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THE MONIST.

THE ARCHITECTURE OF THEORIES.

OF the fifty or hundred systems of philosophy that have been advanced at different times of the world's history, perhaps the larger number have been, not so much results of historical evolution, as happy thoughts which have accidently occurred to their authors. An idea which has been found interesting and fruitful has been adopted, developed, and forced to yield explanations of all sorts of phenomena. The English have been particularly given to this way of philosophising; witness, Hobbes, Hartley, Berkeley, James Mill. Nor has it been by any means useless labor; it shows us what the true nature and value of the ideas developed are, and in that way affords serviceable materials for philosophy. Just as if a man, being seized with the conviction that paper was a good material to make things of, were to go to work to build a *papier mâché* house, with roof of roofing-paper, foundations of pasteboard, windows of paraffined paper, chimneys, bath tubs, locks, etc., all of different forms of paper, his experiment would probably afford valuable lessons to builders, while it would certainly make a detestable house, so those one-idea'd philosophies are exceedingly interesting and instructive, and yet are quite unsound.

The remaining systems of philosophy have been of the nature of reforms, sometimes amounting to radical revolutions, suggested by certain difficulties which have been found to beset systems previously in vogue; and such ought certainly to be in large part the motive of any new theory. This is like partially rebuilding a house.

The faults that have been committed are, first, that the dilapidations have generally not been sufficiently thoroughgoing, and second, that not sufficient pains has been taken to bring the additions into deep harmony with the really sound parts of the old structure.

When a man is about to build a house, what a power of thinking he has to do, before he can safely break ground! With what pains he has to excogitate the precise wants that are to be supplied! What a study to ascertain the most available and suitable materials, to determine the mode of construction to which those materials are best adapted, and to answer a hundred such questions! Now without riding the metaphor too far, I think we may safely say that the studies preliminary to the construction of a great theory should be at least as deliberate and thorough as those that are preliminary to the building of a dwelling-house.

That systems ought to be constructed architectonically has been preached since Kant, but I do not think the full import of the maxim has by any means been apprehended. What I would recommend is that every person who wishes to form an opinion concerning fundamental problems, should first of all make a complete survey of human knowledge, should take note of all the valuable ideas in each branch of science, should observe in just what respect each has been successful and where it has failed, in order that in the light of the thorough acquaintance so attained of the available materials for a philosophical theory and of the nature and strength of each, he may proceed to the study of what the problem of philosophy consists in, and of the proper way of solving it. I must not be understood as endeavoring to state fully all that these preparatory studies should embrace; on the contrary, I purposely slur over many points, in order to give emphasis to one special recommendation, namely, to make a systematic study of the conceptions out of which a philosophical theory may be built, in order to ascertain what place each conception may fitly occupy in such a theory, and to what uses it is adapted.

The adequate treatment of this single point would fill a volume, but I shall endeavor to illustrate my meaning by glancing at several sciences and indicating conceptions in them serviceable for philosophy. As to the results to which long studies thus commenced have led me, I shall just give a hint at their nature.

We may begin with dynamics,—field in our day of perhaps the grandest conquest human science has ever made,—I mean the law of the conservation of energy. But let us revert to the first step taken by modern scientific thought,—and a great stride it was,—the inauguration of dynamics by Galileo. A modern physicist on examining Galileo's works is surprised to find how little experiment had to do with the establishment of the foundations of mechanics. His principal appeal is to common sense and *il lume naturale*. He always assumes that the true theory will be found to be a simple and natural one. And we can see why it should indeed be so in dynamics. For instance, a body left to its own inertia, moves in a straight line, and a straight line appears to us the simplest of curves. In *itself*, no curve is simpler than another. A system of straight lines has intersections precisely corresponding to those of a system of like parabolas similarly placed, or to those of any one of an infinity of systems of curves. But the straight line appears to us simple, because, as Euclid says, it lies evenly between its extremities; that is, because viewed endwise it appears as a point. That is, again, because light moves in straight lines. Now, light moves in straight lines because of the part which the straight line plays in the laws of dynamics. Thus it is that our minds having been formed under the influence of phenomena governed by the laws of mechanics, certain conceptions entering into those laws become implanted in our minds, so that we readily guess at what the laws are. Without such a natural prompting, having to search blindfold for a law which would suit the phenomena, our chance of finding it would be as one to infinity. The further physical studies depart from phenomena which have directly influenced the growth of the mind, the less we can expect to find the laws which govern them "simple," that is, composed of a few conceptions natural to our minds.

The researches of Galileo, followed up by Huygens and others, led to those modern conceptions of *Force* and *Law*, which have revolutionised the intellectual world. The great attention given to mechanics in the seventeenth century soon so emphasised these conceptions as to give rise to the Mechanical Philosophy, or doctrine that all the phenomena of the physical universe are to be explained upon mechanical principles. Newton's great discovery imparted a new impetus to this tendency. The old notion that heat consists in an agitation of corpuscles was now applied to the explanation of the chief properties of gases. The first suggestion in this direction was that the pressure of gases is explained by the battering of the particles against the walls of the containing vessel, which explained Boyle's law of the compressibility of air. Later, the expansion of gases, Avogadro's chemical law, the diffusion and viscosity of gases, and the action of Crookes's radiometer were shown to be consequences of the same kinetical theory; but other phenomena, such as the ratio of the specific heat at constant volume to that at constant pressure require additional hypotheses, which we have little reason to suppose are simple, so that we find ourselves quite afloat. In like manner with regard to light, that it consists of vibrations was almost proved by the phenomena of diffraction, while those of polarisation showed the excursions of the particles to be perpendicular to the line of propagation; but the phenomena of dispersion, etc., require additional hypotheses which may be very complicated. Thus, the further progress of molecular speculation appears quite uncertain. If hypotheses are to be tried haphazard, or simply because they will suit certain phenomena, it will occupy the mathematical physicists of the world say half a century on the average to bring each theory to the test, and since the number of possible theories may go up into the trillions, only one of which can be true, we have little prospect of making further solid additions to the subject in our time. When we come to atoms, the presumption in favor of a simple law seems very slender. There is room for serious doubt whether the fundamental laws of mechanics hold good for single atoms, and it seems quite likely that they are capable of motion in more than three dimensions.

To find out much more about molecules and atoms, we must search out a natural history of laws of nature, which may fulfil that function which the presumption in favor of simple laws fulfilled in the early days of dynamics, by showing us what kind of laws we have to expect and by answering such questions as this: Can we with reasonable prospect of not wasting time, try the supposition that atoms attract one another inversely as the seventh power of their distances, or can we not? To suppose universal laws of nature capable of being apprehended by the mind and yet having no reason for their special forms, but standing inexplicable and irrational, is hardly a justifiable position. Uniformities are precisely the sort of facts that need to be accounted for. That a pitched coin should sometimes turn up heads and sometimes tails calls for no particular explanation; but if it shows heads every time, we wish to know how this result has been brought about. Law is *par excellence* the thing that wants a reason.

Now the only possible way of accounting for the laws of nature and for uniformity in general is to suppose them results of evolution. This supposes them not to be absolute, not to be obeyed precisely. It makes an element of indeterminacy, spontaneity, or absolute chance in nature. Just as, when we attempt to verify any physical law, we find our observations cannot be precisely satisfied by it, and rightly attribute the discrepancy to errors of observation, so we must suppose far more minute discrepancies to exist owing to the imperfect cogency of the law itself, to a certain swerving of the facts from any definite formula.

Mr. Herbert Spencer wishes to explain evolution upon mechanical principles. This is illogical, for four reasons. First, because the principle of evolution requires no extraneous cause; since the tendency to growth can be supposed itself to have grown from an infinitesimal germ accidentally started. Second, because law ought more than anything else to be supposed a result of evolution. Third, because exact law obviously never can produce heterogeneity out of homogeneity; and arbitrary heterogeneity is the feature of the universe the most manifest and characteristic. Fourth, because the law of the conservation of energy is equivalent to the proposition that all operations governed by mechanical laws are reversible; so that an immediate corollary from it is that growth is not explicable by those laws, even if they be not violated in the process of growth.

In short, Spencer is not a philosophical evolutionist, but only a half-evolutionist,—or, if you will, only a semi-Spencerian. Now philosophy requires thoroughgoing evolutionism or none.

The theory of Darwin was that evolution had been brought about by the action of two factors: first, heredity, as a principle making offspring nearly resemble their parents, while yet giving room for "sporting," or accidental variations,—for very slight variations often, for wider ones rarely; and, second, the destruction of breeds or races that are unable to keep the birth rate up to the death rate. This Darwinian principle is plainly capable of great generalisation. Wherever there are large numbers of objects, having a tendency to retain certain characters unaltered, this tendency, however, not being absolute but giving room for chance variations, then, if the amount of variation is absolutely limited in certain directions by the destruction of everything which reaches those limits, there will be a gradual tendency to change in directions of departure from them. Thus, if a million players sit down to bet at an even game, since one after another will get ruined, the average wealth of those who remain will perpetually increase. Here is indubitably a genuine formula of possible evolution, whether its operation accounts for much or little in the development of animal and vegetable species.

The Lamarckian theory also supposes that the development of species has taken place by a long series of insensible changes, but it supposes that those changes have taken place during the lives of the individuals, in consequence of effort and exercise, and that reproduction plays no part in the process except in preserving these modifications. Thus, the Lamarckian theory only explains the development of characters for which individuals strive, while the Darwinian theory only explains the production of characters really beneficial to the race, though these may be fatal to individuals.* But more broadly and philosophically conceived, Darwinian evolution is evolution by the operation of chance, and the destruction of

* The neo-Darwinian, Weismann, has shown that mortality would almost necessarily result from the action of the Darwinian principle.

bad results, while Lamarckian evolution is evolution by the effect of habit and effort.

A third theory of evolution is that of Mr. Clarence King. The testimony of monuments and of rocks is that species are unmodified or scarcely modified, under ordinary circumstances, but are rapidly altered after cataclysms or rapid geological changes. Under novel circumstances, we often see animals and plants sporting excessively in reproduction, and sometimes even undergoing transformations during individual life, phenomena no doubt due partly to the enfeeblement of vitality from the breaking up of habitual modes of life, partly to changed food, partly to direct specific influence of the element in which the organism is immersed. If evolution has been brought about in this way, not only have its single steps not been insensible, as both Darwinians and Lamarckians suppose, but they are furthermore neither haphazard on the one hand, nor yet determined by an inward striving on the other, but on the contrary are effects of the changed environment, and have a positive general tendency to adapt the organism to that environment, since variation will particularly affect organs at once enfeebled and stimulated. This mode of evolution, by external forces and the breaking up of habits, seems to be called for by some of the broadest and most important facts of biology and paleontology; while it certainly has been the chief factor in the historical evolution of institutions as in that of ideas; and cannot possibly be refused a very prominent place in the process of evolution of the universe in general.

Passing to psychology, we find the elementary phenomena of mind fall into three categories. First, we have Feelings, comprising all that is immediately present, such as pain, blue, cheerfulness, the feeling that arises when we contemplate a consistent theory, etc. A feeling is a state of mind having its own living quality, independent of any other state of mind. Or, a feeling is an element of consciousness which might conceivably override every other state until it monopolised the mind, although such a rudimentary state cannot actually be realised, and would not properly be consciousness. Still, it is conceivable, or supposable, that the quality of blue should usurp the whole mind, to the exclusion of the ideas of shape, extension, contrast, commencement and cessation, and all other ideas, whatsoever. A feeling is necessarily perfectly simple, *in itself*, for if it had parts these would also be in the mind, whenever the whole was present, and thus the whole could not monopolise the mind.*

Besides Feelings, we have Sensations of reaction; as when a person blindfold suddenly runs against a post, when we make a muscular effort, or when any feeling gives way to a new feeling. Suppose I had nothing in my mind but a feeling of blue, which were suddenly to give place to a feeling of red; then, at the instant of transition there would be a shock, a sense of reaction, my blue life being transmuted into red life. If I were further endowed with a memory, that sense would continue for some time, and there would also be a peculiar feeling or sentiment connected with it. This last feeling might endure (conceivably I mean) after the memory of the occurrence and the feelings of blue and red had passed away. But the *sensation* of reaction cannot exist except in the actual presence of the two feelings blue and red to which it relates. Wherever we have two feelings and pay attention to a relation between them of whatever kind, there is the sensation of which I am speaking. But the sense of action and reaction has two types: it may either be a perception of relation between two ideas, or it may be a sense of action and reaction between feeling and something out of feeling. And this sense of external reaction again has two forms; for it is either a sense of something happening to us, by no act of ours, we being passive in the matter, or it is a sense of resistance, that is, of our expending feeling upon something without. The sense of reaction is thus a sense of connection or comparison between feelings, either, *A*, between one feeling and another, or *B*, between feeling and its absence or lower degree; and under *B* we have, First, the sense of the access of feeling, and Second, the sense of remission of feeling.

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* A feeling may certainly be compound, but only in virtue of a perception which is not that feeling nor any feeling at all.

Very different both from feelings and from reaction-sensations or disturbances of feeling are general conceptions. When we think, we are conscious that a connection between feelings is determined by a general rule, we are aware of being governed by a habit. Intellectual power is nothing but facility in taking habits and in following them in cases essentially analogous to, but in non-essentials widely remote from, the normal cases of connections of feelings under which those habits were formed.

The one primary and fundamental law of mental action consists in a tendency to generalisation. Feeling tends to spread; connections between feelings awaken feelings; neighboring feelings become assimilated; ideas are apt to reproduce themselves. These are so many formulations of the one law of the growth of mind. When a disturbance of feeling takes place, we have a consciousness of gain, the gain of experience; and a new disturbance will be apt to assimilate itself to the one that preceded it. Feelings, by being excited, become more easily excited, especially in the ways in which they have previously been excited. The consciousness of such a habit constitutes a general conception.

The cloudiness of psychological notions may be corrected by connecting them with physiological conceptions. Feeling may be supposed to exist, wherever a nerve-cell is in an excited condition. The disturbance of feeling, or sense of reaction, accompanies the transmission of disturbance between nerve-cells or from a nerve-cell to a muscle-cell or the external stimulation of a nerve-cell. General conceptions arise upon the formation of habits in the nerve-matter, which are molecular changes consequent upon its activity and probably connected with its nutrition.

The law of habit exhibits a striking contrast to all physical laws in the character of its commands. A physical law is absolute. What it requires is an exact relation. Thus, a physical force introduces into a motion a component motion to be combined with the rest by the parallelogram of forces; but the component motion must actually take place exactly as required by the law of force. On the other hand, no exact conformity is required by the mental law. Nay, exact conformity would be in downright conflict with the law; since it would instantly crystallise thought and prevent all further formation of habit. The law of mind only makes a given feeling *more likely* to arise. It thus resembles the "non-conservative" forces of physics, such as viscosity and the like, which are due to statistical uniformities in the chance encounters of trillions of molecules.

The old dualistic notion of mind and matter, so prominent in Cartesianism, as two radically different kinds of substance, will hardly find defenders to-day. Rejecting this, we are driven to some form of hylopathy, otherwise called monism. Then the question arises whether physical laws on the one hand, and the psychical law on the other are to be taken—

(*A*) as independent, a doctrine often called *monism*, but which I would name *neutralism;* or,

(*B*) the psychical law as derived and special, the physical law alone as primordial, which is *materialism;* or,

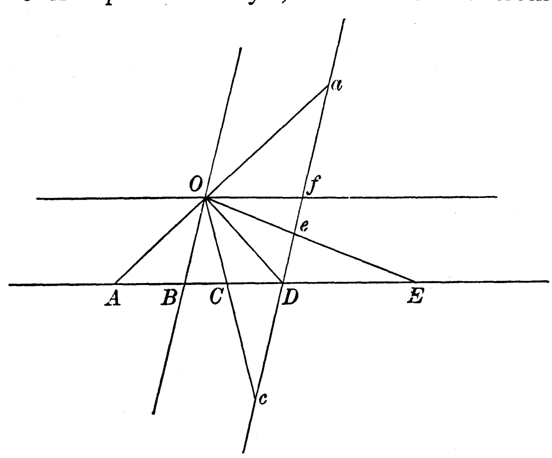
(*C*) the physical law as derived and special, the psychical law alone as primordial, which is *idealism*.

The materialistic doctrine seems to me quite as repugnant to scientific logic as to common sense; since it requires us to suppose that a certain kind of mechanism will feel, which would be a hypothesis absolutely irreducible to reason,—an ultimate, inexplicable regularity; while the only possible justification of any theory is that it should make things clear and reasonable.

Neutralism is sufficiently condemned by the logical maxim known as Ockham's razor, i. e., that not more independent elements are to be supposed than necessary. By placing the inward and outward aspects of substance on a par, it seems to render both primordial.

The one intelligible theory of the universe is that of objective idealism, that matter is effete mind, inveterate habits becoming physical laws. But before this can be accepted it must show itself capable of explaining the tridimensionality of space, the laws of motion, and the general characteristics of the universe, with mathematical clearness and precision; for no less should be demanded of every Philosophy.

Modern mathematics is replete with ideas which may be applied to philosophy. I can only notice one or two. The manner in which mathematicians generalise is very instructive. Thus, painters are accustomed to think of a picture as consisting geometrically of the intersections of its plane by rays of light from the natural objects to the eye. But geometers use a generalised perspective. For instance, in the figure let *O* be the eye, let *A B C D E* be the edgewise view of any plane, and let *a f e D c* be the edgewise view of another plane. The geometers draw rays through *O* cutting both these planes, and treat the points of intersection of each ray with one plane as representing the point of intersection of the same ray with the other plane. Thus, *e* represents *E*, in the painter's way. *D* represents itself. *C* is represented by *c*, which is further from the eye; and *A* is represented by *a* which is on the other side of the eye. Such generalisation is not bound down to sensuous images. Further, according to this mode of representation every point on one plane represents a point on the other, and every point on the latter is represented by a point on the former. But how about the point *f* which is in a direction from *O* parallel to the represented plane, and how about the point *B* which is in a direction parallel to the representing plane? Some will say that these are exceptions; but modern mathematics does not allow exceptions which can be annulled by generalisation. As a point moves from *C* to *D* and thence to *E* and off toward infinity, the corresponding point on the other plane moves from *c* to *D* and thence to *e* and toward *f*. But this second point can pass through *f* to *a*; and when it is there the first point has arrived at *A*. We therefore say that the first point has passed *through infinity*, and that every line joins in to itself somewhat like an oval. Geometers



talk of the parts of lines at an infinite distance as points. This is a kind of generalisation very efficient in mathematics.

Modern views of measurement have a philosophical aspect. There is an indefinite number of systems of measuring along a line; thus, a perspective representation of a scale on one line may be taken to measure another, although of course such measurements will not agree with what we call the distances of points on the latter line. To establish a system of measurement on a line we must assign a distinct number to each point of it, and for this purpose we shall plainly have to suppose the numbers carried out into an infinite number of places of decimals. These numbers must be ranged along the line in unbroken sequence. Further, in order that such a scale of numbers should be of any use, it must be capable of being shifted into new positions, each number continuing to be attached to a single distinct point. Now it is found that if this is true for "imaginary" as well as for real points (an expression which I cannot stop to elucidate), any such shifting will necessarily leave two numbers attached to the same points as before. So that when the scale is moved over the line by any continuous series of shiftings of one kind, there are two points which no numbers on the scale can ever reach, except the numbers fixed there. This pair of points, thus unattainable in measurement, is called the Absolute. These two points may be distinct and real, or they may coincide, or they may be both imaginary. As an example of a linear quantity with a double absolute we may take probability, which ranges from an unattainable absolute certainty *against* a proposition to an equally unattainable absolute certainty *for* it. A line, according to ordinary notions, we have seen is a linear quantity where the two points at infinity coincide. A velocity is another example. A train going with infinite velocity from Chicago to New York would be at all the points on the line at the very same instant, and if the time of transit were reduced to less than nothing it would be moving in the other direction. An angle is a familiar example of a mode of magnitude with no real immeasurable values. One of the questions philosophy has to consider is whether the development of the universe is like the increase of an angle, so that it proceeds forever without tending toward anything unattained, which I take to be the Epicurean view, or whether the universe sprang from a chaos in the infinitely distant past to tend toward something different in the infinitely distant future, or whether the universe sprang from nothing in the past to go on indefinitely toward a point in the infinitely distant future, which, were it attained, would be the mere nothing from which it set out.

The doctrine of the absolute applied to space comes to this, that either—

First, space is, as Euclid teaches, both *unlimited* and *immeasurable*, so that the infinitely distant parts of any plane seen in perspective appear as a straight line, in which case the sum of the three angles of a triangle amounts to 180°; or,

Second, space is *immeasurable* but *limited*, so that the infinitely distant parts of any plane seen in perspective appear as a circle, beyond which all is blackness, and in this case the sum of the three angles of a triangle is less than 180° by an amount proportional to the area of the triangle; or,

Third, space is *unlimited* but *finite*, (like the surface of a sphere,) so that it has no infinitely distant parts; but a finite journey along any straight line would bring one back to his original position, and looking off with an unobstructed view one would see the back of his own head enormously magnified, in which case the sum of the three angles of a triangle exceeds 180° by an amount proportional to the area.

Which of these three hypotheses is true we know not. The largest triangles we can measure are such as have the earth's orbit for base, and the distance of a fixed star for altitude. The angular magnitude resulting from subtracting the sum of the two angles at the base of such a triangle from 180° is called the star's *parallax*. The parallaxes of only about forty stars have been measured as yet. Two of them come out negative, that of Arided (α Cycni), a star of magnitude 1½, which is $-0.''082$, according to C. A. F. Peters, and that of a star of magnitude 7¾, known as Piazzi III 422, which is $-0.''045$ according to R. S. Ball. But these negative parallaxes are undoubtedly to be attributed to errors of observation; for the probable error of such a determination is about $\pm 0.''075$, and it would be strange indeed if we were to be able to see, as it were, more than half way round space, without being able to see stars with larger negative parallaxes. Indeed, the very fact that of all the parallaxes measured only two come out negative would be a strong argument that the smallest parallaxes really amount to $+0.''1$, were it not for the reflexion that the publication of other negative parallaxes may have been suppressed. I think we may feel confident that the parallax of the furthest star lies somewhere between $-0.''05$ and $+0.''15$, and within another century our grandchildren will surely know whether the three angles of a triangle are greater or less than 180°,—that they are *exactly* that amount is what nobody ever can be justified in concluding. It is true that according to the axioms of geometry the sum of the three sides of a triangle are precisely 180°; but these axioms are now exploded, and geometers confess that they, as geometers, know not the slightest reason for supposing them to be precisely true. They are expressions of our inborn conception of space, and as such are entitled to credit, so far as their truth could have influenced the formation of the mind. But that affords not the slightest reason for supposing them exact.

Now, metaphysics has always been the ape of mathematics. Geometry suggested the idea of a demonstrative system of absolutely certain philosophical principles; and the ideas of the metaphysicians have at all times been in large part drawn from mathematics. The metaphysical axioms are imitations of the geometrical axioms; and now that the latter have been thrown overboard, without doubt the former will be sent after them. It is evident, for instance, that we can have no reason to think that every phenomenon in all its minutest details is precisely determined by law. That there is an arbitrary element in the universe we see,—namely, its variety. This variety must be attributed to spontaneity in some form.

Had I more space, I now ought to show how important for philosophy is the mathematical conception of continuity. Most of what is true in Hegel is a darkling glimmer of a conception which the mathematicians had long before made pretty clear, and which recent researches have still further illustrated.

Among the many principles of Logic which find their application in Philosophy, I can here only mention one. Three conceptions are perpetually turning up at every point in every theory of logic, and in the most rounded systems they occur in connection with one another. They are conceptions so very broad and consequently indefinite that they are hard to seize and may be easily overlooked. I call them the conceptions of First, Second, Third. First is the conception of being or existing independent of anything else. Second is the conception of being relative to, the conception of reaction with, something else. Third is the conception of mediation, whereby a first and second are brought into relation. To illustrate these ideas, I will show how they enter into those we have been considering. The origin of things, considered not as leading to anything, but in itself, contains the idea of First, the end of things that of Second, the process mediating between them that of Third. A philosophy which emphasises the idea of the One, is generally a dualistic philosophy in which the conception of Second receives exaggerated attention ; for this One (though of course involving the idea of First) is always the other of a manifold which is not one. The idea of the Many, because variety is arbitrariness and arbitrariness is repudiation of any Secondness, has for its principal component the conception of First. In psychology Feeling is First, Sense of reaction Second, General conception Third, or mediation. In biology, the idea of arbitrary sporting is First, heredity is Second, the process whereby the accidental characters become fixed is Third. Chance is First, Law is Second, the tendency to take habits is Third. Mind is First, Matter is Second, Evolution is Third.

Such are the materials out of which chiefly a philosophical theory ought to be built, in order to represent the state of knowledge to which the nineteenth century has brought us. Without going into other important questions of philosophical architectonic, we can readily foresee what sort of a metaphysics would appropriately be constructed from those conceptions. Like some of the most ancient and some of the most recent speculations it would be a Cosmogonic

Philosophy. It would suppose that in the beginning,—infinitely remote,—there was a chaos of unpersonalised feeling, which being without connection or regularity would properly be without existence. This feeling, sporting here and there in pure arbitrariness, would have started the germ of a generalising tendency. Its other sportings would be evanescent, but this would have a growing virtue. Thus, the tendency to habit would be started; and from this with the other principles of evolution all the regularities of the universe would be evolved. At any time, however, an element of pure chance survives and will remain until the world becomes an absolutely perfect, rational, and symmetrical system, in which mind is at last crystallised in the infinitely distant future.

That idea has been worked out by me with elaboration. It accounts for the main features of the universe as we know it,—the characters of time, space, matter, force, gravitation, electricity, etc. It predicts many more things which new observations can alone bring to the test. May some future student go over this ground again, and have the leisure to give his results to the world.

CHARLES S. PEIRCE.

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THE MONIST.

THE DOCTRINE OF NECESSITY EXAMINED.

IN *The Monist* for January, 1891, I endeavored to show what elementary ideas ought to enter into our view of the universe. I may mention that on those considerations I had already grounded a cosmical theory, and from it had deduced a considerable number of consequences capable of being compared with experience. This comparison is now in progress, but under existing circumstances must occupy many years.

I propose here to examine the common belief that every single fact in the universe is precisely determined by law. It must not be supposed that this is a doctrine accepted everywhere and at all times by all rational men. Its first advocate appears to have been Democritus the atomist, who was led to it, as we are informed, by reflecting upon the "impenetrability, translation, and impact of matter (*ἀντιτυπία καὶ φορὰ καὶ πληγὴ τῆς ὕλης*)." That is to say, having restricted his attention to a field where no influence other than mechanical constraint could possibly come before his notice, he straightway jumped to the conclusion that throughout the universe that was the sole principle of action,—a style of reasoning so usual in our day with men not unreflecting as to be more than excusable in the infancy of thought. But Epicurus, in revising the atomic doctrine and repairing its defences, found himself obliged to suppose that atoms swerve from their courses by spontaneous chance ; and thereby he conferred upon the theory life and entelechy.

For we now see clearly that the peculiar function of the molecular hypothesis in physics is to open an entry for the calculus of probabilities. Already, the prince of philosophers had repeatedly and emphatically condemned the dictum of Democritus (especially in the "Physics," Book II, chapters iv, v, vi), holding that events come to pass in three ways, namely, (1) by external compulsion, or the action of efficient causes, (2) by virtue of an inward nature, or the influence of final causes, and (3) irregularly without definite cause, but just by absolute chance; and this doctrine is of the inmost essence of Aristotelianism. It affords, at any rate, a valuable enumeration of the possible ways in which anything can be supposed to have come about. The freedom of the will, too, was admitted both by Aristotle and by Epicurus. But the Stoa, which in every department seized upon the most tangible, hard, and lifeless element, and blindly denied the existence of every other, which, for example, impugned the validity of the inductive method and wished to fill its place with the *reductio ad absurdum*, very naturally became the one school of ancient philosophy to stand by a strict necessitarianism, thus returning to the single principle of Democritus that Epicurus had been unable to swallow. Necessitarianism and materialism with the Stoics went hand in hand, as by affinity they should. At the revival of learning, Stoicism met with considerable favor, partly because it departed just enough from Aristotle to give it the spice of novelty, and partly because its superficialities well adapted it for acceptance by students of literature and art who wanted their philosophy drawn mild. Afterwards, the great discoveries in mechanics inspired the hope that mechanical principles might suffice to explain the universe; and though without logical justification, this hope has since been continually stimulated by subsequent advances in physics. Nevertheless, the doctrine was in too evident conflict with the freedom of the will and with miracles to be generally acceptable, at first. But meantime there arose that most widely spread of philosophical blunders, the notion that associationalism belongs intrinsically to the materialistic family of doctrines; and thus was evolved the theory of motives; and libertarianism became weakened. At present, historical criticism has almost exploded the miracles, great and small; so that the doctrine of necessity has never been in so great vogue as now.

The proposition in question is that the state of things existing at any time, together with certain immutable laws, completely determine the state of things at every other time (for a limitation to *future* time is indefensible). Thus, given the state of the universe in the original nebula, and given the laws of mechanics, a sufficiently powerful mind could deduce from these data the precise form of every curlicue of every letter I am now writing.

Whoever holds that every act of the will as well as every idea of the mind is under the rigid governance of a necessity co-ordinated with that of the physical world, will logically be carried to the proposition that minds are part of the physical world in such a sense that the laws of mechanics determine everything that happens according to immutable attractions and repulsions. In that case, that instantaneous state of things from which every other state of things is calculable consists in the positions and velocities of all the particles at any instant. This, the usual and most logical form of necessitarianism, is called the mechanical philosophy.

When I have asked thinking men what reason they had to believe that every fact in the universe is precisely determined by law, the first answer has usually been that the proposition is a "presupposition" or postulate of scientific reasoning. Well, if that is the best that can be said for it, the belief is doomed. Suppose it be "postulated": that does not make it true, nor so much as afford the slightest rational motive for yielding it any credence. It is as if a man should come to borrow money, and when asked for his security, should reply he "postulated" the loan. To "postulate" a proposition is no more than to hope it is true. There are, indeed, practical emergencies in which we act upon assumptions of certain propositions as true, because if they are not so, it can make no difference how we act. But all such propositions I take to be hypotheses of individual facts. For it is manifest that no universal principle can in its universality be compromised in a special case or can be requisite for the validity of any ordinary inference. To say, for instance, that the demonstration by Archimedes of the property of the lever would fall to the ground if men were endowed with free-will, is extravagant; yet this is implied by those who make a proposition incompatible with the freedom of the will the postulate of all inference. Considering, too, that the conclusions of science make no pretence to being more than probable, and considering that a probable inference can at most only suppose something to be most frequently, or otherwise approximately, true, but never that anything is precisely true without exception throughout the universe, we see how far this proposition in truth is from being so postulated.

But the whole notion of a postulate being involved in reasoning appertains to a by-gone and false conception of logic. Non-deductive, or ampliative inference is of three kinds: induction, hypothesis, and analogy. If there be any other modes, they must be extremely unusual and highly complicated, and may be assumed with little doubt to be of the same nature as those enumerated. For induction, hypothesis, and analogy, as far as their ampliative character goes, that is, so far as they conclude something not implied in the premises, depend upon one principle and involve the same procedure. All are essentially inferences from sampling. Suppose a ship arrives in Liverpool laden with wheat in bulk. Suppose that by some machinery the whole cargo be stirred up with great thoroughness. Suppose that twenty-seven thimblefuls be taken equally from the forward, midships, and aft parts, from the starboard, centre, and larboard parts, and from the top, half depth, and lower parts of her hold, and that these being mixed and the grains counted, four fifths of the latter are found to be of quality *A*. Then we infer, experientially and provisionally, that approximately four fifths of all the grain in the cargo is of the same quality. I say we infer this *experientially* and *provisionally*. By saying that we infer it *experientially*, I mean that our conclusion makes no pretension to knowledge of wheat-in-itself, our ἀλήθεια, as the derivation of that word implies, has nothing to do with *latent* wheat. We are dealing only with the matter of possible experience,—experience in the full acceptation of the term as something not merely affecting the senses but also as the subject of thought. If there be any wheat hidden on the ship, so that it can neither turn up in the sample nor be heard of subsequently from purchasers,—or if it be half-hidden, so that it may, indeed, turn up, but is less likely to do so than the rest,—or if it can affect our senses and our pockets, but from some strange cause or causelessness cannot be reasoned about,—all such wheat is to be excluded (or have only its proportional weight) in calculating that true proportion of quality A, to which our inference seeks to approximate. By saying that we draw the inference *provisionally*, I mean that we do not hold that we have reached any assigned degree of approximation as yet, but only hold that if our experience be indefinitely extended, and if every fact of whatever nature, as fast as it presents itself, be duly applied, according to the inductive method, in correcting the inferred ratio, then our approximation will become indefinitely close in the long run; that is to say, close to the experience *to come* (not merely close by the exhaustion of a finite collection) so that if experience in general is to fluctuate irregularly to and fro, in a manner to deprive the ratio sought of all definite value, we shall be able to find out approximately within what limits it fluctuates, and if, after having one definite value, it changes and assumes another, we shall be able to find that out, and in short, whatever may be the variations of this ratio in experience, experience indefinitely extended will enable us to detect them, so as to predict rightly, at last, what its ultimate value may be, if it have any ultimate value, or what the ultimate law of succession of values may be, if there be any such ultimate law, or that it ultimately fluctuates irregularly within certain limits, if it do so ultimately fluctuate. Now our inference, claiming to be no more than thus experiential and provisional, manifestly involves no postulate whatever.

For what is a postulate? It is the formulation of a material fact which we are not entitled to assume as a premise, but the truth of which is requisite to the validity of an inference. Any fact, then, which might be supposed postulated, must either be such that it would ultimately present itself in experience, or not. If it will present itself, we need not postulate it now in our provisional inference, since we shall ultimately be entitled to use it as a premise. But if it never would present itself in experience, our conclusion is valid but for the possibility of this fact being otherwise than assumed, that is, it is valid as far as possible experience goes, and that is all that we claim. Thus, every postulate is cut off, either by the provisionality or by the experientiality of our inference. For instance, it has been said that induction postulates that, if an indefinite succession of samples be drawn, examined, and thrown back each before the next is drawn, then in the long run every grain will be drawn as often as any other, that is to say postulates that the ratio of the numbers of times in which any two are drawn will indefinitely approximate to unity. But no such postulate is made; for if, on the one hand, we are to have no other experience of the wheat than from such drawings, it is the ratio that presents itself in those drawings and not the ratio which belongs to the wheat in its latent existence that we are endeavoring to determine; while if, on the other hand, there is some other mode by which the wheat is to come under our knowledge, equivalent to another kind of sampling, so that after all our care in stirring up the wheat, some experiential grains will present themselves in the first sampling operation more often than others in the long run, this very singular fact will be sure to get discovered by the inductive method, which must avail itself of every sort of experience; and our inference, which was only provisional, corrects itself at last. Again, it has been said, that induction postulates that under like circumstances like events will happen, and that this postulate is at bottom the same as the principle of universal causation. But this is a blunder, or *bévue*, due to thinking exclusively of inductions where the concluded ratio is either 1 or 0. If any such proposition were postulated, it would be that under like circumstances (the circumstances of drawing the different samples) different events occur in the same proportions in all the different sets,—a proposition which is false and even absurd. But in truth no such thing is postulated, the experiential character of the inference reducing the condition of validity to this, that if a certain result does not occur, the opposite result will be manifested, a condition assured by the provisionality of the inference. But it may be asked whether it is not conceivable that every instance of a certain class destined to be ever employed as a datum of induction should have one character, while every instance destined not to be so employed should have the opposite character. The answer is that in that case, the instances excluded from being subjects of reasoning would not be experienced in the full sense of the word, but would be among these *latent* individuals of which our conclusion does not pretend to speak.

To this account of the rationale of induction I know of but one objection worth mention: it is that I thus fail to deduce the full degree of force which this mode of inference in fact possesses; that according to my view, no matter how thorough and elaborate the stirring and mixing process had been, the examination of a single handful of grain would not give me any assurance, sufficient to risk money upon, that the next handful would not greatly modify the concluded value of the ratio under inquiry, while, in fact, the assurance would be very high that this ratio was not greatly in error. If the true ratio of grains of quality *A* were 0·80 and the handful contained a thousand grains, nine such handfuls out of every ten would contain from 780 to 820 grains of quality *A*. The answer to this is that the calculation given is correct when we know that the units of this handful and the quality inquired into have the normal independence of one another, if for instance the stirring has been complete and the character sampled for has been settled upon in advance of the examination of the sample. But in so far as these conditions are not known to be complied with, the above figures cease to be applicable. Random sampling and predesignation of the character sampled for should always be striven after in inductive reasoning, but when they cannot be attained, so long as it is conducted honestly, the inference retains some value. When we cannot ascertain how the sampling has been done or the sample-character selected, induction still has the essential validity which my present account of it shows it to have.

I do not think a man who combines a willingness to be convinced with a power of appreciating an argument upon a difficult subject can resist the reasons which have been given to show that the principle of universal necessity cannot be defended as being a postulate of reasoning. But then the question immediately arises whether it is not proved to be true, or at least rendered highly probable, by observation of nature.

Still, this question ought not long to arrest a person accustomed to reflect upon the force of scientific reasoning. For the essence of the necessitarian position is that certain continuous quantities have certain exact values. Now, how can observation determine the value of such a quantity with a probable error absolutely *nil?* To one who is behind the scenes, and knows that the most refined comparisons of masses, lengths, and angles, far surpassing in precision all other measurements, yet fall behind the accuracy of bank-accounts, and that the ordinary determinations of physical constants, such as appear from month to month in the journals, are about on a par with an upholsterer's measurements of carpets and curtains, the idea of mathematical exactitude being demonstrated in the laboratory will appear simply ridiculous. There is a recognised method of estimating the probable magnitudes of errors in physics,—the method of least squares. It is universally admitted that this method makes the errors smaller than they really are; yet even according to that theory an error indefinitely small is indefinitely improbable; so that any statement to the effect that a certain continuous quantity has a certain exact value, if well-founded at all, must be founded on something other than observation.

Still, I am obliged to admit that this rule is subject to a certain qualification. Namely, it only applies to continuous* quantity. Now, certain kinds of continuous quantity are discontinuous at one or at two limits, and for such limits the rule must be modified. Thus, the length of a line cannot be less than zero. Suppose, then, the question arises how long a line a certain person had drawn from a marked point on a piece of paper. If no line at all can be seen, the observed length is zero; and the only conclusion this observation warrants is that the length of the line is less than the smallest length visible with the optical power employed. But indirect observations,—for example, that the person supposed to have drawn the line was never within fifty feet of the paper,—may make it probable that no line at all was made, so that the concluded length will be

* *Continuous* is not exactly the right word, but I let it go to avoid a long and irrelevant discussion.

strictly zero. In like manner, experience no doubt would warrant the conclusion that there is absolutely *no* indigo in a given ear of wheat, and absolutely *no* attar in a given lichen. But such inferences can only be rendered valid by positive experiential evidence, direct or remote, and cannot rest upon a mere inability to detect the quantity in question. We have reason to think there is no indigo in the wheat, because we have remarked that wherever indigo is produced it is produced in considerable quantities, to mention only one argument. We have reason to think there is no attar in the lichen, because essential oils seem to be in general peculiar to single species. If the question had been whether there was iron in the wheat or the lichen, though chemical analysis should fail to detect its presence, we should think some of it probably was there, since iron is almost everywhere. Without any such information, one way or the other, we could only abstain from any opinion as to the presence of the substance in question. It cannot, I conceive, be maintained that we are in any *better* position than this in regard to the presence of the element of chance or spontaneous departures from law in nature.

Those observations which are generally adduced in favor of mechanical causation simply prove that there is an element of regularity in nature, and have no bearing whatever upon the question of whether such regularity is exact and universal, or not. Nay, in regard to this *exactitude*, all observation is directly *opposed* to it; and the most that can be said is that a good deal of this observation can be explained away. Try to verify any law of nature, and you will find that the more precise your observations, the more certain they will be to show irregular departures from the law. We are accustomed to ascribe these, and I do not say wrongly, to errors of observation; yet we cannot usually account for such errors in any antecedently probable way. Trace their causes back far enough, and you will be forced to admit they are always due to arbitrary determination, or chance.

But it may be asked whether if there were an element of real chance in the universe it must not occasionally be productive of signal effects such as could not pass unobserved. In answer to this question, without stopping to point out that there is an abundance of great events which one might be tempted to suppose were of that nature, it will be simplest to remark that physicists hold that the particles of gases are moving about irregularly, substantially as if by real chance, and that by the principles of probabilities there must occasionally happen to be concentrations of heat in the gases contrary to the second law of thermodynamics, and these concentrations, occurring in explosive mixtures, must sometimes have tremendous effects. Here, then, is in substance the very situation supposed ; yet no phenomena ever have resulted which we are forced to attribute to such chance concentration of heat, or which anybody, wise or foolish, has ever dreamed of accounting for in that manner.

In view of all these considerations, I do not believe that anybody, not in a state of casehardened ignorance respecting the logic of science, can maintain that the precise and universal conformity of facts to law is clearly proved, or even rendered particularly probable, by any observations hitherto made. In this way, the determined advocate of exact regularity will soon find himself driven to *a priori* reasons to support his thesis. These received such a socdolager from Stuart Mill in his Examination of Hamilton, that holding to them now seems to me to denote a high degree of imperviousness to reason ; so that I shall pass them by with little notice.

To say that we cannot help believing a given proposition is no argument, but it is a conclusive fact if it be true ; and with the substitution of "I" for "we," it is true in the mouths of several classes of minds, the blindly passionate, the unreflecting and ignorant, and the person who has overwhelming evidence before his eyes. But that which has been inconceivable to-day has often turned out indisputable on the morrow. Inability to conceive is only a stage through which every man must pass in regard to a number of beliefs,—unless endowed with extraordinary obstinacy and obtuseness. His understanding is enslaved to some blind compulsion which a vigorous mind is pretty sure soon to cast off.

Some seek to back up the *a priori* position with empirical arguments. They say that the exact regularity of the world is a natural belief, and that natural beliefs have generally been confirmed by experience. There is some reason in this. Natural beliefs, however, if they generally have a foundation of truth, also require correction and purification from natural illusions. The principles of mechanics are undoubtedly natural beliefs; but, for all that, the early formulations of them were exceedingly erroneous. The general approximation to truth in natural beliefs is, in fact, a case of the general adaptation of genetic products to recognisable utilities or ends. Now, the adaptations of nature, beautiful and often marvellous as they verily are, are never found to be quite perfect; so that the argument is quite *against* the absolute exactitude of any natural belief, including that of the principle of causation.

Another argument, or convenient commonplace, is that absolute chance is *inconceivable*. This word has eight current significations. The Century Dictionary enumerates six. Those who talk like this will hardly be persuaded to say in what sense they mean that chance is inconceivable. Should they do so, it would easily be shown either that they have no sufficient reason for the statement or that the inconceivability is of a kind which does not prove that chance is non-existent.

Another *a priori* argument is that chance is unintelligible; that is to say, while it may perhaps be conceivable, it does not disclose to the eye of reason the how or why of things; and since a hypothesis can only be justified so far as it renders some phenomenon intelligible, we never can have any right to suppose absolute chance to enter into the production of anything in nature. This argument may be considered in connection with two others. Namely, instead of going so far as to say that the supposition of chance can *never* properly be used to explain any observed fact, it may be alleged merely that no facts are known which such a supposition could in any way help in explaining. Or again, the allegation being still further weakened, it may be said that since departures from law are not unmistakably observed, chance is not a *vera causa*, and ought not unnecessarily to be introduced into a hypothesis.

These are no mean arguments, and require us to examine the matter a little more closely. Come, my superior opponent, let me learn from your wisdom. It seems to me that every throw of sixes with a pair of dice is a manifest instance of chance.

"While you would hold a throw of deuce-ace to be brought about by necessity?" [The opponent's supposed remarks are placed in quotation marks.]

Clearly one throw is as much chance as another.

"Do you think throws of dice are of a different nature from other events?"

I see that I must say that *all* the diversity and specificalness of events is attributable to chance.

"Would you, then, deny that there is any regularity in the world?"

That is clearly undeniable. I must acknowledge there is an approximate regularity, and that every event is influenced by it. But the diversification, specificalness, and irregularity of things I suppose is chance. A throw of sixes appears to me a case in which this element is particularly obtrusive.

"If you reflect more deeply, you will come to see that *chance* is only a name for a cause that is unknown to us."

Do you mean that we have no idea whatever what kind of causes could bring about a throw of sixes?

"On the contrary, each die moves under the influence of precise mechanical laws."

But it appears to me that it is not these *laws* which made the die turn up sixes; for these laws act just the same when other throws come up. The chance lies in the diversity of throws; and this diversity cannot be due to laws which are immutable.

"The diversity is due to the diverse circumstances under which the laws act. The dice lie differently in the box, and the motion given to the box is different. These are the unknown causes which produce the throws, and to which we give the name of chance; not the mechanical law which regulates the operation of these causes. You see you are already beginning to think more clearly about this subject."

Does the operation of mechanical law not increase the diversity?

"Properly not. You must know that the instantaneous state of a system of particles is defined by six times as many numbers as there are particles, three for the co-ordinates of each particle's position, and three more for the components of its velocity. This number of numbers, which expresses the amount of diversity in the system, remains the same at all times. There may be, to be sure, some kind of relation between the co-ordinates and component velocities of the different particles, by means of which the state of the system might be expressed by a smaller number of numbers. But, if this is the case, a precisely corresponding relationship must exist between the co-ordinates and component velocities at any other time, though it may doubtless be a relation less obvious to us. Thus, the intrinsic complexity of the system is the same at all times."

Very well, my obliging opponent, we have now reached an issue. You think all the arbitrary specifications of the universe were introduced in one dose, in the beginning, if there was a beginning, and that the variety and complication of nature has always been just as much as it is now. But I, for my part, think that the diversification, the specification, has been continually taking place. Should you condescend to ask me why I so think, I should give my reasons as follows:

1) Question any science which deals with the course of time. Consider the life of an individual animal or plant, or of a mind. Glance at the history of states, of institutions, of language, of ideas. Examine the successions of forms shown by paleontology, the history of the globe as set forth in geology, of what the astronomer is able to make out concerning the changes of stellar systems. Everywhere the main fact is growth and increasing complexity. Death and corruption are mere accidents or secondary phenomena. Among some of the lower organisms, it is a moot point with biologists whether there be anything which ought to be called death. Races, at any rate, do not die out except under unfavorable circumstances. From these broad and ubiquitous facts we may fairly infer, by the most unexceptionable logic, that there is probably in nature some agency by which the complexity and diversity of things can be increased; and that consequently the rule of mechanical necessity meets in some way with interference.

2) By thus admitting pure spontaneity or life as a character of the universe, acting always and everywhere though restrained

within narrow bounds by law, producing infinitesimal departures from law continually, and great ones with infinite infrequency, I account for all the variety and diversity of the universe, in the only sense in which the really *sui generis* and new can be said to be accounted for. The ordinary view has to admit the inexhaustible mulitudinous variety of the world, has to admit that its mechanical law cannot account for this in the least, that variety can spring only from spontaneity, and yet denies without any evidence or reason the existence of this spontaneity, or else shoves it back to the beginning of time and supposes it dead ever since. The superior logic of my view appears to me not easily controverted.

3) When I ask the necessitarian how he would explain the diversity and irregularity of the universe, he replies to me out of the treasury of his wisdom that irregularity is something which from the nature of things we must not seek to explain. Abashed at this, I seek to cover my confusion by asking how he would explain the uniformity and regularity of the universe, whereupon he tells me that the laws of nature are immutable and ultimate facts, and no account is to be given of them. But my hypothesis of spontaneity does explain irregularity, in a certain sense; that is, it explains the general fact of irregularity, though not, of course, what each lawless event is to be. At the same time, by thus loosening the bond of necessity, it gives room for the influence of another kind of causation, such as seems to be operative in the mind in the formation of associations, and enables us to understand how the uniformity of nature could have been brought about. That single events should be hard and unintelligible, logic will permit without difficulty: we do not expect to make the shock of a personally experienced earthquake appear natural and reasonable by any amount of cogitation. But logic does expect things *general* to be understandable. To say that there is a universal law, and that it is a hard, ultimate, unintelligible fact, the why and wherefore of which can never be inquired into, at this a sound logic will revolt; and will pass over at once to a method of philosophising which does not thus barricade the road of discovery.

4) Necessitarianism cannot logically stop short of making the

whole action of the mind a part of the physical universe. Our notion that we decide what we are going to do, if as the necessitarian says, it has been calculable since the earliest times, is reduced to illusion. Indeed, consciousness in general thus becomes a mere illusory aspect of a material system. What we call red, green, and violet are in reality only different rates of vibration. The sole reality is the distribution of qualities of matter in space and time. Brain-matter is protoplasm in a certain degree and kind of complication,—a certain arrangement of mechanical particles. Its feeling is but an inward aspect, a phantom. For, from the positions and velocities of the particles at any one instant, and the knowledge of the immutable forces, the positions at all other times are calculable; so that the universe of space, time, and matter is a rounded system uninterfered with from elsewhere. But from the state of feeling at any instant, there is no reason to suppose the states of feeling at all other instants are thus exactly calculable; so that feeling is, as I said, a mere fragmentary and illusive aspect of the universe. This is the way, then, that necessitarianism has to make up its accounts. It enters consciousness under the head of sundries, as a forgotten trifle; its scheme of the universe would be more satisfactory if this little fact could be dropped out of sight. On the other hand, by supposing the rigid exactitude of causation to yield, I care not how little, —be it but by a strictly infinitesimal amount,—we gain room to insert mind into our scheme, and to put it into the place where it is needed, into the position which, as the sole self-intelligible thing, it is entitled to occupy, that of the fountain of existence; and in so doing we resolve the problem of the connection of soul and body.

5) But I must leave undeveloped the chief of my reasons, and can only adumbrate it. The hypothesis of chance-spotaneity is one whose inevitable consequences are capable of being traced out with mathematical precision into considerable detail. Much of this I have done and find the consequences to agree with observed facts to an extent which seems to me remarkable. But the matter and methods of reasoning are novel, and I have no right to promise that other mathematicians shall find my deductions as satisfactory as I myself do, so that the strongest reason for my belief must for the

present remain a private reason of my own, and cannot influence others. I mention it to explain my own position; and partly to indicate to future mathematical speculators a veritable goldmine, should time and circumstances and the abridger of all joys prevent my opening it to the world.

If now I, in my turn, inquire of the necessitarian why he prefers to suppose that all specification goes back to the beginning of things, he will answer me with one of those last three arguments which I left unanswered.

First, he may say that chance is a thing absolutely unintelligible, and therefore that we never can be entitled to make such a supposition. But does not this objection smack of naïve impudence? It is not mine, it is his own conception of the universe which leads abruptly up to hard, ultimate, inexplicable, immutable law, on the one hand, and to inexplicable specification and diversification of circumstances on the other. My view, on the contrary, hypothetises nothing at all, unless it be hypothesis to say that all specification came about in some sense, and is not to be accepted as unaccountable. To undertake to account for anything by saying boldly that it is due to chance would, indeed, be futile. But this I do not do. I make use of chance chiefly to make room for a principle of generalisation, or tendency to form habits, which I hold has produced all regularities. The mechanical philosopher leaves the whole specification of the world utterly unaccounted for, which is pretty nearly as bad as to boldly attribute it to chance. I attribute it altogether to chance, it is true, but to chance in the form of a spontaneity which is to some degree regular. It seems to me clear at any rate that one of these two positions must be taken, or else specification must be supposed due to a spontaneity which develops itself in a certain and not in a chance way, by an objective logic like that of Hegel. This last way I leave as an open possibility, for the present; for it is as much opposed to the necessitarian scheme of existence as my own theory is.

Secondly the necessitarian may say there are, at any rate, no observed phenomena which the hypothesis of chance could aid in explaining. In reply, I point first to the phenomenon of growth and developing complexity, which appears to be universal, and which though it may possibly be an affair of mechanism perhaps, certainly presents all the appearance of increasing diversification. Then, there is variety itself, beyond comparison the most obtrusive character of the universe: no mechanism can account for this. Then, there is the very fact the necessitarian most insists upon, the regularity of the universe which for him serves only to block the road of inquiry. Then, there are the regular relationships between the laws of nature,—similarities and comparative characters, which appeal to our intelligence as its cousins, and call upon us for a reason. Finally, there is consciousness, feeling, a patent fact enough, but a very inconvenient one to the mechanical philosopher.

Thirdly, the necessitarian may say that chance is not a *vera causa*, that we cannot know positively there is any such element in the universe. But the doctrine of the *vera causa* has nothing to do with elementary conceptions. Pushed to that extreme, it at once cuts off belief in the existence of a material universe; and without that necessitarianism could hardly maintain its ground. Besides, variety is a fact which must be admitted; and the theory of chance merely consists in supposing this diversification does not antedate all time. Moreover, the avoidance of hypotheses involving causes nowhere positively known to act—is only a recommendation of logic, not a positive command. It cannot be formulated in any precise terms without at once betraying its untenable character,—I mean as rigid rule, for as a recommendation it is wholesome enough.

I believe I have thus subjected to fair examination all the important reasons for adhering to the theory of universal necessity, and have shown their nullity. I earnestly beg that whoever may detect any flaw in my reasoning will point it out to me, either privately or publicly; for if I am wrong, it much concerns me to be set right speedily. If my argument remains unrefuted, it will be time, I think, to doubt the absolute truth of the principle of universal law; and when once such a doubt has obtained a living root in any man's mind, my cause with him, I am persuaded, is gained.

C. S. PEIRCE.

THE LAW OF MIND.

IN an article published in *The Monist* for January 1891, I endeavored to show what ideas ought to form the warp of a system of philosophy, and particularly emphasised that of absolute chance. In the number of April 1892, I argued further in favor of that way of thinking, which it will be convenient to christen *tychism* (from τύχη, chance). A serious student of philosophy will be in no haste to accept or reject this doctrine; but he will see in it one of the chief attitudes which speculative thought may take, feeling that it is not for an individual, nor for an age, to pronounce upon a fundamental question of philosophy. That is a task for a whole era to work out. I have begun by showing that *tychism* must give birth to an evolutionary cosmology, in which all the regularities of nature and of mind are regarded as products of growth, and to a Schelling-fashioned idealism which holds matter to be mere specialised and partially deadened mind. I may mention, for the benefit of those who are curious in studying mental biographies, that I was born and reared in the neighborhood of Concord,—I mean in Cambridge,—at the time when Emerson, Hedge, and their friends were disseminating the ideas that they had caught from Schelling, and Schelling from Plotinus, from Boehm, or from God knows what minds stricken with the monstrous mysticism of the East. But the atmosphere of Cambridge held many an antiseptic against Concord transcendentalism; and I am not conscious of having contracted any of that virus. Nevertheless, it is probable that some cultured bacilli, some benignant form of the disease was implanted in my soul, unawares, and that now, after long incubation, it comes to the surface, modified by mathematical conceptions and by training in physical investigations.

The next step in the study of cosmology must be to examine the general law of mental action. In doing this, I shall for the time drop my tychism out of view, in order to allow a free and independent expansion to another conception signalised in my first *Monist*-paper as one of the most indispensable to philosophy, though it was not there dwelt upon ; I mean the idea of continuity. The tendency to regard continuity, in the sense in which I shall define it, as an idea of prime importance in philosophy may conveniently be termed *synechism*. The present paper is intended chiefly to show what synechism is, and what it leads to. I attempted, a good many years ago, to develop this doctrine in the *Journal of Speculative Philosophy* (Vol. III.); but I am able now to improve upon that exposition, in which I was a little blinded by nominalistic prepossessions. I refer to it, because students may possibly find that some points not sufficiently explained in the present paper are cleared up in those earlier ones.

WHAT THE LAW IS.

Logical analysis applied to mental phenomena shows that there is but one law of mind, namely, that ideas tend to spread continuously and to affect certain others which stand to them in a peculiar relation of affectibility. In this spreading they lose intensity, and especially the power of affecting others, but gain generality and become welded with other ideas.

I set down this formula at the beginning, for convenience ; and now proceed to comment upon it.

INDIVIDUALITY OF IDEAS.

We are accustomed to speak of ideas as reproduced, as passed from mind to mind, as similar or dissimilar to one another, and, in short, as if they were substantial things ; nor can any reasonable objection be raised to such expressions. But taking the word "idea" in the sense of an event in an individual consciousness, it is clear that an idea once past is gone forever, and any supposed recurrence of it is another idea. These two ideas are not present in the same state of consciousness, and therefore cannot possibly be compared. To say, therefore, that they are similar can only mean that an occult power from the depths of the soul forces us to connect them in our thoughts after they are both no more. We may note, here, in passing that of the two generally recognised principles of association, contiguity and similarity, the former is a connection due to a power without, the latter a connection due to a power within.

But what can it mean to say that ideas wholly past are thought of at all, any longer? They are utterly unknowable. What distinct meaning can attach to saying that an idea in the past in any way affects an idea in the future, from which it is completely detached? A phrase between the assertion and the denial of which there can in no case be any sensible difference is mere gibberish.

I will not dwell further upon this point, because it is a commonplace of philosophy.

CONTINUITY OF IDEAS.

We have here before us a question of difficulty, analogous to the question of nominalism and realism. But when once it has been clearly formulated, logic leaves room for one answer only. How can a past idea be present? Can it be present vicariously? To a certain extent, perhaps; but not merely so; for then the question would arise how the past idea can be related to its vicarious representation. The relation, being between ideas, can only exist in some consciousness: now that past idea was in no consciousness but that past consciousness that alone contained it; and that did not embrace the vicarious idea.

Some minds will here jump to the conclusion that a past idea cannot in any sense be present. But that is hasty and illogical. How extravagant, too, to pronounce our whole knowledge of the past to be mere delusion! Yet it would seem that the past is as completely beyond the bonds of possible experience as a Kantian thing-in-itself.

How can a past idea be present? Not vicariously. Then, only by direct perception. In other words, to be present, it must be *ipso facto* present. That is, it cannot be wholly past; it can only be going, infinitesimally past, less past than any assignable past date. We are thus brought to the conclusion that the present is connected with the past by a series of real infinitesimal steps.

It has already been suggested by psychologists that consciousness necessarily embraces an interval of time. But if a finite time be meant, the opinion is not tenable. If the sensation that precedes the present by half a second were still immediately before me, then, on the same principle the sensation preceding that would be immediately present, and so on *ad infinitum*. Now, since there is a time, say a year, at the end of which an idea is no longer *ipso facto* present, it follows that this is true of any finite interval, however short.

But yet consciousness must essentially cover an interval of time; for if it did not, we could gain no knowledge of time, and not merely no veracious cognition of it, but no conception whatever. We are, therefore, forced to say that we are immediately conscious through an infinitesimal interval of time.

This is all that is requisite. For, in this infinitesimal interval, not only is consciousness continuous in a subjective sense, that is, considered as a subject or substance having the attribute of duration; but also, because it is immediate consciousness, its object is *ipso facto* continuous. In fact, this infinitesimally spread-out consciousness is a direct feeling of its contents as spread out. This will be further elucidated below. In an infinitesimal interval we directly perceive the temporal sequence of its beginning, middle, and end,—not, of course, in the way of recognition, for recognition is only of the past, but in the way of immediate feeling. Now upon this interval follows another, whose beginning is the middle of the former, and whose middle is the end of the former. Here, we have an immediate perception of the temporal sequence of its beginning, middle, and end, or say of the second, third, and fourth instants. From these two immediate perceptions, we gain a mediate, or inferential, perception of the relation of all four instants. This mediate perception is objectively, or as to the object represented, spread over the four instants; but subjectively, or as itself the subject of duration, it is completely embraced in the second moment. [The reader will observe that I use the word *instant* to mean a point of time, and *moment* to mean an infinitesimal duration.] If it is objected that, upon the theory proposed, we must have more than a mediate perception of the succession of the four instants, I grant it; for the sum of the two infinitesimal intervals is itself infinitesimal, so that it is immediately perceived. It is immediately perceived in the whole interval, but only mediately perceived in the last two thirds of the interval. Now, let there be an indefinite succession of these inferential acts of comparative perception; and it is plain that the last moment will contain objectively the whole series. Let there be, not merely an indefinite succession, but a continuous flow of inference through a finite time; and the result will be a mediate objective consciousness of the whole time in the last moment. In this last moment, the whole series will be recognised, or known as known before, except only the last moment, which of course will be absolutely unrecognisable to itself. Indeed, even this last moment will be recognised like the rest, or, at least be just beginning to be so. There is a little *elenchus*, or appearance of contradiction, here, which the ordinary logic of reflection quite suffices to resolve.

INFINITY AND CONTINUITY, IN GENERAL.

Most of the mathematicians who during the last two generations have treated the differential calculus have been of the opinion that an infinitesimal quantity is an absurdity; although, with their habitual caution, they have often added "or, at any rate, the conception of an infinitesimal is so difficult, that we practically cannot reason about it with confidence and security." Accordingly, the doctrine of limits has been invented to evade the difficulty, or, as some say, to explain the signification of the word "infinitesimal." This doctrine, in one form or another, is taught in all the text-books, though in some of them only as an alternative view of the matter; it answers well enough the purposes of calculation, though even in that application it has its difficulties.

The illumination of the subject by a strict notation for the logic of relatives had shown me clearly and evidently that the idea of an infinitesimal involves no contradiction, before I became acquainted with the writings of Dr. Georg Cantor (though many of these had already appeared in the *Mathematische Annalen* and in *Borchardt's Journal*, if not yet in the *Acta Mathematica*, all mathematical journals of the first distinction), in which the same view is defended with extraordinary genius and penetrating logic.

The prevalent opinion is that finite numbers are the only ones that we can reason about, at least, in any ordinary mode of reasoning, or, as some authors express it, they are the only numbers that can be reasoned about mathematically. But this is an irrational prejudice. I long ago showed that finite collections are distinguished from infinite ones only by one circumstance and its consequences, namely, that to them is applicable a peculiar and unusual mode of reasoning called by its discoverer, De Morgan, the "syllogism of transposed quantity."

Balzac, in the introduction of his *Physiologie du mariage*, remarks that every young Frenchman boasts of having seduced some Frenchwoman. Now, as a woman can only be seduced once, and there are no more Frenchwomen than Frenchmen, it follows, if these boasts are true, that no French women escape seduction. If their number be finite, the reasoning holds. But since the population is continually increasing, and the seduced are on the average younger than the seducers, the conclusion need not be true. In like manner, De Morgan, as an actuary, might have argued that if an insurance company pays to its insured on an average more than they have ever paid it, including interest, it must lose money. But every modern actuary would see a fallacy in that, since the business is continually on the increase. But should war, or other cataclysm, cause the class of insured to be a finite one, the conclusion would turn out painfully correct, after all. The above two reasonings are examples of the syllogism of transposed quantity.

The proposition that finite and infinite collections are distinguished by the applicability to the former of the syllogism of transposed quantity ought to be regarded as the basal one of scientific arithmetic.

If a person does not know how to reason logically, and I must say that a great many fairly good mathematicians,—yea distinguished ones,—fall under this category, but simply uses a rule of thumb in blindly drawing inferences like other inferences that have turned out well, he will, of course, be continually falling into error about infinite numbers. The truth is such people do not reason, at all. But for the few who do reason, reasoning about infinite numbers is easier than about finite numbers, because the complicated syllogism of transposed quantity is not called for. For example, that the whole is greater than its part is not an axiom, as that eminently bad reasoner, Euclid, made it to be. It is a theorem readily proved by means of a syllogism of transposed quantity, but not otherwise. Of finite collections it is true, of infinite collections false. Thus, a part of the whole numbers are even numbers. Yet the even numbers are no fewer than all the numbers ; an evident proposition since if every number in the whole series of whole numbers be doubled, the result will be the series of even numbers.

1, 2, 3, 4, 5, 6, etc.
2, 4, 6, 8, 10, 12, etc.

So for every number there is a distinct even number. In fact, there are as many distinct doubles of numbers as there are of distinct numbers. But the doubles of numbers are all even numbers.

In truth, of infinite collections there are but two grades of magnitude, the *endless* and the *innumerable*. Just as a finite collection is distinguished from an infinite one by the applicability to it of a special mode of reasoning, the syllogism of transposed quantity, so, as I showed in the paper last referred to, a numerable collection is distinguished from an innumerable one by the applicability to it of a certain mode of reasoning, the Fermatian inference, or, as it is sometimes improperly termed, "mathematical induction."

As an example of this reasoning, Euler's demonstration of the binomial theorem for integral powers may be given. The theorem is that $(x+y)^n$, where n is a whole number, may be expanded into the sum of a series of terms of which the first is $x^n y^0$ and each of the others is derived from the next preceding by diminishing the exponent of x by 1 and multiplying by that exponent and at the same time increasing the exponent of y by 1 and dividing by that increased exponent. Now, suppose this proposition to be true for a certain exponent, $n = M$, then it must also be true for $n = M + 1$.

For let one of the terms in the expansion of $(x+y)^M$ be written $Ax^p y^q$. Then, this term with the two following will be

$$A\,x^p y^q + A\frac{p}{q+1}\,x^{p-1}y^{q+1} + A\frac{p}{q+1}\frac{p-1}{q+2}\,x^{p-2}y^{q+2}$$

Now, when $(x+y)^M$ is multiplied by $x+y$ to give $(x+y)^{M+1}$, we multiply first by x and then by y instead of by x and add the two results. When we multiply by x, the second of the above three terms will be the only one giving a term involving $x^p y^{q+1}$ and the third will be the only one giving a term in $x^{p-1}y^{q+2}$; and when we multiply by y the first will be the only term giving a term in $x^p y^{q+1}$, and the second will be the only term giving a term in $x^{p-1}y^{q+2}$. Hence, adding like terms, we find that the coefficient of $x^p y^{q+1}$ in the expansion of $(x+y)^{M+1}$ will be the sum of the coefficients of the first two of the above three terms, and that the coefficient of $x^{p-1}y^{q+2}$ will be the sum of the coefficients of the last two terms. Hence, two successive terms in the expansion of $(x+y)^{M+1}$ will be

$$A\left\{1+\frac{p}{q+1}\right\}x^p y^{q+1} + A\frac{p}{q+1}\left\{1+\frac{p-1}{q+2}\right\}x^{p-1}y^{q+2}$$

$$= A\frac{p+q+1}{q+1}\,x^p y^{q+1} + A\frac{p+q+1}{q+1}\cdot\frac{p}{q+2}\,x^{p-1}y^{q+2}.$$

It is, thus, seen that the succession of terms follows the rule. Thus if any integral power follows the rule, so also does the next higher power. But the first power obviously follows the rule. Hence, all powers do so.

Such reasoning holds good of any collection of objects capable of being ranged in a series which though it may be endless, can be numbered so that each member of it receives a definite integral number. For instance, all the whole numbers constitute such a numerable collection. Again, all numbers resulting from operating according to any definite rule with any finite number of whole numbers form such a collection. For they may be arranged in a series thus. Let F be the symbol of operation. First operate on 1, giving F(1). Then, operate on a second 1, giving F(1,1). Next, introduce 2, giving 3rd, F(2); 4th, F(2,1); 5th, F(1,2); 6th, F(2,2). Next use a third variable giving 7th, F(1,1,1); 8th, F(2,1,1); 9th, F(1,2,1); 10th, F(2,2,1); 11th, F(1,1,2); 12th, F(2,1,2); 13th, F(1,2,2);

14th, F(2, 2, 2). Next introduce 3, and so on, alternately introducing new variables and new figures; and in this way it is plain that every arrangement of integral values of the variables will receive a numbered place in the series.*

The class of endless but numerable collections (so called because they can be so ranged that to each one corresponds a distinct whole number) is very large. But there are collections which are certainly innumerable. Such is the collection of all numbers to which endless series of decimals are capable of approximating. It has been recognised since the time of Euclid that certain numbers are surd or incommensurable, and are not exactly expressible by any finite series of decimals, nor by a circulating decimal. Such is the ratio of the circumference of a circle to its diameter, which we know is nearly 3.1415926. The calculation of this number has been carried to over 700 figures without the slightest appearance of regularity in their sequence. The demonstrations that this and many other numbers are incommensurable are perfect. That the entire collection of incommensurable numbers is innumerable has been clearly proved by Cantor. I omit the demonstration; but it is easy to see that to discriminate one from some other would, in general, require the use of an endless series of numbers. Now if they cannot be exactly expressed and discriminated, clearly they cannot be ranged in a linear series.

It is evident that there are as many points on a line or in an interval of time as there are of real numbers in all. These are, therefore, innumerable collections. Many mathematicians have incautiously assumed that the points on a surface or in a solid are more than those on a line. But this has been refuted by Cantor. Indeed, it is obvious that for every set of values of coördinates there is a single distinct number. Suppose, for instance, the values of the coördinates all lie between 0 and + 1. Then if we compose a number by putting in the first decimal place the first figure of the first coördinate, in the second the first figure of the second coördi-

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* This proposition is substantially the same as a theorem of Cantor, though it is enunciated in a much more general form.

nate, and so on, and when the first figures are all dealt out go on to the second figures in like manner, it is plain that the values of the coördinates can be read off from the single resulting number, so that a triad or tetrad of numbers, each having innumerable values, has no more values than a single incommensurable number.

Were the number of dimensions infinite, this would fail; and the collection of infinite sets of numbers having each innumerable variations, might, therefore, be greater than the simple innumerable collection, and might be called *endlessly infinite*. The single individuals of such a collection could not, however, be designated, even approximately, so that this is indeed a magnitude concerning which it would be possible to reason only in the most general way, if at all.

Although there are but two grades of magnitudes of infinite collections, yet when certain conditions are imposed upon the order in which individuals are taken, distinctions of magnitude arise from that cause. Thus, if a simply endless series be doubled by separating each unit into two parts, the successive first parts and also the second parts being taken in the same order as the units from which they are derived, this double endless series will, so long as it is taken in that order, appear as twice as large as the original series. In like manner the product of two innumerable collections, that is, the collection of possible pairs composed of one individual of each, if the order of continuity is to be maintained, is, by virtue of that order, infinitely greater than either of the component collections.

We now come to the difficult question, What is continuity? Kant confounds it with infinite divisibility, saying that the essential character of a continuous series is that between any two members of it a third can always be found. This is an analysis beautifully clear and definite; but unfortunately, it breaks down under the first test. For according to this, the entire series of rational fractions arranged in the order of their magnitude, would be an infinite series, although the rational fractions are numerable, while the points of a line are innumerable. Nay, worse yet, if from that series of fractions any two with all that lie between them be excised, and any number of such finite gaps he made, Kant's definition is still true of the series, though it has lost all appearance of continuity.

Cantor defines a continuous series as one which is *concatenated* and *perfect*. By a concatenated series, he means such a one that if any two points are given in it, and any finite distance, however small, it is possible to proceed from the first point to the second through a succession of points of the series each at a distance from the preceding one less than the given distance. This is true of the series of rational fractions ranged in the order of their magnitude. By a perfect series, he means one which contains every point such that there is no distance so small that this point has not an infinity of points of the series within that distance of it. This is true of the series of numbers between 0 and 1 capable of being expressed by decimals in which only the digits 0 and 1 occur.

It must be granted that Cantor's definition includes every series that is continuous; nor can it be objected that it includes any important or indubitable case of a series not continuous. Nevertheless, it has some serious defects. In the first place, it turns upon metrical considerations; while the distinction between a continuous and a discontinuous series is manifestly non-metrical. In the next place, a perfect series is defined as one containing "every point" of a certain description. But no positive idea is conveyed of what all the points are: that is definition by negation, and cannot be admitted. If that sort of thing were allowed, it would be very easy to say, at once, that the continuous linear series of points is one which contains every point of the line between its extremities. Finally, Cantor's definition does not convey a distinct notion of what the components of the conception of continuity are. It ingeniously wraps up its properties in two separate parcels, but does not display them to our intelligence.

Kant's definition expresses one simple property of a continuum; but it allows of gaps in the series. To mend the definition, it is only necessary to notice how these gaps can occur. Let us suppose, then, a linear series of points extending from a point, *A*, to a point, *B*, having a gap from *B* to a third point, *C*, and thence extending to a final limit, *D*; and let us suppose this series conforms to Kant's definition. Then, of the two points, *B* and *C*, one or both must be excluded from the series; for otherwise, by the definition, there would be points between them. That is, if the series contains C, though it contains all the points up to B, it cannot contain B. What is required, therefore, is to state in non-metrical terms that if a series of points up to a limit is included in a continuum the limit is included. It may be remarked that this is the property of a continuum to which Aristotle's attention seems to have been directed when he defines a continuum as something whose parts have a common limit. The property may be exactly stated as follows: If a linear series of points is continuous between two points, A and D, and if an endless series of points be taken, the first of them between A and D and each of the others between the last preceding one and D, then there is a point of the continuous series between all that endless series of points and D, and such that every other point of which this is true lies between this point and D. For example, take any number between 0 and 1, as 0.1; then, any number between 0.1 and 1, as 0.11; then any number between 0.11 and 1, as 0.111; and so on, without end. Then, because the series of real numbers between 0 and 1 is continuous, there must be a *least* real number, greater than every number of that endless series. This property, which may be called the Aristotelicity of the series, together with Kant's property, or its Kanticity, completes the definition of a continuous series.

The property of Aristotelicity may be roughly stated thus: a continuum contains the end point belonging to every endless series of points which it contains. An obvious corollary is that every continuum contains its limits. But in using this principle it is necessary to observe that a series may be continuous except in this, that it omits one or both of the limits.

Our ideas will find expression more conveniently if, instead of points upon a line, we speak of real numbers. Every real number is, in one sense, the limit of a series, for it can be indefinitely approximated to. Whether every real number is a limit of a *regular* series may perhaps be open to doubt. But the series referred to in the definition of Aristotelicity must be understood as including all series whether regular or not. Consequently, it is implied that between any two points an innumerable series of points can be taken.

Every number whose expression in decimals requires but a finite number of places of decimals is commensurable. Therefore, incommensurable numbers suppose an infinitieth place of decimals. The word infinitesimal is simply the Latin form of infinitieth; that is, it is an ordinal formed from *infinitum*, as centesimal from *centum*. Thus, continuity supposes infinitesimal quantities. There is nothing contradictory about the idea of such quantities. In adding and multiplying them the continuity must not be broken up, and consequently they are precisely like any other quantities, except that neither the syllogism of transposed quantity, nor the Fermatian inference applies to them.

If A is a finite quantity and i an infinitesimal, then in a certain sense we may write $A + i = A$. That is to say, this is so for all purposes of measurement. But this principle must not be applied except to get rid of *all* the terms in the highest order of infinitesimals present. As a mathematician, I prefer the method of infinitesimals to that of limits, as far easier and less infested with snares. Indeed, the latter, as stated in some books, involves propositions that are false; but this is not the case with the forms of the method used by Cauchy, Duhamel, and others. As they understand the doctrine of limits, it involves the notion of continuity, and therefore contains in another shape the very same ideas as the doctrine of infinitesimals.

Let us now consider an aspect of the Aristotelical principle which is particularly important in philosophy. Suppose a surface to be part red and part blue; so that every point on it is either red or blue, and, of course, no part can be both red and blue. What, then, is the color of the boundary line between the red and the blue? The answer is that red or blue, to exist at all, must be spread over a surface; and the color of the surface is the color of the surface in the immediate neighborhood of the point. I purposely use a vague form of expression. Now, as the parts of the surface in the immediate neighborhood of any ordinary point upon a curved boundary are half of them red and half blue, it follows that the boundary is half red and half blue. In like manner, we find it necessary to hold that consciousness essentially occupies time; and what is present to the mind at any ordinary instant, is what is present during a moment in which that instant occurs. Thus, the present is half past and half to come. Again, the color of the parts of a surface at any finite distance from a point, has nothing to do with its color just at that point; and, in the parallel, the feeling at any finite interval from the present has nothing to do with the present feeling, except vicariously. Take another case: the velocity of a particle at any instant of time is its mean velocity during an infinitesimal instant in which that time is contained. Just so my immediate feeling is my feeling through an infinitesimal duration containing the present instant.

ANALYSIS OF TIME

One of the most marked features about the law of mind is that it makes time to have a definite direction of flow from past to future. The relation of past to future is, in reference to the law of mind, different from the relation of future to past. This makes one of the great contrasts between the law of mind and the law of physical force, where there is no more distinction between the two opposite directions in time than between moving northward and moving southward.

In order, therefore, to analyse the law of mind, we must begin by asking what the flow of time consists in. Now, we find that in reference to any individual state of feeling, all others are of two classes, those which affect this one (or have a tendency to affect it, and what this means we shall inquire shortly), and those which do not. The present is affectible by the past but not by the future.

Moreover, if state A is affected by state B, and state B by state C, then A is affected by state C, though not so much so. It follows, that if A is affectible by B, B is not affectible by A.

If, of two states, each is absolutely unaffectible by the other, they are to be regarded as parts of the same state. They are contemporaneous.

To say that a state is *between* two states means that it affects one and is affected by the other. Between any two states in this sense lies an innumerable series of states affecting one another; and if a state lies between a given state and any other state which can be reached by inserting states between this state and any third state, these inserted states not immediately affecting or being affected by either, then the second state mentioned immediately affects or is affected by the first, in the sense that in the one the other is *ipso facto* present in a reduced degree.

These propositions involve a definition of time and of its flow. Over and above this definition they involve a doctrine, namely, that every state of feeling is affectible by every earlier state.

THAT FEELINGS HAVE INTENSIVE CONTINUITY.

Time with its continuity logically involves some other kind of continuity than its own. Time, as the universal form of change, cannot exist unless there is something to undergo change, and to undergo a change continuous in time, there must be a continuity of changeable qualities. Of the continuity of intrinsic qualities of feeling we can now form but a feeble conception. The development of the human mind has practically extinguished all feelings, except a few sporadic kinds, sound, colors, smells, warmth, etc., which now appear to be disconnected and disparate. In the case of colors, there is a tridimensional spread of feelings. Originally, all feelings may have been connected in the same way, and the presumption is that the number of dimensions was endless. For development essentially involves a limitation of possibilities. But given a number of dimensions of feeling, all possible varieties are obtainable by varying the intensities of the different elements. Accordingly, time logically supposes a continuous range of intensity in feeling. It follows, then, from the definition of continuity, that when any particular kind of feeling is present, an infinitesimal continuum of all feelings differing infinitesimally from that is present.

THAT FEELINGS HAVE SPATIAL EXTENSION.

Consider a gob of protoplasm, say an amœba or a slime-mould. It does not differ in any radical way from the contents of a nerve-cell, though its functions may be less specialised. There is no doubt that this slime-mould, or this amœba, or at any rate some similar mass of protoplasm feels. That is to say, it feels when it is in its excited condition. But note how it behaves. When the whole is quiescent and rigid, a place upon it is irritated. Just at this point, an active motion is set up, and this gradually spreads to other parts. In this action, no unity nor relation to a nucleus, or other unitary organ can be discerned. It is a mere amorphous continuum of protoplasm, with feeling passing from one part to another. Nor is there anything like a wave-motion. The activity does not advance to new parts, just as fast as it leaves old parts. Rather, in the beginning, it dies out at a slower rate than that at which it spreads. And while the process is going on, by exciting the mass at another point, a second quite independent state of excitation will be set up. In some places, neither excitation will exist, in others each separately, in still other places, both effects will be added together. Whatever there is in the whole phenomenon to make us think there is feeling in such a mass of protoplasm,—*feeling*, but plainly no *personality*,—goes logically to show that that feeling has a subjective, or substantial, spatial extension, as the excited state has. This is, no doubt, a difficult idea to seize, for the reason that it is a subjective, not an objective, extension. It is not that we have a feeling of bigness; though Professor James, perhaps rightly, teaches that we have. It is that the feeling, as a subject of inhesion, is big. Moreover, our own feelings are focused in attention to such a degree that we are not aware that ideas are not brought to an absolute unity; just as nobody not instructed by special experiment has any idea how very, very little of the field of vision is distinct. Still, we all know how the attention wanders about among our feelings; and this fact shows that those feelings that are not coordinated in attention have a reciprocal externality, although they are present at the same time. But we must not tax introspection to make a phenomenon manifest which essentially involves externality.

Since space is continuous, it follows that there must be an immediate community of feeling between parts of mind infinitesimally near together. Without this, I believe it would have been impossible for minds external to one another, ever to become coördinated, and equally impossible for any coördination to be established in the action of the nerve-matter of one brain.

AFFECTIONS OF IDEAS.

But we are met by the question what is meant by saying that one idea affects another. The unravelment of this problem requires us to trace out phenomena a little further.

Three elements go to make up an idea. The first is its intrinsic quality as a feeling. The second is the energy with which it affects other ideas, an energy which is infinite in the here-and-nowness of immediate sensation, finite and relative in the recency of the past. The third element is the tendency of an idea to bring along other ideas with it.

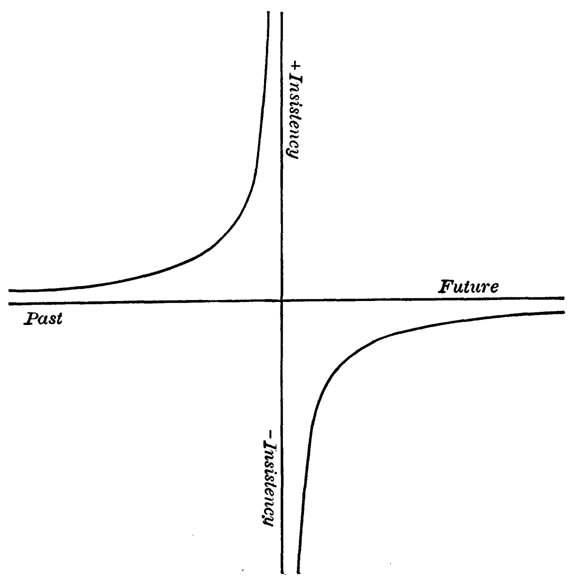
As an idea spreads, its power of affecting other ideas gets rapidly reduced; but its intrinsic quality remains nearly unchanged. It is long years now since I last saw a cardinal in his robes; and my memory of their color has become much dimmed. The color itself, however, is not remembered as dim. I have no inclination to call it a dull red. Thus, the intrinsic quality remains little changed; yet more accurate observation will show a slight reduction of it. The third element, on the other hand, has increased. As well as I can recollect, it seems to me the cardinals I used to see wore robes more scarlet than vermilion is, and highly luminous. Still, I know the color commonly called cardinal is on the crimson side of vermilion and of quite moderate luminosity, and the original idea calls up so many other hues with it, and asserts itself so feebly, that I am unable any longer to isolate it.

A finite interval of time generally contains an innumerable series of feelings; and when these become welded together in association, the result is a general idea. For we have just seen how by continuous spreading an idea becomes generalised.

The first character of a general idea so resulting is that it is living feeling. A continuum of this feeling, infinitesimal in duration, but still embracing innumerable parts, and also, though infinitesimal, entirely unlimited, is immediately present. And in its absence of boundedness a vague possibility of more than is present is directly felt.

Second, in the presence of this continuity of feeling, nominalistic maxims appear futile. There is no doubt about one idea affecting another, when we can directly perceive the one gradually modified and shaping itself into the other. Nor can there any longer be any difficulty about one idea resembling another, when we can pass along the continuous field of quality from one to the other and back again to the point which we had marked.

Third, consider the insistency of an idea. The insistency of a past idea with reference to the present is a quantity which is less the further back that past idea is, and rises to infinity as the past idea



is brought up into coincidence with the present. Here we must make one of those inductive applications of the law of continuity which have produced such great results in all the positive sciences. We must extend the law of insistency into the future. Plainly, the insistency of a future idea with reference to the present is a quantity affected by the minus sign; for it is the present that affects the future, if there be any effect, not the future that affects the present. Accordingly, the curve of insistency is a sort of equilateral hyperbola. [See the figure.] Such a conception is none the less mathematical, that its quantification cannot now be exactly specified.

Now consider the induction which we have here been led into. This curve says that feeling which has not yet emerged into immediate consciousness is already affectible and already affected. In fact, this is habit, by virtue of which an idea is brought up into present consciousness by a bond that had already been established between it, and another idea while it was still *in futuro*.

We can now see what the affection of one idea by another consists in. It is that the affected idea is attached as a logical predicate to the affecting idea as subject. So when a feeling emerges into immediate consciousness, it always appears as a modification of a more or less general object already in the mind. The word suggestion is well adapted to expressing this relation. The future is suggested by, or rather is influenced by the suggestions of, the past.

IDEAS CANNOT BE CONNECTED EXCEPT BY CONTINUITY.

That ideas can nowise be connected without continuity is sufficiently evident to one who reflects upon the matter. But still the opinion may be entertained that after continuity has once made the connection of ideas possible, then they may get to be connected in other modes than through continuity. Certainly, I cannot see how anyone can deny that the infinite diversity of the universe, which we call chance, may bring ideas into proximity which are not associated in one general idea. It may do this many times. But then the law of continuous spreading will produce a mental association; and this I suppose is an abridged statement of the way the universe has been evolved. But if I am asked whether a blind ἀνάγκη cannot bring ideas together, first I point out that it would not remain blind. There being a continuous connection between the ideas, they would infallibly become associated in a living, feeling, and perceiving general idea. Next, I cannot see what the mustness or necessity of this ἀνάγκη would consist in. In the absolute uniformity of the phenomenon, says the nominalist. Absolute is well put in; for if it merely happened so three times in succession, or three million times in succession, in the absence of any reason, the coincidence could only be attributed to chance. But absolute uniformity must extend over the whole infinite future; and it is idle to talk of that except as an idea. No; I think we can only hold that wherever ideas come together they tend to weld into general ideas; and wherever they are generally connected, general ideas govern the connection; and these general ideas are living feelings spread out.

MENTAL LAW FOLLOWS THE FORMS OF LOGIC.

The three main classes of logical inference are Deduction, Induction, and Hypothesis. These correspond to three chief modes of action of the human soul. In deduction the mind is under the dominion of a habit or association by virtue of which a general idea suggests in each case a corresponding reaction. But a certain sensation is seen to involve that idea. Consequently, that sensation is followed by that reaction. That is the way the hind legs of a frog, separated from the rest of the body, reason, when you pinch them. It is the lowest form of psychical manifestation.

By induction, a habit becomes established. Certain sensations, all involving one general idea, are followed each by the same reaction; and an association becomes established, whereby that general idea gets to be followed uniformly by that reaction.

Habit is that specialisation of the law of mind whereby a general idea gains the power of exciting reactions. But in order that the general idea should attain all its functionality, it is necessary, also, that it should become suggestible by sensations. That is accomplished by a psychical process having the form of hypothetic inference. By hypothetic inference, I mean, as I have explained in other writings, an induction from qualities. For example, I know that the kind of man known and classed as a "mugwump" has certain characteristics. He has a high self-respect and places great value upon social distinction. He laments the great part that rowdyism and unrefined good-fellowship play in the dealings of American politicians with their constituency. He thinks that the reform which would follow from the abandonment of the system by which the distribution of offices is made to strengthen party organisations and a return to the original and essential conception of office-filling would be found an unmixed good. He holds that monetary considerations should usually be the decisive ones in questions of public policy.

He respects the principle of individualism and of *laisser-faire* as the greatest agency of civilisation. These views, among others, I know to be obtrusive marks of a "mugwump." Now, suppose I casually meet a man in a railway-train, and falling into conversation find that he holds opinions of this sort; I am naturally led to suppose that he is a "mugwump." That is hypothetic inference. That is to say, a number of readily verifiable marks of a mugwump being selected, I find this man has these, and infer that he has all the other characters which go to make a thinker of that stripe. Or let us suppose that I meet a man of a semi-clerical appearance and a sub-pharisaical sniff, who appears to look at things from the point of view of a rather wooden dualism. He cites several texts of scripture and always with particular attention to their logical implications; and he exhibits a sternness, almost amounting to vindictiveness, toward evil-doers, in general. I readily conclude that he is a minister of a certain denomination. Now the mind acts in a way similar to this, every time we acquire a power of coördinating reactions in a peculiar way, as in performing any act requiring skill. Thus, most persons have a difficulty in moving the two hands simultaneously and in opposite directions through two parallel circles nearly in the medial plane of the body. To learn to do this, it is necessary to attend, first, to the different actions in different parts of the motion, when suddenly a general conception of the action springs up and it becomes perfectly easy. We think the motion we are trying to do involves this action, and this, and this. Then, the general idea comes which unites all those actions, and thereupon the desire to perform the motion calls up the general idea. The same mental process is many times employed whenever we are learning to speak a language or are acquiring any sort of skill.

Thus, by induction, a number of sensations followed by one reaction become united under one general idea followed by the same reaction; while by the hypothetic process, a number of reactions called for by one occasion get united in a general idea which is called out by the same occasion. By deduction, the habit fulfils its function of calling out certain reactions on certain occasions.

UNCERTAINTY OF MENTAL ACTION.

The inductive and hypothetic forms of inference are essentially probable inferences, not necessary; while deduction may be either necessary or probable.

But no mental action seems to be necessary or invariable in its character. In whatever manner the mind has reacted under a given sensation, in that manner it is the more likely to react again; were this, however, an absolute necessity, habits would become wooden and ineradicable, and no room being left for the formation of new habits, intellectual life would come to a speedy close. Thus, the uncertainty of the mental law is no mere defect of it, but is on the contrary of its essence. The truth is, the mind is not subject to "law," in the same rigid sense that matter is. It only experiences gentle forces which merely render it more likely to act in a given way than it otherwise would be. There always remains a certain amount of arbitrary spontaneity in its action, without which it would be dead.

Some psychologists think to reconcile the uncertainty of reactions with the principle of necessary causation by means of the law of fatigue. Truly for a *law*, this law of fatigue is a little lawless. I think it is merely a case of the general principle that an idea in spreading loses its insistency. Put me tarragon into my salad, when I have not tasted it for years, and I exclaim "What nectar is this!" But add it to every dish I taste for week after week, and a habit of expectation has been created; and in thus spreading into habit, the sensation makes hardly any more impression upon me; or, if it be noticed, it is on a new side from which it appears as rather a bore. The doctrine that fatigue is one of the primordial phenomena of mind I am much disposed to doubt. It seems a somewhat little thing to be allowed as an exception to the great principle of mental uniformisation. For this reason, I prefer to explain it in the manner here indicated, as a special case of that great principle. To consider it as something distinct in its nature, certainly somewhat strengthens the necessitarian position; but even if it be distinct, the hypothesis that all the variety and apparent arbitrariness of mental action ought to be explained away in favor of absolute determinism does not seem to me to recommend itself to a sober and sound judgment, which seeks the guidance of observed facts and not that of prepossessions.

RESTATEMENT OF THE LAW

Let me now try to gather up all these odds and ends of commentary and restate the law of mind, in a unitary way.

First, then, we find that when we regard ideas from a nominalistic, individualistic, sensualistic way, the simplest facts of mind become utterly meaningless. That one idea should resemble another or influence another, or that one state of mind should so much as be thought of in another is, from that standpoint, sheer nonsense.

Second, by this and other means we are driven to perceive, what is quite evident of itself, that instantaneous feelings flow together into a continuum of feeling, which has in a modified degree the peculiar vivacity of feeling and has gained generality. And in reference to such general ideas, or continua of feeling, the difficulties about resemblance and suggestion and reference to the external, cease to have any force.

Third, these general ideas are not mere words, nor do they consist in this, that certain concrete facts will every time happen under certain descriptions of conditions; but they are just as much, or rather far more, living realities than the feelings themselves out of which they are concreted. And to say that mental phenomena are governed by law does not mean merely that they are describable by a general formula; but that there is a living idea, a conscious continuum of feeling, which pervades them, and to which they are docile.

Fourth, this supreme law, which is the celestial and living harmony, does not so much as demand that the special ideas shall surrender their peculiar arbitrariness and caprice entirely; for that would be self-destructive. It only requires that they shall influence and be influenced by one another.

Fifth, in what measure this unification acts, seems to be regulated only by special rules; or, at least, we cannot in our present knowledge say how far it goes. But it may be said that, judging by appearances, the amount of arbitrariness in the phenomena of human minds is neither altogether trifling nor very prominent.

PERSONALITY.

Having thus endeavored to state the law of mind, in general, I descend to the consideration of a particular phenomenon which is remarkably prominent in our own consciousnesses, that of personality. A strong light is thrown upon this subject by recent observations of double and multiple personality. The theory which at one time seemed plausible that two persons in one body corresponded to the two halves of the brain will, I take it, now be universally acknowledged to be insufficient. But that which these cases make quite manifest is that personality is some kind of coördination or connection of ideas. Not much to say, this, perhaps. Yet when we consider that, according to the principle which we are tracing out, a connection between ideas is itself a general idea, and that a general idea is a living feeling, it is plain that we have at least taken an appreciable step toward the understanding of personality. This personality, like any general idea, is not a thing to be apprehended in an instant. It has to be lived in time; nor can any finite time embrace it in all its fulness. Yet in each infinitesimal interval it is present and living, though specially colored by the immediate feelings of that moment. Personality, so far as it is apprehended in a moment, is immediate self-consciousness.

But the word coördination implies somewhat more than this; it implies a teleological harmony in ideas, and in the case of personality this teleology is more than a mere purposive pursuit of a predeterminate end; it is a developmental teleology. This is personal character. A general idea, living and conscious now, it is already determinative of acts in the future to an extent to which it is not now conscious.

This reference to the future is an essential element of personality. Were the ends of a person already explicit, there would be no room for development, for growth, for life; and consequently there would be no personality. The mere carrying out of predetermined purposes is mechanical. This remark has an application to the philosophy of religion. It is that a genuine evolutionary philosophy, that is, one that makes the principle of growth a primordial element of the universe, is so far from being antagonistic to the idea of a personal creator, that it is really inseparable from that idea; while a necessitarian religion is in an altogether false position and is destined to become disintegrated. But a pseudo-evolutionism which enthrones mechanical law above the principle of growth, is at once scientifically unsatisfactory, as giving no possible hint of how the universe has come about, and hostile to all hopes of personal relations to God.

COMMUNICATION.

Consistently with the doctrine laid down in the beginning of this paper, I am bound to maintain that an idea can only be affected by an idea in continuous connection with it. By anything but an idea, it cannot be affected at all. This obliges me to say, as I do say, on other grounds, that what we call matter is not completely dead, but is merely mind hide-bound with habits. It still retains the element of diversification; and in that diversification there is life. When an idea is conveyed from one mind to another, it is by forms of combination of the diverse elements of nature, say by some curious symmetry, or by some union of a tender color with a refined odor. To such forms the law of mechanical energy has no application. If they are eternal, it is in the spirit they embody; and their origin cannot be accounted for by any mechanical necessity. They are embodied ideas; and so only can they convey ideas. Precisely how primary sensations, as colors and tones, are excited, we cannot tell, in the present state of psychology. But in our ignorance, I think that we are at liberty to suppose that they arise in essentially the same manner as the other feelings, called secondary. As far as sight and hearing are in question, we know that they are only excited by vibrations of inconceivable complexity; and the chemical senses are probably not more simple. Even the least psychical of peripheral sensations, that of pressure, has in its excitation conditions which, though apparently simple, are seen to be complicated enough when we consider the molecules and their attractions. The principle with which I set out requires me to maintain that these feelings are communicated to the nerves by continuity, so that there must be something like them in the excitants themselves. If this seems extravagant, it is to be remembered that it is the sole possible way of reaching any explanation of sensation, which otherwise must be pronounced a general fact absolutely inexplicable and ultimate. Now absolute inexplicability is a hypothesis which sound logic refuses under any circumstances to justify.

I may be asked whether my theory would be favorable or otherwise to telepathy. I have no decided answer to give to this. At first sight, it seems unfavorable. Yet there may be other modes of continuous connection between minds other than those of time and space.

The recognition by one person of another's personality takes place by means to some extent identical with the means by which he is conscious of his own personality. The idea of the second personality, which is as much as to say that second personality itself, enters within the field of direct consciousness of the first person, and is as immediately perceived as his ego, though less strongly. At the same time, the opposition between the two persons is perceived, so that the externality of the second is recognised.

The psychological phenomena of intercommunication between two minds have been unfortunately little studied. So that it is impossible to say, for certain, whether they are favorable to this theory or not. But the very extraordinary insight which some persons are able to gain of others from indications so slight that it is difficult to ascertain what they are, is certainly rendered more comprehensible by the view here taken.

A difficulty which confronts the synechistic philosophy is this. In considering personality, that philosophy is forced to accept the doctrine of a personal God; but in considering communication, it cannot but admit that if there is a personal God, we must have a direct perception of that person and indeed be in personal communication with him. Now, if that be the case, the question arises how it is possible that the existence of this being should ever have been doubted by anybody. The only answer that I can at present make is that facts that stand before our face and eyes and stare us in the face are far from being, in all cases, the ones most easily discerned. That has been remarked from time immemorial.

CONCLUSION.

I have thus developed as well as I could in a little space the *synechistic* philosophy, as applied to mind. I think that I have succeeded in making it clear that this doctrine gives room for explanations of many facts which without it are absolutely and hopelessly inexplicable; and further that it carries along with it the following doctrines: 1st, a logical realism of the most pronounced type; 2nd, objective idealism; 3rd, tychism, with its consequent thoroughgoing evolutionism. We also notice that the doctrine presents no hindrances to spiritual influences, such as some philosophies are felt to do.

C. S. PEIRCE.

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THE MONIST.

MAN'S GLASSY ESSENCE.

IN *The Monist* for January, 1891, I tried to show what conceptions ought to form the brick and mortar of a philosophical system. Chief among these was that of absolute chance for which I argued again in last April's number.* In July, I applied another fundamental idea, that of continuity, to the law of mind. Next in order, I have to elucidate, from the point of view chosen, the relation between the psychical and physical aspects of a substance.

The first step towards this ought, I think, to be the framing of a molecular theory of protoplasm. But before doing that, it seems indispensable to glance at the constitution of matter, in general. We shall, thus, unavoidably make a long detour; but, after all, our pains will not be wasted, for the problems of the papers that are to follow in the series will call for the consideration of the same question.

All physicists are rightly agreed the evidence is overwhelming which shows all sensible matter is composed of molecules in swift motion and exerting enormous mutual attractions, and perhaps repulsions, too. Even Sir William Thomson, Lord Kelvin, who wishes to explode action at a distance and return to the doctrine of a plenum, not only speaks of molecules, but undertakes to assign definite

* I am rejoiced to find, since my last paper was printed, that a philosopher as subtle and profound as Dr. Edmund Montgomery has long been arguing for the same element in the universe. Other world-renowned thinkers, as M. Renouvier and M. Delbœuf, appear to share this opinion.

magnitudes to them. The brilliant Judge Stallo, a man who did not always rightly estimate his own qualities in accepting tasks for himself, declared war upon the atomic theory in a book well worth careful perusal. To the old arguments in favor of atoms which he found in Fechner's monograph, he was able to make replies of considerable force, though they were not sufficient to destroy those arguments. But against modern proofs he made no headway at all. These set out from the mechanical theory of heat. Rumford's experiments showed that heat is not a substance. Joule demonstrated that it was a form of energy. The heating of gases under constant volume, and other facts instanced by Rankine, proved that it could not be an energy of strain. This drove physicists to the conclusion that it was a mode of motion. Then it was remembered that John Bernoulli had shown that the pressure of gases could be accounted for by assuming their molecules to be moving uniformly in rectilinear paths. The same hypothesis was now seen to account for Avogadro's law, that in equal volumes of different kinds of gases exposed to the same pressure and temperature are contained equal numbers of molecules. Shortly after, it was found to account for the laws of diffusion and viscosity of gases, and for the numerical relation between these properties. Finally, Crookes's radiometer furnished the last link in the strongest chain of evidence which supports any physical hypothesis.

Such being the constitution of gases, liquids must clearly be bodies in which the molecules wander in curvilinear paths, while in solids they move in orbits or quasi-orbits. (See my definition *solid* II, 1, in the "Century Dictionary.")

We see that the resistance to compression and to interpenetration between sensible bodies is, by one of the prime propositions of the molecular theory, due in large measure to the kinetical energy of the particles, which must be supposed to be quite remote from one another, on the average, even in solids. This resistance is no doubt influenced by finite attractions and repulsions between the molecules. All the impenetrability of bodies which we can observe is, therefore, a limited impenetrability due to kinetic and positional energy. This being the case, we have no logical right to suppose that absolute impenetrability, or the exclusive occupancy of space, belongs to molecules or to atoms. It is an unwarranted hypothesis, not a *vera causa.** Unless we are to give up the theory of energy, finite positional attractions and repulsions between molecules must be admitted. Absolute impenetrability would amount to an infinite repulsion at a certain distance. No analogy of known phenomena exists to excuse such a wanton violation of the principle of continuity as such a hypothesis is. In short, we are logically bound to adopt the Boscovichian idea that an atom is simply a distribution of component potential energy throughout space, (this distribution being absolutely rigid,) combined with inertia. The potential energy belongs to two molecules, and is to be conceived as different between molecules *A* and *B* from what it is between molecules *A* and *C*. The distribution of energy is not necessarily spherical. Nay, a molecule may conceivably have more than one centre ; it may even have a central curve, returning into itself. But I do not think there are any observed facts pointing to such multiple or linear centres. On the other hand, many facts relating to crystals, especially those observed by Voigt,† go to show that the distribution of energy is harmonical but not concentric. We can easily calculate the forces which such atoms must exert upon one another by considering‡ that they are equivalent to aggregations of pairs of electrically positive and negative points infinitely near to one another. About such an atom there would be regions of positive and of negative potential, and the number and distribution of such regions would determine the valency of the atom, a number which it is easy to see would in many cases be somewhat indeterminate. I must not dwell further upon this hypothesis, at present. In another paper, its consequences will be further considered.

I cannot assume that the students of philosophy who read this magazine are thoroughly versed in modern molecular physics, and

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* By a *vera causa*, in the logic of science, is meant a state of things known to exist in some cases and supposed to exist in other cases, because it would account for observed phenomena.

† Wiedemann, *Annalen*, 1887–1889.

‡ See Maxwell on Spherical Harmonics, in his *Electricity and Magnetism*.

therefore it is proper to mention that the governing principle in this branch of science is Clausius's law of the virial. I will first state the law, and then explain the peculiar terms of the statement. This statement is that the total kinetic energy of the particles of a system in stationary motion is equal to the total virial. By a *system* is here meant a number of particles acting upon one another.* Stationary motion is a quasi-orbital motion among a system of particles so that none of them are removed to indefinitely great distances nor acquire indefinitely great velocities. The kinetic energy of a particle is the work which would be required to bring it to rest, independently of any forces which may be acting upon it. The virial of a pair of particles is half the work which the force which actually operates between them would do if, being independent of the distance, it were to bring them together. The equation of the virial is

$$\tfrac{1}{2}\Sigma mv^2 = \tfrac{1}{2}\Sigma\Sigma Rr.$$

Here m is the mass of a particle, v its velocity, R is the attraction between two particles, and r is the distance between them. The sign Σ on the left hand side signifies that the values of mv^2 are to be summed for all the particles, and $\Sigma\Sigma$ on the right hand side signifies that the values of Rr are to be summed for all the pairs of particles. If there is an external pressure P (as from the atmosphere) upon the system, and the volume of vacant space within the boundary of that pressure is V, then the virial must be understood as including $\tfrac{3}{2}PV$, so that the equation is

$$\tfrac{1}{2}\Sigma mv^2 = \tfrac{3}{2}PV + \tfrac{1}{2}\Sigma\Sigma Rr.$$

There is strong (if not demonstrative) reason for thinking that the temperature of any body above the absolute zero (—273° C.), is proportional to the average kinetic energy of its molecules, or say $a\theta$,

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* The word *system* has three peculiar meanings in mathematics. (*A*.) It means an orderly exposition of the truths of astronomy, and hence a theory of the motions of the stars; as the Ptolemaic *system*, the Copernican *system*. This is much like the sense in which we speak of the Calvinistic *system* of theology, the Kantian *system* of philosophy, etc. (*B*.) It means the aggregate of the planets considered as all moving in somewhat the same way, as the solar *system;* and hence any aggregate of particles moving under mutual forces. (*C*.) It means a number of forces acting simultaneously upon a number of particles.

where a is a constant and θ is the absolute temperature. Hence, we may write the equation

$$a\theta = \tfrac{1}{2}\overline{mv^2} = \tfrac{3}{2}P\overline{V} + \tfrac{1}{2}\overline{\Sigma Rr}$$

where the heavy lines above the different expressions signify that the average values for single molecules are to be taken. In 1872, a student in the University of Leyden, Van der Waals, propounded in his thesis for the doctorate a specialisation of the equation of the virial which has since attracted great attention. Namely, he writes it

$$a\theta = (P + \frac{c}{V^2})\,(V - b).$$

The quantity b is the volume of a molecule, which he supposes to be an impenetrable body, and all the virtue of the equation lies in this term which makes the equation a cubic in V, which is required to account for the shape of certain isothermal curves.* But if the idea of an impenetrable atom is illogical, that of an impenetrable molecule is almost absurd. For the kinetical theory of matter teaches us that a molecule is like a solar system or star-cluster in miniature. Unless we suppose that in all heating of gases and vapors internal work is performed upon the molecules, implying that their atoms are at considerable distances, the whole kinetical theory of gases falls to the ground. As for the term added to P, there is no more than a partial and roughly approximative justification for it. Namely, let us imagine two spheres described round a particle as their centre, the radius of the larger being so great as to include all the particles whose action upon the centre is sensible, while the radius of the smaller is so large that a good many molecules are included within it. The possibility of describing such a sphere as the outer one implies that the attraction of the particles varies at some distances inversely as some higher power of the distance than the cube, or, to speak more clearly, that the attraction multiplied by the cube of the distance diminishes as the distance increases; for the number of particles at a given distance from any

* But, in fact, an inspection of these curves is sufficient to show that they are of a higher degree than the third. For they have the line $V = 0$, or some line V a constant for an asymptote, while for small values of P, the values of $d^2P/(dV)^2$ are positive.

one particle is proportionate to the square of that distance and each of these gives a term of the virial which is the product of the attraction into the distance. Consequently unless the attraction multiplied by the cube of the distance diminished so rapidly with the distance as soon to become insensible, no such outer sphere as is supposed could be described. However, ordinary experience shows that such a sphere is possible; and consequently there must be distances at which the attraction does thus rapidly diminish as the distance increases. The two spheres, then, being so drawn, consider the virial of the central particle due to the particles between them. Let the density of the substance be increased, say, N times. Then, for every term, Rr, of the virial before the condensation, there will be N terms of the same magnitude after the condensation. Hence, the virial of each particle will be proportional to the density, and the equation of the virial becomes

$$a\theta = P\overline{V} + \frac{c}{\overline{V}}.$$

This omits the virial within the inner sphere, the radius of which is so taken that within that distance the number of particles is not proportional to the number in a large sphere. For Van der Waals this radius is the diameter of his hard molecules, which assumption gives his equation. But it is plain that the attraction between the molecules must to a certain extent modify their distribution, unless some pecular conditions are fulfilled. The equation of Van der Waals can be approximately true therefore only for a gas. In a solid or liquid condition, in which the removal of a small amount of pressure has little effect on the volume, and where consequently the virial must be much greater than $P\overline{V}$, the virial must increase with the volume. For suppose we had a substance in a critical condition in which an increase of the volume would diminish the virial more than it would increase $\frac{3}{2}P\overline{V}$. If we were forcibly to diminish the volume of such a substance, when the temperature became equalised, the pressure which it could withstand would be less than before, and it would be still further condensed, and this would go on indefinitely until a condition were reached in which an increase of volume would increase $\frac{3}{2}P\overline{V}$ more than it would decrease the virial.

In the case of solids, at least, P may be zero; so that the state reached would be one in which the virial increases with the volume, or the attraction between the particles does not increase so fast with a diminution of their distance as it would if the attraction were inversely as the distance.

Almost contemporaneously with Van der Waals's paper, another remarkable thesis for the doctorate was presented at Paris by Amagat. It related to the elasticity and expansion of gases, and to this subject the superb experimenter, its author, has devoted his whole subsequent life. Especially interesting are his observations of the volumes of ethylene and of carbonic acid at temperatures from 20° to 100° and at pressures ranging from an ounce to 5000 pounds to the square inch. As soon as Amagat had obtained these results, he remarked that the "coefficient of expansion at constant volume," as it is absurdly called, that is, the rate of variation of the pressure with the temperature, was very nearly constant for each volume. This accords with the equation of the virial, which gives

$$\frac{dp}{d\theta}=\frac{a}{\overline{V}}-\frac{d\Sigma\overline{Rr}}{d\theta}.$$

Now, the virial must be nearly independent of the temperature, and therefore the last term almost disappears. The virial would not be quite independent of the temperature, because if the temperature (i. e. the square of the velocity of the molecules) is lowered, and the pressure correspondingly lowered, so as to make the volume the same, the attractions of the molecules will have more time to produce their effects, and consequently, the pairs of molecules the closest together will be held together longer and closer; so that the virial will generally be increased by a decrease of temperature. Now, Amagat's experiments do show an excessively minute effect of this sort, at least, when the volumes are not too small. However, the observations are well enough satisfied by assuming the "coefficient of expansion at constant volume" to consist wholly of the first term, $a/\overline{V}$. Thus, Amagat's experiments enable us to determine the values of a and thence to calculate the virial; and this we find varies for carbonic acid gas nearly inversely to $\overline{V}^{0.9}$. There is, thus, a rough approximation to satisfying Van der Waals's equation. But the most interesting result of Amagat's experiments, for our purpose at any rate, is that the quantity *a*, though nearly constant for any one volume, differs considerably with the volume, nearly doubling when the volume is reduced fivefold. This can only indicate that the mean kinetic energy of a given mass of the gas for a given temperature is greater the more the gas is compressed. But the laws of mechanics appear to enjoin that the mean kinetic energy of a moving particle shall be constant at any given temperature. The only escape from contradiction, then, is to suppose that the mean mass of a moving particle diminishes upon the condensation of the gas. In other words, many of the molecules are dissociated, or broken up into atoms or sub-molecules. The idea that dissociation should be favored by diminishing the volume will be pronounced by physicists, at first blush, as contrary to all our experience. But it must be remembered that the circumstances we are speaking of, that of a gas under fifty or more atmospheres pressure, are also unusual. That the "coefficient of expansion under constant volume" when multiplied by the volumes should increase with a decrement of the volume is also quite contrary to ordinary experience; yet it undoubtedly takes place in all gases under great pressure. Again, the doctrine of Arrhenius* is now generally accepted, that the molecular conductivity of an electrolyte is proportional to the dissociation of ions. Now the molecular conductivity of a fused electrolyte is usually superior to that of a solution. Here is a case, then, in which diminution of volume is accompanied by increased dissociation.

The truth is that several different kinds of dissociation have to be distinguished. In the first place, there is the dissociation of a chemical molecule to form chemical molecules under the regular action of chemical laws. This may be a double decomposition, as when iodhydric acid is dissociated, according to the formula

$$HI + HI = HH + II;$$

or, it may be a simple decomposition, as when pentachloride of phosphorus is dissociated according to the formula

$$PCl_5 = PCl_3 + ClCl.$$

* Anticipated by Clausius as long ago as 1857; and by Williamson in 1851.

All these dissociations require, according to the laws of thermo-chemistry, an elevated temperature. In the second place, there is the dissociation of a physically polymerous molecule, that is, of several chemical molecules joined by physical attractions. This I am inclined to suppose is a common concomitant of the heating of solids and liquids; for in these bodies there is no increase of compressibility with the temperature at all comparable with the increase of the expansibility. But, in the third place, there is the dissociation with which we are now concerned, which must be supposed to be a throwing off of unsaturated sub-molecules or atoms from the molecule. The molecule may, as I have said, be roughly likened to a solar system. As such, molecules are able to produce perturbations of one another's internal motions; and in this way a planet, i. e. a sub-molecule, will occasionally get thrown off and wander about by itself, till it finds another unsaturated sub-molecule with which it can unite. Such dissociation by perturbation will naturally be favored by the proximity of the molecules to one another.

Let us now pass to the consideration of that special substance, or rather class of substances, whose properties form the chief subject of botany and of zoölogy, as truly as those of the silicates form the chief subject of mineralogy: I mean the life-slimes, or protoplasm. Let us begin by cataloguing the general characters of these slimes. They one and all exist in two states of aggregation, a solid or nearly solid state and a liquid or nearly liquid state; but they do not pass from the former to the latter by ordinary fusion. They are readily decomposed by heat, especially in the liquid state; nor will they bear any considerable degree of cold. All their vital actions take place at temperatures very little below the point of decomposition. This extreme instability is one of numerous facts which demonstrate the chemical complexity of protoplasm. Every chemist will agree that they are far more complicated than the albumens. Now, albumen is estimated to contain in each molecule about a thousand atoms; so that it is natural to suppose that the protoplasms contain several thousands. We know that while they are chiefly composed of oxygen, hydrogen, carbon, and nitrogen, a large number of other elements enter into living bodies in small proportions; and it is likely that most of these enter into the composition of protoplasms. Now, since the numbers of chemical varieties increase at an enormous rate with the number of atoms per molecule, so that there are certainly hundreds of thousands of substances whose molecules contain twenty atoms or fewer, we may well suppose that the number of protoplasmic substances runs into the billions or trillions. Professor Cayley has given a mathematical theory of "trees," with a view of throwing a light upon such questions; and in that light the estimate of trillions (in the English sense) seems immoderately moderate. It is true that an opinion has been emitted, and defended among biologists, that there is but one kind of protoplasm; but the observations of biologists, themselves, have almost exploded that hypothesis, which from a chemical standpoint appears utterly incredible. The anticipation of the chemist would decidedly be that enough different chemical substances having protoplasmic characters might be formed to account, not only for the differences between nerve-slime and muscle-slime, between whale-slime and lion-slime, but also for those minuter pervasive variations which characterise different breeds and single individuals.

Protoplasm, when quiescent, is, broadly speaking, solid; but when it is disturbed in an appropriate way, or sometimes even spontaneously without external disturbance, it becomes, broadly speaking, liquid. A moner in this state is seen under the microscope to have streams within its matter; a slime-mould slowly flows by force of gravity. The liquefaction starts from the point of disturbance and spreads through the mass. This spreading, however, is not uniform in all directions; on the contrary it takes at one time one course, at another another, through the homogeneous mass, in a manner that seems a little mysterious. The cause of disturbance being removed, these motions gradually (with higher kinds of protoplasm, quickly) cease, and the slime returns to its solid condition.

The liquefaction of protoplasm is accompanied by a mechanical phenomenon. Namely, some kinds exhibit a tendency to draw themselves up into a globular form. This happens particularly with the contents of muscle-cells. The prevalent opinion, founded on some of the most exquisite experimental investigations that the history of science can show, is undoubtedly that the contraction of muscle-cells is due to osmotic pressure; and it must be allowed that that is a factor in producing the effect. But it does not seem to me that it satisfactorily accounts even for the phenomena of muscular contraction; and besides, even naked slimes often draw up in the same way. In this case, we seem to recognise an increase of the surface-tension. In some cases, too, the reverse action takes place, extraordinary pseudopodia being put forth, as if the surface-tension were diminished in spots. Indeed, such a slime always has a sort of skin, due no doubt to surface-tension, and this seems to give way at the point where a pseudopodium is put forth.

Long-continued or frequently repeated liquefaction of the protoplasm results in an obstinate retention of the solid state, which we call fatigue. On the other hand repose in this state, if not too much prolonged, restores the liquefiability. These are both important functions.

The life-slimes have, further, the peculiar property of growing. Crystals also grow; their growth, however, consists merely in attracting matter like their own from the circumambient fluid. To suppose the growth of protoplasm of the same nature, would be to suppose this substance to be spontaneously generated in copious supplies wherever food is in solution. Certainly, it must be granted that protoplasm is but a chemical substance, and that there is no reason why it should not be formed synthetically like any other chemical substance. Indeed, Clifford has clearly shown that we have overwhelming evidence that it is so formed. But to say that such formation is as regular and frequent as the assimilation of food is quite another matter. It is more consonant with the facts of observation to suppose that assimilated protoplasm is formed at the instant of assimilation, under the influence of the protoplasm already present. For each slime in its growth preserves its distinctive characters with wonderful truth, nerve-slime growing nerve-slime and muscle-slime muscle-slime, lion-slime growing lion-slime, and all the varieties of breeds and even individual characters being preserved in the growth. Now it is too much to suppose there are billions of different kinds of protoplasm floating about wherever there is food.

The frequent liquefaction of protoplasm increases its power of assimilating food; so much so, indeed, that it is questionable whether in the solid form it possesses this power.

The life-slime wastes as well as grows; and this too takes place chiefly if not exclusively in its liquid phases.

Closely connected with growth is reproduction; and though in higher forms this is a specialised function, it is universally true that wherever there is protoplasm, there is, will be, or has been a power of reproducing that same kind of protoplasm in a separated organism. Reproduction seems to involve the union of two sexes; though it is not demonstrable that this is always requisite.

Another physical property of protoplasm is that of taking habits. The course which the spread of liquefaction has taken in the past is rendered thereby more likely to be taken in the future; although there is no absolute certainty that the same path will be followed again.

Very extraordinary, certainly, are all these properties of protoplasm; as extraordinary as indubitable. But the one which has next to be mentioned, while equally undeniable, is infinitely more wonderful. It is that protoplasm feels. We have no direct evidence that this is true of protoplasm universally, and certainly some kinds feel far more than others. But there is a fair analogical inference that all protoplasm feels. It not only feels but exercises all the functions of mind.

Such are the properties of protoplasm. The problem is to find a hypothesis of the molecular constitution of this compound which will account for these properties, one and all.

Some of them are obvious results of the excessively complicated constitution of the protoplasm molecule. All very complicated substances are unstable; and plainly a molecule of several thousand atoms may be separated in many ways into two parts in each of which the polar chemical forces are very nearly saturated. In the solid protoplasm, as in other solids, the molecules must be supposed to be moving as it were in orbits, or, at least, so as not to wander indefinitely. But this solid cannot be melted, for the same reason that starch cannot be melted; because an amount of heat insufficient to make the entire molecules wander is sufficient to break them up completely and cause them to form new and simpler molecules. But when one of the molecules is disturbed, even if it be not quite thrown out of its orbit at first, sub-molecules of perhaps several hundred atoms each are thrown off from it. These will soon acquire the same mean kinetic energy as the others, and therefore velocities several times as great. They will naturally begin to wander, and in wandering will perturb a great many other molecules and cause them in their turn to behave like the one originally deranged. So many molecules will thus be broken up, that even those that are intact will no longer be restrained within orbits, but will wander about freely. This is the usual condition of a liquid, as modern chemists understand it; for in all electrolytic liquids there is considerable dissociation.

But this process necessarily chills the substance, not merely on account of the heat of chemical combination, but still more because the number of separate particles being greatly increased, the mean kinetic energy must be less. The substance being a bad conductor, this heat is not at once restored. Now the particles moving more slowly, the attractions between them have time to take effect, and they approach the condition of equilibrium. But their dynamic equilibrium is found in the restoration of the solid condition, which therefore takes place, if the disturbance is not kept up.

When a body is in the solid condition, most of its molecules must be moving at the same rate, or, at least, at certain regular sets of rates; otherwise the orbital motion would not be preserved. The distances of neighboring molecules must always be kept between a certain maximum and a certain minimum value. But if, without absorption of heat, the body be thrown into a liquid condition, the distances of neighboring molecules will be far more unequally distributed, and an effect upon the virial will result. The chilling of protoplasm upon its liquefaction must also be taken into account. The ordinary effect will no doubt be to increase the cohesion and with that the surface-tension, so that the mass will tend to draw itself up. But in special cases, the virial will be increased so much that the surface-tension will be diminished at points where the temperature is first restored. In that case, the outer film will give way and the tension at other places will aid in causing the general fluid to be poured out at those points, forming pseudopodia.

When the protoplasm is in a liquid state, and then only, a solution of food is able to penetrate its mass by diffusion. The protoplasm is then considerably dissociated ; and so is the food, like all dissolved matter. If then the separated and unsaturated sub-molecules of the food happen to be of the same chemical species as sub-molecules of the protoplasm, they may unite with other sub-molecules of the protoplasm to form new molecules, in such a fashion that when the solid state is resumed, there may be more molecules of protoplasm than there were at the beginning. It is like the jack-knife whose blade and handle, after having been severally lost and replaced, were found and put together to make a new knife.

We have seen that protoplasm is chilled by liquefaction, and that this brings it back to the solid state, when the heat is recovered. This series of operations must be very rapid in the case of nerve-slime and even of muscle-slime, and may account for the unsteady or vibratory character of their action. Of course, if assimilation takes place, the heat of combination, which is probably trifling, is gained. On the other hand, if work is done, whether by nerve or by muscle, loss of energy must take place. In the case of the muscle, the mode by which the instantaneous part of the fatigue is brought about is easily traced out. If when the muscle contracts it be under stress, it will contract less than it otherwise would do, and there will be a loss of heat. It is like an engine which should work by dissolving salt in water and using the contraction during the solution to lift a weight, the salt being recovered afterwards by distillation. But the major part of fatigue has nothing to do with the correlation of forces. A man must labor hard to do in a quarter of an hour the work which draws from him enough heat to cool his body by a single degree. Meantime, he will be getting heated, he will be pouring out extra products of combustion, perspiration, etc., and he will be driving the blood at an accelerated rate through minute tubes at great expense. Yet all this will have little to do with his fatigue. He may sit quietly at his table writing, doing practically no physical work at all, and yet in a few hours be terribly fagged. This seems to be owing to the deranged sub-molecules of the nerve-slime not having had time to settle back into their proper combinations. When such sub-molecules are thrown out, as they must be from time to time, there is so much waste of material.

In order that a sub-molecule of food may be thoroughly and firmly assimilated into a broken molecule of protoplasm, it is necessary not only that it should have precisely the right chemical composition, but also that it should be at precisely the right spot at the right time and should be moving in precisely the right direction with precisely the right velocity. If all these conditions are not fulfilled, it will be more loosely retained than the other parts of the molecule; and every time it comes round into the situation in which it was drawn in, relatively to the other parts of that molecule and to such others as were near enough to be factors in the action, it will be in special danger of being thrown out again. Thus, when a partial liquefaction of the protoplasm takes place many times to about the same extent, it will, each time, be pretty nearly the same molecules that were last drawn in that are now thrown out. They will be thrown out, too, in about the same way, as to position, direction of motion, and velocity, in which they were drawn in; and this will be in about the same course that the ones last before them were thrown out. Not exactly, however; for the very cause of their being thrown off so easily is their not having fulfilled precisely the conditions of stable retention. Thus, the law of habit is accounted for, and with it its peculiar characteristic of not acting with exactitude.

It seems to me that this explanation of habit, aside from the question of its truth or falsity, has a certain value as an addition to our little store of mechanical examples of actions analogous to habit. All the others, so far as I know, are either statical or else involve forces which, taking only the sensible motions into account, violate the law of energy. It is so with the stream that wears its own bed. Here, the sand is carried to its most stable situation and left there. The law of energy forbids this; for when anything reaches a position of stable equilibrium, its momentum will be at a maximum, so that it can according to this law only be left at rest in an unstable situation. In all the statical illustrations, too, things are brought into certain states and left there. A garment receives folds and keeps them; that is, its limit of elasticity is exceeded. This failure to spring back is again an apparent violation of the law of energy; for the substance will not only not spring back of itself (which might be due to an unstable equilibrium being reached) but will not even do so when an impulse that way is applied to it. Accordingly, Professor James says "the phenomena of habit . . . are due to the plasticity of the . . . materials." Now, plasticity of materials means the having of a low limit of elasticity. (See the "Century Dictionary," under *solid.*) But the hypothetical constitution of protoplasm here proposed involves no forces but attractions and repulsions strictly following the law of energy. The action here, that is, the throwing of an atom out of its orbit in a molecule, and the entering of a new atom into nearly, but not quite the same orbit, is somewhat similar to the molecular actions which may be supposed to take place in a solid strained beyond its limit of elasticity. Namely, in that case certain molecules must be thrown out of their orbits, to settle down again shortly after into new orbits. In short, the plastic solid resembles protoplasm in being partially and temporarily liquefied by a slight mechanical force. But the taking of a set by a solid body has but a moderate resemblance to the taking of a habit, inasmuch as the characteristic feature of the latter, its inexactitude and want of complete determinacy, is not so marked in the former, if it can be said to be present there, at all.

The truth is that though the molecular explanation of habit is pretty vague on the mathematical side, there can be no doubt that systems of atoms having polar forces would act substantially in that manner, and the explanation is even too satisfactory to suit the convenience of an advocate of tychism. For it may fairly be urged that since the phenomena of habit may thus result from a purely mechanical arrangement, it is unnecessary to suppose that habit-taking is a primordial principle of the universe. But one fact remains unexplained mechanically, which concerns not only the facts of habit, but all cases of actions apparently violating the law of energy; it is that all these phenomena depend upon aggregations of trillions of molecules in one and the same condition and neighborhood; and it is by no means clear how they could have all been brought and left in the same place and state by any conservative forces. But let the mechanical explanation be as perfect as it may, the state of things which it supposes presents evidence of a primordial habit-taking tendency. For it shows us like things acting in like ways because they are alike. Now, those who insist on the doctrine of necessity will for the most part insist that the physical world is entirely individual. Yet law involves an element of generality. Now to say that generality is primordial, but generalisation not, is like saying that diversity is primordial but diversification not. It turns logic upside down. At any rate, it is clear that nothing but a principle of habit, itself due to the growth by habit of an infinitesimal chance tendency toward habit-taking, is the only bridge that can span the chasm between the chance-medley of chaos and the cosmos of order and law.

I shall not attempt a molecular explanation of the phenomena of reproduction, because that would require a subsidiary hypothesis, and carry me away from my main object. Such phenomena, universally diffused though they be, appear to depend upon special conditions; and we do not find that all protoplasm has reproductive powers.

But what is to be said of the property of feeling? If consciousness belongs to all protoplasm, by what mechanical constitution is this to be accounted for? The slime is nothing but a chemical compound. There is no inherent impossibility in its being formed synthetically in the laboratory, out of its chemical elements; and if it were so made, it would present all the characters of natural protoplasm. No doubt, then, it would feel. To hesitate to admit this would be puerile and ultra-puerile. By what element of the molecular arrangement, then, would that feeling be caused? This question cannot be evaded or pooh-poohed. Protoplasm certainly does feel; and unless we are to accept a weak dualism, the property must be shown to arise from some peculiarity of the mechanical system. Yet the attempt to deduce it from the three laws of mechanics, applied to never so ingenious a mechanical contrivance, would obviously be futile. It can never be explained, unless we admit that physical events are but degraded or undeveloped forms of psychical events. But once grant that the phenomena of matter are but the result of the sensibly complete sway of habits upon mind, and it only remains to explain why in the protoplasm these habits are to some slight extent broken up, so that according to the law of mind, in that special clause of it sometimes called the principle of accommodation,* feeling becomes intensified. Now the manner in which habits generally get broken up is this. Reactions usually terminate in the removal of a stimulus ; for the excitation continues as long as the stimulus is present. Accordingly, habits are general ways of behavior which are associated with the removal of stimuli. But when the expected removal of the stimulus fails to occur, the excitation continues and increases, and non-habitual reactions take place ; and these tend to weaken the habit. If, then, we suppose that matter never does obey its ideal laws with absolute precision, but that there are almost insensible fortuitous departures from regularity, these will produce, in general, equally minute effects. But protoplasm is in an excessively unstable condition ; and it is the characteristic of unstable equilibrium, that near that point excessively minute causes may produce startlingly large effects. Here then, the usual departures from regularity will be followed by others that are very great ; and the large fortuitous departures from law so produced, will tend still further to break up the laws, supposing that these are of the nature of habits. Now, this breaking up of habit and renewed fortuitous spontaneity will, according to the law of mind, be accompanied by an intensification of feeling. The nerve-protoplasm is, without doubt, in the most unstable condition of any kind of matter ; and consequently, there the resulting feeling is the most manifest.

Thus we see that the idealist has no need to dread a mechan-

* "Physiologically, . . . accommodation means the breaking up of a habit. . . . Psychologically, it means reviving consciousness." Baldwin, *Psychology*, Part III ch. i., § 5.

ical theory of life. On the contrary, such a theory, fully developed, is bound to call in a tychistic idealism as its indispensable adjunct. Wherever chance-spontaneity is found, there, in the same proportion, feeling exists. In fact, chance is but the outward aspect of that which within itself is feeling. I long ago showed that real existence, or thing-ness, consists in regularities. So, that primeval chaos in which there was no regularity was mere nothing, from a physical aspect. Yet it was not a blank zero; for there was an intensity of consciousness there in comparison with which all that we ever feel is but as the struggling of a molecule or two to throw off a little of the force of law to an endless and innumerable diversity of chance utterly unlimited.

But after some atoms of the protoplasm have thus become partially emancipated from law, what happens next to them? To understand this, we have to remember that no mental tendency is so easily strengthened by the action of habit as is the tendency to take habits. Now, in the higher kinds of protoplasm, especially, the atoms in question have not only long belonged to one molecule or another of the particular mass of slime of which they are parts; but before that, they were constituents of food of a protoplasmic constitution. During all this time, they have been liable to lose habits and to recover them again; so that now, when the stimulus is removed, and the foregone habits tend to reassert themselves, they do so in the case of such atoms with great promptness. Indeed, the return is so prompt that there is nothing but the feeling to show conclusively that the bonds of law have ever been relaxed.

In short, diversification is the vestige of chance-spontaneity; and wherever diversity is increasing, there chance must be operative. On the other hand, wherever uniformity is increasing, habit must be operative. But wherever actions take place under an established uniformity, there so much feeling as there may be takes the mode of a sense of reaction. That is the manner in which I am led to define the relation between the fundamental elements of consciousness and their physical equivalents.

It remains to consider the physical relations of general ideas. It may be well here to reflect that if matter has no existence except as a specialisation of mind, it follows that whatever affects matter according to regular laws is itself matter. But all mind is directly or indirectly connected with all matter, and acts in a more or less regular way; so that all mind more or less partakes of the nature of matter. Hence, it would be a mistake to conceive of the psychical and the physical aspects of matter as two aspects absolutely distinct. Viewing a thing from the outside, considering its relations of action and reaction with other things, it appears as matter. Viewing it from the inside, looking at its immediate character as feeling, it appears as consciousness. These two views are combined when we remember that mechanical laws are nothing but acquired habits, like all the regularities of mind, including the tendency to take habits, itself; and that this action of habit is nothing but generalisation, and generalisation is nothing but the spreading of feelings. But the question is, how do general ideas appear in the molecular theory of protoplasm?

The consciousness of a habit involves a general idea. In each action of that habit certain atoms get thrown out of their orbit, and replaced by others. Upon all the different occasions it is different atoms that are thrown off, but they are analogous from a physical point of view, and there is an inward sense of their being analogous. Every time one of the associated feelings recurs, there is a more or less vague sense that there are others, that it has a general character, and of about what this general character is. We ought not, I think, to hold that in protoplasm habit never acts in any other than the particular way suggested above. On the contrary, if habit be a primary property of mind, it must be equally so of matter, as a kind of mind. We can hardly refuse to admit that wherever chance motions have general characters, there is a tendency for this generality to spread and to perfect itself. In that case, a general idea is a certain modification of consciousness which accompanies any regularity or general relation between chance actions.

The consciousness of a general idea has a certain "unity of the ego," in it, which is identical when it passes from one mind to another. It is, therefore, quite analogous to a person; and, indeed, a person is only a particular kind of general idea. Long ago, in the

Journal of Speculative Philosophy (Vol. III, p. 156), I pointed out that a person is nothing but a symbol involving a general idea; but my views were, then, too nominalistic to enable me to see that every general idea has the unified living feeling of a person.

All that is necessary, upon this theory, to the existence of a person is that the feelings out of which he is constructed should be in close enough connection to influence one another. Here we can draw a consequence which it may be possible to submit to experimental test. Namely, if this be the case, there should be something like personal consciousness in bodies of men who are in intimate and intensely sympathetic communion. It is true that when the generalisation of feeling has been carried so far as to include all within a person, a stopping-place, in a certain sense, has been attained; and further generalisation will have a less lively character. But we must not think it will cease. *Esprit de corps*, national sentiment, sym-pathy, are no mere metaphors. None of us can fully realise what the minds of corporations are, any more than one of my brain-cells can know what the whole brain is thinking. But the law of mind clearly points to the existence of such personalities, and there are many ordinary observations which, if they were critically examined and supplemented by special experiments, might, as first appearances promise, give evidence of the influence of such greater persons upon individuals. It is often remarked that on one day half a dozen people, strangers to one another, will take it into their heads to do one and the same strange deed, whether it be a physical experiment, a crime, or an act of virtue. When the thirty thousand young people of the society for Christian Endeavor were in New York, there seemed to me to be some mysterious diffusion of sweetness and light. If such a fact is capable of being made out anywhere, it should be in the church. The Christians have always been ready to risk their lives for the sake of having prayers in common, of getting together and praying simultaneously with great energy, and especially for their common body, for "the whole state of Christ's church militant here in earth," as one of the missals has it. This practice they have been keeping up everywhere, weekly, for many centuries. Surely, a personality ought to have developed in that church, in that "bride of Christ," as they call it, or else there is a strange break in the action of mind, and I shall have to acknowledge my views are much mistaken. Would not the societies for psychical research be more likely to break through the clouds, in seeking evidences of such corporate personality, than in seeking evidences of telepathy, which, upon the same theory, should be a far weaker phenomenon ?

C. S. Peirce.

EVOLUTIONARY LOVE.

AT FIRST BLUSH. COUNTER-GOSPELS.

PHILOSOPHY, when just escaping from its golden pupa-skin, mythology, proclaimed the great evolutionary agency of the universe to be Love. Or, since this pirate-lingo, English, is poor in such-like words, let us say Eros, the exuberance-love. Afterwards, Empedocles set up passionate-love and hate as the two coördinate powers of the universe. In some passages, kindness is the word. But certainly, in any sense in which it has an opposite, to be senior partner of that opposite, is the highest position that love can attain. Nevertheless, the ontological gospeller, in whose days those views were familiar topics, made the One Supreme Being, by whom all things have been made out of nothing, to be cherishing-love. What, then, can he say to hate? Never mind, at this time, what the scribe of the apocalypse, if he were John, stung at length by persecution into a rage unable to distinguish suggestions of evil from visions of heaven, and so become the Slanderer of God to men, may have dreamed. The question is rather what the sane John thought, or ought to have thought, in order to carry out his idea consistently. His statement that God is love seems aimed at that saying of Ecclesiastes that we cannot tell whether God bears us love or hatred. "Nay," says John, "we can tell, and very simply! We know and have trusted the love which God hath in us. God is love." There is no logic in this, unless it means that God loves all men. In the preceding paragraph, he had said, "God is light and in him is no darkness at all." We are to understand, then, that as darkness is merely the defect of light, so hatred and evil are mere imperfect stages of ἀγάπη and ἀγαθόν, love and loveliness. This concords with that utterance reported in John's Gospel: "God sent not the Son into the world to judge the world; but that the world should through him be saved. He that believeth on him is not judged: he that believeth not hath been judged already . . . And this is the judgment, that the light is come into the world, and that men loved darkness rather than the light." That is to say, God visits no punishment on them; they punish themselves, by their natural affinity for the defective. Thus, the love that God is, is not a love of which hatred is the contrary; otherwise Satan would be a coördinate power; but it is a love which embraces hatred as an imperfect stage of it, an Anteros—yea, even needs hatred and hatefulness as its object. For self-love is no love; so if God's self is love, that which he loves must be defect of love; just as a luminary can light up only that which otherwise would be dark. Henry James, the Swedenborgian, says: "It is no doubt very tolerable finite or creaturely love to love one's own in another, to love another for his conformity to one's self: but nothing can be in more flagrant contrast with the creative Love, all whose tenderness *ex vi termini* must be reserved only for what intrinsically is most bitterly hostile and negative to itself." This is from "Substance and Shadow: an Essay on the Physics of Creation." It is a pity he had not filled his pages with things like this, as he was able easily to do, instead of scolding at his reader and at people generally, until the physics of creation was wellnigh forgot. I must deduct, however, from what I just wrote: obviously no genius could make his every sentence as sublime as one which discloses for the problem of evil its everlasting solution.

The movement of love is circular, at one and the same impulse projecting creations into independency and drawing them into harmony. This seems complicated when stated so; but it is fully summed up in the simple formula we call the Golden Rule. This does not, of course, say, Do everything possible to gratify the egoistic impulses of others, but it says, Sacrifice your own perfection to the perfectionment of your neighbor. Nor must it for a moment be confounded with the Benthamite, or Helvetian, or Beccarian motto, Act for the greatest good of the greatest number. Love is not directed to abstractions but to persons; not to persons we do not know, nor to numbers of people, but to our own dear ones, our family and neighbors. "Our neighbor," we remember, is one whom we live near, not locally perhaps, but in life and feeling.

Everybody can see that the statement of St. John is the formula of an evolutionary philosophy, which teaches that growth comes only from love, from--I will not say self-*sacrifice*, but from the ardent impulse to fulfil another's highest impulse. Suppose, for example, that I have an idea that interests me. It is my creation. It is my creature; for as shown in last July's *Monist*, it is a little person. I love it; and I will sink myself in perfecting it. It is not by dealing out cold justice to the circle of my ideas that I can make them grow, but by cherishing and tending them as I would the flowers in my garden. The philosophy we draw from John's gospel is that this is the way mind develops; and as for the cosmos, only so far as it yet is mind, and so has life, is it capable of further evolution. Love, recognising germs of loveliness in the hateful, gradually warms it into life, and makes it lovely. That is the sort of evolution which every careful student of my essay "The Law of Mind," must see that *synechism* calls for.

The nineteenth century is now fast sinking into the grave, and we all begin to review its doings and to think what character it is destined to bear as compared with other centuries in the minds of future historians. It will be called, I guess, the Economical Century; for political economy has more direct relations with all the branches of its activity than has any other science. Well, political economy has its formula of redemption, too. It is this: Intelligence in the service of greed ensures the justest prices, the fairest contracts, the most enlightened conduct of all the dealings between men, and leads to the *summum bonum*, food in plenty and perfect comfort. Food for whom? Why, for the greedy master of intelligence. I do not mean to say that this is one of the legitimate conclusions of political economy, the scientific character of which I fully acknowledge. But the study of doctrines, themselves true, will often temporarily encourage generalisations extremely false, as the study of physics has encouraged necessitarianism. What I say, then, is that the great attention paid to economical questions during our century has induced an exaggeration of the beneficial effects of greed and of the unfortunate results of sentiment, until there has resulted a philosophy which comes unwittingly to this, that greed is the great agent in the elevation of the human race and in the evolution of the universe.

I open a handbook of political economy,—the most typical and middling one I have at hand,—and there find some remarks of which I will here make a brief analysis. I omit qualifications, sops thrown to Cerberus, phrases to placate Christian prejudice, trappings which serve to hide from author and reader alike the ugly nakedness of the greed-god. But I have surveyed my position. The author enumerates "three motives to human action:

The love of self;

The love of a limited class having common interests and feelings with one's self;

The love of mankind at large."

Remark, at the outset, what obsequious title is bestowed on greed,—"the love of self." Love! The second motive *is* love. In place of "a limited class" put "certain persons," and you have a fair description. Taking "class" in the old-fashioned sense, a weak kind of love is described. In the sequel, there seems to be some haziness as to the delimitation of this motive. By the love of mankind at large, the author does not mean that deep, subconscious passion that is properly so called; but merely public-spirit, perhaps little more than a fidget about pushing ideas. The author proceeds to a comparative estimate of the worth of these motives. Greed, says he, but using, of course, another word, "is not so great an evil as is commonly supposed . . . Every man can promote his own interests a great deal more effectively than he can promote any one else's, or than any one else can promote his." Besides, as he remarks on another page, the more miserly a man is, the more good he does. The second motive "is the most dangerous one to which society is exposed." Love is all very pretty: "no higher or purer source of human happiness exists." (Ahem!) But it is a "source of enduring injury," and, in short, should be overruled by something wiser. What is this wiser motive ? We shall see.

As for public spirit, it is rendered nugatory by the "difficulties in the way of its effective operation." For example, it might suggest putting checks upon the fecundity of the poor and the vicious; and "no measure of repression would be too severe," in the case of criminals. The hint is broad. But unfortunately, you cannot induce legislatures to take such measures, owing to the pestiferous "tender sentiments of man towards man." It thus appears, that public-spirit, or Benthamism, is not strong enough to be the effective tutor of love, (I am skipping to another page,) which must therefore be handed over to "the motives which animate men in the pursuit of wealth," in which alone we can confide, and which "are in the highest degree beneficent."* Yes, in the "highest degree" without exception are they beneficent to the being upon whom all their blessings are poured out, namely, the Self, whose "sole object," says the writer in accumulating wealth is his individual "sustenance and enjoyment." Plainly, the author holds the notion that some other motive might be in a higher degree beneficent even for the man's self to be a paradox wanting in good sense. He seeks to gloze and modify his doctrine; but he lets the perspicacious reader see what his animating principle is; and when, holding the opinions I have repeated, he at the same time acknowledges that society could not exist upon a basis of intelligent greed alone, he simply pigeon-holes himself as one of the eclectics of inharmonious opinions. He wants his mammon flavored with a *soupçon* of god.

The economists accuse those to whom the enunciation of their atrocious villainies communicates a thrill of horror of being *sentimentalists*. It may be so: I willingly confess to having some tincture of sentimentalism in me, God be thanked! Ever since the French Revolution brought this leaning of thought into ill-repute,—and not altogether undeservedly, I must admit, true, beautiful, and good as

* How can a writer have any respect for science, as such, who is capable of confounding with the scientific propositions of political economy, which have nothing to say concerning what is "beneficent," such brummagem generalisations as this?

that great movement was,—it has been the tradition to picture sentimentalists as persons incapable of logical thought and unwilling to look facts in the eyes. This tradition may be classed with the French tradition that an Englishman says *godam* at every second sentence, the English tradition that an American talks about "Britishers," and the American tradition that a Frenchman carries forms of etiquette to an inconvenient extreme, in short with all those traditions which survive simply because the men who use their eyes and ears are few and far between. Doubtless some excuse there was for all those opinions in days gone by; and sentimentalism, when it was the fashionable amusement to spend one's evenings in a flood of tears over a woeful performance on a candle-litten stage, sometimes made itself a little ridiculous. But what after all is sentimentalism? It is an *ism*, a doctrine, namely, the doctrine that great respect should be paid to the natural judgments of the sensible heart. This is what sentimentalism precisely is; and I entreat the reader to consider whether to contemn it is not of all blasphemies the most degrading. Yet the nineteenth century has steadily contemned it, because it brought about the Reign of Terror. That it did so is true. Still, the whole question is one of *how much*. The reign of terror was very bad; but now the Gradgrind banner has been this century long flaunting in the face of heaven, with an insolence to provoke the very skies to scowl and rumble. Soon a flash and quick peal will shake economists quite out of their complacency, too late. The twentieth century, in its latter half, shall surely see the deluge-tempest burst upon the social order,—to clear upon a world as deep in ruin as that greed-philosophy has long plunged it into guilt. No post-thermidorian high jinks then!

So a miser is a beneficent power in a community, is he? With the same reason precisely, only in a much higher degree, you might pronounce the Wall Street sharp to be a good angel, who takes money from heedless persons not likely to guard it properly, who wrecks feeble enterprises better stopped, and who administers wholesome lessons to unwary scientific men, by passing worthless checks upon them,—as you did, the other day, to me, my millionaire Master in glomery, when you thought you saw your way to using my process without paying for it, and of so bequeathing to your children something to boast of their father about,—and who by a thousand wiles puts money at the service of intelligent greed, in his own person. Bernard Mandeville, in his "Fable of the Bees," maintains that private vices of all descriptions are public benefits, and proves it, too, quite as cogently as the economist proves his point concerning the miser. He even argues, with no slight force, that but for vice civilisation would never have existed. In the same spirit, it has been strongly maintained and is to-day widely believed that all acts of charity and benevolence, private and public, go seriously to degrade the human race.

The "Origin of Species" of Darwin merely extends politico-economical views of progress to the entire realm of animal and vegetable life. The vast majority of our contemporary naturalists hold the opinion that the true cause of those exquisite and marvellous adaptations of nature for which, when I was a boy, men used to extol the divine wisdom is that creatures are so crowded together that those of them that happen to have the slightest advantage force those less pushing into situations unfavorable to multiplication or even kill them before they reach the age of reproduction. Among animals, the mere mechanical individualism is vastly reënforced as a power making for good by the animal's ruthless greed. As Darwin puts it on his title-page, it is the struggle for existence; and he should have added for his motto: Every individual for himself, and the Devil take the hindmost! Jesus, in his sermon on the Mount, expressed a different opinion.

Here, then, is the issue. The gospel of Christ says that progress comes from every individual merging his individuality in sympathy with his neighbors. On the other side, the conviction of the nineteenth century is that progress takes place by virtue of every individual's striving for himself with all his might and trampling his neighbor under foot whenever he gets a chance to do so. This may accurately be called the Gospel of Greed.

Much is to be said on both sides. I have not concealed, I could not conceal, my own passionate predilection. Such a confession will probably shock my scientific brethren. Yet the strong feeling is in itself, I think, an argument of some weight in favor of the agapastic theory of evolution,—so far as it may be presumed to bespeak the normal judgment of the Sensible Heart. Certainly, if it were possible to believe in agapasm without believing it warmly, that fact would be an argument against the truth of the doctrine. At any rate, since the warmth of feeling exists, it should on every account be candidly confessed ; especially since it creates a liability to one-sidedness on my part against which it behooves my readers and me to be severally on our guard.

SECOND THOUGHTS. IRENICA.

Let us try to define the logical affinities of the different theories of evolution. Natural selection, as conceived by Darwin, is a mode of evolution in which the only positive agent of change in the whole passage from moner to man is fortuitous variation. To secure advance in a definite direction chance has to be seconded by some action that shall hinder the propagation of some varieties or stimulate that of others. In natural selection, strictly so called, it is the crowding out of the weak. In sexual selection, it is the attraction of beauty, mainly.

The "Origin of Species" was published toward the end of the year 1859. The preceding years since 1846 had been one of the most productive seasons,—or if extended so as to cover the great book we are considering, *the* most productive period of equal length in the entire history of science from its beginnings until now. The idea that chance begets order, which is one of the corner-stones of modern physics (although Dr. Carus considers it "the weakest point in Mr. Peirce's system,") was at that time put into its clearest light. Quetelet had opened the discussion by his "Letters on the Application of Probabilities to the Moral and Political Sciences," a work which deeply impressed the best minds of that day, and to which Sir John Herschel had drawn general attention in Great Britain. In 1857, the first volume of Buckle's "History of Civilisation" had created a tremendous sensation, owing to the use he made of this same idea. Meantime, the "statistical method" had, under that very name, been applied with brilliant success to molecular physics. Dr.

John Herapath, an English chemist, had in 1847 outlined the kinetical theory of gases in his "Mathematical Physics"; and the interest the theory excited had been refreshed in 1856 by notable memoirs by Clausius and Krönig. In the very summer preceding Darwin's publication, Maxwell had read before the British Association the first and most important of his researches on this subject. The consequence was that the idea that fortuitous events may result in a physical law, and further that this is the way in which those laws which appear to conflict with the principle of the conservation of energy are to be explained, had taken a strong hold upon the minds of all who were abreast of the leaders of thought. By such minds, it was inevitable that the "Origin of Species," whose teaching was simply the application of the same principle to the explanation of another "non-conservative" action, that of organic development, should be hailed and welcomed. The sublime discovery of the conservation of energy by Helmholtz in 1847, and that of the mechanical theory of heat by Clausius and by Rankine, independently, in 1850, had decidedly overawed all those who might have been inclined to sneer at physical science. Thereafter a belated poet still harping upon "science peddling with the names of things" would fail of his effect. Mechanism was now known to be all, or very nearly so. All this time, utilitarianism,—that improved substitute for the Gospel,—was in its fullest feather; and was a natural ally of an individualistic theory. Dean Mansell's injudicious advocacy had led to mutiny among the bondsmen of Sir William Hamilton, and the nominalism of Mill had profited accordingly; and although the real science that Darwin was leading men to was sure some day to give a death-blow to the sham-science of Mill, yet there were several elements of the Darwinian theory which were sure to charm the followers of Mill. Another thing: anæsthetics had been in use for thirteen years. Already, people's acquaintance with suffering had dropped off very much; and as a consequence, that unlovely hardness by which our times are so contrasted with those that immediately preceded them, had already set in, and inclined people to relish a ruthless theory. The reader would quite mistake the drift of what I am saying if he were to understand me as wishing to suggest that any of those things (except perhaps Malthus) influenced Darwin himself. What I mean is that his hypothesis, while without dispute one of the most ingenious and pretty ever devised, and while argued with a wealth of knowledge, a strength of logic, a charm of rhetoric, and above all with a certain magnetic genuineness that was almost irresistible, did not appear, at first, at all near to being proved; and to a sober mind its case looks less hopeful now than it did twenty years ago; but the extraordinarily favorable reception it met with was plainly owing, in large measure, to its ideas being those toward which the age was favorably disposed, especially, because of the encouragement it gave to the greed-philosophy.

Diametrically opposed to evolution by chance, are those theories which attribute all progress to an inward necessary principle, or other form of necessity. Many naturalists have thought that if an egg is destined to go through a certain series of embryological transformations, from which it is perfectly certain not to deviate, and if in geological time almost exactly the same forms appear successively, one replacing another in the same order, the strong presumption is that this latter succession was as predeterminate and certain to take place as the former. So, Nägeli, for instance, conceives that it somehow follows from the first law of motion and the peculiar, but unknown, molecular constitution of protoplasm, that forms must complicate themselves more and more. Kölliker makes one form generate another after a certain maturation has been accomplished. Weismann, too, though he calls himself a Darwinian, holds that nothing is due to chance, but that all forms are simple mechanical resultants of the heredity from two parents.* It is very noticeable that all these different sectaries seek to import into their science a mechanical necessity to which the facts that come under their observation do not point. Those geologists who think that the variation of species is due to cataclasmic alterations of climate or of the chemical constitution of the air and water are also making mechanical necessity chief factor of evolution.

* I am happy to find that Dr. Carus, too, ranks Weismann among the opponents of Darwin, notwithstanding his flying that flag.

Evolution by sporting and evolution by mechanical necessity are conceptions warring against one another. A third method, which supersedes their strife, lies enwrapped in the theory of Lamarck. According to his view, all that distinguishes the highest organic forms from the most rudimentary has been brought about by little hypertrophies or atrophies which have affected individuals early in their lives, and have been transmitted to their offspring. Such a transmission of acquired characters is of the general nature of habit-taking, and this is the representative and derivative within the physiological domain of the law of mind. Its action is essentially dissimilar to that of a physical force; and that is the secret of the repugnance of such necessitarians as Weismann to admitting its existence. The Lamarckians further suppose that although some of the modifications of form so transmitted were originally due to mechanical causes, yet the chief factors of their first production were the straining of endeavor and the overgrowth superinduced by exercise, together with the opposite actions. Now, endeavor, since it is directed toward an end, is essentially psychical, even though it be sometimes unconscious; and the growth due to exercise, as I argued in my last paper, follows a law of a character quite contrary to that of mechanics.

Lamarckian evolution is thus evolution by the force of habit.—That sentence slipped off my pen while one of those neighbors whose function in the social cosmos seems to be that of an Interrupter, was asking me a question. Of course, it is nonsense. Habit is mere inertia, a resting on one's oars, not a propulsion. Now it is energetic projaculation (lucky there is such a word, or this untried hand might have been put to inventing one) by which in the typical instances of Lamarckian evolution the new elements of form are first created. Habit, however, forces them to take practical shapes, compatible with the structures they affect, and in the form of heredity and otherwise, gradually replaces the spontaneous energy that sustains them. Thus, habit plays a double part; it serves to establish the new features, and also to bring them into harmony with the general morphology and function of the animals and plants to which they belong. But if the reader will now kindly give himself the trouble of turning back a page or two, he will see that this account of Lamarckian evolution coincides with the general description of the action of love, to which, I suppose, he yielded his assent.

Remembering that all matter is really mind, remembering, too, the continuity of mind, let us ask what aspect Lamarckian evolution takes on within the domain of consciousness. Direct endeavor can achieve almost nothing. It is as easy by taking thought to add a cubit to one's stature, as it is to produce an idea acceptable to any of the Muses by merely straining for it, before it is ready to come. We haunt in vain the sacred well and throne of Mnemosyne; the deeper workings of the spirit take place in their own slow way, without our connivance. Let but their bugle sound, and we may then make our effort, sure of an oblation for the altar of whatsoever divinity its savor gratifies. Besides this inward process, there is the operation of the environment, which goes to break up habits destined to be broken up and so to render the mind lively. Everybody knows that the long continuance of a routine of habit makes us lethargic, while a succession of surprises wonderfully brightens the ideas. Where there is a motion, where history is a-making, there is the focus of mental activity, and it has been said that the arts and sciences reside within the temple of Janus, waking when that is open, but slumbering when it is closed. Few psychologists have perceived how fundamental a fact this is. A portion of mind abundantly commissured to other portions works almost mechanically. It sinks to the condition of a railway junction. But a portion of mind almost isolated, a spiritual peninsula, or *cul-de-sac*, is like a railway terminus. Now mental commissures are habits. Where they abound, originality is not needed and is not found; but where they are in defect, spontaneity is set free. Thus, the first step in the Lamarckian evolution of mind is the putting of sundry thoughts into situations in which they are free to play. As to growth by exercise, I have already shown, in discussing "Man's Glassy Essence," in last October's *Monist*, what its *modus operandi* must be conceived to be, at least, until a second equally definite hypothesis shall have been offered. Namely, it consists of the flying asunder of molecules, and the reparation of the parts by new matter. It is, thus, a sort of reproduction. It takes place only during exercise, because the activity of protoplasm consists in the molecular disturbance which is its necessary condition. Growth by exercise takes place also in the mind. Indeed, that is what it is to *learn*. But the most perfect illustration is the development of a philosophical idea by being put into practice. The conception which appeared, at first, as unitary, splits up into special cases; and into each of these new thought must enter to make a practicable idea. This new thought, however, follows pretty closely the model of the parent conception; and thus a homogeneous development takes place. The parallel between this and the course of molecular occurrences is apparent. Patient attention will be able to trace all these elements in the transaction called learning.

Three modes of evolution have thus been brought before us; evolution by fortuitous variation, evolution by mechanical necessity, and evolution by creative love. We may term them *tychastic* evolution, or *tychasm*, *anancastic* evolution, or *anancasm*, and *agapastic* evolution, or *agapasm*. The doctrines which represent these as severally of principal importance, we may term *tychasticism*, *anancasticism*, and *agapasticism*. On the other hand the mere propositions that absolute chance, mechanical necessity, and the law of love, are severally operative in the cosmos, may receive the names of *tychism*, *anancism*, and *agapism*.

All three modes of evolution are composed of the same general elements. Agapasm exhibits them the most clearly. The good result is here brought to pass, first, by the bestowal of spontaneous energy by the parent upon the offspring, and, second, by the disposition of the latter to catch the general idea of those about it and thus to subserve the general purpose. In order to express the relation that tychasm and anancasm bear to agapasm, let me borrow a word from geometry. An ellipse crossed by a straight line is a sort of cubic curve; for a cubic is a curve which is cut thrice by a straight line; now a straight line might cut the ellipse twice and its associated straight line a third time. Still the ellipse with the straight line across it would not have the characteristics of a cubic. It would have, for instance, no contrary flexure, which no true cubic wants; and it would have two nodes, which no true cubic has. The geometers say that it is a *degenerate* cubic. Just so, tychasm and anancasm are degenerate forms of agapasm.

Men who seek to reconcile the Darwinian idea with Christianity will remark that tychastic evolution, like the agapastic, depends upon a reproductive creation, the forms preserved being those that use the spontaneity conferred upon them in such wise as to be drawn into harmony with their original, quite after the Christian scheme. Very good! This only shows that just as love cannot have a contrary, but must embrace what is most opposed to it, as a degenerate case of it, so tychasm is a kind of agapasm. Only, in the tychastic evolution progress is solely owing to the distribution of the napkin-hidden talent of the rejected servant among those not rejected, just as ruined gamesters leave their money on the table to make those not yet ruined so much the richer. It makes the felicity of the lambs just the damnation of the goats, transposed to the other side of the equation. In genuine agapasm, on the other hand, advance takes place by virtue of a positive sympathy among the created springing from continuity of mind. This is the idea which tychasticism knows not how to manage.

The anancasticist might here interpose, claiming that the mode of evolution for which he contends agrees with agapasm at the point at which tychasm departs from it. For it makes development go through certain phases, having its inevitable ebbs and flows, yet tending on the whole to a foreordained perfection. Bare existence by this its destiny betrays an intrinsic affinity for the good. Herein, it must be admitted, anancasm shows itself to be in a broad acception a species of agapasm. Some forms of it might easily be mistaken for the genuine agapasm. The Hegelian philosophy is such an anancasticism. With its revelatory religion, with its synechism (however imperfectly set forth), with its "reflection," the whole idea of the theory is superb, almost sublime. Yet, after all, living freedom is practically omitted from its method. The whole movement is that of a vast engine, impelled by a *vis a tergo*, with a blind and mysterious fate of arriving at a lofty goal. I mean that such an engine it *would* be, if it really worked; but in point of fact, it is a Keely motor. Grant that it really acts as it professes to act, and there is nothing to do but accept the philosophy. But never was there seen such an example of a long chain of reasoning,—shall I say with a flaw in every link?—no, with every link a handful of sand, squeezed into shape in a dream. Or say, it is a pasteboard model of a philosophy that in reality does not exist. If we use the one precious thing it contains, the idea of it, introducing the tychism which the arbitrariness of its every step suggests, and make that the support of a vital freedom which is the breath of the spirit of love, we may be able to produce that genuine agapasticism, at which Hegel was aiming.

A THIRD ASPECT. DISCRIMINATION.

In the very nature of things, the line of demarcation between the three modes of evolution is not perfectly sharp. That does not prevent its being quite real; perhaps it is rather a mark of its reality. There is in the nature of things no sharp line of demarcation between the three fundamental colors, red, green, and violet. But for all that they are really different. The main question is whether three radically different evolutionary elements have been operative; and the second question is what are the most striking characteristics of whatever elements have been operative.

I propose to devote a few pages to a very slight examination of these questions in their relation to the historical development of human thought. I first formulate for the reader's convenience the briefest possible definitions of the three conceivable modes of development of thought, distinguishing also two varieties of anancasm and three of agapasm. The tychastic development of thought, then, will consist in slight departures from habitual ideas in different directions indifferently, quite purposeless and quite unconstrained whether by outward circumstances or by force of logic, these new departures being followed by unforeseen results which tend to fix some of them as habits more than others. The anancastic development of thought will consist of new ideas adopted without foreseeing whither they tend, but having a character determined by causes either external to the mind, such as changed circumstances of life, or internal to the mind as logical developments of ideas already accepted, such as generalisations. The agapastic development of thought is the adoption of certain mental tendencies, not altogether heedlessly, as in tychasm, nor quite blindly by the mere force of circumstances or of logic, as in anancasm, but by an immediate attraction for the idea itself, whose nature is divined before the mind possesses it, by the power of sympathy, that is, by virtue of the continuity of mind ; and this mental tendency may be of three varieties, as follows. First, it may affect a whole people or community in its collective personality, and be thence communicated to such individuals as are in powerfully sympathetic connection with the collective people, although they may be intellectually incapable of attaining the idea by their private understandings or even perhaps of consciously apprehending it. Second, it may affect a private person directly, yet so that he is only enabled to apprehend the idea, or to appreciate its attractiveness, by virtue of his sympathy with his neighbors, under the influence of a striking experience or development of thought. The conversion of St. Paul may be taken as an example of what is meant. Third, it may affect an individual, independently of his human affections, by virtue of an attraction it exercises upon his mind, even before he has comprehended it. This is the phenomenon which has been well called the *divination* of genius ; for it is due to the continuity between the man's mind and the Most High.

Let us next consider by means of what tests we can discriminate between these different categories of evolution. No absolute criterion is possible in the nature of things, since in the nature of things there is no sharp line of demarcation between the different classes. Nevertheless, quantitative symptoms may be found by which a sagacious and sympathetic judge of human nature may be able to estimate the approximate proportions in which the different kinds of influence are commingled.

So far as the historical evolution of human thought has been tychastic, it should have proceeded by insensible or minute steps ; for such is the nature of chances when so multiplied as to show phenomena of regularity. For example, assume that of the native-born white adult males of the United States in 1880, one fourth part were below 5 feet 4 inches in stature and one fourth part above 5 feet 8 inches. Then by the principles of probability, among the whole population, we should expect

216	under	4	feet	6	inches,		216	above	6	feet	6	inches.	
48	"	4	"	5	"		48	"	6	"	7	"	
9	"	4	"	4	"		9	"	6	"	8	"	
less than 2	"	4	"	3	"		less than 2	"	6	"	9	"	

I set down these figures to show how insignificantly few are the cases in which anything very far out of the common run presents itself by chance. Though the stature of only every second man is included within the four inches between 5 feet 4 inches and 5 feet 8 inches, yet if this interval be extended by thrice four inches above and below, it will embrace all our 8 millions odd of native-born adult white males (of 1880), except only 9 taller and 9 shorter.

The test of minute variation, if *not* satisfied, absolutely negatives tychasm. If it *is* satisfied, we shall find that it negatives anancasm but not agapasm. We want a positive test, satisfied by tychasm, only. Now wherever we find men's thought taking by imperceptible degrees a turn contrary to the purposes which animate them, in spite of their highest impulses, there, we may safely conclude, there has been a tychastic action.

Students of the history of mind there be of an erudition to fill an imperfect scholar like me with envy edulcorated by joyous admiration, who maintain that ideas when just started are and can be little more than freaks, since they cannot yet have been critically examined, and further that everywhere and at all times progress has been so gradual that it is difficult to make out distinctly what original step any given man has taken. It would follow that tychasm has been the sole method of intellectual development. I have to confess I cannot read history so; I cannot help thinking that while tychasm has sometimes been operative, at others great steps covering nearly the same ground and made by different men independently, have been mistaken for a succession of small steps, and further that students have been reluctant to admit a real entitative "spirit" of an age or of a people, under the mistaken and unscrutinised impression that they should thus be opening the door to wild and unnatural hypotheses. I find, on the contrary, that, however it may be with the education of individual minds, the historical development of thought has seldom been of a tychastic nature, and exclusively in backward and barbarising movements. I desire to speak with the extreme modesty which befits a student of logic who is required to survey so very wide a field of human thought that he can cover it only by a reconnaisance, to which only the greatest skill and most adroit methods can impart any value at all; but, after all, I can only express my own opinions and not those of anybody else; and in my humble judgment, the largest example of tychasm is afforded by the history of Christianity, from about its establishment by Constantine, to, say, the time the of Irish monasteries, an era or eon of about 500 years. Undoubtedly the external circumstance which more than all others at first inclined men to accept Christianity in its loveliness and tenderness, was the fearful extent to which society was broken up into units by the unmitigated greed and hard-heartedness into which the Romans had seduced the world. And yet it was that very same fact, more than any other external circumstance, that fostered that bitterness against the wicked world of which the primitive Gospel of Mark contains not a single trace. At least, I do not detect it in the remark about the blasphemy against the Holy Ghost, where nothing is said about vengeance, nor even in that speech where the closing lines of Isaiah are quoted, about the worm and the fire that feed upon the "carcasses of the men that have transgressed against me." But little by little the bitterness increases until in the last book of the New Testament, its poor distracted author represents that all the time Christ was talking about having come to save the world, the secret design was to catch the entire human race, with the exception of a paltry 144000, and souse them all in brimstone lake, and as the smoke of their torment went up for ever and ever, to turn and remark, "There is no curse any more." Would it be an insensible smirk or a fiendish grin that should accompany such an utterance? I wish I could believe St. John did not write it; but it is his gospel which tells about the "resurrection unto condemnation,"—that is of men's being resuscitated just for the sake of torturing them;—and, at any rate, the Revelation is a very ancient composition. One can understand that the early Christians were like men trying with all their might to climb a steep declivity of smooth wet clay; the deepest and truest element of their life, animating both heart and head, was universal love; but they were continually, and against their wills, slipping into a party spirit, every slip serving as a precedent, in a fashion but too familiar to every man. This party feeling insensibly grew until by about A. D. 330 the lustre of the pristine integrity that in St. Mark reflects the white spirit of light was so far tarnished that Eusebius, (the Jared Sparks of that day,) in the preface to his History, could announce his intention of exaggerating everything that tended to the glory of the church and of suppressing whatever might disgrace it. His Latin contemporary Lactantius is worse, still; and so the darkling went on increasing until before the end of the century the great library of Alexandria was destroyed by Theophilus,* until Gregory the Great, two centuries later, burnt the great library of Rome, proclaiming that "Ignorance is the mother of devotion," (which is true, just as oppression and injustice is the mother of spirituality,) until a sober description of the state of the church would be a thing our not too nice newspapers would treat as "unfit for publication." All this movement is shown by the application of the test given above to have been tychastic. Another very much like it on a small scale, only a hundred times swifter, for the study of which there are documents by the library-full, is to be found in the history of the French Revolution.

Anancastic evolution advances by successive strides with pauses between. The reason is that in this process a habit of thought having been overthrown is supplanted by the next strongest. Now this next strongest is sure to be widely disparate from the first, and as often as not is its direct contrary. It reminds one of our old rule of making the second candidate vice-president. This character, therefore, clearly distinguishes anancasm from tychasm. The character which distinguishes it from agapasm is its purposelessness. But external and internal anancasm have to be examined separately.

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* See *Draper's History of Intellectual Development*, chap. x.

Development under the pressure of external circumstances, or cataclasmine evolution, is in most cases unmistakable enough. It has numberless degrees of intensity, from the brute force, the plain war, which has more than once turned the current of the world's thought, down to the hard fact of evidence, or what has been taken for it, which has been known to convince men by hordes. The only hesitation that can subsist in the presence of such a history is a quantitative one. Never are external influences the only ones which affect the mind, and therefore it must be a matter of judgment for which it would scarcely be worth while to attempt to set rules, whether a given movement is to be regarded as principally governed from without or not. In the rise of medieval thought, I mean scholasticism and the synchronistic art developments, undoubtedly the crusades and the discovery of the writings of Aristotle were powerful influences. The development of scholasticism from Roscellin to Albertus Magnus closely follows the successive steps in the knowledge of Aristotle. Prantl thinks that that is the whole story, and few men have thumbed more books than Carl Prantl. He has done good solid work, notwithstanding his slap-dash judgments. But we shall never make so much as a good beginning of comprehending scholasticism until the whole has been systematically explored and digested by a company of students regularly organised and held under rule for that purpose. But as for the period we are now specially considering, that which synchronised the Romanesque architecture, the literature is easily mastered. It does not quite justify Prantl's dicta as to the slavish dependence of these authors upon their authorities. Moreover, they kept a definite purpose steadily before their minds, throughout all their studies. I am, therefore, unable to offer this period of scholasticism as an example of pure external anancasm, which seems to be the fluorine of the intellectual elements. Perhaps the recent Japanese reception of western ideas is the purest instance of it in history. Yet in combination with other elements, nothing is commoner. If the development of ideas under the influence of the study of external facts be considered as external anancasm,—it is on the border between the external and the internal forms,—it is, of course, the principal thing in modern learning. But Whewell, whose masterly comprehension of the history of science critics have been too ignorant properly to appreciate, clearly shows that it is far from being the overwhelmingly preponderant influence, even there.

Internal anancasm, or logical groping, which advances upon a predestined line without being able to foresee whither it is to be carried nor to steer its course, this is the rule of development of philosophy. Hegel first made the world understand this; and he seeks to make logic not merely the subjective guide and monitor of thought, which was all it had been ambitioning before, but to be the very mainspring of thinking, and not merely of individual thinking but of discussion, of the history of the development of thought, of all history, of all development. This involves a positive, clearly demonstrable error. Let the logic in question be of whatever kind it may, a logic of necessary inference or a logic of probable inference, (the theory might perhaps be shaped to fit either,) in any case it supposes that logic is sufficient of itself to determine what conclusion follows from given premises; for unless it will do so much, it will not suffice to explain why an individual train of reasoning should take just the course it does take, to say nothing of other kinds of development. It thus supposes that from given premises, only one conclusion can logically be drawn, and that there is no scope at all for free choice. That from given premises only one conclusion can logically be drawn, is one of the false notions which have come from logicians' confining their attention to that Nantucket of thought, the logic of non-relative terms. In the logic of relatives, it does not hold good.

One remark occurs to me. If the evolution of history is in considerable part of the nature of internal anancasm, it resembles the development of individual men; and just as 33 years is a rough but natural unit of time for individuals, being the average age at which man has issue, so there should be an approximate period at the end of which one great historical movement ought to be likely to be supplanted by another. Let us see if we can make out anything of the kind. Take the governmental development of Rome as being sufficiently long and set down the principal dates.

B. C. 753, Foundation of Rome.
B. C. 510, Expulsion of the Tarquins.
B. C. 27, Octavius assumes title Augustus.
A. D. 476, End of Western Empire.
A. D. 962, Holy Roman Empire.
A. D. 1453, Fall of Constantinople.

The last event was one of the most significant in history, especially for Italy. The intervals are 243, 483, 502, 486, 491 years. All are rather curiously near equal, except the first which is half the others. Successive reigns of kings would not commonly be so near equal. Let us set down a few dates in the history of thought.

B. C. 585, Eclipse of Thales. Beginning of Greek philosophy.
A. D. 30, The crucifixion.
A. D. 529, Closing of Athenian schools. End of Greek philosophy.
A. D. 1125, (Approximate) Rise of the Universities of Bologna and Paris.
A. D. 1543, Publication of the "De Revolutionibus" of Copernicus. Beginning of Modern Science.

The intervals are 615, 499, 596, 418, years. In the history of metaphysics, we may take the following:

B. C. 322, Death of Aristotle.
A. D. 1274, Death of Aquinas.
A. D. 1804, Death of Kant.

The intervals are 1595 and 530 years. The former is about thrice the latter.

From these figures, no conclusion can fairly be drawn. At the same time, they suggest that perhaps there may be a rough natural era of about 500 years. Should there be any independent evidence of this, the intervals noticed may gain some significance.

The agapastic development of thought should, if it exists, be distinguished by its purposive character, this purpose being the development of an idea. We should have a direct agapic or sympathetic comprehension and recognition of it, by virtue of the continuity of thought. I here take it for granted that such continuity of thought has been sufficiently proved by the arguments used in my paper on the "Law of Mind" in *The Monist* of last July. Even if those arguments are not quite convincing in themselves, yet if they are reënforced by an apparent agapasm in the history of thought, the two propositions will lend one another mutual aid. The reader will, I trust, be too well grounded in logic to mistake such mutual support for a vicious circle in reasoning. If it could be shown directly that there is such an entity as the "spirit of an age" or of a people, and that mere individual intelligence will not account for all the phenomena, this would be proof enough at once of agapasticism and of synechism. I must acknowledge that I am unable to produce a cogent demonstration of this; but I am, I believe, able to adduce such arguments as will serve to confirm those which have been drawn from other facts. I believe that all the greatest achievements of mind have been beyond the powers of unaided individuals; and I find, apart from the support this opinion receives from synechistic considerations, and from the purposive character of many great movements, direct reason for so thinking in the sublimity of the ideas and in their occurring simultaneously and independently to a number of individuals of no extraordinary general powers. The pointed Gothic architecture in several of its developments appears to me to be of such a character. All attempts to imitate it by modern architects of the greatest learning and genius appear flat and tame, and are felt by their authors to be so. Yet at the time the style was living, there was quite an abundance of men capable of producing works of this kind of gigantic sublimity and power. In more than one case, extant documents show that the cathedral chapters, in the selection of architects, treated high artistic genius as a secondary consideration, as if there were no lack of persons able to supply that; and the results justify their confidence. Were individuals in general, then, in those ages possessed of such lofty natures and high intellect? Such an opinion would break down under the first examination.

How many times have men now in middle life seen great discoveries made independently and almost simultaneously! The first instance I remember was the prediction of a planet exterior to Uranus by Leverrier and Adams. One hardly knows to whom the principle of the conservation of energy ought to be attributed, although it may reasonably be considered as the greatest discovery science has ever made. The mechanical theory of heat was set forth by Rankine and by Clausius during the same month of February, 1850; and there are eminent men who attribute this great step to Thomson.* The kinetical theory of gases, after being started by John Bernoulli and long buried in oblivion, was reinvented and applied to the explanation not merely of the laws of Boyle, Charles, and Avogadro, but also of diffusion and viscosity, by at least three modern physicists separately. It is well known that the doctrine of natural selection was presented by Wallace and by Darwin at the same meeting of the British Association; and Darwin in his "Historical Sketch" prefixed to the later editions of his book shows that both were anticipated by obscure forerunners. The method of spectrum analysis was claimed for Swan as well as for Kirchhoff, and there were others who perhaps had still better claims. The authorship of the Periodical Law of the Chemical Elements is disputed between a Russian, a German, and an Englishman; although there is no room for doubt that the principal merit belongs to the first. These are nearly all the greatest discoveries of our times. It is the same with the inventions. It may not be surprising that the telegraph should have been independently made by several inventors, because it was an easy corollary from scientific facts well made out before. But it was not so with the telephone and other inventions. Ether, the first anæsthetic, was introduced independently by three different New England physicians. Now ether had been a common article for a century. It had been in one of the pharmacopœias three centuries before. It is quite incredible that its anæsthetic property should not have been known; it was known. It had probably passed from mouth to ear as a secret from the days of Basil Valentine; but for long it had been a secret of the Punchinello kind. In New England, for many years, boys had used it for amusement. Why then had it not been put to its serious use? No reason can be given, except that the motive to do so was not strong enough. The motives to doing so could only have been desire for gain and philanthropy. About 1846, the

* Thomson, himself, in his article *Heat* in the *Encyclopedia Britannica*, never once mentions the name of Clausius.

date of the introduction, philanthropy was undoubtedly in an unusually active condition. That sensibility, or sentimentalism, which had been introduced in the previous century, had undergone a ripening process, in consequence of which, though now less intense than it had previously been, it was more likely to influence unreflecting people than it had ever been. All three of the ether-claimants had probably been influenced by the desire for gain; but nevertheless they were certainly not insensible to the agapic influences.

I doubt if any of the great discoveries ought, properly, to be considered as altogether individual achievements; and I think many will share this doubt. Yet, if not, what an argument for the continuity of mind, and for agapasticism is here! I do not wish to be very strenuous. If thinkers will only be persuaded to lay aside their prejudices and apply themselves to studying the evidences of this doctrine, I shall be fully content to await the final decision.

CHARLES S. PEIRCE.

REPLY TO THE NECESSITARIANS.

REJOINDER TO DR. CARUS.

§ 1. In *The Monist* for January, 1891, and in the number for April, 1892, I attacked the doctrine that every event is precisely determined by law. Like everybody else, I admit that there is regularity: I go further; I maintain the existence of law as something *real and general.* But I hold there is no reason to think that there are general formulæ to which the phenomena of nature *always* conform, or to which they *precisely* conform. At the end of my second paper, the partisans of the doctrine of necessity were courteously challenged and besought to attempt to answer my arguments. This, so far as I can learn, Dr. Carus alone, in *The Monist* of July and October, 1892, has publicly vouchsafed to do. For this I owe him my particular thanks and a careful rejoinder.

§ 2. I number the paragraphs of his papers consecutively. The following index shows the pages on which those paragraphs commence, and the numbered sections of this rejoinder in which they are noticed.

DR. CARUS		REJOINDER		DR. CARUS		REJOINDER		DR. CARUS		REJOINDER	
Vol. II,	p.560	¶ 1		Vol. II,	p.566	¶ 21	§13,14	Vol. II,	p.571	¶ 41	
"	"	2	§28	"	"	22	3.12.22	"	"	42	
"	"	3	29	"	"	23	12	"	"	43	
"	561	4	23,27	"	"	24	3.13.14	"	572	44	25
"	"	5	21	"	567	25	8	"	"	45	
"	"	6	18	"	"	26	8	"	"	46	5.23
"	"	7	18	"	"	27	8.22	"	573	47	
"	562	8	18*bis.*	"	"	28	20.22	"	"	48	7
"	"	9	21	"	"	29	12	"	"	49	7
"	"	10		"	368	30	22	"	"	50	7
"	563	11	12.21.22	"	"	31	22 *bis*	"	574	51	7.25
"	"	12	19	"	"	32	22	"	"	52	
"	"	13	19	"	569	33	22	"	"	53	25
"	564	14	3.12*bis*.22	"	"	34	22	"	"	54	
"	"	15	8 *ter*, 16	"	"	35	8	"	"	55	25 *bis*
"	565	16	21	"	"	36	29	"	575	56	17
"	"	17	3	"	"	37		"	"	57	17 *bis*
"	"	18	3	"	570	38		"	"	58	17.25
"	"	19	13 note	"	"	39		"	"	59	17
"	566	20	14	"	"	40		"	"	60	17

DR. CARUS		REJOINDER		DR. CARUS		REJOINDER		DR. CARUS		REJOINDER	
Vol. II,	p.576	¶61	§7.17.25*bis*	Vol. III.	p.73	¶110	§ 6	Vol. III,	p.85	¶159	§15.24
"	"	62	17.21	"	"	111	6	"	"	160	15.24
"	"	63	17	"	74	112	6	"	86	161	15 *bis*
"	577	64	17 *bis*	"	"	113	6.11	"	"	162	15
"	"	65	26	"	"	114	6	"	"	163	5.14
"	"	66	25	"	"	115	6	"	"	164	14
"	"	67	25 *bis*	"	75	116	6.7	"	87	165	4.14
"	578	68	6.7	"	"	117	6	"	"	166	14
"	"	69	17	"	76	118		"	"	167	14
"	579	70	6	"	"	119	9	"	"	168	27
"	"	71	3	"	"	120	6.16	"	"	169	27
"	580	72	3	"	"	121	14	"	88	170	27
"	"	73	3	"	"	122	11	"	"	171	27
"	"	74	3	"	77	123	3	"	"	172	27
"	"	75	3	"	"	124	10	"	"	173	
"	581	76	3	"	"	125	7	"	"	174	
"	"	77	3.11	"	78	126	8	"	89	175	27
"	"	78		"	"	127	8	"	"	176	27 *bis*
"	"	79	21	"	"	128	29	"	"	177	27
"	582	80		"	79	129		"	"	178	27
"	"	81	29	"	"	130		"	"	179	
"	"	82	29	"	"	131		"	"	180	30
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"	"	85	27	"	"	134		"	90	183	30
"	"	86		"	80	135		"	"	184	
"	69	87	12	"	"	136		"	"	185	27
"	"	88	5.15*bis*	"	"	137	15	"	"	186	27
"	"	89		"	"	138	15	"	91	187	27
"	"	90		"	"	139	5.15	"	"	188	
"	"	91	5	"	81	140	15	"	"	189	
"	"	92	5	"	"	141	15	"	"	190	
"	70	93	5	"	"	142	15.25	"	92	191	4
"	"	94	5	"	"	143	15.24	"	"	192	27 *bis*
"	"	95	3	"	"	144	15 *bis*	"	"	193	20.27 *bis*
"	"	96	4.11	"	"	145	15 *bis*	"	93	194	
"	71	97		"	"	146	15 *b's*	"	"	195	8
"	"	98		"	"	147	15 *ter*	"	"	196	8
"	"	99		"	82	148	15	"	94	197	
"	"	100		"	"	149	12.15	"	"	198	6
"	72	101		"	"	150	15 *bis.* 25	"	"	199	30
"	"	102	11	"	83	151	15.24	"	"	200	
"	"	103	6	"	"	152	15.24	"	"	201	
"	"	104		"	"	153	15.24	"	95	202	
"	"	105		"	"	154	15.24	"	"	203	20.29
"	73	106	11	"	"	155	15.24	"	"	204	29
"	"	107	6	"	84	156	15 bis	"	96	205	20
"	"	108	6	"	"	157	15	"	"	206	
"	"	109	6	"	"	158	15.24	"	"	207	

§ 3. Dr. Carus's philosophy is hard to understand. Some phrases which he frequently uses lead the reader to imagine that he is listening to an old-fashioned Königsberg Kantian. What, then, is our surprise when we find (¶ 14) that he sneers at the Kantian, Sir William Hamilton (whom he calls Mr. Hamilton) as having "no adequate conception of the *a priori.*" In his "Ursache, Grund und Zweck" (1883), an admirably clear and systematic exposition of much of his thought, he takes a Schleiermacherian view of the *a priori.* He admits it to be founded in the universal conditions of cognition ; but he thinks it is among the objective rather than the subjective conditions. This is an opinion to which Hamilton is also at times inclined. It is a weak conception, unless the whole distinction between the inward and the outward world be reformed in the light of agapastic and synechistic ontology. For to deny that the *a priori* is subjective is to remove its essential character; and to make it both subjective and objective (otherwise than in the sense in which Kant himself makes it objective) is uncalled for, and is cut off by Ockham's razor. But when synechism has united the two worlds, this view gains new life.

Another thing which has astonished me is Dr. Carus's extravagant laudation (¶ 17) of Venn's highly enlightened and remarkably bright-thinking, yet blundering little book, "The Logic of Chance."* This is the way he speaks of it: "This admirable work, we will make bold to say, marks a new epoch in the study of logic." He adds that it "paves the way which Mr. Peirce has actually followed." But the question of the nature of probability had long before that publication engaged the attention of some of the most powerful intellects in England; and my opinion concerning it was fully made up before I saw the book. I do not think I learned anything from that except a classification of the philosophies of probability. However, after all his eulogy, Dr. Carus only uses the book to quote from it Mill's rewording of Kant's definition of causation, which he would better have quoted direct.

Let me say, not to Dr. Carus, but to the younger generation of readers, that if they imagine that Hamilton, because he is antiquated, is not worth reading, they are much mistaken. The Scotistic elements of his philosophy, and his method in the notes on Reid are especially worthy of attention. As for Mill, though his philosophy was not profound, it is, at least in his "Examination of Hamilton," admirably set forth. Whoever wishes to appeal to the American

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* J. S. Mill had in the first edition of his *Logic* decisively taken an objective cenception of chance and probability; but in his second edition he had become puzzled and had retracted, leaving that chapter, and with it his whole logic, a melancholy wreck, over which the qualified reader sighs, "And this once seemed intelligible!" Venn in the first edition of his book set forth the same objective conception with great clearness, and for that he was entitled to high praise, notwithstanding his manifest inadequacy to the problems treated. But in his second edition, he too has fallen away from his first and correct view, and has adopted a theory which I shall some day show to be untenable. Venn's whole method in logic, as well as his system, is in my opinion of the weakest.

philosophical mind needs to be quite familiar with the writings of these two men.

Dr. Carus himself accepts all that I hold for erroneous in Kant's definition of causation as universal and necessary sequence. Mill merely substitutes the exacter words *invariable* for "universal," and *unconditional* for "necessary."* In giving his form of the definition, Mill shows why it is not applicable to the sequence of day and night, namely, that that is not necessary. Yet Dr. Carus writes (¶ 18) of this very same sequence as if it came under Mill's definition !†

Again, why should he make it "the immortal merit of the great Scotchman" (¶ 22), that is, of Hume, that he admitted the truth of Leibniz's principle?

The famed puzzle of causation is peculiarly understood by Dr. Carus. The difficulties which the perusal of Hume suggested to the mind of Kant,‡ were such as belonged to all categories, or general conceptions of the understanding. The precritical Kant inherited a very decided nominalism from Leibniz and Wolf; and the puzzle for him was simply the usual difficulty that plagues nominalism when it finds itself confronted with a reality which has an element of generality. Necessity is, I need hardly say, but a particular variety of universality. But Dr. Carus (¶ 24) passes over this, to dwell upon an entirely different objection to causation, namely, that it seems to be a creation out of nothing, and a miracle.

I find myself equally at cross-purposes with him, when in ¶¶ 71–77, he speaks of the prevalent views of logicians concerning *comprehension*. This word, in logic, measures the amount of predicates or marks attached to a conception; but Dr. Carus's criticisms seem to be based upon the idea that by comprehension is meant

* Mill often did good service in substituting precise terms for ambiguous ones; as when in speaking of mathematical conclusions he prefers to say they are legitimate deductions rather than that they are necessary.

† In his *Ursache, Grund und Zweck*, Dr. Carus alludes to this passage. But he prefers the treatment of the question by Reid, whom he calls Mill's opponent (*Gegner*).

‡ It is of comparatively little consequence what Hume really meant. The main interest is in what Kant thought he meant.

logical breadth, or the amount of subjects to which the conception is applicable.

I am simply gravelled by his remarks (¶ 95) concerning sundry English words.

No more do I know what to make of his praise (¶ 123) of the German translation of a French phrase used in the theory of functions, meaning *univocally determined.*

§ 4. One habit which goes far to obscure Dr. Carus's meaning is that whenever he finds his opinion at variance with a familiar saying, instead of rejecting that formula, he retains it and changes the meaning. This is calculated to throw the whole discussion into confusion. Thus, nothing is more certain than that the so-called "law of identity," or A is A, was intended to express the fact that every term is predicable of itself. But Dr. Carus, simply because he finds that "meaningless and useless" (¶ 96), thinks himself authorised to confuse the terminology of logic by making this formula, $A = A$, under the same old name, mean that things to which the same name is applicable are for some purpose equivalent.

In like manner, he changes the meaning of the word *freedom* (¶ 165), so that the distinction between those who maintain and those who oppose the freedom of the will may, in words, disappear. It seems scarcely defensible for a thoroughgoing necessitarian, such as he is, to fly the flag of Free Will.

He, also, changes the meaning of *spontaneity* so far that, according to him, "masses gravitate spontaneously" (¶ 191), and so pretends that his doctrine does not suppress the spontaneity of nature!

§ 5. There are other questions of terminology in which I am unable to agree with Dr. Carus. Thus, when I define necessitarianism as "the theory that the will is subject to the general mechanical law of cause and effect," Dr. Carus (¶ 139) wishes to delete "mechanical." But the result would be to define a doctrine to which the advocates of free will would generally subscribe, as readily as their opponents. In order properly to limit the definition, it is quite requisite to exclude "free causation." By "mechanical" causation, I mean a causation entirely determinative, like that of dynamics, but not necessarily operating upon matter.

Dr. Carus mentions (¶ 84) that there are several different ideas to which the term necessity is applied. It seems to me that what lies at the bottom of all of them is the experience of reaction against one's will. In the simplest form, this gives the sense of reality. Dr. Carus himself admits (¶ 46) that reality involves the idea of inevitable fate. Yet philosophical necessity is a special case of universality. But the universality, or better, the generality, of a pure form involves no necessity. It is only when the form is materialised that the distinction between necessity and freedom makes itself plain. These ideas are, therefore, as it seems to me, of a mixed nature. Dr. Carus (¶¶ 91–94) insists that by the necessary, he wishes to be understood to mean in all cases the *inevitable.* This is the idea of *fate,* and is not the conception which determinists usually attach to the term necessity. Yet he does not appear to be quite consistent. At one time (¶ 88), he carefully distinguishes necessity from fate. At another time (¶ 163), every element of compulsion is to be excluded from the conception of necessity.

§ 6. One important key to Dr. Carus's opinions is the recognition of the fact that, like many other philosophers, he is a nominalist tinctured with realistic opinions.

He says (¶ 103), that "there is no need of discussing the truism that, properly speaking, there is no absolute sameness." Now, upon the nominalistic theory, there is not only no absolute or numerical identity, but there are not even any real agreements or likenesses between individuals; for likeness consists merely in the calling of several individuals by one name, or (in some sytems) in their exciting one idea. On the other hand, upon the realistic theory, the fact that identity is a relation of reason does not in the least prevent it from being real. On that theory, it is real unless it is *false* that anything is itself. Thus, upon either theory, identity is just as real as similarity. But Dr. Carus, being a nominalist leaning toward realism, is inclined to make dynamical relations real, and second-intentional ones unreal. This opinion, I think, is a transitional one.

The declaration (¶ 198) that "natural laws are simply a description of nature as nature is," and that "the facts of nature express the character of nature," are nominalistic. But in another place (¶¶ 107–116), he says distinctly that uniformities are real.

He says (¶ 70), "Mr. Peirce attempts to explain natural laws as if they were concrete and *single facts*." This is eminently nominalistic. The nominalist alone makes this sharp distinction between the abstract and the concrete,* which must not be confounded with Hegel's distinction for which the same words are used. The nominalist alone falls into the absurdity of talking of "single facts," or *individual generals*. Yet Dr. Carus says (¶ 68) that natural laws describe the facts of nature *sub specie aeternitatis*. Now I understand Spinoza to be a realist. In ¶ 117 he considers it "settled" "that there are samenesses." This is realistic. But in ¶ 120, he holds "the whole business of science to be to systematise the samenesses of experience," which is nominalistic.

§ 7. Dr. Carus seems to be in some doubt as to how far evolutionism ought to be carried. In ¶¶ 48–51, he seems to side with my contention that it should be thoroughgoing. In ¶ 116, he makes intellect an evolution from feeling. Yet he is sometimes (¶ 125) "inclined" to say the world never was a chaos; he sometimes (¶ 61) thinks it weak to suppose that real chance begets order; and he sometimes (¶ 68) goes so far as to pronounce eternity to be the *conditio sine qua non* of natural law.

§ 8. Every reader of *The Monist* knows that our good editor's great word is "formal law." The clearest statement he has ever made of this doctrine I find in the following two sentences (¶ 127):

"The *a priori* systems of thought are . . . . constructions raised out of the recognition of the formal, i. e. relational samenesses that appear in experience All possibilities of a certain class of relations can be exhausted and formulated in theorems." †

This is perspicuous. For example, of pairs, we can easily show that there are but two forms $A:A$ and $A:B$. This proposition,

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* Along with the distinction, I would of course do away with this use of the words *abstract* and *concrete* to which no clear idea can be attached, as far as I can see.

† I cannot but disapprove of this use of the word "construction" to mean a studied theory, because the word is imperatively required in the theory of cognition to denote a mathematical diagram framed according to a general precept.

—theorem if you will,—exhausts the possibilities. If we make believe there is no danger of falling into error in mathematical reasoning,—and one danger, though not, perhaps, a very serious one, *is* eliminated,—then this proposition is absolutely certain. But I will say, at once, that such a proposition is not, in a proper sense, synthetic. It is a mere corollary from the definition of a pair. Moreover, its application to experience, or to possible experience, opens the door to probability, and shuts out absolute necessity and certainty, *in toto*.

Concerning points like this, Dr. Carus, in company with the general body of thinkers, is laboring under a great disadvantage from not understanding the logic of relatives. It is a subject I have been studying for a great many years, and I feel and know that I have an important report that I ought to make upon it. This branch of logic is, however, so abstruse, that I have never been able to find the leisure to translate my conclusions into a form in which their significance would be manifest even to a powerful thinker whose thoughts had not long been turned in that direction. I shall succeed in doing so, whenever I can find myself in a situation where I need think of nothing else for months, and not before. That may not be for thirty years; but I believe it is the intention of providence that it should be. Meantime, I will testify, and the reader can take my testimony for what he thinks it is worth, that all deductive reasoning, except that kind which is so childishly simple that acute minds have doubted whether there was any reasoning there,—I mean non-relative syllogism,—requires an act of choice; because from a given premise, several conclusions,—in some cases an infinite number,—can be drawn. Hence, Dr. Carus is altogether too hasty in his confidence (¶¶ 195, 196) that general thinking machines "are not impossibilities." An act of original and arbitrary determination would be required; and it seems almost evident that no machine could perform such an act except within narrow limits, thought out beforehand and embodied in its construction. More over, positive observation is called for in all inference, even the simplest,—though in deduction it is only observation of an object of imagination. Moreover, a peculiar act which may properly be called *abstraction** is usually required, consisting in seizing evanescent elements of thought and holding them before the mind as "substantive" objects, to borrow a phrase from William James. At the same time, the process I am describing, that is, relative deduction, is perfectly general and demonstrative, and depends upon the truth of the assumed premises, and not, like inductive reasoning, upon the manner in which those premises present themselves.

But the application of the logic of relatives shows that the propositions of arithmetic, which Dr. Carus usually adduces as examples of formal law (¶ 15), are, in fact, only corollaries from definitions. They are certain only as applied to ideal constructions, and in such application, they are merely analytical.

The truth is our ideas about the distinction between analytical and synthetical judgments is much modified by the logic of relatives, and by the logic of probable inference. An analytical proposition is a definition or a proposition *deducible* from definitions; a synthetical proposition is a proposition not analytical. Deduction, or analytical reasoning, is, as I have shown in my "Theory of Probable Reasoning," a reasoning in which the conclusion follows (necessarily, or probably) from the state of things expressed in the premises, in contradistinction to scientific, or synthetical, reasoning, which is a reasoning in which the conclusion follows probably and approximately from the premises, owing to the conditions under which the latter have been observed, or otherwise ascertained. The two classes of reasoning present, besides, some other contrasts that need not be insisted upon in this place. They also present some significant resemblances. Deduction is really a matter of perception and of experimentation, just as induction and hypothetic inference are; only, the perception and experimentation are concerned with imaginary objects instead of with real ones. The operations of perception and of experimentation are subject to error, and therefore it is only in a Pickwickian sense that mathematical reasoning can be said to be perfectly certain. It is so, only under the condition that no error

* I apply this term because it is essentially like the passage from the concrete "virtuous" to the abstract "virtue," or from the concrete "white" (adjective) to the abstract "whiteness," or "white" (substantive).

creeps into it: yet, after all, it is susceptible of attaining a practical certainty. So, for that matter, is scientific reasoning; but not so readily. Again, mathematics brings to light results as truly occult * and unexpected as those of chemistry; only they are results dependent upon the action of reason in the depths of our own consciousness, instead of being dependent, like those of chemistry, upon the action of Cosmical Reason, or Law. Or, stating the matter under another aspect, analytical reasoning depends upon associations of similarity, synthetical reasoning upon associations of contiguity. The logic of relatives, which justifies these assertions, shows accordingly that deductive reasoning is really quite different from what it was supposed by Kant to be; and this explains how it is that he and others have taken various mathematical propositions to be synthetical which in their ideal sense, as propositions of pure mathematics, are in truth only analytical.

Descending from things I can demonstrate to things of which various facts, in the light of those demonstrations, fully persuade me, I will say that in my opinion there are many synthetical propositions which, if not *a priori* in Dr. Carus's sense, are, at least, innate (notwithstanding his frequent denials of this, as in ¶ 15) though he is quite right in saying that their abstract and distinct formulation comes very late (¶ 126). But turn the facts as I will, I cannot see that they afford the slightest reason for thinking that such propositions are ever absolutely universal, exact, or necessary in their truth. On the contrary, the principles of probable inference show this to be impossible.

Dr. Carus adduces the instance of a geometrical proposition, namely, "that two congruent regular tetrahedrons, when put together, will form a hexahedron." (¶ 25.) This, he says, seems to be "a very wonderful thing"; for why should not a larger tetrahedron be formed, just as two heaps of flour make a large heap of flour? Yet, he continues, the probability that the two tetrahedrons

* I can never use this word without thinking of the explanation of it given by Petrus Peregrinus in his *Epistole de Magnete*. He says that physical properties are occult in the sense that they are only brought out by experimentation, and are not to be deduced from admixtures of *hot* and *cold*, *moist* and *dry*.

do always make a hexahedron is 1, "which means certainty" (¶ 27). But as it happens, the proposition, in the form stated is quite erroneous. What is true is this. If two tetrahedra are so placed that one face of each is coincident with one face of the other, while all the other faces are inclined to one another, and if of the 8 faces, the 2 that are coincident are not counted, there remain to be counted $8 - 2 = 6$ faces. But there is nothing more wonderful about this than that $8 - 2 = 6$, which is an easy corollary from definitions. Very few propositions in mathematics that appear "marvellous" will hold water; and those few excite our astonishment only because the real complexity of the conditions are masked in an intuitional presentation of them.

Dr. Carus holds (¶ 15) that formal knowledge is absolutely universal, exact, and necessary. In some cases, as where he says that, given the number of dimensions of space, the entire geometry could be deduced (¶ 35), the boasted infallibility will prove on examination to be downright error. In all other cases, the propositions only relate to ideal constructions, and their applicability to the real world is at the best doubtful, and, as I think, false; while in their ideal purity, they are not synthetical.

Thus, my good friend and antagonist holds that the combination of oxygen and hydrogen to produce water is not "different in principle" from that of the tetrahedra to produce a hexahedron (¶ 26). There is all the difference between the ideal and the real; which to my Scotistic mind is very important. But this is not the only passage in which he speaks as if form were the principle of individuation.

§ 9. Dr. Carus's position is even weaker than that of Kant, who makes space, for example, a necessary form of thought (in a broad sense of that term). But Dr. Carus appears to consider space as an absolute reality. For he says (¶ 119) that "every single point of space has its special and individual qualities." Here again form is made the principle of individuation; whence the queer phrase, "individual qualities."

§ 10. Dr. Carus argues that whatever is unequivocally determinate is necessary. (¶ 124.) Were the determination spoken of real dynamic determination, this would be a mere truism. But the expression used, *eindeutig bestimmt*, merely expresses a mathematical determination, and therefore no real necessity ensues. The equation

$$x^2-23x+132\tfrac{1}{4}=0$$

determines x to be either 11·477 or 11·523. In this sense, x has necessarily one value or the other. The equation

$$x^2-12x+6=0$$

determines x to be either 11·477 or 0·523. Together, the two equations uniquely determine x to be 11·477. This shows how much that argument amounts to.

§ 11. By "sameness," Dr. Carus means equivalence for a given purpose. (¶¶ 102, 106.) By the "idea of sameness," he means (¶¶ 77, 96) the principle that things having a common character are for some purpose equivalent. This, he says, "has a solid basis in the facts of experience." By a "world of sameness" (¶ 113), he seems to mean one in which any two given concrete things are in some respect equivalent. He argues (¶ 122) that a "world of sameness is a world in which necessity rules." I do not see this. It seems to me so bald a *non sequitur*, that I cannot but suppose the thought escapes my apprehension. If there were anything in the argument, it would seem to be a marvellously expeditious way of settling the whole dispute ; and therefore it would have been worth the trouble of stating, so as to bring it within the purview of minds like mine.

§ 12. My candid opponent sometimes endorses emphatically the Leibnizian principle. "Necessitarianism must be founded on something other than observation. Observation is *a posteriori*; it has reference to single facts, to particulars; yet the doctrine of necessity . . . is of universal application. The doctrine of necessity . . . is of an *a priori* nature." (¶ 11.) "Millions of single experiences . . . cannot establish a solid belief in necessity." (¶ 14.) "No amount of experience is sufficient to constitute causation by a mere synthesis of sequences." (¶ 22.) "Millions of millions of cases" constitute "no proof" that a proposition "is always so." (¶ 29.)

Nevertheless, he holds that the law of "the conservation of matter and energy" so conclusively proves necessary causation, that the obstinacy of Hume, himself, could not have withstood the argument. (¶ 23.) One wonders, then, what is supposed to prove this "law of the conservation of matter and energy," if no amount of experience can prove it.

But the *a priori* itself can "be based on the firm ground of experience." (¶ 14.) In that case, it is not prior to experience, after all! "The idea of necessity is based upon the conception of sameness, and . . . the existence of sameness is a fact of experience." (¶ 87.) If absolute necessity can be irrefragably demonstrated from the fact that two things are alike, it is a pity Dr. Carus should not state this demonstration in a form that I, and men like me, can understand. That would be more to the purpose than merely saying it can be proved. Absolute chance is rejected as "involving a violation of laws well established by *positive evidence.*" (¶ 149.)

All these *denials* that absolute necessity can be established and absolute chance refuted by experiential evidence, mixed with as clear *assertions* of the same things, when taken together, have the appearance of an attempt, as the politicians say, to "straddle" the question.

§ 13. But the ingenious Doctor seeks to bolster up necessity by introducing the confused notion of "causation."

I do not know where the idea originated that a cause is an instantaneous state of things, perfectly determinative of every subsequent state. It seems to be at the bottom of Kant's discourse on the subject; yet it accords neither with the original conception of a cause, nor with the principles of mechanics. The original idea of an efficient cause is that of an agent, more or less like a man. It is prior to the effect, in the sense of having come into being before the latter; but it is not transformed into the effect. In this sense, it may happen that an event is a cause of a subsequent event; seldom, however, is it the principal cause. Far less are events the only causes. The modern mechanical conception, on the other hand, is that the relative positions of particles determine their accelerations at the instants when they occupy those positions. In other words, if the positions of all the particles are given at *two* instants (together with the law of force), then the positions at all other instants may be deduced.* This doctrine conflicts with Kant's second analogy of experience, as interpreted by him, in no less than four essential particulars. In the first place, far from involving any principle that could properly be termed generation, or *Erzeugung*, which is Kant's word for the sequence of effect from cause, the modern mechanical doctrine is a doctrine of persistence, and, as I have repeatedly explained, positively prohibits any real growth. In the second place, one state of things (i. e. one configuration of the system) is not sufficient to determine a second; it is two that determine a third. To whomsoever may think that this is an inconsiderable divergence of opinions, let me say, study the logic of relatives, and you will think so no longer. In the third place, the two determining configurations, according to mechanics, may be taken at almost any two instants, and the determined configuration be taken at any third instant we like. *There is no mechanical truth in saying that the past determines the future, rather than the future the past.* We habitually follow tradition in continuing to use that form of expression, but every mathematician knows that it is nothing but a form of expression. We continue, for convenience, to talk of mechanical phenomena as if they were regulated, in the same manner in which our intentions regulate our actions (which is essentially a determination of the future by the past), although we are quite aware that it is not really so. Remark how Kant reasons:

> "If it is a necessary law of our sensibility, and consequently a *formal condition* of all perceptions, that the preceding time determines the following, (since I can only come to the following through the preceding,) then is it also an indispensable *law of the empirical representation* of the time-series that the appearances of the preceding time determine every occurrence in the following."

What this leads to is a causality like that of mental phenomena, where it *is* the past which determines the future, and *not* (in

* It follows as a corollary from this that if the positions of the particles at any one instant, together with the velocities at that instant, and the law of force, are given, the positions at all instants can be calculated. Of course, to give the positions and velocities at one instant, is a special case of the giving of the positions at two instants. The two instants may be such that there will be more than one solution of the problem; but this is an insignificant detail.

the same sense) the reverse; but the doctrine of the conservation of energy consists precisely in the denial that anything like this occurs in the domain of physics. Had Kant studied the psychological phenomena more attentively and generalised them more broadly, he would have seen that in the mind causation is not absolute, but follows such a curve as is traced in my essay towards "The Law of Mind" (*The Monist*, Vol. II, 550). Does our judicious editor deem it ungracious to find fault with Kant for not doing so much more than he did, considering what that hero-like achievement was? We must seem to carp, as long as thinkers can hold that achievement for sufficient. In the fourth place, Kant's "Analogy" ignores that continuity which is the life-blood of mathematical thought. He deals with those awkward chunks of phenomenon, called "events." He represents one such "event" as determined by certain others, definitely, while the rest have nothing to do with it. It is impossible to cement such thought as this into hermetic continuity with the refined conceptions of modern dynamics. The statement that every instantaneous state of things determines precisely all subsequent states, and not at all any previous states, could, I rather think, be shown to involve a contradiction.

The notion which Dr. Carus holds of a cause seems to be that it is a state, embracing all the positions and velocities of all the masses at one instant, the effect being a similar state for any subsequent instant. (¶¶ 21, 24.) This breaks at once with common parlance, with dynamics, and with philosophical logic. In common parlance, we do not say that the position and upward velocity of a missile is the cause of its being at a subsequent instant lower down and moving with a greater downward velocity.* In dynamics, it is the fixed force, gravitation, or whatever else, together with those relative positions of the bodies that determine the intensity and direction of the forces, that is regarded as the cause. But these causes are not previous to, but simultaneous with, their effects, which are the instantaneous accelerations. Finally, logic opposes our calling

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* It would seem to follow from his notion that in uniform motion each minute's motion is the cause of that of the next. Yet he says (¶ 19) "there is no cause that is equal to its effect."

one of two states which equally determine one another (as any two states of a system do, if the velocities are taken to be included in these states) *the* determinator, or cause, simply because of the circumstance that it precedes the other in time,—a circumstance that is upon the principles of dynamics plainly insignificant and irrelevant.

Everybody will make slips in the use of words that have been on his lips from before the time when he learned to think; but the practice which I endeavor to follow in regard to the word *cause*, is to use it in the Aristotelian sense of an *efficient cause*, in all its crudeness. In short, I refuse to use it at all as a philosophical word. When my conception is of a dynamical character, I endeavor to employ the accepted terminology of dynamics;* and when my idea is a more general and logical one, I prefer to speak of the *explanation*.

§ 14. Dr. Carus thinks the element of necessity in causation can be demonstrated by considering the process as a transformation. "It is a sequence of two states which belong together as an initial and final aspect of one and the same event." (¶ 21. Compare ¶¶ 20, 24.) He neglects to explain how he brings under this formula the inward causation of the will and character, as set forth by him in ¶¶ 163–167.

It is unnecessary for me to reply, at length, to an argument so manifestly inconclusive. On the one hand, it conflicts with the principle that absolute necessity cannot be proved from experience; and on the other hand, it leaves room for an imperfect necessity.

Professor Tait has done an ill office to thought in countenancing the idea that the conservation of energy is of the same nature as the "conservation," or rather perduration, of matter. Dr. Carus says (¶ 121) that

> "The law of the conservation of matter and energy rests upon the experience (corroborated by experiments) that causation is transformation. It states that the total amount of matter and the total amount of energy remain constant. There is no creation out of nothing and no conversion of something into nothing."

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* But, as I have elsewhere said, I should like to persuade mathematicians to speak of "positional energy" as *Kinetic potency*, the *vis viva* as *Kinetic energy*, and the total "energy" as the *Kinetic entelechy*.

The historical part of this statement contains only a small grain of truth; but that I will not stop to criticise. The point I wish to make is that the law of the conservation of energy is here represented under a false aspect. The true substance of the law is that the accelerations, or rates of change, of the motions of the particles at any instant depend solely on their relative positions at those instants. The equation which expresses the law under this form is a differential equation of the second order; that is, it involves the rates of change of the rates of change of positions, together with the positions themselves. Now, because of the purely analytic proposition of the differential calculus that

$$D_t{}^2 s = \tfrac{1}{2} D_s (D_t s)^2,$$

the first integral of the differential equation of the second order, that is, the differential equation of the first order which expresses the same state of things, equates half the sum of the masses, each multiplied by the square of its velocity, to a function of the relative positions of the particles *plus* an arbitrary constant.* In order to fix our ideas, let us take a very simple example, that of a single particle accelerated towards an infinite plane, at a rate proportional to the n^{th} power of its distance from the plane. In this case, if s be the distance, the second differential equation will be

$$D_t{}^2 s = -a s^n,$$

and the first integral of it will be

$$(D_t s)^2 = -\frac{2a}{n+1} s^{n+1} + C.$$

By the first law of motion, and the Pythagorean proposition, the part of the velocity-square depending on the horizontal component is also constant.

The arbitrary constant, C, plainly has its genesis in the fact that forces do not determine velocities, but only accelerations. Its value will be fixed as soon as the velocity at any instant is known. This quantity would exist, just the same, and be independent of the time, and would therefore be "conserved" whether the forces were

* The differential equation being an ordinary, not a partial one, this is an absolute constant, determined by initial (or final, or any instantaneous) conditions.

"conservative," that is, simply positional, or not. Now, this constant is the energy; or rather, the energy is composed of this constant increased by another which is absolutely indeterminable, being merely supposed large enough to make the sum positive.

Thus, the law of energy does not prescribe that the total amount of energy shall remain constant; for this would be so in any case by virtue of the second law of motion; but what it prescribes is that the total energy diminished by the living force shall give a remainder which depends upon the relative positions of the particles and not upon the time or the velocities. It is also to be noticed that the energy has no particular magnitude, or quantity. Furthermore, in transformations of kinetical energy into positional energy, and the reverse, the different portions of energy do not retain their identity, any more than, in book-keeping, the identity of the amounts of different items is preserved. In short, the conservation of energy, (I do not mean the *law* of conservation,) is a mere result of algebra. Very different is it with the "conservation" of matter. For, in the first place, the total mass is a perfectly definite quantity; and, in the second place, in all its transformations, not only is the *total* amount constant, but all the different parts preserve their identity. To speak, therefore, of "the conservation of matter and energy," is to assimilate facts of essentially contrary natures; and to say that the law of the conservation of energy makes the total amount of energy constant is to attribute to this law a phenomenon really due to another law, and to overlook what this law really does determine, namely, *that the total energy less the kinetic energy gives a remainder which is exclusively positional.*

§ 15. Dr. Carus does not make it clear what he means by *chance.* He does, indeed, say (¶¶ 145, 146):

"What is chance?

"Chance is any event not especially intended, either not calculated, or, with a given and limited stock of knowledge, incalculable."

This defines what he means by a chance event, in the concrete; what he understands by probability, we are left to conjecture. But from what he says in ¶ 147, I infer that he regards it as dependent upon the state of our ignorance, and therefore nothing real.

I am, therefore, much puzzled when I find him expressing a conviction (¶¶ 88, 156) that chance plays an important part in the real world. He explains very distinctly that "when we call a throw of the dice pure chance, we mean that the incidents which condition the turning up of these or those special faces of the dice have not been, or cannot be, calculated." (¶ 147.) This is the commonest, because the shallowest, philosophy of chance. Even Venn might teach him better than that. However, according to that view, when he writes of "the important part that chance plays in the world,—not absolute chance . . . . but that same chance of which the throw of the die is a typical instance" (¶ 88), he can be understood to mean no more than that many things happen which we are not in condition to calculate or predict. This is not playing a part *in the world*, one would say—at least, not in the natural world ; it is only playing a part in our ignorance.

Dr. Carus frequently uses phrases which make us suspect he penetrates deeper. Thus, he says, "we do not believe in absolute chance, but we believe in chance" (¶ 144); and again, "Every man is the architect of his own fortune—but not entirely. There are sometimes coincidences determining the fates of men." (¶ 161.) But when we remark the consecution of ¶¶ 137–162, we feel pretty sure he really sees no further. To do so would have been to perceive that indefinitely varied specificalness *is* chance.

For a long time, I myself strove to make chance that diversity in the universe which laws leave room for, instead of a violation of law, or lawlessness. That was truly believing in chance that was not absolute chance. It was recognising that chance does play a part in the real world, apart from what we may know or be ignorant of. But it was a transitional belief which I have passed through, while Dr. Carus seems not to have reached it.

As for absolute chance, Dr. Carus makes the momentous admission that it is "not unimaginable" (¶ 150). If so, its negation, or absolute necessity, cannot be a formal principle.

§ 16. But it is time for me to leave the consideration of Dr. Carus's system and to take up his strictures upon mine. His philosophy is one eminently enlightened by modern ideas, which it synthetises to an unusual extent. It is distinguished for its freedom from the vice of one-sidedness, and displays every facet of the gem of philosophical inquiry, except the one on which it rests, the question of absolute law. Its prominent faults, which I feel sure must have struck every competent reader, are that it shows little trace of meditation upon the thoughts of the great idealists, and that there is a certain want of congruity between different elements of it. How strangely it sounds, for instance, to find an apriorian, and one who is dinging "formal laws" so perpetually into our ears, one who holds that "in order to weave the woof of the *a posteriori* elements into coherent cloth, we want the warp of the *a priori*" (¶ 15), to find this man declaring for a positivism "which accepts no doctrine, theory, or law, unless it be a formulation of facts," and proclaiming that "the whole business of science is to systematise the samenesses of experience, and to present them in convenient formulas" (¶ 120). Now there is just one way of bringing such warring elements into harmony, and curing the greatest defect of the system,—and it is a way which would also bring the whole into far better concordance with natural science. It is to lop off the heads of all absolute propositions whose subject is not the Absolute, and reduce them to the level of probable and approximate statements. Were that defalcation performed, Dr. Carus's philosophy would, in its general features, offer no violent opposition to my opinions. Moreover, the Doctor has at heart the conciliation of religion and science. I confess such serious concern makes me smile; for I think the atonement he desires is a thing which will come to pass of itself when time is ripe, and that our efforts to hasten it have just that slight effect that our efforts to hasten the ripening of apples on a tree may have. Besides, natural ripening is the best. Let science and religion each have stout faith in itself, and refuse to compromise with alien and secondary purposes, but push the development of its own thought on its own line; and then, when reconcilement comes,—as come it surely will,—it will have a positive value, and be an unmixed good. But since our accomplished editor thinks himself called upon to assist in this birth of time, let me ask him whether of all the conditions of such peace, the first is not that religious thought should abandon that extravagant absoluteness of assertion which is proper to the state of intellectual infancy, but which it has so long been too timid to let go? This pragmatical and unneeded absoluteness it is which is most deeply contrary to the method, the results, and the whole spirit of science; and no error can be greater than to fancy that science, or scientific men, rest upon it or readily tolerate it.

§ 17. Dr. Carus (¶¶ 56–64) condemns my method of investigation as contrary to that by which science has been advanced; and holds that a radically different, and thoroughly positivistic method is requisite,—a method so intensely positivistic as to exclude all originality. I suppose he will not object to my forming an opinion concerning the methods of science. I was brought up in an atmosphere of scientific inquiry, and have all my life chiefly lived among scientific men. For the last thirty years, the study which has constantly been before my mind has been upon the nature, strength, and history of methods of scientific thought. I have no space here to argue the question. In its logical aspect, I have partly considered it in various publications; and in its historical aspect, I have long been engaged upon a treatise about it. My critic says (¶ 57) that I am "very positivistic in my logic of science." This is a singular misapprehension. Few of the great scientific minds with whom I have come into personal contact, and from whom I endeavored to learn were disposed to contemn originality or the ideal part of the mind's work in investigation; and those few, it was easy to see, really breathed an atmosphere of ideas which were so incessantly present that they were unconscious of them. Were I to name those of my teachers who were most positivistic in theory, a smile would be excited. My own historical studies, which have been somewhat minutely critical, have, on the whole confirmed the views of Whewell, the only man of philosophical power conjoined with scientific training who had made a comprehensive survey of the whole course of science, that progress in science depends upon the observation of the right facts by minds *furnished with appropriate ideas.* Finally, my long investigation of the logical process of scientific reasoning led me many years ago to the conclusion that science is nothing but a development of our natural instincts. So much for my *theory* of scientific logic. It is as totally opposed as anything can be to Dr. Carus's theory (¶ 69, note; and "Ursache, Grund und Zweck," p. 2) that originality is out of place in science.

But in my *practice* of scientific reasoning, Dr. Carus accuses me of being what he calls a "constructionist"; that is, a theoriser unguided by indications from observation or accepted facts. To a mind upon whom that celebrated and splendid chapter of Kant upon the architectonical method failed to make a deep impression, I may appear so; but *travesty* is in truth hardly too strong a word to describe the account of my method by Dr. Carus.

Perhaps exaggeration is not without its value. If so, let me sum up the method Dr. Carus recommends. Eschew originality, is its pious formula; do not think for yourself, nor countenance results obtained by original minds. Distrust them; they are not safe men. Leave originality to mathematicians and their breed, to poets, and to all those who seek the sad notoriety of having unsettled belief.* Flee all philosophies which smack of this aberrant nineteenth century.† This theory of Dr. Carus condemns itself; for it is highly original, and soars into the free ether untrammelled by historic facts.

Keppler comes very close to realising my ideal of the scientific method; and he is one of the few thinkers who have taken their readers fully into their confidence as to what their method really has been.‡ I should not feel justified in inflicting upon mine an autobiographical account of my own course of thought; but some things Dr. Carus's accusation forces me to mention. My method of attacking all problems has ever been to begin with an historical and rational inquiry into the special method adapted to the special problem. This is the essence of my architectonical proceeding upon which Dr. Carus has commented very severely. To look an inch before one's nose involves originality: therefore, it is wrong to have a conscious method. But further, in regard to philosophy, not only

* Dr. Carus calls attention to the connection between my doctrine of the fixation of opinion and his anti-originalism.

† Dr. Carus passes a sweeping judgment on Post-Kantian philosophy, as being original.

‡ This was a remark of my father's.

the methods, but the elementary ideas which are to enter into those methods, should be subjected to careful preliminary examination. This, especially, Dr. Carus finds very unscientific. (¶ 64, and elsewhere.) It is, undoubtedly, the most characteristic feature of my procedure. Certainly it was not a notion hastily or irreflectively caught up; but is the maturest fruit of a lifetime of reflection upon the methods of science, including those of philosophy; and if it shall be found that one contribution to thought on my part has proved of permanent service, that, I expect, will be the one. This method in no wise teaches that the method and materials for thought are not to be modified in the course of the study of the subject-matter. But instead of taking ideas at haphazard, or being satisfied with those that have been handed down from the good old times, as a mind keenly alive to the dangers of originality would have done, I have undertaken to make a systematic survey of human knowledge (a very slight sketch of which composed the substance of my paper on the "Architecture of Theories,") in order to find what ideas have, as a fact, proved most fruitful, and to observe the special utilities they have severally fulfilled. A subsidiary object of this survey was to note what the great obstacles are to-day in the way of the further advance of the different branches of science. In my "Architecture of Theories," I never professed to do more than make a slight sketch of a small portion of my preliminary studies, devoting thirteen lines to some hints as to the nature of the results. In the four following papers I have given a selection of a few of these results. Among those which remain to be reported are some of much more immediate importance than any of those hitherto set forth. If anybody has been surprised to find my subsequent papers developing thoughts which they were unable to foresee from my first, it is only what I warned people from the outset that they would find to happen. Nor have the greatest of these surprises yet been reached.*

The next series of facts reviewed was that of the history of philosophy. I waded right into this fearful slough of "originality," in

* A person in the last *Monist*, breaks in upon my series of articles to foretell what the "issues of synechism" will be. Were he able to do so, it would certainly be the height of ill-manners thus to take the words out of my mouth.

order to gather what seemed to throw a light upon the subject. Finally, I reviewed the general facts of the universe.

I now found myself forced by a great many different indications to the conclusion that an evolutionary philosophy of some kind must be accepted,—including among such philosophies systems like those of Aristotle and of Hegel. From this point the reasoning was more rapid. Evolution had been a prominent study for half a generation; and much light had been thrown upon the conditions for a fruitful evolutionary philosophy. The first question was, how far shall this evolution go back? What shall we suppose *not* to be a product of growth? I fancy it is this cautious reflectiveness of my procedure which especially displeases Dr. Carus. It is not positivistic: it is architectonic. But the answer to the question was not far to seek. If an evolutionary explanation is to be adopted, philosophy, logic, and the economy of research all dictate that in the first essay, at least, that style of explanation be carried as far back as explanation is called for. What elements of the universe require no explanation? This was a simple question, capable of being decided by logic with as much facility and certainty as a suitable problem is solved by differential calculus. Being, and the uniformity in which being consists, require to be explained. The only thing that does not require it is non-existent spontaneity. This was soon seen to mean absolute chance. The conclusion so reached was clinched by a careful reëxamination of the office of chance in science generally, and especially in the doctrines of evolution. Arrived at this point, the next question was, what is the principle by which the development is to proceed? It was a difficult inquiry, and involved researches from different points of view.

But I will not trouble the reader with further autobiographical details. I have given enough to show that my method has neither been in theory purely empirical, nor in practice mere brain-spinning; and that, in short, my friend Dr. Carus's account of it has been as incorrect as can be.

§ 18. The learned doctor (¶¶ 6, 7, 8) pronounces me to be an imitator of David Hume, or, at least, classes my opinions as closely allied to his. Yet be it known that never, during the thirty years in which I have been writing on philosophical questions, have I failed in my allegiance to realistic opinions and to certain Scotistic ideas; while all that Hume has to say is said at the instance and in the interest of the extremest nominalism. Moreover, instead of being a purely negative critic, like Hume, seeking to annul a fundamental conception generally admitted, I am a positive critic, pleading for the admission to a place in our scheme of the universe for an idea generally rejected. In the first paper of this series, in which I gave a preliminary sketch of such of my ideas as could be so presented, I carefully recorded my opposition to all philosophies which deny the reality of the Absolute, and asserted that "the one intelligible theory of the universe is that of objective idealism, that matter is effete mind." This is as much as to say that I am a Schellingian, of some stripe; so that, on the whole, I do not think Dr. Carus has made a very happy hit in likening me to Hume, to whose whole method and style of philosophising I have always been perhaps too intensely averse. Yet, notwithstanding my present disclaimer, I have little doubt apriorians will continue to describe me as belonging to the sceptical school. They have their wonderful ways of arriving at truth, without stooping to confront their conclusions with facts; and it is amusing to see how sincerely they are convinced that nobody can have science at heart, without denying all they uphold.*

My opponent has a habit of throwing out surprising opinions without the least attempt to illuminate them with the effulgence of reason. Thus he says (¶ 8): "If Kant's answer to Hume had been satisfactory, Mr. Peirce would probably not have renewed the attack." What attack? All that Hume attacked I defend, namely, law as a reality. How could a defence of that which I defend as essen-

* As I am writing, I am shown a letter, in which the writer says: "Peirce with all his materialistic ideas, yet," etc I never promulgated a materialistic idea in my life. The writer simply assumes that science is materialistic. As I am correcting the proofs, I notice that Mr. B. C. Burt, in his new *History of Modern Philosophy*, sets me down as sceptical, though doubtfully. There are a good many inaccuracies in the work. This was inevitable in a first edition. But the ingenious plan of the book admirably adapts it to the wants of just that class of students who cannot understand that no repertory of facts ever can be trusted implicitly.

tial to my position, cause me to surrender that position, namely, that real regularity is imperfect? In any sense in which Hume could have admitted the possibility of law, it must be precisely followed; since its existence could consist only in the conformity of facts unto it. But perhaps Dr. Carus means that if one question had been completely settled, I should probably have confined myself to talking about that, instead of broaching a new one.

§ 19. Another misunderstanding of my position on the part of Dr. Carus (¶¶ 12, 13) is simply due to "boldly" having been twice printed where the reading should have been "baldly," in my paper on "The Doctrine of Necessity." (*The Monist*, Vol. II, p. 336, lines 20 and 25.) I wish printers would learn that I never use the word *bold*. I have so little of the quality, that I don't know what it means. As I read the "revise," as usual, it was presumably my fault that the *erratum* occurred. At any rate, had my meaning been clearly expressed, the proof-reader would not have been misled by my defective chirography. What I was trying to say was, in substance, this: Absolute chance is a hypothesis; and, like every hypothesis, can only be defended as explaining certain phenomena.* Yet to suppose that an event is brought about by absolute chance is utterly illogical, since as a hypothesis it could only be admitted on the ground of its explaining observed facts; now from mere non-law nothing necessarily follows, and therefore nothing can be explained; for to explain a fact is to show that it is a necessary, or, at least, a probable, result from another fact, known or supposed. Why is not this a complete refutation of the theory of absolute chance? *Answer*: because the *existence* of absolute chance, as well as many of its characters, are not themselves absolute chances, or sporadic events, unsubject to general law. On the contrary, these things *are* general laws. Everybody is familiar with the fact that chance has laws, and that statistical results follow therefrom. Very well: I do not propose to explain anything as due to the action of

* Its being hypothetical will not prevent its being established with a very high degree of certainty. Thus, all history is of the nature of hypothesis; since its facts cannot be directly observed, but are only supposed to be true to account for the characters of the monuments and other documents.

chance, that is, as being lawless. I do not countenance the idea that Bible stories, for instance, show that nature's laws were violated;—though they may help to show that nature's laws are not so mechanical as we are accustomed to think. But I only propose to explain the regularities of nature as consequences of the only uniformity, or general fact, there was in the chaos, namely, the general absence of any determinate law.* In fact, after the first step is taken, I only use *chance* to give room for the development of law by means of the law of habits.

§ 20. In ¶ 28, I read: "Mr. Peirce does not object to necessity in certain cases; he objects to necessity being a universal feature of the world." This is correctly stated, and so it is in ¶ 203. I object to necessity being universal, as well as to its ever being exact. In short, I object to absolute universality, absolute exactitude, absolute necessity, being attributed to any proposition that does not deal with the A and the Ω, in the which I do not include any object of ordinary knowledge. But it is careless to write (¶ 193) that I "describe the domain of mind as the absence of law." Is not one of my papers entitled "The Law of Mind"? It is true that I make the law of mind essentially different in its mode of action from the law of mechanics, inasmuch as it requires its own violation; but it is law, not chance uncontrolled. That it is not "an undetermined and indeterminable sporting" should have been obvious from my expressly stating that its ultimate result must be the entire elimination of chance from the universe. That directly negatives the adjective "indeterminable," and hence also the adjective "undetermined." Still more unwarranted is the statement (¶ 205) that I deny "that there are samenesses in this world." If the slightest excuse for such an accusation can be found in all my writings I shall be mightily surprised.

§ 21. Dr. Carus fully admits (¶ 9) the justice of my first reply to the argument that necessity is postulated in all scientific reasoning, which reply is that to postulate necessity does not make it true.

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* Somebody may notice that I here admit a proposition as absolutely true. Undoubtedly; because it relates to the Absolute.

As this reply, if correct is complete, Dr. Carus was bound after that admission to drop the postulate-argument in favor of necessity.* But he takes no notice, at all, of my four-page argument to show that scientific reasoning does *not* postulate absolute universality, exactitude, or necessity (*The Monist*, Vol. II, pp. 324–327); but calmly asserts, four or five times over (¶¶ 5, 11, 16, 62, 79), without one scintilla of argumentation, that that postulate *is* made, and uses this as an argument in favor of necessity.

§ 22. He also fully admits (¶¶ 11, 14, 22) the justice of my argument that the absoluteness of universality, exactitude, and necessity, cannot be proved, nor rendered probable, by arguments from observation. That argument consisted in assuming that all arguments from observation are probable arguments, and in showing that probable inferences are always affected with probable errors.

Had I deemed it requisite, I might easily have fortified that argument by a more profound analysis of scientific reasoning. Such an analysis I had formerly given in my "Theory of Probable Inference" (in "Studies in Logic," Boston : Little and Brown).

But, notwithstanding his admissions, Dr. Carus sets up his *ipse dixit* against my argumentation. "We deny most positively," says the editorial Elohim, "that the calculus of probabilities is applicable to the order of the world, as to whether it may or may not be universal." (¶¶ 27, 31.)

To support this, he cites (¶¶ 31–34) four passages from articles written by me sixteen years ago. I hope my mind has not been stationary during all these years; yet there is little in those old articles which I now think positively erroneous, and nothing in the passages cited. My present views had, at that time, already begun to urge themselves on my mind; but they were not ripe for public avowal. In the first of the passages cited, I express the opinion, which I first uttered in my earlier lectures before the Lowell Institute, in 1866, afterwards in the *Popular Science Monthly* in 1877, in

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* Indeed, to admit that reply is all but to admit the non-absolute grade of necessity.

still fuller elaboration in my "Theory of Probable Inference" in 1882, and maintain now as strongly as ever, that no definite probability can be assigned to any general arrangement of nature. To speak of an *antecedent* probability would imply that there was a statistical science of different universes; and a *deduced* probability requires an antecedent probability for one of its data.* This consideration only goes to fortify my present position, that we cannot conclude from observed facts with any degree of probability, and therefore *a fortiori* not with certainty, that any proposition is absolutely universal, exact, or necessary. In the absence of any weight of probability in favor of any particular exact statement, the formal presumption is altogether against any one out of innumerable possible statements of that kind.

The second passage cited is one in which I argue that the universe is not a chaos, or chance-medley. Now Dr. Carus admits (¶ 28) that I do not to-day maintain that it is a chance-medley.

The third passage cited is this: "A contradiction is involved in the very idea of a chance-world." This is in entire harmony with my present position that "a chaos . . . . being without connection or regularity would properly be without existence." ("Architecture of Theories," *The Monist*, Vol. I, p. 176.)

The fourth passage is to the effect that "the interest which the uniformities of nature have for an animal measures his place in the scale of intelligence." This I still believe.

So much for my supposed contradictions. If I am not mistaken, our amiable editor, whose admirable editorship springs so largely out of his amiability, in copying out these passages was really not half so much intent on showing me to be wrong at present, as on showing me to have been right formerly. However hard he hits, he contrives to honey his sockdologers, and sincerely cares more to make the reader admire his antagonist when he is right than to condemn him when he is wrong. There is a touch of art in this that proclaims the born editor, and which I can hardly hope to imitate.

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* I rightly go somewhat further in my *Theory of Probable Inference*; but that has no bearing on the present discussion.

Though Dr. Carus admits over and over again that necessity cannot be based on observation, he often slips back to the idea that it can be so based. He says, (¶ 30) that "form is a quality of this world, not of some samples of it, but throughout, so far as we know of existence in even the most superficial way." But does he not see that all we *do* know, and all we *shall* to-morrow, or at any date know, is nothing but a sample of our possible experience,—nay, is but a sample of what we are in the future to have already experienced? I have characterised inductive inference as reasoning from samples; but the most usual way of sampling a class is by examining *all* the instances of it that have come under our observation, or which we can at once collect.

§ 23. Dr. Carus (¶¶ 44, 46) holds that from my social theory of reality, namely, that the real is the idea in which the community ultimately settles down, the existence of something inevitable is to be inferred. I confess I never anticipated that anybody would urge that. I thought just the reverse might be objected, namely, that all absoluteness was removed from reality by that theory; and it was many years ago that, in my "Theory of Probable Inference," I admitted the obvious justice, as it seemed to me, of that objection. We cannot be quite sure that the community ever will settle down to an unalterable conclusion upon any given question. Even if they do so for the most part, we have no reason to think the unanimity will be quite complete, nor can we rationally presume any overwhelming *consensus* of opinion will be reached upon every question. All that we are entitled to assume is in the form of a *hope* that such conclusion may be substantially reached concerning the particular questions with which our inquiries are busied.

Such, at least, are the results to which the consideration of the doctrine of probability brings my mind irresistibly. So that, the social theory of reality, far from being incompatible with tychism, inevitably leads up to that form of philosophy. Socialistic, or as I prefer to term it, agapastic ontology seems to me likely to find favor with many minds at an early day, because it is a natural path by which the nominalist may be led into the realistic ways of thought, ways toward which many facts and inward forces impel him. It is well, therefore, to call attention to the circumstances that the realism to which it leads is a doctrine which declares general truths to be real,—independent of the opinions of any particular collection of minds,—but not to be destined, in a strictly universal, exact, and sure acceptation, to be so settled, and established. Now to assert that general truths are objectively real, but to deny that they are strictly universal, exact, and certain, is to embrace the doctrine of absolute chance. Thus it is that the agapastic ontologist who endeavors to escape tychism will find himself "led into" that "inextricable confusion" which Dr. Carus (¶ 4) has taken a contract to show that I am led into.

§ 24. Conservatism is wholesome and necessary; the most convinced radical must admit the wisdom of it, in the abstract; and a conservative will be in no haste to espouse the doctrine of absolute chance. I, myself, pondered over it for long years before doing so. But I am persuaded, at length, that mankind will before very long take up with it; and I do not believe philosophers will be found tagging on to the tail of the general procession.

My little dialogue between the tychist and the necessitarian (*The Monist,* Vol. II, pp. 331–333) seems to have represented pretty fairly the views of the latter; for Dr. Carus, in ¶¶ 151–155, does little more than reiterate them, without much, if at all, reinforcing them. His ¶¶ 158–160 merely work out, in a form perhaps not quite clear, what is manifest from the elementary principles of dynamics, and was considered in my dialogue.

His arguments in this connection, apart from those already noticed, are that absolute chance is something which if it existed would require explanation, that the manifold specificalness of nature is explained by law without any aid from chance, and that absolute chance if it existed, in the sense in which it is supposed to exist in my chaos, could not possibly breed law as supposed by me. To the consideration of these arguments I proceed to apply myself.

§ 25. One of the architectonic—and, therefore, I suppose, by Dr. Carus considered as highly reprehensible—features of my theory, is that, instead of saying off-hand what elements strike me as requiring explanation and what as not doing so, which seems to be his way, I have devoted a long time to the study of the whole logical doctrine of explanation, and of the history of explanations, and have based upon the general principles so ascertained my conclusions as to what things do and what do not require to be explained.

Dr. Carus (¶ 67) defines *explanation* as a description of a special process of nature in such a way that the process is recognised as a transformation. This I cannot quite grant. First, I cannot admit that "special processes of nature" are the only things to be explained. For instance, if I were to meet a gentleman who seemed to conform scrupulously to all the usages of good society, except that he wore to an evening party an emerald satin vest, that would be a fact calling for explanation, although it would not be a "special process of nature." Second, I cannot admit that an explanation is a description of the fact explained. It is true that in the setting forth of some explanations, it is convenient to restate the fact explained, so as to set it under another aspect; but even in these cases, the statement of *other* facts is essential. In all cases, it is *other facts*, usually hypothetical, which constitute the explanation; and the process of explaining is a process by which from those other facts the fact to be explained is shown to follow as a consequence, by virtue of a general principle, or otherwise. Thus, a "special process of nature" calling for explanation is the circumstance that the planet Mars, while moving in a general way from west to east among the fixed stars, yet retrogrades a part of the time, so as to describe loops in the heavens. The explanation is that Mars revolves in one approximate circle and we in another. Again, it has been stated that a warm spring in Europe is usually followed by a cool autumn, and the explanation has been offered that so many more icebergs than usual are liberated during a warm spring, that they subsequently lower sensibly the temperature of Europe. I care little whether the fact and the explanation are correct or no. The case illustrates, at any rate, my point that an explanation is a special fact, supposed or known, from which the fact to be explained follows as a consequence. Third, I cannot admit that every description which recognises the fact described as a transformation is an explanation; far less that "it is complete and exhaustive" (¶ 67).

A magician transforms a watch into a dove. Recognise it as a transformation and the trick is explained, is it? This is delightfully facile. Describe the change from a caterpillar to a butterfly as a transformation, and does that explain it? Fourth, I cannot admit that every explanation recognises the fact explained as a transformation. The explanation of the loops in the motion of Mars is not of that nature. But I willingly recognise in Dr. Carus's definition an attempt,—more or less successful,—to formulate one of the great offices of scientific inquiry, that of bridging over the gap between the familiar and the unfamiliar.

Explanation, however, properly speaking, is the replacement of a complex predicate, or one which seems improbable or extraordinary, by a simple predicate from which the complex predicate follows on known principles. In like manner, a *reason*, in one sense, is the replacement of a multiple subject of an observational proposition by a general subject, which by the very conditions of the special experience is predicable of the multiple subject.* Such a reason may be called an explanation in a loose sense.

Accordingly, that which alone requires an explanation is a coincidence.

Hence, I say that a uniformity, or law, is *par excellence*, the thing that requires explanation. And Dr. Carus (¶ 51) admits that this "is perfectly true."

But I cannot imagine anything further from the truth than his statement (¶ 66) that "the only thing in the world of which we cannot and need not give account is the existence of facts itself." I should say, on the contrary, that the existence of facts is the only thing of which we need give account. Forms may indulge in whatever eccentricities they please in the world of dreams, without responsibility; but when they attempt that kind of thing in the world of real existence, they must expect to have their conduct inquired into. But should Dr. Carus reply that I mistake his meaning, that it is only "being in general" (¶ 66) that he holds unaccountable, I

* Dr. Carus, in his *Ursache, Grund und Zweck*, well says that *reasons* are discovered by induction, in the strict sense. It is often admitted that *causes* can only be inferred by hypothetic reasoning.

reply that this is simply expressing scepticism as to the possibility and need of philosophy. In a certain sense, my theory of reality, namely that reality is the dynamical reaction of certain forms upon the mind of the community, is a proposed explanation of being in general; and be it remarked that the mind of the community, itself, is the thing the nature of whose being this explanation first of all puts upon an idealistic footing.

Chance, according to me, or irregularity,—that is, the absence of any coincidence,—calls for no explanation. If *law* calls for a particular explanation, as Dr. Carus admits it does, surely the mere absence of law calls for no further explanation than is afforded by the mere absence of any particular circumstance necessitating the result. An explanation is the conception of a fact as a necessary result, thereby accounting for the coincidence it presents. It would be highly absurd to say that the absence of any definite character, must be accounted for, as if it were a peculiar phenomenon, simply because the imperfection of language leads us so to talk of it. Quite unfounded, therefore, is Dr. Carus's opinion that "chance needs exactly as much explanation as anything else" (¶ 53);—an opinion which, so far as I can see, rests on no defensible principle.

Equally hasty is his oft repeated objection (¶¶ 55, 58, 61) that my absolute chance is something ultimate and inexplicable. I go back to a chaos so irregular that in strictness the word existence is not applicable to its merely germinal state of being; and here I reach a region in which the objection to ultimate causes loses its force. But I do not stop there. Even this nothingness, though it antecedes the infinitely distant absolute beginning of time, is traced back to a nothingness more rudimentary still, in which there is no variety, but only an indefinite specificability, which is nothing but a tendency to the diversification of the nothing, while leaving it as nothing as it was before. What objectionable ultimacy is here? The objection to an ultimate consists in its raising a barrier across the path of inquiry, in its specifying a phenomenon at which questions must stop, contrary to the postulate, or hope, of logic. But what question to which any meaning can be attached am I forbidding by my absolute chance? If what is demanded is a theological backing, or rational antecedent, to the chaos, that my theory fully supplies. The chaos is a state of intensest feeling, although, memory and habit being totally absent, it is sheer nothing still. Feeling has existence only so far as it is welded into feeling. Now the welding of this feeling to the great whole of feeling is accomplished only by the reflection of a later date. In itself, therefore, it is nothing; but in its relation to the end, it is everything.

More unreasonable yet is Dr. Carus's pretension, that the manifold specificalness, which is what I mean by chance, is capable of explanation (¶¶ 142, 143) by his own philosophic method. He may explain one particularity by another, of course; but to explain specificalness itself, would be to show that a specific predicate is a necessary consequence of a generic one, or that a whole is without ambiguity a part of its part. Remark, reader, at this point, that chance, whether it be absolute or not, is not the mere creature of our ignorance. It is that diversity and variety of things and events which law does not prevent. Such is that real chance upon which the kinetical theory of gases, and the doctrines of political economy, depend. To say that it is not absolute is to say that it,—this diversity, this specificalness,—can be explained as a consequence of law. But this, as we have seen, is logically absurd.

Dr. Carus admits that absolute chance is "not unimaginable" (¶ 150). Chance itself pours in at every avenue of sense: it is of all things the most obtrusive. That it is absolute, is the most manifest of all intellectual perceptions. That it is a being, living and conscious, is what all the dullness that belongs to ratiocination's self can scarce muster hardihood to deny.

Almost as unthinking is the objection (¶ 61) that absolute chance could never beget order. I have noticed elsewhere the historic oblivescence of this objection. Must I once again repeat that the tendency to take habits, being itself a habit, has *eo ipso* a tendency to grow; so that only a slightest germ is needed? A realist, such as I am, can find no difficulty in the production of that first infinitesimal germ of habit-taking by chance, provided he thinks chance could act at all. This seems, at first blush, to be explaining something as a chance-result. But exact analysis will show it is not so.

In like manner, when the eminent thinker who does me the honor to notice my speculation, objects that I do not, after all, escape making law absolute, since the tendency to take habits which I propose to make universal is itself a law, I confess I can find only words without ideas in the objection. Law is a word found convenient, I grant, in describing that tendency ; but is there no difference between a law the essence of which is to be inviolable (which is the nominalistic conception of mechanical law, whose being, they say, lies in its action) and that mental law the violation of which is so included in its essence that unless it were violated it would cease to exist? In my essay, "The Law of Mind," I have so described that law. In so describing it, I make it a law, but not an absolute law ; and thus I clearly escape the contradiction attributed to me.

§ 26. In my attack on "The Doctrine of Necessity," I offered four positive arguments for believing in real chance. They were as follows :

1. The general prevalence of growth, which seems to be opposed to the conservation of energy.

2. The variety of the universe, which is chance, and is manifestly inexplicable.

3. Law, which requires to be explained, and like everything which is to be explained must be explained by something else, that is, by non-law or real chance.

4. Feeling, for which room cannot be found if the conservation of energy is maintained.

In a brief conversation I had with him, my friend remarked (and if it was an inconsiderate concession, I certainly do not wish to hold him to it) that while the theory of tychism had some attractive features, its weakness consisted in the absence of any positive reasons in its favor. I infer from this that I did not properly state the above four arguments. I therefore desire once more to call attention to them, especially in their relations to one another.

Mathematicians are familiar with the theorem that if a system of particles is subject only to positional forces, it is such that if at any instant the velocities were all suddenly reversed, without being altered in quantity, the whole previous history of the system would be repeated in inverse succession. Hence, when physicists find themselves confronted with a phenomenon which takes place only in one order of succession and never in the reverse order,—of which no better illustration could be found than the phenomena of growth, for nobody ever heard of an animal growing back into an egg,—they always take refuge in the laws of probability as preventive of the velocities ever getting so reversed. To understand my argument number 1, it is necessary to make this method of escape from apparent violations of the law of energy quite familiar to oneself. For example, according to the law of energy, it seems to follow (and by the aid of the accepted theory of light it does follow) that if a prism, or a grating, disperses white light into a spectrum, then the colors of the spectrum falling upon the prism or grating at the same angles, and in the same proportions, will be recombined into white light; and, everybody knows that this does in fact happen. Nevertheless, the usual and prevalent effect of prisms and gratings is to produce colored spectra. Why? Evidently, because, by the principles of probability, it will rarely happen that colored lights converging from different directions will fall at just the right angles and in just the right proportions to be recombined into white light. So, when physicists meet with the phenomena of frictional and viscous resistance to a body in motion, although, according to their doctrine, if the molecules were to move with the same velocities in opposite directions the moving body would be accelerated, yet they say that the laws of probability, applied to the trillions of molecules concerned, render this practically certain not to occur. I do no more, then, than follow the usual method of the physicists, in calling in chance to explain the apparent violation of the law of energy which is presented by the phenomena of growth: only instead of chance as they understand it, I call in absolute chance. For many months, I endeavored to satisfy the data of the case with ordinary *quasi* chance; but it would not do. I believe that in a broad view of the universe, a simulation of a given elementary mode of action can hardly be explained except by supposing the genuine mode of action somewhere has place. If it is improbable that colored lights should fall together in just such a way as to give a white ray, is it not an equally extraordinary thing that they should all be generated in such a way as to produce a white ray? If it is incredible that trillions of molecules in a fluid should strike a solid body moving through it so as to accelerate it, is it not marvellous that trillions of trillions of molecules all alike should ever have got so segregated as to create a state of things in which they should be practically certain to retard the body? It is far from easy to understand how mere positional forces could ever have brought about those vast congregations of similar atoms which we suppose to exist in every mass of gas, and by which we account for the apparent violations of the law of energy in the phenomena of the viscosity of the gas. There is no difficulty in seeing how sulphuric acid acting on marble may produce an aggregation of molecules of carbonic anhydride, because there are similar aggregations in the acid and in the marble; but how were such aggregations brought about in the first place? I will not go so far as to say that such a result is manifestly impossible with positional forces alone; but I do say that we cannot help suspecting that the simulated violation of the law of energy has a real violation of the same law as its ultimate explanation. Now, growth *appears* to violate the law of energy. To explain it, we must, at least, suppose a simulated, or *quasi*, chance, such as Darwin calls in to produce his fortuitous variations from strict heredity. It may be there is no real violation of the law, and no real chance; but even if there be nothing of the sort in the immediate phenomenon, can the conditions upon which the phenomenon depends have been brought about except by real chance? It is conceivable, again, that the law of the conservation of forces is not strictly accurate, and that, nevertheless, there is no absolute chance. But I think so much has been done to put the law of the conservation of forces upon the level of the other mechanical laws, that when one is led to entertain a serious doubt of the exactitude of that, one will be inclined to question the others.

Besides, few psychologists will deny the very intimate connection which seems to subsist between the law, or *quasi*-law, of growth and the law of habit, which is the principal, if not (as I hold it to be) the sole, law of mental action. Now, this law of habit seems to be quite radically different in its general form from mechanical law, inasmuch as it would at once cease to operate if it were rigidly obeyed ; since in that case all habits would at once become so fixed as to give room for no further formation of habits. In this point of view, then, growth seems to indicate a positive violation of law.

Let us now consider argument number 3 : and remark how it fortifies number 1. Physical laws that appear to be radically different yet present some striking analogies. Electrical force appears to be polar. Its polarity is explained away by Franklin's one-fluid theory, but in that view the force is a repulsion. Now, gravitation is an attraction, and is, therefore, essentially different from electricity. Yet both vary inversely as the square of this distance. Radiation, likewise follows the same formula. In this last case, the formula, in one aspect of it, follows from the conservation of energy. In another aspect of it, it results from the principle of probability, and does not hold good, in a certain sense, when the light is concentrated by a lens free from spherical aberration. But neither the conservation of energy nor the principle of probability seems to afford any possible explanation of the application of this theory to gravitation nor to electricity. How, then, are such analogies to be explained? The law of the conservation of energy and that of the perduration of matter present so striking an analogy that it has blinded some powerful intellects to their radically different nature. The law of action and reaction, again, has often been stated as the law of the conservation of momentum. Yet it is not only an independent law, but is even of a contrary nature, inasmuch as it is only the algebraical sum of opposite momenta that is "conserved."* How is this striking analogy between three fundamental laws to be explained? Consider the still more obvious analogy between space and time. Newton argues that the laws of mechanics prove space and time to be absolute entities. Leibniz, on the other hand, takes them as laws of nature. Either view calls for an explanation of the

* The conservation of a vortex, which consists of the preservation of a certain charater of motion by the same particles, though derived from the coöperation of other laws, is, in form, quite different.

analogy between them, which no such reflection as the impossibility of motion without that analogy can supply. Kant's theory seems to hint at the possibility of an explanation from both being derived from the nature of the same mind. Any three orthogonal directions* in space are exactly alike, yet are dynamically independent.

These things call for explanation ; yet no explanation of them can be given, if the laws are fundamentally original and absolute.

Moreover, law itself calls for explanation. But how is it to be explained if it is as fundamentally original and absolute as it is commonly supposed to be? Yet if it is not so absolute, there is such a phenomenon as absolute chance.

Thus, the chance which growth calls for is now seen to be absolute, not *quasi* chance.

Now consider argument number 2. The variety of the universe so far as it consists of unlikenesses between things calls for no explanation. But so far as it is a general character, it ought to be explained. The manifold diversity or specificalness, in general, which we see whenever and wherever we open our eyes, constitutes its liveliness, or vivacity. The perception of it is a direct, though darkling, perception of God. Further explanation in that direction is uncalled for. But the question is, whether this manifold specificalness was put into the universe at the outset, whether God created the universe in the infinitely distant past and has left it to its own machinery ever since, or whether there is an incessant influx of specificalness. Some of us are evolutionists ; that is, we are so impressed with the pervasiveness of growth, whose course seems only here and there to be interrupted, that it seems to us that the universe as a whole, so far as anything can possibly be conceived or logically opined of the whole, should be conceived as growing. But others say, though parts of the universe simulate growth at intervals, yet there really is no growth on the whole,—no passage from a simpler to a more complex state of things, no increasing diversity.

Now, my argument is that, according to the principles of logic,

* In speaking of directions, we assume the Euclidean hypothesis that the angles of a triangle are equal to two right angles.

we never have a right to conclude that anything is absolutely inexplicable or unaccountable. For such a conclusion goes beyond what can be directly observed, and we have no right to conclude what goes beyond what we observe, except so far as it explains or accounts for what we observe. But it is no explanation or account of a fact to pronounce it inexplicable or unaccountable, or to pronounce any other fact so. Now, to say no process of diversification takes place in nature leaves the infinite diversity of nature unaccounted for; while to say the diversity is the result of a general tendency to diversification is a perfectly logical probable inference. Suppose there be a general tendency to diversification; what would be the consequence? Evidently, a high degree of diversity. But this is just what we find in nature. It does not answer the purpose to say there is diversity because God made it so, for we cannot tell what God would do, nor penetrate his counsels. We see what He *does* do, and nothing more. For the same reason one cannot logically infer the existence of God; one can only know Him by direct perception.

It is to be noted that a general tendency to diversification does not explain diversity in its specific characters; nor is this called for. Neither can such a tendency explain any specific fact. Any attempt to make use of the principle in that manner would be utterly illogical. But it can be used to explain universal facts, just as quasi-chance is used to explain statistical facts. Now, the diversity of nature is a universal fact.

To explain diversity is to go behind the chaos, to the original undiversified nothing. Diversificacity was the first germ.

Argument No. 4 was, upon its negative side, sufficiently well presented in my "Doctrine of Necessity Examined." Mechanical causation, if absolute, leaves nothing for consciousness to do in the world of matter; and if the world of mind is merely a transcript of that of matter, there is nothing for consciousness to do even in the mental realm. The account of matters would be better, if it could be left out of account. But the positive part of the argument, showing what can be done to reinstate consciousness as a factor of the universe when once tychism is admitted, is reinforced in the later papers. This ought to commend itself to Dr. Carus, who shows himself fully alive to the importance of that part of the task of science which consists in bridging gaps. But consciousness, for the reason just stated, is not to be so reinstated without tychism ; nor can the work be accomplished by assigning to the mind an occult power, as in two theories to be considered in the section following this. As might be anticipated, (and a presumption of this kind is rarely falsified in metaphysics,) to bridge the gap synechism is required. Supposing matter to be but mind under the slavery of inveterate habit, the law of mind still applies to it. According to that law, consciousness subsides as habit becomes established, and is excited again at the breaking up of habit. But the highest quality of mind involves a great readiness to take habits, and a great readiness to lose them ; and this implies a degree of feeling neither very intense nor very feeble.

I have noticed above (§ 7) Dr. Carus's dubious attitude toward the first argument. I considered in the last section his attempted reply to the second. To the third argument, he replies (¶ 65) that law ought to be accounted for by the principle of sufficient reason. But, of course, that principle cannot recommend itself to me, a realist; for it is nothing but the lame attempt of a nominalist to wriggle out of his difficulties. Reasons explain nothing, except upon some theistic hypothesis which may be pardoned to the yearning heart of man, but which must appear doubtful in the eyes of philosophy, since it comes to this, that Tom, Dick, and Harry are competent to pry into the counsels of the Most High, and can invite in their cousins and sweethearts and sweethearts' cousins to look over the original designs of the Ancient of Days.

§ 27. My fourth argument it is which seems to have made most impression upon Dr. Carus's mind (¶ 85), and his reply is rather elaborate.

While embracing unequivocally the necessitarian dogma, equally for mind and for matter (¶ 193), Dr. Carus wishes utterly to repudiate materialism and the mechanical philosophy (¶ 133). To facilitate his, thus, walking the slack-rope, he makes (¶ 168) a division of events into "(1) mechanical, (2) physical, (3) chemical, (4) physiological, and (5) psychical events." The first three (¶¶ 169–171) are merely distinguished by the magnitude of the moving masses, so that, for philosophical purposes, they do not differ at all. As for physiological events, though he devotes a paragraph (¶ 172) to their definition, he utterly fails to distinguish them from the mechanical (including the physical and chemical) on the one hand, or from the psychical on the other. Dr. Carus seems to think (¶ 176) that by this division he has separated himself entirely from the materialists; but this is an illusion, for nobody denies the existence of feelings.

The truth is, he distinctly enrolls himself in the mechanical army when he asserts that mental laws are of the same necessitarian character as mechanical laws (¶ 193). The only question that remains as to his position is whether he is a materialist or not. He instances (¶ 185) the case of a general receiving a written dispatch and being stimulated into great activity by its perusal, and causing great motions to be made and missiles to be sped in consequence. Now, the dilemma is this. Will Dr. Carus, on the one hand, say that the motion of those missiles was determined by mechanical laws alone, in which case, it would only be necessary to state all the positions and velocities of particles concerned, a hundred years before, to determine just how those bullets would move and, consequently, whether the guns were to be fired or not, and this would constitute him a materialist, or will he say that the laws of motion do not suffice to determine motions of matter, in which case, since they formally certainly do so suffice, they must be *violated*, and he will be giving to mind a direct dynamical power which is open to every objection that can be urged against tychism?

Now admire the decision with which he cuts the Gordian knot!

"THERE ARE NO PURELY MECHANICAL PHENOMENA." (¶ 175.)

That is,

"*The laws of motion* ARE *applicable to and will explain all motions.*" (¶ 177.)

But hold!

"The mechanical philosopher . . . . feels warranted in the hope that . . . . the actions of man . . . . can be explained by the laws of motion . . . . We may anticipate that this conclusion will prove ERRONEOUS. *And so it is.*" (¶ 176.)

At the same time,

"NO OBJECTION CAN BE MADE *to the possibility of explaining the delicate motions* *in the nervous substance of the brain by the laws of molar or molecular mechanics.*" (¶ 178.)

Yet,

"*The simplest* psychical reflexes, including those *physiological reflexes* which we must suppose to have originated by conscious adaptation . . . . CANNOT *be explained from mechanical* or physical *laws* alone." (¶ 186.)

However,

"*We do* NOT *say that there are motions* . . . . in the brain . . . . *which form exceptions to the laws of mechanics.*" (¶ 187.)

Nevertheless,

"The brain-atoms are possessed of the same spontaneity as the atoms of a gravitating stone. Yet there is present an additional feature; there are present states of awareness . . . . Neither states of awareness nor their meanings can be weighed on any scales, be they ever so delicate, nor are they determinable in foot-pounds." (¶ 192.)

Clearness is the first merit of a philosopher; and what ¶ 192 comes to is crystal-clear. Dr. Carus wants to have the three laws of motion always obeyed; but he wishes the forces between the molecules to be varied according to the momentary states of awareness. All right: he is entitled to suppose whatever he likes, so long as the supposition is self-consistent, as this supposition is. It conflicts with the law of energy, it is true; for that law is that the forces depend on the situations of the particles alone, and not on the time. It is liable to give rise to perpetual motion. It was intended, no doubt, to be an improvement on my molecular theory of protoplasm, earlier in the same number. It escapes materialism. It supposes a direct dynamical action between mind and matter, such as has not been supposed by any eminent philosopher that I know of for centuries. I am sorry to say that it shows a dangerous leaning toward originality. The argument for thus rejecting the law of the conservation of energy, I leave to others to be weighed. It seems to suppose a much larger falsification of that law than my doctrine; but it is a pretty clever attempt to escape my conclusions. It rejects what has to be rejected, the law of the conservation of energy; and is far more intelligent than the theory of those (like Oliver and Lodge) who wish to give to mind a power of deflecting atoms, which would satisfy the conservation of energy while violating the law of action and reaction. If it can have due consideration, I doubt not it will accelerate the acceptance of my views. Meantime, I do not see where that "inextricable confusion" into which I was to be led is to come in. (¶ 4.)

§ 28. Little more requires to be noticed in Dr. Carus's articles. He admits (¶ 2) that indeterminism is the more natural belief, which is no slight argument in its favor.

§ 29. The remarks upon the theological bearings of the theories, if they are found somewhat wide of the mark, are explained by the haste of the editor to show just what all the affiliations of my views were, before I had had time to explain what those views are. The remarks to which I refer will be found in ¶¶ 3, 36, 81, 82, 83, 128, 203, 204. They are worth putting together.

§ 30. The doctrine of symbolism, to which Dr. Carus has recourse, seems to be similar to that of my essay "Some Consequences of Four Incapacities" (*Journal of Speculative Philosophy*, II.) (¶¶ 180, 183, 199.) On this head, I can only approve of his ideas.

§ 31. It is true that I wrote many definitions for one of the "encyclopedic lexicons." But they were necessarily rather vaguely expressed, in order to include the popular use of terms, and in some cases were modified by proof-readers or editors; and for reasons not needful here to explain, they are hardly such as I should give in a Philosophical Dictionary proper.

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THE LOGIC OF RELATIVES.

§ 1. *Three Grades of Clearness.*—The third volume of Professor Schröder's *Exact Logic*,[1] which volume bears separately the title I have chosen for this paper, is exciting some interest even in this country. There are in America a few inquirers into logic, sincere and diligent, who are not of the genus that buries its head in the sand,—men who devote their thoughts to the study with a view to learning something that they do not yet know, and not for the sake of upholding orthodoxy, or any other foregone conclusion. For them this article is written as a kind of popular exposition of the work that is now being done in the field of logic. To them I desire to convey some idea of what the new logic is, how two "algebras," that is, systems of diagrammatical representation by means of letters and other characters, more or less analogous to those of the algebra of arithmetic, have been invented for the study of the logic of relatives, and how Schröder uses one of these (with some aid from the other and from other notations) to solve some interesting problems of reasoning. I also wish to illustrate one other of several important uses to which the new logic may be put. To this end I must first clearly show what a relation is.

Now there are three grades of clearness in our apprehensions of the meanings of words. The first consists in the connexion of

[1] *Algebra und Logik der Relative.* Leipsic: B. G. Teubner. 1895. Price, 16 M.

the word with familiar experience. In that sense, we all have a clear idea of what *reality* is and what *force* is,—even those who talk so glibly of mental force being correlated with the physical forces. The second grade consists in the abstract definition, depending upon an analysis of just what it is that makes the word applicable. An example of defective apprehension in this grade is Professor Tait's holding (in an appendix to the reprint of his Britannica article, *Mechanics*) that energy is "objective" (meaning it is a substance), because it is permanent, or "persistent." For independence of time does not of itself suffice to make a substance; it is also requisite that the aggregant parts should always preserve their identity, which is not the case in the transformations of energy. The third grade of clearness consists in such a representation of the idea that fruitful reasoning can be made to turn upon it, and that it can be applied to the resolution of difficult practical problems.

§ 2. *Of the term Relation in its first Grade of Clearness.*—An essential part of speech, the Preposition, exists for the purpose of expressing relations. Essential it is, in that no language can exist without prepositions, either as separate words placed before or after their objects, as case-declensions, as syntactical arrangements of words, or some equivalent forms. Such words as "brother," "slayer," "at the time," "alongside," "not," "characteristic property" are relational words, or *relatives*, in this sense, that each of them *becomes a general name when another general name is affixed to it as object.* In the Indo-European languages, in Greek, for example, the so-called genitive case (an inapt phrase like most of the terminology of grammar) is, very roughly speaking, the form most proper to the attached name. By such attachments, we get such names as "brother *of* Napoleon," "slayer *of* giants," "*ἐπὶ Ἐλλισσαίου*, at the time of Elias," "*παρὰ ἀλλήλων*, alongside *of* each other," "not guilty," "a characteristic property *of* gallium." *Not* is a relative because it means "other than"; *scarcely*, though a relational word of highly complex meaning, is not a relative. It has, however, to be treated in the logic of relatives. Other relatives do not become general names until two or more names have been thus affixed. Thus, "giver to the city" is just such a relative as the preceding; for "giver to the city of a statue of himself" is a complete general name (that is, there might be several such humble admirers of themselves, though there be but one, as yet); but "giver" requires *two* names to be attached to it, before it becomes a complete name. The dative case is a somewhat usual form for the second object. The archaic instrumental and locative cases were serviceable for third and fourth objects.

Our European languages are peculiar in their marked differentiation of common nouns from verbs. *Proper* nouns must exist in all languages; and so must such "pronouns," or indicative words, as *this*, *that*, *something*, *anything*. But it is probably true that in the great majority of the tongues of men, distinctive common nouns either do not exist or are exceptional formations. In their meaning as they stand in sentences, and in many comparatively widely-studied languages, common nouns are akin to participles, as being mere inflexions of verbs. If a language has a verb meaning "is a man," a noun "man" becomes a superfluity. For all men are mortals is perfectly expressed by "Anything either is-a-man not or is-a-mortal." Some man is a miser is expressed by "Something both is-a-man and is-a-miser." The best treatment of the logic of relatives, as I contend, will dispense altogether with class names and only use such verbs. A verb requiring an object or objects to complete the sense may be called a *complete relative*.

A verb by itself signifies a mere dream, an imagination unattached to any particular occasion. It calls up in the mind an *icon*. A *relative* is just that, an icon, or image, without attachments to experience, without "a local habitation and a name," but with indications of the need of such attachments.

An indexical word, such as a proper noun or demonstrative or selective pronoun, has force to draw the attention of the listener to some hecceity common to the experience of speaker and listener. By a hecceity, I mean, some element of existence which, not merely by the likeness between its different apparitions, but by an inward force of identity, manifesting itself in the continuity of its apparition throughout time and in space, is distinct from everything else, and is thus fit (as it can in no other way be) to receive a proper name or to be indicated as *this* or *that*. Contrast this with the signification of the verb, which is sometimes in my thought, sometimes in yours, and which has no other identity than the agreement between its several manifestations. That is what we call an abstraction or idea. The nominalists say it is a *mere* name. Strike out the "mere," and this opinion is approximately true. The realists say it *is* real. Substitute for "is," *may be*, that is, *is* provided experience and reason shall, as their final upshot, uphold the truth of the particular predicate, and the natural existence of the law it expresses, and this is likewise true. It is certainly a great mistake to look upon an idea, merely because it has not the mode of existence of a hecceity, as a lifeless thing.

The proposition, or sentence, signifies that an eternal fitness, or truth, a permanent conditional force, or law, attaches certain hecceities to certain parts of an idea. Thus, take the idea of "buying by—of—from—in exchange for—." This has four places where hecceities, denoted by indexical words, may be attached. The proposition "A buys B from C at the price D," signifies an eternal, irrefragable, conditional force gradually compelling those attachments in the opinions of inquiring minds.

Whether or not there be in the reality any definite separation between the hecceity-element and the idea-element is a question of metaphysics, not of logic. But it is certain that in the expression of a fact we have a considerable range of choice as to how much we will denote by the indexical and how much signify by iconic words. Thus, we have stated "all men are mortal" in such a form that there is but one index. But we may also state it thus: "Taking anything, either it possesses not humanity or it possesses mortality." Here "humanity" and "mortality" are really proper names, or purely denotative signs, of familiar ideas. Accordingly, as here stated, there are three indices. Mathematical reasoning largely depends on this treatment of ideas as things; for it aids in the iconic representation of the whole fact. Yet for some purposes it is disadvantageous. These truths will find illustration in § 13 below.

Any portion of a proposition expressing ideas but requiring something to be attached to it in order to complete the sense, is in a general way relational. But it is only a *relative* in case the attachment of indexical signs will suffice to make it a proposition, or, at least, a complete general name. Such a word as *exceedingly* or *previously* is relational, but is not a relative, because significant words require to be added to it to make complete sense.

§ 3. *Of Relation in the Second Grade of Clearness.*—Is relation anything more than a connexion between two things? For example, can we not state that A gives B to C without using any other relational phrase than that one thing is connected with another? Let us try. We have the general idea of *giving*. Connected with it are the general ideas of *giver*, *gift*, and "*donee.*" We have also a particular transaction connected with no general idea except through that of giving. We have a first party connected with this transaction and also with the general idea of giver. We have a second party connected with that transaction, and also with the general idea of "donee." We have a subject connected with that transaction and also with the general idea of gift. A is the only hecceity directly connected with the first party; C is the only hecceity directly connected with the second party, B is the only hecceity directly connected with the subject. Does not this long statement amount to this, that A gives B to C?

In order to have a distinct conception of Relation, it is necessary not merely to answer this question but to comprehend the reason of the answer. I shall answer it in the negative. For, in the first place, if relation were nothing but connexion of two things, all things would be connected. For certainly, if we say that A is unconnected with B, that non-connexion is a relation between A and B. Besides, it is evident that any two things whatever make a pair. Everything, then, is equally related to everything else, if mere connexion be all there is in relation. But that which is equally and necessarily true of everything is no positive fact, at all. This would reduce relation, considered as simple connexion between two things, to nothing, unless we take refuge in saying that relation *in general* is indeed nothing, but that *modes* of relation are something. If, however, these different modes of relation are different modes of connexion, relation ceases to be simple bare connexion. Going back, however, to the example of the last paragraph, it will be pointed out that the peculiarity of the mode of connexion of A with the transaction consists in A's being in connexion with an element connected with the transaction, which element is connected with the peculiar general idea of a *giver*. It will, therefore, be said, by those who attempt to defend an affirmative answer to our question, that the peculiarity of a mode of connexion consists in this, that that connexion is indirect and takes place through something which is connected with a peculiar general idea. But I say that is no answer at all; for if all things are equally connected, nothing can be more connected with one idea than with another. This is unanswerable. Still, the affirmative side may modify their position somewhat. They may say, we grant that it is necessary to recognise that relation is something more than connexion; it is *positive* connexion. Granting that all things are connected, still all are not positively connected. The various modes of relationship are, then, explained as above. But to this I reply: you propose to make the peculiarity of the connexion of A with the transaction depend (no matter by what machinery) upon that connexion having a positive connexion with the idea of a giver. But "positive connexion" is not enough; the relation of the general idea is quite peculiar. In order that it may be characterised, it must, on your principles, be made indirect, taking place through something which is itself connected with a general idea. But this last connexion is again more than a mere general positive connexion. The same device must be resorted to, and so on *ad infinitum*. In short, you are guilty of a *circulus in definiendo*. You make the relation of any two things consist in their connexion being connected with a general idea. But that last connexion is, on your own principles, itself a *relation*, and you are thus defining relation by relation; and if for the second occurrence you substitute the definition, you have to repeat the substitution *ad infinitum*.

The affirmative position has consequently again to be modified. But, instead of further tracing possible tergiversations, let us directly establish one or two positive positions. In the first place, I say that every relationship concerns some definite number of correlates. Some relations have such properties that this fact is concealed. Thus, any number of men may be brothers. Still, brotherhood is a relation between pairs. If A, B, and C are all brothers, this is merely the consequence of the three relations, A is brother of B, B is brother of C, C is brother of A. Try to construct a relation which shall exist either between two or between three things such as "—is either a brother or betrayer of—to—." You can only make sense of it by somehow interpreting the dual relation as a triple one. We may express this as saying that every relation has a definite number of blanks to be filled by indices, or otherwise. In the case of the majority of relatives, these blanks are qualitatively different from one another. These qualities are thereby communicated to the connexions.

In a complete proposition there are no blanks. It may be called a *medad*, or *medadic relative*, from *μηδαμός*, none, and -*άδα* the accusative ending of such words as *μόνας, δυάς, τριάς, τετράς*, etc.[1] A non-relative name with a substantive verb, as "—is a man," or "man that is—," or "—'s manhood" has one blank; it is a *monad*, or *monadic relative*. An ordinary relative with an active verb as "—is a lover of—" or "the loving by—of—" has two blanks; it is a *dyad*, or *dyadic relative*. A higher relative similarly treated has a plurality of blanks. It may be called a *polyad*. The rank of a relative among these may be called its *adinity*, that is, the peculiar quality of the number it embodies.

A *relative*, then, may be defined as the equivalent of a word or phrase which, either as it is (when I term it a *complete* relative), or else when the verb "is" is attached to it (and if it wants such attachment, I term it a *nominal* relative), becomes a sentence with some number of proper names left blank. A *relationship*, or *fundamentum relationis*, is a fact relative to a number of objects, consid-

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[1] The Pythagoreans, who seem first to have used these words, probably attached a patronymic signification to the termination. A *triad* was derivative of *three*, etc.

ered apart from those objects, as if, after the statement of the fact, the designations of those objects had been erased. A *relation* is a relationship considered as something that may be said to be true of one of the objects, the others being separated from the relationship yet kept in view. Thus, for each relationship there are as many relations as there are blanks. For example, corresponding to the relationship which consists in one thing loving another there are two relations, that of loving and that of being loved by. There is a nominal relative for each of these relations, as "lover of—," and "loved by—." These nominal relatives belonging to one relationship, are in their relation to one another termed *correlatives*. In the case of a dyad, the two correlatives, and the corresponding relations are said, each to be the *converse* of the other. The objects whose designations fill the blanks of a complete relative are called the *correlates*. The correlate to which a nominal relative is attributed is called the *relate*.

In the statement of a relationship, the designations of the correlates ought to be considered as so many *logical subjects* and the relative itself as the *predicate*. The entire set of logical subjects may also be considered as a *collective subject*, of which the statement of the relationship is *predicate*.

§ 4. *Of Relation in the third Grade of Clearness.*—Mr. A. B. Kempe has published in the *Philosophical Transactions* a profound and masterly "Memoir on the Theory of Mathematical Form," which treats of the representation of relationships by "Graphs," which is Clifford's name for a diagram, consisting of spots and lines, in imitation of the chemical diagrams showing the constitution of compounds. Mr. Kempe seems to consider a relationship to be nothing but a complex of bare connexions of pairs of objects, the opinion refuted in the last section. Accordingly, while I have learned much from the study of his memoir, I am obliged to modify what I have found there so much that it will not be convenient to cite it; because long explanations of the relation of my views to his would become necessary if I did so.

A chemical atom is quite like a relative in having a definite number of loose ends or "unsaturated bonds," corresponding to the blanks of the relative. In a chemical molecule, each loose end of one atom is joined to a loose end, which it is assumed must belong to some other atom, although in the vapor of mercury, in argon, etc., two loose ends of the same atom would seem to be joined; and why pronounce such hermaphrodism impossible? Thus the chemical molecule is a *medad,* like a complete proposition. Regarding proper names and other indices, after an "is" has been attached to them, as monads, they, together with other monads, correspond to the two series of chemical elements, H, Li, Na, K, Rb, Cs, etc., and Fl, Cl, Br, I. The dyadic relatives correspond to the two series, Mg, Ca, Sr, Ba, etc., and O, S, Se, Te, etc. The triadic relatives correspond to the two series B, Al, Zn, In, Tl, etc., and N, P, As, Sb, Bi, etc. Tetradic relatives are, as we shall see, a superfluity; they correspond to the series C, Si, Ti, Sn, Ta, etc. The proposition "John gives John to John" corresponds in

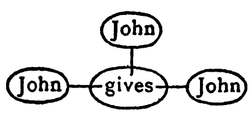

Fig. 1.

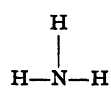

Fig. 2.

its constitution, as Figs. 1 and 2 show, precisely to ammonia.

But beyond this point the analogy ceases to be striking. In fact, the analogy with the ruling theory of chemical compounds quite breaks down. Yet I cannot resist the temptation to pursue it. After all, any analogy, however fanciful, which serves to focus attention upon matters which might otherwise escape observation is valuable. A chemical compound might be expected to be quite as much like a proposition as like an algebraical invariant; and the brooding upon chemical graphs has hatched out an important theory in invariants. Fifty years ago, when I was first studying chemistry, the theory was that every compound consisted of two oppositely electrified atoms or radicles; and in like manner every compound radicle consisted of two opposite atoms or radicles. The argument to this effect was that chemical attraction is evidently between things unlike one another and evidently has a saturation point; and further that we observe that it is the elements the most extremely unlike which attract one another. Lothar Meyer's curve having for its ordinates the atomic volumes of the elements and for its abscissas their atomic weights tends to support the opinion that elements strongly to attract one another must have opposite characters; for we see that it is the elements on the steepest downward slopes of that curve which have the strongest attractions for the elements on the steepest upward inclines. But when chemists became convinced of the doctrine of valency, that is, that every element has a fixed number of loose ends, and when they consequently began to write graphs for compounds, it seems to have been assumed that this necessitated an abandonment of the position that atoms and radicles combine by opposition of characters, which had further been weakened by the refutation of some mistaken arguments in its favor. But if chemistry is of no aid to logic, logic here comes in to enlighten chemistry. For in logic, the medad must always be composed of one part having a negative, or antecedental, character, and another part of a positive, or consequental, character; and if either of these parts is compound its constituents are similarly related to one another. Yet this does not, at all, interfere with the doctrine that each relative has a definite number of blanks or loose ends. We shall find that, in logic, the negative character is a character of reversion in this sense, that if the negative part of a medad is compound, *its* negative part has, on the whole, a positive character. We shall also find, that if the negative part of a medad is compound, the bond joining its positive and negative parts has its character reversed, just as those relatives themselves have.

Several propositions are in this last paragraph stated about logical medads which now must be shown to be true. In the first place, although it be granted that every relative has a definite number of blanks, or loose ends, yet it would seem, at first sight, that there is no need of each of these joining no more than one other. For instance, taking the triad "—kills—to gratify—," why may not the three loose ends all join in one node and then be connected with the loose end of the monad "John is—" as in Fig. 3 making the proposition "John it is that kills what is John to gratify what is John"? The answer is, that a little exercise of generalising power will show that such a four-way node is really a tetradic relative,

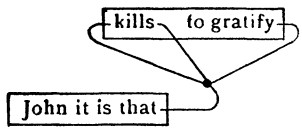

Fig. 3.

which may be expressed in words thus, "—is identical with—and with—and with—"; so that the medad is really equivalent to that

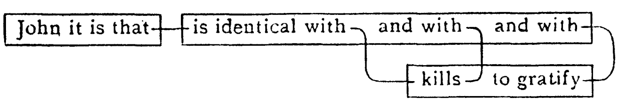

Fig. 4.

of Fig. 4, which corresponds to prussic acid as shown in Fig. 5.

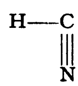

Fig. 5.

Thus, it becomes plain that every node of bonds is equivalent to a relative; and the doctrine of valency is established for us in logic.

We have next to inquire into the proposition that in every combination of relatives there is a negative and a positive constituent. This is a corollary from the general logical doctrine of the illative character of the copula, a doctrine precisely opposed to the opinion of the quantification of the predicate. A satisfactory discussion of this fundamental question would require a whole article. I will only say in outline that it can be positively demonstrated in several ways that a proposition of the form "man = rational animal," is a compound of propositions each of a form which may be stated thus: "*Every* man (if there be any) is a rational animal" or "Men are *exclusively* (if anything) rational animals." Moreover, it must be acknowledged that the illative relation (that expressed by "therefore") is the most important of logical relations, the be-all and the end-all of the rest. It can be demonstrated that formal logic needs no other elementary logical relation than this;

but that with a symbol for this and symbols of relatives, including monads, and with a mode of representing the attachments of them, all syllogistic may be developed, far more perfectly than any advocate of the quantified predicate ever developed it, and in short in a way which leaves nothing to be desired. This in fact *will* be virtually shown in the present paper. It can further be shown that no other copula will of itself suffice for all purposes. Consequently, the copula of equality ought to be regarded as merely derivative.

Now, in studying the logic of relatives we must sedulously avoid the error of regarding it as a highly specialised doctrine. It is, on the contrary, nothing but formal logic generalised to the very tiptop. In accordance with this view, or rather with this theorem (for it is susceptible of positive demonstration), we must regard the *relative copula*, which is the bond between two blanks of relatives, as only a generalisation of the ordinary copula, and thus of the "*ergo.*" When we say that from the proposition A the proposition B necessarily follows, we say that "the truth of A in *every way* in which it can exist at all is the truth of B," or otherwise stated "A is true *only* in so far as B is true." This is the very same relation which we express when we say that "*every* man is mortal," or "men are *exclusively* mortals." For this is the same as to say, "Take anything whatever, M; then, if M is a man, it follows necessarily that M is mortal." This mode of junction is essentially the same as that between the relatives in the compound relative "lover, in *every* way in which it may be a lover at all, of a servant," or, otherwise expressed, "lover (if at all) *exclusively* of servants." For to say that "Tom is a lover (if at all) only of servants of Dick," is the same as to say "Take anything whatever, M; then, if M is loved by Tom, M is a servant of Dick," or "everything there may be that is loved by Tom is a servant of Dick."

Now it is to be observed that the illative relation is not simply convertible; that is to say, that "from A necessarily follows B" does not necessarily imply that "from B necessarily follows A." Among the vagaries of some German logicians of some of the inexact schools, the convertibility of illation (like almost every other imaginable absurdity) has been maintained; but all the other inexact schools deny it, and exact logic condemns it, at once. Consequently, the copula of inclusion, which is but the *ergo* freed from the accident of asserting the truth of its antecedent, is equally inconvertible. For though "men include only mortals," it does not follow that "mortals include *only* men," but, on the contrary, what follows is "mortals include *all* men." Consequently, again, the fundamental *relative copula* is inconvertible. That is, because "Tom loves (if anybody) only a servant (or servants) of Dick," it does not follow that "Dick is served (if at all) only by somebody loved by Tom," but, on the contrary, what follows is "Dick is master of *every* person (there may be) who is loved by Tom." We thus see clearly, first, that, as the fundamental relative copula, we must take that particular mode of junction; secondly, that that mode is at bottom the mode of junction of the *ergo*, and so joins a relative of antecedental character to a relative of consequental character; and, thirdly, that that copula is inconvertible, so that the two kinds of constituents are of opposite characters. There are, no doubt, convertible modes of junction of relatives, as in "lover of a servant;[1] but it will be shown below that these are complex and indirect in their constitution.

[1] Professor Schröder proposes to substitute the word "symmetry" for *convertibility*, and to speak of *simply convertible* modes of junction as "symmetrical.' Such an example of wanton disregard of the admirable traditional terminology of logic, were it widely followed, would result in utter uncertainty as to what any

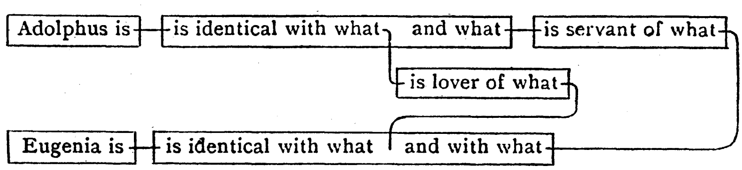

Fig. 6.

writer on logic might mean to say, and would thus be utterly fatal to all our efforts to render logic exact. Professor Schröder denies that the mode of junction in "lover of a servant" is "symmetrical," which word in practice he makes synonymous with "commutative," applying it only to such junctions as that between "lover" and "servant" in "Adolphus is at once lover and servant of Eugenia." Commutativity depends on one or more polyadic relatives having two like blanks as shown in Fig. 6.

It remains to be shown that the antecedent part of a medad has a negative, or reversed, character, and how this, in case it be compound, affects both its relatives and their bonds. But since this matter is best studied in examples, I will first explain how I propose to draw the logical graphs.

It is necessary to use, as the sign of the relative copula, some symbol which shall distinguish the antecedent from the consequent; and since, if the antecedent is compound (owing to the very character which I am about to demonstrate, namely, its reversing the characters of the relatives and the bonds it contains), it is very important to know just how much is included in that antecedent, while it is a matter of comparative indifference how much is included in the consequent (though it is simply everything not in the antecedent), and since further (for the same reason) it is important to know how many antecedents, each after the first a part of another, contain a given relative or copula, I find it best to make the line which joins antecedent and consequent encircle the whole of the former. Letters of the alphabet may be used as abbreviations of complete relatives; and the proper number of bonds may be attached to each. If one of these is encircled, that circle must have a bond corresponding to each bond of the encircled letter. Chemists sometimes write above atoms Roman numerals to indicate their *adinities;* but I do not think this necessary. Fig. 7 shows, in a com-

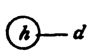

Fig. 7.

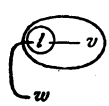

Fig. 8.

plete medad, my sign of the relative copula. Here, *h* is the monad "—is a man," and *d* is the monad "—is mortal." The antecedent is completely enclosed, and the meaning is "Anything whatever, if it be a man, is mortal." If the circle encloses a dyadic or polyadic relative, it must, of course, have a tail for every bond of that relative. Thus, in Fig. 8, *l* is the dyad "—loves—," and it is important to remark that the bond to the left is the lover and that to the right is the loved. Monads are the only relatives for which we need not be attentive to the positions of attachment of the bonds. In this figure, *w* is the monad "—is wise," and *v* is the monad "—is virtuous." The *l* and *v* are enclosed in a large common circle. Had this not been done, the medad could not be read (as far as any rules yet given show), because it would not consist of antecedent and consequent. As it is, we begin the reading of the medad at the bond connecting antecedent and consequent. Every bond of a logical graph denotes a hecceity; and every unencircled bond (as this one is) stands for any hecceity the reader may choose from the universe. This medad evidently refers to the universe of men. Hence the interpretation begins: "Let M be any man you please." We proceed along this bond in the direction of the antecedent, and on entering the circle of the antecedent we say: "If M be." We then enter the inner circle. Now, entering a circle means a relation to *every*. Accordingly we add "whatever." Traversing *l* from left to right, we say "lover." (Had it been from right to left we should have read it "loved.") Leaving the circle is the mark of a relation "only to," which words we add. Coming to *v* we say "what is virtuous." Thus our antecedent reads: "Let M be any man you please. If M be whatever it may that is lover only to the virtuous." We now return to the consequent and read, "M is wise." Thus the whole means, "Whoever loves only the virtuous is wise."

As another example, take the graph of Fig. 9, where *l* has the

Fig. 9.

same meaning as before and *m* is the dyad "—is mother of—." Suppose we start with the left hand bond. We begin with saying "Whatever." Since cutting this bond does not sever the medad, we proceed at once to read the whole as an unconditional statement and we add to our "whatever" "there is." We can now move round the ring of the medad either clockwise or counter-clockwise. Taking the last way, we come to *l* from the left hand and therefore add "is a lover." Moving on, we enter the circle round *m;* and entering a circle is a sign that we must say "of *every thing* that." Since we pass through *m* backwards we do not read "is mother" but "is mothered" or "has for mother." Then, since we pass *out* of the circle we should have to add "only"; but coming back, as we do, to the starting point, we need only say "that same thing." Thus, the interpretation is "Whatever there is, is lover of everything that has for mother that same thing," or "Every woman loves everything of which she is mother." Starting at the same point and going round the other way, the reading would be "Everybody is mother (if at all) only of what is loved by herself." Starting on the right and proceeding clockwise, "Everything is loved by every mother of itself." Proceeding counter-clockwise, "Everything has for mothers only lovers of itself."

Triple relatives afford no particular difficulty. Thus, in Fig. 10, *w* and *v* have the same significations as before; *r* is the monad, "—is a reward," and *g* is the triad "—gives ꞁ to —." It can be read either

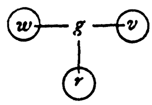

Fig. 10.

"Whatever is wise gives every reward to every virtuous person," or "Every virtuous person has every reward given to him by everybody that is wise," or "Every reward is given by everybody who is wise to every virtuous person."

A few more examples will be instructive. Fig. 11, where A is the proper name Alexander means "Alexander loves only the virtuous," i. e., "Take anybody you please; then, if he be Alexander and if he loves anybody, this latter is virtuous."

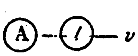

Fig. 11.

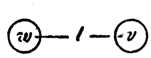

Fig. 12.

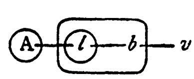

Fig. 13.

If you attempt, in reading this medad, to start to the right of *l*, you fall into difficulty, because your antecedent does not then consist of an antecedent and consequent, but of two circles joined by a bond, a combination to be considered below. But Fig. 12 may be read with equal ease on whichever side of *l* you begin, whether as "whoever is wise loves everybody that is virtuous," or "whoever is virtuous is loved by everybody that is wise." If in Fig. 13

-b- be the dyad "—is a benefactor of—," the medad reads, "Alexander stands only to virtuous persons in the relation of loving only their benefactors."

Fig. 14, where *-s-* is the dyad "—is a servant of —" may be read, according to the above principles, in the several ways following:

"Whoever stands to any person in the relation of lover to none but his servants benefits him."

"Every person stands only to a person benefited by him in the relation of a lover only of a servant of that person."

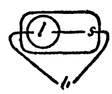

Fig. 14.

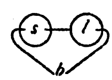

Fig. 15.

"Every person, M, is benefactor of everybody who stands to M in the relation of being served by everybody loved by him."

"Every person, N, is benefited by everybody who stands to N in the relation of loving only servants of him."

"Every person, N, stands only to a benefactor of N in the relation of being served by everybody loved by him."

"Take any two persons, M and N. If, then, N is served by every lover of M, N is benefited by M."

Fig. 15 represents a medad which means, "Every servant of any person, is a benefactor of whomever may be loved by that person." Equivalent statements easily read off from the graphs are as follows:

"Anybody, M, no matter who, is servant (if at all) only of somebody who loves (if at all) only persons benefited by M."

"Anybody, no matter who, stands to every master of him in the relation of benefactor of whatever person may be loved by him."

"Anybody, no matter who, stands to whoever loves him in the relation of being benefited by whatever servant he may have."

"Anybody, N, is loved (if at all) only by a person who is served (if at all) only by benefactors of N."

"Anybody, no matter who, loves (if at all) only persons benefited by all servants of his."

"Anybody, no matter who, is served (if at all) only by benefactors of everybody loved by him."

I will now give an example containing triadic relatives, but no monads. Let *p* be "—prevents—from communicating with—," the second blank being represented by a bond from the right of *p* and the third by a bond from below *p*. Let *β* mean "—would betray—to—," the arrangement of bonds being the same as with *p*. Then, Fig. 16 means that "whoever loves only persons who pre-

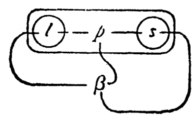

Fig. 16.

vent every servant of any person, A, from communicating with any person, B, would betray B to A." I will only notice one equivalent statement, viz.: "Take any three persons, A, B, C, no matter who. Then, either C betrays B to A, or else two persons, M and N, can be found, such that M does not prevent N from communicating with B, although M is loved by C and N is a servant of A."

This last interpretation is an example of the method which is, by far, the plainest and most unmistakable of any in complicated cases. The rule for producing it is as follows:

1. Assign a letter of the alphabet to denote the hecceity represented by each bond.[1]

2. Begin by saying: "Take any things you please, namely," and name the letters representing bonds not encircled; then add, "Then suitably select objects, namely," and name the letters representing bonds each once encircled; then add, "Then take any things you please, namely," and name the letters representing bonds each twice encircled. Proceed in this way until all the letters

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[1] In my method of graphs, the spots represent the relatives, their bonds the hecceities; while in Mr. Kempe's method, the spots represent the objects, whether individuals or abstract ideas, while their bonds represent the relations. Hence, my own exclusive employment of bonds between pairs of spots does not, in the least, conflict with my argument that in Mr. Kempe's method such bonds are insufficient.

representing bonds have been named, no letter being named until all those encircled fewer times have been named; and each hecceity corresponding to a letter encircled odd times is to be suitably chosen according to the intent of the assertor of the medad proposition, while each hecceity corresponding to a bond encircled even times is to be taken as the interpreter or the opponent of the proposition pleases.

3. Declare that you are about to make statements concerning certain propositions, to which, for the sake of convenience, you will assign numbers in advance of enunciating them or stating their relations to one another. These numbers are to be formed in the following way. There is to be a number for each letter of the medad (that is for those which form spots of the graph, not for the letters assigned by clause 1 of this rule to the bonds), and also a number for each circle round more than one letter; and the first figure of that number is to be a 1 or a 2, according as the letter or the circle is in the principal antecedent or the principal consequent; the second figure is to be 1 or 2, according as the letter or the circle belongs to the antecedent or the consequent of the principal antecedent or consequent, and so on.

Declare that one or other of those propositions whose numbers contain no 1 before the last figure is true. Declare that each of those propositions whose numbers contain an odd number of 1's before the last figure consists in the assertion that *some one* or another of the propositions whose numbers commence with its number is true. For example, 11 consists in the assertion that either 111 or 1121 or 1122 is true, supposing that these are the only propositions whose numbers commence with 11. Declare that each of those propositions whose numbers contain an even number of 1's (or none) before the last figure consists in the assertion that *every one of* the propositions whose numbers commence with its number is true. Thus, 12 consists in the assertion that 121, 1221, 1222 are all true, provided those are the only propositions whose numbers commence with 12. The process described in this clause will be abridged except in excessively complicated cases.

4. Finally, you are to enunciate all those numbered proposi-

tions which correspond to single letters. Namely, each proposition whose number contains an even number of 1's, will consist in affirming the relative of the spot-letter to which that number corresponds after filling each blank with that bond-letter which by clause 1 of this rule was assigned to the bond at that blank. But if the number of the proposition contains an odd number of 1's, the relative, with its blanks filled in the same way, is to be denied.

In order to illustrate this rule, I will restate the meanings of the medads of Figs. 7–16, in all the formality of the rule; although such formality is uncalled for and awkward, except in far more complicated cases.

Fig. 7. Let A be anything you please. There are two propositions, 1 and 2, one of which is true. Proposition 1 is, that A is not a man. Proposition 2 is, that A is mortal. More simply, Whatever A may be, either A is not a man or A is mortal.

Fig. 8. Let A be anybody you please. Then, I will find a person, B, so that either proposition 1 or proposition 2 shall be true. Proposition 1 asserts that both propositions 11 and 12 are true. Proposition 11 is that A loves B. Proposition 12 is that B is not virtuous. Proposition 2 is that A is wise. More simply, Take anybody, A, you please. Then, either A is wise, or else a person, B, can be found such that B is not virtuous and A loves B.

Fig. 9. Let A and B be any persons you please. Then, either proposition 1 or proposition 2 is true. Proposition 1 is that A is not a mother of B. Proposition 2 is that A loves B. More simply, whatever two persons A and B may be, either A is not a mother of B or A loves B.

Fig. 10. Let A, B, C be any three things you please. Then, one of the propositions numbered, 1, 21, 221, 222 is true. Proposition 1 is that A is not wise. Proposition 21 is that B is not a reward. Proposition 221 is that C is not virtuous. Proposition 222 is that A gives B to C. More simply, take any three things, A, B, C, you please. Then, either A is not wise, or B is not a reward, or C is not virtuous, or A gives B to C.

Fig. 11. Take any two persons, A and B, you please. Then, one of the propositions 1, 21, 22 is true. 1 is that A is not Alex-

ander. 21 is that A does not love B. Proposition 3 is that B is virtuous.

Fig. 12. Take any two persons, A and B. Then, one of the propositions 1, 21, 22 is true. 1 is that A is not wise. 21 is that B is not virtuous. 22 is that A loves B.

Fig. 13. Take any two persons, A and C. Then a person, B can be found such that one of the propositions 1, 21, 22 is true. Proposition 21 asserts that both 211 and 212 are true. Proposition 1 that A is not Alexander. Proposition 211 is that A loves B. Proposition 212 is that B does not benefit C. Proposition 22 is that C is virtuous. More simply, taking any two persons, A and C, either A is not Alexander, or C is virtuous, or there is some person, B, who is loved by A without benefiting C.

Fig. 14. Take any two persons, A and B, and I will then select a person C. Either proposition 1 or proposition 2 is true. Proposition 1 is that both 11 and 12 are true. Proposition 11 is that A loves C. Proposition 12 is that C is not a servant of B. Proposition 2 is that A benefits B. More simply, of any two persons, A and B, either A benefits the other, B, or else there is a person, C, who is loved by A but is not a servant of B.

Fig. 15. Take any three persons, A, B, C. Then one of the propositions 1, 21, 22 is true. 1 is that A is not a servant of B; 21 is that B is not a lover of C; 22 is that A benefits C.

Fig. 16. Take any three persons, A, B, C. Then I can so select D and E, that one of the propositions 1 or 2 is true. 1 is that 11 and 121 and 122 are all true. 11 is that A loves D, 121 is that E is a servant of C, 122 is that D does not prevent E from communicating with B. 2 is that A betrays B to C.

I have preferred to give these examples rather than fill my pages with a dry abstract demonstration of the correctness of the rule. If the reader requires such a proof, he can easily construct it. This rule makes evident the reversing effect of the encirclements, not only upon the "quality" of the relatives as affirmative or negative, but also upon the selection of the hecceities as performable by advocate or opponent of the proposition, as well as upon the conjunctions of the propositions as disjunctive or conjunctive, or (to avoid this absurd grammatical terminology) as alternative or simultaneous.

It is a curious example of the degree to which the thoughts of logicians have been tied down to the accidents of the particular language they happened to write (mostly Latin), that while they hold it for an axiom that two *nots* annul one another, it was left for me to say as late as 1867[1] that *some* in formal logic ought to be understood, and could be understood, so that *some-some* should mean *any*. I suppose that were ordinary speech of any authority as to the forms of logic, in the overwhelming majority of human tongues two negatives intensify one another. And it is plain that if "not" be conceived as less than anything, what is less than that is *a fortiori* not. On the other hand, although *some* is conceived in our languages as *more than none*, so that two "somes" intensify one another, yet what it ought to signify for the purposes of syllogistic is that, instead of the selection of the instance being left,—as it is, when we say "any man is not good,"—to the opponent of the proposition, when we say "some man is not good," this selection is transferred to the opponent's opponent, that is to the defender of the proposition. Repeat the some, and the selection goes to the opponent's opponent's opponent, that is, to the opponent again, and it becomes equivalent to *any*. In more formal statement, to say "Every man is mortal," or "Any man is mortal," is to say, "A man, as suitable as any to prove the proposition false, is mortal," while "Some man is mortal" is equivalent to "A man, as suitable as any to prove the proposition *not* false, is mortal." "Some-some man is mortal" is accordingly "A man, as suitable as any to prove the proposition *not not*-false, is mortal."

In like manner, encircled $2N+1$ times, a disjunctive conjunction of propositions becomes a copulative conjunction. Here, the case is altogether similar. Encircled even times, the statement is that some one (or more) of the propositions is true ; encircled odd times, the statement is that any one of the propositions is true.

[1] "On the Natural Classification of Arguments." *Proceedings of the American Academy of Arts and Sciences.*

The negative of "lover of every servant" is "non-lover of some servant." The negative of "lover every way (that it is a lover) of a servant" is "lover some way of a non-servant."

The general nature of a relative and of a medad has now been made clear. At any rate, it will become so, if the reader carefully goes through with the explanations. We have not, however, as yet shown how every kind of proposition can be graphically expressed, nor under what conditions a medad is necessarily true. For that purpose it will be necessary to study certain special logical relatives.

§ 5. *Triads the primitive relatives.*—That out of triads all polyads can be constructed is made plain by Fig. 17.

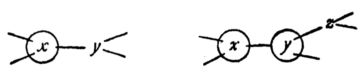

Fig. 17.

Fig. 18 shows that from two triads a dyad can be made. Fig. 19 shows that from one triad a monad can be made. Fig. 20 shows

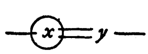

Fig. 18.

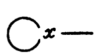

Fig. 19.

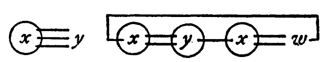

Fig. 20.

that from any even number of triads a medad can be made. In general, the union of a μ-ad and a ν-ad gives a $(\mu + \nu - 2\lambda)$-ad, where λ is the number of bonds of union. This formula shows that *artiads*, or even-ads, can produce only *artiads*. But any perissid, or odd-ad (except a monad), can by repetition produce a relative of any *adinity*.

Since the principal object of a notation for relatives is not to produce a handy *calculus* for the solution of special logical problems, but to help the study of logical principles, the study of logical graphs from that point of view must be postponed to a future occasion. For present purposes that notation is best which carries analysis the furthest, and presents the smallest number of unanalyzed forms. It will be best, then, to use single letters for relatives of some one definite and odd number of blanks. We naturally choose three as the smallest number which will answer the purpose.

We shall, therefore, substitute for such a dyad as "—is lover of—" some such triad as "—is coexistent with | and a lover of—." If, then, we make *-w-* to signify "—is coexistent with | and with —," that which we have hitherto written as in Fig. 12 will be written as in Fig. 21. But having once recognised that such a mode

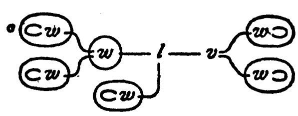

Fig. 21.

of writing is possible, we can continue to use our former methods, provided we now consider them as abbreviations.

The logical doctrine of this section, must, we may remark, find its application in metaphysics, if we are to accept the Kantian principle that metaphysical conceptions mirror those of formal logic.

§ 6. *Relatives of Second Intention.*—The general method of graphical representation of propositions has now been given in all its essential elements, except, of course, that we have not, as yet, studied any truths concerning special relatives; for to do so would seem, at first, to be "extralogical." Logic in this stage of its development may be called *paradisaical logic*, because it represents the state of Man's cognition before the Fall. For although, with this apparatus, it is easy to write propositions necessarily true, it is absolutely impossible to write any which is necessarily false, or, in any way which that stage of logic affords, to find out that anything is false. The mind has not as yet eaten of the fruit of the Tree of Knowledge of Truth and Falsity. Probably it will not be doubted that every child in its mental development necessarily passes through a stage in which he has some ideas, but yet has never recognised that an idea may be erroneous; and a stage that every child necessarily passes through must have been formerly passed through by the race in its adult development. It may be doubted whether many of the lower animals have any clear and steady conception of falsehood; for their instincts work so unerringly that there is little to force it upon their attention. Yet plainly without a knowledge of falsehood no development of discursive reason can take place.

This paradisaical logic appears in the study of non-relative formal logic. But *there* no possible avenue appears by which the knowledge of falsehood could be brought into this Garden of Eden except by the arbitrary and inexplicable introduction of the Serpent in the guise of a proposition necessarily false. The logic of relatives, affords such an avenue, and *that*, the very avenue by which in actual development, this stage of logic supervenes. It is the avenue of experience and logical reflexion.

By *logical* reflexion, I mean the observation of thoughts in their expressions. Aquinas remarked that this sort of reflexion is requisite to furnish us with those ideas which, from lack of contrast, ordinary external experience fails to bring into prominence. He called such ideas *second intentions*. It is by means of *relatives of second intention* that the general method of logical representation is to find completion.

Let ⋈ signify that "—is { neither— / nor— ." Then Fig. 22 means

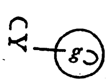

Fig. 22.

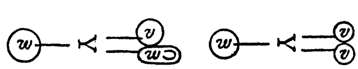

Fig. 23.

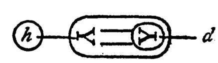

Fig. 24.

that taking any two things whatever, either the one is neither itself nor the other (putting it out of the question as an absurdity), or the other is a non-giver of something to that thing. That is, nothing gives all things, each to itself. Thus, the existence of any gen-

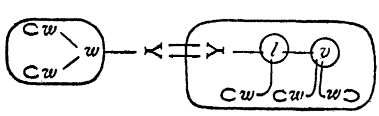

Fig. 25.

eral description of thing can be denied. Either medad of Fig. 23 means no wise men are virtuous. Fig. 24 is equivalent to Fig. 7. Fig. 25 means "each wise man is a lover of something virtuous."

Thus we see that this mode of junction,—lover of some virtuous,—which seems so simple,—is really complex. Fig. 26 means "some

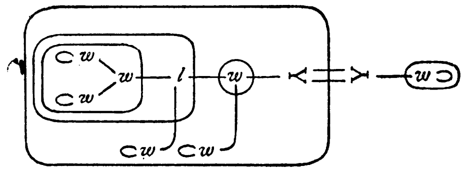

Fig. 26.

one thing is loved by all wise men." Fig. 27 means that every man is either wise or virtuous. Fig. 28 means that every man is both wise and virtuous.

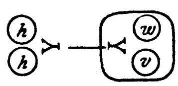

Fig. 27.

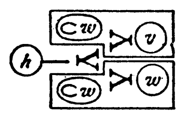

Fig. 28.

These explanations need not be carried further to show that we have here a perfectly efficient and highly analytical method of representing relations.

§ 7. *The Algebra of Dyadic Relatives.*—Although the primitive relatives are triadic, yet they may be represented with but little violence by means of dyadic relatives, provided we allow several attachments to one blank. For instance, A gives B to C, may be represented by saying A is the first party in the transaction D, B is subject of D, C is second party of D, D is a giving by the first party of the subject to the second party. Triadic relatives cannot conveniently be represented on one line of writing. These considerations led me to invent the algebra of dyadic relatives as a tolerably convenient substitute in many cases for the graphical method of representation. In place of the one "operation," or mode of conjunction of graphical method, there are in this algebra four operations.

For the purpose of this algebra, I entirely discard the idea that every compound relative consists of an antecedent and a consequent part. I consider the circle round the antecedent as a mere sign of negation, for which in the algebra I substitute an *obelus* over that antecedent. The line between antecedent and consequent, I treat as a sign of an "operation" by itself. It signifies that anything whatever being taken as correlate of the first written member,—antecedent or consequent,—and as first relate of the second written member, either the one or the other is to be accepted. Thus in place of the relative of Fig. 29 signifying that "taking anything whatever, M, either—is not a lover of M, or M is a benefactor of —," that is "—is a lover only of a benefactor of—," I write

$$\bar{l} \dagger b.$$

Or if it happens to be read the other way, putting a short mark over any letters to signify that relate and correlate are interchanged, I write the same thing

$$\breve{b} \dagger \breve{\bar{l}}.$$

This operation, which may, at need, be denoted by a dagger in print, to which I give a scorpion-tail curve in its cursive form, I call *relative addition.*

The relative "—stands to everything which is a benefactor of — in the relation of servant of every lover of his," shows,

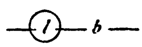

Fig. 29.

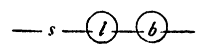

Fig. 30.

as written in Fig. 30, an unencircled bond between s and l. The junction of the l and the b may therefore be regarded as direct. Stating the relative so as to make this direct junction prominent, it is "—is servant of everything that is a lover of a benefactor of—." In the algebra, as far as already explained, "lover of a benefactor" would be written

$$\overline{\bar{l} \dagger \bar{b}}$$

that is, not a non-lover of every benefactor, or not a lover only of non-benefactors. This mode of junction, I call, in the algebra, the operation of *relative multiplication,* and write it

$$l b.$$

We have, then, the purely formal, or meaningless, equation

$$l b = \overline{\bar{l} \dagger \bar{b}}.$$

And in like manner, as a consequence of this,

$$l \dagger b = \overline{\bar{l}\bar{b}}.$$

That is to say, "To say that A is a lover of everything but benefactors of B," or "A is a non-lover only of benefactors of B," is the same as to say that A is not a non-lover of a non-benefactor of B.

To express in the algebra the relative of Fig. 31

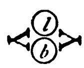

Fig. 31.

or "— is both a lover and a benefactor of —," I write

$$l \cdot b,$$

calling this "the operation of *non-relative multiplication*." To express "— is either a lover or a benefactor of —," which might be written

$$\overline{\bar{l} \cdot \bar{b}},$$

I write

$$l + b,$$

calling this the operation of *non-relative addition*, or more accurately, of *aggregation*. These last two operations belong to the Boolian algebra of non-relative logic. They are De Morgan's operations of composition and aggregation. Boole himself did not use the last, but in place of it an operation more properly termed addition which gives no interpretable result when the aggregants have any common aggregant. Mr. Venn still holds out for Boole's operation, and there are weighty considerations in its favor. In my opinion, the decision between the two operations should depend upon whether the quantified predicate is rejected (when aggregation should be used), or accepted (when Boole's strict addition should be used).

The use of these four operations necessitates continual resort to parentheses, brackets, and braces to show how far the different compound relatives extend. It also becomes desirable to have a "copula of inclusion," or the sign of "is exclusively (if anything)." For this purpose I have since 1870 employed the sign $-\!\!<$ (intended for an improved $\leqq$). It is easily made in the composing room from a dash followed by $<$, and in its cursive form is struck off in two rapid strokes, thus $\prec$. Its meaning is exemplified in the formula

$$w \prec v$$

"anybody who is wise (if any there be) is exclusively found among the virtuous." We also require in this algebra the signs of relatives of second intention

0, "— is inconsistent with —," ∞, "— is coexistent with —,"
T, "— is other than —," 1, "— is identical with."

The algebra has a moderate amount of power in skilful hands; but its great defect is the vast multitude of purely formal propositions which it brings along. The most significant of these are

$$s(l \,\text{ɟ}\, b) \prec s l \,\text{ɟ}\, b$$

and

$$(l \,\text{ɟ}\, b)s \prec l \,\text{ɟ}\, bs.$$

That is, whatever is a servant of something which is a lover of everything but benefactors is a servant-of-a-lover to everything but benefactors, etc.

Professor Schröder attaches, as it seems to me, too high a value to this algebra. That which is in his eyes the greatest recommendation of it is to me scarcely a merit, namely that it enables us to express in the outward guise of an equation propositions whose real meaning is much simpler than that of an equation.

§ 8. *General algebra of logic.*—Besides the algebra just described, I have invented another which seems to me much more valuable. It expresses with the utmost facility everything which can be expressed by a graph, and frequently much more clearly than the unabridged graphs described above. The method of using it in the solution of special problems has also been fully developed by me.

In this algebra every proposition consists of two parts, its quantifiers and its Boolian. The Boolian consists of a number of relatives united by a non-relative multiplication and aggregation. No relative operations are required (though they can be introduced if desired). Each elementary relative is represented by a letter on the line of writing with subjacent indices to denote the hecceities which fill its blanks. An obelus is drawn over such a relative to deny it.

To the left of the Boolian are written the quantifiers. Each of these is a Π or a Σ with one of the indices written subjacent to it, to signify that in the Boolian every object in the universe is to be imaged substituted successively for that index and the non-relative product (if the quantifier is Π) or the aggregate (if the quantifier is Σ) of the results taken. The order of the quantifiers is, of course, material. Thus

$$\Pi_i \Sigma_j l_{ij} = (l_{11} \dotplus l_{12} \dotplus l_{13} \dotplus \text{etc.}) \cdot (l_{21} \dotplus l_{22} \dotplus l_{23} \dotplus \text{etc.}) \cdot \text{etc.}$$

will mean anything loves something. But

$$\Sigma_j \Pi_i l_{ij} = l_{11} \cdot l_{21} \cdot l_{31} \cdot \text{etc.} \dotplus l_{12} \cdot l_{22} \cdot l_{32} \cdot \text{etc.} \dotplus l_{13} \cdot l_{23} \cdot l_{33} \cdot \text{etc.} \dotplus \text{etc.}$$

will mean something is loved by all things.

This algebra, which has but two operations, and those easily manageable, is, in my opinion, the most convenient apparatus for the study of difficult logical problems, although the graphical method is capable of such modification as to render it substantially as convenient on the average. Nor would I refuse to avail myself of the algebra of dyadic relatives in the simpler cases in which it is easily handled.

§ 9. *Method of Calculating with the General Algebra.*—My rules for working this algebra, the fruit of long experience with applying it to a great variety of genuine inquiries, have never been published. Nor can I here do more than state such as the beginner will be likely to require.

A number of premises being given, it is required to know the most important conclusions of a certain description which can be drawn from them. The first step will be to express the premises by means of the general algebra, taking care to use entirely different letters *as indices* in the different premises.

These premises are then to be copulated (or, in Whewell's phrase, colligated), i. e., non-relatively multiplied together, by multiplying their Boolians and writing before the product all the quantifiers. The relative order of the quantifiers of each premise must (in general) be undisturbed; but the relative order of quantifiers of different premises is arbitrary. The student ought to place Σ's as far to the left and Π's as far to the right as possible. Different arrangements of the quantifiers will lead to different conclusions from the premises. It sometimes happens that each of several arrangements leads to a conclusion which could not easily be reached from any other arrangement.

The premises, being so copulated, become one copulated premise. This copulated premise is next to be logically multiplied into itself any number of times, the indices being different in all the different factors. For there will be certain conclusions which I call conclusions of the first order, which can be drawn from the copulated premise without such involution, certain others, which I call inferences of the second order, which can be drawn from its square, etc. But after involution has been carried to a certain point, higher powers will only lead to inferences of subsidiary importance. The student will get a just idea of this matter by considering the rise and decline of interest in the theorems of any mathematical theory, such as geometry or the theory of numbers, as the fundamental hypotheses are applied more and more times in the demonstrations. The number of factors in the copulated premise, which embraces *all* the hypotheses that either theory assumes, is not great. Yet from this premise many thousand conclusions have already been drawn in the case of geometry and hundreds in the case of the theory of numbers. New conclusions are now coming in faster than ever before. From the nature of logic they can never be exhausted. But as time goes on the conclusions become more special and less important. It is true that mathematics, as a whole, does not become more special nor its late discoveries less important, because there is a growth of the hypotheses. Up to a certain degree, the importance of the conclusions increases with their "order." Thus, in geometry, there is nothing worth mention of the first order, and hardly of the second. But there is a great falling off in the importance of conclusions in the theories mentioned long before the fiftieth order has been reached.

This involution having been performed, the next step will be the identification (occasionally the diversification) of certain indices. The rule is, that any index quantified with a Π can be transmitted, throughout the Boolian, into any other index whose quantifier stands to the left of its own, which now becomes useless, since it refers to nothing in the Boolian. For example, in

$$\Sigma_i \Pi_j \, l_{ij}$$

which in the Algebra of Dyadic Relatives would be written $\infty(l \dagger 0)$, we can identify $\dagger$ with i and write

$$\Sigma_i \, l_{ii}$$

which in the other algebra becomes $\infty(l \cdot \mathrm{I})\infty$.

That done, the Boolian is to be manipulated according to any of the methods of non-relative Boolian algebra, and the conclusion is read off.

But it is only in the simplest cases that the above operations suffice. Relatives of second intention will often have to be introduced; and their peculiar properties must be attended to. Those of 0 and ∞ are covered by the rules of non-relative Boolian algebra; but it is not so with I and T. We have, for example, to observe that

$$\Pi_i \, x_i + y_i = \Pi_i \, \Pi_j \, x_i + \mathsf{T}_{ij} + y_i .$$

$$\Sigma_i \, x_i \cdot y_i = \Sigma_i \, \Sigma_j \, x_i \cdot \mathrm{I}_{ij} \cdot y_j .$$

Exceedingly important are the relatives signifying "— is a quality of —" and "— is a relation of — to —." It may be said that mathematical reasoning (which is the only deductive reasoning, if not absolutely, at least eminently) almost entirely turns on the consideration of abstractions as if they were objects. The protest of nominalism against such hypostatisation, although, if it knew how to formulate itself, it would be justified as against much of the empty disputation of the medieval Dunces, yet, as it was and is formulated, is simply a protest against the only kind of thinking that has ever advanced human culture. Nobody will work long with the logic of relatives,—unless he restricts the problems of his studies very much,—without seeing that this is true.

§ 10. *Schröder's Conception of Logical Problems.*.—Of my own labors in the logic of relatives since my last publication in 1884, I intend to give a slight hint in § 13. But I desire to give some idea of a part of the contents of Schröder's last volume. In doing so, I shall adhere to my own notation; for I cannot accept Professor Schröder's proposed innovations. I shall give my reasons in detail for this dissent in the *Bulletin of the American Mathematical Society.* I will here only indicate their general nature. I have no objection whatever to the creation of a new system of signs *ab ovo,* if anybody can propose such a system sufficiently recommending itself. But *that* Professor Schröder does not attempt. He wishes his notation to have the support of existing habits and conventions, while proposing a measure of reform in the present usage. For that he must obtain general consent. Now it seems to me quite certain that no such general agreement can be obtained without the strictest deference to the principle of priority. Without that, new notations can only lead to confusion thrice confounded. The experience of biologists in regard to the nomenclature of their genera and other groups shows that this is so. I believe that their experience shows that the only way to secure uniformity in regard to conventions of this sort, is to accept for each operation and relative the sign definitively recommended by the person who introduced that operation or relative into the Boolian algebra, unless there are the most *substantial* reasons for dissatisfaction with the meaning of the sign. Objections of lesser magnitude may justify slight modifications of signs; as I modify Jevons's ·|· to ✢, by uniting the two dots by a connecting line, and as I so far yield to Schröder's objections to using ∞ for the sign of whatever is, as to resort to the similarly shaped sign of Aries ♈ (especially as a notation of some power is obtained by using all the signs of the Zodiac in the same sense, as I shall show elsewhere). In my opinion, Professor Schröder alleges no sufficient reason for a single one of his innovations; and I further consider them as *positively* objectionable.

The volume consists of thirty-one long sections filling six hundred and fifty pages. I can, therefore, not attempt to do more than to exemplify its contents by specimens of the work selected as particularly interesting. Professor Schröder chiefly occupies himself with what he calls "solution-problems," in which it is required to deduce from a given proposition an *equation* of which one member consists in a certain relative determined in advance, while the other member shall not contain that relative. He rightly remarks that such problems often involve problems of elimination.

While I am not at all disposed to deny that the so-called "solution-problems," consisting in the ascertainment of the general forms of relatives which satisfy given conditions, are often of considerable importance, I cannot admit that the interest of logical study centres in them. I hold that it is usually much more to the purpose to express in the simplest way what a given premise discloses in regard to the constitution of a relative, whether that simplest expression is of the nature of an equation or not. Thus, one of Schröder's problems is, "Given $x \prec a$, required x,"—for instance, knowing that an opossum is a marsupial, give a description of the opossum. The so-called solution is $\Sigma_u = x u \cdot a$, or opossums embrace precisely what is common to marsupials and to some other class. In my judgment $x \prec a$ might with great propriety be called the solution of $\Sigma_u = x u \cdot a$. When the information contained in a proposition is not of the nature of an equation, why should we, by circumlocutions, insist upon expressing it in the form of an equation?

Professor Schröder attaches great importance to the generality of solutions. In my opinion, this is a mistake. It is not merely that he insists that solutions shall be *complete*, as for example when we require *every root* of a numerical equation, but further that they shall all be embraced under one algebraical expression. Upon that he insists and with that he is satisfied. Whether or not the "solution" is such as to exhibit anything of the real constitution of the relative which forms the first member of the equation he does not seem to care; at least, there is no apparent consideration of the question of how such a result can be secured.

Pure mathematics always selects for the subjects of its studies manifolds of perfect homogeneity; and thence it comes that for the problems which first present themselves general solutions are possible, which notwithstanding their generality, guide us at once to all the particular solutions. But even in pure mathematics the class of problems which are capable of solutions at once general and useful is an exceedingly limited one. All others have to be treated by subdivision of cases. That is what meets us everywhere in higher algebra. As for general solutions, they are for the most part trivial,—like the well-known and obvious test for a prime number that the continued product of all lesser numbers increased by 1 shall be divisible by that number. Only in those cases in which a general solution points the way to the particular solutions is it valuable; for it is only the particular solutions which picture to the mind the solution of a problem; and a form of words which fails to produce a definite picture in the mind is meaningless.

Professor Schröder endeavors to give the most general formula of a logical problem. It is in dealing with such very general and fundamental matters that the exact logician is most in danger of violating his own principles of exactitude. To seek a formula for all logical problems is to ask what it is, in general terms, that men inquire. To answer that question, my own logical proceeding would be to note that it asks what the essence of a question, in general, is. Now a question is a rational contrivance or device, and in order to understand any rational contrivance, experience shows that the best way is to begin by considering what circumstances of need prompted the contrivance, and then upon what general principle its action is designed to fill that need. Applying this general experience to the case before us, we remark that every question is prompted by some need,—that is, by some unsatisfactory condition of things, and that the object of asking the question is to fill that need by bringing reason to bear upon it and to do this by a hypnotically suggestive indication of that to which the mind has to apply itself. I do not know that I have ever, before this minute, considered the question what is the most general formulation of a problem in general; for I do not find much virtue in general formulæ. Nor do I think my answer to this question affords any particularly precious suggestion. But its ordinary character makes it all the better an illustration of the manner—or one of the manners—in which an exact logician may attack, off-hand, a suddenly sprung question. A question, I say, is an indication suggestive (in the hypnotic sense) of what has to be thought about in order to satisfy some more or less pressing want. Ideas like those of this statement, and not talk about φx, and "roots," and the like, must, in my opinion, form the staple of a logical analysis and useful description of a problem, in general. I am none the less a mathematical logician for that. If of two students of the theory of numbers one should insist upon considering numbers as expressed in a system of notation like the Arabic (though using now one number as base of the numeration, and now another), while the other student should maintain that all that was foreign to the theory of numbers, which ought not to consider upon what system the numbers with which it deals are expressed, those two students would, to my apprehension, occupy positions analogous to that of Schröder and mine in regard to this matter of the formulation of the problems of logic; and supposing the student who wished to consider the forms of expression of numbers were to accuse the other of being wanting in the spirit of an arithmetician, that charge would be unjust in quite the same way in which it would be unjust to charge me with deficiency in the mathematical spirit on account of my regarding the conceptions of "values," and "roots," and all that as very special ideas, which can only lumber up the field of consciousness with such hindrances as it is the very end and aim of that diagrammatic method of thinking that characterises the mathematician to get rid of.

But different questions are so very unlike that the only way to get much idea of the nature of a problem is to consider the different cases separately. There are in the first place questions about needs and their fulfillment which are not directly affected by the asking of the questions. A very good example is a chess problem. You have only to experiment in the imagination just as you would do on the board if it were permitted to touch the men, and if your experiments are intelligently conducted and are carried far enough, the solution required must be discovered. In other cases, the need to which the question relates is nothing but the intellectual need of having that question answered. It may happen that questions of this kind can likewise be answered by imaginary experimentation; but the more usual case requires real experimentation. The need is of one or other of two kinds. In the one class of cases we experience on several occasions to which our own deliberate action gave a common character, an excitation of one and the same novel idea or sensation, and the need is that a large number of propositions having the same novel consequent but different antecedents, should be replaced by one proposition which brings in the novel element, so that the others shall appear as mere consequences of every day facts with a single novel one. We may express this intellectual need in a brief phrase as the need of synthetising a multitude of subjects. It is the need of *generalisation*. In another class of cases, we find in some new thing, or new situation, a great number of characters, the same as would naturally present themselves as consequences of a hypothetical state of things, and the need is that the large number of novel propositions with one subject or antecedent should be replaced by a single novel proposition, namely that the new thing or new occasion belongs to the hypothetical class, from which all those other novelties shall follow as mere consequences of matters of course. This intellectual need, briefly stated, is the need of synthetising a multitude of predicates. It is the need of *theory*. Every problem, then, is either a problem of consequences, a problem of generalisation, or a problem of theory. This statement illustrates how special solutions are the only ones which directly mean anything or embody any knowledge; and general solutions are only useful when they happen to suggest what the special solutions will be.

Professor Schröder entertains very different ideas upon these matters. The general problem, according to him, is, "Given the proposition $Fx = 0$, required the 'value' of x_0," that is, an expression not containing x which can be equated to x. This 'value' must be the "general root," that is, it must, under one general description, cover every possible object which fulfils a given condition. This, by the way, is the simplest explanation of what Schröder means by a "solution-problem"; it is the problem to find that form of relative which necessarily fulfils a given condition and in which every relative that fulfils that condition can be expressed. Schröder shows that the solution of such a problem can be put into the form $\sum_u [x = fu]$, which means that a suitable logical function (f) of *any* relative, u, no matter what, will satisfy the condition $\mathrm{F}x = 0$; and that nothing which is not equivalent to such a function will satisfy that condition. He further shows, what is very significant, that the solution may be required to satisfy the "adventitious condition" $fx = x$. This fact about the adventitious condition is all that prevents me from rating the value of the whole discussion as far from high.

Professor Schröder next produces what he calls "the rigorous solution" of the general question. This promises something very fine,—the rigorously correct resolution of everything that ever could (but for this knowledge) puzzle the human mind. It is true that it supposes that a particular relative has been found which shall satisfy the condition $\mathrm{F}x = 0$. But that is seldom difficult to find. Either 0, or ∞, or some other trivial solution commonly offers itself. Supposing, then, that a be this particular solution, that is, that $\mathrm{F}a = 0$, the "rigorous solution" is

$$x = fu = a \cdot \infty(\mathrm{F}u)\infty \dagger u \cdot (0 \prec \overline{\mathrm{F}u} \prec 0).$$

That is, it is such a function of u that when u satisfies the condition $\mathrm{F}u = 0$, $fu = u$; but when u does not satisfy this condition $fu = a$. Now $\mathrm{F}a = 0$.

Since Professor Schröder carries his algebraicity so very far, and talks of "roots," "values," "solutions," etc., when, even in my opinion, with my bias towards algebra, such phrases are out of place, let us see how this "rigorous solution" would stand the climate of numerical algebra. What should we say of a man who professed to give rigorous general solutions of algebraic equations of every degree (a problem included, of course, under Professor Schröder's general problem)? Take the equation $x^5 + \mathrm{A}x^4 + \mathrm{B}x^3 + \mathrm{C}x^2 + \mathrm{D}x + \mathrm{E} = 0$. Multiplying by $x - a$ we get

$$x^6 + (\mathrm{A}-a)x^5 + (\mathrm{B}-a\mathrm{A})x^4 + (\mathrm{C}-a\mathrm{B})x^3 + (\mathrm{D}-a\mathrm{C})x^2 + (\mathrm{E}-a\mathrm{D}) x - a\mathrm{E} = 0$$

The roots of this equation are precisely the same as those of the proposed quintic together with the additional root $x = a$. Hence, if we solve the sectic we thereby solve the quintic. Now, our

Schröderian solver would say, "There is a certain function, fu, every value of which, no matter what be the value of the variable, is a root of the sextic. And this function is formed by a direct operation. Namely, for all values of u which satisfy the equation

$$u^6+(A-a)u^5+(B-aA)u^4+(C-aB)u^3+(D-aC)u^2+(E-aD)u-aE=0$$

$fu=u$, while for all other values, $fu=a$.

Then, $x=fu$ is the expression of every root of the sextic and of nothing else. It is safe to say that Professor Schröder would pronounce a pretender to algebraical power who should talk in that fashion to be a proper subject for *surveillance* if not for confinement in an asylum. Yet he would only be applying Professor Schröder's "rigorous solution," neither more nor less. It is true that Schröder considers this solution as somewhat unsatisfactory; but he fails to state any principle according to which it should be so. Nor does he hold it too unsatisfactory to be frequently resorted to in the course of the volume. The *invention* of this solution exhibits in a high degree that very effective ingenuity which the *solution itself* so utterly lacks, owing to its resting on no correct conception of the nature of problems in general and of their solutions and of the meaning of a proposition.

§ 11. *Professor Schröder's Pentagrammatical Notation.*—Professor Schröder's greatest success in the logic of relatives, is due precisely to his having, in regard to certain questions, proceeded by the separation of cases, quite abandoning the glittering generalities of the algebra of dyadic relatives. As his greatest success, I reckon his solutions of "inverse row and column problems" in § 16, resting upon an investigation in § 15 of the relations of various compound relatives which end in 0, ∞, I, and T. The investigations of § 15 might perfectly well have been carried through without any other instrument than the algebra of dyadic relatives. This course would have had certain advantages, such as that of exhibiting the principles on which the formulæ rest. But directness of proof would not have been of the number of those advantages; this is on the contrary decidedly with the notation invented and used by Professor Schröder. This notation may be called *pentagrammatic*, since it denotes a relative by a row of 5 characters. Imagine a list to be made of all the objects in the universe. Second, imagine a switch-board, consisting of a horizontal strip of brass for each object (these strips being fastened on a wall at a little distance one over another according to the order of the objects in the list) together with a vertical strip of brass for each object (these strips being fastened a little forward of the others, and being arranged in the same order), with holes at all the intersections, so that when a brass plug is inserted in any hole, the object corresponding to the horizontal brass strip can act in some way upon the object corresponding to the vertical brass strip. In order then, by means of this switch-board, to get an analogue of any dyadic relative, a lover of —," we insert plugs so that A and B, being any two objects, A can act on B, if and only if A is a lover of B. Now in Professor Schröder's pentagrammatic notation, the first of the five characters denoting any logical function of a primitive relative, *a*, refers to those horizontal strips, all whose holes are plugged in the representation of *a* (or, as we may say for short, "in *a*"), the second refers to those horizontal strips, each of which has in *a* every hole plugged but one. This one, not necessarily the same for all such strips, may be denoted by A. The third character refers to those horizontal strips which in *a* have several holes plugged, and several empty. The full holes (different, it may be, in the different horizontal strips) may be denoted by β. The fourth character refers to those horizontal strips which in *a* have, each of them, but one hole plugged, generally a different hole in each. This one plugged hole may be denoted by Γ. The fifth character will refer to those rows each of which in *a* has all its holes empty. Then, *a* will be denoted by $\infty\,\bar{\mathrm{A}}\,\beta\,\Gamma\,0$; and $\bar{a}$ by $0\,\mathrm{A}\,\bar{\beta}\,\bar{\Gamma}\,\infty$; for in $\bar{a}$, all the holes must be filled that are void in *a*, and *vice versa*. Consequently $\bar{a}\mathsf{T} = 0\,\bar{\mathrm{A}}\,\infty\;\infty\;\infty$. This shall be shown as soon as we have first examined the pentagrammatic symbol for *a*. This symbol divides *a* into four aggregants, viz:

$$a = (a \prec 0) \boldsymbol{+} a \cdot [(a \prec \mathsf{1}) \cdot \bar{a}]\,\mathsf{T} \boldsymbol{+} a \cdot a\mathsf{T} \cdot (\bar{a} \cdot \bar{a}\mathsf{T})\mathsf{T} \boldsymbol{+} a \cdot (\bar{a} \prec \mathsf{1})$$

In order to prove, by the algebra itself that this equation holds, we remark that $a = a \cdot b \boldsymbol{+} a \cdot \bar{b}$, whatever *b* may be. For *b*, substitute

$(a \dagger 0)$. Then, $a \dagger 0 \prec a \dagger \mathrm{T}$; but $a \dagger \mathrm{T} = a$. Hence, $a \cdot b = a \dagger 0$. $a \cdot \bar{b} = a \cdot \bar{a}\,\infty = a \cdot \bar{a}(\mathrm{I} \dotplus \mathrm{T}) = a \cdot (\bar{a}\mathrm{I} \dotplus \bar{a}\mathrm{T})$. But $\bar{a}\mathrm{I} = \bar{a}$, and $a \cdot \bar{a} = 0$. Hence $a \cdot \bar{b} = a \cdot \bar{a}\mathrm{T}$. Thus $a = a \dagger 0 \dotplus a \cdot \bar{a}\mathrm{T}$. Now, in $\bar{a} = \bar{a} \cdot c \dotplus \bar{a} \cdot \bar{c}$, substitute for c, $a \dagger \mathrm{I}$. This gives $\bar{a} = (a \dagger \mathrm{I}) \cdot \bar{a} \dotplus \bar{a}\mathrm{T} \cdot \bar{a}$; and thus, $a = a \dagger 0 \dotplus a \cdot [(a \dagger \mathrm{I}) \cdot \bar{a}]\,\mathrm{T} \dotplus a \cdot (\bar{a}\mathrm{T} \cdot \bar{a})\mathrm{T}$. Finally, $a = a \cdot a\mathrm{T} \dotplus a \cdot (\bar{a} \dagger \mathrm{I})$. But $a \cdot (\bar{a} \dagger \mathrm{I}) = a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (\bar{a}\mathrm{T} \cdot \bar{a})\mathrm{T} \dotplus a \cdot (\bar{a} \dagger \mathrm{I}) \cdot \{[(a \dagger \mathrm{I}) \dotplus a] \dagger \mathrm{I}\}$.

And

$$
\begin{aligned}
a \cdot (\bar{a} \dagger \mathrm{I}) \cdot \{[(a \dagger \mathrm{I}) \dotplus a] \dagger \mathrm{I}\} &= a \cdot \{\bar{a} \cdot [(a \dagger \mathrm{I}) \dotplus a] \dagger \mathrm{I}\} && \text{(by distribution)} \\
&= a \cdot [\bar{a} \cdot (a \dagger \mathrm{I}) \dagger \mathrm{I}] && (\text{since } \bar{a} \cdot a = 0) \\
&= a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (a \dagger \mathrm{I} \dagger \mathrm{I}) && \text{(by distribution)} \\
&= a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (a \dagger 0) && \text{(if more than 2 things exist)} \\
&= a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (a \dagger \mathrm{I} \cdot \mathrm{T}) && (\text{since } 0 = \mathrm{I} \cdot \mathrm{T}) \\
&= a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (a \dagger \mathrm{I}) \cdot (a \dagger \mathrm{T}) && \text{(by distribution)} \\
&= a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (a \dagger \mathrm{I}) && (\text{since } a \dagger \mathrm{T} = a) \\
&= a \cdot (\bar{a} \cdot a \dagger \mathrm{I}) && \text{(by distribution)} \\
&= a \cdot (0 \dagger \mathrm{I}) && (\text{since } \bar{a} \cdot a = 0) \\
&= a \cdot 0 && \text{(if more than 1 object exists)} \\
&= 0.
\end{aligned}
$$

So that $a \cdot (\bar{a} \dagger \mathrm{I}) = a \cdot (\bar{a} \dagger \mathrm{I}) \cdot (\bar{a}\mathrm{T} \cdot \bar{a})\mathrm{T}$ and thus

$$a = a \dagger 0 \dotplus a \cdot [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \dotplus a \cdot a\mathrm{T}\,(\bar{a}\mathrm{T} \cdot \bar{a})\,\mathrm{T} \dotplus a \cdot (\bar{a} \dagger \mathrm{I}).$$

This is the meaning of the symbol $\infty \bar{\mathrm{A}} \beta \Gamma 0$.

We, now, at length, return, as promised to the examination of $\bar{a}\mathrm{T}$. First, $a \dagger 0 \prec \overline{\bar{a}\mathrm{T}} \dagger 0$. For $\overline{\bar{a}\mathrm{T}} = a \dagger \mathrm{I}$ and $a \dagger \mathrm{I} \dagger 0 = a \dagger (\mathrm{I} \dagger 0) = a \dagger 0$. Hence the first character in the pentagrammatic symbol for $\bar{a}\mathrm{T}$ must be 0. Second $a \cdot [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \prec \bar{a}\mathrm{T} \cdot [(\bar{a}\mathrm{T} \dagger \mathrm{I}) \cdot \overline{\bar{a}\mathrm{T}}]\mathrm{T}$. For it is plain that $a \cdot [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \prec [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \prec \bar{a}\mathrm{T}$. Also $\bar{a} \prec \bar{a}\,\infty \prec \bar{a}(\mathrm{T} \dagger \mathrm{I}) \prec \bar{a}\mathrm{T} \dagger \mathrm{I}$. Hence $[(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \prec [(a \dagger \mathrm{I}) \cdot (\bar{a}\mathrm{T} \dagger \mathrm{I})]\mathrm{T}$. But $a \dagger \mathrm{I} = \overline{\bar{a}\mathrm{T}}$. Hence, $a \cdot [(a \dagger \mathrm{I}) \cdot \bar{a}]\mathrm{T} \prec \bar{a}\mathrm{T} \cdot [(\bar{a}\mathrm{T} \dagger \mathrm{I}) \cdot \overline{\bar{a}\mathrm{T}}]\mathrm{T}$. Hence, the second character in the pentagrammatic sign for $\bar{a}\mathrm{T}$, is the same as that of a. Thirdly $a \cdot a\mathrm{T} \cdot (\bar{a}\mathrm{T} \cdot \bar{a})\mathrm{T} \prec \bar{a}\mathrm{T} \dagger 0$. For $\bar{a} \prec \bar{a}\mathrm{I} \prec \bar{a}(\mathrm{T} \dagger \mathrm{I}) \prec \bar{a}\mathrm{T} \dagger \mathrm{I}$. Hence $(\bar{a} \cdot \bar{a}\mathrm{T})\mathrm{T} \prec [(\bar{a}\mathrm{T} \dagger \mathrm{I}) \cdot (\bar{a}\mathrm{T} \dagger \mathrm{T})]\,\mathrm{T} \prec (\bar{a}\mathrm{T} \dagger \mathrm{I} \cdot \mathrm{T})\mathrm{T} \prec (\bar{a}\mathrm{T} \dagger 0)\mathrm{T} \prec \bar{a}\mathrm{T} \dagger 0\mathrm{T} \prec \bar{a}\mathrm{T} \dagger 0$. Consequently, the third character of the pentagrammatic symbol of $\bar{a}\mathrm{T}$ must be ∞.

Fourthly, $a\cdot(\bar{a}\dagger\mathrm{I})\prec\bar{a}\mathrm{T}\dagger 0$. For we have just seen that $\bar{a}\prec\bar{a}\mathrm{T}\dagger\mathrm{I}$. Hence $\bar{a}\dagger\mathrm{I}\prec\bar{a}\mathrm{T}\dagger\mathrm{I}\dagger\mathrm{I}$. But $\mathrm{I}\dagger\mathrm{I}=0$ if there is more than one object in the universe. Hence $\bar{a}\dagger\mathrm{I}\prec\bar{a}\mathrm{T}\dagger 0$. Consequently, the fourth character of the pentagrammatic formula for $\bar{a}\mathrm{T}$ is ∞. Finally, $\bar{a}\dagger 0\prec\bar{a}\mathrm{T}\dagger 0$. For $\bar{a}\dagger 0\prec\bar{a}\dagger 0\dagger 0\prec\bar{a}\dagger\mathrm{I}\cdot\mathrm{T}\dagger 0\prec(\bar{a}\dagger\mathrm{I})\cdot(\bar{a}\dagger\mathrm{T})\dagger 0\prec\bar{a}\dagger\mathrm{I}\dagger 0\prec\bar{a}\mathrm{T}\dagger 0$. Hence the fifth character of the pentagram of $\bar{a}\mathrm{T}$ is ∞. In fine, that pentagram is $0\bar{\mathrm{A}}\infty\infty\infty$. Professor Schröder obtains this result more directly by means of a special calculus of the pentagrammatic notation. In that way, he obtains, in § 15, a vast number of formulæ, which in § 16 are applied in the first place with great success to the solution of such problems as this: Required a form of relation in which everything stands to something but nothing to everything. The author finds instantaneously that every relative signifying such a relation must be reducible to the form $\bar{u}\infty\cdot u + \mathrm{I}\cdot(u\dagger 0 + \bar{u}\dagger 0)$. In fact, the first term of this expression $\bar{u}\infty\cdot u$, for which $\bar{u}\infty\cdot u\infty$ might as well be written, embraces all the relatives in question. For let $\bar{u}$ be any such relative. Then, $u=\bar{u}\infty\cdot u$. The second term is added, curiously enough, merely to *exclude other relations*. For if u is such a relative that something is u to everything or to nothing, then that something would be in the relation $\bar{u}\infty\cdot u$ to nothing. To give it a correlate the second term is added; and since all the relatives are already included, it matters not what that correlate be, so long as the second term does not exclude any of the required relatives which are included under the first term. Let v be any relative of the kind required, then $v\cdot(u\dagger 0 + \bar{u}\dagger 0)$ will answer for the second term. If we had no letter expressing a relation known to be of the required kind, the problem would be impossible. Fortunately, both I and T are of that kind. Of course, the negative of such a relative is itself such a relative; so that

$$(u\dagger 0\,\bar{u}\dagger 0)\cdot(v + u\infty\cdot\bar{u}\infty)$$

would be an equivalent form, equally with

$$(u\dagger 0 + \bar{u}\dagger 0)\cdot v + u\infty\cdot\bar{u}\infty.$$

§ 16 concludes with some examples of eliminations of great apparent complexity. In the first of these we have given $x=$

$(\bar{u} \curlywedge \mathrm{I})\infty \boldsymbol{+} u$; and it is required to eliminate u. We have, however, instantly $u \prec x$

$$(\bar{u} \curlywedge \mathrm{I})\infty \prec x$$

Whence, immediately,

$$(\bar{x} \curlywedge \mathrm{I})\infty \prec x,$$

or

$$\infty \prec (x \cdot x\mathsf{T})\infty .$$

The next example, the most complicated, requires u to be eliminated from the equation

$$x = \bar{u} \curlywedge 0 \boldsymbol{+} (u \curlywedge \mathrm{I})\infty \cdot \bar{u}\mathsf{T} \boldsymbol{+} (u \curlywedge \mathrm{I}) \cdot \bar{u} \boldsymbol{+} (\bar{u} \curlywedge \mathrm{I}) \cdot u \boldsymbol{+} (u\mathsf{T} \cdot \bar{u}\mathsf{T} \curlywedge 0) \cdot \bar{u},$$

He performs the elimination by means of the pentagrammatic notation very easily as follows: Putting $u = \infty \bar{\mathrm{A}} \beta \Gamma 0$

$$\begin{array}{rl}
\bar{u} \curlywedge 0 & = 0\ 0\ 0\ 0\ \infty \\
(u \curlywedge \mathrm{I})\infty \cdot \bar{u}\mathsf{T} & = 0\ \bar{\mathrm{A}}\ 0\ 0\ 0 \\
(u \curlywedge \mathrm{I}) \cdot \bar{u} & = 0\ \mathrm{A}\ 0\ 0\ 0 \\
(\bar{u} \curlywedge \mathrm{I}) \cdot u & = 0\ 0\ 0\ \Gamma\ 0 \\
(u\mathsf{T} \cdot \bar{u}\mathsf{T} \curlywedge 0) \cdot \bar{u} & = 0\ 0\ \bar{\beta}\ 0\ 0 \\
\hline
\text{sum} & \ \ 0\ \infty\ \bar{\beta}\ \Gamma\ \infty
\end{array}$$

Thus, x is of the form ∞ -$\beta \Gamma 0$, which has been found in former problems to imply $x \curlywedge \mathrm{I} \prec x$.

Without the pentagrammatical notation this elimination would prove troublesome, although with that as a guide it could easily be obtained by the algebra alone.

§ 12. *Professor Schröder's Iconic Solution of* $x \prec \varphi x$.

Another valuable result obtained by Professor Schröder is the solutions of the problem

$$x \prec \varphi x.$$

Namely, he shows that

$$x = f^{\infty} u$$

where

$$fu = u \cdot \varphi u$$

[Of course, by contraposition, this gives for the solution of $\varphi x \prec x$ $x = f^{\infty} u$ where $fu = u \boldsymbol{+} \varphi u$.] The correctness of this solution will appear upon a moment's reflexion; and nearly all the useful solutions in the volume are cases under this.

It happens very frequently that the iteration of the functional operation is unnecessary, because it has no effect.

Suppose, for example, that we desire the general form of a "transitive" relative, that is, such a one, x, that

$$x\,x \prec x.$$

In this case, since $1 \prec l \dagger \breve{l}$ whatever l may be, we have

$$x \prec x\,1 \prec x\,(x \dagger \breve{x}) \prec x\,x \dagger \breve{x} \prec x \dagger \breve{x},$$

or

$$x \prec x \dagger \breve{x}$$

If, then,

$$fu = u \cdot (u \dagger \breve{u}),$$

we have

$$x = f^{\infty}\,u.$$

Here,

$$fu \prec u\,;$$

so that

$$f^{\infty}\,u \prec fu.$$

Also,

$$f^2 u = fu \cdot (fu \dagger \breve{\overline{fu}}) = u \cdot (u \dagger \breve{u}) \cdot [u \cdot (u \dagger \breve{u}) \dagger (\breve{u} \boldsymbol{+} u\,\breve{u})]$$
$$= u \cdot (u \dagger \breve{u}) \cdot [uf(1 \boldsymbol{+} u)\,\breve{u}] \cdot [u \dagger \breve{u} \dagger (1 \boldsymbol{+} u)\,\breve{u}].$$

Now

$$fu = u\cdot(u \dagger \breve{u}) = u\cdot(u \dagger \breve{u})\cdot(u \dagger \breve{u})\cdot(u \dagger \breve{u}) = u\cdot(u \dagger \breve{u})\cdot(u \dagger 1\breve{u})\cdot(u \dagger 1\breve{u})$$
$$\prec u\cdot(u \dagger \breve{u})\cdot[u \dagger (1 \boldsymbol{+} u)\breve{u}]\cdot[u \dagger (\breve{u} \dagger u)\breve{u}] \prec$$
$$\prec u\cdot(u \dagger \breve{u})\cdot[u \dagger (1 \boldsymbol{+} u)\breve{u}]\cdot(u \dagger \breve{u} \dagger u\,\breve{u})$$
$$\prec u\cdot(u \dagger \breve{u})\cdot[u \dagger (1 \boldsymbol{+} u)\breve{u}]\cdot[u \dagger \breve{u} \dagger (1 \boldsymbol{+} u)\breve{u}] \prec f^2 u.$$

Thus $fu = f^{\infty} u$; and

$$x = \sum_u u \cdot (u \dagger \breve{u})$$

This is a truly iconic result; that is, it shows us what the constitution of a transitive relative really is. It shows us that transitiveness always depends upon inclusion; for to say that A is $l \dagger \breve{l}$ of B is to say that the things loved by B are included among those loved by A. The factor $u \dagger \breve{u}$ is transitive by itself; for

$$(u \dagger \breve{u})(u \dagger \breve{u}) \prec u \dagger \breve{u}\,u \dagger \breve{u} \prec u \dagger \mathrm{T} \dagger \breve{u} \prec u \dagger \breve{u}.$$

The effect of the other factor, u, of the form for the general transitive is merely in certain cases to exclude universal identity, and thus to extend the class of relatives represented by $u \dagger \breve{u}$ so as to include those of which it is not true that $1 \prec x$. Here we have an instance of restriction having the effect of extension, that is, restriction of special relatives extends the class of relatives represented. This does not take place in all cases, but only where certain relatives can be represented in more than one way.

Indicating, for a moment, the copula by a dash, the typical and fundamental syllogism is

$$A — B \quad B — C$$
$$\therefore A — C.$$

That is to say, the principle of this syllogism enters into every syllogism. But to say that this is a valid syllogism is merely to say that the copula expresses a transitive relation. Hence, when we now find that transitiveness always depends upon inclusion, the initial analysis by which the copula of inclusion was taken as the general one is fully confirmed. For the chief end of formal logic is the representation of the syllogism.

§ 13. *Introduction to the Logic of Quantity.*—The great importance of the idea of quantity in demonstrative reasoning seems to me not yet sufficiently explained. It appears, however, to be connected with the circumstance that the relations of being greater than and of being at least as great as are transitive relations. Still, a satisfactory evolutionary logic of mathematics remains a desideratum. I intend to take up that problem in a future paper. Meantime the development of projective geometry and of geometrical topics has shown that there are at least two large mathematical theories of continuity into which the idea of continuous *quantity*, in the usual sense of that word, does not enter at all. For projective geometry Schubert has developed an algebraical calculus which has a most remarkable affinity to the Boolian algebra of logic. It is, however, imperfect, in that it only gives imaginary points, rays, and planes, without deciding whether they are real or not. This defect cannot be remedied until topology—or, as I prefer to call it, mathematical topics—has been further developed and its logic accurately analysed. To do this ought to be one of the first tasks of exact logicians. But before that can be accomplished, a perfectly satisfactory logical account of the conception of continuity is required. This involves the definition of a certain kind of infinity; and in order to make that quite clear, it is requisite to begin by developing the logical doctrine of infinite multitude. This doctrine still remains, after the works of Cantor, Dedekind, and others, in an inchoate condition. For example, such a question remains unanswered as the following: Is it, or is it not, logically possible for two collections to be so multitudinous that neither can be put into a one-to-one correspondence with a part or the whole of the other? To resolve this problem demands, not a mere *application* of logic, but a further *development* of the conception of logical possibility.

I formerly defined the possible as that which in a given state of information (real or feigned) we do not know not to be true. But this definition to-day seems to me only a twisted phrase which, by means of two negatives, conceals an anacoluthon. We know in advance of experience that certain things are not true, because we see they are impossible. Thus, if a chemist tests the contents of a hundred bottles for fluorine, and finds it present in the majority, and if another chemist tests them for oxygen and finds it in the majority, and if each of them reports his result to me, it will be useless for them to come to me together and say that they know infallibly that fluorine and oxygen cannot be present in the same bottle; for I see that such infallibility is *impossible*. I know it is not true, because I satisfy myself that there is no room for it even in that ideal world of which the real world is but a fragment. I need no sensible experimentation, because ideal experimentation establishes a much broader answer to the question than sensible experimentation could give. It has come about through the agencies of development that man is endowed with intelligence of such a nature that he can by ideal experiments ascertain that in a certain universe of logical possibility certain combinations occur while others do not occur. Of those which occur in the ideal world some do and some do not occur in the real world; but all that occur in the real world occur also in the ideal world. For the real world is the world of sensible experience, and it is a part of the process of sensible experience to locate its facts in the world of ideas. This is what I mean by saying that the sensible world is but a fragment of the ideal world. In respect to the ideal world we are virtually omniscient; that is to say, there is nothing but lack of time, of perseverance, and of activity of mind to prevent our making the requisite experiments to ascertain positively whether a given combination occurs or not. Thus, every proposition about the ideal world can be ascertained to be either true or false. A description of thing which occurs in that world is *possible, in the substantive logical sense.* Very many writers assert that everything is logically possible which involves no contradiction. Let us call that sort of logical possibility, *essential,* or *formal,* logical possibility. It is not the only logical possibility; for in this sense, two propositions contradictory of one another may both be severally possible, although their combination is not possible. But in the *substantive* sense, the contradictory of a possible proposition is impossible, because we are virtually omniscient in regard to the ideal world. For example, there is no contradiction in supposing that only four, or any other number, of independent atoms exist. But it is made clear to us by ideal experimentation, that five atoms are to be found in the ideal world. Whether all five are to be found in the sensible world or not, to say that there are only four in the ideal world is a proposition absolutely to be rejected, notwithstanding its involving no contradiction.

It would be a great mistake to suppose that ideal experimentation can be performed without danger of error; but by the exercise of care and industry this danger may be reduced indefinitely. In sensible experimentation, no care can always avoid error. The results of induction from sensible experimentation are to afford some ratio of frequency with which a given consequence follows given conditions in the existing order of experience. In induction from ideal experimentation, no particular order of experience is forced upon us; and consequently no such numerical ratio is deducible. We are confined to a dichotomy: the result either is that some description of thing occurs or that it does not occur. For example, we cannot say that one number in every three is divisible by three and one in every five is divisible by five. This is, indeed, so if we choose to arrange the numbers in the order of counting; but if we arrange them with reference to their prime factors, just as many are divisible by one prime as by another. I mean, for instance, when they are arranged as follows:

1, 2, 4, 8, etc.	5, 10, 20, 40, etc.	7, 14, 28, 56, etc.	35, 70, etc.
3, 6, 12, 24, etc.	15, 30, 60, 120, etc.	21, 42, 84, 168, etc.	105, 210, etc.
9, 18, 36, 72, etc.	45, 90, 180, 360, etc.	etc.	etc.
27, 54, 108, 16, etc.	135, 270, 540, 1080, etc.		
etc.	etc.		

Thus, dichotomy rules the ideal world. Plato, therefore, for whom that world alone was real, showed that insight into concepts but dimly apprehended that has always characterised philosophers of the first order, in holding dichotomy to be the only truthful mode of division. Lofty moral sense consists in regarding, not indeed *the*, but yet *an*, ideal world as in some sense the only real one; and hence it is that stern moralists are always inclined to dual distinctions.

Ideal experimentation has one or other of two forms of results. It either proves that $\Sigma_i m_i$, a particular proposition true of the ideal world, and going on, finds $\Sigma_j \bar{m}_j$ also true; that is, that m and $\bar{m}$ are both possible, or it succeeds in its induction and shows the universal proposition $\Pi_i m_i$ to be true of the ideal world; that is that $\bar{m}$ is *necessary* and m *impossible.*

Every result of an ideal induction clothes itself, in our modes of thinking, in the dress of a *contradiction.* It is an anacoluthon to say that a proposition is impossible *because* it is selfcontradictory. It rather is thought so as to appear selfcontradictory, because the ideal induction has shown it to be impossible. But the result is that in the absence of any interfering contradiction every particular proposition is possible in the substantive logical sense, and its contradictory universal proposition is impossible. But where contradiction interferes this is reversed.

In former publications I have given the appellation of *universal* or *particular* to a proposition according as its *first* quantifier is Π or Σ. But the study of substantive logical possibility has led me to substitute the appellations *negative* and *affirmative* in this sense, and to call a proposition *universal* or *particular* according as its *last* quantifier is Π or Σ. For letting l be any relative, one or other of the two propositions

$$\Pi_i \Sigma_j l_{ij} \qquad \Sigma_i \Pi_j \bar{l}_{ij}$$

and one or other of the two propositions

$$\Pi_j \Sigma_i \bar{l}_{ij} \qquad \Sigma_j \Pi_i l_{ij}$$

are true, while the other one of each pair is false. Now, in the absence of any peculiar property of the special relative l, the two similar forms $\Sigma_i \Pi_j \bar{l}_{ij}$ and $\Sigma_j \Pi_i l_{ij}$ must be equally possible in the substantive logical sense. But these two propositions cannot both be true. Hence, both must be false in the ideal world, in the absence of any constraining contradiction. Accordingly, these ought to be regarded as universal propositions, and their contradictions, $\Pi_i \Sigma_j l_{ij}$ and $\Pi_j \Sigma_i \bar{l}_{ij}$, as particular propositions.

There are two opposite points of view, each having its logical value, from one of which, of two quantifiers of the same proposition, the preceding is more important than the following, while from the other point of view the reverse is the case. Accordingly, we may say that an affirmative proposition is particular in a secondary way, and that a particular proposition is affirmative in a secondary way.

If an index is not quantified at all, the proposition is, with reference to that index, *singular*. To ascertain whether or not such a proposition is true of the ideal world, it must be shown to depend upon some universal or particular proposition.

If some of the quantifiers refer not to hecceities, having in themselves no general characters except the logical characters of identity, diversity, etc., but refer to *characters*, whether non-relative or relative, these alone are to be considered in determining the "quantity" of an ideal proposition as universal or particular. For anything whatever is true of *some* character, unless that proposition be downright absurd; while nothing is true of *all* characters except what is formally necessary. Consider, for example, a dyadic relation. This is nothing but an aggregation of pairs. Now any two hecceities may in either order form a pair; and any aggregate whatever of such pairs will form *some* dyadic relation. Hence, we may totally disregard the manner in which the hecceities are connected in determining the possibility of a hypothesis about *some* dyadic relation.

Characters have themselves characters, such as importance, obviousness, complexity, and the like. If some of the quantified indices denote such characters of characters, they will, in reference to a purely ideal world be paramount in determining the quantity of the proposition as universal or particular.

All quantitative comparison depends upon a *correspondence*. A correspondence is a relation which every subject[1] of one collection bears to a subject of another collection, to which no other is in the same relation. That is to say, the relative "corresponds to" has

$$\sum_u u\cdot(1 \dagger \bar{u})$$

not merely as its *form*, but as its *definition*. This relative is transitive; for its relative product into itself is

$$[\sum_u u\cdot(1\dagger\bar{u})][\sum_v v\cdot(1\dagger\bar{v})] \prec \sum_u\sum_v uv\cdot(1\dagger\bar{u})(1\dagger\bar{v})$$
$$\prec \sum_u\sum_v uv\cdot(1\dagger\bar{u}\dagger\bar{v}) \prec \sum_u\sum_v uv\cdot(1\dagger\overline{uv}) \prec \sum_w w\cdot(1\dagger\bar{w})$$

But it is to be observed that if the P's, the Q's, and the R's are three collections, it does not follow because every P corresponds to an R, and every Q corresponds to an R that every object of the aggregate collection $P + Q$ corresponds to an R. The *dictum de omni* in external appearance fails here. For P may be $[u\cdot(1\dagger\bar{u})]R$ and Q may be $[v\cdot(1\dagger\bar{v})]R$; but the aggregate of these is not $[(u+v)\cdot(1\dagger\overline{u+v})]R$, which equals $[(u+v)\cdot(1\dagger\bar{u})\cdot(1\dagger\bar{v})]R$. The aggregate of the two first is $\{(u\dagger v)\cdot[v\cdot(1\dagger\bar{v})+1\dagger\bar{u}].[u\cdot(1\dagger\bar{u})+1\dagger\bar{v}]\}R$, which is obviously too broad to be necessarily included under the other expression. Correspondence is, therefore, not a relation between the subjects of one collection and those of another, but between the collections themselves. Let $q_{\alpha i}$ mean that i is a subject of the collection, α, and let $r_{\beta jk}$ mean that j stands in the relation β to k. Then, to say that the collection P corresponds to the collection Q, or, as it is sometimes expressed, that "*for* every

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[1] I prefer to speak of a member of a collection as a *subject* of it rather than as an *object* of it; for in this way I bring to mind the fact that the collection is virtually a quality or class-character.

subject of Q there is a subject of P," is to make the assertion expressed by

$$\Sigma_\beta \Pi_i \Sigma_j \Pi_k \bar{q}_{Pi} \maltese r_{\beta ij} \cdot (1_{ik} \maltese \bar{r}_{\beta kj}) \cdot q_{Qj}.$$

In the algebra of dual relatives this may be written

$$\Sigma_\beta P \prec \bar{q} \dagger [r_\beta \cdot (1 \dagger \bar{r}_\beta)] \breve{q} Q.$$

The transitivity is evident; for

$$\Sigma_\beta \Sigma_\gamma \bar{q} \dagger [r_\beta \cdot (1 \dagger \bar{r}_\beta)] \breve{q} \{ \bar{q} \dagger [r_\gamma \cdot (1 \dagger \bar{r}_\gamma)] \breve{q} \}$$
$$\prec \Sigma_\beta \Sigma_\gamma \bar{q} \dagger [r_\beta \cdot (1 \dagger \bar{r}_\beta)] \{ \breve{q} \bar{q} \dagger [r_\gamma \cdot (1 \dagger \bar{r}_\gamma)] \breve{q} \}$$
$$\prec \Sigma_\beta \Sigma_\gamma \bar{q} \dagger [r_\beta \cdot (1 \dagger \bar{r}_\beta)] \{ \mathrm{T} \dagger [r_\gamma \cdot (1 \dagger \bar{r}_\gamma)] \breve{q} \}$$
$$\prec \Sigma_\beta \Sigma_\gamma \bar{q} \dagger [r_\beta \cdot (1 \dagger \bar{r}_\beta)] [r_\gamma \cdot (1 \dagger \bar{r}_\gamma)] \breve{q}$$
$$\prec \Sigma_\beta \Sigma_\gamma \bar{q} \dagger [r_\beta r_\gamma \cdot (1 \dagger \bar{r}_\beta \dagger \bar{r}_\gamma)] \breve{q}$$
$$\prec \Sigma_\delta \bar{q} \dagger [r_\delta \cdot (1 \dagger \bar{r}_\delta)] \breve{q}.*$$

Not only is the relative of correspondence transitive, but it also possesses what may be called *antithetic transitivity*. Namely, if c be the relative, not only is $cc \prec c$ but also $c \prec c \dagger c$. To demonstrate this very important proposition is, however, far from easy. The quantifiers of the assertion that for every subject of one character there is a subject of another are $\Sigma_\beta \Pi_i \Sigma_j \Pi_k$. Hence, the proposition is particular and will be true in the ideal world, except in case a positive contradiction is involved.

Let us see how such contradiction can arise. The assertion that for every subject of P there is a subject of Q is

$$\Sigma_\beta \Pi_i \Sigma_j \Pi_k \bar{q}_{Pi} \maltese r_{\beta ij} \cdot (1_{ik} \maltese \bar{r}_{\beta ki}) \cdot q_{Qj}.$$

This cannot vanish if the first aggregant term does not vanish, that is, if $\Pi_i q_{Pi}$ or there is no subject of P. It cannot vanish if everything is a subject of Q. For in that case, the last factor of the latter aggregant disappears, and substituting 1 for r_β the second aggregant becomes ∞. The expression cannot vanish if every subject of P is a subject of Q. For when 1 is substituted for r_β, we get

$$\Pi_i \bar{q}_{Pi} \maltese q_{Qi}.$$

If P has but a single individual subject and Q has a subject, for every P there is a Q. For in this case we have only to take for β

*It must be remembered that to a person familiar with the algebra all such series of steps become evident at first glance.

the relation of the subject of P to any one of the subjects of Q. But if P has more than one subject, and Q has but one, the expression above vanishes. For let 1 and 2 be the two subjects of P. Substituting 1 for i, we get

$$\Pi_k \, r_{\beta 1j} \cdot (1_{1k} + \bar{r}_{\beta kj}) \cdot q_{Qj}$$

Substituting 2 for i we get

$$\Pi_k \, r_{\beta 2j} \cdot (1_{2k} + \bar{r}_{\beta kj}) \cdot q_{Qj}$$

Multiplying these

$$\Pi_k \, \Pi_k \, r_{\beta 1j} \cdot r_{\beta 2j} \cdot (1_{1k} + \bar{r}_{\beta kj}) \cdot (1_{2k'} + \bar{r}_{\beta kj}) \cdot q_{Qi}$$

Substituting 2 for k and 1 for k', this gives

$$r_{\beta 1j} \cdot r_{\beta 2j} \cdot \bar{r}_{\beta 2j} \cdot {}_{\beta 1j} \cdot Q_{qi}$$

which involves two contradictions.

It is to be remarked that although if every subject of P is a subject of Q, then for every subject of P there is a subject of Q, yet it does not follow that if the subjects of P are a part only of the subjects of Q, that there is then not a subject of P for every subject of Q. For example, numbering 2, 4, 6, etc., as the 1st, 2nd, 3rd, etc., of the even numbers, there is an even number for every whole number, although the even numbers form but a part of the whole numbers.

It is now requisite, in order to prove that $c \prec c \dagger c$, to draw three propositions from the doctrine of substantive logical possibility. The first is that given any relation, there is a possible relation which differs from the given relation only in excluding any of the pairs we may choose to exclude. Suppose, for instance, that for every subject of P there is a subject of Q, that is that

$$\Sigma_\beta \; \breve{q} \, \mathrm{P} \prec [r_\beta \cdot (1 \dagger \bar{r}_\beta)] \breve{q} \, \mathrm{Q}.$$

The factor $(1 \dagger \bar{r}_\beta)$ here has the effect of allowing each correlate but one relate. Each relate is, however, allowed any number of correlates. If we exclude all but one of these, the one retained being, if possible, a subject of Q, we have a possible relation, β', such that

$$\Sigma_{\beta'} \; \breve{q} \, \mathrm{P} \prec [r_{\beta'} \cdot (1 \dagger \bar{r}_{\beta'}) \cdot (\bar{r}_{\beta'} \dagger 1)] \breve{q} \, \mathrm{Q}.$$

The second proposition of substantive logical possibility is that whatever is true of *some* of a class is true of the whole of *some* class. That is, if we accept a proposition of the form $\Sigma_i \, a_i \cdot b_i$, we can write

$$\Sigma_\gamma \, \Pi_i \, \bar{q}_{\gamma i} + \bar{a}_i + b_i,$$

though this will generally fail positively to assert, in itself, what is implied, that the collection j excludes whatever is a but not b, and includes something in common with a. There are, however, cases in which this implication is easily made plain.

Applying these two principles to the relation of correspondence, we get a new statement of the assertion that for every P there is a Q. Namely, if we write $a_{\alpha i}$ to signify that i is a relate of the relative r_α to some correlate, that is if $a_{\alpha i} = (i \prec r_\alpha \infty)$, if we write $b_{\alpha j}$ to signify that ʒ is a correlate of the relative r_α to some relate, that is if $b_{\alpha j} = (j \prec \breve{r}_\alpha \infty)$, and if we write $p_{c\alpha}$ to signify that r_α is an aggregate of the relative r_c, that is, if $p_{c\alpha} = (r_\alpha \prec r_c)$, then the proposition that for every subject of P there is a subject of Q may be put in the form,

$$\Sigma_c \, \Sigma_\gamma \, \Pi_x \, \Pi_y \, \Sigma_\delta \, \Sigma_\epsilon \, \Pi_\alpha \, \Sigma_i \, \Sigma_j \, \Pi_\beta \, \Pi_u \, \Pi_v$$
$$[\bar{p}_{c\alpha} \boldsymbol{+} a_{\alpha i} \cdot q_{\mathrm{P}i} \cdot b_{\alpha j} \cdot q_{\mathrm{Q}j} \cdot q_{\gamma j} \cdot (\bar{a}_{\alpha u} \boldsymbol{+} 1_{iu}) \cdot (\bar{b}_{\alpha v} \boldsymbol{+} 1_{jv}) \cdot (\bar{p}_{c\beta} \boldsymbol{+} 1_{\alpha\beta} \boldsymbol{+} \bar{a}_{\beta i} \cdot$$
$$b_{\beta j})] \cdot (\bar{q}_{\mathrm{P}x} \boldsymbol{+} a_{\delta x} \cdot p_{c\delta}) \cdot (\bar{q}_{\mathrm{Q}y} \boldsymbol{+} \bar{q}_{\gamma y} \boldsymbol{+} b_{\epsilon y} \cdot p_{c\epsilon}).$$

This states that there is a collection of pairs, c, any single pair of which, α, has for its sole first subject a subject of P, and for its sole second subject a subject of Q which is at the same time a subject of a collection, j, and that no two pairs of the collection, c, have the same first subject or the same second subject, and that every subject of P is a first subject of some pair of this collection, c, and every subject of Q which is at the same time a subject of γ is a second subject of some pair of the same collection, c.

The third proposition of the doctrine of substantive logical possibility of which we have need is that all hecceities are alike in respect to their capacity for entering into possible pairs. Consequently, all the objects of any collection whatever may be severally and distinctly paired with all the objects of a collection which shall either be wholly contained in, or else shall entirely contain, any other collection whatever. Consequently,

$$\Pi_{\mathrm{P}} \, \Pi_{\mathrm{Q}} \, \Sigma_c \, \Sigma_\delta \, \Pi_x \, \Sigma_\delta \, \Pi_y \, \Sigma_\delta \, \Pi_\alpha \, \Sigma_i \, \Sigma_j \, \Pi_n \, \Pi_v \, \Pi_\beta \, \Pi_m \, \Pi_n$$
$$[\bar{p}_{c\alpha} \boldsymbol{+} a_{\alpha i} \cdot q_{\mathrm{P}i} \cdot b_{\alpha j} \cdot q_{\delta j} \cdot (\bar{a}_{\alpha u} \boldsymbol{+} 1_{uu}) \cdot (b_{\alpha v} \boldsymbol{+} 1_{vj}) \cdot (\bar{p}_{c\beta} \boldsymbol{+} 1_{\alpha\beta} \boldsymbol{+} \bar{a}_{\beta i} \cdot$$
$$b_{\beta j})] \cdot (\bar{q}_{px} \boldsymbol{+} a_{\delta x} \cdot p_{c\delta}) \cdot (\bar{q}_{\delta y} \boldsymbol{+} b_{\epsilon y} \cdot p_{c\epsilon}) \cdot (\bar{q}_{\delta m} \boldsymbol{+} q_{\mathrm{Q}m} \boldsymbol{+} \bar{q}_{\mathrm{Q}n} \boldsymbol{+} q_{\delta n}).$$

Although the above three propositions belong to a system of doctrine not universally recognised, yet I believe their truth is unquestionable. Suppose, now, that it is not true that for every subject of P there is a subject of Q. Then, in the last formula, $\Pi_m \bar{q}_{\delta m} \dagger q_{Qm} \prec 0$. This leaves for the last factor $\Pi_n \bar{q}_{Qn} \dagger q_{\delta n}$, and then the formula expresses that for every subject of Q there is a subject of P. In other words, we have demonstrated the important proposition that *two collections cannot be disparate in respect to correspondence*, but that for every subject of the one there must be a subject of the other.

The theorem $c \prec c \mathfrak{J} c$ is now established; for since of any two collections one corresponds to the other, we have $\infty \prec c \dagger \check{c}$ or (non-relatively multiplying by $\check{c}$) $\check{c} \prec c$. Hence, $c \prec 1 c \prec (\check{c} \mathfrak{J} c)$ $c \prec \check{c} \mathfrak{J} c c \prec c \mathfrak{J} c c$; and, by the transitive principle $c c \prec c$, we finally obtain $c \prec c \mathfrak{J} c$.

Thus is established the conception of *multitude*. Namely, if for every subject of P there is a subject of Q, while there is not for every subject of Q a subject of P, the *multitude* of Q is said to be *greater* than that of P. But if for every subject of each collection there is a subject of the other, the *multitudes* of the two collections are said to be *equal* the one to the other. We may create a scale of objects, one for every group of equal collections. Calling these objects *arithms*, the first arithm will belong to 0 considered as a collection, the second to individuals, etc. Calling a collection the counting of which can be completed an *enumerable* collection, the multitude of any enumerable collection equals that of the arithms that precede its arithm. Calling a collection whose multitude equals that of all the arithms of enumerable collection a *denumerable* collection (because its subjects can all be distinguished by ordinal numbers, though the counting of it cannot be completed), the arithms preceding the arithm of denumerable collections form a denumerable collection. More multitudinous collections are greater than the collections of arithms which precede their arithm.

Let there be a denumerable collection, say the cardinal numbers; and let there be two houses. Let there be a collection of children, each of whom wishes to have those numbers placed in some way into those houses, no two children wishing for the same distribution, but every distribution being wished for by some child. Then, as Dr. George Cantor has proved, the collection of children is greater in multitude than the collection of numbers. Let a collection equal in multitude to that collection of children be called an *abnumeral* collection *of the first dignity*. The real numbers (surd and rational) constitute such a collection.

I now ask, suppose that for every way of placing the subjects of one collection in two houses, there is a way of placing the subjects of another collection in two houses, does it follow that for every subject of the former collection there is a subject of the latter? In order to answer this, I first ask whether the multitude of possible ways of placing the subjects of a collection in two houses can equal the multitude of those subjects. If so, let there be such a multitude of children. Then, each having but one wish, they can among them wish for every possible distribution of themselves among two houses. Then, however they may actually be distributed, some child will be perfectly contented. But ask each child which house he wishes himself to be in, and put every child in the house where he does not want to be. Then, no child would be content. Consequently, it is absurd to suppose that any collection can equal in multitude the possible ways of distributing its subjects in two houses.

Accordingly, the multitude of ways of placing a collection of objects abnumeral of the first dignity into two houses is still greater in multitude than that multitude, and may be called abnumeral of the second dignity. There will be a denumerable succession of such dignities. But there cannot be any multitude of an infinite dignity; for if there were, the multitude of ways of distributing it into two houses would be no greater than itself.[1]

[1] Inasmuch as the above theorem is, as I believe, quite opposed to the opinion prevalent among students of Cantor, and they may suspect that some fallacy lurks in the reasoning about wishes, I shall here give a second proof of a part of the theorem, namely that there is an endless succession of infinite multitudes related to one another as above stated, a relation entirely different, by the way, from those of the orders of infinity used in the calculus. I shall not be able to prove by this

We thus not only answer the question proposed, and show that of two unequal multitudes the multitude of ways of distributing the greater is the greater; but we obtain the entire scale of collectional

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second method, as is proved in the text, that there are no higher multitudes, and in particular no maximum multitude.

The ways of distributing a collection into two houses are equal to the possible combinations of members of that collection (including zero); for these combinations are simply the aggregates of individuals put into either one of the houses in the different modes of distribution. Hence, the proposition is that the combinations of whole numbers are more multitudinous than the whole numbers, that the combinations of combinations of whole numbers are still more multitudinous, the combinations of combinations of combinations again more multitudinous, and so on without end.

I assume the previously proved proposition that of any two collections there is one which can be placed in one-to-one correspondence with a part or the whole of the other. This obviously amounts to saying that the members of any collection can be arranged in a linear series such that of any two different members one comes later in the series than the other.

A part may be equal to the whole; as the even numbers are equal in multitude to all the numbers (since every number has a double distinct from the doubles of all other numbers, and that double is an even number). Hence, it does not follow that because one collection can be placed in one-to-one correspondence to a part of another, it is less than that other, that is, that it cannot also, by a rearrangement, be placed in one-to-one correspondence with the whole. This makes an inconvenience in reasoning which can be overcome in a manner I proceed to describe.

Let a collection be arranged in a linear series. Then, let us speak of a *section* of that series, meaning the aggregate of all the members which are later than (or as late as) one *assignable* member and at the same time earlier than (or as early as) a second *assignable* member. Let us call a series *simple* if it cannot be severed into sections each equal in multitude to the whole. A series not simple itself may be conceivably severed into *simple sections*, or it may be so arranged that it cannot be so severed (for example the series of rational fractions arranged in the order of their magnitudes). But suppose two collections to be each ranged in a linear series, and suppose one of them, A, is in one-to-one correspondence with a part of the other B. If now the latter series, B, can be severed into simple sections, in each of which it is possible to find a member at least as early in the series as any member of that section that is in correspondence with a member of the other collection A, and also a member at least as late in the series as any member of that section that is in correspondence with any member of the other collection, and if it is also possible to find a section of the series, B, equal to the whole series, B, in which it is possible to find a member *later* than any member that is in correspondence with any member of the collection, A, then I say that the collection, B, is greater than the collection, A. This is so obvious that I think the demonstration may be omitted.

Now, imagine two infinite collections, the α's and the β's, of which the β's are the more multitudinous. I propose to prove that the possible combinations of β's are more multitudinous than the possible combinations of α's. For let the pairs of conjugate combinations (meaning by conjugate combinations a pair each of which includes every member of the whole collection which the other excludes) of the β's be arranged in a linear series; and those of the α's in another linear series. Let the order of the pairs in each of the two series be subject to the rule that if of two pairs one contains a combination composed of fewer members than either combination of the other pair, it shall precede the latter in the series. Let the order of the pairs in the series of pairs of combinations of β's be further determined by the rule that where the first rule does not decide, one of two pairs shall precede the other whose smaller combination (this rule not applying where one combinations are equal) contains fewer β's which are in correspondence with α's in one fixed correspondence of all the α's with a part of the β's.

In this fixed correspondence each α has its β, while there is an infinitely greater multitude of β's without α's than with. Let the two series of pairs of combinations quantity, which we find to consist of two equal parts (that is two parts whose multitudes of grades are equal), the one finite, the other infinite. Corresponding to the multitude of 0 on the finite scale is the abnumeral of 0 dignity, which is the denumerable, on the infinite scale, etc.

So much of the general logical doctrine of quantity has been here given, in order to illustrate the power of the logic of relatives in enabling us to treat with unerring confidence the most difficult conceptions, before which mathematicians have heretofore shrunk appalled.

I had been desirous of examining Professor Schröder's developments concerning individuals and individual pairs; but owing to the length this paper has already reached, I must remit that to some future occasion.

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NEW YORK.

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be so placed in correspondence that every pair of unequal combinations of α's is placed in correspondence with that pair of combinations of β's of which the smaller contains only the β's corresponding in the fixed correspondence to the smaller combination of α's; and let every pair of equal combinations of α's be put into correspondence with a pair of β's of which the smaller contains only the β's belonging in the fixed correspondence to one of the combinations of α's.

Then it is evident that each series will generally consist of an infinite multitude of simple sections. In none of these will the combinations be more multitudinous than those of the β's. In some, the combinations of α's will be equal to those of the β's; but in an infinitely greater multitude of such simple sections and each of these infinitely more multitudinous, the combinations of β's will be infinitely more multitudinous than those of the α's. Hence it is evident that the combinations of the β's will on the whole be infinitely more multitudinous than those of the α's.

That is if the multitude of finite numbers be a, and $2^a = b$, $2^b = c$, $2^c = d$, etc $a < b < c < d <$ etc. *ad infinitum.*

It may be remarked that the *finite* combinations of finite whole numbers form no larger a multitude than the finite whole numbers themselves. But there are nfinite collections of finite whole numbers; and it is these which are infinitely more numerous than those numbers themselves.

THE REGENERATED LOGIC.

THE appearance of Schroeder's *Exact Logic*[1] has afforded much gratification to all those homely thinkers who deem the common practice of designating propositions as "unquestionable," "undoubtedly true," "beyond dispute," etc., which are known to the writer who so designates them to be doubted, or perhaps even to be disputed, by persons who with good mental capacities have spent ten or more years of earnest endeavor in fitting themselves to judge of matters such as those to which the propositions in question relate, to be no less heinous an act than a trifling with veracity, and who opine that questions of logic ought *not* to be decided upon philosophical principles, but on the contrary, that questions of philosophy ought to be decided upon logical principles, these having been themselves settled upon principles derived from the only science in which there has never been a prolonged dispute relating to the proper objects of that science. Among those homely thinkers the writer of this review is content to be classed.

Why should we be so much gratified by the appearance of a single book? Do we anticipate that this work is to convince the philosophical world? By no means; because we well know that prevalent philosophical opinions are not formed upon the above principles, nor upon any approach to them. A recent little paper by an eminent psychologist concludes with the remark that the ver-

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[1] *Vorlesungen über die Algebra der Logik* (Exakte Logik). Von Dr. Ernst Schröder, Ord. Professor der Mathematik an der technischen Hochschule zu Karlsruhe in Baden. Dritter Band. *Algebra und Logik der Relative*. Leipsic: B. G. Teubner. 1895. Price, 16 M.

dict of a majority of four of a jury, provided the individual members would form their judgments independently, would have greater probability of being true than the unanimous verdict now is. Certainly, this may be assented to ; for the present verdict is not so much an opinion as a resultant of psychical and physical forces. But the remark seemed to me a pretty large concession from a man imbued with the idea of the value of modern opinion about philosophical questions formed according to that scientific method which the Germans and their admirers regard as the method of modern science,—I mean, that method which puts great stress upon co-operation and solidarity of research even in the early stages of a branch of science, when independence of thought is the wholesome attitude, and gregarious thought is really sure to be wrong. For, as regards the verdict of German *university professors*, which, excepting at epochs of transition, has always presented a tolerable approach to unanimity upon the greater part of fundamental questions, it has always been made up as nearly as possible in the same way that the verdict of a jury is made up. Psychical forces, such as the spirit of the age, early inculcations, the spirit of loyal discipline in the general body, and that power by virtue of which one man bears down another in a negotiation, together with such physical forces as those of hunger and cold, are the forces which are mainly operative in bringing these philosophers into line ; and none of these forces have any direct relation to reason. Now, these men write the larger number of those books which are so thorough and solid that every serious inquirer feels that he is obliged to read them ; and his time is so engrossed by their perusal that his mind has not the leisure to digest their ideas and to reject them. Besides, he is somewhat overawed by their learning and thoroughness. This is the way in which certain opinions—or rather a certain verdict—becomes prevalent among philosophical thinkers everywhere ; and reason takes hardly the leading part in the performance. It is true, that from time to time, this prevalent verdict becomes altered, in consequence of its being in too violent opposition with the changed spirit of the age; and the logic of history will usually cause such a change to be an advance toward truth in some respect. But this process is so slow, that it is not to be expected that any rational opinion about logic will become prevalent among philosophers within a generation, at least.

Nevertheless, hereafter, the man who sets up to be a logician without having gone carefully through Schroeder's Logic will be tormented by the burning brand of *false pretender* in his conscience, until he has performed that task; and that task he cannot perform without acquiring habits of exact thinking which shall render the most of the absurdities which have hitherto been scattered over even the best of the German treatises upon logic impossible for him. Some amelioration of future treatises, therefore, though it will leave enough that is absurd, is to be expected; but it is not to be expected that those who form their opinions about logic or philosophy rationally, and therefore not gregariously, will ever comprise the majority even of philosophers. But opinions thus formed, and among such those formed by thoroughly informed and educated minds, are the only ones which need cause the homely thinker any misgiving concerning his own.

It is a remarkable historical fact that there is a branch of science in which there has never been a prolonged dispute concerning the proper objects of that science. It is the mathematics. Mistakes in mathematics occur not infrequently, and not being detected give rise to false doctrine, which may continue a long time. Thus, a mistake in the evaluation of a definite integral by Laplace, in his *Mécanique céleste*, led to an erroneous doctrine about the motion of the moon which remained undetected for nearly half a century. But after the question had once been raised, all dispute was brought to a close within a year. So, several demonstrations in the first book of Euclid, notably that of the 16th proposition, are vitiated by the erroneous assumption that a part is necessarily less than its whole. These remained undetected until after the theory of the non-Euclidean geometry had been completely worked out; but since that time, no mathematician has defended them; nor could any competent mathematician do so, in view of Georg Cantor's, or even of Cauchy's discoveries. Incessant disputations have, indeed, been kept up by a horde of undisciplined minds about quadratures, cyclotomy, the theory of parallels, rotation, attraction, etc. But the disputants are one and all men who cannot discuss any mathematical problem without betraying their want of mathematical power and their gross ignorance of mathematics at every step. Again, there have been prolonged disputes among real mathematicians concerning questions which were not mathematical or which had not been put into mathematical form. Instances of the former class are the old dispute about the measure of force, and that lately active concerning the number of constants of an elastic body; and there have been sundry such disputes about mathematical physics and probabilities. Instances of the latter class are the disputes about the validity of reasonings concerning divergent series, imaginaries, and infinitesimals. But the fact remains that concerning strictly mathematical questions, and among mathematicians who could be considered at all competent, there has never been a single prolonged dispute.

It does not seem worth while to run through the history of science for the sake of the easy demonstration that there is no other extensive branch of knowledge of which the same can be said.

Nor is the reason for this immunity of mathematics far to seek. It arises from the fact that the objects which the mathematician observes and to which his conclusions relate are objects of his mind's own creation. Hence, although his proceeding is not infallible,—which is shown by the comparative frequency with which mistakes are committed and allowed,—yet it is so easy to repeat the inductions upon new instances, which can be created at pleasure, and extreme cases can so readily be found by which to test the accuracy of the processes, that when attention has once been directed to a process of reasoning suspected of being faulty, it is soon put beyond all dispute either as correct or as incorrect.

Hence, we homely thinkers believe that, considering the immense amount of disputation there has always been concerning the doctrines of logic, and especially concerning those which would otherwise be applicable to settle disputes concerning the accuracy of reasonings in metaphysics, the safest way is to appeal for our logical principles to the science of mathematics, where error can only long go unexploded on condition of its not being suspected.

This double assertion, first, that logic ought to draw upon mathematics for control of disputed principles, and second that ontological philosophy ought in like manner to draw upon logic, is a case under a general assertion which was made by Auguste Comte, namely, that the sciences may be arranged in a series with reference to the abstractness of their objects; and that each science draws regulating principles from those superior to it in abstractness, while drawing data for its inductions from the sciences inferior to it in abstractness. So far as the sciences can be arranged in such a scale, these relationships must hold good. For if anything is true of a whole genus of objects, this truth may be adopted as a principle in studying every species of that genus. While whatever is true of a species will form a datum for the discovery of the wider truth which holds of the whole genus. Substantially the following scheme of the sciences is given in the *Century Dictionary*:

MATHEMATICS

Philosophy { Logic, Metaphysics.

Science of Time	Geometry
Nomological Psychics	Nomological Physics { Molar, Molecular, Ethereal
Classificatory Psychics	Classificatory Physics { Chemistry, Biology, or the chemistry of protoplasms
Descriptive Psychics	Descriptive Physics

PRACTICAL SCIENCE.

Perhaps each psychical branch ought to be placed above the corresponding physical branch. However, only the first three branches concern us here.

Mathematics is the most abstract of all the sciences. For it makes no external observations, nor asserts anything as a real fact. When the mathematician deals with facts, they become for him mere "hypotheses"; for with their truth he refuses to concern himself. The whole science of mathematics is a science of hypotheses; so that nothing could be more completely abstracted from concrete reality. Philosophy is not quite so abstract. For though it makes no *special* observations, as every other positive science does, yet it does deal with reality. It confines itself, however, to the universal phenomena of experience; and these are, generally speaking, sufficiently revealed in the ordinary observations of every-day life. I would even grant that philosophy, in the strictest sense, confines itself to such observations as *must* be open to every intelligence which can learn from experience. Here and there, however, metaphysics avails itself of one of the grander generalisations of physics, or more often of psychics, not as a governing principle, but as a mere datum for a still more sweeping generalisation. But logic is much more abstract even than metaphysics. For it does not concern itself with any facts not implied in the supposition of an unlimited applicability of language.

Mathematics is not a positive science; for the mathematician holds himself free to say that A is B or that A is not B, the only obligation upon him being, that as long as he says A is B, he is to hold to it, consistently. But logic begins to be a positive science; since there are some things in regard to which the logician is not free to suppose that they are or are not; but acknowledges a compulsion upon him to assert the one and deny the other. Thus, the logician is forced by positive observation to admit that there is such a thing as doubt, that some propositions are false, etc. But with this compulsion comes a corresponding responsibility upon him not to admit anything which he is not forced to admit.

Logic may be defined as the science of the laws of the stable establishment of beliefs. Then, *exact* logic will be that doctrine of the conditions of establishment of stable belief which rests upon perfectly undoubted observations and upon mathematical, that is, upon *diagrammatical*, or, *iconic*, thought. We, who are sectaries of "exact" logic, and of "exact" philosophy, in general, maintain that those who follow such methods will, so far as they follow them, escape all error except such as will be speedily corrected after it is once suspected. For example, the opinions of Professor Schröder and of the present writer diverge as much as those of two "exact" logicians well can; and yet, I think, either of us would acknowledge that, however serious he may hold the errors of the other to be, those errors are, in the first place, trifling in comparison with the original and definite advance which their author has, by the "exact" method, been able to make in logic, that in the second place, they are trifling as compared with the errors, obscurities, and negative faults of any of those who do not follow that method, and in the third place, that they are chiefly, if not wholly, due to their author not having found a way to the application of diagrammatical thought to the particular department of logic in which they occur.

"Exact" logic, in its widest sense, will (as I apprehend) consist of three parts. For it will be necessary, first of all, to study those properties of beliefs which belong to them as beliefs, irrespective of their stability. This will amount to what Duns Scotus called *speculative grammar*. For it must analyse an assertion into its essential elements, independently of the structure of the language in which it may happen to be expressed. It will also divide assertions into categories according to their essential differences. The second part will consider to what conditions an assertion must conform in order that it may correspond to the "reality," that is, in order that the belief it expresses may be stable. This is what is more particularly understood by the word *logic*. It must consider, first, *necessary*, and second, *probable* reasoning. Thirdly, the general doctrine must embrace the study of those general conditions under which a problem presents itself for solution and those under which one question leads on to another. As this completes a triad of studies, or trivium, we might, not inappropriately, term the last study *Speculative rhetoric*. This division was proposed in 1867 by me, but I have often designated this third part as *objective logic*.

Dr. Schröder's Logic is not intended to cover all this ground. It is not, indeed, as yet complete; and over five hundred pages may be expected yet to appear. But of the seventeen hundred and sixty-six pages which are now before the public, only an introduction of one hundred and twenty-five pages rapidly examines the speculative grammar, while all the rest, together with all that is promised, is restricted to the deductive branch of logic proper. By the phrase "exact logic" upon his title-page, he means logic treated algebraically. Although such treatment is an aid to exact logic, as defined on the last page, it is certainly not synonymous with it. The principal utility of the algebraic treatment is stated by him with admirable terseness: it is "to set this discipline free from the fetters in which language, by force of custom, has bound the human mind." Upon the algebra may, however, be based a calculus, by the aid of which we may in certain difficult problems facilitate the drawing of accurate conclusions. A number of such applications have already been made; and mathematics has thus been enriched with new theorems. But the applications are not so frequent as to make the elaboration of a facile calculus one of the most pressing desiderata of the study. Professor Schröder has done a great deal in this direction; and of course his results are most welcome, even if they be not precisely what we should most have preferred to gain.

The introduction, which relates to first principles, while containing many excellent observations, is somewhat fragmentary and wanting in a unifying idea; and it makes logic too much a matter of feeling. It cannot be said to belong to exact logic in any sense. Thus, under β (Vol. I., p. 2) the reader is told that the sciences have to suppose, not only that their objects really exist, but also that they are knowable and that for every question there is a true answer and but one. But, in the first place, it seems more exact to say that in the discussion of one question nothing at all concerning a wholly unrelated question can be implied. And, in the second place, as to an inquiry presupposing that there is some one truth, what can this possibly mean except it be that there is one destined upshot to inquiry with reference to the question in hand,—one result, which when reached will never be overthrown? Undoubtedly, we hope that this, *or something approximating to this*, is so, or we should not trouble ourselves to make the inquiry. But we do not necessarily have much confidence that it *is* so. Still less need we think it is so about the *majority* of the questions with which we concern ourselves. But in so exaggerating the presupposition, both in regard to its universality, its precision, and the amount of belief there need be in it, Schröder merely falls into an error common to almost all philosophers about all sorts of "presuppositions." Schröder (under ε, p. 5) undertakes to define a contradiction in terms without having first made an ultimate analysis of the proposition. The result is a definition of the usual peripatetic type; that is, it affords no analysis of the conception whatever. It amounts to making the contradiction in terms an ultimate unanalysable relation between two propositions,—a sort of blind reaction between them. He goes on (under δ, p. 9) to define, after Sigwart, logical consequentiality, as *a compulsion of thought*. Of course, he at once endeavors to avoid the dangerous consequences of this theory, by various qualifications. But all that is to no purpose. Exact logic will say that *C*'s following logically from *A* is a state of *things* which no impotence of thought can alone bring about, unless there is also an impotence of existence for *A* to be a fact without *C* being a fact. Indeed, as long as this latter impotence exists and can be ascertained, it makes little or no odds whether the former impotence exists or not. And the last anchor-hold of logic he makes (under ι) to lie in the correctness of a feeling! If the reader asks *why* so subjective a view of logic is adopted, the answer seems to be (under β, p. 2), that in this way Sigwart escapes the necessity of founding logic upon the theory of cognition. By the theory of cognition is usually meant an explanation of the possibility of knowledge drawn from principles of psychology. Now, the only sound psychology being a special science, which ought itself to be based upon a well-grounded logic, it is indeed a vicious circle to make logic rest upon a theory of cognition so understood. But there is a much more general doctrine to which the name theory of cognition might be applied. Namely, it is that speculative grammar, or analysis of the nature of assertion, which rests upon observations, indeed, but upon observations of the rudest kind, open to the eye of every attentive person who is familiar with the use of language, and which, we may be sure, no rational being, able to converse at all with his fellows, and so to express a doubt of anything, will ever have any doubt. Now, proof does not consist in giving superfluous and superpossible certainty to that which nobody ever did or ever will doubt, but in removing doubts which do, or at least might at some time, arise. A man first comes to the study of logic with an immense multitude of opinions upon a vast variety of topics; and they are held with a degree of confidence, upon which, after he has studied logic, he comes to look back with no little amusement. There remains, however, a small minority of opinions that logic never shakes; and among these are certain observations about assertions. The student would never have had a desire to learn logic if he had not paid some little attention to assertion, so as at least to attach a definite signification to assertion. So that, if he has not thought more accurately about assertions, he must at least be conscious, in some out-of-focus fashion, of certain properties of assertion. When he comes to the study, if he has a good teacher, these already dimly recognised facts will be placed before him in accurate formulation, and will be accepted as soon as he can clearly apprehend their statements.

Let us see what some of these are. When an assertion is made, there really is some speaker, writer, or other sign-maker who delivers it; and he supposes there is, or will be, some hearer, reader, or other interpreter who will receive it. It may be a stranger upon a different planet, an æon later; or it may be that very same man as he will be a second after. In any case, the deliverer makes signals to the receiver. Some of these signs (or at least one of them) are supposed to excite in the mind of the receiver familiar images, pictures, or, we might almost say, *dreams*,—that is, reminiscences of sights, sounds, feelings, tastes, smells, or other sensations, now quite detached from the original circumstances of their first occurrence, so that they are free to be attached to new occasions. The deliverer is able to call up these images at will (with more or less effort) in his own mind; and he supposes the receiver can do the same. For instance, tramps have the habit of carrying bits of chalk and making marks on the fences to indicate the habits of the people that live there for the benefit of other tramps who may come on later. If in this way a tramp leaves an assertion that the people are stingy, he supposes the reader of the signal will have met stingy people before, and will be able to call up an image of such a person attachable to a person whose acquaintance he has not yet made. Not only is the outward significant word or mark a sign, but the image which it is expected to excite in the mind of the receiver will likewise be a sign,—a sign by resemblance, or, as we say, an *icon*,—of the similar image in the mind of the deliverer, and through that also a sign of the real quality of the thing. This icon is called the *predicate* of the assertion. But instead of a single *icon*, or sign by resemblance of a familiar image or "dream," evocable at will, there may be a complexus of such icons, forming a composite image of which the whole is not familiar. But though the whole is not familiar, yet not only are the parts familiar images, but there will also be a familiar image of its mode of composition. In fact, two types of complication will be sufficient. For example, one may be conjunctive and the other disjunctive combination. Conjunctive combination is when two images are both to be used at once; and disjunctive when one or other is to be used. (This is not the most scientific selection of types; but it will answer the present purpose.) The sort of idea which an icon embodies, if it be such that it can convey any positive information, being applicable to some things but not to others, is called a *first intention*. The idea embodied by an icon which cannot of itself convey any information, being applicable to everything or to nothing, but which may, nevertheless, be useful in modifying other icons, is called a *second intention*.

The assertion which the deliverer seeks to convey to the mind of the receiver relates to some object or objects which have forced themselves upon his attention; and he will miss his mark altogether unless he can succeed in forcing those very same objects upon the attention of the receiver. No icon can accomplish this, because an icon does not relate to any particular thing; nor does its idea strenuously force itself upon the mind, but often requires an effort to call it up. Some such sign as the word *this*, or *that*, or *hullo*, or *hi*, which awakens and directs attention must be employed. A sign which denotes a thing by forcing it upon the attention is called an *index*. An index does not describe the qualities of its object. An object, in so far as it is denoted by an index, having *thisness*, and distinguishing itself from other things by its continuous identity and forcefulness, but not by any distinguishing characters, may be called a *hecceity*. A hecceity in its relation to the assertion is a *subject* thereof. An assertion may have a multitude of subjects; but to that we shall return presently.

Neither the predicate, nor the subjects, nor both together, can make an *assertion*. The assertion represents a compulsion which experience, meaning the course of life, brings upon the deliverer to attach the predicate to the subjects as a sign of them taken in a particular way. This compulsion strikes him at a certain instant; and he remains under it forever after. It is, therefore, different from the temporary force which the hecceities exert upon his attention. This new compulsion may pass out of mind for the time being; but it continues just the same, and will act whenever the occasion arises, that is, whenever those particular hecceities and that first intention are called to mind together. It is, therefore, a permanent conditional force, or *law*. The deliverer thus requires a kind of sign which shall signify a law that to objects of indices an icon appertains as sign of them in a given way. Such a sign has been called a *symbol*. It is the *copula* of the assertion.

Returning to the subjects, it is to be remarked that the assertion may contain the suggestion, or request, that the receiver *do* something with them. For instance, it may be that he is first to take any one, no matter what, and apply it in a certain way to the icon, that he is then to take another, perhaps this time a suitably chosen one, and apply that to the icon, etc. For example, suppose the assertion is: "Some woman is adored by all catholics." The constituent icons are, in the probable understanding of this assertion, three, that of a woman, that of a person, *A*, adoring another, *B*, and that of a non-catholic. We combine the two last disjunctively, identifying the non-catholic with *A*; and then we combine this compound with the first icon conjunctively, identifying the woman with *B*. The result is the icon expressed by, "*B* is a woman, and moreover, either *A* adores *B* or else *A* is a non-catholic." The subjects are all the things in the real world past and present. From these the receiver of the assertion is suitably to choose one to occupy the place of *B*; and then it matters not what one he takes for *A*. A suitably chosen object is a woman, and any object, no matter what, adores her, unless that object be a non-catholic.

This is forced upon the deliverer by experience; and it is by no idiosyncrasy of his; so that it will be forced equally upon the receiver.

Such is the meaning of one typical assertion. An assertion of *logical necessity* is simply one in which the subjects are the objects of any collection, no matter what. The consequence is, that the icon, which can be called up at will, need only to be called up, and the receiver need only ascertain by experiment whether he can distribute any set of indices in the assigned way so as to make the assertion false, in order to put the truth of the assertion to the test. For example, suppose the assertion of logical necessity is the assertion that from the proposition, "Some woman is adored by all catholics," it logically follows that "Every catholic adores some woman." That is as much as to say that, for every imaginable set of subjects, either it is false that some woman is adored by all catholics or it is true that every catholic adores some woman. We try the experiment. In order to avoid making it false that some woman is adored by all catholics, we must choose our set of indices so that there shall be one of them, B, such that, taking any one, A, no matter what, B is a woman, and moreover either A adores B or else A is a non-catholic. But that being the case, no matter what index, A, we may take, either A is a non-catholic or else an index can be found, namely, B, such that B is a woman, and A adores B. We see, then, by this experiment, that it is impossible so to take the set of indices that the proposition of consecution shall be false. The experiment may, it is true, have involved some blunder; but it is so easy to repeat it indefinitely, that we readily acquire any desired degree of certitude for the result.

It will be observed that this explanation of logical certitude depends upon the fact of speculative grammar that the predicate of a proposition, being essentially of an ideal nature, can be called into the only kind of existence of which it is capable, at will.

A not unimportant dispute has raged for many years as to whether hypothetical propositions (by which, according to the traditional terminology, I mean any compound propositions, and not merely those *conditional* propositions to which, since Kant, the term has often been restricted) and categorical propositions are one in essence. Roughly speaking, English logicians maintain the affirmative, Germans the negative. Professor Schröder is in the camp of the latter, I in that of the former.

I have maintained since 1867 that there is but one primary and fundamental logical relation, that of illation, expressed by *ergo*. A proposition, for me, is but an argumentation divested of the assertoriness of its premise and conclusion. This makes every proposition a conditional proposition at bottom. In like manner a "term," or class-name, is for me nothing but a proposition with its indices or subjects left blank, or indefinite. The common noun happens to have a very distinctive character in the Indo-European languages. In most other tongues it is not sharply discriminated from a verb or participle. "Man," if it can be said to mean anything by itself, means "what I am thinking of is a man." This doctrine, which is in harmony with the above theory of signs, gives a great unity to logic; but Professor Schröder holds it to be very erroneous.

Cicero and other ancient writers mention a great dispute between two logicians, Diodorus and Philo, in regard to the significance of conditional propositions. This dispute has continued to our own day. The Diodoran view seems to be the one which is natural to the minds of those, at least, who speak the European languages. How it may be with other languages has not been reported. The difficulty with this view is that nobody seems to have succeeded in making any clear statement of it that is not open to doubt as to its justice, and that is not pretty complicated. The Philonian view has been preferred by the greatest logicians. Its advantage is that it is perfectly intelligible and simple. Its disadvantage is that it produces results which seem offensive to common sense.

In order to explain these positions, it is best to mention that *possibility* may be understood in many senses; but they may all be embraced under the definition that that is possible which, in a certain state of information, is not known to be false. By varying the supposed state of information all the varieties of possibility are obtained. Thus, *essential* possibility is that which supposes nothing to be known except logical rules. *Substantive* possibility, on the other hand, supposes a state of omniscience. Now the Philonian logicians have always insisted upon beginning the study of conditional propositions by considering what such a proposition means in a state of omniscience; and the Diodorans have, perhaps not very adroitly, commonly assented to this order of procedure. Duns Scotus terms such a conditional proposition a "*consequentia simplex de inesse.*" According to the Philonians, "If it is now lightening it will thunder," understood as a consequence *de inesse*, means "It is either not now lightening or it will soon thunder." According to Diodorus, and most of his followers (who seem here to fall into a logical trap), it means it is now lightening and it will soon thunder.

Although the Philonian views lead to such inconveniences as that it is true, as a consequence *de inesse*, that if the Devil were elected president of the United States, it would prove highly conducive to the spiritual welfare of the people (because he will not be elected), yet both Professor Schröder and I prefer to build the algebra of relatives upon this conception of the conditional proposition. The inconvenience, after all, ceases to seem important, when we reflect that, no matter what the conditional proposition be understood to mean, it can always be expressed by a complexus of Philonian conditionals and denials of conditionals. It may, however, be suspected that the Diodoran view has suffered from incompetent advocacy, and that if it were modified somewhat, it might prove the preferable one.

The consequence *de inesse*, "if A is true, then B is true," is expressed by letting i denote the actual state of things, A_i mean that in the actual state of things A is true, and B_i mean that in the actual state of things B is true, and then saying "If A_i is true then B_i is true," or, what is the same thing, "Either A_i is not true or B_i is true." But an *ordinary* Philonian conditional is expressed by saying, "In *any* possible state of things, i, either A_i is not true, or B_i is true."

Now let us express the categorical proposition, "Every man is wise." Here, we let m_i mean that the individual object i is a man, and w_i mean that the individual object i is wise. Then, we assert that, "taking any individual of the universe, i, no matter what, either that object, i, is not a man or that object, i, is wise"; that is, whatever is a man is wise. That is, "whatever i can indicate, either m_i is not true or w_i is true. The conditional and categorical propositions are expressed in precisely the same form ; and there is absolutely no difference, to my mind, between them. The *form* of relationship is the same.

I find it difficult to state Professor Schröder's objection to this, because I cannot find any clear-cut, unitary conception governing his opinion. More than once in his first volume promises are held out that § 28, the opening section of the second volume, shall make the matter plain. But when the second volume was published, all we found in that section was, as far as repeated examination has enabled me to see, as follows. First, hypothetical propositions, unlike categoricals, essentially involve the idea of time. When this is eliminated from the assertion, they relate only to two possibilities, what always is and what never is. Second, a categorical is always either true or false ; but a hypothetical is either true, false, or meaningless. Thus, "this proposition is false" is meaningless ; and another example is, "the weather will clear as soon as there is enough sky to cut a pair of trousers." Third, the supposition of negation is forced upon us in the study of hypotheticals, never in that of categoricals. Such are Schröder's arguments, to which I proceed to reply.

As to the idea of time, it *may* be introduced ; but to say that the range of possibility in hypotheticals is always a unidimensional continuum is incorrect. "If you alone trump a trick in whist, you take it." The possibilities are that each of the four players plays any one of the four suits. There are 2^{16} different possibilities. Certainly, the universe in hypotheticals is far more frequently finite than in categoricals. Besides, it is an *ignoratio elenchi* to drag in time, when no logician of the English camp has ever alleged anything about propositions involving time. That is not the question.

Every proposition is either true or false, and something not a proposition, when considered as a proposition, is, from the Philonian point of view, true. To be objectionable, a proposition must assert something; if it is merely neutral, it is not positively objectionable, that is, it is not false. "This proposition is false," far from being meaningless, is self-contradictory. That is, it means two irreconcilable things. That it involves contradiction (that is, leads to contradiction if supposed true), is easily proved. For if it be true, it is true; while if it be true, it is false. Every proposition besides what it explicitly asserts, tacitly implies its own truth. The proposition is not true unless *both*, what it explicitly asserts and what it tacitly implies, are true. This proposition, being self-contradictory, is false; and hence, what it explicitly asserts is true. But what it tacitly implies (its own truth) is false. The difficulty about the proposition concerning the piece of blue sky is not a logical one, at all. It is no more senseless than any proposition about a "red odor" which might be a term of a categorical.

The fact stated about negation is only true of the sorts of propositions which are commonly put into categorical and hypothetical shapes, and has nothing to do with the essence of the propositions. In a paper "On the Validity of the Laws of Logic" in the *Journal of Speculative Philosophy*, Vol, II., I have given a sophistical argument that black is white, which shows in the domain of categoricals the phenomena to which Professor Schröder refers as peculiar to hypotheticals.

The *consequentia de inesse* is, of course, the extreme case where the conditional proposition loses all its proper signification, owing to the absence of any range of possibilities. The conditional proper is, "In any possible case, i, either A_i is not true, or B_i is true." In the consequence *de inesse* the meaning sinks to, "In the true state of things, i, either A_i is not true or B_i is true."

My general algebra of logic (which is not that algebra of dual relations, likewise mine, which Professor Schröder prefers, although in his last volume he often uses this general algebra) consists in simply attaching indices to the letters of an expression in the Boolian algebra, making what I term a Boolian, and prefixing to this a series of "quantifiers," which are the letters Π and Σ, each with an index attached to it. Such a quantifier signifies that every individual of the universe is to be substituted for the index the Π or Σ carries, and that the non-relative product or aggregate of the results is to be taken.

Properly to express an ordinary conditional proposition the quantifier Π is required. In 1880, three years before I developed that general algebra, I published a paper containing a chapter on the algebra of the copula (a subject I have since worked out completely in manuscript). I there noticed the necessity of such quantifiers properly to express conditional propositions; but the algebra of quantifiers not being at hand, I contented myself with considering consequences *de inesse*. Some apparently paradoxical results were obtained. Now Professor Schröder seems to accept these results as holding good in the general theory of hypotheticals; and then, since such results are in strong contrast with the doctrine of categoricals, he infers, in § 45 of his Vol. II., a great difference between hypotheticals and categoricals. But the truth simply is that such hypotheticals want the characteristic feature of conditionals, that of a range of possibilities.

In connexion with this point, I must call attention to a mere algebraical difference between Schröder and me. I retain Boole's idea that there are but two *values* in the system of logical quantity. This harmonises with my use of the general algebra. Any two numbers may be selected to represent those values. I prefer 0 and a positive logarithmic ∞. To express that something is A and something is not A, I write:

$$\infty = \Sigma_i A_i \qquad \infty = \Sigma_j A_j$$

or, what is the same thing:

$$\Sigma_i A_i > 0 \qquad \Sigma_j A_j > 0.$$

I have no objection to writing, *as a mere abbreviation*, which may, however, lead to difficulties, if not *interpreted*:

$$A > 0 \qquad \dot{A} > 0.$$

But Professor Schröder understands these formulæ literally, and accordingly *rejects* Boole's conception of two values. He does not seem to understand my mode of apprehending the matter; and hence considers it a great limitation of my system that I restrict myself to two values. In fact, it is a mere difference of algebraical form of conception. I very much prefer the Boolian idea as more simple, and more in harmony with the general algebra of logic.

Somewhat intimately connected with the question of the relation between categoricals and hypotheticals is that of the quantification of the predicate. This is the doctrine that identity, or equality, is the fundamental relation involved in the copula. Holding as I do that the fundamental relation of logic is the illative relation, and that only in special cases does the premise follow from the conclusion, I have in a consistent and thoroughgoing manner opposed the doctrine of the quantification of the predicate. Schröder seems to admit some of my arguments; but still he has a very strong *penchant* for the equation.

Were I not opposed to the quantification of the predicate, I should agree with Venn that it was a mistake to replace Boole's operation of addition by the operation of aggregation, as most Boolians now do. I should consider the "principle of duality" rather an argument *against* than *for* our modern practice. The algebra of dual relatives would be almost identical with the theory of matrices were addition retained; and this would be a great advantage.

It is Schröder's predilection for equations which motives his preference for the algebra of dual relatives, namely, the fact that in that algebra, even a simple undetermined inequality can be expressed as an equation. I think, too, that that algebra has merits; it certainly has uses to which Schröder seldom puts it. Yet, after all, it has too much formalism to greatly delight me,—too many bushels of chaff *per* grain of wheat. I think Professor Schröder likes algebraic formalism better, or dislikes it less, than I.

He looks at the problems of logic through the spectacles of equations, and he formulates them, from that point of view, as he thinks, with great generality; but, as I think, in a narrow spirit. The great thing, with him, is to solve a proposition, and get a *value* of x, that is, an equation of which x forms one member without occurring in the other. How far such equation is *iconic*, that is, has a meaning, or exhibits the constitution of x, he hardly seems to care. He prefers general values to particular roots. Why? I should think the particular root alone of service, for most purposes, unless the general expressions were such that particular roots could be deduced from it,—particular instances, I mean, *showing* the constitution of x. In most instances, a profitable solution of a mathematical problem must consist, in my opinion, of an exhaustive examination of special cases; and quite exceptional are those fortunate problems which mathematicians naturally prefer to study, where the enumeration of special cases, together with the pertinent truths about them, flow so naturally from the general statement as not to require separate examination.

I am very far from denying the interest and value of the problems to which Professor Schröder has applied himself; though there are others to which I turn by preference. Certainly, he has treated his problems with admirable power and clearness. I cannot in this place enter into the elementary explanations which would be necessary to illustrate this for more than a score of readers.

In respect to individuals, both non-relative and pairs, he has added some fundamental propositions to those which had been published. But he is very much mistaken in supposing that I have expressed contrary views. He simply mistakes my meaning.

In regard to algebraical signs, I cannot accept any of Professor Schröder's proposals except this one. While it would be a serious hindrance to the promulgation of the new doctrine to insist on new types being cut, and while I, therefore, think my own course in using the dagger as the sign of relative addition must be continued, yet I have always given that sign in its cursive form a scorpion-tail curve to the left; and it would be finical to insist on one form of curve rather than another. In almost all other cases, in my judgment, Professor Schröder's signs can never be generally received, because they are at war with a principle, the general character of which is such that Professor Schröder would be the last of all men to wish to violate it, a principle which the biologists have been led to adopt in regard to their systematic nomenclature. It is that priority must be respected, or all will fall into chaos. I will not enter further into this matter in this article.

Of what use does this new logical doctrine promise to be? The first service it may be expected to render is that of correcting a considerable number of hasty assumptions about logic which have been allowed to affect philosophy. In the next place, if Kant has shown that metaphysical conceptions spring from formal logic, this great generalisation upon formal logic must lead to a new apprehension of the metaphysical conceptions which shall render them more adequate to the needs of science. In short, "exact" logic will prove a stepping-stone to "exact" metaphysics. In the next place, it must immensely widen our logical notions. For example, a class consisting of a lot of things jumbled higgledy-piggledy must now be seen to be but a degenerate form of the more general idea of a *system*. Generalisation, which has hitherto meant passing to a larger class, must mean taking in the conception of the whole system of which we see but a fragment, etc., etc. In the next place, it is already evident to those who know what has already been made out, that that speculative rhetoric, or objective logic, mentioned at the beginning of this article, is destined to grow into a colossal doctrine which may be expected to lead to most important philosophical conclusions. Finally, the calculus of the new logic, which is applicable to everything, will certainly be applied to settle certain logical questions of extreme difficulty relating to the foundations of mathematics. Whether or not it can lead to any method of discovering methods in mathematics it is difficult to say. Such a thing is conceivable.

It is now more than thirty years since my first published contribution to "exact" logic. Among other serious studies, this has received a part of my attention ever since. I have contemplated it in all sorts of perspectives and have often reviewed my reasons for believing in its importance. My confidence that the key of philosophy is here, is stronger than ever after reading Schröder's last volume. One thing which helps to make me feel that we are developing a living science, and not a dead doctrine, is the healthy mental independence it fosters, as evidenced, for example, in the divergence between Professor Schröder's opinions and mine. There is no bovine nor ovine gregariousness here. But Professor Schröder and

I have a common method which we shall ultimately succeed in applying to our differences, and we shall settle them to our common satisfaction ; and when that method is pouring in upon us new and incontrovertible positively valuable results, it will be as nothing to either of us to confess that where he had not yet been able to apply that method he has fallen into error.

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WHAT PRAGMATISM IS.

THE writer of this article has been led by much experience to believe that every physicist, and every chemist, and, in short, every master in any department of experimental science, has had his mind moulded by his life in the laboratory to a degree that is little suspected. The experimentalist himself can hardly be fully aware of it, for the reason that the men whose intellects he really knows about are much like himself in this respect. With intellects of widely different training from his own, whose education has largely been a thing learned out of books, he will never become inwardly intimate, be he on ever so familiar terms with them; for he and they are as oil and water, and though they be shaken up together, it is remarkable how quickly they will go their several mental ways, without having gained more than a faint flavor from the association. Were those other men only to take skilful soundings of the experimentalist's mind,—which is just what they are unqualified to do, for the most part,—they would soon discover that, excepting perhaps upon topics where his mind is trammelled by personal feeling or by his bringing up, his disposition is to think of everything just as everything is thought of in the laboratory, that is, as a question of experimentation. Of course, no living man possesses in their fullness all the attributes characteristic of his type: it is not the typical doctor whom you will see every day driven in buggy or coupé, nor is it the typical pedagogue that will be met with in the first school-room you enter. But when you have found, or ideally constructed upon a basis of observation, the typical experimentalist, you will find that whatever assertion you may make to him, he will either understand as meaning that if a given prescription for an experiment ever can be and ever is carried out in act, an experience of a given description will result, or else he will see no sense at all in what you say. If you talk to him as Mr. Balfour talked not long ago to the British Association, saying that "the physicist seeks for something deeper than the laws connecting possible objects of experience," that "his object is a physical reality" unrevealed in experiments, and that the existence of such non-experiential reality "is the unalterable faith of science," to all such ontological meaning you will find the experimentalist mind to be color-blind. What adds to that confidence in this which the writer owes to his conversations with experimentalists is that he himself may almost be said to have inhabited a laboratory from the age of six until long past maturity; and having all his life associated mostly with experimentalists, it has always been with a confident sense of understanding them and of being understood by them.

That laboratory life did not prevent the writer (who here and in what follows simply exemplifies the experimentalist type) from becoming interested in methods of thinking; and when he came to read metaphysics, although much of it seemed to him loosely reasoned and determined by accidental prepossessions, yet in the writings of some philosophers, especially Kant, Berkeley, and Spinoza, he sometimes came upon strains of thought that recalled the ways of thinking of the laboratory, so that he felt he might trust to them; all of which has been true of other laboratory-men.

Endeavoring, as a man of that type naturally would, to formulate what he so approved, he framed the theory that a *conception,* that is, the rational purport of a word or other expression, lies exclusively in its conceivable bearing upon the conduct of life; so that, since obviously nothing that might not result from experiment can have any direct bearing upon conduct, if one can define accurately all the conceivable experimental phenomena which the affirmation or denial of a concept could imply, one will have therein a complete definition of the concept, and *there is absolutely nothing* *more in it.* For this doctrine he invented the name *pragmatism.* Some of his friends wished him to call it *practicism* or *practicalism* (perhaps on the ground that πρακτικός is better Greek than πραγματικός). But for one who had learned philosophy out of Kant, as the writer, along with nineteen out of every twenty experimentalists who have turned to philosophy, had done, and who still thought in Kantian terms most readily, *praktisch* and *pragmatisch* were as far apart as the two poles, the former belonging in a region of thought where no mind of the experimentalist type can ever make sure of solid ground under his feet, the latter expressing relation to some definite human purpose. Now quite the most striking feature of the new theory was its recognition of an inseparable connection between rational cognition and rational purpose; and that consideration it was which determined the preference for the name *pragmatism.*

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Concerning the matter of philosophical nomenclature, there are a few plain considerations, which the writer has for many years longed to submit to the deliberate judgment of those few fellow-students of philosophy, who deplore the present state of that study, and who are intent upon rescuing it therefrom and bringing it to a condition like that of the natural sciences, where investigators, instead of contemning each the work of most of the others as misdirected from beginning to end, co-operate, stand upon one another's shoulders, and multiply incontestible results; where every observation is repeated, and isolated observations go for little; where every hypothesis that merits attention is subjected to severe but fair examination, and only after the predictions to which it leads have been remarkably borne out by experience is trusted at all, and even then only provisionally; where a radically false step is rarely taken, even the most faulty of those theories which gain wide credence being true in their main experiential predictions. To those students, it is submitted that no study can become scientific in the sense described, until it provides itself with a suitable technical nomenclature, whose every term has a single definite meaning universally accepted among students of the subject, and whose vocables have no such sweetness or charms as might tempt loose writers to abuse them,—which is a virtue of scientific nomenclature too little appreciated. It is submitted that the experience of those sciences which have conquered the greatest difficulties of terminology, which are unquestionably the taxonomic sciences, chemistry, mineralogy, botany, zoölogy, has conclusively shown that the one only way in which the requisite unanimity and requisite ruptures with individual habits and preferences can be brought about is so to shape the canons of terminology that they shall gain the support of *moral principle* and of every man's sense of decency; and that, in particular, (under defined restrictions,) the general feeling shall be that he who introduces a new conception into philosophy is under an obligation to invent acceptable terms to express it, and that when he has done so, the duty of his fellow-students is to accept those terms, and to resent any wresting of them from their original meanings, as not only a gross discourtesy to him to whom philosophy was indebted for each conception, but also as an injury to philosophy itself; and furthermore, that once a conception has been supplied with suitable and sufficient words for its expression, no other *technical* terms denoting the same things, considered in the same relations, should be countenanced. Should this suggestion find favor, it might be deemed needful that the philosophians in congress assembled should adopt, after due deliberation, convenient canons to limit the application of the principle. Thus, just as is done in chemistry, it might be wise to assign fixed meanings to certain prefixes and suffixes. For example, it might be agreed, perhaps, that the prefix *prope-* should mark a broad and rather indefinite extension of the meaning of the term to which it was prefixed; the name of a doctrine would naturally end in *-ism,* while *-icism* might mark a more strictly defined acception of that doctrine, etc. Then again, just as in biology no account is taken of terms antedating Linnæus, so in philosophy it might be found best not to go back of the scholastic terminology. To illustrate another sort of limitation, it has probably never happened that any philosopher has attempted to give a general name to his own doctrine without that name's soon acquiring in common philosophical usage, a signification much broader than was originally intended. Thus, special systems go by the names Kantianism, Benthamism, Comtianism, Spencerianism, etc., while transcendentalism, utilitarianism, positivism, evolutionism, synthetic philosophy, etc. have irrevocably and very conveniently been elevated to broader governments.

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After awaiting in vain, for a good many years, some particularly opportune conjuncture of circumstances that might serve to recommend his notions of the ethics of terminology, the writer has now, at last, dragged them in over head and shoulders, on an occasion when he has no specific proposal to offer nor any feeling but satisfaction at the course usage has run without any canons or resolutions of a congress. His word "pragmatism" has gained general recognition in a generalised sense that seems to argue power of growth and vitality. The famed psychologist, James, first took it up, seeing that his "radical empiricism" substantially answered to the writer's definition of pragmatism, albeit with a certain difference in the point of view. Next, the admirably clear and brilliant thinker, Mr. Ferdinand C. S. Schiller, casting about for a more attractive name for the "anthropomorphism" of his *Riddle of the Sphinx,* lit, in that most remarkable paper of his on *Axioms as Postulates,* upon the same designation "pragmatism," which in its original sense was in generic agreement with his own doctrine, for which he has since found the more appropriate specification "humanism," while he still retains "pragmatism" in a somewhat wider sense. So far all went happily. But at present, the word begins to be met with occasionally in the literary journals, where it gets abused in the merciless way that words have to expect when they fall into literary clutches. Sometimes the manners of the British have effloresced in scolding at the word as ill-chosen, —ill-chosen, that is, to express some meaning that it was rather designed to exclude. So then, the writer, finding his bantling "pragmatism" so promoted, feels that it is time to kiss his child good-by and relinquish it to its higher destiny; while to serve the precise purpose of expressing the original definition, he begs to announce the birth of the word "pragmaticism," which is ugly enough to be safe from kidnappers.[2]

Much as the writer has gained from the perusal of what other pragmatists have written, he still thinks there is a decisive advantage in his original conception of the doctrine. From this original form every truth that follows from any of the other forms can be deduced, while some errors can be avoided into which other pragmatists have fallen. The original view appears, too, to be a more compact and unitary conception than the others. But its capital merit, in the writer's eyes, is that it more readily connects itself with a critical proof of its truth. Quite in accord with the logical order of investigation, it usually happens that one first forms an hypothesis that seems more and more reasonable the further one examines into it, but that only a good deal later gets crowned with an adequate proof. The present writer having had the pragmatist theory under consideration for many years longer than most of its adherents, would naturally have given more attention to the proof of it. At any rate, in endeavoring to explain pragmatism, he may be excused for confining himself to that form of it that he knows best. In the present article there will be space only to explain just what this doctrine, (which, in such hands as it has now fallen into, may probably play a pretty prominent part in the philosophical discussions of the next coming years,) really consists in. Should the exposition be found to interest readers of *The Monist*, they would certainly be much more interested in a second article which would give some samples of the manifold applications of pragmaticism (assuming it to be true) to the solution of problems of different kinds. After that, readers might be prepared to take an interest in a proof

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[2] To show how recent the general use of the word "pragmatism" is, the writer may mention that, to the best of his belief, he never used it in copy for the press before to-day, except by particular request, in *Baldwin's Dictionary*. Toward the end of 1890, when this part of the *Century Dictionary* appeared, he did not deem that the word had sufficient status to appear in that work. But he has used it continually in philosophical conversation since, perhaps, the mid-seventies.

that the doctrine is true,—a proof which seems to the writer to leave no reasonable doubt on the subject, and to be the one contribution of value that he has to make to philosophy. For it would essentially involve the establishment of the truth of synechism.

The bare definition of pragmaticism could convey no satisfactory comprehension of it to the most apprehensive of minds, but requires the commentary to be given below. Moreover, this definition takes no notice of one or two other doctrines without the previous acceptance (or virtual acceptance) of which pragmaticism itself would be a nullity. They are included as a part of the pragmatism of Schiller, but the present writer prefers not to mingle different propositions. The preliminary propositions had better be stated forthwith.

The difficulty in doing this is that no formal list of them has ever been made. They might all be included under the vague maxim, "Dismiss make-believes." Philosophers of very diverse stripes propose that philosophy shall take its start from one or another state of mind in which no man, least of all a beginner in philosophy, actually is. One proposes that you shall begin by doubting everything, and says that there is only one thing that you cannot doubt, as if doubting were "as easy as lying." Another proposes that we should begin by observing "the first impressions of sense," forgetting that our very percepts are the results of cognitive elaboration. But in truth, there is but one state of mind from which you can "set out," namely, the very state of mind in which you actually find yourself at the time you do "set out,"—a state in which you are laden with an immense mass of cognition already formed, of which you cannot divest yourself if you would; and who knows whether, if you could, you would not have made all knowledge impossible to yourself? Do you call it *doubting* to write down on a piece of paper that you doubt? If so, doubt has nothing to do with any serious business. But do not make believe; if pedantry has not eaten all the reality out of you, recognise, as you must, that there is much that you do not doubt, in the least. Now that which you do not at all doubt, you must and do regard as infallible, absolute truth. Here breaks in Mr. Make Believe: "What! Do you mean to say that one is to believe what is not true, or that what a man does not doubt is *ipso facto* true?" No, but unless he can make a thing white and black at once, *he* has to regard what he does not doubt as absolutely true. Now you, *per hypothesiu,* are that man. "But you tell me there are scores of things I do not doubt. I really cannot persuade myself that there is not some one of them about which I am mistaken." You are adducing one of your make-believe facts, which, even if it were established, would only go to show that doubt has a *limen,* that is, is only called into being by a certain finite stimulus. You only puzzle yourself by talking of this metaphysical "truth" and metaphysical "falsity," that you know nothing about. All you have any dealings with are your doubts and beliefs,[3] with the course of life that forces new beliefs upon you and gives you power to doubt old beliefs. If your terms " truth" and "falsity" are taken in such senses as to be definable in terms of doubt and belief and the course of experience, (as for example they would be, if you were to define the "truth" as that to a belief in which belief would tend if it were to tend indefinitely toward absolute fixity,) well and good: in that case, you are only talking about doubt and belief. But if by truth and falsity you mean something not definable in terms of doubt and belief in any way, then you are talking of entities of whose existence you can know nothing, and which Ockham's razor would clean shave off. Your problems would be greatly simplified, if, instead of saying that you want to know the "Truth," you were simply to say that you want to attain a state of belief unassailable by doubt.

Belief is not a momentary mode of consciousness; it is a habit of mind essentially enduring for some time, and mostly (at least) unconscious; and like other habits, it is, (until it meets with some surprise that begins its dissolution,) perfectly self-satisfied. Doubt is of an altogether contrary genus. It is not a habit, but the privation of a habit. Now a privation of a habit, in order to be anything

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[3] It is necessary to say that "belief" is throughout used merely as the name of the contrary to doubt, without regard to grades of certainty nor to the nature of the proposition held for true, i. e. "believed."

at all, must be a condition of erratic activity that in some way must get superseded by a habit.

Among the things which the reader, as a rational person, does not doubt, is that he not merely has habits, but also can exert a measure of self-control over his future actions; which means, however, *not* that he can impart to them any arbitrarily assignable character, but, on the contrary, that a process of self-preparation will tend to impart to action, (when the occasion for it shall arise,) one fixed character, which is indicated and perhaps roughly measured by the absence (or slightness) of the feeling of self-reproach, which subsequent reflection will induce. Now, this subsequent reflection is part of the self-preparation for action on the next occasion. Consequently, there is a tendency, as action is repeated again and again, for the action to approximate indefinitely toward the perfection of that fixed character, which would be marked by entire absence of self-reproach. The more closely this is approached, the less room for self-control there will be; and where no self-control is possible there will be no self-reproach.

These phenomena seem to be the fundamental characteristics which distinguish a rational being. Blame, in every case, appears to be a modification, often accomplished by a transference, or "projection," of the primary feeling of self-reproach. Accordingly, we never blame anybody for what had been beyond his power of previous self-control. Now, thinking is a species of conduct which is largely subject to self-control. In all their features, (which there is no room to describe here,) logical self-control is a perfect mirror of ethical self-control,—unless it be rather a species under that genus. In accordance with this, what you cannot in the least help believing is not, justly speaking, wrong belief. In other words, for you it is the absolute truth. True, it is conceivable that what you cannot help believing to-day, you might find you thoroughly disbelieve to-morrow. But then there is a certain distinction between things you "cannot" do, merely in the sense that nothing stimulates you to the great effort and endeavors that would be required, and things you cannot do because in their own nature they are insusceptible of being put into practice. In every stage of your excogitations, there is something of which you can only say, "I cannot think otherwise," and your experientially based hypothesis is that the impossibility is of the second kind.

There is no reason why "thought," in what has just been said, should be taken in that narrow sense in which silence and darkness are favorable to thought. It should rather be understood as covering all rational life, so that an experiment shall be an operation of thought. Of course, that ultimate state of habit to which the action of self-control ultimately tends, where no room is left for further self-control, is, in the case of thought, the state of fixed belief, or perfect knowledge.

Two things here are all-important to assure oneself of and to remember. The first is that a person is not absolutely an individual. His thoughts are what he is "saying to himself," that is, is saying to that other self that is just coming into life in the flow of time. When one reasons, it is that critical self that one is trying to persuade; and all thought whatsoever is a sign, and is mostly of the nature of language. The second thing to remember is that the man's circle of society, (however widely or narrowly this phrase may be understood,) is a sort of loosely compacted person, in some respects of higher rank than the person of an individual organism. It is these two things alone that render it possible for you,—but only in the abstract, and in a Pickwickian sense,—to distinguish between absolute truth and what you do not doubt.

Let us now hasten to the exposition of pragmaticism itself. Here it will be convenient to imagine that somebody to whom the doctrine is new, but of rather preternatural perspicacity, asks questions of a pragmaticist. Everything that might give a dramatic illusion must be stripped off, so that the result will be a sort of cross between a dialogue and a catechism, but a good deal liker the latter,—something rather painfully reminiscent of *Mangnall's Historical Questions.*

Questioner: I am astounded at your definition of your pragmatism, because only last year I was assured by a person above all suspicion of warping the truth,—himself a pragmatist,—that your doctrine precisely was "that a conception is to be tested by its practical effects." You must surely, then, have entirely changed your definition very recently.

Pragmatist: If you will turn to Vols. VI and VII of the *Revue Philosophique,* or to the *Popular Science Monthly* for November 1877 and January 1878, you will be able to judge for yourself whether the interpretation you mention was not then clearly excluded. The exact wording of the English enunciation, (changing only the first person into the second,) was: "Consider what effects that might conceivably have practical bearings you conceive the object of your conception to have. Then your conception of those effects is the WHOLE of your conception of the object."

Questioner: Well, what reason have you for asserting that this is so?

Pragmatist: That is what I specially desire to tell you. But the question had better be postponed until you clearly understand what those reasons profess to prove.

Questioner: What, then, is the *raison d'être* of the doctrine? What advantage is expected from it?

Pragmatist: It will serve to show that almost every proposition of ontological metaphysics is either meaningless gibberish,—one word being defined by other words, and they by still others, without any real conception ever being reached,—or else is downright absurd; so that all such rubbish being swept away, what will remain of philosophy will be a series of problems capable of investigation by the observational methods of the true sciences,—the truth about which can be reached without those interminable misunderstandings and disputes which have made the highest of the positive sciences a mere amusement for idle intellects, a sort of chess,—idle pleasure its purpose, and reading out of a book its method. In this regard, pragmaticism is a species of prope-positivism. But what distinguishes it from other species is, first, its retention of a purified philosophy; secondly, its full acceptance of the main body of our instinctive beliefs; and thirdly, its strenuous insistence upon the truth of scholastic realism, (or a close approximation to that, well-stated by the late Dr. Francis Ellingwood Abbot in the Introduction to his *Scientific Theism*). So, instead of merely jeering at metaphysics, like other prope-positivists, whether by long drawn-out parodies or otherwise, the pragmaticist extracts from it a precious essence, which will serve to give life and light to cosmology and physics. At the same time, the moral applications of the doctrine are positive and potent; and there are many other uses of it not easily classed. On another occasion, instances may be given to show that it really has these effects.

Questioner: I hardly need to be convinced that your doctrine would wipe out metaphysics. Is it not as obvious that it must wipe out every proposition of science and everything that bears on the conduct of life? For you say that the only meaning that, for you, any assertion bears is that a certain experiment has resulted in a certain way: Nothing else but an experiment enters into the meaning. Tell me, then, how can an experiment, in itself, reveal anything more than that something once happened to an individual object and that subsequently some other individual event occurred?

Pragmatist: That question is, indeed, to the purpose,—the purpose being to correct any misapprehensions of pragmaticism. You speak of an experiment in itself, emphasising "*in itself.*" You evidently think of each experiment as isolated from every other. It has not, for example, occurred to you, one might venture to surmise, that every connected series of experiments constitutes a single collective experiment. What are the essential ingredients of an experiment? First, of course, an experimenter of flesh and blood. Secondly, a verifiable hypothesis. This is a proposition[4] relating to the universe environing the experimenter, or to some well-known part of it and affirming or denying of this only some experimental possibility or impossibility. The third indispensable ingredient is a sincere doubt in the experimenter's mind as to the truth of that

[4] The writer, like most English logicians, invariably uses the word *proposition,* not as the Germans define their equivalent, *Satz,* as the language-expression of a judgment (*Urtheil*), but as that which is related to any assertion, whether mental and self-addressed or outwardly expressed, just as any possibility is related to its actualisation. The difficulty of the, at best, difficult problem of the essential nature of a Proposition has been increased, for the Germans, by their *Urtheil,* confounding, under one designation, the mental *assertion* with the *assertible.*

hypothesis. Passing over several ingredients on which we need not dwell, the purpose, the plan, and the resolve, we come to the act of choice by which the experimenter singles out certain identifiable objcts to be operated upon. The next is the external (or quasi-external) ACT by which he modifies those objects. Next, comes the subsequent *reaction* of the world upon the experimenter in a perception; and finally, his recognition of the teaching of the experiment. While the two chief parts of the event itself are the action and the reaction, yet the unity of essence of the experiment lies in its purpose and plan, the ingredients passed over in the enumeration.

Another thing: in representing the pragmaticist as making rational meaning to consist in an experiment, (which you speak of as an event in the past,) you strikingly fail to catch his attitude of mind. Indeed, it is not in an experiment, but in *experimental phenomena,* that rational meaning is said to consist. When an experimentalist speaks of a *phenomenon,* such as "Hall's phenomenon," "Zeemann's phenomenon" and its modification, "Michelson's phenomenon," or "the chess-board phenomenon," he does not mean any particular event that did happen to somebody in the dead past, but what *surely will* happen to everybody in the living future who shall fulfil certain conditions. The phenomenon consists in the fact that when an experimentalist shall come to *act* according to a certain scheme that he has in mind, then will something else happen, and shatter the doubts of sceptics, like the celestial fire upon the altar of Elijah.

And do not overlook the fact that the pragmaticist maxim says nothing of single experiments or of single experimental phenomena, (for what is conditionally true *in futuro* can hardly be singular,) but only speaks of *general kinds* of experimental phenomena. Its adherent does not shrink from speaking of general objects as real, since whatever is true represents a real. Now the laws of nature are true.

The rational meaning of every proposition lies in the future. How so? The meaning of a proposition is itself a proposition. Indeed, it is no other than the very proposition of which it is the meaning: it is a translation of it. But of the myriads of forms into which a proposition may be translated, what is that one which is to be called its very meaning? It is, according to the pragmaticist, that form in which the proposition becomes applicable to human conduct, not in these or those special circumstances, nor when one entertains this or that special design, but that form which is most directly applicable to self-control under every situation, and to every purpose. This is why he locates the meaning in future time; for future conduct is the only conduct that is subject to self-control. But in order that that form of the proposition which is to be taken as its meaning should be applicable to every situation and to every purpose upon which the proposition has any bearing, it must be simply the general description of all the experimental phenomena which the assertion of the proposition virtually predicts. For an experimental phenomenon is the fact asserted by the proposition that action of a certain description will have a certain kind of experimental result; and experimental results are the only results that can affect human conduct. No doubt, some unchanging idea may come to influence a man more than it had done; but only because some experience equivalent to an experiment has brought its truth home to him more intimately than before. Whenever a man acts purposively, he acts under a belief in some experimental phenomenon. Consequently, the sum of the experimental phenomena that a proposition implies makes up its entire bearing upon human conduct. Your question, then, of how a pragmaticist can attribute any meaning to any assertion other than that of a single occurrence is substantially answered.

Questioner: I see that pragmaticism is a thorough-going phenomenalism. Only why should you limit yourself to the phenomena of experimental science rather than embrace all observational science? Experiment, after all, is an uncommunicative informant. It never expiates: it only answers "yes" or "no"; or rather it usually snaps out "No!" or, at best, only utters an inarticulate grunt for the negation of its "no." The typical experimentalist is not much of an observer. It is the student of natural history to whom nature opens the treasury of her confidence, while she treats the cross-examining experimentalist with the reserve he merits. Why should your phenomenalism sound the meagre jews-harp of experiment rather than the glorious organ of observation?

Pragmaticist: Because pragmaticism is not definable as "thorough-going phenomenalism," although the latter doctrine may be a kind of pragmatism. The *richness* of phenomena lies in their sensuous quality. Pragmaticism does not intend to define the phenomenal equivalents of words and general ideas, but, on the contrary, eliminates their sential element, and endeavors to define the rational purport, and this it finds in the purposive bearing of the word or proposition in question.

Questioner: Well, if you choose so to make Doing the Be-all and the End-all of human life, why do you not make meaning to consist simply in doing? Doing has to be done at a certain time upon a certain object. Individual objects and single events cover all reality, as everybody knows, and as a practicalist ought to be the first to insist. Yet, your meaning, as you have described it, is *general*. Thus, it is of the nature of a mere word and not a reality. You say yourself that your meaning of a proposition is only the same proposition in another dress. But a practical man's meaning is the very thing he means. What do you make to be the meaning of "George Washington"?

Pragmaticist: Forcibly put! A good half dozen of your points must certainly be admitted. It must be admitted, in the first place, that if pragmaticism really made Doing to be the Be-all and the End-all of life, that would be its death. For to say that we live for the mere sake of action, as action, regardless of the thought it carries out, would be to say that there is no such thing as rational purport. Secondly, it must be admitted that every proposition professes to be true of a certain real individual object, often the environing universe. Thirdly, it must be admitted that pragmaticism fails to furnish any translation or meaning of a proper name, or other designation of an individual object. Fourthly, the pragmaticistic meaning is undoubtedly general; and it is equally indisputable that the general is of the nature of a word or sign. Fifthly, it must be admitted that individuals alone exist; and sixthly, it may be admitted that the very meaning of a word or significant object ought to be the very essence of reality of what it signifies. But when, those admissions having been unreservedly made, you find the pragmaticist still constrained most earnestly to deny the force of your objection, you ought to infer that there is some consideration that has escaped you. Putting the admissions together, you will perceive that the pragmaticist grants that a proper name, (although it is not customary to say that it has a *meaning*,) has a certain denotative function peculiar, in each case, to that name and its equivalents; and that he grants that every assertion contains such a denotative or pointing-out function. In its peculiar individuality, the pragmaticist excludes this from the rational purport of the assertion, although *the like* of it, being common to all assertions, and so, being general and not individual, may enter into the pragmaticistic purport. Whatever exists, *ex-sists,* that is, really acts upon other existents, so obtains a self-identity, and is definitely individual. As to the general, it will be a help to thought to notice that there are two ways of being general. A statue of a soldier on some village monument, in his overcoat and with his musket, is for each of a hundred families the image of its uncle, its sacrifice to the union. That statue, then, though it is itself single, represents any one man of whom a certain predicate may be true. It is *objectively* general. The word "soldier," whether spoken or written, is general in the same way; while the name, "George Washington," is not so. But each of these two terms remains one and the same noun, whether it be spoken or written. and whenever and wherever it be spoken or written. This noun is not an existent thing: it is a *type,* or *form,* to which objects, both those that are externally existent and those which are imagined, may *conform,* but which none of them can exactly be. This is subjective generality. The pragmaticistic purport is general in both ways.

As to reality, one finds it defined in various ways; but if that principle of terminological ethics that was proposed be accepted, the equivocal language will soon disappear. For *realis* and *realitas* are not ancient words. They were invented to be terms of philosophy in the thirteenth century, and the meaning they were intended to express is perfectly clear. That is *real* which has such and such characters, whether anybody thinks it to have those characters or not. At any rate, that is the sense in which the pragmaticist uses the word. Now, just as conduct controlled by ethical reason tends toward fixing certain habits of conduct, the nature of which, (as to illustrate the meaning, peaceable habits and not quarrelsome habits,) does not depend upon any accidental circumstances, and *in that sense,* may be said to be *destined;* so, thought, controlled by a rational experimental logic, tends to the fixation of certain opinions, equally destined, the nature of which will be the same in the end, however the perversity of thought of whole generations may cause the postponement of the ultimate fixation. If this be so, as every man of us virtually assumes that it is, in regard to each matter the truth of which he seriously discusses, then, according to the adopted definition of "real," the state of things which will be believed in that ultimate opinion is real. But, for the most part, such opinions will be general. Consequently, *some* general objects are real. (Of course, nobody ever thought that *all* generals were real; but the scholastics used to assume that generals were real when they had hardly any, or quite no, experiential evidence to support their assumption; and their fault lay just there, and not in holding that generals could be real.) One is struck with the inexactitude of thought even of analysts of power, when they touch upon modes of being. One will meet, for example, the virtual assumption that what is relative to thought cannot be real. But why not, exactly? *Red* is relative to sight, but the fact that this or that is in that relation to vision that we call being red is not *itself* relative to sight; it is a real fact.

Not only may generals be real, but they may also be *physically efficient,* not in every metaphysical sense, but in the common-sense acception in which human purposes are physically efficient. Aside from metaphysical nonsense, no sane man doubts that if I feel the air in my study to be stuffy, that thought may cause the window to be opened. My thought, be it granted, was an individual event. But what determined it to take the particular determination it did, was in part the general fact that stuffy air is unwholesome, and in part other *Forms,* concerning which Dr. Carus has caused so many men to reflect to advantage,—or rather, *by* which, and the general truth concerning which Dr. Carus's mind was determined to the forcible enunciation of so much truth. For truths, on the average, have a greater tendency to get believed than falsities have. Were it otherwise, considering that there are myriads of false hypotheses to account for any given phenomenon, against one sole true one (or if you will have it so, against every true one,) the first step toward genuine knowledge must have been next door to a miracle. So, then, when my window was opened, because of the truth that stuffy air is malsain, a physical effort was brought into existence by the efficiency of a general and non-existent truth. This has a droll sound because it is unfamiliar; but exact analysis is with it and not against it; and it has besides, the immense advantage of not blinding us to great facts,—such as that the ideas "justice" and "truth" are, notwithstanding the iniquity of the world, the mightiest of the forces that move it. Generality is, indeed, an indispensable ingredient of reality; for mere individual existence or actuality without any regularity whatever is a nullity. Chaos is pure nothing.

That which any true proposition asserts is *real,* in the sense of being as it is regardless of what you or I may think about it. Let this proposition be a general conditional proposition as to the future, and it is a real general such as is calculated really to influence human conduct; and such the pragmaticist holds to be the rational purport of every concept.

Accordingly, the pragmaticist does not make the *summum bonum* to consist in action, but makes it to consist in that process of evolution whereby the existent comes more and more to embody those generals which were just now said to be *destined,* which is what we strive to express in calling them *reasonable.* In its higher stages, evolution takes place more and more largely through self-control, and this gives the pragmaticist a sort of justification for making the rational purport to be general.

There is much more in elucidation of pragmaticism that might be said to advantage, were it not for the dread of fatiguing the reader. It might, for example, have been well to show clearly that the pragmaticist does not attribute any different essential mode of being to an event in the future from that which he would attribute to a similar event in the past, but only that the practical attitude of the thinker toward the two is different. It would also have been well to show that the pragmaticist does not make Forms to be the *only* realities in the world, any more than he makes the reasonable purport of a word to be the only kind of meaning there is. These things are, however, implicitly involved in what has been said. There is only one remark concerning the pragmaticist's conception of the relation of his formula to the first principles of logic which need detain the reader.

Aristotle's definition of universal predication, which is usually designated, (like a papal bull or writ of court, from its opening words,) as the *Dictum de omni,* may be translated as follows: "We call a predication, (be it affirmative or negative,) *universal,* when, and only when, there is nothing among the existent individuals to which the subject affirmatively belongs, but to which the predicate will not likewise be referred (affirmatively or negatively, according as the universal predication is affirmative or negative)." The Greek is: **λέγομεν τὸ κατὰ παντὸς κατηγορεῖσθαι ὅταν μηδὲν ᾖ λαβεῖν τῶν τοῦ ὑποκειμένου καθ' οὗ θάτερον οὐ λεχθήσεται· καὶ τὸ κατὰ μηδενὸς ὡσαύτως.** The important words "existent individuals" have been introduced into the translation (which English idiom would not here permit to be literal); but it is plain that existent individuals were what Aristotle meant. The other departures from literalness only serve to give modern English forms of expression. Now, it is well known that propositions in formal logic go in pairs, the two of one pair being convertible into another by the interchange of the ideas of antecedent and consequent, subject and predicate, etc. The parallelism extends so far that it is often assumed to be perfect; but it is not quite so. The proper mate of this sort to the *Dictum de omni* is the following definition of affirmative predication: We call a predication *affirmative,* (be it universal or particular,) when, and only when, there is nothing among the sensational effects that belong universally to the predicate which will not be, (universally or particularly, according as the affirmative predication is universal or particular,) said to belong to the subject. Now, this is substantially the essential proposition of pragmaticism. Of course, its parallelism to the *dictum de omni* will only be admitted by a person who admits the truth of pragmaticism.

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Suffer me to add one word more on this point. For if one cares at all to know what the pragmaticist theory consists in, one must understand that there is no other part of it to which the pragmaticist attaches quite as much importance as he does to the recognition in his doctrine of the utter inadequacy of action or volition or even of resolve or actual purpose, as materials out of which to construct a conditional purpose or the concept of conditional purpose. Had a purposed article concerning the principle of continuity and synthetising the ideas of the other articles of a series in the early volumes of *The Monist* ever been written, it would have appeared how, with thorough consistency, that theory involved the recognition that continuity is an indispensable element of reality, and that continuity is simply what generality becomes in the logic of relatives, and thus, like generality, and more than generality, is an affair of thought, and is the essence of thought. Yet even in its truncated condition, an extra-intelligent reader might discern that the theory of those cosmological articles made reality to consist in something more than feeling and action could supply, inasmuch as the primeval chaos, where those two elements were present, was explicitly shown to be pure nothing. Now, the motive for alluding to that theory just here is, that in this way one can put in a strong light a position which the pragmaticist holds and must hold, whether that cosmological theory be ultimately sustained or exploded, namely, that the third category,—the category of thought, representation, triadic relation, mediation, genuine thirdness, thirdness as such,—is an essential ingredient of reality, yet does not by itself constitute reality, since this category, (which in that cosmology appears as the element of habit,) can have no concrete being without action, as a separate object on which to work its government, just as action cannot exist without the immediate being of feeling on which to act. The truth is that pragmaticism is closely allied to the Hegelian absolute idealism, from which, however, it is sundered by its vigorous denial that the third category, (which Hegel degrades to a mere stage of thinking,) suffices to make the world, or is even so much as self-sufficient. Had Hegel, instead of regarding the first two stages with his smile of contempt, held on to them as independent or distinct elements of the triune Reality, pragmaticists might have looked up to him as the great vindicator of their truth. (Of course, the external trappings of his doctrine are only here and there of much significance.) For pragmaticism belongs essentially to the triadic class of philosophical doctrines, and is much more essentially so than Hegelianism is. (Indeed, in one passage, at least, Hegel alludes to the triadic form of his exposition as to a mere fashion of dress.)

C. S. PEIRCE.

MILFORD, PA., September, 1904.

POSTSCRIPT. During the last five months, I have met with references to several objections to the above opinions, but not having been able to obtain the text of these objections, I do not think I ought to attempt to answer them. If gentlemen who attack either pragmatism in general or the variety of it which I entertain would only send me copies of what they write, more important readers they could easily find, but they could find none who would examine their arguments with a more grateful avidity for truth not yet apprehended, nor any who would be more sensible of their courtesy.

C. S. P.

Feb. 9, 1905.

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THE MONIST

ISSUES OF PRAGMATICISM.

PRAGMATICISM was originally enounced[1] in the form of a maxim, as follows: Consider what effects that might *conceivably* have practical bearings you *conceive* the objects of your *conception* to have. Then, your *conception* of those effects is the whole of your *conception* of the object.

I will restate this in other words, since ofttimes one can thus eliminate some unsuspected source of perplexity to the reader. This time it shall be in the indicative mood, as follows: The entire intellectual purport of any symbol consists in the total of all general modes of rational conduct which, conditionally upon all the possible different circumstances and desires, would ensue upon the acceptance of the symbol.

Two doctrines that were defended by the writer about nine years before the formulation of pragmaticism may be treated as consequences of the latter belief. One of these may be called Critical Common-sensism. It is a variety of the Philosophy of Common Sense, but is marked by six distinctive characters, which had better be enumerated at once.

Character I. Critical Common-sensism admits that there not only are indubitable propositions but also that there are indubitable inferences. In one sense, anything evident is indubitable; but the propositions and inferences which Critical Common-Sensism holds to be original, in the sense one cannot "go behind" them (as the

[1] *Popular Science Monthly*, XII, 293; for Jan. 1878. An introductory article opens the volume, in the number for Nov., 1877.

lawyers say) are indubitable in the sense of being acritical. The term "reasoning" ought to be confined to such fixation of one belief by another as is reasonable, deliberate, self-controlled. A reasoning must be conscious; and this consciousness is not mere "immediate consciousness," which (as I argued in 1868, *J. Spec. Phil.,* Vol. II) is simple Feeling viewed from another side, but is in its ultimate nature (meaning in that characteristic element of it that is not reducible to anything simpler), a sense of taking a habit, or disposition to respond to a given kind of stimulus in a given kind of way. As to the nature of that, some *éclaircissements* will appear below and again in my third paper, on the Basis of Pragmaticism. But the secret of rational consciousness is not so much to be sought in the study of this one peculiar nucleolus, as in the review of the process of self-control in its entirety. The machinery of logical self-control works on the same plan as does moral self-control, in multiform detail. The greatest difference, perhaps, is that the latter serves to inhibit mad puttings forth of energy, while the former most characteristically insures us against the quandary of Buridan's ass. The formation of habits under imaginary action (see the paper of Jan., 1878, p. 290 at the top) is one of the most essential ingredients of both; but in the logical process the imagination takes far wider flights, proportioned to the generality of the field of inquiry, being bounded in pure mathematics solely by the limits of its own powers, while in the moral process we consider only situations that may be apprehended or anticipated. For in moral life we are chiefly solicitous about our conduct and its inner springs, and the approval of conscience, while in intellectual life there is a tendency to value existence as the vehicle of forms. Certain obvious features of the phenomena of self-control (and especially of habit), can be expressed compactly and without any hypothetical addition, except what we distinctly rate as imagery, by saying that we have an occult nature of which and of its contents we can only judge by the conduct that it determines, and by phenomena of that conduct. All will assent to that (or all but the extreme nominalist), but anti-synechistic thinkers wind themselves up in a facticious snarl by falsifying the phenomena in representing consciousness to be, as it were, a skin, a separate tissue, overlying an unconscious region of the occult nature, mind, soul, or physiological basis. It appears to me that in the present state of our knowledge a sound methodeutic prescribes that, in adhesion to the appearances, the difference is only relative and the demarcation not precise.

According to the maxim of Pragmaticism, to say that determination affects our occult nature is to say that it is capable of affecting deliberate conduct; and since we are conscious of what we do deliberately, we are conscious *habitualiter* of whatever hides in the depths of our nature; and it is presumable (and *only* presumable,[2] although curious instances are on record), that a sufficiently energetic effort of attention would bring it out. Consequently, to say that an operation of the mind is controlled is to say that it is, in a special sense, a conscious operation; and this no doubt is the consciousness of reasoning. For this theory requires that in reasoning we should be conscious, not only of the conclusion, and of our deliberate approval of it, but also of its being the result of the premiss from which it does result, and furthermore that the inference is one of a possible class of inferences which conform to one guiding principle. Now in fact we find a well-marked class of mental operations, clearly of a different nature from any others which do possess just these properties. They alone deserve to be called *reasonings;* and if the reasoner is conscious, even vaguely, of what his guiding principle is, his reasoning should be called a *logical argumentation*. There are, however, cases in which we are conscious that a belief has been determined by another given belief, but are not conscious that it proceeds on any general principle. Such is St. Augustine's *"cogito, ergo sum."* Such a process should be called, not a reasoning but an *acritical inference*. Again, there are cases in which one belief is determined by another, without our being at all aware of it. These should be called *associational suggestions of belief*.

Now the theory of Pragmaticism was originally based, as anybody will see who examines the papers of Nov. 1877 and Jan. 1878,

[2] But see the experiments of J. Jastrow and me "On Slight Differences of Sensation" in the *Memoirs of the National Academy of Sciences*. Vol. III.

upon a study of that experience of the phenomena of self-control which is common to all grown men and women; and it seems evident that to some extent, at least, it must always be so based. For it is to conceptions of deliberate conduct that Pragmaticism would trace the intellectual purport of symbols; and deliberate conduct is self-controlled conduct. Now control may itself be controlled, criticism itself subjected to criticism; and ideally there is no obvious definite limit to the sequence. But if one seriously inquires whether it is possible that a completed series of actual efforts should have been endless or beginningless, (I will spare the reader the discussion), I think he can only conclude that (with some vagueness as to what constitutes an effort) this must be regarded as impossible. It will be found to follow that there are, besides perceptual judgments, original (i. e. indubitable because uncriticized) beliefs of a general and recurrent kind, as well as indubitable acritical inferences.

It is important for the reader to satisfy himself that genuine doubt always has an external origin, usually from surprise; and that it is as impossible for a man to create in himself a genuine doubt by such an act of the will as would suffice to imagine the condition of a mathematical theorem, as it would be for him to give himself a genuine surprise by a simple act of the will.

I beg my reader also to believe that it would be impossible for me to put into these articles over two per cent. of the pertinent thought which would be necessary in order to present the subject as I have worked it out. I can only make a small selection of what it seems most desirable to submit to his judgment. Not only must all steps be omitted which he can be expected to supply for himself, but unfortunately much more that may cause him difficulty.

Character II. I do not remember that any of the old Scotch philosophers ever undertook to draw up a complete list of the original beliefs, but they certainly thought it a feasible thing, and that the list would hold good for the minds of all men from Adam down. For in those days Adam was an undoubted historical personage. Before any waft of the air of evolution had reached those coasts how could they think otherwise? When I first wrote, we were hardly orientated in the new ideas, and my impression was that the indubitable propositions changed with a thinking man from year to year. I made some studies preparatory to an investigation of the rapidity of these changes, but the matter was neglected, and it has been only during the last two years that I have completed a provisional inquiry which shows me that the changes are so slight from generation to generation, though not imperceptible even in that short period, that I thought to own my adhesion, under inevitable modification, to the opinion of that subtle but well-balanced intellect, Thomas Reid, in the matter of Common Sense (as well as in regard to immediate perception, along with Kant).[3]

Character III. The Scotch philosophers recognized that the original beliefs, and the same thing is at least equally true of the acritical inferences, were of the general nature of instincts. But little as we know about instincts, even now, we are much better acquainted with them than were the men of the XVIIIth century. We know, for example, that they can be somewhat modified in a very short time. The great facts have always been known; such as that instinct seldom errs, while reason goes wrong nearly half the time, if not more frequently. But one thing the Scotch failed to recognize is that the original beliefs only remain indubitable in their application to affairs that resemble those of a primitive mode of life. It is, for example, quite open to reasonable doubt whether the motions of electrons are confined to three dimensions, although it is good methodeutic to presume that they are until some evidence to the contrary is forthcoming. On the other hand, as soon as we find that a belief shows symptoms of being instinctive, although it may seem to be dubitable, we must suspect that experiment would show that it is not really so; for in our artificial life, especially in that of a student, no mistake is more likely than that of taking a paper-doubt for the genuine metal. Take, for example, the belief in the criminality of incest. Biology will doubtless testify that the practice is unadvisable; but surely nothing that it has to say could

[3] I wish I might hope, after finishing some more difficult work, to be able to resume this study and to go to the bottom of the subject, which needs the qualities of age and does not call upon the powers of youth. A great range of reading is necessary; for it is the belief men *betray* and not that which they *parade* which has to be studied.

warrant the intensity of our sentiment about it. When, however, we consider the thrill of horror which the idea excites in us, we find reason in that to consider it to be an instinct; and from that we may infer that if some rationalistic brother and sister were to marry, they would find that the conviction of horrible guilt could not be shaken off.

In contrast to this may be placed the belief that suicide is to be classed as murder. There are two pretty sure signs that this is not an instinctive belief. One is that it is substantially confined to the Christian world. The other is that when it comes to the point of actual self-debate, this belief seems to be completely expunged and ex-sponged from the mind. In reply to these powerful arguments, the main points urged are the authority of the fathers of the church and the undoubtedly intense instinctive clinging to life. The latter phenomenon is, however, entirely irrelevant. For though it is a wrench to part with life, which has its charms at the very worst, just as it is to part with a tooth, yet there is no *moral* element in it whatever. As to the Christian tradition, it may be explained by the circumstances of the early Church. For Christianity, the most terribly earnest and most intolerant of religions,—[See *The Book of Revelations of St. John the Divine,*]—and it remained so until diluted with civilization,—recognized no morality as worthy of an instant's consideration except Christian morality. Now the early Church had need of martyrs, i. e., witnesses, and if any man had done with life, it was abominable infidelity to leave it otherwise than as a witness to its power. This belief, then, should be set down as dubitable; and it will no sooner have been pronounced dubitable, than Reason will stamp it as false.

The Scotch School appear to have no such distinction, concerning the limitations of indubitability and the consequent limitations of the jurisdiction of original belief.

Character IV. By all odds, the most distinctive character of the Critical Common-sensist, in contrast to the old Scotch philosopher, lies in his insistence that the acritically indubitable is invariably vague.

Logicians have been at fault in giving Vagueness the go-by, so far as not even to analyze it. The present writer has done his best to work out the Stechiology (or Stoicheiology), Critic, and Methodeutic of the subject, but can here only give a definition or two with some proposals respecting terminology.

Accurate writers have apparently made a distinction between the *definite* and the *determinate*. A subject is *determinate* in respect to any character which inheres in it or is (universally and affirmatively) predicated of it, as well as in respect to the negative of such character, these being the very same respect. In all other respects it is *indeterminate*. The *definite* shall be defined presently. A sign (under which designation I place every kind of thought, and not alone external signs,) that is in any respect objectively indeterminate (i. e. whose object is undetermined by the sign itself) is objectively *general* in so far as it extends to the interpreter the privilege of carrying its determination further.[4] *Example*: "Man is mortal." To the question, What man? the reply is that the proposition explicitly leaves it to you to apply its assertion to what man or men you will. A sign that is objectively indeterminate in any respect is objectively *vague* in so far as it reserves further determination to be made in some other conceivable sign, or at least does not appoint the interpreter as its deputy in this office. *Example*: "A man whom I could mention seems to be a little conceited." The *suggestion* here is that the man in view is the person addressed; but the utterer does not authorize such an interpretation or *any* other application of what she says. She can still say, if she likes, that she does *not* mean the person addressed. Every utterance naturally leaves the right of further exposition in the utterer; and

[4] Hamilton and a few other logicians understood the subject of a universal proposition in the collective sense; but every person who is well-read in logic is familiar with many passages in which the leading logicians explain with an iteration that would be superfluous if all readers were intelligent, that such a subject is distributively not collectively general. A term denoting a collection is singular, and such a term is an "abstraction" or product of the operation of hypostatic abstraction as truly as is the name of the essence. "Mankind" is quite as much an abstraction and *ens rationis* as is "humanity." Indeed, every object of a conception is either a signate individual or some kind of indeterminate individual. Nouns in the plural are usually distributive and general; common nouns in the singular are usually indefinite.

therefore, in so far as a sign is indeterminate, it is vague, unless it is expressly or by a well-understood convention rendered general. Usually, an affirmative predication covers *generally* every essential character of the predicate, while a negative predication *vaguely* denies some essential character. In another sense, honest people, when not joking, intend to make the meaning of their words determinate, so that there shall be no latitude of interpretation at all. That is to say, the character of their meaning consists in the implications and non-implications of their words; and they intend to fix what is implied and what is not implied. They believe that they succeed in doing so, and if their chat is about the theory of numbers, perhaps they may. But the further their topics are from such presciss, or "abstract," subjects, the less possibility is there of such precision of speech. In so far as the implication is not determinate, it is usually left vague; but there are cases where an unwillingness to dwell on disagreeable subjects causes the utterer to leave the determination of the implication to the interpreter; as if one says, "That creature is filthy, in every sense of the term."

Perhaps a more scientific pair of definitions would be that anything is *general* in so far as the principle of excluded middle does not apply to it and is *vague* in so far as the principle of contradiction does not apply to it. Thus, although it is true that "Any proposition you please, *once you have determined its identity,* is either true or false"; yet *so long as it remains indeterminate and so without identity,* it need neither be true that any proposition you please is true, nor that any proposition you please is false. So likewise, while it is false that "A proposition *whose identity I have determined* is both true and false," yet until it is determinate, it may be true that a proposition is true and that a propostion is false.

In those respects in which a sign is not vague, it is said to be *definite,* and also with a slightly different mode of application, to be *precise,* a meaning probably due to *præcisus* having been applied to curt denials and refusals. It has been the well-established, ordinary sense of *precise* since the Plantagenets; and it were much to be desired that this word, with its derivatives *precision, precisive,* etc., should, in the dialect of philosophy, be restricted to this sense.

To express the act of *rendering precise* (though usually only in reference to numbers, dates, and the like,) the French have the verb *préciser,* which, after the analogy of *décider,* should have been *précider.* Would it not be a useful addition to our English terminology of logic, to adopt the verb *to precide,* to express the general sense, to render precise? Our older logicians with salutary boldness seem to have created for their service the verb *to prescind,* the corresponding Latin word meaning only to "cut off at the end," while the English word means to suppose without supposing some more or less determinately indicated accompaniment. In geometry, for example, we "prescind" shape from color, which is precisely the same thing as to "abstract" color from shape, although very many writers employ the verb "to abstract" so as to make it the equivalent of "prescind." But whether it was the invention or the courage of our philosophical ancestors which exhausted itself in the manufacture of the verb "prescind," the curious fact is that instead of forming from it the noun *prescission,* they took pattern from the French logicians in putting the word *precision* to this second use. About the same time[5] [See Watts. *Logick,* 1725, I, vi, 9 *ad fin.*] the adjective *precisive* was introduced to signify what *prescissive* would have more unmistakably conveyed. If we desire to rescue the good ship Philosophy for the service of Science from the hands of lawless rovers of the sea of literature, we shall do well to keep prescind, presciss, prescission, and prescissive on the one hand, to refer to dissection in hypothesis, while precide, precise, precision, and precisive are used so as to refer exclusively to an expression of determination which is made either full or free for the interpreter. We shall thus do much to relieve the stem "abstract" from staggering under the double burden of conveying the idea of prescission as well as the unrelated and very important idea of the creation of *ens rationis* out of an *ἔπος πτερόεν,*—to filch the phrase to furnish a name for an expression of non-substantive thought,—an opera-

[5] But unfortunately it has not been in the writer's power to consult the *Oxford Dictionary* concerning these words; so that probably some of the statements in the text might be corrected with the aid of that work.

tion that has been treated as a subject of ridicule,—this hypostatic abstraction,—but which gives mathematics half its power.

The purely formal conception that the three affections of terms, *determination, generality,* and *vagueness* form a group dividing a category of what Kant calls "functions of judgment" will be passed by as unimportant by those who have yet to learn how important a part purely formal conceptions may play in philosophy. Without stopping to discuss this, it may be pointed out that the "quantity" of propositions in logic, that is, the distribution of the *first* subject[6], is either *singular* (that is, determinate, which renders it substantially negligible in formal logic), or *universal* (that is, general), or *particular* (as the mediæval logicians say, that is, vague or *indefinite*). It is a curious fact that in the logic of relations it is the first and last quantifiers of a proposition that are of chief importance. To affirm of anything that it is a horse is to yield to it *every* essential character of a horse: to deny of anything that it is a horse is vaguely to refuse to it *some* one or more of those essential characters of the horse. There are, however, predicates that are unanalyzable in a given state of intelligence and experience. These are, therefore, determinately affirmed or denied. Thus, this same group of concepts reappears. Affirmation and denial are in themselves unaffected by these concepts, but it is to be remarked that there are cases in which we can have an apparently definite idea of a border line between affirmation and negation. Thus, a point of a surface may be in a region of that surface, or out of it, or on its boundary. This gives us an indirect and vague conception of an intermediary between affirmation and denial in general, and consequently of an intermediate, or nascent state, between determination and indetermination. There must be a similar intermediacy between generality and vagueness. Indeed, in an article in the seventh volume of

[6] Thus returning to the writer's original nomenclature, in despite of *Monist* VII, 209, where an obviously defective argument was regarded as sufficient to determine a mere matter of terminology. But the Quality of propositions is there regarded from a point of view which seems extrinsic. I have not had time, however, to re-explore all the ramifications of this difficult question by the aid of existential graphs, and the statement in the text about the last quantifier may need modification.

The Monist, pp. 205-217, there lies just beneath the surface of what is explicitly said, the idea of an endless series of such *intermediacies.* We shall find below some application for these reflections.

Character V. The Critical Common-sensist will be further distinguished from the old Scotch philosopher by the great value he attaches to doubt, provided only that it be the weighty and noble metal itself, and no counterfeit nor paper substitute. He is not content to ask himself whether he does doubt, but he invents a plan for attaining to doubt, elaborates it in detail, and then puts it into practice, although this may involve a solid month of hard work; and it is only after having gone through such an examination that he will pronounce a belief to be indubitable. Moreover, he fully acknowledges that even then it may be that some of his indubitable beliefs may be proved false.

The Critical Common-sensist holds that there is less danger to heuretic science in believing too little than in believing too much. Yet for all that, the consequences to heuretics of believing too little may be no less than disaster.

Character VI. Critical Common-sensism may fairly lay claim to this title for two sorts of reasons; namely, that on the one hand it subjects four opinions to rigid criticism: its own; that of the Scotch school; that of those who would base logic or metaphysics on psychology or any other special science, the least tenable of all the philosophical opinions that have any vogue; and that of Kant; while on the other hand it has besides some claim to be called Critical from the fact that it is but a modification of Kantism. The present writer was a pure Kantist until he was forced by successive steps into Pragmaticism. The Kantist has only to abjure from the bottom of his heart the proposition that a thing-in- itself can, however indirectly, be conceived; and then correct the details of Kant's doctrine accordingly, and he will find himself to have become a Critical Common-sensist.

Another doctrine which is involved in Pragmaticism as an essential consequence of it, but which the writer defended (*J. Spec. Phil.,* Vol. II, p. 155 *ad fin.* 1868, and *N. Am. Rev.,* Vol. CXIII, pp. 449-472, 1871), before he had formulated, even in his own mind, the principle of pragmaticism, is the scholastic doctrine of realism. This is usually defined as the opinion that there are real objects that are general, among the number being the modes of determination of existent singulars, if, indeed, these be not the only such objects. But the belief in this can hardly escape being accompanied by the acknowledgment that there are, besides, real *vagues*, and especially real possibilities. For possibility being the denial of a necessity, which is a kind of generality, is vague like any other contradiction of a general. Indeed, it is the reality of some possibilities that pragmaticism is most concerned to insist upon. The article of Jan. 1878 endeavored to gloze over this point as unsuited to the exoteric public addressed; or perhaps the writer wavered in his own mind. He said that if a diamond were to be formed in a bed of cotton-wool, and were to be consumed there without ever having been pressed upon by any hard edge or point, it would be merely a question of nomenclature whether that diamond should be said to have been hard or not. No doubt, this is true, except for the abominable falsehood in the word MERELY, implying that symbols are unreal. Nomenclature involves classification; and classification is true or false, and the generals to which it refers are either reals in the one case, or figments in the other. For if the reader will turn to the original maxim of pragmaticism at the beginning of this article, he will see that the question is, not what *did* happen, but whether it would have been well to engage in any line of conduct whose successful issue depended upon whether that diamond *would* resist an attempt to scratch it, or whether all other logical means of determining how it ought to be classed *would* lead to the conclusion which, to quote the very words of that article, would be "the belief which alone could be the result of investigation carried *sufficiently far.*" Pragmaticism makes the ultimate intellectual purport of what you please to consist in conceived conditional resolutions, or their substance; and therefore, the conditional propositions, with their hypothetical antecedents, in which such resolutions consist, being of the ultimate nature of meaning, must be capable of being true, that is, of expressing whatever there be which is such as the proposition expresses, independently of being thought to be so in any judgment, or being represented to be so in any other symbol of any man or men. But that amounts to saying that possibility is sometimes of a real kind.

Fully to understand this, it will be needful to analyze modality, and ascertain in what it consists. In the simplest case, the most subjective meaning, if a person does not know that a proposition is false, he calls it *possible.* If, however, he knows that it is *true,* it is much more than possible. Restricting the word to its characteristic applicability, a state of things has the Modality of the possible,—that is, of the merely possible,—only in case the contradictory state of things is likewise possible, which proves possibility to be the vague modality. One who knows that Harvard University has an office in State Street, Boston, and has impression that it is at No. 30, but yet suspects that 50 is the number, would say "I think it is at No. 30, but it *may be* at No. 50," or "it *is possibly* at No. 50." Thereupon, another, who does not doubt his recollection, might chime in, "It *actually is* at No. 50," or simply "it *is* at No. 50," or " it *is* at No. 50, *de inesse.*" Thereupon, the person who had first asked, what the number was might say, "Since you are so positive, it *must be* at No. 50," for "I know the first figure is 5. So, since you are both certain the second is a 0, why 50 it *necessarily is.*" That is to say, in this most subjective kind of Modality, that which is known by direct recollection is in the Mode of *Actuality,* the determinate mode. But when knowledge is indeterminate among alternatives, either there is one state of things which alone accords with them all, when this is in the Mode of *Necessity,* or there is more than one state of things that no knowledge excludes, when each of these is in the Mode of *Possibility.*

Other kinds of subjective Modality refer to a Sign or Representamen which is assumed to be true, but which does not include the Utterer's (i. e. the speaker's, writer's, thinker's or other symbolizer's) total knowledge, the different Modes being distinguished very much as above. There are other cases, however, in which, justifiably or not, we certainly think of Modality as objective. A man says, "I *can* go to the seashore if I like." Here is implied, to be sure, his ignorance of how he will decide to act. But this is not the point of the assertion. It is that the complete determination of conduct in the *act* not yet having taken place, the further determination of it belongs to the subject of the action regardless of external circumstances. If he had said, "I *must* go where my employers may send me," it would imply that the function of such further determination lay elsewhere. In "You *may* do so and so," and "You *must* do so," the "may" has the same force as "can," except that in the one case freedom from particular circumstances is in question, and in the other freedom from a law or edict. Hence the phrase, "You *may* if you *can*." I must say that it is difficult for me to preserve my respect for the competence of a philosopher whose dull logic, not penetrating beneath the surface, leaves him to regard such phrases as misrepresentations of the truth. So an act of hypostatic abstraction which in itself is no violation of logic, however it may lend itself to a dress of superstition, may regard the collective tendencies to variableness in the world, under the name of Chance, as at one time having their way, and at another time overcome by the element of order; so that, for example, a superstitious cashier, impressed by a bad dream, may say to himself of a Monday morning, "*May be,* the bank has been robbed." No doubt, he recognizes his total ignorance in the matter. But besides that, he has in mind the absence of any particular cause which should protect his bank more than others that are robbed from time to time. He thinks of the variety in the universe as vaguely analogous to the indecision of a person, and borrows from that analogy the garb of his thought. At the other extreme stand those who declare as inspired, (for they have no rational proof of what they allege), that an actuary's advice to an insurance company is based on nothing at all but ignorance.

There is another example of objective possibility: "A pair of intersecting rays, i. e., unlimited straight lines conceived as movable objects, *can* (or *may*) move, without ceasing to intersect, so that one and the same hyperboloid shall be completely covered by the track of each of them." How shall we interpret this, remembering that the object spoken of, the pair of rays, is a pure creation of the Utterer's imagination, although it is required (and, indeed, forced) to conform to the laws of space? Some minds will be better satisfied with a more subjective, or nominalistic, others with a more objective, realistic interpretation. But it must be confessed on all hands that whatever degree or kind of reality belongs to pure space belongs to the substance of that proposition, which merely expresses a property of space.

Let us now take up the case of that diamond which, having been crystallized upon a cushion of jeweler's cotton, was accidentally consumed by fire before the crystal of corundum that had been sent for had had time to arrive, and indeed without being subjected to any other pressure than that of the atmosphere and its own weight. The question is, was that diamond *really* hard? It is certain that no discernible *actual* fact determined it to be so. But is its hardness not, nevertheless, a *real* fact? To say, as the article of Jan. 1878 seems to intend, that it is just as an arbitrary "usage of speech" chooses to arrange its thoughts, is as much as to decide against the reality of the property, since the real is that which is such as it is regardless of how it is, at any time, thought to be. Remember that this diamond's condition is not an isolated fact. There is no such thing; and an isolated fact could hardly be real. It is an unsevered, though presciss part of the unitary fact of nature. Being a diamond, it was a mass of pure carbon, in the form of a more or less transparent crystal, (brittle, and of facile octahedral cleavage, unless it was of an unheard of variety), which, if not trimmed after one of the fashions in which diamonds may be trimmed, took the shape of an octahedron, apparently regular (I need not go into minutiæ), with grooved edges, and probably with some curved faces. Without being subjected to any considerable pressure, it could be found to be insoluble, very highly refractive, showing under radium rays (and perhaps under "dark light" and X-rays) a peculiar bluish phosphorescence, having as high a specific gravity as realgar or orpiment, and giving off during its combustion less heat than any other form of carbon would have done. From some of these properties hardness is believed to be inseparable. For like it they bespeak the high polemerization of the molecule. But however this may be, how can the hardness of all other diamonds fail to bespeak *some* real relation among the diamonds without which a piece of carbon would not be a diamond? Is it not a monstrous perversion of the word and concept *real* to say that the accident of the non-arrival of the corundum prevented the hardness of the diamond from having the *reality* which it otherwise, with little doubt, would have had?

At the same time, we must dismiss the idea that the occult state of things (be it a relation among atoms or something else), which constitutes the reality of a diamond's hardness can possibly consist in anything but in the truth of a general conditional proposition. For to what else does the entire teaching of chemistry relate except to the "behavior" of different possible kinds of material substance? And in what does that behavior consist except that if a substance of a certain kind should be exposed to an agency of a certain kind, a certain kind of sensible result *would* ensue, according to our experiences hitherto. As for the pragmaticist, it is precisely his position that nothing else than this can be so much as *meant* by saying that an object possesses a character. He is therefore obliged to subscribe to the doctrine of a real Modality, including real Necessity and real Possibility.

A good question, for the purpose of illustrating the nature of Pragmaticism, is, What is Time? It is not proposed to attack those most difficult problems connected with the psychology, the epistemology, or the metaphysics of Time, although it will be taken for granted, as it must be according to what has been said, that Time is real. The reader is only invited to the humbler question of what we mean by Time, and not of every kind of meaning attached to Past, Present, and Future either. Certain peculiar feelings are associated with the three general determinations of Time; but those are to be sedulously put out of view. That the reference of events to Time is irresistible will be recognized; but as to how it may differ from other kinds of irresistibility is a question not here to be considered. The question to be considered is simply, What is the intellectual purport of the Past, Present, and Future? It can only be treated with the utmost brevity.

That Time is a particular variety of objective Modality is too obvious for argumentation. The Past consists of the sum of *faits accomplis,* and this Accomplishment is the Existential Mode of Time. For the Past really acts upon us, and *that* it does, not at all in the way in which a Law or Principle influences us, but precisely as an Existent object acts. For instance, when a *Nova Stella* bursts out in the heavens, it acts upon one's eyes just as a light struck in the dark by one's own hands would; and yet it is an event which happened before the Pyramids were built. A neophyte may remark that its reaching the eyes, which is all we know, happens but a fraction of a second before we know it. But a moment's consideration will show him that he is losing sight of the question, which is not whether the distant Past can act upon us *immediately,* but whether it acts upon us just as any Existent does. The instance adduced (certainly a commonplace enough fact), proves conclusively that the mode of the Past is that of Actuality. Nothing of the sort is true of the Future, to compass the understanding of which it is indispensable that the reader should divest himself of his Necessitarianism,—at best, but a scientific theory,—and return to the Common-sense State of Nature. Do you never say to yourself, "I *can* do this or that as well to-morrow as to-day"? Your Necessitarianism is a theoretical pseudo-belief,—a make-believe belief,—that such a sentence does not express the real truth. That is only to stick to proclaiming the unreality of that Time, of which you are invited, be it reality or figment, to consider the meaning. You need not fear to compromise your darling theory by looking out at its windows. Be it true in theory or not, the unsophisticated conception is that everything in the Future is either *destined,* i. e. necessitated already, or is *undecided,* the contingent future of Aristotle. In other words, it is not Actual, since it does not act except through the idea of it, that is, as a law acts; but is either Necessary or Possible, which are of the same mode since (as remarked above) Negation being outside the category of modality cannot produce a variation in Modality. As for the Present instant, it is so inscrutable that I wonder whether no sceptic has ever attacked its reality. I can fancy one of them dipping his pen in his blackest ink to commence the assault, and then suddenly reflecting that his entire life is in the Present,—the "living present," as we say, this instant when all hopes and fears concerning it come to their end, this Living Death in which we are born anew. It is plainly that Nascent State between the Determinate and the Indeterminate that was noticed above.

Pragmaticism consists in holding that the purport of any concept is its conceived bearing upon our conduct. How, then, does the Past bear upon conduct? The answer is self-evident: whenever we set out to do anything, we "go upon," we base our conduct on facts already known, and for these we can only draw upon our memory. It is true that we may institute a new investigation for the purpose; but its discoveries will only become applicable to conduct after they have been made and reduced to a memorial maxim. In short, the Past is the store-house of all our knowledge.

When we say that we know that some state of things exists, we mean that it used to exist, whether just long enough for the news to reach the brain and be retransmitted to tongue or pen, or longer ago. Thus, from whatever point of view we contemplate the Past, it appears as the Existential Mode of Time.

How does the Future bear upon conduct? The answer is that future facts are the only facts that we can, in a measure, control; and whatever there may be in the Future that is not amenable to control are the things that we *shall* be able to infer, or *should* be able to infer under favorable circumstances. There may be questions concerning which the pedulum of opinion never would cease to oscillate, however favorable circumstances may be. But if so, those questions are *ipso facto* not *real* questions, that is to say, are questions to which there is no true answer to be given. It is natural to use the future tense (and the conditional mood is but a mollified future) in drawing a conclusion or in stating a consequence. "If two unlimited straight lines in one plane and crossed by a third making the sum ... then these straight lines *will* meet on the side, etc." It cannot be denied that acritical inferences may refer to the Past in its capacity as past; but according to Pragmaticism, the conclusion of a Reasoning power must refer to the Future. For its meaning refers to conduct, and since it is a reasoned conclusion must refer to deliberate conduct, which is controllable conduct. But the only controllable conduct is Future conduct. As for that part of the Past that lies beyond memory, the Pragmaticist doctrine is that the meaning of its being believed to be in connection with the Past consists in the acceptance as truth of the conception that we ought to conduct ourselves according to it (like the meaning of any other belief). Thus, a belief that Christopher Columbus discovered America really refers to the future. It is more difficult, it must be confessed, to account for beliefs that rest upon the double evidence of feeble but direct memory and upon rational inference. The difficulty does not seem insuperable; but it must be passed by.

What is the bearing of the Present instant upon conduct?

Introspection is wholly a matter of inference. One is immediately conscious of his Feelings, no doubt; but not that they are feelings of an *ego*. The *self* is only inferred. There is no time in the Present for any inference at all, least of all for inference concerning that very instant. Consequently the present object must be an external object, if there be any objective reference in it. The attitude of the Present is either conative or perceptive. Supposing it to be perceptive, the perception must be immediately known as external,—not indeed in the sense in which a hallucination is *not* external, but in the sense of being present regardless of the perceiver's will or wish. Now this kind of externality is conative externality. Consequently, the attitude of the present instant (according to the testimony of Common Sense, which is plainly adopted throughout) can only be a Conative attitude. The consciousness of the present is then that of a struggle over what shall be; and thus we emerge from the study with a confirmed belief that it is the Nascent State of the Actual.

But how is Temporal Modality distinguished from other Objective Modality? Not by any general character since Time is unique and *sui generis*. In other words there is only one Time. Sufficient attention has hardly been called to the surpassing truth of this for Time as compared with its truth for Space. Time, therefore, can only be identified by brute compulsion. But we must not go further.

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tems of thought constructed by science, and even for those universes that will never be created. The eternal norm of the divine principle is actualized in nature, but it is above nature, because nature is, as it were, only a single instance of the infinite possibilities to which it applies or could apply. Thus God is not limited to the cosmic order of nature, but is supernatural in the literal sense of the word. No system of things or beings, either real or imaginary, could exist without him, except perhaps a chaos; yet even that would be so filled with self-contradictory inconsistencies or irrational unthinkables that (if it be a true chaos) it would be thinkable only as a vague idea; it could not stand, for it would be doomed to die even before an attempt were made to call it into existence.

There is no prophet that preaches the superpersonal God more plainly than mathematics, and the magic squares are like a magic mirror which reflects a ray of the symmetry of the divine norm immanent in all things, in the immeasurable immensity of the cosmos not less than in the mysterious depths of the human mind.

EDITOR.

MR. PETERSON'S PROPOSED DISCUSSION.

Very valuable ideas ofttimes appear so obvious, when once set forth, that high laudation of their inventors would invite ridicule. Such, we are told, was the notion that obseded C. Colombo, and such is Mr. Peterson's proposal to start in *The Monist* a discussion of philosophical terminology. It may be a very simple proposal, but nobody, as far as one careful reader of *The Monist* remembers, had made it before; and its utility to students of phenomenology, normative science, and metaphysics will have a high co-efficient in its proportionality to the advantage they take of it. Duty calls upon us to contribute, each one what he can that will be useful, whether in the way of question or in that of answer. It seems likely that in my life-time of study I may have learned something of the way to investigate questions such as Mr. Peterson puts; and if so, here is an opportunity to be of aid to other students.

Experience, the first term concerning which Mr. Peterson asks for light, is somewhat remarkable for having been employed as nearly as possible in the same sense from Polus the Acragentine (i. e. native of Girgenti) sophist down to Avenarius and Haeckel.

As my first step in investigating its meaning, I should look out its equivalent *empeiria*,[1] in Bonitz's *Index Aristotelicus.* For every serious student of philosophy ought to be able to read the common dialect of Greek at sight, and needs on his shelves the Berlin Aristotle, in the fifth volume of which is that index. On looking out *empeiria* there, what first strikes one is that it is not a very common word with Aristotle, nor yet an unusual one, since Bonitz cites something over a dozen passages in which it occurs. The first (*Post. Anal.* II, xix) runs: "From sense are engendered memories, and from multiplied memory of the same thing is engendered experience; for many memories make up a single experience." Waitz (*Organon,* II, 429,) has a minute note on this passage. Another passage to which the Index refers (*Nic. Ethics,* VI, viii,) is thus translated by Stewart in his valuable "Notes" on the work: "If we ask why a boy may be a mathematician, but cannot understand philosophy or natural science, we find that it is because the truths of mathematics are abstract" [a bad explanation but that does not affect the evidence as to the meaning of *empeiria,*] "whereas the principles of philosophy and natural science are reached through long *experience.* A boy does not realize the meaning of the principles of philosophy and natural science, but merely repeats by rote the formulae used to express them." In the *Politics* (A, xi) Aristotle remarks that theorizing is free, while experience is necessitated, and goes on to speak of experience with live stock, etc. In another place in the *Politics* (E, ix) he says that the military commander of greatest experience in strategy is to be preferred, even though his habit of peculation be known; while for the chief of police, or for a treasurer, experience is of no account in comparison with integrity. But the cynosural passage is the first chapter of Book A of the *Metaphysics;* and here he remarks (as he likewise does in the *Ethics,*) that experience is a knowledge (*gnosis*)[2] of singulars. Therein Aristotle's language differs from that of the Socrates of Plato, with whom *empeiria* is the skill that results from long dealings with any matter. Aristotle never intended to say that there is no other cognition of singulars than in experience; for that would directly contradict his doctrine that experience is a mass of memories relating to the same subject. His remark was, however, understood in the Middle Ages to be a *definition* of experience, and was repeated as such, a blunder that was not so unnatural as it would have been if the scholastic doctors had dealt

[1] ἐμπειρία. [2] γνῶσις.

with direct experience. The teachings of the *Aristotelic Index* having been exhausted, I turn to Harper's *Latin Lexicon,* which informs me that no writer of the Golden Age used *experientia* in the general sense, though that acception became common in the Silver Age, especially with Tacitus. The next work that I personally should consult would be my own notes collected during more than forty years. I always carry a pad of the size of a Post Card, of thick papers, (50 in a pad, enough to last for two days, at least); and on these I note whatever elements of experience may reach me. I keep these in drawers and boxes like the card catalogue of a library. I arrange and rearrange them from time to time. It is a treasure more valuable than a policy of insurance. I probably have near two hundred thousand such notes. But in order to bring what I have to say to a close, I will quote from the definition of experience given by the father of modern experiential philosophy, Dr. John Locke. In the *Essay concerning Humane*[3] *Understanding,* (II, i, 2) we read (and the italics are in the original): "Whence has [the mind] all the materials of reason and knowledge? To this I answer, in one word, from *experience*: in that our knowledge is founded, and from that it ultimately derives itself. Our observation employed either about *external sensible objects, or about the internal operations of our minds, perceived and reflected on by ourselves, is that which supplies our understanding with materials of thinking.*" This definition so formally stated, by such an authority, quite peerless for our present purpose, should be accepted as definite and as a landmark that it would be a crime to displace or disturb. For in order that philosophy should become a successful science, it must, like biology, have its own vocabulary; and as in biology, it must be the rule that whoever wishes to introduce a new concept is to invent a new word to express it. This is no suggestion of the moment. I am, for my humble part, maturely convinced that philosophy will never be upon the road to sound results until we dismiss our affection for old words and our dislike of newfangled words, and make its vocabulary over after the fashion of taxonomic zoölogy and botany. I limit my recommendation to technical terms; for I can pretend to no competence to give advice about *belles-lettres.* Yet even there I perceive that people read old authors, and admire them for saying what they never meant to say; because the modern readers forget that two or three centuries ago words still familiar

[3] *Humane* and *human* were one and the same word in Locke's day.

suggested quite different ideas from those the same words now suggest.

But somebody may object that Locke's definition is vague, being founded on a misconception of the nature of perception. Suppose, the objector will say, that a newborn male infant were to be brought up among a colony of men on a desert island, without ever having seen a woman and barely having heard of such a creature. Suppose that, arrived at the age of twenty, he were to meet on the beach a Pacific Island woman who had swum over from another island. Would not the irresistible, the only possible cognition he could have of this creature be strongly colored by his own instincts? It would be the ineluctable result of "*observation employed concerning an external sensible object.*" The word "experience," however, is employed by Locke chiefly to enable him to say that human cognitions are inscribed by the individual's life-history upon a *tabula rasa,* and are not, like those of the lower animals, gifts of inborn instinct. His definition is vague for the reason that he never realized how important the innate element of our directest perceptions really is.

To such an objector I might say, My dear fellow, you must be joking; for under the guise of an objection you reinforce what I was saying with a new argument for restricting the use of the word "experience" to the expression of that vague idea which Locke so well defines. You make it plain that a distinct word is wanted, or rather two distinct words, to express the two more precise concepts which you suggest. The idea of the word "experience," was to refer to that which is forced upon a man's recognition, will-he nill-he, and shapes his thoughts to something quite different from what they naturally would have been. But the philosophers of experience, like many of other schools, forget to how great a degree it is true that the universe is all of a piece, and that we are all of us natural products, naturally partaking of the characteristics that are found everywhere through nature. It is in some measure nonsensical to talk of a man's nature as opposed to what perceptions force him to think. True, man continually finds himself resisted, both in his active desires and in that passive inertia of thought which causes any new phenomenon to give him a shock of surprise. You may think of an element of knowledge which thus resists his superficial tendencies; but to express precisely that idea you must have a new word: it will not answer the purpose to call it *experience.* You may also reflect that every man's environment is in some measure unfavorable to his development; and so far as this affects his cognitive development, you have there an element that is opposed to the man's nature. But surely the word *experience* would be ill-chosen to express that.

But I am encroaching far too much upon the space of this number, and am taking too much advantage of our good editor's indulgence. I did wish to consider what element of his philosophy Comte had specially in mind in christening it *Positive*. He plainly meant that it should be unlike the metaphysical thought which kneads over and over what we know already, and would be like the sort of material which is furnished by a microscope or by an archæologist's spade. I hope Mr. Peterson's suggestion may bring a whole crop of fruit.

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PROLEGOMENA TO AN APOLOGY FOR PRAGMATICISM.

COME on, my Reader, and let us construct a diagram to illustrate the general course of thought; I mean a System of diagrammatization by means of which any course of thought can be represented with exactitude.

"But why do that, when the thought itself is present to us?" Such, substantially, has been the interrogative objection raised by more than one or two superior intelligences, among whom I single out an eminent and glorious General.

Recluse that I am, I was not ready with the counter-question, which should have run, "General, you make use of maps during a campaign, I believe. But why should you do so, when the country they represent is right there?" Thereupon, had he replied that he found details in the maps that were so far from being "right there," that they were within the enemy's lines, I ought to have pressed the question, "Am I right, then, in understanding that, if you were thoroughly and perfectly familiar with the country, as, for example, if it lay just about the scenes of your childhood, no map of it would then be of the smallest use to you in laying out your detailed plans?" To that he could only have rejoined, "No, I do not say that, since I might probably desire the maps to stick pins into, so as to mark each anticipated day's change in the situations of the two armies." To that again, my surrejoinder should have been, "Well, General, that precisely corresponds to the advantages of a diagram of the course of a discussion. Indeed, just there, where you have so clearly pointed it out, lies the advantage of diagrams in general. Namely, if I may try to state the matter after you, one can make exact experiments upon uniform diagrams; and when one does so, one must keep a bright lookout for unintended and unexpected changes thereby brought about in the relations of different significant parts of the diagram to one another. Such operations upon diagrams, whether external or imaginary, take the place of the experiments upon real things that one performs in chemical and physical research. Chemists have ere now, I need not say, described experimentation as the putting of questions to Nature. Just so, experiments upon diagrams are questions put to the Nature of the relations concerned." The General would here, may be, have suggested, (if I may emulate illustrious warriors in reviewing my encounters in afterthought,) that there is a good deal of difference between experiments like the chemist's, which are trials made upon the very substance whose behavior is in question, and experiments made upon diagrams, these latter having no physical connection with the things they represent. The proper response to that, and the only proper one, making a point that a novice in logic would be apt to miss, would be this: "You are entirely right in saying that the chemist experiments upon the very object of investigation, albeit, after the experiment is made, the particular sample he operated upon could very well be thrown away, as having no further interest. For it was not the particular sample that the chemist was investigating; it was the molecular *structure*. Now he was long ago in possession of overwhelming proof that all samples of the same molecular structure react chemically in exactly the same way; so that one sample is all one with another. But the object of the chemist's research, that upon which he experiments, and to which the question he puts to Nature relates, is the Molecular Structure, which in all his samples has as complete an identity as it is in the nature of Molecular Structure ever to possess. Accordingly, he does, as you say, experiment upon the Very Object under investigation. But if you stop a moment to consider it, you will acknowledge, I think, that you slipped in implying that it is otherwise with experiments made upon diagrams. For what is there the Object of Investigation? It is the *form of a relation.* Now this Form of Relation is the very form of the relation between the two corresponding parts of the diagram. For example, let f_1 and f_2 be the two distances of the two foci of a lens from the lens. Then,

$$\frac{1}{f_1}+\frac{1}{f_2}=\frac{1}{f_0}.$$

This equation is a diagram of the form of the relation between the two focal distances and the principal focal distance; and the conventions of algebra (and all diagrams, nay all pictures, depend upon conventions) in conjunction with the writing of the equation, establish a relation between the very *letters* f_1, f_2, f_0, regardless of their significance, the form of which relation is the *Very Same* as the form of the relation between the three focal distances that these letters denote. This is a truth quite beyond dispute. Thus, this algebraic Diagram presents to our observation the very, identical object of mathematical research, that is, the Form of the harmonic mean, which the equation aids one to study. [But do not let me be understood as saying that a Form possesses, itself, Identity in the strict sense; that is, what the logicians, translating ἀριθμῷ, call "numerical identity."]

Not only is it true that by experimentation upon some diagram an experimental proof can be obtained of every necessary conclusion from any given Copulate of Premisses, but, what is more, no "necessary" conclusion is any more apodictic than inductive reasoning becomes from the moment when experimentation can be multiplied *ad libitum* at no more cost than a summons before the imagination. I might furnish a regular proof of this, and am dissuaded from doing so now and here only by the exigency of space, the ineluctable length of the requisite explanations, and particularly by the present disposition of logicians to accept as sufficient F. A. Lange's persuasive and brilliant, albeit defective and in parts even erroneous, apology for it. Under these circumstances, I will content myself with a rapid sketch of my proof. First, an analysis of the essence of a sign, (stretching that word to its widest limits, as *anything which, being determined by an object. determines an interpretation to determination, through it. by the same object,*) leads to a proof that every sign is determined by its object, either first, by partaking in the characters of the object, when I call the sign an *Icon*; secondly, by being really and in its individual existence connected with the individual object, when I call the sign an *Index*; thirdly, by more or less approximate certainty that it will be interpreted as denoting the object, in consequence of a habit [which term I use as including a natural disposition], when I call the sign a *Symbol*.* I next examine into the different efficiencies and inefficiencies of these three kinds of signs in aiding the ascertainment of truth. A Symbol incorporates a habit, and is indispensable to the application of any *intellectual* habit, *at least*. Moreover, Symbols afford the means of thinking about thoughts in ways in which we could not otherwise think of them. They enable us, for example, to create Abstractions, without which we should lack

* In the original publication of this division, in 1867, the term "representamen" was employed in the sense of a sign in general, while "sign" was taken as a synonym of *index*, and an *Icon* was termed a "likeness."

a great engine of discovery. These enable us to count, they teach us that collections are individuals [individual = individual object], and in many respects they are the very warp of reason. But since symbols rest exclusively on habits already definitely formed but not furnishing any observation even of themselves, and since knowledge is habit, they do not enable us to add to our knowledge even so much as a necessary consequent, unless by means of a definite preformed habit. Indices, on the other hand, furnish positive assurance of the reality and the nearness of their Objects. But with the assurance there goes no insight into the nature of those Objects. The same Perceptible may, however, function doubly as a Sign. That footprint that Robinson Crusoe found in the sand, and which has been stamped in the granite of fame, was an Index to him that some creature was on his island, and at the same time, as a Symbol, called up the idea of a man. Each *Icon* partakes of some more or less overt character of its object. They, one and all, partake of the most overt character of all lies and deceptions, — their Overtness. Yet they have more to do with the living character of truth than have either Symbols or Indices. The Icon does not stand unequivocally for this or that existing thing, as the Index does. Its Object may be a pure fiction, as to its existence. Much less is its Object necessarily a thing of a sort habitually met with. But there is one assurance that the Icon does afford in the highest degree. Namely, that which is displayed before the mind's gaze,—the Form of the Icon, which is also its object,—must be *logically possible*. This division of Signs is only one of ten different divisions of Signs which I have found it necessary more especially to study. I do not say that they are all satisfactorily definite in my mind. They seem to be all trichotomies, which form an attribute to the essentially triadic nature of a Sign. I mean because three things are concerned in the functioning of a Sign; the Sign itself, its Object, and its Interpretant. I cannot discuss all these divisions in this article; and it can well be believed that the whole nature of reasoning cannot be fully exposed from the consideration of one point of view among ten. That which we can learn from this division is of what sort a Sign must be to represent the sort of Object that reasoning is concerned with. Now reasoning has to make its conclusion manifest. Therefore, it must be chiefly concerned with forms, which are the chief objects of rational insight. Accordingly, Icons are specially requisite for reasoning. A Diagram is mainly an Icon, and an Icon of intelligible relations. It is true that what must be is not to be learned by simple inspection of anything. But when we talk of deductive reasoning being necessary, we do not mean, of course, that it is infallible. But precisely what we do mean is that the conclusion follows from the form of the relations set forth in the premiss. Now since a diagram, though it will ordinarily have Symbolide Features, as well as features approaching the nature of Indices, is nevertheless in the main an Icon of the forms of relations in the constitution of its Object, the appropriateness of it for the representation of necessary inference is easily seen. But since you may, perhaps, be puzzled to understand how an Icon can exhibit a necessity—a Must-be,—I will here give, as an example of its doing so, my proof that the single members of no collection or plural, are as many as are the collections it includes, each reckoned as a single object, or, in other words, that there can be no relation in which every collection composed of members of a given collection should (taken collectively as a single object,) stand to some member of the latter collection to which no other such included collection of the following proposition, namely: that, taking any collection or plural, whatsoever, be it finite or infinite, and calling this the *given collection;* and considering all the collections, or plurals, each of which is composed of some of the individual members of the given collection (but including along with these *Nothing* which is to be here as a collection having no members at all; and also including the single members of the given collection, conceived as so many collections each of a single member), and calling these the *involved collections;* the proposition is that there is no possible relation in which each involved collection, (considered as a single object,) stands to a member of the given collection, without any other of the involved collections standing in the same relation to that same member of the given collection so stands. This purely symbolic statement can be rendered much more perspicuous by the introduction of Indices, as follows. The proposition is that no matter what collection C may be, and no matter what relation R may be, there must be some collection, *c′*, composed exclusively of members of C, which does not stand in the relation R to any member, *k*, of C, unless some other collection, *c″*, likewise composed of members of C, stands in the same relation R to the same *k*. The theorem is important in the doctrine of multitude, since it is the same as to say that any collection, no matter how great, is less multitudinous than the collection of possible collections composed exclusively of members of it; although formerly this was assumed to be false of some infinite collections. The demonstration begins by insisting that, if the proposition be false, there must be some definite relation of which it is false. Assume, then, that the letter *R* is an index of any one such relation you please. Next divide the members of C into four classes as follows:

Class I is to consist of all those members of C (if there be any such) to each of which no collection of members of C stands in the relation *R*.

Class II is to consist of all those members of C to each of which one and only one collection of members of C stands in the relation *R*; and this class has two subclasses, as follows:

Sub-Class 1 is to consist of whatever members of Class II there may be each of which is contained in that one collection of members of C that is in the relation, *R*, to it.

Sub-Class 2 is to consist of whatever members of Class II there may be none of which is contained in that one collection of members of C that is in the relation *R* to it.

Class III is to consist of all those members of C, if there be any such, to each of which more than one collection of members of C are in the relation *R*.

This division is complete; but everybody would consider the easy diagrammatical proof that it is so as needless to the point of nonsense, implicitly relying on a Symbol in his memory which assures him that every Division of such construction is complete.

I ought already to have mentioned that, throughout the enunciation and demonstration of the proposition to be proved, the term "collection included in the given collection" is to be taken in a peculiar sense to be presently defined. It follows that there is one "possible collection" that is included in every other, that is, which excludes whatever any other excludes. Namely, this is the "possible collection" which includes only the Sphinxes, which is the same that includes only the Basilisks, and is identical with the "possible collection" of all the Centaurs, the unique and ubiquitous collection called "Nothing," which has no member at all. If you object to this use of the term "collection," you will please substitute for it, throughout the enunciation and the demonstration, any other designation of the same object. I prefix the adjective "possible," though I must confess it does not express my meaning, merely to indicate that I extend the term "collection" to Nothing, which, of course, has no existence. Were the suggested objection to be persisted in by those *soi-disant* reasoners who refuse to think at all about the object of this or that description, on the ground that it is "inconceivable," I should not stop to ask them how they could say that, when that involves thinking of it in the very same breath, but should simply say that for them it would be necessary to except collections consisting of single individuals. Some of these mighty intellects refuse to allow the use of any name to denote single individuals and also plural collections along with them; and for them the proposition ceases to be true of pairs. If they would not allow pairs to be denoted by any term that included all higher collections, the proposition would cease to be true of triplets and so on. In short, by restricting the meaning of "possible collection," the proposition may be rendered false of *small* collections. No general formal restriction can render it false of *greater* collections.

I shall now assume that you will permit me to use the term "possible collection" according to the following definition. A "possible collection" is an *ens rationis* of such a nature that the definite plural of any noun, or possible noun of definite signification, (as "the A's," "the B's," etc) denotes one, and only one, "possible collection" in any one perfectly definite state of the universe; and there is a certain relation between some "possible collections," expressed by saying that one "possible collection" *includes* another (or the same) "possible collection," and if, and only if, of two nouns one is universally and affirmatively predicable of the other in any one perfectly definite state of the universe, then the "possible collection" denoted by the definite plural of the former *includes* whatever "possible collection" is *included* by the "possible collection" denoted by the definite plural of the latter, and of any two different "possible collections," one or other must *include* something not *included by* the other.

A diagram of the definition of "possible collection" being compared with a diagram embracing whatever members of subclasses 1 and 2 that it may, excluding all the rest, will now assure us that any such aggregate is a possible collection of members of the class C, no matter what individuals of Classes I and III be included or excluded in the aggregate along with those members of Class II, if any there be in the aggregate.

We shall select, then, a single possible collection of members of C to which we give the proper name *c*, and this possible collection shall be one which contains no individual of Subclass 1, but contains whatever individual there may be of Subclass 2. We then ask whether or not it is true that *c* stands in the relation *R* to a member of C to which no other possible collection of members of C stands in the same relation; or, to put this question into a more convenient shape, we ask, Is there any member of the Class C to which *c* and no other possible collection of members of C stands in the relation *R*? If there be such a member or members of C, let us give one of them the proper name T. Then T must belong to one of our four divisions of this class. That is,

either T belongs to Class I, (but that cannot be since by the definition of Class I, to no member of this class is any possible collection of members of C in the relation *R*);

or T belongs to Subclass 1, (but that cannot be, since by the definition of that subclass, every member of it is a member of the only possible collection of members of C that is *R* to it, which possible collection cannot be *c*, because *c* is only known to us by a description which forbids its containing any member of Subclass 1. Now it is *c*, and *c* only, that is in the relation *R* to T);

or T belongs to Subclass 2, (but that cannot be, since by the definition of that subclass, no member of it is a member of the only possible collection of members of C that is *R* to it, which possible collection cannot be *c*, because the description by which alone *c* can be recognized makes it contain every member of Subclass 2. Now it is *c* only that is in the relation *R* to T);

or T belongs to Class III (but this cannot be, since to every member of that class, by the definition of it, more than one collection of members of C stand in the relation *R*, while to T only one collection, namely, *c*, stands in that relation).

Thus, T belongs to none of the classes of members of C, and consequently is not a member of C. Consequently, there is no such member of C; that is, no member of C to which *c*, and no other possible collection of members of C, stands in the relation *R*. But *c* is the proper name we were at liberty to give to whatever possible collection of members of C we pleased. Hence, there is no possible collection of members of C that stands in the relation *R* to a member of the class C to which no other possible collection of members of C stands in this relation *R*. But *R* is the name of *any* relation we please, and C is any class we please. It is, therefore, proved that no matter what class be chosen, or what relation be chosen, there will be some possible collection of members of that class (in the sense in which Nothing is such a collection) which does not stand in that relation to any member of that class to which no other such possible collection stands in the same relation.

When I was a boy, my logical bent caused me to take pleasure in tracing out upon a map of an imaginary labyrynth one path after another in hopes of finding my way to a central compartment. The operation we have just gone through is essentially of the same sort, and if we are to recognize the one as essentially performed by experimentation upon a diagram, so must we recognize that the other is performed. The demonstration just traced out brings home to us very strongly, also, the convenience of so constructing our diagram as to afford a clear view of the mode of connection of its parts, and of its composition at each stage of our operations upon it. Such convenience is obtained in the diagrams of algebra. In logic, however, the desirability of convenience in threading our way through complications is much less than in mathematics, while there is another desideratum which the mathematician as such does not feel. The mathematician wants to reach the conclusion, and his interest in the process is merely as a means to reach similar conclusions. The logician does not care what the result may be; his desire is to understand the nature of the process by which it is reached. The mathematician seeks the speediest and most abridged of secure methods; the logician wishes to make each smallest step of the process stand out distinctly, so that its nature may be understood. He wants his diagram to be, above all, as analytical as possible.

In view of this, I beg leave, Reader, as an Introduction to my defence of pragmatism, to bring before you a very simple system of diagrammatization of propositions which I term the System of Existential Graphs. For, by means of this, I shall be able almost immediately to deduce some important truths of logic, little understood hitherto, and closely connected with the truth of pragmaticism; while discussions of other points of logical doctrine, which concern pragmaticism but are not directly settled by this system, are nevertheless much facilitated by reference to it.

By a *graph,* (a word overworked of late years,) I, for my part, following my friends Clifford and Sylvester, the introducers of the term, understand in general a diagram composed principally of spots and of lines connecting certain of the spots. But I trust it will be pardoned to me that, when I am discussing Existential Graphs, without having the least business with other Graphs, I often omit the differentiating adjective and refer to an Existential Graph as a Graph simply. But you will ask, and I am plainly bound to say, precisely what kind of a Sign an Existential Graph, or as I abbreviate that phrase here, a *Graph*, is. In order to answer this I must make reference to two different ways of dividing all Signs. It is no slight task, when one sets out from none too clear a notion of what a Sign is,—and you will, I am sure, Reader, have noticed that my definition of a Sign is not convincingly distinct,—to establish a single vividly distinct division of all Signs. The one division which I have already given has cost more labor than I should care to confess. But I certainly could not tell you what sort of a Sign an Existential Graph is, without reference to two other divisions of Signs. It is true that one of these involves none but the most superficial considerations, while the other, though a hundredfold more difficult, resting as it must for a clear comprehension of it upon the profoundest secrets of the structure of Signs, yet happens to be extremely familiar to every student of logic. But I must remember, Reader, that your conceptions may penetrate far deeper than mine; and it is to be devoutly hoped they may. Consequently, I ought to give such hints as I conveniently can, of my notions of the structure of Signs, even if they are not strictly needed to express my notions of Existential Graphs.

I have already noted that a Sign has an Object and an Interpretant, the latter being that which the Sign produces in the Quasi-mind that is the Interpreter by determining the latter to a feeling, to an exertion, or to a Sign, which determination is the Interpretant. But it remains to point out that there are usually two Objects, and more than two Interpretants. Namely, we have to distinguish the Immediate Object, which is the Object as the Sign itself represents it, and whose Being is thus dependent upon the Representation of it in the Sign, from the Dynamical Object, which is the Reality which by some means contrives to determine the Sign to its Representation. In regard to the Interpretant we have equally to distinguish, in the first place, the Immediate Interpretant, which is the interpretant as it is revealed in the right understanding of the Sign itself, and is ordinarily called the *meaning* of the sign; while in the second place, we have to take note of the Dynamical Interpretant which is the actual effect which the Sign, as a Sign, really determines. Finally there is what I provisionally term the Final Interpretant, which refers to the manner in which the Sign tends to represent itself to be related to its Object. I confess that my own conception of this third interpretant is not yet quite free from mist. Of the ten divisions of signs which have seemed to me to call for my special study, six turn on the characters of an Interpretant and three on the characters of the Object. Thus the division into Icons, Indices, and Symbols depends upon the different possible relations of a Sign to its Dynamical Object. Only one division is concerned with the nature of the Sign itself, and this I now proceed to state.

A common mode of estimating the amount of matter in a MS. or printed book is to count the number of words.* There will ordinarily be about twenty *the*s on a page, and of course they count as twenty words. In another sense of the word "word," however, there is but one word "the" in the English language; and it is impossible that this word should lie visibly on a page or be heard in any voice,

* Dr. Edward Eggleston originated the method.

for the reason that it is not a Single thing or Single event. It does not exist; it only determines things that do exist. Such a definitely significant Form, I propose to term a *Type*. A Single event which happens once and whose identity is limited to that one happening or a Single object or thing which is in some single place at any one instant of time, such event or thing being significant only as occurring just when and where it does, such as this or that word on a single line of a single page of a single copy of a book, I will venture to call a *Token*. An indefinite significant character such as a tone of voice can neither be called a Type nor a Token. I propose to call such a Sign a *Tone*. In order that a Type may be used, it has to be embodied in a Token which shall be a sign of the Type, and thereby of the object the Type signifies. I propose to call such a Token of a Type an *Instance* of the Type. Thus, there may be twenty Instances of the Type "the" on a page. The term (Existential) *Graph* will be taken in the sense of a Type; and the act of embodying it in a *Graph-Instance* will be termed *scribing* the Graph (not the Instance), whether the Instance be written, drawn, or incised. A mere blank place is a Graph-Instance, and the Blank *per se* is a Graph; but I shall ask you to assume that it has the peculiarity that it cannot be abolished from any Area on which it is scribed, as long as that Area exists.

A familiar logical triplet is Term, Proposition, Argument. In order to make this a division of all signs, the first two members have to be much widened. By a *Seme*, I shall mean anything which serves for any purpose as a substitute for an object of which it is, in some sense, a representative or Sign. The logical Term, which is a class-name, is a Seme. Thus, the term "The mortality of man" is a Seme. By a *Pheme* I mean a Sign which is equivalent to a grammatical sentence, whether it be Interrogative, Imperative, or Assertory. In any case, such a

Sign is intended to have some sort of compulsive effect on the Interpreter of it. As the third member of the triplet, I sometimes use the word *Delome* (pronounce dee-loam, from δήλωμα), though *Argument* would answer well enough. It is a Sign which has the Form of tending to act upon the Interpreter through his own self-control, representing a process of change in thoughts or signs, as if to induce this change in the Interpreter.

A Graph is a Pheme, and in my use hitherto, at least, a Proposition. An Argument is represented by a series of Graphs.

The Immediate Object of all knowledge and all thought is, in the last analysis, the Percept. This doctrine in no wise conflicts with Pragmaticism, which holds that the Immediate Interpretant of all thought proper is Conduct. Nothing is more indispensable to a sound epistemology than a crystal-clear discrimination between the Object and the Interpretant of knowledge; very much as nothing is more indispensable to sound notions of geography than a crystal-clear discrimination between north latitude and south latitude; and the one discrimination is not more rudimentary than the other. That we are conscious of our Percepts is a theory that seems to me to be beyond dispute; but it is not a fact of Immediate Perception. A fact of Immediate Perception is not a Percept, nor any part of a Percept; a Percept is a Seme, while a fact of Immediate Perception or rather the Perceptual Judgment of which such fact is the Immediate Interpretant, is a Pheme that is the direct Dynamical Interpretant of the Percept, and of which the Percept is the Dynamical Object, and is with some considerable difficulty, (as the history of psychology shows,) distinguished from the Immediate Object, though the distinction is highly significant. But not to interrupt our train of thought, let us go on to note that while the Immediate Object of a Percept is excessively vague, yet natural thought makes up for that lack, (as it almost amounts to,) as follows. A late Dynamical Interpretant of the whole complex of Percepts is the Seme of a Perceptual Universe that is represented in instinctive thought as determining the original Immediate Object of every Percept. Of course, I must be understood as talking not psychology, but the logic of mental operations. Subsequent Interpretants furnish new Semes of Universes resulting from various adjunctions to the Perceptual Universe. They are, however, all of them, Interpretants of Percepts.

Finally, and in particular, we get a Seme of that highest of all Universes which is regarded as the Object of every true Proposition, and which, if we name it all, we call by the somewhat misleading title of "The Truth."

That said, let us go back and ask this question: How is it that the Percept, which is a Seme, has for its direct Dynamical Interpretant the Perceptual Judgment, which is a Pheme? For that is not the usual way with Semes, certainly. All the examples that happen to occur to me at this moment of such action of Semes are instances of Percepts, though doubtless there are others. Since not all Percepts act with equal energy in this way, the instances may be none the less instructive for being Percepts. However, Reader, I beg you will think this matter out for yourself, and then you can see,—I wish I could,—whether your independently formed opinion does not fall in with mine. My opinion is that a pure perceptual Icon,—and many really *great* psychologists have evidently thought that Perception is a passing of images before the mind's eye, much as if one were walking through a picture-gallery,—could not have a Pheme for its direct Dynamical Interpretant. I desire, for more than one reason, to tell you *why* I think so, although that you should to-day appreciate my reasons seems to be out of the question. Still, I wish you to understand me so far as to know that, mistaken though I be, I am not so sunk in intellectual night as to be dealing lightly with philosophic Truth when I aver that weighty reasons have moved me to the adoption of my opinion; and I am also anxious that it should be understood that those reasons have not been psychological at all, but are purely logical. My reason, then, briefly stated and abridged, is that it would be *illogical* for a pure Icon to have a Pheme for its Interpretant, and I hold it to be impossible for thought not subject to self-control, as a Perceptual Judgment manifestly is not, to be illogical. I dare say this reason may excite your derision or disgust, or both; and if it does, I think none the worse of your intelligence. You probably opine, in the first place, that there is no meaning in saying that thought which draws no Conclusion is illogical, and that, at any rate, there is no standard by which I can judge whether such thought is logical or not; and in the second place, you probably think that, if self-control has any essential and important relation to logic, which I guess you either deny or strongly doubt, it can only be that it is that which makes thought *logical*, or else which establishes the distinction between the logical and the illogical, and that in any event it has to be such as it is, and would be logical, or illogical, or both, or neither, whatever course it should take. But though an Interpretant is not necessarily a Conclusion, yet a Conclusion is necessarily an Interpretant. So that if an Interpretant is not subject to the rules of Conclusions there is nothing monstrous in my thinking it is subject to some generalization of such rules. For any evolution of thought, whether it leads to a Conclusion or not, there is a certain normal course, which is to be determined by considerations not in the least psychological, and which I wish to expound in my next article; and while I entirely agree, in opposition to distinguished logicians, that normality can be no criterion for what I call rationalistic reasoning, such as alone is admissible in science, yet it is precisely the criterion of instinctive or common-sense reasoning, which, within its own field, is much more trustworthy than rationalistic reasoning. In my opinion, it is self-control which makes any other than the normal course of thought possible, just as nothing else makes any other than the normal course of action possible; and just as it is precisely that that gives room for an ought-to-be of conduct, I mean Morality, so it equally gives room for an ought-to-be of thought, which is Right Reason; and where there is no self-control, nothing but the normal is possible. If your reflections have led you to a different conclusion from mine, I can still hope that when you come to read my next article, in which I shall endeavor to show what the forms of thought are, in general and in some detail, you may yet find that I have not missed the truth.

But supposing that I am right, as I probably shall be in the opinions of *some* readers, how then is the Perceptual Judgment to be explained? In reply, I note that a Percept cannot be dismissed at will, even from memory. Much less can a person prevent himself from perceiving that which, as we say, stares him in the face. Moreover, the evidence is overwhelming that the perceiver is aware of this compulsion upon him; and if I cannot say for certain how this knowledge comes to him, it is not that I cannot conceive how it could come to him, but that, there being several ways in which this might happen, it is difficult to say which of those ways actually is followed. But that discussion belongs to psychology; and I will not enter upon it. Suffice it to say that the perceiver is aware of being compelled to perceive what he perceives. Now existence means precisely the exercise of compulsion. Consequently, whatever feature of the percept is brought into relief by some association and thus attains a logical position like that of the observational premiss of an explaining Abduction,* the attribution of Existence to it in the Perceptual Judgment is virtually and in an extended sense, a logical Abductive Inference nearly approximating to necessary inference. But my next paper will throw a flood of light upon the logical affiliation of the Proposition and the Pheme generally, to coercion.

That conception of Aristotle which is embodied for us in the cognate origin of the terms *actuality* and *activity* is one of the most deeply illuminating products of Greek thinking. Activity implies a generalization of *effort;* and effort is a two-sided idea, effort and resistance being inseparable, and therefore the idea of Actuality has also a dyadic form.

No cognition and no Sign is absolutely precise, not even a Percept; and indefiniteness is of two kinds, indefiniteness as to what is the Object of the Sign, and indefiniteness as to its Interpretant, or indefiniteness in Breadth and in Depth. Indefiniteness in Breadth may be either Implicit or Explicit. What this means is best conveyed in an example. The word *donation* is indefinite as to who makes the gift, what he gives, and to whom he gives it. But it calls no attention, itself, to this indefiniteness. The word *gives* refers to the same sort of fact, but its meaning is such that that meaning is felt to be incomplete unless those items are, at least formally, specified; as they are in "Somebody gives something to some person (real or artificial)." An ordinary Proposition ingeniously contrives to convey novel information through Signs whose significance depends entirely on the interpreter's familiarity with them; and this it does by means of a "Predicate," i. e,, a term explicitly indefinite in breadth,

* Abduction, in the sense I give the word, is any reasoning of a large class of which the provisional adoption of an explanatory hypothesis is the type. But it includes processes of thought which lead only to the suggestion of questions to be considered, and includes much besides.

and defining its breadth by means of "Subjects," or terms whose breadths are somewhat definite, but whose informative depth (i. e., all the depth except an essential superficies) is indefinite, while conversely the depth of the Subjects is in a measure defined by the Predicate. A Predicate is either non-relative, or *a monad,* that is, is explicitly indefinite in one extensive respect, as is "black"; or it is a dyadic relative, or dyad, such as "kills," or it is a polyadic relative, such as "gives." These things must be diagrammatized in our system.

Something more needs to be added under the same head. You will observe that under the term "Subject" I include, not only the subject nominative, but also what the grammarians call the direct and the indirect object, together, in some cases, with nouns governed by prepositions. Yet there is a sense in which we can continue to say that a Proposition has but one Subject, for example, in the proposition, "Napoleon ceded Louisiana to the United States," we may regard as the Subject the ordered triplet, "Napoleon,—Louisiana,—the United States," and as the Predicate, "has for its first member, the agent, or party of the first part, for its second member the object, and for its third member the party of the second part of one and the same act of cession." The view that there are three subjects is, however, preferable for most purposes, in view of its being so much more analytical, as will soon appear.

All general, or definable, Words, whether in the sense of Types or of Tokens, are certainly Symbols. That is to say, they denote the objects that they do by virtue only of there being a habit that associates their signification with them. As to Proper Names, there might perhaps be a difference of opinion, especially if the Tokens are meant. But they should probably be regarded as Indices, since the actual connection (as we listen to talk,) of Instances of the same typical words with the same Objects, alone causes them to be interpreted as denoting those Objects. Excepting, if necessary, propositions in which all the subjects are such signs as these, no proposition can be expressed without the use of Indices.* If, for example, a man remarks, "Why, it is raining!" it is only by some such *circumstances* as that he is now standing here looking out at a window as he speaks, which would serve as an Index (not, however, as a Symbol,) that he is speaking of this place at this time, whereby we can be assured that he cannot be speaking of the weather on the satellite of Proeyon, fifty centuries ago. Nor are Symbols and Indices together generally enough. The arrangement of the words in the sentence, for instance, must serve as *Icons,* in order that the sentence may be understood. The chief need for the Icons is in order to show the Forms of the synthesis of the elements of thought. For in precision of speech, Icons can represent nothing but Forms and Feelings. That is why Diagrams are indispensable in all Mathematics, from Vulgar Arithmetic up, and in Logic are almost so. For Reasoning, nay, Logic generally, hinges entirely on Forms. You, Reader, will not need to be told that a regularly stated Syllogism is a Diagram; and if you take at random a half dozen out of the hundred odd logicians who plume themselves upon not belonging to the sect of Formal Logic, and if from this latter sect you take another half dozen at random, you will find that in proportion as the former avoid diagrams, they utilize the syntactical Form of their sentences. No pure Icons represent anything but Forms; no pure Forms are represented by anything but Icons. As for Indices, their utility especially shines where other Signs fail. Extreme precision being desired in the description of a red color, should I call it vermillion, I may be criti-

* Strictly pure Symbols can signify only things familiar, and those only in so far as they are familiar.

cized on the ground that vermillion differently prepared has quite different hues, and thus I may be driven to the use of the color-wheel, when I shall have to Indicate four disks individually, or I may say in what proportions light of a given wave-length is to be mixed with white light to produce the color I mean. The wave-length being stated in fractions of a micron, or millionth of a meter, is referred through an Index to two lines on an individual bar in the Pavillon de Breteuil, at a given temperature and under a pressure measured against gravity at a certain station and (strictly) at a given date, while the mixture with white, after white has been fixed by an Index of an individual light, will require at least one new Index. But of superior importance in Logic is the use of Indices to denote Categories and Universes,* which are classes that, being enormously large, very promiscuous, and known but in small part, cannot be satisfactorily defined, and therefore can only be denoted by Indices. Such, to give but a single instance, is the collection of all things in the Physical Universe. If anybody, your little son for example, who is such an assiduous researcher, always asking, What is the Truth, (*Τί ἐστιν ἀλήθεια;*) but like "jesting Pilate," will not always stay for an answer, should ask you what the Universe of things physical is, you may, if convenient, take him to the Rigi-Kulm, and about sunset, point out all that is to be seen of Mountains, Forests, Lakes, Castles, Towns, and then, as the stars come out, all there is to be seen in the heavens, and all that though not seen, is reasonably conjectured to be there; and then tell him, "Imagine that what is to be seen in a city back yard to grow to all you can see here, and then let this grow in the same proportion as many times as there are trees in sight from

* I use the term *Universe* in a sense which excludes many of the socalled "universes of discourse" of which Boole, De Morgan, and many subsequent logicians speak, but which, being perfectly definable, would in the present system be denoted by the aid of a graph.

here, and what you would finally have would be harder to find in the Universe than the finest needle in America's yearly crop of hay." But such methods are perfectly futile: Universes cannot be described.

Oh, I overhear what you are saying, O Reader: that a Universe and a Category are not at all the same thing; a Universe being a receptacle or class of Subjects, and a Category being a mode of Predication, or class of Predicates. I never said they were the same thing; but whether you describe the two correctly is a question for careful study.

Let us begin with the question of Univeres. It is rather a question of an advisable point of view than of the truth of a doctrine. A logical universe is, no doubt, a collection of *logical* subjects, but not necessarily of metaphysical Subjects, or "substances"; for it may be composed of characters, of elementary facts, etc. See my definition in Baldwin's Dictionary. Let us first try whether we may not assume that there is but one kind of Subjects which are either existing things or else quite fictitious. Let it be asserted that there is some married woman who will commit suicide in case her husband fails in business. Surely that is a very different proposition from the assertion that some married woman will commit suicide if all married men fail in business. Yet if nothing is real but existing things, then, since in the former proposition nothing whatever is said as to what the lady will or will not do if her husband does *not* fail in business, and since of a given married couple this can only be false if the fact is contrary to the assertion, it follows it can only be false if the husband *does* fail in business and if the wife then fails to commit suicide. But the proposition only says that there is *some* married couple of which the wife is of that temper. Consequently, there are only two ways in which the proposition can be false, namely, first, by there not being any married couple, and secondly, by *every* married man failing in business while *no* married woman commits suicide. Consequently, all that is required to make the proposition true is that there should either be some married man who does not fail in business, or else some married woman who commits suicide. That is, the proposition amounts merely to asserting that there is a married woman who will commit suicide if *every* married man fails in business. The equivalence of these two propositions is the absurd result of admitting no reality but existence. If, however, we suppose that to say that a woman will suicide if her husband fails, means that every *possible* course of events would either be one in which the husband would not fail or one in which the wife would commit suicide, then, to make that false it will not be requisite for the husband actually to fail, but it will suffice that there are *possible* circumstances under which he would fail, while yet his wife would not commit suicide. Now you will observe that there is a great difference between the two following propositions:

1st, There is some *one* married woman who under all possible conditions would commit suicide or else her husband would not have failed.

2nd, Under all possible circumstances there is some married woman *or other* who would commit suicide, or else her husband would not have failed.

The former of these is what is really meant by saying that there is some married woman who would commit suicide if her husband were to fail, while the latter is what the denial of any possible circumstances except those that really take place logically leads to interpreting, (or virtually interpreting,) the Proposition as asserting.

In other places, I have given many other reasons for my firm belief that there are real possibilities. I also think, however, that, in addition to actuality and possibility, a *third* mode of reality must be recognized in that which, as the gipsy fortune-tellers express it, is "sure to come true," or, as we may say, is *destined*,* although I do not mean to assert that this is affirmation rather than the negation of this Mode of Reality. I do not see by what confusion of thought anybody can persuade himself that he does not believe that to-morrow is destined to come. The point is that it is to-day really true that to-morrow the sun will rise; or that, even if it does not, the clocks or *something*, will go on. For if it be not real it can only be fiction: a Proposition is either True or False. But we are too apt to confound destiny with the impossibility of the opposite. I see no impossibility in the sudden stoppage of everything. In order to show the difference, I remind you that, "impossibility" is that which, for example, describes the mode of falsity of the idea that there should be a collection of objects so multitudinous that there would not be characters enough in the universe of characters to distinguish all those things from one another. Is there anything of that sort about the stoppage of all motion? There is, perhaps, a *law of nature* against it; but that is all. However, I will postpone the consideration of that point. Let us, at least, *provide* for such a mode of being in our system of diagrammatization, since it *may* turn out to be needed and, as I think, surely will.

I will proceed to explain why, although I am not prepared to deny that every proposition can be represented, and that I must say, for the most part very conveniently, under your view that the Universes are receptacles of the Subjects alone, I, nevertheless, cannot deem that mode of analyzing propositions to be satisfactory.

*I take it that anything may fairly be said to be *destined* which is sure to come about although there is no necessitating reason for it. Thus, a pair of dice, thrown often enough, will be sure to turn up sixes some time, although there is no necessity that they should. The probability that they will is 1 : that is all. *Fate* is that special kind of *destiny* by which events are supposed to be brought about *under definite circumstances* which involve no necessitating cause for those occurrences.

And to begin with, I trust you will all agree with me that no analysis, whether in logic, in chemistry, or in any other science, is satisfactory, unless it be thorough, that is, unless it separates the compound into components each entirely homogeneous in itself, and therefore free from the smallest admixture of any of the others. It follows that in the Proposition, "Some Jew is shrewd," the Predicate is " Jew-that-is-shrewd," and the Subject is *Something,* while in the proposition "Every Christian is meek," the Predicate is "Either not Christian or else meek," while the Subject is *Anything;* unless, indeed, we find reason to prefer to say that this Proposition means, "It is false to say that a person is Christian of whom it is false to say that he is meek." In this last mode of analysis, when a Singular Subject is not in question (which case will be examined later,) the only Subject is *Something.* Either of these two modes of analysis quite clear the Subject from any Predicative ingredients; and at first sight, either seems quite favorable to the view that it is only the Subjects which belong to the Universes. Let us, however, consider the following two forms of propositions:

1. Any adept alchemist could produce a philosopher's stone of some kind or other,
2. There is one kind of philosopher's stone that any adept alchemist could produce.

We can express these on the principle that the Universes are receptacles of Subjects as follows:

1. The Interpreter having selected any individual he likes, and called it A, an object B can be found, such that, Either A would not be an adept alchemist, or B would be a philosopher's stone of some kind, and A could produce B.
2. Something B might be found, such that, no matter what the Interpreter might select and call A, B would be a philosopher's stone of some kind, while

either A would not be an adept alchemist, or else A could produce B.

In these forms there are two Universes, the one of individuals selected at pleasure by the interpreter of the proposition, the other of suitable objects.

I will now express the same two propositions on the principle that each Universe consists, not of Subjects, but the one of True assertions, the other of False, but each to the effect that there is something of a given description.

1. This is false: That something, P, is an adept alchemist and that this is false, that while something, S, is a philosopher's stone of some kind, P could produce S.
2. This is true: That something, S, is a philosopher's stone of some kind; and this is false, that something, P, is an adept alchemist while this is false, that P could produce S.

Here, the whole proposition is mostly made up of the truth or falsity of assertions that a thing of this or that description exists, the only conjunction being "and." That this method is highly analytic is manifest. Now since our whole intention is to produce a method for the perfect analysis of propositions, the superiority of this method over the other for our purpose is undeniable. Moreover, in order to illustrate how that other might lead to false logic, I will tack the predicate of No. 2, in its objectionable form upon the subject of No. 1 in the same form, and *vice versa*. I shall thus obtain two propositions which that method represents as being as simple as are Nos. 1 and 2. We shall see whether they are so. Here they are:

3. The Interpreter having designated any object to be called A, an object B may be found such that

   B is a philosopher's stone of some kind, while either A is not an adept alchemist or else A could produce B.

4. Something, B, may be found, such that, no matter what the interpreter may select, and call A,
   Either A would not be an adept alchemist, or B would be a philosopher's stone of some kind, and A could produce B.

Proposition 3 may be expressed in ordinary language thus: There is a kind of philosopher's stone, and if there be any adept alchemist, he could produce a philosopher's stone of some kind. That is, No. 3 differs from No. 1 only in adding that there is a kind of philosopher's stone. It differs from No. 2 in not saying that any two adepts could produce the same kind of stone, (nor that any adept could produce any existing kind,) while No. 2 asserts that some kind is both existent and could be made by every adept.

Proposition 4, in ordinary language, is: If there be (or were) an adept alchemist, there is (or would be) a kind of philosopher's stone that any adept could produce. This asserts the substance of No. 2, but only conditionally upon the existence of an adept; but it asserts, what No. 1 does not, that all adepts could produce some one kind of stone, and this is precisely the difference between No. 4 and No. 1.

To me it seems plain that the propositions 3 and 4 are both less simple than No. 1 and less simple than No. 2, each adding some thing to one of the pair first given and asserting the other conditionally. Yet the method of treating the Universes as receptacles for the metaphysical Subjects only, involves as a consequence the representation of 3 and 4 as quite on a par with 1 and 2.

It remains to show that the other method does not carry this error with it. It is the states of things affirmed or denied that are contained in the universes, then, the propositions become as follows:

3. This is true: that there is a philosopher's stone of some kind, S, and that it is false that there is an

adept, A, and that it is false that A could produce a philosopher's stone of some kind, S'. [Where it is neither asserted nor denied that S and S' are the same, thus distinguishing this from 2.]

4. This is false: That there is an adept, A, and that this is false: That there is a stone of a kind, S, and this is false: That there is an adept, A', and that this is false: That A' could produce a stone of the kind S. [Where again it is neither asserted nor denied that A and A' are identical, but the point is that this proposition holds even if they are not identical, thus distinguishing this from 1.]

These forms exhibit the greater complexity of Propositions 3 and 4, by showing that they really relate to *three* individuals each; that is to say, 3 to two possible different kinds of stone, as well as to an adept; and 4 to two possible different adepts, and to a kind of stone. Indeed, the two forms of statement of 3 and 4 on the other theory of the universes are absolutely identical in meaning with the following different forms on the same theory. Now it is, to say the least, a serious fault in a method of analysis that it can yield two analyses so different of one and the same compound.

3. An object, B, can be found, such that whatever object the interpreter may select and call A, an object, B', can thereupon be found such that B is an existing kind of philosopher's stone, and either A would not be an adept or else B' is a kind of philosopher's stone such as A could produce.

4. Whatever individual the Interpreter may choose to call A, an object, B, may be found, such that whatever individual the Interpreter may choose to call A', Either A is not an adept or B is an existing kind of philosopher's stone, and either A' is not an adept or else A' could produce a stone of the kind B.

But while my forms are perfectly analytic, the need of diagrams to exhibit their meaning to the eye (better than merely giving a separate line to every proposition said to be false,) is painfully obtrusive.*

I will now say a few words about what you have called Categories, but for which I prefer the designation Predicaments, and which you have explained as predicates of predicates. That wonderful operation of hypostatic abstraction by which we seem to create *entia rationis* that are, nevertheless, sometimes real, furnish us the means of turning predicates from being signs that we think or think *through,* into being subjects thought of. We thus think of the thought-sign itself, making it the object of another thought-sign. Thereupon, we can repeat the operation of hypostatic abstraction, and from these second intentions derive third intentions. Does this series proceed endlessly? I think not. What then are the characters of its different members? My thoughts on this subject are not yet harvested. I will only say that the subject concerns Logic, but that the divisions so obtained must not be confounded with the different Modes of Being; Actuality, Possibility, Destiny [or Freedom from Destiny]. On the contrary, the succession of Predicates of Predicates is different in the different Modes of Being. Meantime, it will be proper that in our system of diagrammatization we should provide for the division, whenever needed, of each of our three Universes of modes of reality into *Realms* for the different Predicaments.

All the various meanings of the word "Mind," Logical, Metaphysical, and Psychological, are apt to be confounded more or less, partly because considerable logical acumen is required to distinguish some of them, and because of

* In correcting the proofs, a good while after the above was written, I am obliged to confess that in some places the reasoning is erroneous; and a much simpler argument would have supported the same conclusion more justly; though some weight ought to be accorded to my argument here, on the whole.

the lack of any machinery to support the thought in doing so, partly because they are so many, and partly because (owing to these causes,) they are all called by one word, "mind." In one of the narrowest and most concrete of its logical meanings, a Mind is that Seme of The Truth, whose determinations become Immediate Interpretants of all other Signs whose Dynamical Interpretants are dynamically connected. In our Diagram the same thing which represents The Truth must be regarded as in another way representing the Mind, and indeed, as being the Quasi-mind of all the Signs represented on the Diagram. For any set of Signs which are so connected that a complex of two of them can have one interpretant, must be Determinations of one Sign which is a *Quasi-mind.*

Thought is not necessarily connected with a brain. It appears in the work of bees, of crystals, and throughout the purely physical world; and one can no more deny that it is really there, than that the colors, the shapes, etc. of objects are really there. Consistently adhere to that unwarrantable denial, and you will be driven to some form of idealistic nominalism akin to Fichte's. Not only is thought in the organic world, but it develops there. But as there cannot be a General without Instances embodying it, so there cannot be thought without Signs. We must here give "Sign" a very wide sense, no doubt, but not too wide a sense to come within our definition. Admitting that connected Signs must have a Quasi-mind, it may further be declared that there can be no isolated sign. Moreover, signs require at least two Quasi-minds; a *Quasi-utterer* and a *Quasi-interpreter;* and although these two are at one (i. e. *are* one mind) in the sign itself, they must nevertheless be distinct. In the Sign they are, so to say, *welded.* Accordingly, it is not merely a fact of human Psychology, but a necessity of Logic, that every logical evolution of thought should be dialogic. You may say that all this is loose talk; and I admit that, as it stands, it has a large infusion of arbitrariness. It might be filled out with argument so as to remove the greater part of this fault; but in the first place, such an expansion would require a volume,—and an uninviting one; and in the second place. what I have been saying is only to be applied to a slight determination of our system of diagrammatization, which it will only slightly affect; so that, should it be incorrect, the utmost *certain* effect will be a danger that our system *may* not represent every variety of non-human thought.

There now seems to remain no reason why we should not proceed forthwith to formulate and agree upon

THE CONVENTIONS
Determining the Forms and Interpretations of
Existential Graphs.

Convention the First: Of the Agency of the Scripture. We are to imagine that two parties* collaborate in composing a Pheme, and in operating upon this so as to develop a Delome. [Provision shall be made in these Conventions for expressing every kind of Pheme as a Graph;† and it is certain that the Method could be applied to aid the development and analysis of any kind of purposive thought. But hitherto no Graphs have been studied but such as are Propositions; so that, in the resulting uncertainty as to what modifications of the Conventions might be required for other applications, they have mostly been here stated as if they were only applicable to the expression of Phemes and the working out of necessary conclusions.]

The two collaborating parties shall be called the *Graphist* and the *Interpreter.* The Graphist shall responsibly scribe each original Graph and each addition to it, with the proper indications of the Modality to be attached to

* They may be two bodies of persons, two persons, or two mental attitudes or states of one person.

† A *Graph* has already been defined on p. 503 *et seq.*

it the *relative Quality** of its position, and every particular of its dependence on and connections with other graphs. The Interpreter is to make such erasures and insertions of the Graph delivered to him by the Graphist as may accord with the *"General Permissions"* deducible from the Conventions and with his own purposes.

Convention the Second: Of the Matter of the Scripture, and the Modality of the Phemes expressed. The matter which the Graph-instances are to determine, and which thereby becomes the *Quasi-mind* in which the Graphist

* The traditional and ancient use of the term propositional *Quality* makes it an affair of the mode of expression solely. For "Socrates is mortal" and "Socrates is immortal" are equally Affirmative, "Socrates is not mortal" and "Socrates is not immortal" are equally Negative, provided "is not" translates *non est.* If, however, "is not" is in Latin *est non,* with no difference of meaning, the proposition is infinitated. Without anything but the merest verbiage to support the supposition that there is any corresponding distinction between different meanings of propositions, Kant insisted on raising the difference of expression to the dignity of a category. In *The Monist,* Vol. VII, p. 209, I gave some reason for considering a relative proposition to be affirmative or negative according as it does or does not unconditionally assert the existence of an indefinite subject. Although at the time of writing that, nine and a half years ago, I was constrained against my inclinations, to make that statement, yet I never heartily embraced that view, and dismissed it from my mind, until after I had drawn up the present statement of the Conventions of Existential Graphs, I found, quite to my surprise, that I had herein taken substantially the same view. That is to say, although I herein speak only of "relative" quality, calling the assertion of any proposition the Affirmation of it, and regarding the denial of it as an assertion *concerning* that proposition as subject, namely, that it is false; which is my distinction of Quality Relative to the proposition either itself Affirmed, or of which the falsity is affirmed, if the Relative Quality of it is Negative, yet since every Graph in itself either recognizes the existence of a familiar Singular subject or asserts something of an indefinite subject asserted to exist in some Universe, it follows that every relatively Affirmative Graph unconditionally asserts or recognizes the occurrence of some description of object in some Universe; while no relatively Negative Graph does this. The logic of a Limited Universe of Marks suggests a different view of Quality, but careful analysis shows that it is in no fundamental conflict with the above.

A question not altogether foreign to the subject of Quality is whether Quality and Modality are of the same general nature. In selecting a mode of representing Modality, which I have not done without much experimentation, I have finally resorted to one which commits itself as little as possible to any particular theory of the nature of Modality, although there are undeniable objections to such a course. If any particular analysis of Modality had appeared to me to be quite evident, I should have endeavored to exhibit it unequivocally. Meantime, my opinion is that the Universe is a Subject of every Proposition, and that any Modality shown by its indefiniteness to be Affirmative, such as Possibility and Intention, is a special determination of the Universe of The Truth. Something of this sort is seen in Negation. For if we say of a Man that he is not sinless, we represent the sinless as having a place only in an ideal universe which, or the part of which that contains the imagined sinless being, we then positively sever from the identity of the man in question.

and Interpreter are at one, being a Seme of *The Truth,* that is, of the widest Universe of Reality, and at the same time, a Pheme of all that is tacitly taken for granted between the Graphist and Interpreter, from the outset of their discussion, shall be a sheet, called the *Phemic Sheet,* upon which signs can be scribed, and from which any that are already scribed in any manner (even though they be

THE TINCTURES.

OF COLOR.

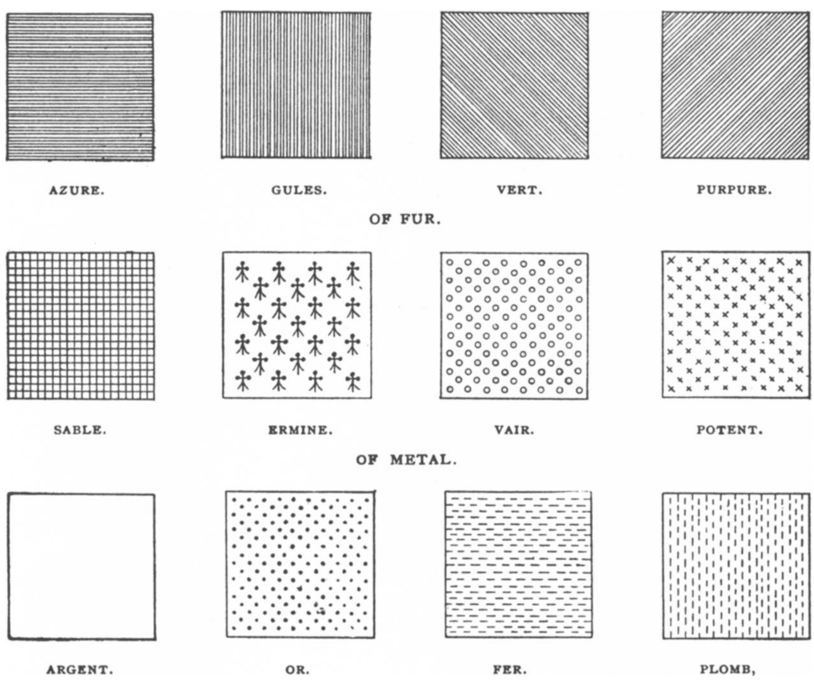

Fig. 1.

incised) *can* be erased. But certain parts of other sheets not having the significance of the Phemic sheet, but on which Graphs can be scribed and erased, shall be sometimes inserted in the Phemic sheet and exposed to view, as the Third Convention shall show. Every part of the exposed surface shall be tinctured in one or another of twelve tinctures. These are divided into three *classes* of four tinctures each, the class-characters being called *Modes of Tincture,* or severally, Color, Fur, and Metal. The tinctures of Colour are Azure, Gules, Vert, and Purpure. Those of Fur are Sable, Ermine, Vair, and Potent. Those of Metal are Argent, Or, Fer, and Plomb. The Tinctures will in practice be represented as in Fig. 1.* The whole of any *continuous* part of the exposed surface in one tincture shall be termed a *Province.* The border of the sheet has one tincture all round; and we may imagine that it was chosen from among twelve, in agreement between the Graphist and the Interpreter at the outset. The province of the border may be called the *March.* Provinces adjacent to the March are to be regarded as overlying it; Provinces adjacent to those Provinces, but not to the March, are to be regarded as overlying the provinces adjacent to the March, and so on. We are to imagine that the Graphist always finds provinces where he needs them.

When any representation of a state of things consisting in the applicability of a given description to an individual or limited set of individuals otherwise indesignate is scribed, the Mode of Tincture of the province on which it is scribed shows whether the Mode of Being which is to be affirmatively or negatively attributed to the state of things described is to be that of Possibility, when Color will be used; or that of Intention, indicated by Fur; or that of Actuality shown by Metal. Special understandings may determine special tinctures to refer to special varieties of the three genera of Modality. Finally, the Mode of Tincture of the March may determine whether the Entire Graph is to be understood as Interrogative, Imperative, or Indicative.

* It is chiefly for the sake of these convenient and familiar modes of representation of Petrosancta, that a modification of heraldic tinctures has been adopted. Vair and Potent here receive less decorative and pictorial Symbols. Fer and Plomb are selected to fill out the quaternion of metals on account of their monosyllabic names.

Convention the Third: Of Areas enclosed within, but severed from, the Phemic Sheet. The Phemic Sheet is to be imagined as lying on the smoother of the two surfaces or sides of a *Leaf,* this side being called the *recto,* and to consist of so much of this side as is continuous with the March. Other parts of the *recto* may be *exposed* to view. Every Graph-instance on the Phemic Sheet is posited unconditionally (unless, according to an agreement between Graphist and Interpreter, the Tincture of its own Province or of the March should indicate a condition); and every Graph-instance on the *recto* is posited affirmatively and, in so far as it is indeterminate, indefinitely.

Should the Graphist desire to negative a Graph, he must scribe it on the *verso,* and then, before delivery to the Interpreter, must make an incision, called a *Cut,* through the Sheet all the way round the Graph-instance to be denied, and must then turn over the excised piece, so as to *expose* its rougher surface carrying the negatived Graph-instance. This reversal of the piece is to be conceived to be an inseparable part of the operation of making a *Cut.** But if the Graph to be negatived includes a Cut, the twice negatived Graph within that Cut must be scribed on the *recto,* and so forth. The part of the exposed surface that is continuous with the part just outside the Cut is called the *Place of the Cut.* A Cut is neither a Graph nor a Graph-instance; but the Cut together with all that it encloses exposed is termed an *Enclosure,* and is conceived to be an Instance of a Graph *scribed* on the Place of the Cut, which is also termed the *Place of the Enclosure.* The surface within the Cut, continuous with the parts just within it, is termed the *Area* of the Cut and of the Enclosure; and the part of the *recto* continuous with the

* I am tempted to say that it is the reversal alone that effects the denial, the Cut merely cutting off the Graph within from assertion concerning the Universe to which the Phemic Sheet refers. But that is not the only possible view, and it would be rash to adopt it definitely, as yet.

March, (i. e., the Phemic Sheet,) is likewise termed an *Area*, namely the Area of the Border. The Copulate of all that is scribed on any one Area, including the Graphs of which the Enclosures whose Place is this Area are Instances, is called the *Entire Graph* of that Area; and any part of the Entire Graph, whether graphically connected with or disconnected from the other parts, provided it might be the Entire Graph of the Sheet, is termed a Partial Graph of the Area.

There may be any number of Cuts, one within another, the Area of one being the Place of the next, and since the Area of each is on the side of the leaf opposite to its Place, it follows that *recto* Areas may be *exposed* which are not parts of the Phemic Sheet. Every Graph-instance on a *recto* Area is affirmatively posited, but is posited conditionally upon whatever may be signified by the Graph on the Place of the Cut of which this Area is the Area. [It follows that Graphs on Areas of different Enclosures on a *verso* Place are only alternatively affirmed, and that while only the Entire Graph of the Area of an Enclosure on a *recto* Place is denied, but not its different Partial Graphs, except alternatively, the Entire Graphs of Areas of different Enclosures on one *recto* Place are copulatively denied.]

Every Graph-instance must lie upon one Area,* although an Enclosure may be a part of it. Graph-instances on different Areas are not to be considered as, nor by any permissible latitude of speech to be called, Parts of one Graph-instance, nor Instances of Parts of one Graph; for it is only Graph-instances on one Area that are called Parts of one Graph-instance, and that only of a Graph-instance

* For, of course, the Graph-instance must be on one sheet; and if part were on the *recto,* and part on the *verso,* it would not be on one continuous sheet. On the other hand, a Graph-instance can perfectly well extend from one Province to another, and even from one *Realm* (or space having one Mode of Tincture) to another. Thus, the Spot, "—is in the relation—to—," may, if the relation is that of an existent object to its purpose, have the first Peg on Metal, the second on Color, and the third on Fur.

on that same Area; for though the Entire Graph on the Area of an enclosure is termed the *Graph of the Enclosure,* it is no Part of the Enclosure and is connected with it only through a denial.

Convention the Fourth: concerning Signs of Individuals and of Individual Identity. A single dot, not too minute, or single congeries of contiguous pretty large dots, whether in the form of a line or surface, when placed on any exposed Area, will refer to a single member of the Universe to which the Tincture of that Area refers, but will not thereby be made to refer determinately to any one. But do not forget that separate dots, or separate aggregates of dots, will not necessarily denote different Objects.

By a *rheme,* or *predicate,* will here be meant a blank form of proposition which might have resulted by striking out certain parts of a proposition, and leaving a *blank* in the place of each, the parts stricken out being such that if each blank were filled with a proper name, a proposition (however nonsensical) would thereby be recomposed. An ordinary predicate of which no analysis is intended to be represented will usually be *written* in abbreviated form, but having a particular point on the periphery of the written form appropriated to each of the blanks that might be filled with a proper name. Such written form with the appropriated points shall be termed a *Spot;* and each appropriated point of its periphery shall be called a *Peg* of the Spot. If a heavy dot is placed at each Peg, the Spot will become a Graph expressing a proposition in which every blank is filled by a word (or concept) denoting an indefinite individual object, "something."

A heavy line shall be considered as a continuum of contiguous dots; and since contiguous dots denote a single individual, such a line without any point of branching will signify the identity of the individuals denoted by its extremities, and the type of such unbranching line shall be the Graph of Identity, any instance of which (on one area, as every Graph-instance must be,) shall be called a *Line of Identity.* The type of a three-way point of such a line (Fig. 2) shall be the *Graph of Teridentity;* and it shall be considered as composed of three contiguous Pegs of a Spot of Identity. An extremity of a Line of Identity not abutting upon another such Line in another area shall be called a *Loose End.* A heavy line, whether confined to one area or not (and therefore not generally being a Graph-instance,) of which two extremities abut upon pegs of spots shall be called a *Ligature.* Two lines cannot abut upon the same peg other than a point of teridentity. [The purpose of this rule is to force the recognition of the demonstrable logical truth that the concept of teridentity is not mere identity. It is identity *and* identity, but this

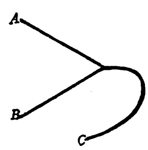

Fig. 2.

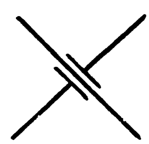

Fig. 3.

"and" is a distinct concept, and is precisely that of teridentity.] A Ligature crossing a Cut is to be interpreted as unchanged in meaning by erasing the part that crosses to the Cut and attaching to the two Loose Ends so produced two Instances of a Proper Name nowhere else used; such a Proper name (for which a capital letter will serve,) being termed a *Selective.* In the interpretation of Selectives it is often necessary to observe the rule which holds throughout the System, that the Interpretation of Existential Graphs must be *endoporeutic*, that is, the application of a Graph on the Area of a Cut will depend on the predetermination of the aplication of that which is on the Plate of the Cut.

In order to avoid the intersection of Lines of Identity, either a Selective may be employed, or a *Bridge,* which is imagined to be a bit of paper ribbon, but will in practice be pictured as in Fig. 3.

Convention the Fifth: Of the Connections of Graph-Instances. Two partial Graph-Instances are said to be *individually and directly connected,* if, and only if, in the Entire Graph, one individually is, either unconditionally or under some condition, and whether affirmatively or negatively, made a Subject of both. Two Graph-Instances connected by a ligature are explicitly and definitely individually and directly connected. Two Graph-Instances in the same Province are thereby explicitly, although indefinitely, individually and directly connected, since both, or one and the negative of the other, or the negative of both, are asserted to be true or false together, that is, under the same circumstances, although these circumstances are not formally defined, but are left to be interpreted according to the nature of the case. Two Graph-instances not in the same Province, though on the same Mode of Tincture are only in so far connected that both are in the same Universe. Two Graph-Instances in different Modes of Tincture are only in so far connected that both, or one and the negative of the other, or the negative of both, are posited as appertaining to the Truth. They cannot be said to have any individual and direct connection. Two Graph-instances that are not individually connected within the innermost Cut which contains them both cannot be so connected at all; and every ligature connecting them is meaningless and may be made or broken.

Relations which do not imply the occurrence in their several universes of all their correlates must not be expressed by Spots or single Graphs,* but all such relations can be expressed in the System.

* It is permissible to have such spots as "possesses the character," "is in the real relation to," but it is not permissible to have such a spot as "can prevent the existence of."

I will now proceed to give a few examples of Existential Graphs in order to illustrate the method of interpretation, and also the *Permissions of Illative Transformation* of them.

If you carefully examine the above conventions, you will find that they are simply the development, and excepting in their insignificant details, the inevitable result of the development of the one convention that if any Graph, A, asserts one state of things to be real and if another graph, B, asserts the same of another state of things, then AB, which results from setting both A and B upon the sheet, shall assert that both states of things are real. This was not the case with my first system of Graphs, described in Vol. VII of *The Monist,* which I now call *Entitative Graphs.* But I was forced to this principle by a series of considerations which ultimately arrayed themselves into an exact logical deduction of all the features of Existential Graphs which do not involve the Tinctures. I have no room for this here; but I state some of the points arrived at somewhat in the order in which they first presented themselves.

In the first place, the most perfectly analytical system of representing propositions must enable us to separate illative transformations into indecomposable parts. Hence, an illative transformation from any proposition, A, to any other, B, must in such a system consist in first transforming A into AB, followed by the transformation of AB into B. For an omission and an insertion appear to be indecomposable transformations and the only indecomposable transformations. That is, if A can be transformed by insertion into AB, and AB by omission in B, the transformation of A into B can be decomposed into an insertion and an omission. Accordingly, since logic has primarily in view argument, and since the conclusiveness of an argument can never be weakened by adding to the premisses nor by subtracting from the conclusion, I thought I ought to take the general form of argument as the basal form of composition of signs in my diagrammatization; and this necessarily took the form of a "scroll," that is (See Figs. 4, 5, 6) a curved line without contrary flexure and returning into itself after once crossing itself, and thus forming an outer and an inner "close." I shall call the outer boundary the *Wall;* and the inner, the *Fence*. In the outer I scribed the Antecedent, in the inner the Consequent, of a Conditional Proposition *de inesse*. The scroll was not taken for this purpose at hap-hazard, but was the result of experiments and reasonings by which I was brought to see that it afforded the most faithful Diagram of such a Proposition. This form once obtained, the logically inevitable

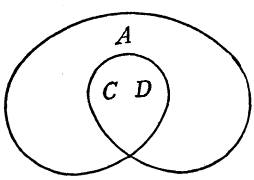

Fig. 4.

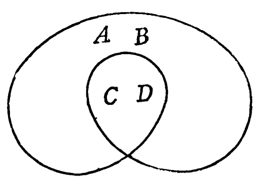

Fig. 5.

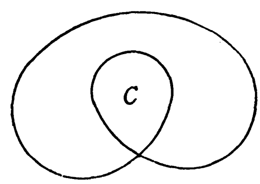

Fig. 6.

development brought me speedily to the System of Existential Graphs. Namely, the idea of the scroll was that Fig. 4, for example, should assert that if A be true (under the actual circumstances), then C and D are both true. This justifies Fig. 5, that if both A and B are true, then both C and D are true, no matter what B may assert, any insertion being permitted in the outer close, and any omission from the inner close. By applying the former clause of this rule to Fig. 6, we see that this scroll with the outer close void, justifies the assertion that if no matter what be true, C is in any case true; so that the two walls of the scroll, when nothing is between them, fall together, collapse, disappear, and leave only the contents of the inner close standing, asserted, in the open field. Supposing, then, that the contents of the inner scroll had been CD, these would have been left standing, both asserted; and we thus return to the principle that writing assertions together on the open sheet asserts them all. Now, Reader, if you will just take pencil and paper and scribe the scroll expressing that if A be true, then it is true that if B be true C and D are true, and compare this with Fig. 5, which amounts to the same thing in meaning, you will see that scroll walls with a void between them collapse even when they belong to different scrolls; and you will further see that a scroll is really nothing but one oval within another. Since a Conditional *de inesse* (unlike other conditionals,) only asserts that either the antecedent is false or the consequent is true, it all but follows that if

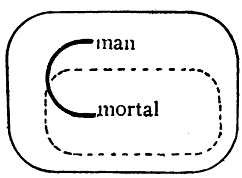

Fig .7.

Fig. 8.

the latter alternative be suppressed by scribing nothing but the antecedent, which may be any proposition, in an oval, that antecedent is thereby denied. The use of a heavy line as a juncture signifying identity is inevitable; and since Fig. 7 must mean that if anything is a man, it is mortal, it will follow that Fig. 8 must mean "Something is a man."

The first permission of illative transformation is now evident as follows:

First Permission, called "The Rule of Deletion and Insertion." Any Graph-Instance can be deleted from any recto *Area, (including the severing of any Line of Identity,) and any Graph-instance can be inserted on any* verso *Area, (including as a Graph-instance the juncture of any two Lines of Identity or Points of Teridentity.)*

The justice of the following will be seen instantly by students of any form of Logical Algebra, and with very little difficulty by others:

Second Permission, called "The Rule of Iteration and Deiteration." Any Graph scribed on any Area may be Iterated in or (if already Iterated,) may be Deiterated by a deletion from that Area or from any other Area included within that. This involves the Permission to distort a line of Identity, at will.

To *iterate* a Graph means to scribe it again, while joining by Ligatures every Peg of the new Instance to the corresponding Peg of the Original Instance. To *deiterate* a Graph is to erase a second Instance of it, of which each Peg is joined by a Ligature to a first Instance of it. One Area is said to be *included within* another if, and only if, it is the Area of a Cut whose Place either is that Area or else, is an Area which, according to this definition, must be regarded as *included within* that other. By this Permission, Fig. 9 may be transformed into Fig. 10, and thence, by Permission No. 1, into Fig. 11.

We now come to the Third Permission, which I shall state in a form which is valid, sufficient for its purpose, and convenient in practice, but which cannot be assumed as an undeduced Permission, for the reason that it allows us to regard the Inner Scroll, after the Scroll is removed, as being a part of the Area on which the Scroll lies. Now this is not strictly either an Insertion or a Deletion; and a perfectly analytical System of Permissions should permit only the indecomposable operations of Insertion and Deletion of Graphs that are simple in expression. The more scientific way would be to substitute for the Second and Third Permissions the following Permission:

If an Area, Υ*, and an Area,* Ω*, be related in any of these four ways,* viz., (1) *If* Υ *and* Ω *are the same Area;* (2) *If* Ω *is the Area of an Enclosure whose Place is* Υ; (3) *If* Ω *is* *the Area of an Enclosure whose Place is the Area of a second Enclosure whose Place is* ϒ; *or* (4) *If* Ω *is the Place of an Enclosure whose Area is vacant except that it is the Place of an Enclosure whose Area is* ϒ, *and except that it may contain ligatures, identifying Pegs in* Ω *with Pegs in* ϒ; *then, if* Ω *be a* recto *area, any simple Graph already scribed upon* ϒ *may be iterated upon* Ω; *while if* Ω *be a* verso *Area, any simple Graph already scribed upon* ϒ *and iterated upon* Ω *may be deiterated by being deleted or abolished from* Ω.

These two Rules (of Deletion and Insertion, and of Iteration and Deiteration) are substantially all the undeduced Permissions needed; the others being either Consequences or Explanations of these. Only, in order that this may be true, it is necessary to assume that all indemonstrable implications of the Blank have from the beginning been scribed upon distant parts of the Phemic Sheet, upon any part of which they may, therefore, be iterated at will. I will give no list of these implications, since it could serve no other purpose than that of warning beginners that necessary propositions not included therein were deducible from the other permissions. I will simply notice two principles the neglect of which might lead to difficulties. One of these is that it is physically impossible to delete or otherwise get rid of a Blank in any Area that contains a Blank, whether alone or along with other Graph-Instances. We may, however, assume that there is one Graph, and only one, an Instance of which entirely fills up an Area, without any Blank. The other principle is that, since a Dot merely asserts that some individual object exists, and is thus one of the implications of the Blank, it may be inserted in any Area; and since the Dot will signify the same thing whatever its size, it may be regarded as an Enclosure whose Area is filled with an Instance of that sole Graph that excludes the Blank. The

Dot, then, denies that Graph, which may, therefore, be understood as the absurd Graph, and its signification may be formulated as "Whatever you please is true." The absurd Graph may also take the form of an Enclosure with its Area entirely Blank, or enclosing only some Instance of a Graph implied in the Blank. These two principles will enable the Graphist to thread his way through some Transformations which might otherwise appear paradoxical and absurd.

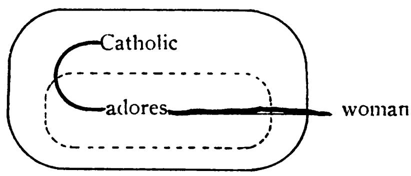

Fig. 9.

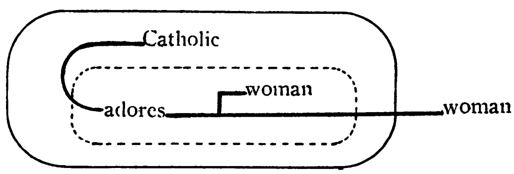

Fig. 10.

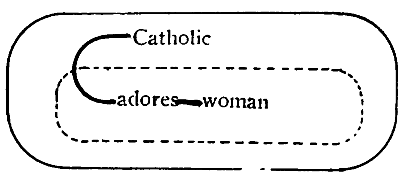

Fig. 11.

Third Permission; called "The Rule of the Double Cut." Two Cuts one within another, with nothing between them, unless it be Ligatures passing from outside the outer Cut to inside the inner one, may be made or abolished on any Area.

Let us now consider the Interpretation of such Ligatures. For that purpose, I first note that the Entire Graph of any *recto* Area is a wholly particular and affirmative

Proposition or Copulation of such Propositions. By "wholly particular," I mean, having for every Subject an indesignate individual. The Entire Graph of any *verso* Area is a wholly universal negative proposition or a disjunction of such propositions.

The first time one hears a Proper Name pronounced, it is but a name, predicated, as one usually gathers, of an existent, or at least historically existent, individual object, of which, or of whom, one almost always gathers some additional information. The next time one hears the name, it is by so much the more definite; and almost every time one hears the name, one gains in familiarity with the object. A Selective is a Proper Name met with by the Interpreter for the first time. But it always occurs twice, and usually on different areas. Now the Interpretation, by Convention No. 3, is to be Endoporeutic, so that it is the outermost occurrence of the Name that is the earliest.

Let us now analyze the interpretation of a Ligature passing through a Cut. Take, for example, the Graph of Fig. 12. The partial Graph on the Place of the Cut asserts that there exists an individual denoted by the extremity of the line of identity on the Cut, which is a millionaire..Call that individual C. Then, since contiguous dots denote the same individual objects, the extremity of the line of identity on the Area of the cut is also C, and the Partial Graph on that Area, asserts that, let the Interpreter choose whatever individual he will, that individual is either not C, or else is not unfortunate. Thus, the Entire Graph asserts that there exists a millionaire who is not unfortunate. Furthermore, the Enclosure lying in the same Argent Province as the "millionaire," it is asserted that this individual's being a millionaire *is connected* with his not being unfortunate. This example

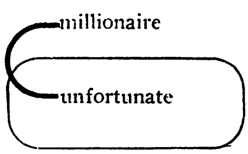

Fig. 12.

shows that the Graphist is permitted to extend any Line of Identity on a recto Area so as to carry an end of it to any Cut in that area. Let us next interpret Fig. 13. It obviously asserts that there exists a Turk who is at once the husband of an Individual denoted by a point on the Cut, which individual we may name U, and is the husband of an Individual, whom we may name V, denoted by another point on the Cut. And the Graph on the Area of the cut, declares that whatever Individual the Interpreter may select either is not, and cannot be, U or is not and cannot be V. Thus, the Entire Graph asserts that there is an existent Turk who is husband of two existent persons; and the "husband," the "Turk" and the enclosure, all being in the same Argent province, although the *Area* of the Enclosure is on color, and thus denies the *possibility* of the identity of U and V, all four predications are true *together,* that is, are true under the same circumstances, which circumstances should be defined by a special convention when anything may turn upon what they are. For the sake of illustrating this, I shall now scribe Fig. 14 all in one province. This may be read, "There is some married woman who will commit suicide in case her husband fails in business." This evidently goes far beyond saying that if every married man fails in business some married woman will commit suicide. Yet note that since the Graph is on Metal it asserts a conditional proposition *de inesse* and only means that there is a married woman whose husband does not fail or else she commits suicide. That, at least, is all it will seem to

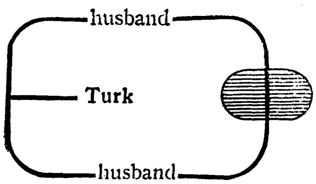

Fig. 13

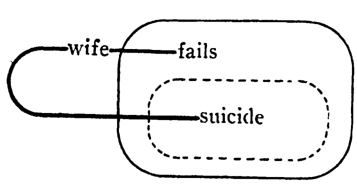

Fig. 14.

mean if we fail to take account of the fact, that being all in one Province, it is said that her suicide is *connected* with his failure. Neglecting that, the proposition only denies that every married man fails, while no married woman commits suicide. The logical principle is that to say that there is some one individual of which one or other of two predicates is true is no more than to say that there either is some individual of which one is true or else there is some individual of which the other is true. Or, to state the matter as an illative permission of the System of Existential Graphs,

Fourth Permission. If the smallest Cut which wholly contains a Ligature connecting two Graphs in different Provinces has its Area on the side of the Leaf opposite to that of the Area of the smallest Cut that contains those two Graphs, then such Ligature may be made or broken at pleasure, as far as these two Graphs are concerned.

Another somewhat curious problem concerning ligatures is to say by what principle it is true, as it evidently is true that the passage of ligatures from without the outer of two Cuts to within the inner of them will not prevent the two from collapsing in case there is no other Graph instance between them. A little study suffices to show that this may depend upon the ligatures' being replaceable by Selectives where they cross the Cuts, and that a Selective is always, at its first occurrence, a new predicate. For it is a principle of Logic that in introducing a new predicate one has a right to assert what one likes concerning it, without any restriction, as long as one implies no assertion concerning anything else. I will leave it to you, Reader, to find out how this principle accounts for the collapse of the two Cuts. Another solution of this problem, not depending on the superfluous device of Selectives is afforded by the second enunciation of the Rule of Iteration and Deiteration; since this permits the Graph of the Inner

Close to be at once iterated on the Phemic Sheet. One may choose between these two methods of solution.

The System of Existential Graphs which I have now sufficiently described,—or, at any rate, have described as well as I know how, leaving the further perfection of it to others,—greatly facilitates the solution of problems of Logic, as will be seen in the sequel, not by any mysterious properties, but simply by substituting for the symbols in which such problems present themselves, concrete visual figures concerning which we have merely to say whether

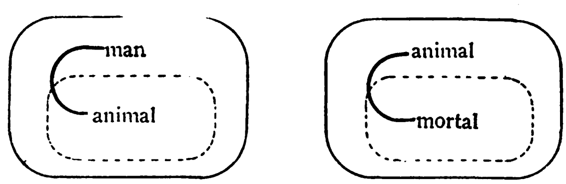

Fig. 15.

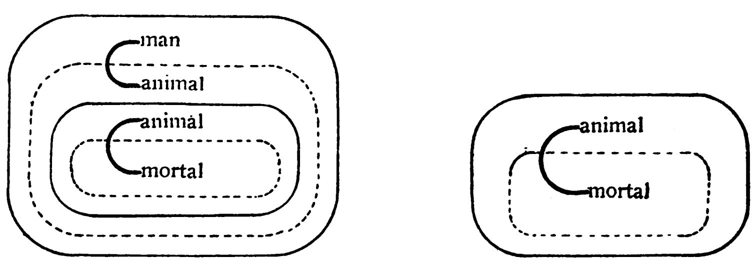

Fig. 16.

or not they admit certain describable relations of their parts. Diagrammatic reasoning is the only really fertile reasoning. If logicians would only embrace this method, we should no longer see attempts to base their science on the fragile foundations of metaphysics or a psychology not based on logical theory; and there would soon be such an advance in logic that every science would feel the benefit of it.

This System may, of course, be applied to the analysis of reasonings. Thus, to separate the syllogistic illation, "Any man would be an animal, and any animal would be mortal; therefore, any man would be mortal," the Premisses are first scribed as in Fig. 15. Then by the rule of Iteration, a first illative transformation gives Fig. 16. Next, by the permission to erase from a recto Area, a

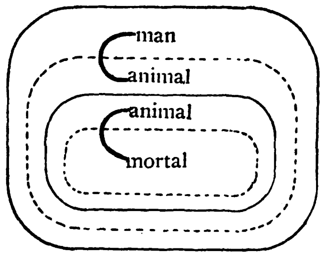

Fig. 17.

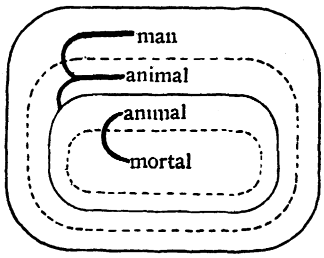

Fig. 18.

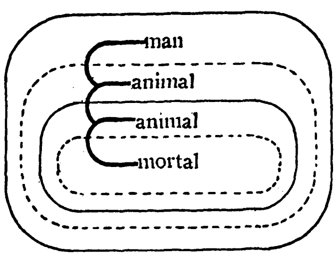

Fig. 19.

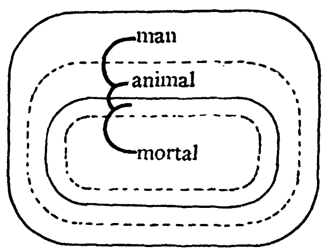

Fig. 20.

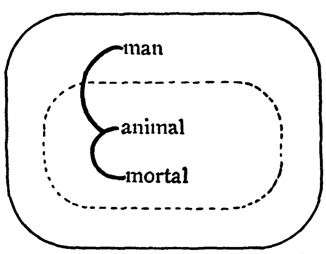

Fig. 21.

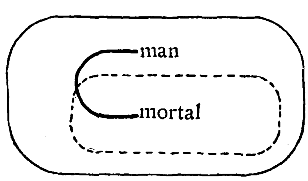

Fig. 22.

second step gives Fig. 17. Then, by the permission to deform a line of Identity on a recto Area, a third step gives Fig. 18. Next, by the permission to insert in a verso Area, a fourth step gives Fig. 19. Next, by Deiteration, a fifth step gives Fig. 20. Next, by the collapse of two Cuts, a sixth step gives Fig. 21; and finally, by omission from a recto Area, a seventh step gives the conclusion Fig. 22. The analysis might have been carried a little further, by means of the Rule of Iteration and Deiteration, so as to increase the number of distinct inferential steps to nine, showing how complex a process the drawing of a syllogistic conclusion really is. On the other hand, it need scarcely be said that there are a number of *deduced* liberties of transformation, by which even much more complicated inferences than a syllogism can be performed at a stroke. For that sort of problem, however, which consists in drawing a conclusion or assuring oneself of its correctness, this System is not particularly adapted. Its true utility is in the assistance it renders,—the support to the mind, by furnishing concrete diagrams upon which to experiment,—in the solution of the most difficult problems of logical theory.

I mentioned on an early page of this paper that this System leads to a different conception of the Proposition and Argument from the traditional view that a Proposition is composed of Names, and that an Argument is composed of Propositions. It is a matter of insignificant detail whether the term Argument be taken in the sense of the Middle Term, in that of the Copulate of Premisses, in that of the setting forth of Premisses and Conclusion, or in that of the representation that the real facts which the premisses assert (together, it may be, with the mode in which those facts have come to light) logically signify the truth of the Conclusion. In any case, when an Argument is brought before us, there is brought to our notice (what appears so clearly in the Illative Transformations of Graphs) a process whereby the Premisses bring forth the Conclusion, not informing the Interpreter of its Truth, but appealing to him to assent thereto. This Process of Transformation, which is evidently the kernel of the matter, is no more built out of Propositions than a motion is built out of positions. The logical relation of the Conclusion to the Premisses might be asserted; but that would not be an Argument, which is essentially intended to be understood as representing what it represents only in virtue of the logical habit which would bring any logical Interpreter to assent to it. We may express this by saying that the Final (or quasi-intended) Interpretant of an Argument represents it as representing its Object after the manner of a Symbol. In an analogous way the relation of Predicate to Subject which is *stated* in a Proposition might be merely described in a Term. But the essence of the Proposition is that it intends, as it were, to be regarded as in an existential relation to its Object, as an Index is, so that its assertion shall be regarded as evidence of the fact. It appears to me that an assertion and a command do not differ essentially in the nature of their Final Interpretants as in their Immediate, and so far as they are effective, in their Dynamical Interpretants; but that is of secondary interest. The Name, or any Seme, is merely a substitute for its Object in one or another capacity in which respect it is all one with the Object. Its Final Interpretant thus represents it as representing its Object after the manner of an Icon, by mere agreement in idea. It thus appears that the difference between the Term, the Proposition, and the Argument, is by no means a difference of complexity, and does not so much consist in structure as in the services they are severally intended to perform.

For that reason, the ways in which Terms and Arguments can be compounded cannot differ greatly from the ways in which Propositions can be compounded. A mystery, or paradox, has always overhung the question of the Composition of Concepts. Namely, if two concepts, A and B, are to be compounded, their composition would seem to be necessarily a third ingredient Concept, C, and the same difficulty will arise as to the Composition of A and C. But the Method of Existential Graphs solves this riddle instantly by showing that, as far as propositions go, and it must evidently be the same with Terms and Arguments, there is but one general way in which their Composition can possibly take place; namely, each component must be indeterminate in some respect or another; and in their composition each determines the other. On the *recto* this is obvious: "Some man is rich" is composed of "Something is a man" and "something is rich," and the two somethings merely explain each other's vagueness in a measure. Two simultaneous independent assertions are still connected in the same manner; for each is in itself vague as to the Universe or the "Province" in which its truth lies, and the two somewhat define each other in this respect. The composition of a Conditional Proposition is to be explained in the same way. The Antecedent is a Sign which is Indefinite as to its Interpretant; the Consequent is a Sign which is Indefinite as to its Object. They supply each the other's lack. Of course, the explanation of the structure of the Conditional gives the explanation of negation; for the negative is simply that from whose Truth it would be true to say that anything you please would follow *de inesse*.

In my next paper, the utility of this diagrammatization of thought in the discussion of the truth of Pragmaticism shall be made to appear.

CHARLES SANTIAGO SANDERS PEIRCE.

MILFORD, PA.

SOME AMAZING MAZES.

"Mazes intricate,
Eccentric, interwov'd, yet regular
Then most, when most irregular they seem."
Milton's Description of the Mystical Angelic Dance.

THE FIRST CURIOSITY.

ABOUT 1860 I cooked up a *mélange* of effects of most of the elementary principles of cyclic arithmetic; and ever since, at the end of some evening's card-play, I have occasionally exhibited it in the form of a "trick" (though there is really no trick about the phenomenon,) with the uniform result of interesting and surprising all the company, albeit their mathematical powers have ranged from a bare sufficiency for an altruistic tolerance of cards up to those of some of the mightiest mathematicians of the age, who assuredly with a little reflection could have unraveled the marvel.

The following shall describe what I do; but you, Reader, must do it too, if you are to appreciate the curiosity of the effect. So be good enough as to take two packets of playing-cards, the one consisting of a complete red suit and the other of a black suit without the king, the cards of each being arranged in regular order in the packet, so that the face-value of every card is equal to its ordinal number in the packet.

☞ N. B. *Throughout all my descriptions of manipulations of cards, it is to be understood, once for* *all, that the observance of the following* STANDING RULES *is taken for granted in all cases where the contrary is not expressly directed*: Firstly, *that a pack or packet of cards* held in the hand *is, unless otherwise directed, to be held with backs up (though not, of course while they are in process of arrangement or rearrangement,) while a pile of cards* FORMED *on the table (in contradistinction to a pile placed, ready formed, on the table, as well as to rows of single cards spread upon the table,) is always to be formed with the faces displayed, and left so until they are gathered up.* Secondly, *that, whether a packet in the hand or a pile on the table be referred to, by the "ordinal, or serial, number" of a single card or of a larger division of the whole is meant its number, counting in the order of succession in the packet or pile, from the card or other part at the* BACK *of the packet or at the* BOTTOM *of the pile as* "Number 1," *to the card or other part at the* FACE *of the packet or the* TOP *of the pile; the ordinal or serial number of this last being equal to the cardinal number of cards (or larger divisions* COUNTED,*) in the whole packet or pile; and the few exceptions to this rule will be noted as they occur;* Thirdly,*that by the "face-value" is meant the number of pips on a plain card, the ace counting as one; while, of the picture-cards, the knave, for which* J *will usually be written, will count as eleven, the queen, or* Q, *as twelve, and the king,* K, *either as thirteen or as the zero of the next suit;* and Fourthly, *that when a number of piles that have been formed upon the table by dealing out the cards, are to be gathered up, the uniform manner of doing so is to be as follows: The first pile to be taken (which pile this is to be will appear in due time,) is to be grasped as a whole and placed* (faces up,) *upon the pile that is to be taken next. Then those two piles are to be grasped as a whole, and placed* (faces up,) *upon the pile that is next to be taken; and so on, until all the* *piles have been gathered up; when, in accordance with the first Standing Rule, the whole packet is to be turned back up.* And note, by the way, that in consequence of the manner in which the piles are gathered, each, after the first, being placed at the back of those already taken, while in observance of the second Standing Rule, we always count places in a packet from the back of it, it follows that the last pile taken will be the first in the regathered packet, while the first taken will become the last, and all the others in the same complementary way, the ordinal numbers of their gathering and those of their places in the regathered packet adding up to one more than the total number of piles.

Of course, while the red packet and the black packet are getting arranged so that the face-value of each card shall also be its ordinal, or serial, number in the packet, the cards must needs be held faces up. But as soon as they have been arranged, the packet of thirteen cards is to be laid on the table, *back up.* You then deal,—for, let me repeat it, Reader, by the inexorable laws of psychology, if you do not actually take cards, (and the U. S. Playing-Card Company's "Fauntleroy" playing cards are the most suitable, although any that run smoothly will do,) and actually go through the processes, the whole description can mean nothing to you;—*you* deal, then, the twelve black cards, one by one, into two piles, the first card being turned to form the bottom of the first pile, the second that of the second pile (on the right hand of the first pile,) the third card going on the first pile again, the fourth on the second, and every following card being placed immediately upon the card whose ordinal, or serial, number in the packet before the deal was two lower than the former's ordinal, or serial, number then was. *The last card, however, is to be exceptionally treated.* Instead of being placed on the top of the second pile according to the rule just given, it is to be placed on the table, face up, and apart from the other cards, to make the bottom card of an isolated pile, to be called the *"discard pile"*; while, in place of it, the first card of the pile of cards of the red suit, which card will, of course, be the ace, is to be placed face up on the top of the second of the two piles formed by the dealing, where that discarded card would naturally have gone. Now you gather up these two piles by grasping the first, or left-hand, pile, placing it, face up, upon the second, or right-hand, pile, and taking up the two together; and you then at once turn the packet back up in compliance with the first standing rule. This whole operation of *firstly,* dealing out into two piles the packet that was at first entirely composed of black cards; but *secondly,* placing the last card, face up, on the discard pile, and *thirdly,* substituting for it the card then at the top of the pile of red cards, by placing this latter, face up, upon the top of the second pile of the deal, and then, *fourthly,* putting the left-hand, or first, pile of the deal, face up, upon the second, and having taken up the whole packet, turning it with its back up,—this whole quadripartite operation, I say, is to be performed, in all, twelve times in succession. My statement that in this operation the last card is treated *exceptionally* was quite correct, since its treatment made an exception to the rule of placing each card on the card that before the deal came two places in advance of it in the packet. Had I said it was treated *irregularly,* I should have written very carelessly, since it is just one of those cases in which a violation of a regularity of a low order establishes a regularity of a much higher order, (if John Milton knew the meaning of the word "regular,")—a pronouncement which must be left for the issue of the performance to ratify; and you shall see, Reader, that the event will ratify it with striking emphasis. Already, we begin to see some regularity in the process, since each of the twelve cards placed on the discard-pile in the twelve performances of the quadripartite operation is seen to belong to the black-suit; so that the packet held in the hand and dealt out, from being originally entirely black, has now become entirely red. Having placed the red king upon the face of this packet, you now lay down the latter in order to have your hands free to manipulate the discard-pile. Holding this discard-pile as the first standing rule directs, you take the cards singly from the top and range them, one by one, from left to right, in a row upon the table, with their backs up. The length of the table from left to right ought to be double that of the row; and this is one of the reasons for preferring cards of a small size. To guard against any mistake, you may take a peek at the seventh card, to make sure that it is the ace, as it should be. The row being formed, I remark to the company, as you should do in substance, that I reserve the right to move as many of these black cards as I please, at any and all times, from one end of the row to the other; but that beyond doing that, I renounce all right to disarrange those cards. Then, taking up the red cards, and holding the packet with its back up, I (and so must you,) request any person to cut it. When he does so, you place the cards he leaves in your hand at the back of the partial packet he removes. This is my proceeding, and must be yours. You then ask some person to say into how many piles (less than thirteen,) the red cards shall be dealt. When he has prescribed the number of piles, you are to hold the packet of red cards back up, and deal cards one by one from the back of it, placing each card on the table face up, and each to the right of the last card dealt. When you have dealt out enough to form the bottom cards of piles to the number commanded, you return to the extreme left-hand pile, *which you are to imagine as lying next to, and to the right of, the extreme right-hand pile*,—as in fact it would come next in clockwise order, if the row were bent down at the ends in the manner shown in Fig. 1, where the piles (here supposed to be eight in all,) are numbered in the order in which their bottom cards are laid down. Indeed, when more than seven piles are ordered, it is not a bad plan actually to arrange them so. So, counting the piles round and round, whether you place them in a circle or not, you place each card on the pile that comes clockwise next after, or to the right of the pile upon which the card next before it was placed (regulating your imagination as above stated,) and so you continue until you have dealt out the whole packet of thirteen cards. You now proceed to gather up the piles according to the Fourth Standing Rule.

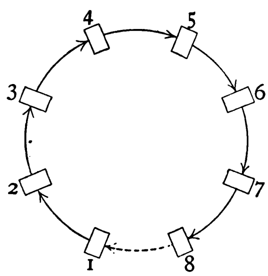

Fig. 1.

That rule, however, does not determine the order of succession in which the piles are to be taken up. I will now give the rule for this. It applies to the dealing of any prime number of cards, or of any number of cards one less than a prime number, into any number of piles less than that prime number. It happens that that form of statement of this rule which is decidedly the most convenient when the number of piles does not exceed seven, as well as when the whole number of cards differs by less than three from some multiple of the number of piles, becomes quite confusing in other cases. A slight modification of it which I will give as a second form of the rule, sometimes greatly mitigates the inconvenience; and it will be well to acquaint yourself with it. But for the most part, when the first form threatens to be confusing, it will be best to resort to that form of the rule which I describe as the third.

For the purpose of this "first curiosity" (indeed, in every case where a prime number of real cards are dealt out,) it matters not what pile you take up first. But in certain cases we shall have occasion to deal out into piles a number of cards, such as 52, which is one less than a prime number. In such case, it will be necessary to add *an imaginary card* to the pack, since a real card would interfere with certain operations. Now imaginary cards, if allowed to get mixed in with real ones, are liable to get lost. Consequently, in such cases, we have to keep the imaginary card constantly at the face of the pack by taking up first the pile on which it is imagined to fall, that is, the pile next to the right of the one on which the last real card falls. I now proceed to state, in its three forms, the rule for determining what pile is to be taken up next after any given pile that has just been taken. It is assumed that the whole pack of cards dealt consists of a prime number of cards; but, of these cards, the last may be an imaginary one, provided the pile on which it is imagined regularly to fall be taken up first.

First Form of the Rule. Count from the place of the extreme right-hand pile, as zero, either way round, clockwise or counterclockwise,—preferably in the shortest way,—to the place of the pile on which the last card, real or imaginary, fell. Then, counting the original places of piles, whether the piles themselves still remain in those places or have already been picked up, from the place of the pile last taken, in the same direction, up to the same number, you will reach the place of the next pile to be taken.

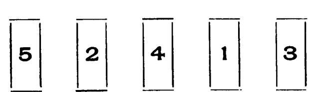

Fig. 2.

Example. If 13 cards are dealt into five piles, the 13th card will fall on the second pile from the extreme right-hand pile going round counter-clockwise. Supposing, then, that the first pile taken is the right-handmost but one, they are all to be taken in the order marked in Fig. 2.

Second Form. Proceed as in the first form of the rule until you have repassed the place of the first pile taken. You will then always find that the place of the last pile taken is nearer to that of some pile, P, previously taken, than it is to the place of that taken immediately before it. Then, the next pile to be taken will be in the same relation of places to the pile taken next after the pile P.

Example. Let 13 cards be dealt into 9 piles. Then the last card will fall on the pile removed 4 places clockwise from the extreme right-hand pile. Then, when you have removed four piles according to the first form of the rule, you will at once perceive, as shown in Fig. 3, (where

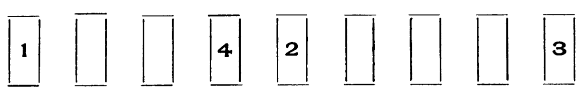

Fig. 3.

it is assumed that the extreme left-hand pile was the one to be taken up first,) that for the rest of the regathering, you have simply to take the pile that stands immediately to the left of the place of the last previous removal but one.

Third Form. In this form of the rule vacant places are not counted, but only the remaining piles, which is sometimes much less confusing. It is requisite, however, carefully to note the place of the pile first taken. You begin as in the first form of the rule; but every time you pass over the place whence the first pile was removed, you diminish the number of your count by one, beginning with the count then in progress; and you adhere to this number until you pass the same place again, and consequently again diminish the number of your count, which will thus ultimately be reduced to one, when you will take every pile you come to.

Example. Let a pack of 52 cards be dealt into 22 piles. The first pile taken up must be the one upon which the imaginary 53d card falls. It is assumed that, before the deal the cards were arranged in suits in the order ◇ ♠ ♡ ♣ and in each suit in the order of their face-values. Then the different columns of Fig. 4 show the cards at the tops of the different piles while the different horizontal rows

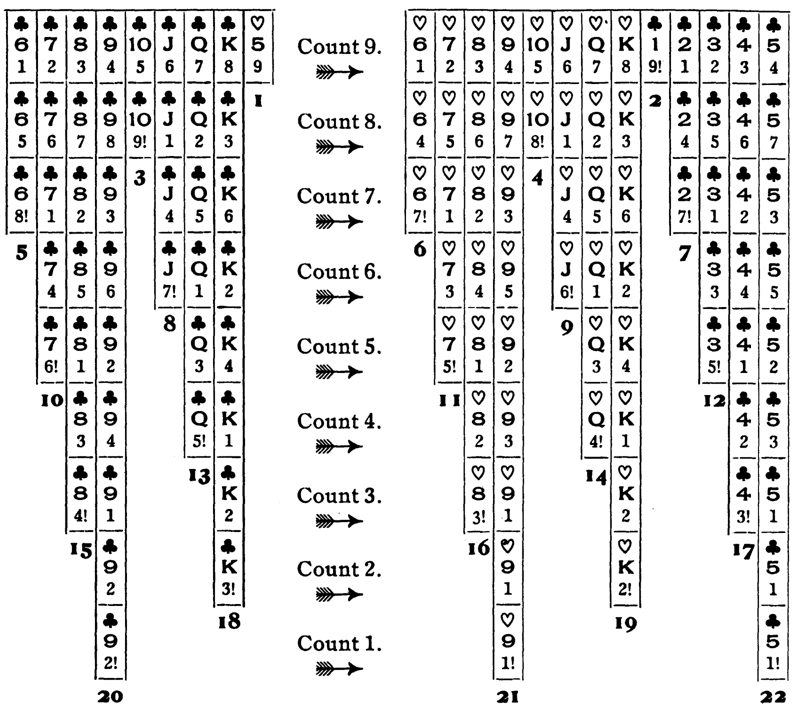

Fig. 4.

show what piles remain, just before you come to count the left-hand-most of the remaining piles, as your countings successively pass through the whole row of piles. The gap between the columns just after the place where the imaginary card is supposed to have fallen, contains the direction thereafter to diminish by one the number of piles you count. Beneath the designations of the top cards are small type numbers which are the numbers in your different countings through the row of piles; and the last number in each count is followed by a note of admiration that is to be understood as a command to gather up that pile. Beneath it is a heavy faced number, which is the ordinal number of that removal.

I hate to bore readers who are capable of exact thought with redundancies; but others often deploy such brilliant talents in not understanding the plainest statements that have no familiar jingle, that I must beg my more active-

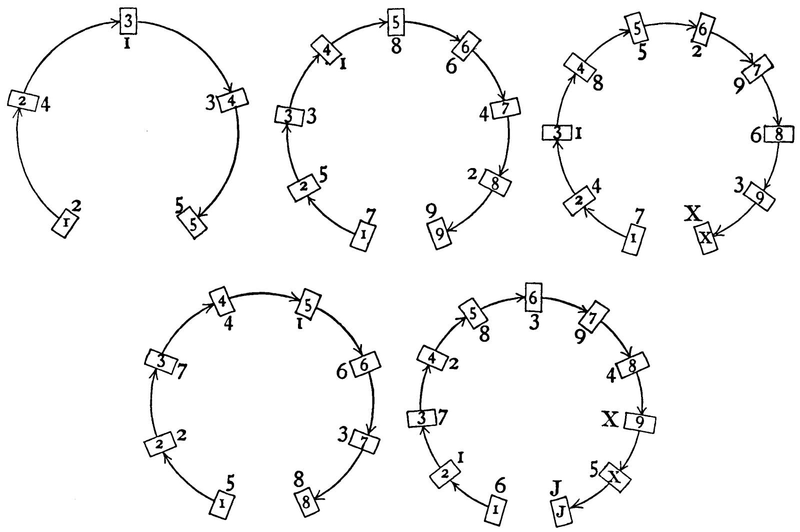

Fig. 5.

minded readers to have patience under the infliction while I exhibit in Fig. 5 the orders in which 5, 8, 9, 10, and 11 piles formed by dealing 13 cards are to be taken up.

When the red cards have thus been regathered, you again hold out the packet to somebody to cut, and again request somebody to say into how many piles they shall be dealt "in order that the mixing may be as thorough as it may." You follow his directions, and regather the piles according to the same rule as before. If your company is not too intelligent, you might venture to ask somebody, before you regather the piles, to say what pile you shall take up first; but this will be presuming a good deal upon the stupidity of the company; for an inference might be drawn which would go far toward destroying the surprise of the result. Nothing absolutely prevents the cards from being cut and dealt any number of times. When the number of piles for the last dealing has been given out, you will have to ascertain what transposition of the black cards is required. There are three alternative ways of doing this, which I proceed to describe.

The best way is to multiply together the numbers of piles of the different dealings of the red cards, subtracting from each product the highest multiple of 13, if there be any, that is less than that product. The result is the cyclical product. By "the different dealings," you here naturally understand those that have taken place since the last shifting of the black row. If a wrong shift has been made, the simplest way to correct it, after new cuttings and dealings, is to resort to a peep at the black ace, and to determining where it ought to be in the third way explained below.

Thus, if the red cards have been dealt into 5 piles and into 3 piles, since 3 times 5 make 15, and 15—13=2, the cyclical product is 2. You now proceed to ascertain how many times 1 has to be cyclically doubled to make that cyclical product. But if 6 doublings do not give it,—which six doublings will give

1 doubling, twice 1 are 2,
2 doublings, twice 2 are 4,
3 doublings, twice 4 are 8,
4 doublings, twice 8 less 13 make 3,
5 doublings, twice 3 are 6,
6 doublings, twice 6 are Q,—

I say if none of the first six doublings gives the cyclical product of the numbers of piles in the dealings, you resort to successive cyclical halvings of 1. The cyclical half of an even number is the simple half; but to get the cyclical half of an odd number, add 7 to half of one less than that number. Thus,

The cyclical half of 1 is $(0 \div 2) + 7 = 7$;
" " " " 7 is $(6 \div 2) + 7 = X$;
" " " " X is 5;
" " " " 5 is $(4 \div 2) + 7 = 9$;
" " " " 9 is $(8 \div 2) + 7 = J$;
" " " " J is $(X \div 2) + 7 = Q$.

If the cyclical product of the numbers of piles in the dealings is one of the first six results of doubling one, you will have (when the time comes,) to bring one card from the right-hand end of the row of black cards to the left-hand end for each such doubling. Thus, if the red cards have twice been dealt into 4 piles, four cards must be brought from the right end to the left end of the row of black cards. For $4 \times 4 - 13 = 3$ and $1 \times 2^4 - 13 = 3$. But if that cyclical product is one of the first six results of successive cyclical halvinngs of one, one card must be carried from the left to the right end of the row of black cards for every halving. Thus, if the red cards have been dealt into 6 and into 8 piles, 4 black cards must be carried from the left-hand end of the row to the right-hand end of the row. $6 \times 8 - 3 \times 13 = 9$ and it takes 4 cyclical halvings to give 9. If the product of the numbers of piles in the dealings is one more than a multiple of 13, the row of black cards is to remain unshifted.

The second way of determining how the black cards are to be transposed is simply, during the last of the dealings, to note what card is laid upon the king. The face-value of this card is the ordinal, or serial place in the row, counting from the left-hand extremity of it, which the ace must be brought to occupy. Now if you remember, as you always ought to do, where the ace is in the row, you will know how many cards to carry from one end to the other so as to bring the ace into that place. But if in the last dealing the king happens to fall at the top of one of the piles, two lines of conduct are open to you. One would be, in regathering the piles, by a pretended awkwardness in taking up the pile that is to be taken next before the one that the king heads, at first to leave its bottom card on the table, so as to get a glimpse of it before you take it up, as you would regularly have done at first; and if the king should happen to be the last card dealt, the card at the back of the packet would be the one for you to get sight of, by a similar imitation blunder. In either case, the card you so aim to get sight of would show the right place for the ace in the row. But if you doubt your ability to be gracefully awkward, it always remains open to you to ask to have the red packet cut again and a number of piles for a new deal to be ordered.

The third way of determining the proper transposition of the black cards is a slight modification of the second. It consists in looking at the card whose back is against the face of the king, when you come to cut the red packet so as to bring the king to the face. [Any practical psychologist, such as a prestigiator must be, can, with the utmost ease look for the card he wants to see, and can inspect it without detection.]

But whichever of these methods you employ, you should not touch the row of black cards until the red cards, having been regathered after the last dealing, you have said something like this: "Now I think that all these dealings and cuttings and exchanges of the last cards have sufficiently mixed up the red cards to give a certain interest to the fact that I am going to show you; namely, that this row of black cards form an index showing where any red card you would like to see is to be found in the red pack. But since there is no black king in the row, of course the place of the red king cannot be indicated; and for that reason, I shall just cut the pack of red cards so as to bring the king to the face of it, and so render any searching for that card needless." You then cut the red cards. That speech is quite important as restraining the minds of the company from reflecting upon the relation between the effect of your cutting and that of theirs. Without much pause you go on to say that you shall leave the row of black cards just as they are, simply putting so many of them from one end of the row to the other. You now ask some one, "Now, what red card would you like to find?" On his naming the face-value of a card, you begin at the left-hand end of the row of black cards and count them aloud and deliberately, pointing to each one as you count it, until you come to the ordinal number which equals the face value of the red card called for; and in case that card is the knave or queen, you call "knave" instead of "eleven" on pointing at the eleventh card, and "queen" on pointing at the last card. When you come to call the number that equals that of the red card called for, you turn the card you are pointing at face up. Suppose it is the six, for example. Then you say, naming the card called for, that that card will be the sixth; or if the card turned up was the knave, you say that the card called for will be "in the knave-place," and so in other cases. You then take up the red packet, and counting them out, aloud and deliberately, from one hand to the other, and from the back toward the face of the packet, when you come to the number that equals the face-value of the black card turned, you turn over this card as soon as you have counted it, and lo! it will be the card called for.

The company never fail to desire to see the thing done again; and on their expressing this wish, after impressing on your memory the present place of the black ace, you have only to hold out the red cards to be cut again, and you again go through the rest of the performance, now abbreviating it by having the cards dealt only once. The third time you do it, since you will now have given them the enjoyment of their little astonishment, there will no longer be any reason for not asking somebody to say what pile you shall take up first, although that will soon lead to their seeing that all the cuttings are entirely nugatory. Still they will not thoroughly understand the phenomenon.

If you wish for an explanation of it, the wish shows that you are not thoroughly grounded in cyclic arithmetic, and that you consequently still have before you the delight of assimilating the first three *Abschnitte* (for that matter the first hundred pages would suffice to reveal the foundations of the present mystery; but I confess I do not particularly admire the first Abschnitt) of Dedekind's lucid and elegant redaction of the unerring Lejeune-Dirichlet's "*Vorlesungen über Zahlentheorie.*" But, perhaps, on another occasion I will myself give a little essay on the subject, "adapted to the meanest capacity," as some of the books of my boyhood used, not too respectfully, to express it.

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MILFORD, PA.

SOME AMAZING MAZES.

A SECOND CURIOSITY.

A phenomenon easier to understand depends on the fact that, in counting round and round a cycle of 53 numbers, $\sqrt{-1} = \pm 30$. (For $30^2 = 900 = 17 \cdot 53 - 1$.) This, likewise, may be exhibited in the form of a "trick." You begin with a pack of 52 playing-cards arranged in regular order. For this purpose, it is necessary to assign ordinal numbers to the four suits. It seems appropriate to number the spade-suit as 1, because its ace carries the maker's trade-mark. I would number the heart-suit 2, because the pips are partially cleft in two; the club-suit 3, because a "club," as the French term *trèfle* reminds us, is a trefoil; and the diamond-suit as 4 or 0, because the pips are quadrilaterals, and counting round and round a cycle of 4, $4 = 0$. But it is convenient, in numbering the cards, to employ the system of arithmetical notation whose base is 13. It will follow that if the cards of each suit are to follow the order 1 2 3 4 5 6 7 8 9 X J Q K, the king of each suit must be numbered as if it were a zero-card of the following suit. The inconvenience of this is very trifling compared with the convenience of directly availing oneself of a regular system of notation; for the exhibitor of the "trick" will have many a "long multiplication" to perform in his head, as will shortly appear. Another slight inconvenience is that the cycle of numeration must be fifty-three, or 4♠, which, or its highest possible multiple, must be subtracted from every product that exceeds 4♠. It is to be remembered that ◇, ♠, ♡, ♣, are used as nothing but other shaped characters for 0, 1, 2, 3, respectively. Thirteen is the base of numeration, but fifty-three, or 4♠, is the cycle of numeration. I adopt ◇, rather then K, as the zero-sign in order to avoid denoting the king of diamonds by ♠ K, etc. In order to exhibit the trick in the highest style, the performer should have this multiplication table

◇ ♠	◇ ♡	◇ ♣	◇ 4	◇ 5	◇ 6	◇ 7	◇ 8	◇ 9	◇ 10	◇ J	◇ Q
◇ ♡	◇ 4	◇ 6	◇ 8	◇ 10	◇ Q	♠ ♠	♠ ♣	♠ 5	♠ 7	♠ 9	♠ J
◇ ♣	◇ 6	◇ 9	◇ Q	♠ ♡	♠ 5	♠ 8	♠ J	♡ ♠	♡ 4	♡ 7	♡ 10
◇ 4	◇ 8	◇ Q	♠ ♣	♠ 7	♠ J	♡ ♡	♡ 6	♡ 10	♣ ♠	♣ 5	♣ 9
◇ 5	◇ 10	♠ ♡	♠ 7	♠ Q	♡ 4	♡ 9	♣ ♠	♣ 6	♣ J	4 ♣ ◇ ♡	4 8 ◇ 7
◇ 6	◇ Q	♠ 5	♠ J	♡ 4	♡ 10	♣ 3	♣ 9	4 ♡ ◇ ♠	4 8 ◇ 7	5 ♠ ♠ ◇	5 7 ♠ 6
◇ 7	♠ ♠	♠ 8	♡ ♡	♡ 9	♣ ♣	♣ 10	4 4 ◇ ♣	4 J ◇ 10	5 5 ♠ 4	5 Q ♠ J	6 6 ♡ 5
◇ 8	♠ ♣	♠ J	♡ 6	♣ ♠	♣ 9	4 4 ◇ ♣	4 Q ◇ J	5 7 ♠ 6	6 ♡ ♡ ♠	6 10 ♡ 9	7 5 ♣ 4
◇ 9	♠ 5	♡ ♠	♡ 10	♣ 6	4 ♡ ◇ ♠	4 J ◇ 10	5 7 ♠ 6	6 ♣ ♡ ♡	6 Q ♡ J	7 8 ♣ 7	8 4 ◇ ♡
◇ 10	♠ 7	♡ 4	♣ ♠	♣ J	4 8 ◇ 7	5 5 ♠ 4	6 ♡ ♡ ♠	6 Q ♡ J	7 9 ♣ 8	8 6 ◇ 4	9 ♣ ♠ ♠
◇ J	♠ 9	♡ 7	♣ 5	4 ♣ ◇ ♡	5 ♠ ♠ ◇	5 Q ♠ J	6 10 ♡ 9	7 8 ♣ 7	8 6 ◇ 4	9 4 ♠ ♡	10 ♡ ♡ ◇
◇ Q	♠ J	♡ 10	♣ 9	4 8 ◇ 7	5 7 ♠ 6	6 6 ♡ 5	7 5 ♣ 4	8 4 ◇ ♡	9 ♣ ♠ ♠	10 ♡ ♡ ◇	J ♠ ♡ Q

by heart in which I have been forced to put 10 in place of x most incongruously simply because I am informed that the latter would transcend the resources of the printing-office.

Yet I do it quite passably without possessing that accomplishment. In those squares of the multiplication-table where two lines are occupied, the upper gives the simple product in tridecimal notation, and the lower the remainder of this after subtracting the highest less multiple of fifty-three, i. e., of 4 ♠.

In order to exhibit the trick, while you are arranging the cards in regular order, you may tell some anecdote which involves some mention of the numbers 5 and 6. For instance, you may illustrate the natural inaptitude of the human animal for mathematics, by saying how all peoples use some multiple of 5 as the base of numeration, because they have 5 fingers on a hand, although any person with any turn for mathematics would see that it would be much simpler, in counting on the fingers, to use 6 as the base of numeration. For having counted 5 on the fingers of one hand, one would simply fold a finger of the other hand for 6, and then make the first finger of the first hand to continue the count. The object of telling this anecdote would be to cause the numbers 5 and 6 to be uppermost in the minds of the company. But you must be very careful not at all to emphasize them; for if you do, you will cause their avoidance. The pack being arranged in regular sequence, you ask the company into how many piles you shall deal them, and if anybody says 5 or 6, deal into that number of piles. If they give some other number, manifest not the slightest shade of preference for one number of piles over another; but have the cards dealt again and again, until you can get for the last card either ♠ x, that is, the ten of the second suit, (i. e., suit number one; since the first suit is numbered ◇, or zero), or ♡ 4, the four of the third suit, or ♠ 6, or ♡ 8. If you cannot influence the company to give you any of the right numbers, after they have ordered several deals, you can say, "Now let me choose a couple of numbers," and by looking through the pack, you will probably find that one or other of those can be brought to the face of the pack in two or three deals. For every deal multiplies the ordinal place of each card by a certain number, counting round and round a cycle of 53. And this multiplier is that number which multiplied by the number of piles in the deal gives +1 or −1 in counting round and round the cycle of 53. For it makes no difference to which end of the pack the card is drawn. After each deal the piles are to be gathered up according to the same rule as in the first "trick," except that the first pile taken must not be the one on which the 52nd card fell, but the one on which the 53rd would have fallen if there had been 53 cards in the pile. The last deal having been made, you lay all the cards now, backs up, in 4 rows of 13 cards in each row, leaving small gaps between the 3rd and 4th and 6th and 7th cards counting from each end, thus:

1	2	3	4	5	6	7	8	9	10	J	Q	K
K	Q	J	10	9	8	7	6	5	4	3	2	1

The object of these gaps is to facilitate the counting of the places from each end, both by yourself and by the company of onlookers. If the first or last card is either ♠ x or ♡ 4, the first card of the pack will form the left-hand end of the top row, and each successive card will be next to the right of the previously laid card, until you come to the end of a row, when the next card will be the extreme left-hand card of the row next below that last formed. But if the first or last card is either ♠ 6 or ♡ 8, you begin at the top of the extreme right-hand column, and lay down the following three cards each under the last, the fifth card forming the head of the column next to the left, and so on, the cards being laid down in successive columns, passing downward in each column, and the successive columns toward the right being formed in regular order.

You now explain to the company, very fully and clearly, that the upper row consists of the places of the diamonds; and you count the places, pointing to each, thus "Ace of diamonds, two of diamonds, three; four, five, six; the seven, a little separated, the eight, nine, and ten, together; then a little gap, and the knave, queen, king of diamonds together. The next row is for the spades in the same regular order, from that end to this," (you will not say "right" and "left," because the spectators will probably be at different sides of the table,) "next the hearts, and last the clubs. Please remember the order of the suits, diamond," (you sweep your finger over the different rows successively) "spades, hearts, and clubs. But," you continue, "those are the places beginning at *that*" (the upper left-hand) "corner. In addition, every card has a *second* place, beginning at *this* opposite corner," (the lower right-hand corner.) "The order is the same; only you count backwards, toward the right in each row; and the order of the suits is the same, diamonds, spades, hearts, clubs; only the places of the diamonds are in the bottom row, the places of the spades next above them, the places of the hearts next above them, and the clubs at the top. These are the regular places for the cards. But owing to their having been dealt out so many times, they are now, of course, all out of both their places." You now request one of the company (not the least intelligent of them,) simply to turn over any card in its place. Suppose he turns up the fifth card in the third row. It will be either the ♡ 3 or ♠ J. Suppose it is the former. Then you say, "Since the three of hearts is in the place of the five of hearts, counting from *that* corner, it follows *of course*" (don't omit this phrase, nor emphasize it; but say it as if what follows were quite a syllogistically evident conclusion,) "that the five of hearts will be in the place of the three of hearts counting from the opposite corner." Thereupon, you count "Spades, hearts: one, two, three," and turn up the card, which, sure enough, will be ♡ 5. "But," you continue, "counting from the first corner, the five of hearts is in the place of the knave of spades, and accordingly, the knave of spades will, of course, be in the place of the five of hearts, counting from the opposite corner." You count, first, to show that ♡ 5 is in the place of ♠ J, and then, always pointing as you count, and counting, first the rows, by giving successively the names of the suits, "diamonds, spades, hearts," and then the places in the row, "one, two, three, four, five," and turning up the card you find it to be, as predicted, the ♠ J. "Now," you continue, "the knave of spades is in the place of the nine of spades counting from the first corner, so that we shall necessarily find the nine of spades in the place of the knave of spades counting from the opposite corner." You count as before, and find your prediction verified. [I will here interrupt the description of the "trick" to remark that the number of different arrangements of the fifty-two cards all possessing this same property is thirty-eight thousand three hundred and eighty-two billions (or millions squared), three hundred and seventy-six thousand two hundred and sixty-six millions, two hundred and forty thousand, $= 6 \times 10 \times 14 \times 18 \times 22 \times 26 \times 30 \times 34 \times 38 \times 42 \times 46 \times 50$, not counting a turning over of the block as altering the arrangement. But of these only one arrangement can be produced by dealing the cards according to our general rule. Either of the four *simplest* arrangements having the property in question will be obtained by first laying out the diamonds in a row so that the values of the cards increase regularly in passing along the row in either direction, then laying out the spades in a parallel row either above or below the diamonds, but leaving space for another row between the diamonds and spades, their values increasing in the counter-direction to the diamonds, then laying out the hearts in a parallel row close upon the other side of the diamonds, their values increasing in the same direction as the spades, and finally laying out the clubs between the diamond-row and the spade-row, their values increasing in the same direction as the former.

Not to let slip an opportunity for a logical remark, let me note that, *in itself considered,* i. e., regardless of their sequence of values, any one arrangement of the cards is as *simple* as any other; just as any continuous line that returns into itself, without crossing or touching itself, or branching, is just as simple, *in itself*, as any other; and relatively to the sequence of values of the cards, only, the arrangement produced in "trick," in which the value of each card is i times the ordinal number of its place, where $i = \pm\sqrt{-1}$, is far simpler than the arrangement just described. But in calling the latter arrangement the "simpler," I use this word in the sense that is most important in logical methodeutic; namely, to mean more facile of human imagination. We form a detailed icon of it in our minds more readily.]

You now promptly turn down again the four cards that have been turned up (for some of the company may have the impression that the proceeding might continue indefinitely; and you do not wish to shatter their pleasing illusions,) and ask how many piles they would like to have the cards dealt in next. If they mention 5 or 6, you say, "Well we will deal them into 5 and 6. Or shall we deal them into 4, 5, 6? Or into 2 and 7? Take your choice." Which ever they choose, you say, "Now in what order shall I make the dealings?" It makes no difference. But how the cards are to be taken up will be described below. After gathering the cards in the mode described in the next paragraph, deal them out, *without turning the cards up.* [I have never tried what I am now describing; but for fear of error, I shall do so before my article goes to press.] After that, you say, "Oh, I don't believe they are sufficiently shuffled. I will milk them." You proceed to do so. That is, holding the pack backs up, you take off the cards now at the top and bottom, and lay them backs up, the card from the bottom remaining at the bottom; and this you repeat 25 times more, thus exhausting the pack. Many persons insist that the proper way of milking the cards is to begin by putting the card that is at the back of the pack at its face; but when I speak of "milking," I mean this *not* to be done. Having milked the pack three times, you count off the four top cards (i. e., the cards that are at the top as you hold the pack with the faces down,) one by one from one hand to the other, putting each card above the last, so as to reverse their positions. You then count the next four into the same receiving hand, *under* the four just taken, so that their relative positions remain the same. The next four are to be counted, one by one, upon the first four, so that their relative positions are reversed, and the next four are to be counted into the receiving hand under those it already holds. So you proceed alternately counting four to the top and four to the bottom of those already in the receiving hand, until the pack is exhausted. You then say, "Now we will play a hand of whist." You allow somebody to cut the cards and deal the pack, as in whist, one by one into four "hands," or packets, turning up the last card for the trump. It will be found that you hold all the trumps, and each of the other players the whole of a plain suit.

I now go back to explain how the cards are to be taken up. If it is decided that the cards are to be dealt into 5 and into 6 piles, (the order of the dealing always being immaterial,) you take them up row by row, in consecutive order, from the upper left-hand to the lower right-hand corner. If they are to be dealt into 4, 5 and 6 piles, or into 2 and 7 piles, in any order, you take them up column by column, from the upper right-hand to the lower left-hand corner. The exact reversal of all the cards in the pack will make no difference in the final result. They may also be taken up in columns and dealt into piles whose product is 14 or 39 (as, for example, into 2 piles and 7 piles, or into 3 piles and 13 piles). They may be taken up in rows and dealt into any number of piles whose product is thirty, or, by the multiplication table is ♡4. The following are some of the sets of numbers whose products, counted round a cycle of 53, equal 30: 6·5; 17·8; 7·5·4·4; 9·7·3; 9·8·7·7; 9·6·6·5; 9·9·5·4; X·8·7; X·9·8·7·6; J·J·2; J·8·4·4; J·5·5·3; Q·X·X·4; Q·X·8·5; Q·7·7·6; K·K·3; ♠X·♠X·4 (decimally, 23·13·4); ♠6·♠4·6; ♠5·◇9·◇X.

The products of the following sets count round a cycle of 53 to −30=23; 4·♠6; 2·7·K; K·Q·X; 8·6·6; 9·8·4; X·X·5; Q·J·7; Q·Q·2; 5·5·5·4; 6·4·4·3; X·9·7·5; J·7·6·2; 11·7·4·3; 13·X·6·2; 13·8·5·3; 7·6·5·5·3; 7·7·7·5·4; 9·7·5·5·2; 11·6·5·4·3; 9·8·8·5·4·4; 8·8·7·7·4·4; 11·8·7·7·2·2; 12·11·9·8·7·6.

The products required to prepare the cards for being laid down column by column are ♠6, decimally expressed, 19; and ♡8, decimally expressed, 34.

The following are some of the sets of numbers whose continued products are 19: 9·8; Q·6; 5·5·5; 6·4·3; J·7·3; 13·6·5; 13·10·3; 8·7·6·4; 9·9·8·6; J·9·5·4; 11·10·9·2; 12·8·7·7; 13·10·8·7; 9·8·8·5·4; 10·7·7·6·5; 10·10·10·10·2; 12·7·7·5·5; 7·4·4·4·3; 13·7·4·4·4·4·4·4; 4·4·4·4·4·4·4·4·4·4·3. The following are sets of numbers whose continued product is 34: ♠4·2; ♠X·K; 29·3; 7·5·4; 9·3·♠♡; 9·9·5; X·7·2·; J·8·4; Q·X·X; 17·11·5; 17·12·9; 19·13·4; 23·11·6: 23·13; 23·17·7; 41·3·2; 5·5·4·4·3; 9·7·7·6·3; 8·6·5·5; 9·9·7·7·2; 13·13·7·2; 17·12·9; 8·4·4·4·4; 2·2·2·2·2·2·2·2·2·2·2·11·10·7·5; 13·12·9; 23·13.

This "trick may be varied in endless ways. For example, you may introduce the derangement that is the inverse of milking. That is, you may pass the cards, one by one, from one hand to the other, placing them alternately att he top and the bottom of the cards held by the receiving hand. Twelve such operations will bring the cards back to their original order. But a pack of 72 cards would be requisite to show all the curious effects of this mode of derangement.

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SOME AMAZING MAZES.

[CONCLUSION.]

EXPLANATION OF CURIOSITY THE FIRST.

You remember that at the end of my description of the card "trick" that made my first curiosity, I half promised to give, some time, an explanation of its rationale. This half promise I proceed to half redeem.

Suppose a prime number, P, of cards to be dealt into S (for *strues*) piles, where $S<P$. (Were $S=P$, it would be impossible to regather the cards, according to the rule given in the description of the "trick.") Then, in each pile, every card that lies directly on another occupied, before the deal, the ordinal, or serial, place in the packet whose number was S more than that of the other; and using Q to denote the integral part of the quotient of the division of P by S, so that $P-QS$ is positive, while $P-(Q+1)S$ is negative (for P being prime, neither can be zero,) and assuming that the piles lie in a horizontal row, and that each card is dealt out upon the pile that is next on the right of the pile on which the last preceding card was dealt, it follows that the left-hand piles, to the number of $P-QS$ of them, contain each $Q+1$ cards, while the $(Q+1)S-P$ piles to the right contain each only Q cards. It is plain, then, that, in each pile, every card above the bottom one is the one that before the dealing stood S places further from the back of the packet than did the card upon which it is placed in dealing. But in what ordinal place in the packet before the dealing did that card stand which after the regathering of the piles comes next in order after the card which just before the regathering of the piles lay at the top of any pile whose ordinal place in the row of piles, counting from the left, may be called the *s*th? In order to answer this question, we have first to consider that the effect of Standing Rule No. IV is that the pile that comes next after any given pile in the order of the regathered packet, counting, as we always do, from back to face, is the pile which was taken up *next before* that

given pile; and of course it is the bottom card of that pile to which our question refers. Now the rule of regathering is that, after taking up any pile we next take up, either the pile that lies P–QS places to the right of it, or else that which lies (Q+1)S–P places to the left of it. In other words, the pile that is taken up *next before* any pile, numbered s from the left of the row, is either the pile numbered s+QS–P (and so lies toward the *left* of pile s) or else is the pile numbered s+(Q+1)S–P (and so lies toward the *right* of pile s). But if pile number s were one of those which contain Q+1 cards each, since these are the first P–QS piles, we should have $s \leqq$P–QS, and the pile taken next before it, if it were to the left of it, would be numbered less than or equal to zero; and there is no such pile. Consequently in that case, that pile taken up next before pile s will be to the right of the pile numbered s, and its number will be s+(Q+1)S–P, which will also have been the number of its bottom card in the packet before the dealing; while, since the bottom card of pile number s was card number s before the dealing, and since this pile contains Q other cards, each originally having occupied a place S further on than the one next below it in the pile, it follows that its top card was, before the dealing, the card whose ordinal number was s+QS. Thus, while every other card of any of the first P–QS piles is followed after the regathering by a card whose original place was numbered S more than its own, the top card of such a pile will then be followed by a card whose original place was S more than its own, *counting round a cycle of* P *cards.* In a similar way, if pile number s contains only Q cards, it is one of the last (Q+1)S–P piles. Then it cannot be that the pile taken up, according to the rule, next before it lay to the right of it; for in that case the number of this previously taken pile would exceed S. It must therefore be pile number s+QS–P; and this will be the original number of its bottom card, while the original number of the top card of pile number s (since this contains only Q cards,) will be s+(Q–1)S. Hence, as before, the top card will be followed after the regathering by a card whose original place would be S greater than its own, but for the subtraction of P in counting round a cycle of P numbers. This rule then holds for all the cards.

It follows that if, after the regathering, the last card, that at the face of the pack or in the P place is the one whose original place may be called the Πth, then any other card, as that whose place after the gathering is the lth, was originally in the Π+lS–mP, where mP is the largest multiple of P that is less than Π+lS. If however, after the regathering, the pack be cut so as to bring the card which was originally the Pth, or last, that is, which was at the face of the pack, back to that same situation, then, since the original places increase by S (round and round a cycle of P places) every time the regathered places increase by 1, it follows that the original place of the card that is first subsequently to that cutting will have been S, that of the second, 2S, etc.; and in general, that of the *l*th will have been $lS-mP$. If the cards had originally been arranged in the order of their face values, the face value of the card in the *l*th place after the cut will be $lS-mP$, which we may briefly express by saying that the dealing into S piles with the subsequent cutting that brings the face card back to its place, "cyclically multiplies the face-value of each card by S," the cycle being P. If after dealing into S piles, another dealing is made into T piles, and another into U piles, etc., after which a cut brings the face card back to its place, the face value of every card will be cyclically multiplied by $S \times T \times U \times$ etc. Moreover, if cuttings were made before each of the dealings, since each cutting only cyclically adds the same number to the place of every card, the cards will still follow after one another according to the same rule; so that the final cutting that restores the face card to its place, annuls the effect of all those previous cuttings.

My hints as to the rationale of the exceptional treatment of the last card in twelve initial deals, and as to the extraordinary relation which results between the orders of succession of the black and of the red cards must be prefaced by some observations on the effects of reiterated dealings into a constant number of piles. What I shall say will apply to a pack of any prime number of cards greater than two; but to convey more definite ideas I shall refer particularly to a suit of 13 cards, each at the outset having its ordinal number in the packet equal to its face-value. The effect of one cyclic multiplication of the face-values by 2, brought about by dealing the suit into 2 piles, regathering, and cutting, if need be, so as to restore the king to the face of the packet, will be to shift all the cards except the king in one circuit. That is, the order before and after the cyclic multiplication being as here shown.

Before the cyclic doubling of
the face values ....1, 2, 3, 4, 5, 6, 7, 8, 9, X, J, Q, K,
After the same ....2, 4, 6, 8, X, Q, 1, 3, 5, 7, 9, J, K,

the 2 takes the place of the 1, the 4 that of the 2, the 8 that of the 4, the 3 that of the 8, the 6 that of the 3, the Q that of the 6, the J that of the Q, the 9 that of the J, the 5 that of the 9, the X that of the

5, the 7 that of the X, and the 1 that of the 7; so that the values are shifted as shown by the arrows on the circumference of the circle of Fig. 6. If 7, instead of 2, be the number of piles into which the thirteen cards are dealt there will be a similar shift round the same circuit, but in the direction opposite to the pointings of the arrows; and if the cards are dealt into 6 or into 11 piles, there will be a shift in a similar single circuit along the sides of the inscribed stellated polygon. But if the 13 cards are dealt into a number of piles other than 2, 6, 7, or 11, the single circuit will break into 2, 3, 4, or 6 separate circuits of shifting. Thus, if the dealing be into 4 or into 10 piles, there will be two such circuits, each along the sides of a hexagon whose vertices are at alternate points along

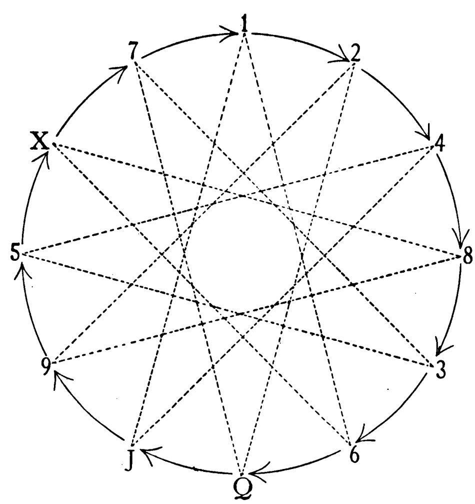

Fig. 6.

the circumference of the circle in the same figure (or, what comes to the same thing, at alternate vertices, along the periphery of the stellated polygon). Dealing into 4 piles makes one round from 1 to 4, from 4 to 3, from 3 to Q, from Q, to 9, from 9 to X, and from X back to 1; while another round is from 2 to 8, from 8 to 6, from 6 to J, from J to 5, from 5 to 7, and from 7 back to 2. Dealing into 5 or into 8 piles will make three circuits each from one vertex to the next one of 3 squares inscribed in the circle. Dealing into 3 or into 9 piles will give 4 circuits round three inscribed equilateral triangles. Finally, dealing into 12 piles, with regathering, etc. according to rule, simply reverses the order so that the ace and queen, the 2 and knave, the 3 and ten, etc. change places.

It has already been made evident that if any prime number, P, of cards, each inscribed with a number, so that, when operations begin this number shall be equal to the ordinal place of the card in the pack, be dealt into any lesser number, S, of piles, and these be re-gathered, etc. according to rule, the effect is cyclically to multiply by S the number inscribed on any card which is identified solely by its resulting ordinal place, that is, to multiply in counting the numbers round and round a cycle of P numbers,—or, to state it otherwise, the ordinary product has the highest lesser multiple of P subtracted from it, though this seems to me to be a needlessly complicated form of conceiving the cyclical product. In counting round and round, the number of numbers in the cycle, the so-called "modulus of the cycle," is the same as zero; so that the product of its multiplication by S is zero; or, regarding the matter in the other way, SP diminished by the largest lesser multiple of P gives P. Consequently, the face card will not change its face-value. Let the dealing etc. be reiterated until it has been performed δ times. The effect will be to multiply the face-values (of cards identified only by their final ordinal places) by S^{δ}. Since this is the same multiplier for all the cards, it follows that when δ attains such a value that the card in any one place, with the exception of the face card of the pack, *which alone retains an unchanging value,* recovers its original value, every one of the P–1 cards of (apparently) changing values equally recovers its original value; and if the values do not shift round a single circuit of P–1 cards, all the circuits must be equal; for otherwise the single number S^{δ} would not fix the values of all the cards. And since zero, or P, is the only number that remains unchanged by a multiplication where the multiplier is not unity (and S is always cyclically greater, that is, more advanced clockwise, than 1 and less than P,) it follows that the moduli of the shifts must all be the same divisor of P–1, and consequently P–1 deals, whatever be the constant number of piles, must restore the original order. The pure arithmetical statement of this result is that S^{P-1}, whenever P is a prime number and S not a multiple of it, must exceed by one some multiple of P. This proposition goes by the name of its discoverer, perhaps the most penetrating mind in the history of mathematics; being known as "Fermat's theorem"; although from our present point of view, it may seem too obvious to be entitled to rank as a "theorem." The books give half a dozen demonstrations of it. It lies at the root of cyclic arithmetic.

Fermat said he possessed a demonstration of his theorem; and there is every reason for believing him; but he did not publish any proof. About 1750, the mathematician König asserted that he held an autograph manuscript of Leibniz containing a proof of the proposition; but it has never been published, so far as I know. Euler, at any rate, first published a proof of it; and Lambert gave a similar one in 1769. Subsequently Euler gave a proof less encumbered with irrelevant considerations; and this second proof is substantially the same as that in Gauss's celebrated "*Disquisitiones Arithmeticae*" of 1801, §49. Several other simple proofs have since been given; but none, I think, better than that derived from the consideration of repeated deals.

But what concerns the curious phenomenon of my little "trick" is not so much Fermat's theorem as it is the more comprehensive fact that, whatever odd prime number, P, the number of cards in the pack may be, there is some number, S, such that in repeated deals into that number of piles, all the numbers less than P shift round a single circuit. I hope and trust, Reader, that you will not take my word for this. If fifty years spent chiefly with books makes my counsel about reading of any value, I would submit for your approbation the following maxims:

I. There are more books that are really worth reading than you will ever be able to read. Confine yourself, therefore, to books worth reading and re-reading; and as far as you can, own the good books that are valuable to you.

II. Always read every book critically. A book may have three kinds of value. First, it may enrich your ideas with the mere possibilities, the mere ideas, that it suggests. Secondly, it may inform you of facts. Thirdly, it may submit, for your approbation, lines of thought and evidences of the reasonable connection of possibilities and facts. Consider carefully the attractiveness of the ideas, the credibility of the assertions, and the strengths of the arguments, and set down your well-matured objections in the margins of your own books.

III. Moreover, procure, in lots of twenty thousand or more, slips of stiff paper of the size of postcards, made up into pads of fifty or so. Have a pad always about you, and note upon one of them anything worthy of note, the subject being stated at the top and reference being made below to available books or to your own note books. If your mind is active, a day will seldom pass when you do not find a dozen items worth such recording; and at the end of twenty years, the slips having been classified and arranged and rearranged, from time to time, you will find yourself in possession of an encyclopædia adapted to your own special wants. It is especially the small points that are thus to be noted; for the large ideas you will carry in your head.

If you are the sort of person to whom anything like this recommends itself, you will want to know what evidence there is of the truth of what I assert, that there is some number of piles into which any prime number of cards must be dealt out one less than that prime number of times before they return to their original order.

If these maxims meet your approval, and you read this screed at all, you will certainly desire to see my proposition proved. At any rate, I shall assume that such is your desire. Very well; proofs can be found in all the books on the subject from the date of Gauss's immortal work down. But all those proofs appear to me to be needlessly involved, and I shall endeavor to proceed in a more straightforward way, which "mehr rechnend zu Werke geht." Indeed, I think I shall render the matter more comprehensible by first examining a few special cases. But at the outset let us state distinctly what it is that is to be proved. It is that if P is any prime number greater than 2, then there must be some number of piles, S, into which a pack of P cards must be dealt (and regathered and cut, according to the rule) P–1 times in order to bring them all round to their original places again. The reason I limit the proposition to primes will presently appear: the reason I limit the primes to those that are greater than 2 is that two cards cannot, in accordance with the rule be dealt, etc., into more than one pile (if you call that dealing); and of course this does not alter the arrangement; and since there is no number of piles less than one, the theorem, in this case, reduces itself to an identical proposition; while if 1 be considered to be a prime number, the proposition is falsified since there is no number of piles into which one card can be dealt and regathered according to the rule, which requires S to be less than P.

Let our first example be that of P=17. Then P–1=16; and unless there be a single circuit of 16 face-values, which my whole present object is to show that there must be, all the circuits must either be one or more sets of 8 circuits of 2 values each, or sets of 4 circuits of 4 values each, or sets of 2 circuits of 8 values each; unless, indeed, we count in, as we ought to do, the case of 16 circuits of 1 value each. This last means that each of the 16 cards retains its face-value after a single deal. It is obtrusively obvious that this can only be when S=1. But since in these hints toward a demonstration of the proposition the particular values of S do not concern us, and had better be dismissed from our minds, we will denote this value of S by S^{xvi}, meaning that it is a value that gives 16 circuits. We will now ask what is the number of piles into which 2 dealings will restore the face-value of every card; or, in other words, will give 8 circuits of 2 values each. Letting x denote that unknown quantity, the number of piles, or the cyclic multiplier, the equation to determine it is $x^2=1$. To many readers two values satisfying this equation will be apparent. But I do not care what they are, further than that the value $x=1$ obviously satisfies the equation $x^2=1$. I do care, however, to show that there can be but two solutions of the equation $x^2=1$. For suppose that $x_1^2=1$ and $x_2^2=1$. Then $x_2^1-x_2^2=(x_1+x_2)\cdot(x_1-x_2)=0$ or equals mP. Now if a multiple of a prime number be separated into two or more factors, one of these, at least, must itself be a multiple of that prime, just as in the algebra of real and of imaginary quantities and in quaternions, if the product of several quantities be zero, one or other of those factors must be zero; and just as in logic, if an assertion consisting of a number of asserted items be false, one or more of these items must be false. In addition, every summand has its own independent effect; but every unit of a product is compounded of units of all the several factors. This is the formal, or purely intellectual, principle at the root of all the reasons for making the number of cards dealt, especially in reiterated dealings, to be a prime. It follows, then, that there are but two numbers of piles dealings into each of which will restore the original arrangement after 2 deals; and one of these is $x=1$; for evidently (bear this in mind,) if $x^a=1$, then also $x^{(ab)}=(x^a)^b=1$. There is then but one number of piles dealings into which shift the values of the cards in eight, and only eight, circuits; and this number we will denote by S^{viii}. Then, reserving x to denote any root of the equation $x^2=1$, and taking ξ to denote that one of the two roots that is not 1, we will take y to denote any number of piles, after dealing into which 4 times, the resulting arrangement of the values will be the original arrangement. That is to say, y will be any root of the cyclic equation $y^4=1$. But $x^4=(x^2)^2=1^2=1$; so that any value of x is a value of y. Let η denote any value of y that is not a value of x; and let us suppose that there are two values of η, which we may denote by $S^{iv}$and S^{xii}. It will be easy to show that there is no third value of η. For $(\eta^2)^2=1$, where η^2 fulfills the definition of x and is thus either 1 or ξ. But the roots of the equation $\eta^2=1$ fulfill the definition of x, whose values are excluded from the definition of η. Hence we can only have $\eta^2=\xi$; and that this has but two roots is proved by the same argument as was used above. Namely, η_1 and η_2 being any two of these, $(\eta_1^2-\eta_2^2)=(\eta_1+\eta_2)\cdot(\eta_1-\eta_2)=0$, so that unless η_1 and η_2 are equal, and $\eta_1-\eta_2=0_1$, then $\eta_1+\eta_2=0$ or η_1 and η_2 are negatives of each other. Now no more than 2 quantities can be each the negative of each of the others. We now pass to the consideration of those numbers of piles into which eight successive dealings result in the original arrangement. Denoting by z any such number, it is defined by the equation $z^8=1$. But every value of y (of which we have seen that there cannot be more than 4,) satisfies this equation, since $y^8=(y^4)^2=1^2=1$. Let ζ denote any value of z which is not a value of y. We may suppose that there are two of these for each of the two values of η, which we will designate as S^{ii}, S^{vi}, S^{x}, S^{xiv}. I need not assert that there are so many; but my argument requires me to prove that there are no more. The equation $(z^2)^4=z^8=1$ shows that z^2 fulfills the definition of y and can therefore have no more than the four values 1, ξ, and the two values of η. Now if $z^2=1$, z can, as we have seen in the case of x, have no other values than $z=1$ and $z=\xi$, both of which are values of y.

If $z^2=\xi$, as we have seen in regard to y, z can have no other values than the two values of η, which are again values of y. Now let us suppose that z has four values, S^{ii}, S^{vi}, S^{x}, and S^{xiv}, that are not values of y; and let us define ζ as any value of z that is not a y. The proof that there can be no more than four ζs is so exactly like the foregoing as to be hardly worth giving. I will relegate it to a paragraph of its own that shall be both eusceptic and euskiptatic;—"what horrors!' I hear from the mouths of those moderns who abominate all manufactures of Hellenic raw materials, like "skip" and "skimp."

We have seen that either $z^2=1$, or $z^2=\xi$, or $z^2=\eta$; and also that, in the first case, either $z=1$ or $z=\xi$, both of which are values of y; and that, in the second case, z has one or other of the two values of η. Accordingly, it only remains that $\zeta^2=\eta$. There are but two values of η and if ζ_1 and ζ_2 are two different values of ζ whose squares are the same value of η, $\zeta_1^2-\zeta_2^2=(\zeta_1+\zeta_2)\cdot(\zeta_1-\zeta_2)=0$. Hence, since $\zeta_1-\zeta_2$ is not zero, it follows that every value of ζ differs from every other value derived from the same η only by being the negative of it. Now no number has two different negatives; and therefore there can be no more than two ζs to every η; and there being no more than two ηs, there can be no more than four ζs.

Now this is the summary of the whole argument: the 17 cards of the pack being consecutively inscribed with numbers from the back to the face of the pack, each number of piles into which they are dealt etc. according to the rule acts as a cyclic multiplier of the face-value of every card. Every such multiplier leaves 0(=17) unchanged, and shifts the other 16 face-values in a number of circuits having the same number of values in each. The possible consequences, excluding the case of a single circuit of 16 values, are the following:

16 circuits of 1 value each can result from but	1	multiplier at the utmost.
8 circuits of 2 values each can result but from	1	other multiplier
4 circuits of 4 values each can result but from	2	other multipliers
2 circuits of 8 values each can result but from	4	other multipliers
In all the number of multipliers that give more than 1 circuit (of all 16 values) is.....	8	at most
But there are in all	16	multipliers
Hence, the number of multipliers that shift the values in 1 circuit of 16 values is.........	8,	at least.

In point of fact, it is precisely 8.

Let us now consider a pack of 31 cards. Here, the zero card not changing its value, there are 30 values which are shifted in one of these ways:

In 30 circuits of 1 value each;
In 15 circuits of 2 values each;
In 10 circuits of 3 values each;
In 6 circuits of 5 values each;
In 5 circuits of 6 values each;
In 3 circuits of 10 values each;
In 2 circuits of 15 values each;
In 1 circuit of 30 values.

I propose to show as before that if we exclude the last case, the others do not account for the effects of so many as 30 different multipliers. In the first place, as in the last example, but one multiplier will give circuits of one value each; and but one other multiple will give circuits of only two values each. We may call the former S^{xxx} and the latter S^{xv}.

The problems of 10 circuits of 3 values each and of 6 circuits of 5 values each can be treated by exactly the same method, 3 and

5 being prime numbers. I shall exhibit in full the solution of the more complicated of the two, leaving the other to the reader.

I propose, then, to show that there are at most but 5 different values which satisfy an equation of the form $s^5=1$. The general idea of my proof will be to assume that there are 5 different values (for it is indifferent to my purpose whether there be so many or not,) and then to show that there is such an equation between these five, that given any four, there is but one value that the fifth can have; that being as much as to say that there are not more than five such values in all. This assumes that every one of the five values differs from every one of the other four; making ten premisses of this kind that have to be introduced. Now to introduce a premiss into a reasoning, is to make some inference which would not necessarily follow if that premiss were not true. Assuming, then, that $s^5=1$, $t^5=1$, $u^5=1$, $v^5=1$, $w^5=1$, are the five assumed equations, I note that the division by one divisor of both sides of an equation necessarily yields equal quotients only if the divisor is known not to be zero. Hence if I divide my equations by $s-t$, by $s-u$, by $s-v$, by $s-w$, by $t-u$, by $t-v$, by $t-w$, by $u-v$, $u-w$, and by $v-w$, I shall certainly introduce the ten premisses that all the five values are different; and with a little ingenuity,—a *very* little, as it turns out,—I ought to reach my legitimate conclusion.

I will begin then by subtracting $t^5=1$ from $s^5=1$, giving $s^5-t^5=0$; and dividing this by $s-t$, and using $\cdot|\cdot$ as the logical sign of disjunction, that is, to mean "or else," I get

(1) $s^4+s^3t+s^2t^2+st^3+t^4=0 \cdot|\cdot s=t.$

By analogy, I can equally write

$$s^4+s^3u+s^2u^2+su^3+u^4=0 \cdot|\cdot s=u.$$

Subtracting the latter of these from the former, I get,

$$s^3(t-u)+s^2(t^2-u^2)+s(t^3-u^3)+t^4-u^4=0 \cdot|\cdot s=t \cdot|\cdot s=u.$$

And dividing this by $t-u$, I obtain

(2) $s^3+s^2(t+u)+s(t^2+tu+u^2)+t^3+t^2u+tu^2+u^3=0 \cdot|\cdot s=t \cdot|\cdot s=u \cdot|\cdot t=u.$

By analogy, I can equally write

$$s^3+s^2(t+v)+s(t^2+tv+v^2)+t^3+t^2v+tv^2+v^3=0 \cdot|\cdot s=t \cdot|\cdot s=v \cdot|\cdot t=v.$$

Subtracting the last equation from the last but one, I get

$$(s^2+st+t^2)(u-v)+(s+t)(u^2-v^2)+u^3-v^3=0 \cdot|\cdot s=t \cdot|\cdot s=u \cdot|\cdot s=v \cdot|\cdot t=u \cdot|\cdot t=v.$$

and dividing by $u-v$, I have

(3) $s^2+st+t^2+(s+t)(u+v)+u^2+uv+v^2=0 \cdot|\cdot s=t \cdot|\cdot s=u \cdot|\cdot s=v \cdot|\cdot t=u \cdot|\cdot t=v \cdot|\cdot u=v.$

By analogy, I can equally write

$$s^2+st+t^2+(s+t)(u+w)+u^2+uw+w^2=0 \cdot|\cdot s=t \cdot|\cdot s=u \cdot|\cdot s=w \cdot|\cdot t=u \cdot|\cdot t=w \cdot|\cdot u=w.$$

Subtracting the last from the last but one, and dividing by $v-w$, I get

$$(4) \quad s+t+u+v+w=0 \cdot|\cdot s=t \cdot|\cdot s=u \cdot|\cdot s=v \cdot|\cdot s=w \cdot|\cdot t=u \cdot|\cdot t=v \cdot|\cdot t=w \cdot|\cdot u=v \cdot|\cdot u=w \cdot|\cdot v=w.$$

This shows at once that there cannot be more than 5 different numbers, which, counting round any prime cycle, all have their 5th powers equal to 1. By a similar process, as you can almost see without slate and pencil, from $x^3=1$, $y^3=1$, $z^3=1$ one can deduce $x+y+z=0 \cdot|\cdot x=y \cdot|\cdot x=z \cdot|\cdot y=z$. The existence of these 5 and these 3 numbers must, for the present, be regarded as problematic, except that we cannot shut our eyes to the fact that 1 is one of the members of each set; as indeed $1^\delta=1$, whatever the exponent may be.

I have numbered some of the equations obtained in the proof that there are no more than 5 fifth roots of unity. You will observe that (1) equates to zero the sum of all possible terms of the fourth degree formed by two roots; that (2) equates to zero the sum of all possible terms of the third degree formed by three roots; that (3) equates to zero the sum of all possible terms of the second degree formed from four roots; and that (4) equates to zero the sum of all possible terms of the first degree formed by all five roots. Now it is plain that if we assume that there are n unequal nth roots of unity, then by subtracting $x_2^n=1$ from $x_1^n=1$, and dividing by x_1-x_2, we shall equate to zero the sum of all possible terms of the $(n-1)$th degree in x_1 and x_2. And if we have proved, in regard to any m of the roots, that (all being unequal,) the sum of all possible terms of the $(n-m+1)$th degree in these roots is equal to zero; then by taking two such equations of the $(n-m+1)$th degree in $m-1$ roots common to the two, with one root in each equation not entering into the other; by subtracting one of these equations from the other, and then dividing by the difference between the two roots which enter each into but one of these equations, we shall get an equation of the $(n-m)$th degree in $m+1$ roots. For $x^n-y^n=(x-y)\cdot\sum_{i=0}^{n-1} x^i y^{n-i-1}$

Accordingly, by repetitions of this process, we shall ultimately find that the sum of the n roots, if there be so many, is 0. This proves that there can be no more than n unequal nth roots of unity in cyclic arithmetic any more than in unlimited real or imaginary arithmetic.

But if the root of unity be of an order not prime but composite, so that it is the root of an equation of the form $x^{pq}=1$, it is evident that it is satisfied by every root of $y^p=1$ and by every root of $y^q=1$; since every power of 1 is 1. Accordingly, exclusive of roots of a lower order, the number of roots of unity of order n, that is, the number of roots of $x^n=1$, additional to those that are roots of unity of lower order cannot be greater than the number of numbers not greater than n and prime to it. A number is said to be prime to a number when they have no other common divisor than 1. I shall write the expression of two or more numbers separated by heavy vertical lines to denote the greatest common divisor of those numbers. Thus, I shall write $12 \mid 18=6$. This vertical line may be considered as a reminiscence of the line that separates numbers in the usual algorithm of the greatest common divisor. A prime number is a number prime to every other number. Consequently, 1 is a prime number. It is the only prime number that is prime to itself; for $p \mid p=p$. The number of numbers not exceeding a number, n, but prime to it is now called the *totient* of n. In the books of the first four fifths of the nineteenth century, the totient of n was denoted by $\phi(n)$; but since the invention of the word *totient*, about 1880, Tn has become the preferable notation. $T1=1$; but if p be a prime not prime to itself $Tp=p-1$. It is quite obvious that the totient of any number, n, whose prime factors not prime to themselves are p', p'', p''', etc. is obtained by subtracting from n the p'th part of it, and then successively from each remainder the p''th, etc. part of it, but not using any prime factor twice. Thus $T4=2$ (for $4 \mid 1=1$ and $4 \mid 3=1$; but $4 \mid 2=2$ and $4 \mid 4=4$); $T6=2$ (for $6-\frac{1}{2}\cdot 6=3$ and $3-\frac{1}{3}3=2$); $T8=4$ (for $8-\frac{1}{2}\cdot 8=4$), $T9=6$, $T10=4$, etc. If $m \mid n=1$, then $Tmn=(Tm)(Tn)$. On the other hand, if p is a prime and m any exponent, $Tp^n=(p-1)p^{n-1}$. A "perfect number" is defined as one which is equal to the sum of its "aliquot parts," that is, of all its divisors except itself; but, in a more philosophical sense, *every* number is a perfect number. That is to say, it is equal to the sum of the totients of *all* its divisors;—a proposition which is perfectly obvious if regarded from the proper point of view. However, since this proposition has some relevancy to the proposition I am endeavoring to prove; namely, that there is some number of piles, dealing into which shifts all the face-values of the cards along a single cycle, I will repeat a pretty demonstration of the former proposition that I find in the books. Having selected any number, m, rule a sheet of paper into columns, a column for each divisor of m; and write these divisors, in increasing order from left to right each at the top of its column as its principal heading. Just beneath this, write in parentheses, as a subsidiary heading to the column, the complementary divisor, i. e., the divisor whose product into the principal heading is the number m; and draw a line under this subsidiary heading. Now, to fill up the columns, run over all the numbers in regular succession, from 1 up to m inclusive, writing each in one column, and in one only; namely in that column which is furthest to the right of all the columns of whose principal headings the number to be written is a multiple. Here, for example, is the table for m=20:

1 (20)	2 (10)	4 (5)	5 (4)	10 (2)	20 (1)
1	2				
3		4	5		
	6				
7		8			
9				10	
11		12			
13	14		15		
		16			
17	18				
19					20

By this means it is obvious that each column will receive all those multiples of the principal heading whose quotients by that heading are prime to the subsidiary heading, and will receive no other numbers. Thus, every column will contain just one number for each number prime to the subsidiary heading but not greater than it; [since no number is entered which exceeds the product of the two headings.] In other words, the number of numbers in each column equals the totient of the subsidiary heading; and since the subsidiary headings are all the divisors, and the total number of numbers entered is m, the sum of the totients of all the divisors of m is m, whatever number m may be. It will be convenient to have a name for this principle; and since, as I remarked, it renders every number a perfect number in a perfected sense of that term, or say a *perfecti perfect* number, I will refer to it as the *rule of perfection.*

According to this, although $x^6=1$ may have 6 roots, yet since x^2, x^3, and x^6 are also roots, by the rule of perfection there can be but $T6=T2\cdot T3=1\cdot 2=2$ numbers of piles into which dealing must be made 6 times successively in order to restore the original arrangement; and similarly for the other divisors. So then the number of ways of dealing (i. e., number of piles into which the cards can be dealt, etc.) which will restore 31 cards to their original order in less than 30 deals cannot exceed $T1+T2+T3+T5+T6+T10+T15$. There are, however, in all 30 ways of dealing; and by the rule of perfection $30=T1+T2+T3+T5+T6+T10+T15+T30$. Hence, there must be $T30=T2 \cdot T3 \cdot T5=1 \cdot 2 \cdot 4=8$ ways of dealing which shift the 30 values in a single circuit. And so with any other prime number than 31. This argument is so near a perfect demonstration that there always must be such ways of dealing that I may leave its perfectionment to the reader.

I do not know of any general rule for ascertaining what the particular numbers of piles are into which the prime number p of cards must be dealt $p-1$ times in order to bring round the original arrangement again. It seems that there is a *Canon Arithmeticus* got out by Jacobi, which gives the numbers for the first 170 primes or so. It was published in the year of my birth; so that it was clearly the purpose of the Eternal that I should have the advantage of it. But that purpose must have been frustrated; for I never saw the book. The *Tables Arithmétiques* of Hoüel (Gauthiers-Villars: 1866. 8vo, pp. 44) gives those numbers for all primes less than 200. From these tables it appears that for about five-eighths of the primes one such number is either 2 or $p-2$. Now as soon as one has been found, it is easy to find the rest which are all the powers of that one whose exponents are prime to $p-1$. In case $p-1$ has few prime factors, the numbers any one of which we seek must be nearly a third, perhaps nearly or quite half of all the $p-1$ numbers; so that ere many trials have been made, one is likely to light upon one of them. Thus if $p=17$, try 2. Now $2^4=16=-1$; so this will not do. Nor will -2. Try 3. We have $3^2=9=-8$; $3^3=27=-7$, $3^4=81=-4$, $3^8=(3^4)^2=(-4)^2=16=-1$. Evidently 3 is one of the numbers and the others are $3^3=-7$, $3^5=-12=5$, $3^7=(3^3)(3^4)=(-7)\cdot(-4)=28=-6$, and the negatives of these. If the prime factors are many, a different procedure may be preferable. Take the case of $p=31$. Here $p-1=2\cdot 3\cdot 5$. Turning to that table of the first nine powers of the first hundred numbers which is given in so many editions of Vega, I find in the column of cubes, $5^3=125=4(31)+1$, and $6^3=216=7\cdot 31-1$; and in the column of 5th powers, I find $3^5=243=8(31)-5$. Consequently, $(3^5)^3=3^{15}=-1$. This renders it *likely* that 3 may be such a number as I seek. $3^2=9$, $3^3=-4$, $3^4=-12$, $3^5=-5$, $3^6=16=-15$, $3^{10}=-6$, $3^{12}=+8$, 3^{15}

$=(3^5)^3=-125=1$. It is evident that 3 is one of the numbers. The other seven are $3^7=3^5\cdot 3^2=-45=-14$, $3^{11}=3\cdot 3^{10}=-18=13$, $3^{13}=3\cdot 3^{12}=24=-7$, $3^{17}=3^{15}\cdot 3^2=-9$; $3^{19}=3^{15}\cdot 3^4=+12$, $3^{23}=3^{19}\cdot 3^4=-144=+11$, $3^{29}=3^{17}\cdot 3^6=(-9)\cdot(-15)-135=+11$.

Since, then, whatever prime number not prime to itself p may be, there are always $T(p-1)$ numbers of which the lowest power equal to 1 (counting round the p cycle,) is the $(p-1)$th and these powers run through all the values of the cycle excepting only $p=0$, it follows that these numbers may appropriately be called *basal* (or *primitive*) roots of the cycle; and that their exponents are true *cyclic logarithms* of all the numbers of the cycle except zero. But since, if b be such a basal root, its $(p-1)$th power, like that of any other number, equals 1 (counting round the p-cycle), it follows that these exponents run round a cycle smaller by one unit than that of their powers; or in other words, the *modulus* of the cycle of logarithms is $p-1$, while the modulus of the cycle of natural numbers is p.*

The cyclic logarithms form an entirely distinct number-system from that of the corresponding natural numbers. For the modulus

* This being the first occasion I have had in this essay to employ the word "'modulus," I will take occasion to say that its general meaning is now well established. It means that signless quantity which measures the magnitude of a quantity and is a factor of it. So that if M and M′ are the moduli of two quantities, $M\mu$ and $M'\mu'$, their product is $MM'\cdot\mu\mu'$, where MM′ is an ordinary product, but $\mu\mu'$ may be a peculiar function. Thus, the absolute value of -2, or 2, is its "modulus," as 3 is of -3; and $(-2)\cdot(-3)=+6$ where $2\times 3=6$ by ordinary multiplication, but $(-1)\times(-1)=+1$ by an extension of ordinary multiplication. So the "modulus" of $A+Bi$, where $i^2=-1$, is $\sqrt{A^2+B^2}$. The tensor of a quaternion and the determinant of a square matrix are other examples of moduli. The cardinal number of numbers in a cycle has no sign and may properly be called the modulus of the cycle. But I sometimes refer to it as "the cycle," for short. The present usage of mathematicians is to use, what seems to me a too involved way of conceiving of cyclic arithmetic which carries with it an irregular use of the word "modulus." Legendre and the earlier writers on cyclic arithmetic conceived of its numbers as signifying the lengths of different steps along a cycle of objects, and thus spoke of 18 as being *equal* to 1 on a cycle of 17, just as we say that the 1st, 15th, 22d, and 29th days of August fall on *the same* day of the week, and just as we say that 270° of longitude west of any meridian and 90° east of it are *the very same* longitude. Gauss, however, introduced a different locution, involving quite another form of thought. Instead of saying that 18 *is*, or *equals*, 1 in counting round a cycle of modulus 17, he prefers to say that 18 and 1 belong to the same *class* of numbers *congruent to* one another for the *modulus* 17. Here the idea of a cycle appears to be rejected in favor of the idea that $(18-1)/17$ is a whole number.

Now I fully admit that the conception of an indefinitely advancing series is involved in that of a cycle, and further that non-cyclical numbers have to be used to some extent in cyclic arithmetic. But at the same time it seems to me that the theoric idea of a cycle ought to take the lead in this branch of mathematics. In particular, I cannot see why the term *cyclic logarithms* is not perfectly correct and far more expressive than Gauss's colorless name of "indices."

of their cycle is composite instead of prime, a circumstance which essentially modifies some of the principles of arithmetic. For example, every natural number of a cycle of prime modulus gives an unequivocal quotient when divided by another. But some numbers in a cycle of composite modulus give two or more quotients when divided by certain others, while others are not divisible without remainders. The whole doctrine shall be set forth here. I will preface it with a statement of the essential differences between the system of all positive finite integers, the system of all real finite integers, and any cyclical system. I omit the Cantorian system, partly because the full explanation of it would be needed and would be long, and partly because there is a doubt whether it really possesses an important character which Cantor attributes to it.

It is singular that though the systems to be defined possess, besides several independent common characters, others in respect to which they differ, yet *all* the properties of each system are necessary consequences of a single principle of immediate sequence. In stating this, I shall abbreviate a frequently recurring phrase of nine syllables by writing, '*m* is A of (or to) *n*,' or even '*m* is A*n*,' to mean that the member, *m*, of the system is in a certain relation of immediate antecedence to the member *n*. I shall express the same thing by writing '*n* is A'd by *m*.' But when I call A an abbreviation, I do not mean to imply that the words "immediately antecedent" express its meaning in a satisfactory way. On the contrary, in part, they suggest something repugnant to its meaning, which must be gathered exclusively from the following definitions of the three kinds of systems:

A *cyclical system* of objects is such a collection of objects that, the expression '*m* is A to *n*' signifying some recognizable relation of *m* to *n*, every member of the system is A to some member or other, and whatever predicate, P, may be, if P is true of no member of the system without being true of some member of it that is A'd by that member, then P is true either of no member or of every member.

The system of all positive whole numbers is a single collection of numbers, the general essential character of which collection is that there is a recognizable relation signified by A, such that every positive integer is A to a positive integer, and there is one, and one only, initial positive integer, 0 (or, if this be excluded, then 1,) such that, whatever predicate P may be, if P is true of no positive integer without being also true of some positive integer to which the former is A, then either this predicate is false of that initial positive integer or else is true of all positive integers.

The system of all real integers is a collection of numbers of which the general essential character is that there is recognizable relation signified by one being A to another, such that every number of the system is both A to a number of the system, and is A'd by a number of the system, and whatever predicate P may be, if this be not true of any number, *n,* of the system without being both true of some number that is A of *n,* and true also of some number that is A'd by *n,* then P is either false of every number of the system or is true of every number of the system.

A *Cantorian* system is essentially a system of objects positively determined by every collection of objects of the system being A to some object of the system, and by a certain object, 0, being a member of the system; while it is negatively determined by the principle that, whatsoever predicate P may be, if P is not true of every member of any collection of the system without being also true of some member that is A'd by that collection, then either P is not true of the member, 0, or it is true of every member of the system.

Now for several reasons, partly for the sake of the logical interest and instruction that will accrue I will proceed to show precisely *how* all the fundamental properties common to cyclical systems follow from my definition. In accordance with the usage of logicians and mathematicians, I shall call this "demonstrating" those properties. The reader must not fall into the error of supposing that, by this expression, I mean *rationally convincing* him that all cyclical systems have these properties; for I know well that he is perfectly cognizant of that already. All I am seeking to convince him of is, 1st, *that,* and 2d, *how,* their truth of all cyclical systems follows from my definition. But in the course of doing so, I shall endeavor to bring to his notice some things well worth knowing concerning necessary reasonings in general. Especially, I shall try to point out errors of logical doctrine which students of the subject who neglect the logic of relations are apt to fall into.

A brace of these errors, are, first, that nothing of importance can be deduced from a single premiss; and secondly, that from two premisses one sole complete conclusion can be drawn. Persons who hold the latter notion cannot have duly considered the paucity of the premisses of arithmetic and the immensity of higher arithmetic, otherwise called the "theory of numbers," itself. As to the former belief, aside from the consideration that whatever follows from two propositions equally follows from the one which results from their copulation, they will have occasion to change their opinion when they come to see what can be deduced from the definition of a cyclic system, which definition is not a copulative proposition.

That couple of logical heresies, being married together, legitimately generates a third more malignant than either; namely, that necessary reasoning takes a course from which it can no more deviate than a good machine can deviate from its proper way of action, and that its future work might conceivably be left to a machine,—some Babbage's analytical engine or some logical machine (of which several have actually been constructed). Even the logic of relations fails to eradicate that notion completely, although it does show that much unexpected truth may often be brought to light by the repeated reintroduction of a premiss already employed; and in fact, this proceeding is carried to great lengths in the development of any considerable branch of mathematics. Although, moreover, the logic of relations shows that the introduction of abstractions,—which nominalists have taken such delight in ridiculing,—is of the greatest service in necessary inference, and further shows that, apart from either of those manoeuvres,—either the iteration of premisses or the introduction of abstractions,—the situations in which the necessary reasoner finds several lines of reasoning open to him are frequent. Nevertheless, in spite of all this, the tendency of the logic of relations itself,—the highest and most rational theory of necessary reasoning yet developed,—is to insinuate the idea that in necessary reasoning one is always limited to a narrow choice between quasi-mechanical processes; so that little room is left for the exercise of invention. Even the great mathematician, Sylvester, perhaps the mind the most exuberant in original ideas of pure mathematics of any since Gauss, was infected with this error; and consequently, conscious of his own inventive power, was led to preface his "Outline Trace of the Theory of Reducible Cyclodes," with a footnote which seems to mean that mathematical conclusions are not always derived by an apodictic procedure of reason. If he meant that a man might, by a happy guess, light upon a truth which might have been made a mathematical conclusion, what he said was a truism. If he meant that the hint of the way of solving a mathematical problem might be derived from any sort of accidental experience, it was equally a matter of course. But the truth is that all genuine mathematical work, except the formation of the initial postulates (if this be regarded as mathematical work,) is necessary reasoning. The mistake of Sylvester and of all who think that necessary reasoning leaves no room for originality,—it is hardly credible however that there is anybody who does not know that mathematics calls for the profoundest invention, the most athletic imagination, and for a power of generalization in comparison to whose every-day performances the most vaunted performances of metaphysical, biological, and cosmological philosophers in this line seem simply puny,—their error, the key of the paradox which they overlook, is that originality is not an attribute of the *matter* of life, present in the whole only so far as it is present in the smallest parts, but is an affair of *form,* of the way in which parts none of which possess it are joined together. Every action of Napoleon was such as a treatise on physiology ought to describe. He walked, ate, slept, worked in his study, rode his horse, talked to his fellows, just as every other man does. But he combined those elements into shapes that have not been matched in modern times. Those who dispute about Free-Will and Necessity commit a similar oversight. Notwithstanding my tychism, I do not believe there is enough of the ingredient of pure chance now left in the universe to account at all for the indisputable fact that mind acts upon matter. I do not believe there is any amount of *immediate* action of that kind sufficient to show itself in any easily discerned way. But one endless series of mental events may be immediately followed by a beginningless series of physical transformations. If, for example, all atoms are vortices in a fluid, and every fluid is composed of atoms, and these are vortices in an underlying fluid, we can imagine one way in which a beginningless series of transformations of energy* might take place in a fraction of a second. Now whether this particular way of solving the paradox happens to be the actual way, or not, it suffices to show us that from the supposed fact that mind acts *immediately* only on mind, and matter *immediately* only on matter, it by no means follows that mind cannot act on matter, and matter on mind, without any *tertium quid.* At any rate, our power of self-control certainly does not reside in the smallest bits of our conduct, but is an effect of building up a character. All supremacy of mind is of the nature of Form.

The plan of a demonstration can obviously not spring up in the mind complete at the outset; since when the plan is perfected, the

* You may well be puzzled, dear Reader, to iconize the consecution of a beginningless series upon an endless series. But you have only to imagine a dot to be placed upon the rim of a half-circle at each point whose angular distance from the beginning of the semicircumference has a positive or nega-

demonstration itself is so. The thought of the plan begins with an act of ἀγχίνοια* which, in consequence of pre-existent associations, brings out the idea of a possible object, this idea not being itself involved in the proposition to be proved. In this idea is discerned that the possibility of its object follows in some way from the condition, general subject, or antecedent of the proposition to be proved, while the known characters of the object of the new idea will, it is perceived, be at least adjuvant to the establishment of the predicate or consequent of that proposition.

I shall term the step of so introducing into a demonstration a new idea not explicitly or directly contained in the premisses of the reasoning or in the condition of the proposition which gets proved by the aid of this introduction, a *theō'ric* step. Two considerable advantages may be expected from such a step besides the demonstration of the proposition itself. In the first place, since it is a part of my definition that it really aids the demonstration, it follows that without some such step the demonstration could not have been effected, or at any rate only in some very peculiar way. Now to propositions which can only be proved by the aid of theoric

tive whole number for its natural tangent. These dots will, then, occur at the following angular distances from the origin of measurement.

ANGULAR DISTANCE		TANGENT
0°	00′	0
45	00	+ 1
63	26	+ 2
71	34	+ 3
75	58	+ 4
78	41	+ 5
80	32	+ 6
81	52	+ 7
82	52	+ 8
83	40	+ 9
84	17	+10
84	48	+11
85	14	+12
85	36	+13
85	55	+14
86	11	+15
86	25	+16
86	38	+17
86	49	+18
86	59	+19
87	08	+20
87	16	+21

ANGULAR DISTANCE		TANGENT
87°	24′	+22
87	31	+23
87	37	+24
87	43	+25

and so on endlessly. But after all positive integer values have been passed through before 90°, (where there will not be any dot,) a beginningless series of dots will succeed, for which the tangents are negative; and then

92°	17′	—25
92	23	—24
92	29	—23
92	36	—22
92	44	—21
92	52	—20

ANGULAR DISTANCE		
93°	01′	—19
93	11	—18
93	21	—17
93	35	—16
93	49	—15
94	05	—14
94	24	—13
94	46	—12
95	12	—11
95	43	—10
96	20	— 9
97	08	— 8
98	08	— 7
99	28	— 6
101	19	— 5
104	02	— 4
108	26	— 3
116	34	— 2
135	09	— 1
180	00	0
225	00	+ 1
etc.		

* See *Charmides,* p. 160A, and the last chapter of the First *Posterior Analytics.*

steps, (or which, at any rate, could *hardly* otherwise be proved,) I propose to restrict the application of the hitherto vague word "*theorem*," calling all others, which are deducible from their premisses by the general principles of logic, by the name of *corollaries*. A theorem, in this sense, once it is proved, almost invariably clears the way to the corollarial or easy theorematic proof of other propositions whose demonstrations had before been beyond the powers of the mathematicians. That is the first secondary advantage of a theoric step. The other such advantage is that when a theoric step has once been invented, it may be imitated, and its analogues applied in proving other propositions. This consideration suggests the propriety of distinguishing between varieties of theorems, although the distinctions cannot be sharply drawn. Moreover, a theorem may pass over into the class of corollaries, in consequence of an improvement in the system of logic. In that case, its new title may be appended to its old one, and it may be called a *theorem-corollary*. There are several such, pointed out by De Morgan, among the theorems of Euclid, to whom they were theorems and are reckoned as such, though to a modern exact logician they are only corollaries. If a proposition requires, indeed, for its demonstration, a theoric step, but only one of a familiar kind, that has become quite a matter of course, it may be called a *theoremation*.* If the needed theoric step is a novel one, the proposition which employs it most fully may be termed a *major theorem*; for even if it does not, as yet, appear particularly important, it is likely eventually to prove so. If the theoric invention is susceptible of wide application, it will be the basis of a mathematical method.

But mathematicians are rather seldom logicians or much interested in logic; for the two habits of mind are directly the reverse of each other; and consequently a mathematician does not care to go to the trouble, (which would often be very considerable,) of ascertaining whether the theoric step he proposes to himself to take is absolutely indispensable or not, so long as he clearly perceives that it will be exceedingly convenient; and the consequence is that many demonstrations introduce theoric steps which relieve the mind and obviate confusing complications without being logically necessary. Such demonstrations prove corollaries more easily by treating them as if they were theorems. They may be called *theoric corollaries,* or if one is not sure that they are so, *theorically proved propositions.*

* *θεωρημάτιον* is entered in L. & S., with a reference to the Diatribes of Epictetus.

I wish a historical study were made of all the remarkable theoric steps and noticeable classes of theoric steps. I do not mean a mere narrative, but a critical examination of just what and of what mode the logical efficacy of the different steps has been. Then, upon this work as a foundation, should be erected a logical classification of theoric steps; and this should be crowned with a new methodeutic of necessary reasoning. My future years,—whatever can have become of them, they do not seem so many now as they used, when, at De Morgan's *Open Sesame,* the Aladdin matmûrah of relative logic had been nearly opened to my mind's eye;—but the remains of them shall, I hope, somehow contribute toward setting such an enterprise on foot. I shall not be so short-sighted as to expect any cut-and-dried rules nor yet any higher sort of contrivance, to supersede in the least that ἀγχίνοια,—that penetrating glance at a problem that directs the mathematician to take his stand at the point from which it may be most advantageously viewed. But I do think that that faculty may be taught to nourish and strengthen itself, and to acquire a skill in fulfilling its office with less of random casting about than it as yet can.

Euclid always begins his presentation of a theorem by a statement of it in *general terms,* which is the form of statement most convenient for applying it. This was called the πρότασις, or *proposition.* To this he invariably appends, by a λέγω, "I say," a translation of it into *singular terms,* each general subject being replaced by a Greek letter that serves as the proper name for a single one of the objects denoted by that general subject. Yet the generality of the statement is not lost nor reduced, since the understanding is that the letter may be regarded as the name of any one of those objects that the student may select. This second statement was called the ἔκθεσις, or *exposition.* Euclid lived at a time when the surpassing importance of Aristotle's *Analytics* was not appreciated. The use, probably by Euclid himself, of the term πρότασις, which in Aristotle's writings means a premiss, to denote the conclusion to be proved, illustrates this, and confirms other reasons for thinking that Euclid was unacquainted with the doctrine of the *Analytics.* The invariable appending by Euclid of an ἔκθεσις to the πρότασις (except in a few cases in which the proposition is expressed in the ecthetic form alone,) inclines me to think that it was, for him, a principle of logic that any general proposition can be so stated; and such a form of statement was always convenient in demonstration; sometimes, necessary. If this surmise be correct, Euclid probably looked upon the function of the ἔκθεσις as that of merely supplying a more convenient form for expressing no more than the πρότασις had already asserted. Yet inasmuch as the πρότασις does not mention those proper names consisting of single letters, the ἔκθεσις certainly does supply ideas that, however obvious they be, are not contained in the πρότασις; so that it must be regarded as taking a little theoric step. The principal theoric step of the demonstration is, however, taken in what immediately follows; namely, in "preparation" for the demonstration, the παρασκευή, usually translated "the construction." The Greek word is applied to any thing got up with some elaboration with a view to its being used in any contemplated undertaking: a near equivalent to a frequent use of it is " apparatus." Euclid's παρασκευή consists of precise directions for drawing certain lines, rarely for spreading out surfaces; for though his work entitled *"Elements,"* appears to have been intended as an introduction to theoretical mathematics in general, (the art of computation being the *métier,* —the 'mister, as Chaucer would say, of the Pythagoreans,) yet Euclid always conceives arithmetical quantities,—even when distinguishing between prime and composite integers,—as being lengths of lines. It was his mania. Those lines which are drawn in the παρασκευή are not only all that are referred to in the condition of the proposition, but also all the additional lines which he is about to consider in order to facilitate the demonstration of which this παρασκευή is thus the soul, since in it the principal theoric step is taken. But the construction of these additional lines is introduced by γάρ, here meaning "for," and sometimes the text does not very sharply separate some parts of the παρασκευή from the next step, the ἀπόδειξις, or demonstration. This latter contains mere corollarial reasoning, though, in consequence of its silently assuming the truth of all that has been previously proved or postulated (which Mr. Gow, in his *Short History of Greek Mathematics,* gives as the reason for Euclid's having called his work Στοίχεια; which seems to me very dubious,) this corollarial reasoning will sometimes be a little puzzling to a student who has not so thoroughly assimilated what went before as to have the approximate proposition ready to his mind. After this, a sentence always using ἄρα, "hence," *"ergo,"* repeats the πρότασις (not often the ἔκθεσις,) so as to impress the proposition on the mind of the student, in its new light and new authority, expressed in the form most convenient in future applications of it. This is called συμπέρασμα, the "conclusion," which sounds highly Aristotelian. Yet the classical use of the verb to signify coming to a final conclusion, rendered this noun inevitable as soon as these neuter abstracts came into the frequent use that they had by Euclid's time. The conclusion always ends with the words ὅπερ ἔδει δεῖξαι, "which had to be shown," *quod erat demonstrandum,* for which Q. E. D. is now put.

I will take at random the 20th proposition of the first book, to illustrate the matter. "In every triangle, any two sides, taken together are always greater than the third.

"For let ΑΒΓ be a triangle. I say that any two sides taken together are greater than the third; ΒΑ and ΑΓ than ΒΓ, ΑΒ and ΒΓ than ΑΓ, and ΒΓ and ΓΑ than ΑΒ.

"For extend ΒΑ to the point Δ, taking ΑΔ equal to ΓΑ, [which he has shown in the 2d proposition always to be possible;] and join Δ to Γ by a straight line.

"Now since ΔΑ is equal to ΑΓ, the angle under ΑΔΓ is equal to that under ΑΓΔ [by the *pons asinorum,*] Hence, the angle under ΒΓΔ will be greater than that under ΑΔΓ. [This is a fallacy of a kind to which Euclid is subject from assuming that every figure drawn according to the παρασκευή will necessarily have its parts related in the same way, when it can only be otherwise if space is finite, which he has never formally adopted as a postulate. In the present case, if ΑΔ is more than half-way round space, the triangle ΑΓΔ will include the triangle ΑΒΓ within it; and then the angle ΒΓΔ will be less than the angle ΑΔΓ.] And since ΔΓΒ is a triangle having the angle under ΒΓΔ greater than that under ΒΔΓ, but the greater side subtends under the greater angle [which is the theorem that had just previously been demonstrated,] therefore ΔΒ is greater than ΒΓ. But ΔΑ is equal to ΑΓ. Therefore, ΒΔ and ΑΓ are greater than ΒΓ. Similarly, we shall [i. e. could] show that ΑΒ and ΒΓ are greater than ΓΑ, and ΒΓ and ΓΑ than ΑΒ.

"In every triangle, then, any two sides joined together are greater than the third, which is what had to be shown."

I will now return to the consideration of cyclical systems, and will begin by expressing my definition of such a system in those Existential Graphs which have been explained in *The Monist* (Vol. XVI, pp. 524–544, where correct the errata given in Vol. XVII, p. 160). In reference to those graphs, it is to be borne in mind that they have not been contrived with a view to being used as a calculus, but on the contrary for a purpose opposed to that. Nevertheless, if any one cares to amuse himself by drawing inferences by machinery, the graphs can be put to this work, and will perform it with a facility about equal to that of my universal algebra of logic and as much beyond that of my algebra of dyadic relatives, of which the lamented Schroeder was so much enamoured. The only other contrivances for the purpose appear to me to be of inferior value, unless it be considered worth while to bring a pasigraphy into use. Such ridiculously exaggerated claims have been made for Peano's system, though not, so far as I am aware, by its author, that I shall prefer to refrain from expressing my opinion of its value. I will only say that if a person chooses to use the graphs to work out difficult inferences with expedition, he must devote some hours daily for a week or two to practice with it; and the most efficacious, instructive, and entertaining practice possible will be gained in working out his own method of using the graphs for his purpose. I will just give these little hints. Some slight shading with a blue pencil of the oddly enclosed areas will conduce to clearness. Abbreviate the parts of the graph that do not concern

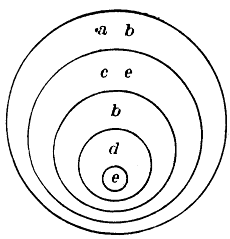

Fig. 7.

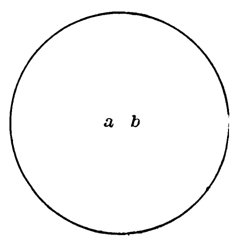

Fig. 8.

your work. Extend the rule of iteration and deiteration, by means of a few theorems which you will readily discover. Do not forget that useful iteration is almost always into an evenly enclosed area, while useful deiteration is, as usually, from an oddly enclosed area. Perform the iteration and the immediately following deiteration at one stroke, in your mind's eye. Do not forget that the ligatures may be considered as graph-instances scribed in the areas where their least enclosed parts lie, and repeated at their attachments. Their intermediate parts may be disregarded. Reflect well on each of the four permissions (especially that curious fourth one,) until you vividly comprehend the why and wherefore of each, and the bearings of each from every point of view that is habitual with you. Do not forget that an enclosure upon whose area there is a vacant cut can everywhere be inserted and erased, while an unenclosed vacant cut declares your initial assumption, first scribed, to have been absurd. You will thus, for example, be enabled to see at a glance that from Fig. 7 can be inferred Fig. 8. The cuts perform two functions; that of denial and that of determining the order of selection of the individual objects denoted by the ligatures. If the outer cuts of any graph form a nest with no spot except in its innermost area, then all that part of the assertion that is therein expressed will need no nest of cuts, but only cuts outside of one another, none of them containing a cut with more than a single spot on it. It will seldom be advisable to apply this to a complicated case, owing to the great number of cuts required; but you should discover and stow away in some sentry-box of your mind whence the beck of any occasion may instantly summon it, the simple rule that expresses all possible complications of this principle. As an example of one of the simplest cases, Fig. 9 and Fig. 10 are seen precisely equivalent.

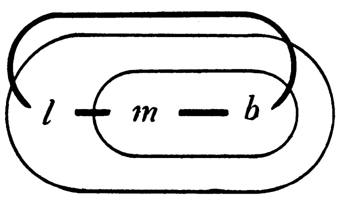

Fig. 9.

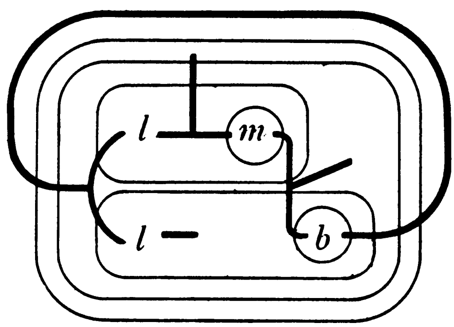

Fig. 10.

Owing to my Existential Graphs having been invented in January of 1897 and not published until October, 1906, it slipped my mind to remark when I finally did print a description of it, what any reader of the volume entitled *Studies in Logic by Members of the Johns Hopkins University*, (Boston: 1882,) might perceive, that in constructing it, I profited by entirely original ideas both of Mrs. and Mr. Fabian Franklin, as well as by a careful study of the remarkable work of O. E. Mitchell, whose early demise the world of exact logic has reason deeply to deplore.

My reason for expressing the definition of a cyclic system in Existential Graphs is that if one learns to think of relations in the forms of those graphs, one gets the most distinct and ecthetically as well as otherwise intellectually, iconic conception of them likely to suggest circumstances of theoric utility, that one can obtain in any way. The aid that the system of graphs thus affords to the process of logical analysis, by virtue of its own analytical purity, is surprisingly great, and reaches further than one would dream. Taught to boys and girls before grammar, to the point of thorough familiarization, it would aid them through all their lives. For there are few important questions that the analysis of ideas does not help to answer. The theoretical value of the graphs, too, depends on this.

Strictly speaking, the term 'definition' has two senses,—Firstly, this term is sometimes quite accurately applied to the composite of characters which are requisite and sufficient to express the signification of the 'definitum,' or predicate defined; but I will distinguish the definition in this sense by calling it the 'definition-term.' Secondly, the word definition is correctly applied to the double assertion that the definition-term's being true of any conceivable object would always be both requisite and sufficient to justify predicating the definitum of that object. I will distinguish the definition in this sense by calling it the 'definition-assertion-pair.' In the present case, as in most cases, it is needless and would be inconvenient to express the entire definition-assertion-pair with strict accuracy, since we only want the definition in order to prove certain existential facts of subjects of which we *assume* that the definitum, 'cyclic-system,' is predicable. We do not care to *prove* that it is predicable, and therefore the assertion that the definitum is predicable of the definition-term is not relevant to our purpose. In the second place, we do not care to meddle with that universe of concepts with which the definition deals; and it would considerably complicate our premisses to no purpose to introduce it. We only care for the predication of the definition-term concerning the definitum so far as it can concern existential facts. All that we care to express in our graph is so much as may be required to deduce every existential fact implied in the existence of a cyclic system.

A cyclic system is a system; and a system is a collection having a regular relation between its members. One member suffices to make a collection, and is requisite to the existence of the collection. The definition, so far as we need it, is then expressed in the graph of Fig. 11. Here K with a "peg" (See *Monist,* Vol. XVI, p. 530) at the side asserts that the object denoted by the peg is a cyclic system. The letter M with one peg at the top and another placed on either side without any distinction of meaning, asserts that the object denoted by the side-peg is a *member* of the system denoted by the top-peg. The letter C, with a peg at the top and another at the side asserts that the object denoted by the top-peg is a relation involved in that relation between all the members which constitutes the entire collection of them as the system that it is, and asserts that the object denoted by the side-peg is such a system. The Roman numerals each having one peg placed at the top or bottom of the numeral and a number of side-pegs equal to the value of the numeral, all these side-pegs being carefully distinguished, is used to express the truth of the proposition resulting from filling the blanks of the rheme denoted by the top or bottom peg, with indefinite signs of objects denoted by the side-pegs taken in their order, all the left-hand pegs being understood to precede all the right-hand pegs, and on each side a higher peg to precede a lower one. With this understanding, the graph of Fig. 11, where for the sake of perspicuity the oddly enclosed, or negating areas are shaded, may be translated into the language of speech in either of the two following equivalent forms (besides many others):

It is false that
there is a cyclic system while it is false that
this system has a member
and involves a relation ("being A to," the bottom peg of II,)
and that it is false that
the system has a member of which it is false that
it is in that relation, A, to a member of the system,
while it is false that
there is a definite predicate, P, (the top or bottom peg of I,) that is true of a member of
the system and is false of a member of the system, and that it is false that
this predicate is true of a member of the system of which it is false that
it is A to a member of the system of which P is true.

This more analytic statement is equivalent to saying that every cyclic system (if there be any,) has a member, and involves a relation called "being A to" (not the graph but perspicuity of speech requires it to be so named,) such that every member of the system is A to a member of the system, and any definite predicate, P, whatsoever, that is at once true of one member of the system and untrue of another is true of some member of the system that is not A to any member of which P is true.

To anybody who has no notion of logic this may seem a queer attempt to explain what is meant by a cyclic system; and it is true that it would be a needlessly involved *verbal* definition; a verbal definition being an explanation of the meaning of a word or phrase intended for a person to whose mind the idea expressed is perfectly distinct. But it is not intended to serve as a *verbal,* but as a *real* definition, that is, to explain to a person to whom the idea may be familiar enough, but who has never picked it to pieces and marked its structure, exactly how the idea is composed. As such, I believe it to be the simplest and most straight-forward explanation possible. When you say that the days of the week "come round in a set of seven," you think of the week everything here expressed of K. I do not mean that all this is *actually* existent in your thought; for thinking no more needs the actual presence in the mind of what is

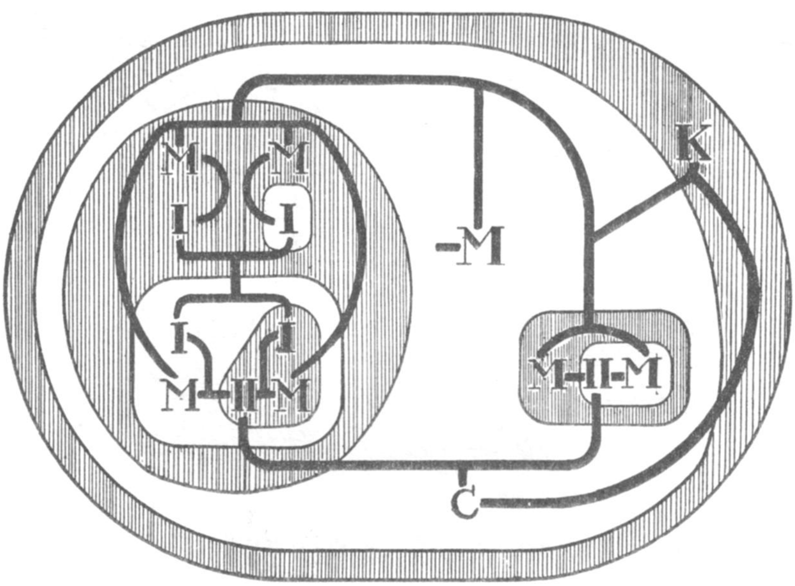

Fig. 11.

thought than knowing the English language means that at every instant while one knows it the whole dictionary is actually present to his mind. Indeed, thinking, if possible, even *less* implies presence to the mind than knowing does; for it is tolerably certain that a mind to whom a word is present with a sense of familiarity knows that word; whereas a mind which being asked to *think* of anything, say a locomotive, simply calls up an image of a locomotive has, in all probability, by bad training, pretty nearly lost the power of thinking; for really to think of the locomotive means to put oneself in readiness to attach to it any of its essential characters that there may be occasion to consider; and this must be done by general signs, not by an image of the object. But the truth of the matter will more fully be brought out as we proceed.

All that we require of the definition may be put into a simpler shape by omitting the letter M, since the interpreter of the graph must well understand that the whole talk of the graphist for the time being, so far as it refers to things and not to the attributes or relations, has reference to the members of a cyclic system. We may consequently use the graph of Fig. 12 in place of Fig. 11.

It will be remarked that the graph of Fig. 12 is no more a definition of a cyclic system than it is of the relation of immediate antecedence; and this is as it should be; for plainly a system cannot

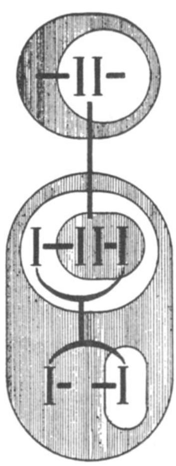

Fig. 12

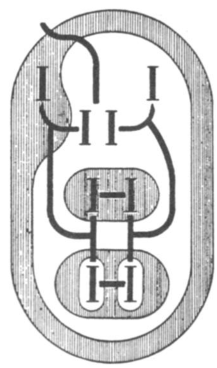

Fig. 13.

be defined, without virtually defining the relation between its members that constitutes it a system.

I will now begin by drawing one of several corollaries that are right at my hand. I am always using the words '*corollary*' and *theorem* in the strict sense of the foregoing definition. This corollary results from the logical principle that to every predicate there is a negative predicate which is true if the former is false, and is false if the former is true. This purely logical principle is expressed in the graph of Fig. 13. Obviously, if any predicate is both true of some member and false of some member of the system, the same will be the case with its negative. Consequently, by the definition, this negative will be true of some member without being true of any to which that member is A; or, in other words, the original predicate will be false of some member without being A to any member of which it is false. Thus, if any predicate is neither true of all nor false of all the members of any cyclic system, but is true of some one and false of some other, there will be two different members of one of which it is true without being true of any to which that member is A, while of the other it is false without being false of any to which that member is A. Or, to put the corollary in a different light, taking any predicate, P, whatsoever, then, in case you can prove that there cannot be more than one exception to the rule that every member of the system resembles some one of those to which it is A in respect to the truth or falsity concerning it of P, then if P be true of one member, it is true of all, and if it be false of one, it is false of all.

I am now going to apply this proposition to a theoric proof of a proposition which is really only a corollary from the definition of a cyclic system. My motive for this departure from good method is that it will afford a good illustration of the advantage of making the selected predicate, P, as special and characteristic of the state of things you are reasoning about as possible. The proposition I am going to prove is, that in any cyclic system that contains more than one member no member will be A to itself. For this purpose I will consider any member of the system you please, and will give it the proper name, N. This ecthetic step is already theoric, but is a matter of course. Another theoric step, not a matter of course, shall consist in my selecting, as the predicate to be considered, "is N." Now if N is A to itself, every member of the system of which this predicate is true (which can be none other than N itself,) will be A to a member of which the predicate is also true; and consequently, by the definition of a cyclic system this predicate cannot be true of one member and false of another. But if there be any other member of the system than N, it will be false of that one. Whence, if N were A to itself and were not the only member of the system, there would be no member of which it would be true that it was N. But by the definition, every cyclic system has some member, and N was chosen as such. So that it must be, either that the system has no other member, or that any member you please, and consequently *every one,* is non-A to itself.

Now what I wanted to point out was that if instead of "is N," I had selected, as my predicate to be considered, "is A to itself," it would merely have followed that since any member that is A to itself is A to a member that is A to itself, by the general definition either every member of the system is A to itself or none is so.

I will now prove that this proposition that no member of a cyclical system is A to itself unless it is the only member of the system is not a theorem, in any strict sense, by proving it corollarially. For this purpose I first prove that no cyclical system, by virtue of the same relation A, involves another as a part, but not the whole of it. For suppose that certain members of a cyclical system form by themselves a cyclical system constituted by the same A-hood. Then, by the part of the definition of a cyclical system that has been expressed as graph in Fig. 11 and in Fig. 12, there is a member of this minor system; and every member of it is A to a member of the major system that is a member of the minor system. Hence, by that same partial definition, the predicate "is a member of the minor system" being true of one is true of all members of the major system. The minor system is, then, the whole of the major system. To go further, I must employ that assertion of the definitum "is a cyclic system" concerning the definition-term, which assertion has not been expressed as a graph, in order to prove, by its conformity with the definition that a single object, having a relation, identity, to itself, that relation conforming to the conditions of the constitutive relation of a cyclical system, must be admitted to be a cyclical system of a single member. If, therefore, one of the members of a cyclical system of more than one member were A to itself, it would be a cyclical system which was a part but not the whole of another cyclical system, which we have seen to be impossible.

I shall now employ the first corollary to prove that every member of a cyclical system is A'd by some member. For take any member you please of any such system you please; and I will assign to it the proper name N. If then N is the only member of the system, by the definition N is A to itself. But if there be another member, it is one of which the predicate "is N" is not true, though there is some member, namely N, of which that predicate is true. Consequently, by that first corollary, there must be a member of which it is not true that it is N which is A to nothing of which this is not true. But, by the definition, every member of a cyclic system is A to some member; and therefore that member which is not A to any member of which "is N" is not true, must be true of a member of which "is N" is true, which, by hypothesis, is only N itself; consequently any member of any cyclic system which one may choose to select is A'd by some member, and by another than itself, if there be another. Q. E. D.

Further investigation of the properties of cyclic systems will need a somewhat more recondite theoric step. Certainly, however, I must not convey the idea that I claim to be quite sure of this. As yet, I have not sufficiently studied the methodeutic of theorematic reasoning. I only have an indistinct apprehension of a principle which seems to me to prove what I say; and I must confess that of all logical habits that of confiding in deductions from vague conceptions is quite the most vicious, since it is just such reasonings that to the intellectual rabble are the most convincing; so that the conclusions get woven into the general common-sense so closely, that it at length seems paradoxical and absurd to deny them, and men of "good sense" cling to them long after they have been clearly disproved. However, whether it be absolutely necessary or not, the only way I see, at present, of demonstrating the remaining properties of a cyclic system is to suppose a predicate to be formed by a process which will seem somewhat complicated. I shall not state what this predicate is, but only suppose it to be formed according to a rule; and even this rule will not be exactly stated but only a description of its provisions will be given. I shall suppose that one member of the system is selected by the rule as one of the class of subjects of which the predicate is true, and that the remaining members of this class shall be taken into it from among the members of the system *one by one,* according to the rule that when the member last taken in is not A to any member already taken in, one and one only of the members of the system not yet taken in to which that last adopted member is A is to be added to the class; and this new addition may, in the same way, require another. If the system were infinite (as we shall soon see that it cannot be,) this might go on endlessly; and so far, we have not seen that this cannot happen. But as soon as it happens that the member last admitted to the class is A to a member already admitted (and consequently that every member admitted to the class is A to an admitted member) the admissions to the class are to be brought to a stop. There are now two supposable cases to be provided for which we shall later find will never occur; but if we did not determine what was to be done if they should (this not being proved impossible) our first proof would involve a *petitio principii.* One is the case in which the finally adopted member is A to a member already having an A that had previously been admitted to the class. The other is the case in which the last (but not necessarily the final) adopted member is not only A'd by the *last previously* adopted member (for the sake of providing which with a member A'd by it, the very last was taken in) but is also A'd by an earlier adopted member. In the latter case, in which the member last adopted, which we may name V, is not only A'd by the last previous one, which we may name U, but is also A'd by a previously adopted member of the class which we may name K, we are to reject from the class all that were admitted after K to U inclusive; so that we revert to what would have been the case, as it might have been, if next after K we had admitted V, to which K is A. We should thus make the class smaller, which we shall soon see could not happen. In the other case, where the last adopted member, which we will name, Z, is A to a previously adopted one, which we will name J, which was not the first member adopted into the class, but is A'd by another, which we will name I, we reject from the class both I and all that were adopted previously to I.

After these supposititious rejections, there is no object of which the predicate, "is a member of the class so formed," is true that is not A of any object of which the same predicate is true, and therefore, by the definition so often appealed to, this predicate cannot be both true of a member of the cyclic system and false of another such member. Now it plainly is true of some member, since the first object taken into it as well as every one subsequently taken into it were members of the cyclic system. Therefore, this predicate cannot be false of any member of the cyclic system. In other words, the class so formed includes all the members of the cyclic system. Consequently, there cannot have been any rejections.

Since there were no rejections, the first member adopted must remain a member of the class; and since we have seen in a former corollary that every member of a cyclic system is A'd by a member of the same system, this first adopted member must be A'd by some member of the system, that is, by some member of the class. But by the rule of formation of the class no member of it except the finally adopted one can be A to a previously adopted member. It follows that there must be a finally adopted one; and by the same rule no member of the class except the first was adopted without there being a *last previously adopted* member. It follows that the succession of adoptions cannot, at any part of it, have been endless. This is one of the most difficult theorems that I had to prove.

Moreover, every member of the class is by the mode of formation A to one, and only to one, member of the class; and consequently the same is true of all the members of every cyclic system.

Moreover, every member of the class except the first was only taken in so as to be A'd by the last, or, at any rate, by one member only; and the first adopted member as we have seen is A'd by the finally adopted member. It cannot be A'd by any other, since by the rule of formation, such another would thereby have become the finally adopted member. Hence, no member of a cyclic system is A'd (in the same sense) by any two members of the system; or no two members are A to the same member.

I have thus, by means of this θεωρία of the formation of a certain kind of class, succeeded in demonstrating, what one might well have doubted, that from the proposition expressed in Fig. 11 follows the double uniqueness of the cyclical relation of A-hood or immediate antecedence. This is the principal, as I think, of those properties that are common and peculiar to cyclical systems. The same theoric step, or a reduplication of it, will enable the reader to prove other properties, common but not peculiar to cyclic systems; and especially that a collection the count of whose members in one order comes to an end can never in any order involve an endless process, whether it comes to an end or does not. There is, by the way, an important logical interest in that mode of succession in which an endless succession, say, of odd numbers, is followed by a beginningless diminishing succession of even numbers. For it shows that two classes of objects may have such a connection with a transitive relation, such as are those of causation, logical implication, etc., that any member of either class is *immediately* in this relation only to a member of the same class, while yet every member of one of the classes may be in this same relation to every member of the other class. Thus, it may be that thought only acts upon thought *immediately*, and matter *immediately* only upon matter; and yet it may be that thought acts on matter and matter upon thought, as in fact is plainly the case, somehow.

In this theoric step, it is noticeable that I have had to embody the idea of *antecedence* generally, in order to prove the properties of cyclical *immediate antecedence*. Any reasoner is always entitled to assume that the mind to which he makes appeal is familiar with the properties of antecedence in general; since if he were not so, he could not even understand what reasoning was at all about. For logical antecedence is an idea which no reasoner can unload or dispense with. It would have been easy to replace, in my demonstrations, all the "previously"s etc. by relations of inference. I have not done so in order not to burden the reader's mind with needlessly intricate forms of thought.

A corollary from what has already been proved is that if we regard the definition of Fig. 12 as the definition of A-hood, or cyclical immediate antecedence, then A-hood is not a single relation but is any one of a class of relations which, if the collection of all the members of the system is not very small, is a large class. For taking any two members of the system, and naming them Y and ϒ, we may form such a relation, that of A′-hood, that whatever is neither Y nor ϒ, nor is A to Y nor to ϒ is A′ to whatever is A'd by it, while whatever is A to Y is A′ to ϒ, whatever is A to ϒ is A′ to Y, whatever is A'd by Y is A″d by ϒ, and whatever is A'd by ϒ is A″d by Y; and then A′ will have the same general properties as A. Thus, if the number of members of a cyclic system is *m*, the number of relations of A-hood is $(m-1)!$ If *m* be seven, the number of A-relations is 720; etc.

There is no relation in a cyclic system exactly answering to general antecedence in a denumeral* system.

As a finitude is a positive complication (as is shown by a form of inference being valid in a finite system that is not elsewhere valid,) so in place of the relation of betweenness which in a linear system endless both ways, which, if those ways are not distinctively characterized, is triadic, we have in a cyclic system a tetradic relation expressible by α with four tails, so that Fig. 14, which means that an object which can, wherever it be in the cycle, pass from its position to that which is *next* to that position, being either A to it or A'd by it, will if at I be *opposite* to an object at J, relatively to any objects at U and at V. That is, such an object cannot move from I to J without passing through U and V. This implies that U is opposite to V relatively to I and J; that no other pair out of the four are opposite to each other relatively to the other pair; and that that way of passing round the cycle in which U is reached next after I is the way in which J is reached next after U, V next after J, and I next after V; while that way in which V is reached next after I is the way in which J is reached next after V, U next after J, and I next after U. This supposes that I, J, U, and V are all different, as those that are opposite must be unless two that are

* See Note at the end of the article.

adjacent are identical, in which case we may understand the relation as always being true and meaningless.

We may modify this relation, so as to render it exact, by de-

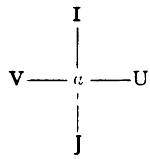

Fig. 14.

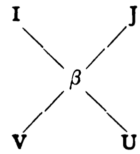

Fig. 15.

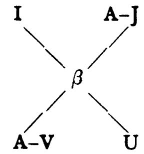

Fig. 16.

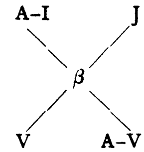

Fig. 17.

fining Fig. 15 as true, if I and J are identical while U and V are also identical; or if I and U are identical while J and V are identical, and also if Fig. 16 or Fig. 17 is true; but as not true unless necessarily so according to these principles. This last clause, by the way, has a very important logical form; but I shall not stop to comment upon it.

It will be observed that if Fig. 15 is true, then one or other of the graphs of Fig. 18 must be true. And if two α-relations hold,

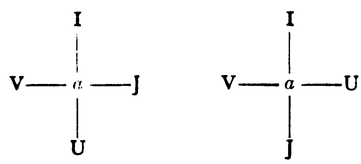

Fig. 18.

having three of their four correlates identical, and not the same pair being opposite in both, then two α-conclusions may be drawn in which the two correlates that only appeared once each in the premis-

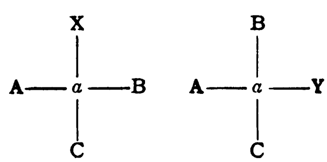

Fig. 19.

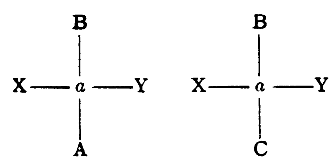

Fig. 20.

ses, appear together, and opposite to one another. Thus, from Fig. 19 may be inferred Fig. 20. The β-relation lends itself to much further inferential procedure. In the first place in Fig. 15, the whole graph may be turned round on the paper so as to bring each correlate into the place of its opposite. It may also be turned through 180° round a vertical axis in the sheet. [It may consequently be turned 180° round a horizontal axis in the sheet.] Moreover, the two correlates on the left, I and V, may be interchanged. [And so, consequently may J and U.] Moreover, from Fig. 21, we can infer Fig. 22. [Whence it follows that from Fig. 23 we can infer Fig

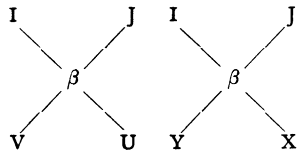

Fig. 21.

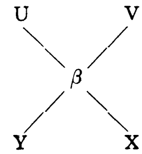

Fig. 22.

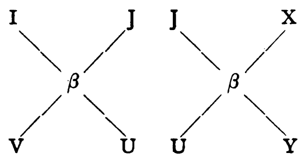

Fig. 23.

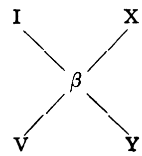

Fig. 24.

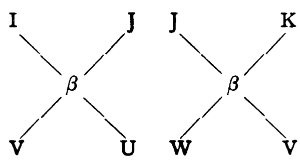

Fig. 25.

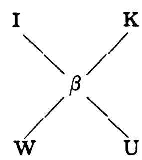

Fig. 26.

24.] Also, from Fig. 25 we can infer Fig. 26. Whence there follow very obviously several transformations. For example, Fig. 27 will be true; and if any three of the four graphs of Fig. 28 are true, so is the other one. It is obvious that the relation β involves cyclical

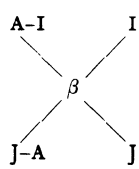

Fig. 27.

addition-subtraction, by its definition. Cyclic arithmetic involves no other *ordinal,* or climacote, numbers than cyclic ordinals. But if we define a *cardinal* number as an adjective essentially applicable, universally and exclusively, to a plural of a single multitude, then even the relations α and β may be said to depend upon the value of a cardinal number; namely, upon the modulus of the cycle; and no cardinal number is cyclic. Dedekind and others consider the pure abstract integers to be ordinal; and in my opinion they are not only right, but might extend the assertion to all real numbers. [But what I mean by an ordinal number precisely must be explained further on.] Nevertheless, the operations of addition, mul-

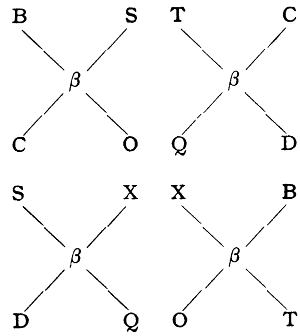

Fig. 28.

tiplication, and involution can be more simply defined if they are regarded as applied to cardinals, that is to multitudes, than if they are regarded in their application to ordinals.

Thus, the sum of two multitudes, M and N, is simply the multitude of a collection composed of the mutually exclusive collections of the multitudes M and N. The ordinal definition, on the other hand, must be that 0+X=X, whatever X may be, while (the ordinal next after Y)+X is the ordinal next after (Y+X). So the product of two multitudes M and N, is simply the multitude of units each composed of a unit of a collection of multitude M and a unit of multitude N; while the ordinal definition must be that 0×0=0 and that X×(the ordinal next after Y) is X+(X·Y) and the ordinal next after X×Y is (X·Y)+Y. So finally the multitude M raised to the power whose exponent is N, is the multitude of ways in which every member of a collection of multitude N can be related in a given

X	Y	Z	X	Y	Z	X	Y	Z
AB			A	B		A		B
X	Y	Z	X	Y	Z	X	Y	Z
B	A			AB			A	B
X	Y	Z	X	Y	Z	X	Y	Z
B		A		B	A			AB

Fig. 29.

way, each to some single member or other of a collection of multitude M. Thus $3^2=9$ because the different configurations of Fig. 29 are nine in number; while $2^3=8$ because the different configurations of Fig. 30 are eight in number. But a definition of involution which

A	B	A	B	A	B	A	B
X Y Z		X Y	Z	X Z	Y	X	Y Z
A	B	A	B	A	B	A	B
Y Z	X	Y	X Z	Z	X Y		X Y Z

Fig. 30.

shall be *purely* ordinal must be quite a complicated affair. We may say, for example, that $X^1=X$ and $X^{1+Y}=X\cdot X^Y$.

In cyclic addition, that is, in the α and β relations, there is but a single cardinal number to be dealt with; and this is fully dealt with in counting round and round the single cycle. But in multiplication there is always another cycle, and thus another cardinal number to be considered, although the modulus of the second cycle is usually such that it is not brought to our attention. But suppose that in a cycle of 72 we multiply the successive integers from zero up by 54. The following will be the result:

$$
\begin{aligned}
0\times 54 &= 0 = 72\\
1\times 54 &= 54 = -18\\
2\times 54 &= 36\\
3\times 54 &= 18\\
4\times 54 &= 72 = 0
\end{aligned}
$$

It will be seen that there is a cycle of modulus 4. Suppose that, instead of 54, we take 27 as the multiplicand. Then we shall have

$$
\begin{aligned}
0\times 27 &= 0 = 72\\
1\times 27 &= 27\\
2\times 27 &= 54 = -18\\
3\times 27 &= 9\\
4\times 27 &= 36\\
5\times 27 &= 63 = -9\\
6\times 27 &= 18\\
7\times 27 &= 45 = -27\\
8\times 27 &= 72 = 0
\end{aligned}
$$

By halving the multiplicand we have doubled the modulus. Suppose, however, that, instead of $\frac{1}{2}\times 54$, we take $\frac{1}{3}54=18$, as the multiplicand. Read the column of successive multiples of 54 upwards, and we shall see that the multiples of 18 have a cycle of modulus 4.

With 6 as the multiplicand we get a cycle of 12 for its multiples, the numbers being as follows:

6, 12, 18, 24, 30, 36, –30, –24, –18, –12, –6, 0

With 2×6 we get a cycle of $\frac{1}{2}$12, every other one. With 4×6 as multiplicand, we get a cycle of $\frac{1}{4}$12=3, with 8×12 as multiplicand, since 3 cannot be halved we still get 3. With 3·6=18 as multiplicand, we get a cycle of $\frac{1}{3}$×12, or every third of the multiples of 6, but with 3·18=54 as modulus, since 4 is not divisible by 3, we still get a cycle of 4. With 6·6=36 as multiplicand, we get every sixth multiple of 6, or two in all, 0 and 36. With 5×6, 7×6, and 11×6 since 12 is not divisible by 5, 7, or 11, we still get a modulus of 12. With 30, the order is as follows:

0, 30, –12, 18, –24, 6, 36, –6, 24, –18, 12, –30, 0.

This principle is obvious: if the multiples of a number N form a cycle of modulus K, and p is a prime number, then the multiples of pN will form a cycle of K/p, provided K is divisible by p; but otherwise, the modulus will remain K. Suppose, then, that the cycle of multiples of 1, that is to say, the cycle of our entire system of numbers is $p^a \cdot q^b$, where p and q are primes, and a and b are any whole numbers. If, then, we multiply 1 by $r^c \cdot s^d \cdot t^e$, where r, s, t are other primes than p and q, the modulus of the cycle of multiples of $r^c \cdot s^d \cdot t^e$ will remain $p^a \cdot q^b$. But every time we multiply this by p we divide the modulus by p, until we have so multiplied it a times. On the other hand, if, instead of multiplying 1 by $r^c \cdot s^d \cdot t^e$, we multiply it by $p^a \cdot q^b$ to get a new multiplicand, the modulus of the cycle of multiples of $p^a \cdot q^b$ will be 1; that is, all multiples will be equal. It will follow by the distributive principle, that $p^a \cdot q^b$ added to any number leaves that number unchanged. That is to say, the modulus of a cycle is the *zero* of that cycle. But right here I must explain what I mean by an *ordinal number.*

Take any enumerable, or finite, collection of distinct objects. Let there be recognized one special relation in which each of them stands to a single one of them, and no two to the same one, and such that any predicate whatsoever that is true of any one of them and is true of the one to which any one of which it is true stands in that relation, is true of all of them. This substantially defines that relation as the relation of "being A'd by." Thereby, that collection is recognized as forming a cyclical system of which those objects are members. But those objects will not in general be numbers of any kind. They may be days of the week or certain meridians of the Globe. But now consider a single "step," or substitution, by which the A of any member of the cyclic system is replaced by the member itself. From what member this step, or substitution began remains indefinite. The "step" still leads to a single member, and the step is a single kind of step even if that member be any member you please, in which case it is not a single, i. e. a singular, but the general member. I will condescend to meet the reader's probably indurated habit of crass nominalist thought by saying that, in the one case, it is a single member not definitely described, and in the other is a single member, left to him to choose; and there is no objection to this, if the member be supposed to be both existent and intelligible, both of which however it need not be. Give this kind of a step a proper name. Next consider in succession all the kinds of step each of which consists in first taking a step of the last previously considered kind and then substituting for the member which it puts in place of another, the member of which that member is A; so that the kinds of steps may be

From the A of a member to that member,
From the A of the A of a member to that member,
From the A of the A of the A of a member to that member,
etc. etc.

Now if each of these has a name, whether pronounced, scribed, or merely thought, those names will come round in a cycle of the same modulus as the original system. They will therefore form a cyclic system, but not a system of objects not essentially ordered, as the original system may have been. This system of names is a cyclic system of numbers. These are ordinal, or climacote, numbers. By ordinal numbers in general I mean names essentially denoting kinds of steps each from any member whatever of a system of objects to, at most, a single object of the system, (i. e., one or another object, depending on what object the step replaces by this other). Thus, as I use the term "ordinal number" I do not mean the absolute first, second, third, etc. member of a row of objects, but rather such as these: the same as, the first after, the second after, the third before, etc. These numbers are certainly "ordinal" in the sense of expressing relative order; yet it might be better to avoid possible misunderstanding by calling them *metrical numbers,* or more specifically, *climacode* or *climacote numbers.*

In order to push further our study of this subject, let us suppose a pack of 72 cards, numbered in order upon their faces, to be dealt into two piles. We will not directly consider those serial face-values, but only their differences. The two piles cannot regularly be reunited, because the difference of successive face-values in each, comes round in a cycle in each pile, the bottom card of the one pile, 1, being 2 more than the top card 71 (counting round the cycle of modulus 72) and that of the other pile also coming round in a cycle. The difference between the face-values of any two cards in either pile is a multiple of 2, the multiplier being the difference of position in that pile. If now we desire so to re-deal the cards of the one pile and the other into any number n of piles, as to produce the same effect as if they had originally been dealt into $2n$ piles, we must first deal the first pile leaving room between every two of the new piles for the piles to be produced by dealing the second pile. If for the number, n, we take 8, we shall get sixteen piles, the first 8 of 5 cards each and the last 5 of 4; and now it is allowable and proper to place each of the first 8 piles on the pile 8 piles further advanced; or equally so to place each of the last 8 piles on the pile 8 piles further advanced, counting round and round the cycle of modulus 16. In either case the cards of each composite pile so formed will form a cycle, successive face-values increasing (round and round the cycle of 72) by 16. The rule for gathering the piles is just the same as that previously given, except that one must confine oneself to piles *of the same set.* For instance if 72 cards, numbered as just described, get in any way dealt into 15 piles, the top cards of the piles will have these values:

61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 58, 59, 60

Now since $15 | 72 = 3$ these are in 5 sets of 3 piles, thus

61,	64,	67,	70,	58,
62,	65,	68,	71,	59,
63,	66,	69,	72,	60.

We shall therefore put the pile headed by 72 on the pile headed by 69, because there is only one pile of the set to the right of the former, and these on the pile headed by 66, and these on that headed by 63, and finally all four on the one headed by 60. So we shall in the next set begin with the pile headed by 71, the last of the larger piles.

We shall thus get the whole pack divided into three portions, and there is absolutely no way of getting them back into a single pack except by *undealing* them, that is by cutting the cards one by one from the three portions in turn, round and round.

This general rule holds in all cases; as much when the entire number of cards is prime as when it is composite. For a prime number is one whose greatest common divisor with any smaller positive integer is 1, while, of course, like any other number, its greatest divisor common to itself is itself.

Having thus fully explained the dealing into any number of piles of any number of cards, prime or composite, I revert, after this almost interminable disquisition, to the subject of cyclic logarithms. I have confined, and shall continue to confine, my study of these to logarithms of numbers whose cycle has a prime modulus. Then, the modulus of the cycle of the logarithms being one less than that of the natural numbers cannot be prime. Still so long as it is a question of employing the logarithms merely to multiply two numbers, the logarithm of the product is simply the sum of the logarithms of multiplier and multiplicand; and in addition it makes no difference whether the modulus be prime or composite. But when it comes to raising numbers to powers or to extracting their roots, the divisors of the number one less than the modulus have to be considered. The modulus being prime, the number one less must be divisible by 2. If 2 be the only prime factor, the modulus must be 3 or 5 or 17 or 65537 or much greater yet. As an example, let us take the modulus 17. Then the following two pairs of tables show the logarithms for the 8 different bases 3, 5, 6, 7, 10, 11, 12, 14.

Nat. nos.	−16	−14	−8	−7	−4	−12	−2	−6	−1	−3	−9	−10	−13	−5	−15	−11	−16
	1	3	9	10	13	5	15	11	16	14	8	7	4	12	2	6	1
Logs.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0

Nat. nos.	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Logs.	0	14	1	12	5	15	11	10	2	3	7	13	4	9	6	8
	−16	−2	−15	−4	−11	−1	−5	−6	−14	−13	−9	−3	−12	−7	−10	−8

Nat. nos.	−16	−12	−9	−11	−4	−3	−15	−7	−1	−5	−8	−6	−13	−14	−2	−10
	1	5	8	6	13	14	2	10	16	12	9	11	4	3	15	7
Logs.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1

Nat. nos.	−16	−15	−14	−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Logs.	0	6	13	12	1	3	15	2	10	7	11	9	4	5	14	8
	−16	−10	−3	−4	−15	−13	−1	−14	−6	−9	−5	−7	−12	−11	−2	−8

Of course, none of the even numbers can be logarithms of a possible base of another system since with a modulus 16 no multiple of an even number can be 1, the logarithm of the base. On the other hand, every odd number is in every system of logarithms the logarithm of some base.

If, instead of 13 cards and 12, the "trick" be done with 17 and

16, say the first eight hearts *increasingly* and then the first eight diamonds *decreasingly,* with the joker or king of hearts to make up 17 and with the first eight spades to correspond with the hearts and the first eight clubs to correspond with the diamonds, laying down the black cards on the table, in *two* rows, one of eight from left to right, and the other below from right to left, after having dealt the black cards 16 times into three piles and every time exchanging the top card of the middle pile for the topmost red card, so as to bring the ace of spades into the right-hand-most place of the upper row, then having done the trick substantially as above described, there is a very pretty way in which you can ask into what *odd* number of piles the *black* cards shall be dealt and then dealing out the red cards, *minus* the extra one 16 times exchanging a card each time for the three court cards and ten of each suit, so as to again render the black ones the index of the places of the red ones. But I leave it to the reader's ingenuity to find out exactly how this is to be done. *Beware of the moduli.*

There is much more to be said on this subject, but I leave it for the reader to investigate.

CHARLES SANTIAGO SANDERS PEIRCE.

MILFORD, PA.

NOTE REFERRED TO ON PAGE 452.

Denumeral is applied to a collection in one-to-one correspondence to a collection in which every member is immediately followed by a single other member, and in which but a single member does not, immediately or mediately, follow any other. A collection is in one-to-one correspondence to another, if, and only if, there is a relation, *r,* such that every member of the first collection is *r* to some member of the second to which no other member of the first is *r,* while to every member of the second some member of the first is *r,* without being *r* to any other member of the second. The positive integers form the most obviously denumeral system. So does the system of all real integers, which, by the way, does not pass through infinity, since infinity itself is not part of the system. So does a Cantorian collection in which the endless series of all positive integers is immediately followed by ω_1, and this by ω_1+1, this by ω_1+2, and so on endlessly, this endless series being immediately followed by $2\omega_1$. Upon this follow an endless series of endless series all positive integer coefficients of ω_1 being exhausted, whereupon immediately follows ω_1^2, and in due course $x\omega_1^2+y\omega_1+z$, where x, y, z, are integers; and so on; in short, any system in which every member can be described so as to distinguish it from every other by a finite number of characters joined together in a finite number of ways, is a denumeral system. For writing the positive whole numbers in any way, most systematically thus:

1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, etc.

it is plain that an infinite square matrix of pairs of such numbers can be arranged in one series, by proceeding along successive bevel lines thus: (1, 1); (1, 10); (10, 1): (1, 11); (10, 10); (11, 1): (1, 100); (10; 11); (11; 10); etc. and consequently whatever can be arranged in such a square can be arranged in one row.

Thus an endless square of quaternions such as the following can be so arranged:

[(1,1) (1,1)] : [(1,1) (1,10)]; [(1,1) (10,1)] : [(1,1) (1,11)]; etc.
[(1,10) (1,1)] : [(1,10) (1,10)]; [(1,10) (10,1)] : [(1,10) (1,11)]; etc.
[(10,1) (1,1)] : [(10,1) (1,10)]; [(10,1) (10,1)] : [(10,1) (1,11)]; etc.
[(1,11) (1,1)] : [(1,11) (1,10)]; [(1,11) (10,1)] : [(1,11) (1,11)]; etc.

Consequently whatever can be arranged in a block of any finite number of dimensions can be arranged in a linear succession. Thus it becomes evident that any collection of objects, every one of which can be distinguished from all others by a finite collection of marks joined in a finite number of ways can be of no greater than the denumeral multitude. (The bearing of this upon Cantor's ω^ω is not very clear to my mind.) But when we come to the collection of all irrational fractions, to exactly distinguish each of which from all others would require an endless series of decimal places, we reach a greater multitude, or grade of maniness, namely, the *first abnumerable multitude*. It is called "abnumerable," to mean that there is, not only no way of counting the single members of such a collection so that, at last, every one will have been counted (in which case the multitude would be *enumerable*), but, further, there is no way of counting them so that every member will after a while get counted (which is the case with the single multitude called *denumeral*). It is called the *first* abnumerable multitude, because it is the smallest of an endless succession of abnumerable multitudes each smaller than the next. For whatever multitude of a collection of single members μ may denote, 2^μ, or the multitude of different collections, in such collection of multitude μ, is always greater than μ. The different members of an abnumerable collection are not capable of being distinguished, each one from all others, by any finite collection of marks or of finite sets of marks. But by the very definition of the first abnumerable multitude, as being the multitude of collections (or we might as well say of denumeral collections) that exist among the members of a denumeral collection, it follows that all the members of a first-abnumerable collection are capable of being ranged in a linear series, and of being so described that, of any two, we can tell which comes earlier in the series. For the two denumeral collections being each serially arranged, so that there is in each a first member and a singular next later member after each member, there will be a definite first member in respect to containing or not containing which the two collections differ, and we may adopt either the rule that the collection that contains, or the rule that the collection that does not contain, this member shall be earlier in the series of collections. Consequently a first-abnumerable collection is capable of having all its members arranged in a linear series. But if we define a *pure* abnumerable collection as a collection of all collections of members of a denumeral collection each of which includes a denumeral collection of those members and excludes a denumeral collection of them, then there will be no two among all such pure abnumerable collections of which one follows next after the other or of which one next precedes the other, according to that rule. For example, among all decimal fractions whose decimal expressions contain each an infinite number of 1s and an infinite number of 0s, but no other figures, it is evident that there will be no two between which others of the same sort are not intermediate in value. What number for instance is next greater or next less than one which has a 1 in every place whose ordinal number is prime and a zero in every place whose ordinal number is composite? ·1110101000101000101000100001 etc. Evidently, there is none; and this being the case, it is evident that all members of a pure second-abnumerable collection, which both contains and excludes among its members first-abnumerable collections formed of the members of a pure first-abnumerable collection, cannot, in any *such* way, be in any linear series. Should further investigation prove that a second-abnumeral multitude can in *no way* be linearly arranged, my former opinion that the common conception of a line implies that there is room upon it for any multitude of points whatsoever will need modification.

Certainly, I am obliged to confess that the ideas of common sense are not sufficiently distinct to render such an implication concerning the continuity of a line evident. But even should it be proved that no collection of higher multitude than the first abnumerable can be linearly arranged, this would be very far from establishing the idea of certain mathematico-logicians that a line consists of points. The question is not a physical one: it is simply whether there can be a consistent conception of a more perfect continuity than the so-called "continuity" of the theory of functions (and of the differential calculus) which makes the continuum a first-abnumerable system of points. It will still remain true, after the supposed demonstration, that no collection of points, each distinct from every other, can make up a line, no matter what relation may subsist between them; and therefore whatever multitude of points be placed upon a line, they leave room for the same multitude that there was room for on the line before placing any points upon it. This would generally be the case if there were room only for the denumeral multitude of points upon the line. As long as there is certainly room for the first denumerable multitude, no denumeral collection can be so placed as to diminish the room, even if, as my opponents seem to think, the line is composed of actual determinate points. But in my view the unoccupied points of a line are mere possibilities of points, and as such are not subject to the law of contradiction, for what merely *can be* may also *not be*. And therefore there is no cutting down of the possibility *merely* by some possibility having been actualized. A man who can see does not become deprived of the power merely by the fact that he has seen.

The argument which seems to me to prove, not only that there is such a conception of continuity as I contend for, but that it is realized in the universe, is that if it were not so, nobody could have any memory. If time, as many have thought, consists of discrete instants, all but the feeling of the present instant would be utterly non-existent. But I have argued this elsewhere. The idea of some psychologists of meeting the difficulties by means of the indefinite phenomenon of the span of consciousness betrays a complete misapprehension of the nature of those difficulties.

Added, 1908, May 26. In going over the proofs of this paper, written nearly a year ago, I can announce that I have, in the interval, taken a considerable stride toward the solution of the question of continuity, having at length clearly and minutely analyzed my own conception of a *perfect continuum* as well as that of an *imperfect continuum,* that is, a continuum having *topical singularities,* or places of lower dimensionality where it is interrupted or divides. These labors are worth recording in a separate paper, if I ever get leisure to write it. Meantime, I will jot down, as well as I briefly can, one or two points. If in an otherwise unoccupied continuum a figure of lower dimensionality be constructed,—such as an oval line on a spheroidal or anchor-ring surface,—either that figure is a part of the continuum or it is not. If it is, it is a topical singularity, and according to my concept of continuity, is a breach of continuity. If it is not, it constitutes no objection to my view that all the parts of a perfect continuum have the same dimensionality as the whole. (Strictly, all the *material,* or *actual,* parts, but I cannot now take the space that minute accuracy would require, which would be many pages.) That being the case, my notion of the essential character of a perfect continuum is the absolute generality with which two rules hold good, 1st, that every part has parts; and 2d, that every sufficiently small part has the same mode of immediate connection with others as every other has. This manifestly vague statement will more clearly convey my idea (though less distinctly,) than the elaborate full explication of it could. In endeavoring to explicate "immediate connection," I seem driven to introduce the idea of time. Now if my definition of continuity involves the notion of immediate connection, and my definition of immediate connection involves the notion of time; and the notion of time involves that of continuity, I am falling into a *circulus in definiendo.* But on analyzing carefully the idea of Time, I find that to say it is continuous is just like saying that the atomic weight of oxygen is 16, meaning that that shall be the standard for all other atomic weights. The one asserts no more of Time than the other asserts concerning the atomic weight of oxygen;—that is, just nothing at all. If we are to suppose the idea of Time is wholly an affair of immediate consciousness, like the idea of royal purple, it cannot be analyzed and the whole inquiry comes to an end. If it can be analyzed, the way to go about the business is to trace out in imagination a course of observation and reflection that might cause the idea (or so much of it as is not mere feeling) to arise in a mind from which it was at first absent. It might arise in such a mind as a hypothesis to account for the seeming violations of the principle of contradiction in all alternating phenomena, the beats of the pulse, breathing, day and night. For though the *idea* would be absent from such a mind, that is not to suppose him blind to the *facts*. His hypothesis would be that we are, somehow, in a situation like that of sailing along a coast in the cabin of a steamboat in a dark night illumined by frequent flashes of lightning, and looking out of the windows. As long as we think the things we see are the same, they seem self-contradictory. But suppose them to be mere aspects, that is, relations to ourselves, and the phenomena are explained by supposing our standpoint to be different in the different flashes. Following out this idea, we soon see that it means nothing at all to say that time is unbroken. For if we all fall into a sleeping-beauty sleep, and *time itself stops during the interruption,* the instant of going to sleep is absolutely unseparated from the instant of waking; and the interruption is merely in our way of thinking, not in time itself. There are many other curious points in my new analysis. Thus, I show that my true continuum might have room only for a denumeral multitude of points, or it might have room for just any abnumeral multitude of which the units are in themselves capable of being put in a linear relationship, or there might be room for all multitudes, supposing no multitude is contrary to a linear arrangement.

ESPOSIZIONE DEL METODO DEI MINIMI QUADRATI.

PER ANNIBALE FERRERO, *Tenente Colonnello di Stato Maggiore, ec. Firenze, 1876.*

BY CHARLES S. PEIRCE, *New York.*

RECENT discussions in this country, of the literature of the method of Least Squares, have passed by without mention the views of the accomplished chief of the geodetical division of the Italian Survey, as set forth in the work above cited, which was first published, in part, in 1871. The subject is here, for the first time, in my opinion, set upon its true and simple basis; at all events the view here taken is far more worthy of attention than most of the proposed proofs of the method.

Lieut. Col. Ferrero begins by considering the principles of the arithmetical mean. A quantity having been directly observed, a number of times, independently, and under like circumstances, the value which might be inferred from the observations is, in the first place, a symmetrical function of the observed quantities; for, if the observations are independent, the order of their occurrences is of no consequence, and the circumstances under which they are taken, differ in no assignable respect, except that of being taken at different times. In the second place, the value inferred must be such a function of the values observed, that when the latter are all equal, the former reduces to this common value. The author calls functions having these two properties, (1st, that of being symmetrical with respect to all the variables, and 2d, that of reducing to the common value of the variables when these are all equal,) *means*. There is a whole class of functions of this sort, such as the arithmetic mean, the geometrical mean, the arithmetic-geometrical mean of Gauss, the quadratic mean,* and many others instanced in the text. It is shown, without difficulty, that these means are continuous functions, and that their value is intermediate between the extreme values of the different variables, when the latter do not differ greatly.

Let o', o'', o''', etc. denote the values given by the observations. Let n denote the number of the observations; let p denote the arithmetical mean;

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* This seems the appropriate name for $\sqrt{\frac{[x^2]}{n}}$.

and let x', x'', x''', etc. denote the excess of the observed values over the arithmetical mean. Then write

$$V = f\,(o', o'', o''', \text{etc.})$$

for any mean of the observations. Develop this function according to powers of x', x'', x''', etc. We have

$$V = f\,(p + x', p + x'', p + x''', \text{etc.})$$
$$= f\,(p, p, p, \text{etc.}) + \frac{dV}{dp}\,(x' + x'' + x''' + \text{etc.}) + \Delta\,;$$

where Δ denotes the terms of higher orders.

Since $$x' + x'' + x''' + \text{etc.} = 0,$$
and $$f\,(p, p, p, \text{etc.}) = p,$$
this reduces to

$$V = p + \Delta.$$

In considering the value of Δ, we may limit ourselves to terms of the second order. As the partial differentials of any species and order, relatively to o', o'', o''', etc. all become equal when x', x'', x''', etc. vanish, we may write

$$\frac{d^2V}{do'^2} = \frac{d^2V}{do''^2} = \frac{d^2V}{do'''^2} = \text{etc.} = \beta$$
$$\frac{d^2V}{do'.do''} = \frac{d^2V}{do''.do'''} = \text{etc.} = \gamma$$

then

$$\Delta = \tfrac{1}{2}\,\beta\,(x'^2 + x''^2 + x'''^2 + \text{etc.}) + \gamma\,(x'x'' + x'x''' + \text{etc.})\,.$$

But the square of $[x] = 0$, gives

$$\Sigma\, x\, x' = -\,\tfrac{1}{2}\,[x^2]\,,$$

so that

$$\Delta = \frac{\beta - \gamma}{2}\,[x^2] = k\,\frac{[x^2]}{n}\,,$$

where k is a quantity which does not increase indefinitely with n. Now, when the observations are good, $\frac{[x^2]}{n}$ is not large, and, therefore, in such a case no mean will differ very much from the arithmetical mean. The latter, being the simplest to deal with, may therefore be used without great disadvantage. Such is, according to Colonel Ferrero, the utmost defence of the principle which can be made to cover all the cases in which it is usual to employ the method; and all further defence of it is more or less limited in its application.

In very many cases, however, it is easy to see that either in regard to the quantity directly observed, or in regard to some function of it, the zero of the scale of measurement, and the unit of the same scale, are both arbitrary. For instance, in photometric observations, this is true of the logarithm of the light. In such cases, considering such function to be the observed quantity, we have there two principles, first proposed, in connection with a really superfluous third one, by Schiaparelli.

1st. The mean to be adopted must be such that if each observed value is multiplied by any constant, the result is increased in the same ratio.

2d. The mean to be adopted must be one which is increased by a constant o, when each observed value is increased by the same constant.

Our author's treatment of these principles is exceedingly neat. Using the same notation as above, write

$$V = p + A_2 + A_3 \ldots + A_n \ldots$$

where A_n is the sum of the terms of the order n in x', x'', x''', etc. The general term A_n is, therefore, of the form $A_n = \alpha\Sigma x^n + \beta\Sigma x'^{n-1}x'' + \gamma\Sigma x'^{n-2}x''^2 \ldots + \zeta\Sigma x'^r x''^s x'''^t \ldots$ where Σ expresses the symmetrical sum of similar terms. In the general term $r + s + t +$ etc. $= n$. Since ζ is evidently a function of p, we may put $\zeta = \phi(p)$, and it remains to find the form of this function. Multiplying every o by c, p is changed to cp, x to cx, and the general term $\zeta\Sigma x'^r x''^s x'''^t$ etc. $= \phi(p)\,\Sigma x'^r x''^s x'''^t$ etc. is changed to $\phi(cp)\,c^n\Sigma x'^r x''^s x'''^t$ etc. Since, therefore, V is changed to cV, we have $\phi(cp)\,c^n = \phi(p)\,c$. Putting $p = 1$, $\phi(c) = \frac{\phi(1)}{c^{n-1}}$. Denoting this numerator by ξ_1, the general term becomes

$$A_n = \frac{1}{p^{n-1}}\left[\alpha_1\Sigma x^n + \ldots + \xi_1\Sigma x'^r x''^s x'''^t \ldots + \ldots\right],$$

where α, ξ, etc., are numerical coëfficients independent of p. From this circumstance it follows that the quantity in square brackets, which may be called A'_n, does not change when the same constant quantity k is added to all the observed quantities o', o'', o''', etc.; for such an addition only increases p by this same constant, and leaves x', x'', x''', etc., unchanged. Thus the mean in question, which may now be written

$$V = p + \frac{A'_2}{p} + \frac{A'_3}{p^2} + \text{etc.},$$

becomes, in consequence of such an addition,

$$V_k = p + k + \frac{A'_2}{p+k} + \frac{A'_3}{(p+k)^2} + \text{etc.}$$

But by principle No. 2, it becomes,

$$V_k = p + k + \frac{A'_2}{p} + \frac{A'_3}{p^2} = \text{etc.}$$

So that, $A'_2 = A'_3 =$ etc. $= 0$, and we have

$$V = p,$$

or the arithmetical mean is the only one which conforms to the given conditions.

Another still more special case, is that contemplated by the demonstrations of Laplace, Poisson, Hagen, Crofton, etc. It is treated by our author, but need not be considered in this notice.

It may be of interest to see how Colonel Ferrero is able, without basing least squares expressly upon the theory of probabilities, to derive the formula for finding mean error. Using always the same notation, he terms

$$m = \sqrt{\frac{[x^2]}{n}}$$

the *mean residual* of the observations.

Suppose, then, that there be an indefinitely great *series of series* of observations of the same quantity, each lesser series consisting of n observations, and each having the same mean residual. Then, there being an infinite number of such series, the mean of their mean results may be taken as the true value, by definition. For the ultimate result of indefinitely continued observation is all that we aim at in sciences of observation. Then the number of the lesser series being q, the result will be

$$V = \frac{[p]}{q}.$$

Adopt the notation

$$\delta = p - V \quad \delta_1 = p_1 - V \quad \delta_2 = p_2 \quad \ldots,$$

then δ, δ_1, δ_2, etc., are the true errors of p, p_1, p_2, etc. Let y'_0, y''_0, y'''_0, etc. be the true errors of the first series of observations, y'_1, y''_1, y'''_1 etc. those of the second series, and so for the others. We have, then, $y = o - V = o - p + \delta = x + \delta$.

Squaring and summing for the nq values of y, we have

$$\Sigma y^2 = \Sigma x^2 + \Sigma\delta^2 + 2\Sigma x \Sigma\delta$$

or, since $\Sigma x = 0$, and $\Sigma\delta = 0$,

$$\Sigma y^2 = \Sigma x^2 + \Sigma\delta^2.$$

Now if η be the quadratic mean of the error of p, we have $\Sigma\delta^2 = nq\eta^2$, and

$$\Sigma y^2 = nqm^2 + nq\eta^2,$$

or the mean error μ of an observation is given by

$$\mu^2 = \frac{\Sigma y^2}{nq} = m^2 + \eta^2.$$

But it is easily shown (from the equality of positive and negative errors) that

$$\eta^2 = \frac{\mu^2}{n}$$

whence

$$\mu = \sqrt{\frac{[x^2]}{n-1}}.$$

With regard to the mode of passing from the principle of the arithmetical mean to the general method of least squares, the best way seems to be first to prove that the solution of the equations

$$a_1 x = n_1$$
$$a_2 x = n_2$$
$$\text{etc.,}$$

is $x = \frac{[an]}{[a^2]}$. This is easy, after the rule for the error of a mean is established. Then, having given the equations

$$a_1 x + b_1 y + c_1 z + \text{etc.} = n_1$$
$$a_2 x + b_2 y + c_2 z + \text{etc.} = n_2;$$

first, consider these as similar to the equations just given; thus,

$$a_1 x = n_1 - b_1 y - c_1 z - \text{etc.,}$$
$$a_2 x = n_2 - b_2 y - c_2 z - \text{etc.,}$$
$$\text{etc.,}$$

whence we obtain the first normal equation,

$$x = \frac{[an_1] - [ab]\,y - [ac]\,z - \text{etc.}}{[a^2]}$$

and the others in a similar way.

The treatise of Colonel Ferrero may be recommended to those desirous of having a thorough practical acquaintance with the method, as decidedly the best and clearest on the subject.