N', P', ... Thus

$$\begin{aligned}\delta U_1 + \delta U_2 = \delta y\left(P - \frac{dQ}{dx} + \frac{d^2R}{dx^2} - \ldots\right) + \delta p\left(Q - \frac{dR}{dx} + \ldots\right) + \ldots \\ + \int\left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \ldots\right)\delta y dx + I\delta y\left(P' - \frac{dQ'}{dx} + \frac{d^2R'}{dx^2} - \ldots\right) \\ + I\delta p\left(Q' - \frac{dR'}{dx} + \ldots\right) + \ldots + I\int\left(N' - \frac{dP'}{dx} + \frac{d^2Q'}{dx^2} - \ldots\right)dx \\ - \delta y\left(IP' - \frac{dIQ'}{dx} + \frac{d^2IR'}{dx^2} - \ldots\right) - \delta p\left(IP' - \frac{dIQ'}{dx} + \ldots\right) - \ldots \\ - \int\left(IN' - \frac{dIP'}{dx} + \frac{d^2IQ'}{dx^2} - \ldots\right)dx.\end{aligned}$$

10. $$V = \phi(s) + c\sqrt{1+p^2},$$

$$\therefore \quad \delta U = \int_0^a \phi'(s)\delta s dx + c\int_0^a \frac{p\delta p}{\sqrt{1+p^2}}dx; \text{ and } \delta s = \int_0^x \frac{p\delta p}{\sqrt{1+p^2}}dx,$$

$$\therefore \quad \delta s = 0 \text{ when } x = 0, \text{ and if } \int \phi'(s)dx = I,$$

$$\delta U = c\int_0^a \frac{p\delta p}{\sqrt{1+p^2}}dx + I\delta s\Big|_0^a - \int_0^a dx\frac{Ip\delta p}{\sqrt{1+p^2}},$$

therefore if the limits of y as well as x be constants,

$$\begin{aligned}\delta U &= \left(\int_0^a \phi'(s)dx + c\right)\int_0^a \frac{p\delta p dx}{\sqrt{1+p^2}} - \int_0^a dx\frac{Ip\delta p}{\sqrt{1+p^2}} \\ &= -c_1\int_0^a dx\delta y\frac{d}{dx}\frac{p}{\sqrt{1+p^2}} + \int_0^a dx\delta y\frac{d}{dx}\frac{Ip}{\sqrt{1+v^2}} \text{ to the first order,}\end{aligned}$$

therefore for a maximum or minimum

$$c_1\frac{d}{dx}\frac{p}{\sqrt{1+p^2}}=\frac{d}{dx}\frac{Ip}{\sqrt{1+p^2}},\ \therefore\ (c_1-I)\frac{d}{dx}\frac{p}{\sqrt{1+p^2}}=\frac{p}{\sqrt{1+p^2}}\cdot\frac{dI}{dx},$$

$\therefore\ (c_1-I)\dfrac{p}{\sqrt{1+p^2}}=c'$, therefore by division

$$\frac{dp}{dx}\cdot\frac{1}{p(1+p^2)}=\frac{p}{c'\sqrt{1+p^2}}\cdot\frac{dI}{dx}\ \text{ or }\ c'\frac{dp}{dx}\cdot\frac{1}{p^2}=\sqrt{1+p^2}\cdot\phi'(s)=\phi'(s)\frac{ds}{dx},$$

$$\therefore\qquad \phi(s)=c''-\frac{c'}{p},\ \therefore\ \frac{dy}{dx}=\frac{c'}{c''-\phi(s)},$$

$$\therefore\qquad \frac{dy}{c'}=\frac{dx}{c''-\phi(s)}=\frac{ds}{\sqrt{c'^2+\{c''-\phi(s)\}^2}}.$$

Hence x and y can be obtained in terms of s, and s be then eliminated; and the four constants which will be involved are given by the facts that $s=0$ at the inferior limits of both x and y, and $s=$ the given value at their superior limits. So if $\phi(s)=s$,

$$\delta U=c\int_0^a\frac{p\delta p}{\sqrt{1+p^2}}dx+x\delta s\Big/_0^a-\int_0^a dx\frac{xp\delta p}{\sqrt{1+p^2}}$$

$$=\int_0^a dx\{c+a-x\}\frac{p\delta p}{\sqrt{1+p^2}}.$$

Hence the problem amounts to this: to find a curve of given length joining given points A, B, so that $\int(a-x)ds$ shall be a maximum or minimum, i.e., that the distance of its centre of gravity from the line $x=a$ shall be a maximum or minimum. This is known to be a catenary, having its directrix parallel to the axis of y. Further, if A be $(0, \kappa_1)$ and B be (a, κ_2), and if $\kappa_2>\kappa_1$ it is geometrically obvious that there will be a maximum or minimum as the arc of the catenary is concave or convex to $x=a$; but if $\kappa_1>\kappa_2$, these results will occur in reverse order. In the former case the longest catenary which can be drawn from B to A will touch the ordinate at A, and therefore, if the given length be greater than is consistent with this, the arc being supposed to be bounded by the ordinates at A and B, a discontinuous solution will be obtained, as in Ex. 3, by producing the ordinate at A. If the ordinate thus produced $=\kappa_2$, the arc will degenerate into two straight lines $=\kappa_2-\kappa_1$ and a: but if the given length be still too great, the ordinate at A may be still further produced to $(0, y_1)$, and the arc made up of the part of ordinate above $A=y_1-\kappa_1$ and an arc of a catenary from B to $(0, y_1)$ *convex* to the axis of x_1 and touching that of y. This may be neatly illustrated statically; vide *Researches*, Chap. XI.

11. The question here is supposed to mean that as at any proposed point of the required curve, the function $u=\{y+(m-x)p\}\{y+(n-x)p\}$ can

only vary by variation of p, the conditions that u is a maximum or minimum may be found by differentiating with regard to p as the only variable. Thus $\frac{du}{dp}=(m-x)\{y+(n-x)p\}+(n-x)\{y+(m-x)p\}$, $\therefore$ for a maximum or minimum $p=\frac{(2x-m-n)y}{2(m-x)(n-x)}$, $\therefore \frac{2p}{y}=\frac{2x-m-n}{(m-x)(n-x)}$,

$$\therefore \qquad y^2=c(m-x)(n-x)\text{; also } \frac{d^2u}{dp^2}=2(m-x)(n-x),$$

therefore u is a maximum or minimum according as $(m-x)(n-x)$ is negative or positive, and y^2 must be positive, therefore u is a maximum or minimum as $y^2=c(m-x)(n-x)$ represents an ellipse or hyperbola, and clearly the proposed point may be any point on one or other of these curves. By varying c the corresponding value of u will vary, but will be constant for the same value of c: for

$$\begin{aligned}4u&=4y^2-4(2x-m-n)py+4(m-x)(n-x)p^2\\&=4c(m-x)(n-x)-2c(2x-m-n)^2+(2x-m-n)^2c=-c(m-n)^2.\end{aligned}$$

12. Here $p\delta y+y\delta p=0$, and $\left(x-\frac{2y}{p}\right)dy+\frac{y^2}{p^2}\delta p=0$, therefore, eliminating δy and δp, $p(xp-2y)=y^2p$, therefore $xp=3y$, or $\frac{p}{y}=\frac{3}{x}$, and therefore $x^3=ay$: thus $y\frac{dy}{dx}=\frac{3x^5}{a^2}$, which $\propto$ with x.

13. If this point be not assumed, then, in Art. 363, it would be necessary to put $V=\frac{1}{\sqrt{y}}\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}^{\frac{1}{2}}$,

$$\therefore \qquad L=\frac{dz}{dx}\div\sqrt{y}\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}^{\frac{1}{2}}=0,\ \therefore\ \frac{dz}{dx}=0,$$

and therefore the curve lies in the vertical plane containing the two given points.

14. If the axis of revolution be axis of x, and extend from $x=0$ to $x=x_1$, then $\int_0^{x_1}2\pi y\sqrt{1+p^2}\,.\,dx$ is constant, and the moment of inertia is $\int_0^{x_1}dx\,.\,\pi y^2\,.\,\frac{y^2\mu}{2}$, and therefore if the solid be homogeneous V may be put $=y^4+ay\sqrt{1+p^2}$, and therefore $K=0$ gives

$$y^4+ay\sqrt{1+p^2}=\frac{ayp^2}{\sqrt{1+p^2}}+c \text{ or } y^4+\frac{ay}{\sqrt{1+p^2}}=c\text{; } \qquad \ldots\ldots\ldots\ldots(1)$$

also the limits of y being unknown,

$$P_0=0=P_1,\ \text{i.e.,}\ \frac{ay_0}{\sqrt{1+p_0^2}}=0=\frac{ay_1}{\sqrt{1+p_1^2}},$$

therefore $y_0=0$ and consequently $c=0$ by (1), unless $p_0=\infty$, which, except when $y_0=0$, would not correspond to a continuous solid of revolution. Thus the generating curve is terminated by the axis, and (1) gives

$$y^3+\frac{a}{\sqrt{1+p^2}}=0,$$

therefore a is negative as y is positive, and the normal

$$n=y\frac{ds}{dx}=y\sqrt{1+p^2}=-\frac{a}{y^2};$$

and $$3y^2=-\frac{ap}{(1+p^2)^{\frac{3}{2}}}\cdot\frac{dp}{dy}=-\frac{a}{(1+p^2)^{\frac{3}{2}}}\cdot\frac{dp}{dx}=-\frac{a}{\rho},$$

and $\therefore$ $$n=3\rho.$$

Here, changing the independent variable from x to y and putting q for $\frac{dx}{dy}$,

V becomes $$V'=y^4q+ay\sqrt{1+q^2},$$

therefore the terms of the second order under the integral sign become

$$\frac{1}{2}ay\frac{d^2\sqrt{1+q^2}}{dq^2}\cdot(\delta q)^2=\frac{ay}{2}\cdot\frac{(\delta q)^2}{(1+q^2)^{\frac{3}{2}}},$$

which is of the same sign as a, that is, negative; and therefore the curve obtained gives a maximum. In effect, the length of the axis is determined from the given area.

15. If the axes of x and y be those of figure and of revolution respectively, a minimum is required of $\int_{x_0}^{x_1}\left(\frac{y^2}{4}+x^2\right)y^2dx\div\int_{x_0}^{x_1}y^2xdx$, while $\int_{x_0}^{x_1}y^2dx$ is constant. Hence, if u, v denote the first two integrals respectively,

$$\frac{\delta u}{v}-\frac{u}{v^2}\delta v+a\,.\,\delta\int_{x_0}^{x_1}y^2dx=0,$$

$$\therefore\quad\int_{x_0}^{x_1}(y^3+2x^2y)\delta y dx-\frac{u}{v}\int_{x_0}^{x_1}2xy\delta y dx+av\int_{x_0}^{x_1}2y\delta y dx=0.$$

To this order of variation $\frac{u}{v}$ and v are constants, and therefore the required curve, if $\frac{u}{v}=l$ and $av=a'$, is $$y^3+2x^2y-2lxy+2a'y=0,$$
or, $y=0$ being inapplicable, $$y^2+2x^2-2lx+2a'=0,$$
which is an ellipse with its minor axis on the axis of x, so that the solid is a part of an oblate spheroid. This result is subject to the condition that $\frac{u}{v}=l$, when the value of y above obtained in terms of l and a' is substituted in u and v, and the integrations are effected. In the solution given in the *History* . . . extending over pages 391 to 397, this condition is proved

to hold; it is also shown that the ellipse meets the axis of x at the lowest point, and that the solution gives a minimum, but x_0 is assumed to vanish, i.e., the axis of revolution to be a diameter of the base of the solid segment.

16. If the axis of revolution be axis of x, the radii of the ends y_0, y_1 and the corresponding limits of x be 0 and x_1 the volume is $\int_0^{x_1} \pi y^2 dx$; and for the pressure at any point inside the vessel (see Besant's *Hydromechanics*, Part I.) $dp = \rho\omega^2 y dy$, therefore $p = \frac{\rho\omega^2 y^2}{2} +$ a constant, but when $y = 0$, $p = 0$, therefore $p = \frac{\rho\omega^2 y^2}{2}$, and therefore the pressure on the curved surface is

$$\int_0^{x_1} \frac{\rho\omega^2 y^2}{2} . 2\pi y \sqrt{1+p^2} dx,$$

and ρ, ω being constants, the pressures on the ends are constant. Hence if the independent variable be changed from x to y, and $q = \frac{dx}{dy}$, V may be written $ay^2 q + y^3 \sqrt{1+q^2}$, therefore $K = 0$ gives

$$\frac{d}{dy}\left\{ay^2 + \frac{y^3 q}{\sqrt{1+q^2}}\right\} = 0, \quad \therefore\ ay^2 + \frac{y^3 q}{\sqrt{1+q^2}} = c;$$

and since the limit x_1 of x is variable

$$\text{the coefficient of } \delta x_1 = 0 = ay_1^2 + \frac{y_1^3 q_1}{\sqrt{1+q_1^2}}, \quad \therefore\ c = 0,$$

therefore the generating curve is given by

$$a\sqrt{1+q^2} = -yq, \text{ or } \frac{dx}{dy} = \pm \frac{a}{\sqrt{y^2 - a^2}},$$

$$\therefore \qquad x + b = \pm a \log (y + \sqrt{y^2 - a^2}),$$

which gives two catenaries of which the axis of x is the directrix, but they only amount to the same thing here, as corresponding to any abscissa the ordinate in one is only *minus* that in the other; and then the conditions that the point $(0, y_0)$ lies on the curve, and that the volume of revolution is given, afford two equations for determining the two constants, and x_1 can be found from the value y_1 of y in the equation to the catenary.

Also, to the second order of variations,

$$\delta V = \frac{1}{2} \frac{y^3}{(1+q^2)^{\frac{3}{2}}} (\delta q)^2,$$

which is positive, and therefore the result corresponds to a minimum.

It may be observed that from the equation $a\sqrt{1+q^2} = -yq$, $\frac{dx}{dy}$ is of constant sign, and therefore the arc of the catenary lies entirely on one side of its axis.

17. If l be the length of the canal, the surface S, C the capacity, and N the normal hydrostatic pressure,

$$S = l\int_{x_0}^{x_1}\sqrt{1+p^2}\,dx,$$

if x, y be horizontal and vertical co-ordinates of any point in the boundary of a transverse section, $C = l\int_{x_0}^{x_1} y\,dx$, and $N = l\int_{x_0}^{x_1} g\rho y\sqrt{1+p^2}\,dx$.

Hence $\quad V = a\sqrt{1+p^2} + by + y\sqrt{1+p^2}$, and $V = Pp + c$,

$$\therefore \quad a\sqrt{1+p^2} + by + y\sqrt{1+p^2} = \frac{ap^2}{\sqrt{1+p^2}} + \frac{yp^2}{\sqrt{1+p^2}} + c \text{ or } \frac{a+y}{\sqrt{1+p^2}} + by = c.$$

Hence

(1) the curve is $\dfrac{a}{\sqrt{1+p^2}} + by = c$, which represents a circle (Art. 379);

(2) the curve is $\dfrac{a+y}{\sqrt{1+p^2}} = c$, a catenary (Art. 380);

(3) the curve is $by + \dfrac{y}{\sqrt{1+p^2}} = c$, but y is supposed zero at the limits of x, therefore $c = 0$ here, and either $y = 0$, which is inapplicable, or

$$1 + p^2 = \frac{1}{b^2}, \quad \therefore \frac{dy}{dx} = \pm\kappa,$$

which represents two straight lines equally inclined to the vertical, in opposite senses.

18. If the axis of y be measured vertically downwards and the axis of x so chosen that the velocity $= \sqrt{2gy}$, the time of describing the curve between the limits x_0 and x_1 of x is $\displaystyle\int_{x_0}^{x_1}\frac{\sqrt{1+p^2}}{\sqrt{2gy}}\,dx$, therefore V may be written $\dfrac{\sqrt{1+p^2}}{y^{\frac{1}{2}}}$, therefore the limits being constant, the variation of the time is $\int \delta V dx$, where

$$\delta V = \left(N - \frac{dP}{dx}\right)\delta y = \left\{-\frac{\sqrt{1+p^2}}{2y^{\frac{3}{2}}} - \frac{d}{dx}\frac{p}{\sqrt{(1+p^2)y}}\right\}\delta y$$

$$= -\delta y\left\{\frac{\sqrt{1+p^2}}{2y^{\frac{3}{2}}} - \frac{p^2}{\sqrt{1+p^2}\,.\,2y^{\frac{3}{2}}} + \frac{q}{\sqrt{y}(1+p^2)^{\frac{3}{2}}}\right\}$$

$$= -\delta y\left\{\frac{1}{2y^{\frac{3}{2}}\sqrt{1+p^2}} + \frac{q}{y^{\frac{1}{2}}(1+p^2)^{\frac{3}{2}}}\right\},$$

but y is positive, and so is q since the curve is convex to the axis of x, and in passing to a lower curve δy is positive therefore δV is negative, and therefore the time is diminished. It follows that the time along any lower curve is less than along an upper curve with the same extremities. Cf. Art. 363.

19. If the wind blow parallel to the axis of x, the rate of sailing may be expressed as a function of p^2, where $p=\frac{dy}{dx}$, since the rate will be the same for the same numerical value of p, whether the wind be on the starboard or larboard quarter or bow, as the case may be. Hence $V=\frac{\sqrt{1+p^2}}{f(p^2)}$, and the variation in the time for a slight change of course, supposing the direction and force of the wind constant throughout, is $\int\frac{dV}{dp}\delta p dx$ taken between proper limits. If the limits be fixed, this gives

$$\frac{d}{dx}\frac{dV}{dp}=0, \text{ and } \therefore \frac{dV}{dp}=c, \text{ or } p=c, \text{ subject to } \frac{d^2V}{dp^2} \text{ being positive,}$$

i.e., the direct course. But if it be supposed that the course may be discontinuous, and therefore divisible into parts in which some at least of the limits are not determined, then the variation in the time is

$$\int\left\{\frac{dV}{dp}\delta p+\frac{1}{2}\frac{d^2V}{dp^2}(\delta p)^2\right\}dx \text{ to the second order,}$$

and therefore for a minimum, as before $\frac{d^2V}{dp^2}$ is positive, but $\frac{dV}{dp}=0$, which gives $p=\pm m$ suppose, for both of which values $\frac{d^2V}{dp^2}$ is the same throughout. This shows that the course may be made up of two straight lines, equally inclined to the direction of the wind. There may be more than one admissible real solution of $\frac{dV}{dp}=0$, and of the corresponding courses and the direct course, that is required which is the least. With such a solution as $p=\pm m$, a straight line must be drawn through one of the given positions corresponding to $p=\pm m$, and another through the other position corresponding to $p=\mp m$: these two straight lines will meet in the point where the tack must be made. It will be seen from a figure that tacking is only of use when the wind is adverse. If the wind be favourable $f(p^2)$ increases as p diminishes, so that the quickest course is the direct one.

20. If the axis of revolution be axis of x, the vertex be at the origin, and b = the radius of the base, the volume $=\int_0^{x_1}\pi y^2dx$, and the resistance, as in Art. 366, $=\int_0^{x_1}\frac{yp^3}{1+p^2}dx$, where x_1 is unknown. The limits of y are 0 and b, and therefore changing the independent variable from x to y, if $\frac{dx}{dy}=q$, V may be put $=2ay^2q+\frac{y}{1+q^2}$, therefore, to the second order inclusive,

$$\delta V=\left\{2ay^2-\frac{2yq}{(1+q^2)^2}\right\}\delta q-\frac{1}{2}(\delta q)^2 \,.\, 2y\frac{1+q^2-4q^2}{(1+q^2)^3}.$$

Hence there will be a minimum when $ay^2 = \frac{yq}{(1+q^2)^2}$, if $3q^2 > 1$ throughout, or when $ay = \frac{p^3}{(1+p^2)^2}$ and $p < \sqrt{3}$, and when $y = 0$, $p = 0$ or ∞, therefore the generating curve must touch the axis of x at the vertex, and be convex to that axis throughout, the inclination thereto of the tangent being $< \frac{\pi}{3}$ at (x_1, b). If now $p = \tan\phi$, $ay = \sin^3\phi\cos\phi$,

$$\therefore \qquad a\frac{dy}{d\phi} = \sin^2\phi(3\cos^2\phi - \sin^2\phi),$$

$$\therefore \qquad a\frac{dx}{d\phi} = \sin\phi\cos\phi(3\cos^2\phi - \sin^2\phi),$$

$\therefore$ if s = arc of the curve, $a\frac{ds}{d\phi} = \sin\phi(3\cos^2\phi - \sin^2\phi) = \sin 3\phi$,

$$\therefore \qquad as = -\tfrac{1}{3}\cos^3\phi + \text{constant},$$

or if $s = 0$ when $\phi = 0$, $\quad 3as = 1 - \cos 3\phi$.

This represents (Art. 111) a hypocycloid, in which the radii of the fixed and moving circles are as 3 : 1. If ϕ_1 be the value of ϕ when $y = b$, the constants a and ϕ_1 are given by the equations

$$ab = \sin^3\phi_1\cos\phi_1$$

and the given volume $= \int_0^{x_1} \pi y^2 dx = \int_0^{\phi_1} \frac{\pi}{a^2}\sin^6\phi\cos^2\phi \,.\, \cos\phi\sin 3\phi \,.\, \frac{d\phi}{a}$

$$= \frac{\pi b^3}{\sin^9\phi_1\cos^3\phi_1}\int_0^{\phi_1}\cos\phi d\phi\{3\sin^7\phi - 3\sin^9\phi - 4\sin^9\phi + 4\sin^{11}\phi\}$$

$$= \frac{\pi b^3}{\sin\phi_1\cos^3\phi_1}\left\{\frac{3}{8} - \frac{7}{10}\sin^2\phi_1 + \frac{1}{3}\sin^4\phi_1\right\}.$$

It may be shown that this has a minimum value $\frac{\pi b^3\sqrt{3}}{5}$, and therefore if this minimum be $>$ than the given volume, the solution is incorrect. As, however, by diminishing b this expression can be made as small as required, in this case the generating curve may be supposed to be made up of the arc of a hypocycloid as before from $y = 0$ to $y = b'$, where b' is sufficiently small, together with a straight line $= b - b'$ from (x_1, b') to (x_1, b). It is further shown in the *Researches*, Chap. X., that the discontinuous solution only holds so long as the volume $< \frac{13}{30}\pi b^3$, and that between the two limiting values of the volume the discontinuous solution gives a less resistance than the continuous solution.