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Episteme 7

**Philosophy of Geometry
from
Riemann to Poincaré**

by
Roberto Torretti

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PHILOSOPHY OF GEOMETRY FROM RIEMANN TO POINCARÉ

by

ROBERTO TORRETTI

Universidad de Puerto Rico

Philosophical concern with geometry has rested on two main points: (i) geometry with its 'long chains of reasons' has been a paradigm of sound systematic knowledge; (ii) as a mathematical theory of space, geometry provides the basic framework for the exact description of physical phenomena and is therefore a focal point for the philosophy of natural science. The development of non-Euclidean geometries in the 19th century brought about a reappraisal of geometry in both respects. This book gives a historical and critical analysis of the philosophical debate on the nature, scope and foundations of geometry from 1850 to 1900, when the conceptual basis was laid for Einstein's geometrodynamical theory of gravitation and cosmology. No such detailed study of the subject has appeared since Bertrand Russell's *Foundations of Geometry* (1897), and yet much of contemporary epistemology is rooted in the philosophy of geometry from Riemann to Poincaré.

Audience:

The book will be of interest to students of history and philosophy of science. It can also be used as a textbook in graduate courses and for supplementary reading in undergraduate courses.

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VOLUME 7

ROBERTO TORRETTI

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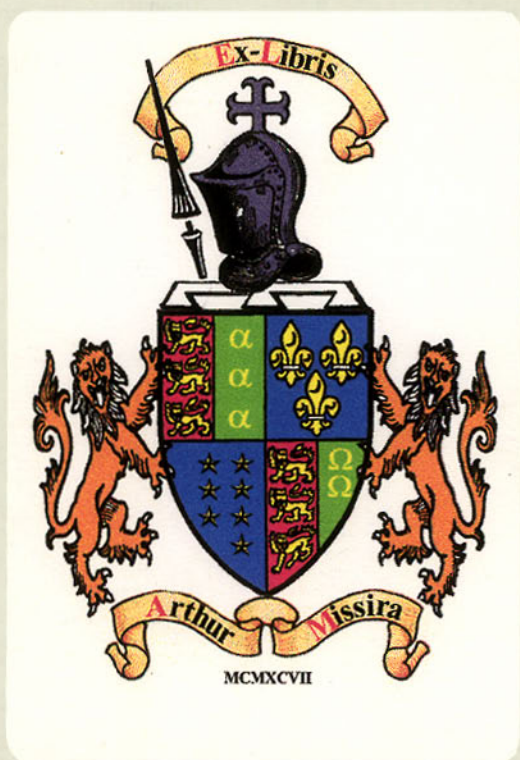
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To Carla

αίγλα διόσδοτος

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PREFACE

Geometry has fascinated philosophers since the days of Thales and Pythagoras. In the 17th and 18th centuries it provided a paradigm of knowledge after which some thinkers tried to pattern their own metaphysical systems. But after the discovery of non-Euclidean geometries in the 19th century, the nature and scope of geometry became a bone of contention. Philosophical concern with geometry increased in the 1920's after Einstein used Riemannian geometry in his theory of gravitation. During the last fifteen or twenty years, renewed interest in the latter theory – prompted by advances in cosmology – has brought geometry once again to the forefront of philosophical discussion.

The issues at stake in the current epistemological debate about geometry can only be understood in the light of history, and, in fact, most recent works on the subject include historical material. In this book, I try to give a selective critical survey of modern philosophy of geometry during its seminal period, which can be said to have begun shortly after 1850 with Riemann's generalized conception of space and to achieve some sort of completion at the turn of the century with Hilbert's axiomatics and Poincaré's conventionalism. The philosophy of geometry of Einstein and his contemporaries will be the subject of another book.

The book is divided into four chapters. Chapter 1 provides background information about the history of science and philosophy. Chapter 2 describes the development of non-Euclidean geometries until the publication of Felix Klein's papers 'On the So-called Non-Euclidean Geometry' in 1871–73. Chapter 3 deals with 19th-century research into the foundations of geometry. Chapter 4 discusses philosophical views about the nature of geometrical knowledge from John Stuart Mill to Henri Poincaré.

Modern philosophy of geometry cannot be separated from investigations concerning fundamental geometrical concepts which have been conducted by professional mathematicians in what are

usually considered to be purely mathematical terms. Thus the work of Bernhard Riemann, Sophus Lie and Moritz Pasch plays a prominent role in the history we shall recount. I have often resorted to 20th-century mathematical concepts for clarifying the thoughts of 19th-century mathematicians. Though this procedure can be questioned from a strictly historical point of view, I find that it favours our philosophical understanding. The Appendix on pp.359ff. defines many of the mathematical concepts used and tells where to find a definition of the rest.

Paragraphs preceded by an asterisk (*) contain supplementary remarks which are generally more important than those relegated to the Notes, but which nevertheless may be omitted without loss of continuity.

References to the literature are given throughout the book in an abbreviated manner. A key to the abbreviations is furnished in the Reference list on pp.420ff. The latter also acknowledges my obligation to many of the writers from whom I have learned. I offer my apologies to those I have omitted, either through inadvertence or because their works were not directly relevant to the present subject.

In writing this book I have been helped by many persons to whom I am deeply grateful. My greatest debt is to Professor Mario Bunge. Plans for the book were examined by him at several stages of its development, and it is clear to me that without his encouragement and advice the book would never have been written. In the course of writing it, I have discussed many difficult or controversial passages with Carla Cordua and the final text owes much to her sound philosophical judgment. Professor Hans Freudenthal has given me his advice on one particular question I submitted to him and has kindly sent me copies of some of his remarkable contributions to geometry and its history and philosophy. The publisher's referees pointed out several mistakes which I hope have been removed. Professors Millard Hansen and Michael Reck and Mr Robert Blohm read substantial parts of the manuscript and have greatly contributed to improve my English. I also owe many valuable indications on style to Dr D.J. Larner. Of course, none of the persons mentioned can be held responsible for the errors which will doubtless still be found in the book, for I have retained many of my views and idioms notwithstanding their objections. Professors Ramón Castilla, Gerhard Knauss and Pedro Salazar, Miss Carola Rosa, Mrs Josefina Santiago and Mr Christian Hermansen have,

at different places, traced and obtained for me some of the sources on which my research rests.

The book was written from September 1974 to July 1976. During half that time, I enjoyed a sabbatical leave; during the other half, I held a John Simon Guggenheim Memorial Fellowship. The John Simon Guggenheim Memorial Foundation has also awarded a subsidy for the publication of the book. I am happy to record my gratitude to the University of Puerto Rico and the John Simon Guggenheim Memorial Foundation for their generous support.

Isla Verde (Puerto Rico), July 1978

CHAPTER 1

BACKGROUND

Modern philosophy of geometry is closely associated with non-Euclidean geometry and may almost be said to stem from it. The long history leading to the discovery of non-Euclidean geometry will be summarized in the first sections of Chapter 2. The present chapter touches upon other aspects of the historical background of our subject, which will be useful in our subsequent discussions. In the first three sections of this chapter, we shall deal with the Greek beginnings of geometry and philosophy, the uses of geometry in Greek and early modern natural science, and the metaphysics of space that was part and parcel of the accepted view of nature from the 17th to the 19th century. In the fourth and last section, we shall discuss the method of coordinates introduced by René Descartes for describing geometrical configurations and relations in space.

1.0.1 *Greek Geometry and Philosophy*

Geometry and philosophy are still called in English and other modern languages by their Hellenic names and for our present purposes we need not seek their origins beyond ancient Greece. It is true that the Greeks themselves liked to trace their sciences back to Oriental sources, and they credited several of their great philosopher-geometers with educational trips to the Middle East. The priestly establishments of the Egyptian and Mesopotamian civilizations had long enjoyed the kind of leisure which Aristotle regarded as a prerequisite of the quest for knowledge,¹ and had developed a variety of notions about things in general which no doubt provided a stimulus and a starting-point for the speculations of the Greeks. But all this traditional Oriental wisdom was quite foreign to the self-assertive, yet self-critical and argumentative method of free individual inquiry the Greeks called philosophy. On the other hand, though the extant monuments of Egyptian mathematics do not suggest that the Greeks could have learnt much from them, Babylonian problem-books of the 17th century B.C. bear witness to a remarkable algebraic ability and

sophistication. Cuneiform texts have also been found which apparently presuppose acquaintance with the fundamental theorem of Euclidean geometry generally known as the Pythagorean theorem. However, the explicit statement of such general propositions is consistently avoided, at least in the documents which have survived, and no attempt is made to order mathematical lore in deductive chains. Yet it is mainly because of the universal scope and the necessary concatenation of its statements that geometry has time and again commanded the attention of philosophers, challenging their epistemological ingenuity and exciting their ontological imagination. It is therefore not unjustified, in a book on the philosophy of geometry, to ignore pre-Greek mathematics and to assume naïvely that both philosophy and geometry were born together, say, in the mind of Thales the Milesian, about 600 B.C. Thales, at any rate, was reportedly the first thinker to derive all things from a single perennial material principle and also the first to demonstrate geometrical theorems.²

It is very likely that the first geometrical demonstrations consisted of diagrams that plainly exhibited the relations they were intended to prove. A good example of this kind of demonstration is given in the mathematical scene in Plato's *Meno*, in which Socrates leads a young uneducated servant to see that the square built on the diagonal of another square is twice as large as the latter.³ It is not difficult to understand how one can arrive at general conclusions by looking at particular diagrams. An intelligent look is not overwhelmed by the rich fullness of its object but pays attention only to some of its features. Any other object which shares these features will also share all those properties and relations which are seen to go with them inevitably.⁴ However, Greek geometers developed, fairly soon after Thales, a different manner of proof, which does not depend on what can be seen by looking at the disposition of lines and points in a diagram or a series of diagrams, but rather on what can be gathered by understanding the meaning of words in a sentence or a set of sentences.⁵ Scholars generally agree that one of the earliest instances of this style of doing mathematics has been preserved almost intact in Book IX, Propositions 21–34 of Euclid's *Elements*. These concern some basic relations between odd and even numbers.⁶ Proposition 21 says that the sum of any multitude of even numbers is even because each summand, being even, has an integral part which is exactly one half of it; hence, the sum of these halves is exactly one half of the

sum of the wholes. Proposition 22 asserts that the sum of an even multitude of odd numbers is even, because each summand minus a unit is even, and the sum of the subtracted units is also even, so that the full sum can be represented as a sum of even numbers. These results lead to the following elegant proof of Proposition 23:

If as many odd numbers as we please be added together, and their multitude be odd, the whole will also be odd. For let as many odd numbers as we please, AB, BC, CD, the multitude of which is odd, be added together; I say that the whole AD is also odd. Let the unit DE be subtracted from CD; therefore the remainder CE is even. But CA is also even; therefore the whole AE is also even. And DE is a unit. Therefore AD is odd.⁷

Though this and the preceding proofs are plainly meant to be illustrated by diagrams in which the integers under consideration are represented by straight segments (Fig. 1), such diagrams will not in the least aid us to visualize the force of the argument. This rests entirely on the meaning of the words *odd*, *even*, *add*, *subtract*, and cannot therefore 'be seen except by thought'.⁸ Had they not adopted this method of exact, forceful, yet unintuitive thinking, Greek mathematicians could never have found out that there are incommensurable magnitudes, such as, for example, pairs of linear segments which cannot both be integral multiples of the same unit segment, no matter how small you choose this to be. For, as B.L. van der Waerden pointedly observes:

When we deal with line segments which one sees and which one measures empirically, it has no sense to ask whether they have or not a common measure; a hair's breadth will fit an integral number of times into every line that is drawn. The question of commensurability makes sense only for line segments which are objects of thought.⁹

The incommensurability of the side and the diagonal of a square was discovered in the second half of the 5th century B.C. An early proof, preserved in Proposition 118 of Book X of Euclid's *Elements*,¹⁰ is directly linked with the theory of odd and even numbers in Book IX. The discovery of incommensurables, a fact which plainly eludes, and may even be said to defy, our imagination, must have powerfully contributed to bring about the preponderance of that decidedly intellectual approach to its subject matter which is perhaps the most remarkable feature of Greek geometry. Such an approach was indeed



Fig. 1.

unavoidable if, as we have just seen, one of the most basic certainties concerning the objects of this science could only be attained by reasoning. After the discovery of incommensurables, geometers were bound to demand a strict proof of every statement in their field (if they were not already inclined to do so before). Now, if a statement can only be *proved* by deriving (inferring, deducing) it from other statements, it is clear that the attempt to prove *all* statements must lead to a vicious circle or to an infinite regress. It is unlikely that Greek mathematicians had a clear perception of the inadmissibility of circular reasoning and infinite regress before Aristotle. But they instinctively avoided these dangers by reasoning always from assumptions which they did not claim to be provable. In a well-known passage of the *Republic*, Plato speaks disparagingly of this feature of mathematical practice: Since geometry and the other mathematical disciplines are thus unable to give a reason (*logon didonai*) for the assumptions (*hupotheseis*) they take for granted, they cannot be said to be genuine sciences (*epistemai*).¹¹ The name *science*, bestowed on them out of habit, should be reserved for dialectic, which “does away with assumptions and advances to the very principle (*auten ten arkhēn*) in order to make her ground secure”.¹² Having been born too late to benefit from Plato’s oral teaching, I find it very difficult to determine how he conceived of the dialectician’s ascent to the unique, transcendent principle which he claimed to be the source of all being and all truth. His great pupil, Aristotle, succeeded in disentangling Plato’s main epistemological insights from his mystical fancies and built a solid, sober theory of science that has tremendously influenced the philosophical understanding of mathematics until quite recently. In Aristotle’s terminology, *science* (*episteme*) is equated with knowledge by demonstration (*apodeixis*).¹³ But it is subordinated to a different kind of knowledge, which Aristotle calls *nous*, a word that literally means *intellect* and that has been rendered as *rational intuition* (G.R.G. Mure) and as *intuitive reason* (W.D. Ross),¹⁴ *Nous* gives us an immediate grasp of *principles* (*arkhai*), that is, of true, necessary, universal propositions, which cannot be demonstrated – except, I presume, by resorting to their own consequences – but which are self-evident and provide the ultimate foundation of all demonstrations in their respective fields of knowledge. Intellection of principles is attained by reflecting on perceived data, but it is definitely not an effect of sense impressions modifying the mind. It

results rather from the spontaneous mental activity that extricates from the mass of particular perceptions the universal patterns that shape them and regulate them. Aristotle, guided perhaps by his sane Greek piety, readily acknowledged that there were many principles. He distinguished two classes of them: *axioms* (*axiomata*), which are known to all men and are common to all sciences because they hold sway over all domains of being, and *theses* (*theseis*), which are proper to a particular science.¹⁵ The foremost example of an Aristotelian axiom is the principle of contradiction: "The same attribute cannot at the same time belong and not belong to the same object in the same respect".¹⁶ Theses are classified into *hypotheses* (*hupotheseis*), that posit the existence or inexistence of something, and *definitions* (*horismoi*), that say what something is.¹⁷ Since circles and infinite regress are no more admissible in definitions than in demonstrations, there must be undefined or primitive terms in every deductive science. There are some indications that Aristotle was aware of this, but he never made this requirement fully explicit.¹⁸

It is very likely that Aristotle developed his theory of science prompted by his understanding of the work of contemporary mathematicians, who sought to organize geometry as a deductive system founded on the least possible number of assumptions. These men were the forerunners of Euclid, whose famous *Elements*, written about 300 B.C., are, in a sense, the fruit of their collective efforts. Euclid's book, on the other hand, has been regarded since late Antiquity as a showpiece of Aristotelian methodology. However, I am not sure that it was originally understood in this way. In particular, it is not at all clear to me that Euclid and his mathematical predecessors actually viewed the unproved assumptions on which they built their deductive systems as true, necessary, self-evident propositions. My doubts are nourished mainly by the philological analyses of K. von Fritz and A. Szabó, who have exhibited significant discrepancies between the terminology of Euclid and that of Aristotle;¹⁹ but they also draw support from philosophical considerations.

The first book of Euclid's *Elements* begins with a list of assumptions classed as *horoi* (definitions), *aitemata* (postulates or demands) and *koinai ennoiai* (common notions). Additional *horoi* are given at the beginning of Books II, III, IV, V, VI, VII and XI. The rest of the *Elements* consists of propositions and problems which are supposedly proved and solved by means of these assumptions. This structure is,

of course, strongly reminiscent of Aristotle's blueprint for a deductive science, a fact that is not surprising, since Euclid's work was probably fashioned after earlier books of 'elements' with which Aristotle himself was familiar. Though Euclid does not call his three kinds of assumptions by the same names chosen by Aristotle for his three types of principles, it seems natural to equate *horoi* with *horismoi*, *koinai ennoiai* with *axiomata* and *aitemata* with *hypotheses*. However, as soon as one examines Euclid's list one cannot help feeling that there must be something wrong with these identifications. Let us take a look at the three parts of that list.

(a) Though Euclid's *horoi* are far from satisfying the requirements imposed on definitions by modern logical theory, they do agree with Aristotelian *horismoi* in so far as each one of them declares, in ordinary language sometimes mingled with previously defined technical terms, the nature of the objects designated by a given expression. Some *horoi* are overdetermined, providing alternative logically non-equivalent characterizations of the same object. Such overdetermination, which makes a *horos* into a synthetic statement, reporting factual information of some sort, would be of course inadmissible in a definition in our modern sense, but may very well occur in an Aristotelian *horismos* that says *what something is*. On the other hand, *horismoi* are not supposed to make existential statements. There is, however, a *horos* in Euclid which, though it is not ostensibly a statement of existence, is invoked in a proof as if it had existential import. This is Definition 4 in Book V, which says that "magnitudes are said to *have a ratio* to one another which are capable, when multiplied, of exceeding one another". This means that whenever two magnitudes a , b , such that a is less than b , do have a ratio to one another, there exists a number n , such that a taken n times is greater than b . Since Definition 4 in Book V does not say that there are magnitudes which actually have a ratio to one another, it does not seem to have any existential implications. But in the proof of Proposition 1 in Book X it is assumed as a matter of course that any two unequal but homogeneous magnitudes (two lengths, two areas, etc.) do have a ratio to one another in the sense of Definition 4 in Book V, and that consequently the smaller one will exceed the larger one when multiplied by a suitable number. The existence of such a pair of magnitudes results immediately from Postulates 1 and 2 (quoted below under (c)), as soon as we are given two points, but these

postulates cannot, by any stretch of the imagination, be understood to involve the existence of a number such as we described above. This existential assumption must be regarded as implicit in Definition 4 in Book V, as this *horos* is conceived and used by Euclid.

(b) The list of nine or more *koinai ennoiai* given in the extant manuscripts of Euclid has been reduced to the following by modern text criticism:

(1) Things which are equal to the same thing are also equal to one another.

(2) If equals are added to equals, the wholes are equal.

(3) If equals are subtracted from equals, the remainders are equal.

(4) Things which coincide when superposed on one another are equal to one another.

(5) The whole is greater than the part.²⁰

In his *Commentary on the First Book of Euclid's Elements*, Proclus gives this same list, but under the heading *axiomata*. Szabó conjectures that this, and not *koinai ennoiai*, was the term originally employed by Euclid himself. This does not imply that he used *axioma* in its technical Aristotelian sense, since the word, as Aristotle noted, was current among mathematicians.²¹ The five statements above are 'common' indeed in the sense that most people would readily acknowledge them, but they are not common to all domains of being. The first three and the fifth apply at any rate to the whole Aristotelian category of quantity and may therefore be regarded as axioms according to some passages in Aristotle.²² But the fourth is a specifically geometrical statement and most probably refers only to figures which can be drawn on a plane.

(c) The *aitemata* or postulates read as follows:

Let it be postulated: [1] to draw a straight line from any point to any point; and [2] to produce a straight line continuously in a straight line; and [3] to describe a circle with any centre and distance; and [4] that all right angles are equal to one another; and [5] that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.²³

Is *aitema* just another name for that what Aristotle called *hypothesis*? This, as the reader will recall, is a self-evident statement of existence concerning the subject-matter of a particular science. The five *aitemata* listed above all pertain specifically to geometry. The fifth can be read as an existential statement.²⁴ The first three, on the

other hand, merely demand that certain constructions be possible (i.e. performable in a presumably unambiguous manner, so that there is, for example, only one way of joining two given points by a straight line). Now, though Greek mathematicians and philosophers would have generally agreed that any existing geometrical entity ought to be constructible, this does not imply that every constructible entity must be regarded as existing. Thus, Aristotle's solution of Zeno's paradoxes depends essentially on the premise that, even though a point can always be determined which divides a given segment into two parts in any assigned proportion, such a point need not exist before it is actually constructed.²⁵ We cannot therefore view the first three Euclidean postulates as straightforward existential statements in an Aristotelian sense. Postulate 4, finally, is not existential in any sense whatsoever.²⁶ Those who have regarded Euclid's geometry as an Aristotelian science have usually considered the first four *aitemata* to be self-evident; but, as we shall see in Part 2.1, the self-evidence of the fifth has often been disputed. It is, at any rate, doubtful that Euclid would have used the expression *eitestho* to introduce what he held to be self-evident truths.²⁷ If a proposition is self-evident one need not *beg* one's reader to *grant* it. The shades of meaning which an educated Greek of the 4th or the 3rd century B.C. would have associated with that expression can be gathered from Aristotle's own use of the related noun *aitema*. In agreement with what apparently was the customary meaning of these words in dialectics, he contrasts *aitema* and *hypothesis*. "Any provable proposition that a teacher assumes without proving it, provided that the pupil accepts it, is a hypothesis; not a hypothesis in an absolute sense, though, but only relatively to the pupil." An *aitema*, on the other hand, is "the contrary of the pupil's opinion, or any provable proposition that is assumed and used without proof".²⁸

In the light of the foregoing remarks, it appears unlikely that Euclid ever regarded his threefold list of assumptions as an inventory of principles in the sense of Aristotle. The common notions he probably judged to be true and even necessary. But I do not believe that the same can be said of the postulates. Some of these are incompatible with the cosmological system developed by Aristotle in good agreement with contemporary astronomy. In the closed Aristotelian world not every straight line can be produced continuously, as required by Postulate 2, and not every point can be the centre of a

circle of any arbitrary radius, as demanded by Postulate 3. Moreover, though Postulate 5 is trivially true in such a world (because the condition that the two lines be produced indefinitely cannot be fulfilled), in the absence of Postulate 2 it cannot yield its most significant consequences. Now, there is no reason to think that Euclid and his immediate predecessors would have opposed the new cosmology, which was indeed at that time a very reasonable scientific conjecture. It would rather seem that, as Aristotle once remarked, Greek mathematicians did not care to determine whether their basic premises were true or not.²⁹ I dare say they assumed them, as mathematicians are wont to do, because of their fruitfulness, that is, their capacity to support a beautiful and expanding theory. Nineteen centuries later, as we shall see below, implicit faith in the literal truth of Euclidean geometry powerfully aided the shift "from a closed world to the infinite universe" and the establishment of the metaphysics of space that was such an important ingredient of the scientific world-view from 1700 to 1900.

*Aristotle was well aware that his finite universe might appear to be incompatible with geometry. But, in his opinion, it was not. "Our account does not rob the mathematicians of their science", he writes, "by disproving the actual existence of the infinite in the direction of increase . . . In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish. It is possible to have divided in the same ratio as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them to have such an infinite instead, while its existence will be in the sphere of real magnitudes." (Aristotle, *Phys.*, 207^b27–34). Aristotle is wrong, however. Let m be a line and P a point outside it, and let (P, m) denote the plane determined by P and m . In a finite world there are infinitely many lines on (P, m) which go through P and do not meet m even if they are produced as far as possible. This fact, which is incompatible with Euclidean geometry, cannot be disproved by reducing all lengths in some fixed proportion, as Aristotle suggests. On the other hand, Aristotle's system of the world is based on the geometrical astronomy of Eudoxus (p.13ff.). This does not make it inconsistent, however, because the geometry of Eudoxian planetary models is that of a spherical surface, which does not depend on the Euclidean postulates that are false or trivial in the Aristotelian world.

Such *two-dimensional* spherical geometry is not really *non-Euclidean*—as some philosophical writers claim (Daniels (1972), Angell (1974))—for it does not rest on the denial of any Euclidean postulate; but it does not presuppose the full Euclidean system and is compatible with its partial negation. See Bolyai, SAS, p.21 (§26).

1.0.2 *Geometry in Greek Natural Science*

Pythagoras of Samos (6th century B.C.), or one of his followers, discovered that musical instruments that produce consonant sounds are related to one another by simple numerical ratios. Encouraged by this momentous discovery, the Pythagoreans sought to establish other correspondences between numbers and natural processes. They believed, in particular, that celestial motions stood to one another in numerical relations, producing a universal consonance or 'cosmic harmony'. Since, as they observed, "all other things appeared in their whole nature to be modelled on numbers",³⁰ they concluded that "the elements of numbers were the elements of things".³¹

The Pythagorean programme for an arithmetical physics came to a sudden end when one member of the school—possibly Hippasus of Metapontum—discovered the existence of incommensurables, that is, of magnitudes which can be constructed geometrically but stand in no conceivable numerical proportion to one another. Since this fact can be rationally proved but cannot be empirically verified (p.3), it is all the more remarkable that it should have sufficed to stop the search for numerical relations in nature, so promisingly initiated by the Pythagoreans. I surmise that they unquestioningly took for granted that bodies, their surfaces and edges, as well as the paths they traverse in their motions, must be conceived geometrically. Hence, they had little use for arithmology in physics after they learned that not all geometrical relations can be expressed numerically.

One of the major achievements of classical Greek mathematics was the creation of a conceptual framework permitting the exact quantitative comparison of geometrical magnitudes even if they happen to be incommensurable. This is set forth in Book V of Euclid's *Elements*, which is generally believed to be the work of Eudoxus of Cnidus (c.408–c.355 B.C.), a contemporary and friend of Plato. Eudoxus' basic idea is stunningly simple, as is often the case with great mathematical inventions. It applies to all kinds of magnitudes or

extensive quantities which can be meaningfully compared as to their size and can be added to one another associatively (e.g. lengths, areas, volumes). Let a and b be two such magnitudes, which fulfil the following conditions: (i) a is equal to or less than b ; (ii) there exists a positive integer k such that ka (that is, a taken k times) exceeds b . Any two magnitudes that, taken in a suitable order, agree with this description will be said to be homogeneous. Let a' , b' be another pair of homogeneous magnitudes of any kind. We say that a is to b in the same ratio as a' is to b' (abbreviated: $a/b = a'/b'$) if, for every pair of positive integers m, n , we have that

$$\begin{aligned} ma < nb & \text{ whenever } ma' < nb', \\ ma = nb & \text{ whenever } ma' = nb', \\ ma > nb & \text{ whenever } ma' > nb'. \end{aligned}$$

We say that a has to b a greater ratio than a' has to b' ($a/b > a'/b'$) if, for some pair of positive integers m, n , we have that $ma > nb$, but $ma' \leq nb'$.³² Using these definitions, we can compare the quantitative relations between any pair of homogeneous magnitudes with that between two straight segments. Eudoxian ratios are linearly ordered by the relation *greater than*; they can be put into a one-one order preserving correspondence with the positive real numbers and one can calculate with them as with the latter. The importance of Eudoxus' innovation for geometry is evident and demands no further comment. It also has a direct bearing on the Pythagorean programme. This had failed because there are relations between things which cannot possibly correspond to relations between numbers. But even if numbers and their ratios were inadequate for representing every conceivable physical relation, one could still expect Eudoxian ratios to do this job. After all, the same reasons that had initially supported physical arithmology could be used to justify the project of a more broadly-conceived mathematical science of nature. In fact, Eudoxus himself, as the founder of Greek mathematical astronomy, set some such project going at least in this branch of physical inquiry. And, in the following centuries, Archimedes of Syracuse (287–212 B.C.) and a few others made lasting contributions to statics and optics. But in his pioneering search for a mathematical representation of ordinary terrestrial phenomena, Archimedes – “suprahumanus Archimedes”, as Galilei would call him – did not gather much of a following until the

17th century A.D. Greek astronomy, on the other hand, constituted a strong scientific tradition that lasted almost uninterruptedly from Eudoxus' time to the end of the 16th century A.D. But as astronomical theory was perfected and developed into the supple predictive instrument used by Ptolemy (2nd century A.D.), it also became inconsistent with the then current understanding of physical processes. Geometry was used to compute future occurrences from past observations but was not expected to give an insight into the workings of nature. Thus notwithstanding its auspicious beginnings, Greek science did not persevere in the pursuit of the modified Pythagorean programme that Eudoxus' theory of ratios clearly suggested.

It is somewhat puzzling that the Greeks should have failed to develop a mathematical physics commensurate with their scientific curiosity and ability. I suspect that this can be traced in part at least to the intellectual influence of Plato. The last statement may sound surprising, since Copernicus, Kepler, and Galilei professed great admiration for Plato and often drew inspiration from his writings. Yet the fact remains that Plato took a stand, clearly and resolutely, against the very possibility of a mathematical science of nature, in a well-known passage of the *Republic*, which the founders of modern science apparently chose to ignore, perhaps because they felt that it did not apply to a world created *ex nihilo* by the Christian God. The passage in question occurs in a long discussion of a statesman's education. Would-be rulers ought to be drawn away from the fleeting sense appearances that capture a man's attention from birth, towards the immutable intelligible principle whence those appearances derive their meagre share of being and of value. The conversion of the soul begins with the study of mathematics. Although the mathematical sciences sorely need a 'dialectical' foundations (p.4), they do procure us our first contact with genuine, that is, exact and changeless, truth. The vulgar believe that mathematical statements are about things we touch or see; but such things lack the permanence and, above all, the definiteness proper to mathematical objects. These are not ideas, in Plato's sense, for there are many of a kind (e.g. many circles), but they are not mere appearances, like the objects we perceive through our senses. They stand somewhere in between but nearer to the former than to the latter. Plato displays a hierarchy of mathematical sciences. After arithmetic, or the science of number, comes geometry, both plane and solid. The science of solids as such (*auta kath' auta*) is

followed naturally by the science of solids in motion. Plato calls it *astronomy*, but he warns his readers that this mathematical science of motion has nothing to do with the stars we see twinkling in the sky, although it bears their name.³³

We should use the broideries in the heaven as illustrations to facilitate that study, just as we might employ, if we met with them, diagrams drawn and elaborated with exceptional skills by Daedalus or some other artist (*demiourgos*); for I take it that anyone acquainted with geometry who saw such diagrams would indeed think them most beautifully finished, but would regard it as ridiculous to study them seriously in the hope of gathering from them true relations of equality, doubleness, or any other ratio. [...] Do you not suppose that a true astronomer will have the same feeling when he looks at the movements of the stars? He will judge that heaven and the things in heaven have been put together by their maker (*demiourgos*) with the utmost beauty of which such works admit. But he will hold it absurd to believe that the proportion which night bears to day, both of these to the month, the month to the year, and the other stars to the sun and moon and to one another can be changeless and subject to no aberrations of any kind, though these things are corporeal and visible; and he will also deem it absurd to seek by all means to grasp their truth.³⁴

Plato obviously countenances a purely mathematical theory of motion, which it would be more appropriate to call kinematics or *phoronomy*. He conceives it quite broadly. "Motion – he says – presents not just one, but many forms. Someone truly wise might list them all, but there are two which are manifest to us."³⁵ One is that which is imperfectly illustrated by celestial motions. The other is the "musical motion" (*enarmonios phora*), studied by Pythagorean acoustics. This science, says Plato, has been justly regarded as astronomy's "sister science". Exact observation – not to mention experiment – is completely out of place here too. Plato pours ridicule on "those gentlemen who tease and torture the strings and rack them on the pegs of the instrument".³⁶ Generally speaking,

if any one attempts to learn anything about the objects of sense, I do not care whether he looks upwards with mouth gaping or downwards with mouth shut; he will never, I maintain, acquire knowledge, because nothing of this sort can be the object of a science.³⁷

Plato's warning to would-be astronomers, that they should not expect heavenly bodies to be excessively punctual, nor spend too much effort in observing them in order to "grasp their truth" was probably aimed at none other than young Eudoxus, who, while the *Republic* was being written, attended Plato's lectures and perhaps mentioned his plan for a mathematical theory of planetary motions.

Eudoxus did not follow the philosopher's advice. He developed a kinematical model of each 'wandering star' or *planet* (including the sun and the moon), which could be used to predict its movements with a good measure of success. All Eudoxian models are built on the same general plan. The planet is supposed to be fixed on the equator of a uniformly rotating sphere whose centre coincides with the centre of the earth. The poles of this sphere are fixed on another sphere, concentric with the former, which rotates uniformly about a different axis. The poles of the second sphere are fixed on a third one, etc. This scheme can be repeated as many times as you wish, but the last sphere must, in any case, rotate with the same uniform speed and about the same axis as the firmament of the fixed stars. Following Aristotle, Eudoxian spheres are usually numbered beginning with this one, so that the sphere on which the planet is fixed is counted last; hereafter, we shall also follow this practice.³⁸ Eudoxus' models of the sun and the moon had three spheres each, those of Mercury, Venus, Mars, Jupiter and Saturn had four. His disciple Callippus added two spheres to the sun, two to the moon and one to each of the first three planets, in order to obtain a better agreement with observed facts.

Eudoxus' models gave a first solution of the problem that was to dominate astronomy until the Keplerian revolution. As stated by Simplicius, the 6th-century commentator of Aristotle, this problem consisted in determining "what uniform, ordered, circular motions must be assumed to account for the observable motions of the so-called wandering stars".³⁹ The requirement that phenomena be 'saved' (that is, accounted for) by means of uniform circular motions was usually justified saying that no other kind of motion could suit the divine perfection of the heavens. But I wonder whether this was really the motivation behind Eudoxus' spherical models. After all, Plato's authoritative opinion should have induced his friend and pupil to look for something a little less perfect. On the other hand, non-circular non-uniform motions would have been practically intractable, with the available mathematical resources, so that, as a matter of fact, Eudoxus' choice of uniform circular motions was the only one he could reasonably have made. His success was acknowledged by Plato, who took a different view of astronomy in his old age. The anonymous Athenian who is Plato's spokesman in the *Laws* bears witness to this change.

It is not easy to take in what I mean, nor yet is it very difficult or a very long business: witness the fact that, although it is not a thing which I learnt when I was young or very long ago, I can now, without taking much time, make it known to you: whereas, if it had been difficult, I, at my age, should never have been able to explain it to you at yours. [. . .] This view which is held about the moon, the sun, and the other stars, to the effect that they wander and go astray (*planatai*), is not correct, but the fact is the very contrary of this. For each of them traverses always the same circular path, not many paths, but one only, though it appears to move in many paths.

The whole path and movement of heaven and of all that is therein is by nature akin to the movement and revolution and calculations of intelligence (*nous*).⁴⁰

But Plato does not recant his former evaluation of nature and of the prospects of natural science. He only concludes, in good agreement with traditional Greek piety, that the heavens must be set apart from the rest of the physical world. The planets would not follow those "wonderful calculations with such exactness" if they were soulless beings, destitute of intelligence.⁴¹ The same point is made more explicitly in the *Epinomis*, a supplement to the *Laws* which many 20th-century scholars have attributed to Plato's pupil, the mathematician Philip of Opus, but which in Antiquity was believed to be the work of Plato himself.

It is not possible that the earth and the heaven, the stars, and the masses as a whole which they comprise should, if they have no soul attached to each body or dwelling in each body, nevertheless accurately describe their orbits in the way they do, year by year, month by month, and day by day.⁴²

The achievements of Eudoxian astronomy were thus used to justify a complete separation of celestial from terrestrial nature. The former could be described with mathematical exactitude because it is populated and ruled by rational souls. But it would be foolish to imitate the mathematical methods of astronomy when we consider the clumsy, unpredictable behaviour of the inanimate objects that surround us.

Eudoxian astronomy does not provide what Galilei or Newton would have called a 'system of the world', because each planet is treated independently of the others. But there are two features common to all the planetary models, which can naturally serve to unify them: (i) all spheres have the same centre, namely, the centre of the earth; (ii) each model includes one sphere which rotates exactly like the heaven of the fixed stars. These two features facilitated the incorporation of Eudoxian spheres into Aristotle's cosmology. Aristotle maintained that there are five kinds of 'simple bodies', namely, fire, air, water, earth and aether. Each of these has a peculiar nature

(*phusis*) or “internal principle of motion and rest”. These bodies being simple, their respective natures prescribe them simple motions: earth and water move naturally downwards, i.e. towards the centre of the world; fire and air, upwards, i.e. away from the centre; aether moves neither downwards nor upwards, but in perfect circles about the centre of the world. All things beneath the moon are combinations or mixtures of the first four simple bodies in different proportions, and are therefore more or less evenly distributed about the centre of the world (hence this happens to be the centre of the earth as well). On the other hand, aether is the only material ingredient of heaven. Heaven consists of a series of concentric, rigid, transparent aethereal spheres, eternally rotating with different uniform speeds, each one about its own axis, which passes through the centre of the world. Since the heavenly spheres are material, they must be nested into one another like a Russian doll. The outermost sphere rotates once a day, from East to West, about the North–South axis. Each of the remaining spheres has its poles fixed on the sphere immediately outside it. Thus each sphere induces its own motion on all the spheres contained in it. Luminous ethereal bodies are fixed on some of the spheres. In fact, all except seven are found on the outermost sphere, which is known for this reason as the sphere of the fixed stars. The next three spheres move, respectively, like the last three spheres in the Eudoxian model of Saturn, Saturn itself being affixed to the equator of the third one. Then come three spheres whose rotations cancel out those of the former three, so that a point on the last one moves like a fixed star. The next three spheres move like the last three spheres of the Eudoxian model of Jupiter, and Jupiter is affixed to the last. Again, their rotations are cancelled by the motions of the following three spheres. This scheme is repeated until we come to the innermost sphere, which is the sphere of the moon. If we base our construction on the planetary models of Callippus (p.14), the foregoing scheme gives a total of 49 spheres.⁴³ Each sphere is endowed with a divine mind that keeps it moving in the same way for all eternity.

Although Aristotle did not hesitate to incorporate in his physical cosmology the latest results of the new mathematical astronomy and may even be said to have devised the former to fit the latter, he did not countenance the use of mathematical methods in other branches of physical inquiry. Physical science (*episteme phusike*) was no longer for him, as it had been for Plato, a contradiction in terms. Indeed, his main concern was to develop a conceptual framework for the

scientific study of the world of becoming as known through the senses. He held in fact that there is no other world than this. But he believed that a strict science is not necessarily an exact science, and that only a boor can demand of a science more precision than its subject matter will admit.⁴⁴ All objects of sense are material in the Aristotelian sense of this word, that is to say, they have a potentiality for becoming other than they are. This, according to Aristotle, is a source of indeterminacy, which must appear as an unavoidable imprecision in scientific concepts. "Exact mathematical speech (*mathematike akribologia*) is not to be demanded in all cases, but only in the case of things which have no matter. Hence it is not the style of natural science; for presumably the whole of nature has matter."⁴⁵

This Aristotelian dictum would, if taken literally, exclude mathematical exactness even from astronomy. But Aristotle does not appear to have suggested that, say, the heavenly spheres were only approximately spherical or that their angular velocities were approximately constant. He probably thought that, since the aether is, so to speak, minimally material – its materiality consisting merely in its disposition to rotate perpetually in the same way about the same point – aethereal things can take a simple geometrical shape and obey a simple kinematic law. But such is not the case of the other material things. Indeed, "physical bodies contain surfaces and volumes, lines and points, and these are the subject matter of mathematics".⁴⁶ Aristotle emphatically rejects the Platonic thesis that mathematical objects are ideal entities, existing apart from their imperfect realizations in the world of sense. But mathematicians do separate them – in thought – from matter and motion; and although "no falsity ensues from this separation" (*oude gignetai pseudos khōrizontōn*)⁴⁷ as far as the abstract objects of mathematics are concerned, one cannot expect that what is true of them will apply unqualifiedly to the concrete objects of physics.

We must not fail to notice the mode of being of the essence [of the object of inquiry] and of its concept, for without this, inquiry is but idle. Of things defined, i.e., of 'whats', some are like 'snub' and some like 'concave'. And these differ because 'snub' is bound up with matter (for what is snub is a concave *nose*), while concavity is independent of sensible matter. If then, all physical objects (*panta ta phusika*) are to be conceived like the snub – e.g. nose, eye, flesh, bone, and in general, animal; leaf, root, bark, and, in general, plant (for none of these can be conceived without reference to motion – their concept always involves matter) –, it is clear how we must seek and define the 'what' in the case of physical objects.⁴⁸

One might contend however that, by referring the natural motions of the four sublunary elements to a dimensionless point, which is also the centre of the heavenly spheres, Aristotle has prepared the ground and implicitly granted the need for a geometrical approach to terrestrial physics. Thus, one could argue, if a lump of earth is dropped right under the moon and there is nothing beneath it, it should move, according to Aristotle, in a perfectly straight line towards the centre of the world. This style of thinking is very familiar to us, but was quite foreign to Aristotle. To someone proposing the foregoing analysis he would probably have objected that, since a void cannot possibly exist, the situation described, in which there is *nothing* beneath our lump of earth, makes no physical sense. In real life, a heavy body must always find its way to its natural resting place by pushing aside other lighter bodies that stand in its path. This ensures that its trajectory will never be rectilinear. Moreover, its actual shape is utterly unpredictable, since it depends on the particular nature of the obstacles that the falling body chances to meet.

The Aristotelian synthesis of mathematical astronomy and physical cosmology broke down very soon, because the planetary models of Eudoxus could not be reconciled with all observed facts. Sosigenes (1st century B.C.) mentions two facts that were known to Eudoxus himself, which his theory was incapable of explaining:

- (i) The sun does not traverse all four quadrants of the Zodiac in the same time – the period from a solstice to the next equinox is not equal with that from that equinox to the other solstice; hence, the angular velocity of the sun about the earth cannot be constant.
- (ii) The apparent luminosity of the planets and the apparent size of the moon are subject to considerable fluctuations; hence their distances from the earth cannot be constant.

Callippus sought to cope with fact (i) by adding two more spheres to Eudoxus' solar model, but fact (ii), of course, was totally incompatible with the Aristotelian system of concentric spheres. Third-century astronomers managed to account, on a first approximation, for both facts by assuming that each planet moves with constant angular speed on an *eccentric*, that is, on a circle whose centre is not the centre of the earth but which contains the latter in its interior. Then, towards the end of that century, some hundred years after the death of Aristotle, Apollonius of Perga (265?–170 B.C.) introduced an

extraordinarily pliable kinematical device: epicyclical motion. Let us define:

A body moves with simple or first degree epicyclical motion if it describes a circle (the epicycle) whose centre moves on another circle (the *deferent*) about a fixed point.

A body moves with n th degree epicyclical motion ($n > 1$) if it describes a circle (the n th epicycle) whose centre moves with $(n - 1)$ th degree epicyclical motion.

n th degree epicyclical motion ($n \geq 1$) is said to be uniform if the body moves with constant angular velocity about the centre of the n th epicycle and the centre of the j th epicycle ($1 \leq j \leq n$) moves with constant angular velocity about the centre of the $(j - 1)$ th epicycle (or the centre of the deferent, if $j = 1$). Epicyclical motion is said to be geocentric (heliocentric) if the centre of the deferent coincides with the centre of the earth (sun).

Hipparchus (1st century B.C.) proved that the trajectory of any planet moving with constant speed on an eccentric will also be described by a body moving with a suitable geocentric uniform simple epicyclical motion. A more breathtaking result, that neither Apollonius nor Hipparchus could prove but that they have surmised, is that every imaginable planetary trajectory can be approximated within any arbitrarily assigned margin of error by some geocentric uniform n th degree epicyclical motion (where n is a positive integer generally depending on the assigned margin of error).⁴⁹ Epicyclical motion furnishes, therefore, a general solution of the main problem of Greek astronomy; to 'save the phenomena' by postulating 'uniform regular circular motions' (p.14). This universal scheme can be adjusted to fit any set of astronomical observations if one chooses the right parameters. However, ancient and medieval astronomers never availed themselves of the full power of Apollonius' invention. Both Ptolemy (2nd century A.D.) and Copernicus (1473–1543), the two acknowledged masters of epicyclical astronomy, postulated eccentric deferents, thereby decreasing in one the number of epicycles needed for each planetary model. Ptolemy also resorted to the infamous hypothesis of the *equant* or 'equalizing point' (*punctum aequans*), which he could, in principle, have dispensed with by suitably increasing the epicycles. This hypothesis is applied by Ptolemy to all the wandering stars except the sun. According to it, all circular motions involved in the epicyclical motion of the star are uniform, except that

of the centre K of the first epicycle. K moves on the deferent with variable speed, but there is a fixed point A , the *equant*, such that the line AK turns about A describing equal angles in equal times. The star's epicyclical motion is therefore not uniform in the sense defined earlier, and this deviation from 'Platonic' orthodoxy was indeed one of Copernicus' chief complaints against Ptolemy. All equants postulated by Ptolemy turn out to be collinear with the centre of the respective deferent and with the centre of the earth. Due to the low eccentricity of the earth's elliptical orbit, the Ptolemaic astronomer could achieve a remarkably accurate representation of the trajectories of the planets without having to postulate many epicycles. Let P stand for Venus, Jupiter or Saturn, and let P' be a fictitious body moving with simple epicyclic motion on an eccentric deferent about the earth, with its angular velocity regulated by a suitably placed equant. D.J. de S. Price has calculated that, if the parameters are chosen optimally, the predicted position of P' will always fall within $6'$ of arc of the observed position of P . This approximation compares favourably with the best precision attained by naked eye astronomers before Tycho Brahe (1546–1601). On the other hand, if P stands for Mars, P' may deviate up to $30'$ from P .⁵⁰

Epicyclical astronomy was the highest achievement of applied mathematics before the advent of modern astronomy and modern mathematical physics in the 17th century. In a sense, it may be said to have cleared the way for them, insofar as it led to the development of many useful mathematical techniques and fostered the habit of dealing with time as with a magnitude or extensive quantity. But the aims and what we might call the epistemic attitude of epicyclical astronomy were diametrically opposed to those of 17th-century science. Epicyclical astronomy produced kinematical models of the planetary motions that could, in principle, be indefinitely adjusted to account for new and better observations. But these models had not the slightest semblance of physical plausibility. From this point of view, epicyclical models compare unfavourably with Eudoxus' homocentric spheres, which had been so aptly integrated by Aristotle into an intelligible cosmos, nicely arranged about the centre of the world. The many centres that regulate celestial motions in epicyclical astronomy – the moving centres of the epicycles, the fixed but empty centres of the deferents, the equants or centres of uniform angular velocities – are arbitrary geometrical points, altogether independent

from the distribution of matter in the universe, and their dynamical significance is all but transparent. Indeed, to anyone who views the heavens as a ballet of angels, the intricate, geometrically sophisticated evolutions of a Ptolemaic planet ought to appear as a worthier display of divine choreography than Aristotle's artless merry-go-round. And yet, many Greek thinkers of late Antiquity and most Arab and Latin philosophers of the Middle Ages were wary of accepting the kinematic models of epicyclical astronomy as a faithful picture of the real motions of the stars and tended to regard them merely as computational device, i.e. as a formalism for predicting (or retrodicting) the future (or past) positions of the heavenly bodies from observed data.

Forgetting or deliberately ignoring that Aristotelian cosmology was itself largely based on the mathematical astronomy of an earlier age, the Andalusian philosopher Averroes (c.1126–c.1198) rejected the astronomical theories of his time because they clashed with the teachings of the Philosopher.

The astronomer must construct an astronomical system such that the celestial motions are yielded by it and that nothing physically impossible is implied. [...] Ptolemy was unable to place astronomy on its true foundations. [...] The epicycle and the eccentric are impossible. We must therefore apply ourselves to a new investigation concerning that genuine astronomy whose foundations are the principles of physics. [...] Actually, in our time, astronomy is non-existent; what we have is something that fits calculation but does not agree with what is.⁵¹

His Jewish countryman Maimonides (1135–1204) speaks more cautiously:

If what Aristotle has stated with regard to natural science is true, there are no epicycles or eccentric circles and everything revolves round the centre of the earth. But in that case how can the various motions of the stars come about? [...] How can one conceive the retrogradation of a star, together with its other motions, without assuming the existence of an epicycle?⁵²

This 'perplexity' motivates Maimonides' agnosticism in astronomical matters. *The heavens are the heavens of the Lord but the earth hath He given to the son of man.* (Psalm 114:16). Man can attain knowledge of sublunary physics, but he cannot expect to grasp "the true reality, the nature, the substance, the form, the motions and the causes of the heavens". These can be known by God alone.⁵³ A similar astronomical agnosticism must have been at the root of the 'as if' philosophy professed by some late medieval Christian writers. John of Jandun (c. 1286–c.1328) stated this viewpoint very neatly. According to him, an

astronomer need only know that *if the epicycles and eccentrics did exist, the celestial motions and the other phenomena would exist as they do now.*

The truth of the conditional is what matters, whether or not such orbits really exist among the heavenly bodies. The assumption of such eccentrics and epicycles is sufficient for the astronomer *qua* astronomer because as such he need not trouble himself with the reason why (*unde*). Provided he has the means of correctly determining the places and motions of the planets, he does not inquire whether or not this means that there really are such orbits as he assumes up in the sky [. . .]. For a consequence can be true even when the antecedent is false.⁵⁴

If we understand this passage literally, we shall conclude that in Jandun's methodology, the epicyclical models employed by astronomers for the calculation of celestial motions are only an aid to knowledge, a sort of scaffolding required for attaining it, which cannot claim to be true; but that the trajectories yielded by those models, i.e. the predicted paths of the bodies that are assumed to move epicyclically, are, or ought to be, the true trajectories of the wandering stars. However, some Renaissance writers, who espoused methodological principles akin to Jandun's, apparently understood them in a more radical sense. Their words suggest that the mathematical models of astronomy need not yield the true celestial trajectories; it is enough that they enable us to predict the course of each star in good agreement with its observed positions. This implies, for instance, that a model for Mercury need not agree with the true path of this planet except during the periods of maximum elongation (maximum apparent separation from the sun), since it can be observed with the naked eye only at such times. One does not need to look far to find the reason for this shift of meaning. Jandun is obviously right if astronomy is content to give an accurate reconstruction of celestial kinematics but does not provide a truly illuminating theory of celestial dynamics. For only a theory that derives the actual motions of the heavenly bodies from their natural properties will enable the astronomer to choose between two kinematically equivalent devices, such as two different combinations of epicycles and eccentrics that yield the same trajectory. But if the astronomer lacks a dynamical theory of celestial motions, he can only rely on actual observations to distinguish between planetary models that predict *different* trajectories. Consequently, in the absence of such a theory, astronomy cannot decide between the many discrepant kinematical hypotheses

that happen to be observationally equivalent. A purely kinematic astronomy, that aimed only at description and prediction, but left to angelology the task of understanding astronomical phenomena, was bound to lead to a retreat from truth.

1.0.3 *Modern Science and the Metaphysical Idea of Space*

Johannes Kepler (1571–1630), bent as he was on learning how things really are, had to break away from this tradition. For him, astronomy “is a part of physics, because it inquires about the causes of natural things and events”.⁵⁵ It does not merely seek to foretell the changing configurations of the heavens, but tries to make them intelligible. “Astronomers should not be free to feign anything whatever without sufficient reason. You ought to be able to give probable reasons for the hypotheses you propose as the true causes of appearances.”⁵⁶ “I offer a celestial physics or philosophy in lieu of Aristotle’s celestial theology or metaphysics”,⁵⁷ he proudly wrote to Brengger after finishing the *Astronomia nova*. But Kepler’s celestial physics is the same as terrestrial physics. There is no essential difference between heaven and earth. Hence, the astronomer’s hypotheses concerning the causes of what happens *there* can be tested *here*. Moreover, in order to understand the phenomena in the sky, one must stop regarding the stars as self-willed beings, and look for the analogies between their behaviour and the more familiar processes of inanimate nature. “My aim”, Kepler wrote in 1605, “is to show that the fabric of the heavens (*coelestis machina*) is to be likened not to a divine animal but rather to a clock (and he who believes that a clock is animated attributes to the work the glory that befits its maker, insofar as nearly all its manifold motions result from a single quite simple, attractive bodily force (*vis magnetica corporalis*), just as in a clock all motion proceeds from a simple weight). And I teach how to bring that physical cause under the rule of numbers and of geometry”.⁵⁸ Geometry is indeed the key to universal physics. The mathematical analysis and reconstruction of phenomena is our only source of insight into the workings of nature. “God always geometrizes.”⁵⁹ “We see that the motions [of the planets] occur in time and place and that the force [that binds them to the sun] emanates from its source and diffuses through *the spaces of the world*. All these are geometrical things. Must not that force be subject also to other geometrical necessities?”⁶⁰

“Geometry furnished God with models for the Creation and was implanted in man, together with God’s own likeness.”⁶¹ “God, who created everything in the world according to the norms of quantity, also gave man a mind that can understand *such things*. For as the eye is made for colours, and the ear for sounds, so is the mind of man created for the intellection, not of anything whatever, but of *quantities*; and it grasps a subject the more correctly, the closer that subject is to pure quantity.”⁶² Similar thoughts were voiced independently, at about the same time, by Galileo Galilei (1564–1642) and thereafter provided the methodological groundwork of early modern science. None of these ideas was wholly new, but not until the 17th century did they become the mainstay of a sustained systematic search for comprehensive physical knowledge.

We need not dwell on Kepler’s long laborious quest for the laws of planetary motion, nor on the subsequent development of the new science of nature. What interests us here is a far-reaching implication of the ontological significance which Kepler and his successors ascribed to geometry. If geometry furnished the model of God’s Creation and if “triangles, squares, circles, spheres, cones, pyramids” are the characters in which Nature’s book is written,⁶³ then every point required for the constructions prescribed by Euclid’s postulates must somehow exist. Neither Kepler nor Galilei drew this inference; they both held to the Aristotelian belief in an outward limit of the world, beyond which there is nothing. But René Descartes (1596–1650) taught that “this world, or the entirety of the corporeal substance has no limits in its extension”, for “wherever we imagine such limits we always not only imagine some indefinitely extended spaces beyond them, but perceive those spaces to be real”.⁶⁴ Indeed, if a limit of the world existed, Descartes ‘second law of nature’ could not be true. For according to this law, a freely moving body will always continue to move in a straight line – thereby perpetually performing the construction demanded by Euclid’s second postulate – and this would be impossible if every distance in the world were less than or equal to a given magnitude. While Descartes cautiously formulated his thesis saying that “the extension of the world is indefinite”,⁶⁵ most scientists and philosophers after him did not hesitate to proclaim the infinitude of extension – though they generally did not equate extension with matter, as Descartes had done.

The set of all points required by Euclid’s postulates, endowed with

all the mutual relations implied by Euclid's theorems, is known in current mathematical parlance as Euclidean space. We may say, therefore, that the assumption that geometry is the basis of physics and that the world is a realization of Euclidean theory implies the existence of Euclidean space. Since a structured point-set is not the sort of thing that one would normally expect to exist 'really' or 'physically' as a self-subsisting entity, modern philosophy was beset with a novel ontological problem, the *problem of space*, which consisted in determining the mode of existence of Euclidean space. This problem was not conceived at first in the clear-cut way in which I have stated it. Not until the 19th century was the word 'space' defined to mean a structured point-set, and even then philosophers were not quick to adopt the new usage. Before that time, 'space' ('*spatium*', '*Raum*') designated an immaterial medium in which the points of geometry were supposed to be actually present or perhaps only potentially discernible (somewhat in the manner in which Aristotle had said that they could be distinguished as limits in material things). The problem of space concerned therefore the ontological status of this medium. Was it a construct, an *ens rationis*, abstracted from matter by the thinking mind? Or did it enjoy real existence independent of matter? The latter alternative need not imply that pure space was a substance or self-subsisting entity, like mind or matter; being immaterial, it could still be conceived as something somehow inherent in the divine or in the human mind. In any case, all philosophers bent on establishing the truth of mathematical physics on solid grounds – and that includes Leibniz and Newton, Malebranche and Kant – implicitly agreed that space was – in Poincaré's words – continuous, infinite, three-dimensional, homogeneous and isotropic,⁶⁶ and that all the points contained or discernible in it satisfy the theorems of Euclidean geometry.

This idea of space is certainly not a part of pre-philosophical common sense. The habit of rendering the Greek words *topos* (place) and *kenon* (void) as *space* has fostered the illusion that some such idea was familiar to Greek philosophers. The only word in classical Greek that can be regarded as equivalent to our word 'space' is *khōra*, in the special metaphysical sense in which it is used in Plato's *Timaeus* (in ordinary Greek, *khōra* meant 'land', 'territory', but also 'the space or room in which a thing is'); and even this equivalence is imperfect.⁶⁷ *Topos* or 'place' cannot by any stretch of the philological

imagination be equated with what the moderns call 'space'. Place is always the place of a body and, as any child knows, it is determined by the body's relationship to other, usually adjacent bodies. Aristotle proposed the following explication of this commonsense notion: The place of a body surrounded or contained by another is "the boundary of the containing body at which it is in contact with the contained body".⁶⁸ The place of a body is, according to this, a surface. The Aristotelian philosopher Strato of Lampsacus (3rd century B.C.) believed that the place of a three-dimensional body must also be three-dimensional and defined it as the interval (*diastema*) between the inner boundaries of the containing body.⁶⁹ John Philoponus (6th century A.D.), commenting on Aristotle's *Physics*, again introduced this definition eight centuries later: "Place is not the boundary of the containing body [...], but a certain three-dimensional incorporeal interval, different from the bodies that fall into it. It is the dimensions alone, devoid of any body. Indeed, with regard to the underlying reality, place and the void are the same."⁷⁰ Strato must also have suggested this identification of place with the void, at least on a cosmic scale, for, according to our sources, he declared the void (*to kenon*) to be "isometric" with the body of the world (*to kosmikon soma*), so that "it is void indeed by its own nature, but it is always filled with bodies, and only in thought can it be regarded as self-subsisting".⁷¹ Although this description is strongly reminiscent of the modern idea of space as an empty receptacle which is occupied by matter, I frankly do not think that we can regard Strato as the first proponent of a concept of absolute space. Not only is his "void" always full, but it is finite, like the "cosmic body" with which it is said to coincide and from which it can be separated "only in thought".

F.M. Cornford (1936) ascribed the invention of the modern idea of space to the 5th-century atomists Leucippus and Democritus, who were the first to introduce the philosophical concept of the void (*to kenon*). According to Cornford, these authors had sought to provide thereby a physical realization of geometrical space. Though I certainly approve of the philosophical purpose of Cornford's paper, which is to show that the modern idea of space is a datable – and dated – figment of philosophy, I am not persuaded by his historical argument. The atomists who, like most 5th-century philosophers, had been strongly impressed by Parmenides, contrasted the atoms and the void as "being" and "not-being", respectively, and never spoke of

the former as of something that occupies a part of the latter. Since the atoms are eternal and uncreated, there can be no question of their 'taking up' or 'being received by' the void. The void surrounds the atoms and these move about in it: but the void is not conceived as an underlying continuum that is partly empty and partly full. While the atomists allowed the void to permeate the infinity of atoms, which coalesced in infinitely many independent systems or *kosmoi* ('worlds'), the Stoic school, founded towards 300 B.C. by Zeno of Citium, maintained that there is but one *kosmos*, which is finite and tightly packed with matter, and is surrounded by a boundless void (*kenon*). The union of void and *kosmos* they called *to pan*, i.e. the All. The Stoic All can be said to contain all the points demanded by Euclid, but there is no evidence that the Stoics had geometry in mind when they developed their doctrine.

A likelier antecedent of the modern connotations of 'space' can be found in the use of *spatium* by Lucretius (98–55 B.C.) in his didactic poem *De rerum natura*. Here, *to kenon* becomes "locus ac spatium quod inane vocamus" (the place and space that we call void – I,426, etc.). The adjective *inanis* ('empty', 'void') immediately suggests the contrast with a "locus ac spatium plenum", a full place and space. If this or a similar expression were used in the poem I would not doubt that Lucretius did conceive *spatium* as a medium that was partly empty and partly filled with bodies. The best examples I have chanced upon are line I,525, where *bodies* are said to hold and fill *places*, not *space* ("[corpora] quae loca complerent quacumque tenerent"); and lines I,526 f., which W.H.D. Rouse translates: "There are therefore definite bodies to mark off empty space from full." The last passage would satisfy my requirements if the translation were exact. But Lucretius wrote

*sunt ergo corpora certa
quae spatium pleno possint distinguere inane,*

and it seems more natural to read *plenum* as a noun – 'the full' – standing for that which bodies are said to mark off from *spatium inane*, 'the void'. On the other hand, several passages contrast, in the best atomist tradition, *body* as such – not space occupied by body – with the *void*, the *empty void*, *empty space* or simply *space*. But even if Lucretius never meant to sing the modern idea of space, some of his hexameters must have conjured it up in the minds of his modern readers.⁷²

Turning now to the medieval background of modern science and philosophy, we find that the better-known scholastic writers believed in the finite world and generally rejected the existence of the void inside or outside it. However, some of them were led to countenance its possibility by the consideration of divine omnipotence, which, they granted, involved the power of annihilating the earth without altering the heavens and of creating another world outside ours. The English mathematician and theologian Thomas Bradwardine (c.1290–1349) found a manner of providing all the points required by geometry by lodging them in God's imagination. God must imagine the site of the world before creating it; and since it is absurd to imagine a limited empty space, what God imagines is the infinite space of geometry. God is said to be eternally present in every part of this infinite imaginary site. "Indeed, He coexists wholly and fully with infinite magnitude and imaginary extension and with each part of it."⁷³ Hasdai Crescas (1340–1410), a Catalonian rabbi, asserted the existence of an infinite vacuum consisting of "three abstract dimensions, divested of body". Such incorporeal dimensions "mean nothing but empty place capable of receiving corporeal dimension", whereby it becomes the place of a body.⁷⁴ But not until the Italian *cinquecento* did such ideas gain currency among Christian writers. The independent subsistence of space as an infinite incorporeal receptacle for all bodies was a common tenet of the natural philosophers Bernardino Telesio (1509–1588), Francesco Patrizzi (1529–1597), Giordano Bruno (1548–1600) and Tommaso Campanella (1568–1639). In Bruno's words, "space is a continuous three-dimensional natural quantity, in which the magnitude of bodies is contained, which is prior by nature to all bodies and subsists without them but indifferently receives them all, and is free from the conditions of action and passion, unmixable, impenetrable, unshapeable, non-locatable, outside all bodies yet encompassing and incomprehensibly containing them all".⁷⁵

The most influential 17th-century solutions of the problem of space are the relationist doctrine of Leibniz (1646–1716) and the absolutist view favoured by English writers, that was incorporated by Newton (1643–1727) into the framework of his mechanics. Leibniz characterized space as the order of coexistence, meaning, I presume, that it is nothing but a mathematical structure embodied in coexisting things (or in their simultaneous states). Leibniz conceived this structure as resting entirely on *distance*, which he apparently regarded as a

physical relation between coexistent things.⁷⁶ Absolutists, on the other hand, conceived of space as “infinite amplitude and mensurability”, existing by itself even “after the removal of corporeal matter out of the world” and before the creation of such matter.⁷⁷ Though this incorporeal entity could not be directly perceived, Newton claimed that motion, or rather acceleration relative to it had tangible effects on bodies.

The problem of space had an important role in the development of Kant's critical philosophy. Kant (1724–1804), ever wary of *Schwärmerei*, rejected real infinite pure self-subsisting space as an *Unding*, that is, a non-entity or chimera.⁷⁸ In his early writings, he upheld a relationist theory of space. There would be no space, he wrote in 1746, if material particles were not the site of forces, with which they act upon each other. For “without force there is no connection (*Verbindung*), without connection there is no order and without order there is no space”.⁷⁹ The dynamic interaction between the particles is held responsible for the structural properties of space. Thus, the fact that space has three dimensions follows from the fact that the forces of interaction are inversely proportional to the square of the distance between the interacting particles. (Note that young Kant, like Leibniz, regards distance as a property of matter, prior to the constitution of space.) This is a contingent fact. A different law of interaction would yield a space having more or less than three dimensions. “A science of all the various possible kinds of space would certainly be the highest geometry that a finite understanding might undertake.”⁸⁰ In 1768, however, Kant came to the conclusion that his relationist views were untenable, because space, far from being an attribute of matter or a construct derived from its consideration, was ontologically prior to spatial things. Some essential properties of the latter, he argued, depend on the manner of their imbedding in universal space.⁸¹ But Kant would not naïvely accept the usual absolutist theory of space, which, to his mind, was laden with absurd implications. He was driven therefore to develop a radically new interpretation of the ontological status of space (and of time).

Kant's ontology of space is, at the same time, an epistemology of geometry. As such, it provided the starting-point and, so to speak, the conceptual setting for many of the philosophical discussions of geometry in the 19th century. We must therefore say a few things about it. Kant first presented his new philosophy of space and time in

the Latin dissertation *On the form and the principles of the sensible and the intelligible world* (1770). It is, in fact, the core of the platonizing theory of the principles of human knowledge outlined in that work. The problems raised by this theory forced its abandonment and led Kant to his vaunted revolution in philosophy. The theory of space and time is presented in the *Transcendental Aesthetic*, the first part of the *Critique of pure reason* (first edition, 1781; second, revised, edition, 1787), almost in the same terms as in the Latin dissertation. A consistent reading of Kant's critical philosophy requires however that those terms be qualified in the light of the next two parts, the *Transcendental Analytic* and the *Transcendental Dialectic*. Since most philosophers, outside the narrow circle of Kant specialists, have paid scarcely any attention to this requirement, the straightforward, precritical philosophy of space and geometry developed by Kant in 1770 has played a much greater role in the history of thought than the subtler, more elusive doctrine that might be gathered from the entire *Critique* of 1781 and 1787 and from his other critical writings.

The doctrine of 1770 follows a familiar metaphysical scheme. The human mind is regarded as a substance that interacts with other substances. The capacity of the human mind to have its state of consciousness (*status repraesentativus*) modified by the active presence of an object is called sensibility (*sensualitas*). The modifications caused thereby are called sensations. The modifying object can be the mind itself or another substance; in the latter case, it is said to be outwardly or externally sensed.⁸² Sensations are a source of knowledge of the sensed object; indeed, they are the only source of direct knowledge of individual objects that is available to man. Such knowledge by direct acquaintance is called by Kant intuition (*intuitus, Anschauung*). Human knowledge of reality rests therefore entirely on sense intuition. Intuitive knowledge of an object is brought about by the combination of sensations arising from the object's presence into a coherent presentation of the object itself. Such a combination of sensations is governed by a "law inherent in the mind"⁸³ of which space is a manifestation. For "space is not something objective and real, neither a substance, nor an attribute, nor a relation, but a subjective and ideal schema for coordinating everything that is externally sensed in any way, which arises from the nature of the mind according to a stable law".⁸⁴ We shall not discuss

here the arguments given by Kant in support of this extraordinary assertion. If they are valid, it follows at once that externally sensed objects are spatial insofar as they are presented to us in sense intuition, but that no spatial properties and relations need be ascribed to them as they exist in themselves, independently of their presentation to the human mind. Kant can therefore uphold the ontological priority of space over bodies without having to admit "that inane fabrication of reason",⁸⁵ real self-subsisting empty infinite space.

We have an idea of universal space which is not, however, a general concept under which all particular spaces are subsumed, but a 'singular representation' that comprises such spaces as its parts. Moreover, in Kant's opinion, the idea of space cannot be fully conveyed by concepts, since such spatial relations as the difference between a glove and its mirror-image can only be felt, not understood. He therefore calls our idea of space an intuition, although it obviously does not acquaint us with a real object. It is said to be a pure intuition, because it does not depend on the sensations that are coordinated in space. Surprisingly enough, this non-conceptual idea, which one would naturally expect to be ineffable, is said to be manifest "in the axioms of geometry and in any mental construction of postulates and problems".

That space has only three dimensions, that there is but one straight line joining two given points, that a circle can be drawn on a plane about a given point with any given radius, etc., these facts cannot be inferred (*concludi*) from some universal notion of space, but can only be perceived (*cemi*) concretely in space itself.⁸⁶

Geometrical propositions are therefore not logically true, and they can be denied without fear of contradiction. Nevertheless, "he who exerts himself to feign in his mind any relations different from those prescribed by space itself, labours in vain, for he is compelled to employ this very idea in support of his fiction".⁸⁷ Kant obviously assumed that "the relations prescribed by space itself" are those stated in Euclid's *Elements*. He was persuaded that his new theory of space guarantees and explains the objective validity of (Euclidean) geometry, i.e. the alleged fact that this science, which is not nourished by experience, is nevertheless true of every imaginable physical object.

Nothing at all can be given to the senses unless it agrees with the primitive axioms of space and their consequences (according to the prescriptions of geometry), even though their principle is purely subjective. Therefore, anything that is thus given will, if

self-consistent, necessarily be consistent with the latter, and the laws of sensibility will be laws of nature *insofar as it can be perceived by the senses* (*quatenus in sensus cadere potest*). Hence nature complies exactly (*ad amussim*) with the precepts of geometry regarding all the properties of space demonstrated in this science, on the strength not of a feigned presupposition (*hypothesis*), but of one that is intuitively given as the subjective condition of all phenomena that nature can exhibit to the senses.⁸⁸

Two aspects of the foregoing doctrine must be modified in order to adjust it to Kant's mature philosophy. According to the latter, human knowledge is restricted to the objects of sense, as they appear to us in space and in time. Outside this context, no property or relation can be cognitively predicated of anything. Therefore, the metaphysical scheme of 1770 is no longer tenable. Space cannot be regarded as an attribute of a substance, the mind, which coordinates the modifications that this substance suffers through the action of other substances. The philosophy of space and time must now rest on an analysis of human experience and its presuppositions as revealed from within. In the light of this analysis, ordinary self-awareness is seen to presuppose the perception of objects in space.⁸⁹ Space does not therefore depend on the human psyche, not at any rate as it is known to us, through its phenomenal manifestation, since it is indeed the latter that requires the prior availability of space. If objective space still is said to be subjective it must be because of the 'egotistic' or – *sit venia verbo* – self-like features of the process through which space itself becomes manifest, of that "progress in time, [which] determines everything, and is not in itself determined by anything else".⁹⁰

The second adjustment that must be introduced into the doctrine of 1770 in order to incorporate it into critical philosophy, is more relevant to geometry. According to the *Critique of pure reason* all connection (*Verbindung*) and hence all ordering of a manifold of sense-data is the work of the understanding,⁹¹ and must therefore be regulated by concepts. Hence, preconceptual intuitive space should no longer be described as "that which causes the manifold of appearance to be intuited as ordered in certain relations", but rather as "that which makes it possible that the manifold of appearance be ordered in certain relations".⁹² This 'form of outer intuition' does not therefore by itself possess the structure described by the propositions of geometry. "Space, represented as *object* (as we are actually required to do in geometry), contains more than the mere form of intuition."⁹³

Space is something so uniform and as to all particular properties so indeterminate, that we should certainly not seek a store of laws of nature in it. Whereas that which determines space to assume the form of a circle or the figures of a cone and a sphere, is the understanding, so far as it contains the ground of the unity of their constructions. The mere universal form of intuition, called space, is therefore the substratum of all intuitions determinable to particular objects, and in it lies, of course, the condition of the possibility and of the variety of those intuitions. But the unity of the objects is entirely determined by the understanding.⁹⁴

Since Kant conceived the “manifold of a priori intuition” called space, not as a mere point-set, but as a (presumably three-dimensional) continuum, we must suppose that he would have expected “the mere form of intuition” to constrain the understanding to bestow a definite topological structure on the object of geometry. But, apart from this, the understanding may freely determine it, subject to no other laws than its own. Since the propositions of classical geometry are not logically necessary, nothing can prevent the understanding from developing a variety of alternative geometries (compatible with the prescribed topology), and using them in physics.

Though this conclusion is clearly implied by the foregoing Kantian texts it is unlikely that we would ever light on them if we did not enjoy the benefit of hindsight, that is, if we had not been familiar with the multiplicity of geometrical systems before reading Kant. When the non-Euclidean geometries became a subject of philosophical debate in the second half of the 19th century, the self-appointed custodians of Kantian orthodoxy were among its fiercest opponents. They dismissed the new geometries as interesting, possibly even useful, intellectual exercises that had nothing to do with the true science of space. For this science – as Kant had taught in 1770, and again in the Transcendental Aesthetic of the *Critique of pure reason* and in the chapter on pure mathematics in the *Prolegomena* – was revealed through pure intuition in full agreement with the *Elements* of Euclid.

1.0.4 *Descartes' Method of Coordinates*

The conception of space as a medium containing every point referred to by the propositions of geometry, naturally motivates the view that regards geometry as the science of space. If space is assumed to exist somehow *in rerum natura*, it is almost inevitable to think of geometry as a natural science, that must determine its object in successive

approximations, under the guidance and control of experience. This is a hard task indeed, for space is a shy god, who shuns the sight of his believers.

The tendency to view geometry as the science of space was greatly strengthened by Descartes' method of coordinates, which revolutionized the treatment of geometrical problems and provided the appropriate instrument for the description of the phenomena of motion in modern physics. Descartes' method, introduced in his *Geometry* (1637), may be roughly described as follows: Each point in space is labelled with an ordered triple of (directed) lengths; their relations can then be determined by investigating quantitative relations between their labels; every line and surface can be defined as the locus of all points whose labels are related by a given equation. Following this approach, the primary objects of geometry are points and their relations, and it is reasonable to define geometry as the science of space if the latter is equated with the set of its points or if it is regarded as a medium that can be analyzed into them.

Descartes' method of coordinates probably contributed more than anything else to shape the views on space and geometry of most 19th-century mathematicians. The two boldest conceptual innovations of the 19th century that we shall subsequently have to discuss, namely, Riemann's theory of manifolds (Sections 2.2.8ff) and Lie's theory of continuous groups (Sections 3.1.4, 3.1.5), can be considered in a sense as natural extensions of that method. It is important, therefore, that we have a clear grasp of its foundations. The crucial step in Descartes' method is the construction of an algebra of (directed) lengths. After this is secured, the labelling of points is an easy and fairly obvious matter. We tend to take that step for granted, because we regard directed lengths as real numbers, i.e. as the elements of a complete ordered field. But Descartes did not have such neat concepts at his disposal and had to work them out for himself. In order to make his construction intelligible to contemporary readers, I shall explain it in my own terms, in agreement with today's standards of precision. But I shall avoid every assumption that does not seem to be clearly involved by Descartes' procedure. Anyhow, the reader will do well to take a look at Descartes' text in Book I of his *Geometry*.

We shall define an algebraic structure on the set of directed lengths or, as we shall prefer to say, of directed linear magnitudes of Euclidean space. This structure will turn out to be that of an ordered

field and, if a strong but historically plausible assumption is allowed, that of a complete ordered field. Let m be a Euclidean straight line, produced to infinity. We use capital letters $A, B, C \dots$ to denote points on m . The reader is presumably familiar with the relation of betweenness that holds for such points. This may be characterized as follows:

(i) If B lies between A and C , then $A \neq B \neq C \neq A$ and B lies between C and A .

(ii) If $A \neq C$, there exist on m points B and D , such that B lies between A and C and C lies between A and D .

(iii) If A, B, C are three different points on m , one and only one of them lies between the other two.

(iv) If A_1, A_2, A_3, A_4 are four different points on m , there is a permutation σ of $\{1, 2, 3, 4\}$ such that $A_{\sigma(2)}$ lies between $A_{\sigma(1)}$ and $A_{\sigma(3)}$ and also between $A_{\sigma(1)}$ and $A_{\sigma(4)}$, while $A_{\sigma(3)}$ lies between $A_{\sigma(1)}$ and $A_{\sigma(4)}$ and also between $A_{\sigma(2)}$ and $A_{\sigma(4)}$.

We shall assume that the Euclidean line m has the following property:

(D) If the points on m all belong to either of two mutually disjoint sets, a_1 and a_2 , which are such that whenever two points P and Q belong to the same set a_i ($i = 1, 2$) every point lying between P and Q also belongs to a_i , there exists a unique point X which lies between each point in $a_1 - \{X\}$ and each point in $a_2 - \{X\}$.

This is the strong assumption that I mentioned above. I said that it is historically plausible because there is every reason to believe that Descartes would have readily admitted it.⁹⁵

Henceforth, we shall write $b(ABC)$ for 'B lies between A and C'. Let O and E be two fixed points on m . We shall refer to O as the 'origin'. We shall now define a linear order on the points of m .⁹⁶ If X and Y are two points of m such that X precedes Y in this linear order, we write ' $X < Y$ '. The linear order is characterized by the following three conditions:

(i) $X < O$ if and only if $b(XOE)$.

(ii) $O < X$ if and only if $b(OXE)$ or $X = E$ or $b(OEX)$.

(iii) $X < Y$ ($X, Y \neq O$) if and only if $b(XOY)$, or $Y < O$ and $b(XYO)$ or $O < X$ and $b(OXY)$.

An ordered pair (X, Y) of points on m will here be called a *directed segment*. We denote it by XY . X and Y are, respectively, the first and the last *endpoint* of XY . XY is *positive* if $X < Y$, *negative* if $Y < X$ and *null* if $X = Y$. Two directed segments $XY, X'Y'$ are *congruent* if

(K1) they are both positive or both negative and the Eudoxian ratios XY/OE and $X'Y'/OE$ are equal, or if (K2) they are both null. These definitions can be extended to any other Euclidean line m' . Choose O' and E' on m' so that the Eudoxian ratio $O'E'/OE$ equals OE/OE . Define linear order on m' as before. Let \mathcal{M} be the set of all Euclidean lines ordered in this way. Two directed segments belonging to the same or to different lines of \mathcal{M} are said to be congruent if they satisfy (K1) or (K2). The reader ought to verify that congruence of directed segments is an equivalence. A *directed linear magnitude* (dlm) is an equivalence class of congruent directed segments. If XY is any directed segment, we let $[XY]$ denote the dlm to which it belongs. $[XY]$ is positive, negative or null if XY is, respectively, positive, negative or null. It will be easily seen that, for every point X on an ordered line m , each dlm has one and only one member whose first endpoint is X and whose last endpoint lies on m . In particular, this is true of the origin O . Consequently, as X ranges over the set of points of m , $[OX]$ ranges over the set of dlm's. Let X and Y be points of m . We say that $[OX]$ is less than $[OY]$ (abbreviated: $[OX] < [OY]$) if and only if $X < Y$. If $[OX] < [OY]$ we also say that $[OY]$ is greater than $[OX]$. The relation $<$ obviously defines a linear ordering of the set of dlm's.

Let Σ denote the set of dlm's 'gauged' by the choice of a segment OE . Σ will be endowed with a field structure. Let $[OX], [OY]$ be two dlm's. Let m be the line through O and X (Fig. 2). There is one and only one member of $[OY]$ whose first endpoint is X and whose last endpoint lies on m . Let W be this last endpoint. We define $[OW]$ to be the *sum* ($[OX] + [OY]$) of $[OX]$ and $[OY]$. The reader should satisfy himself that '+' is an operation on Σ and that $(\Sigma, +)$ is an Abelian group. It should be clear, in particular, that $[OO]$ is the neutral

$$[OW] = [OX] + [OY]$$

$$[OY] = [XW]$$

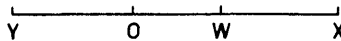


Fig. 2.

$$[OZ] = [OX] \cdot [OY]$$

$$[OE] = [OE']$$

$$[OY] = [OY']$$

$$E'X \parallel Y'Z$$

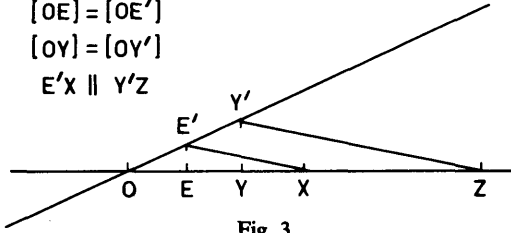


Fig. 3.

element of the group and that $[XO]$ is the inverse $-[OX]$ of $[OX]$. To define the second field operation or *product* of two dlm's we shall use the fact that our ordered lines are embedded in ordinary Euclidean space. Descartes based his own construction on Euclid VI, 2: "If a straight line be drawn parallel to one of the sides of a triangle, it will cut the other sides [or those sides produced] proportionally." Let m be a line ordered as above, with respect to points O and E . Let line m' meet m at O (Fig. 3). Choose E' on m' so that $OE'/OE = OE/OE$. We now define the product $[OX] \cdot [OY]$ of two dlm's $[OX]$, $[OY]$, as follows: Let OY' be the member of $[OY]$ whose first endpoint is O and whose last endpoint lies on m' ; let Z be the point where m meets the parallel to $E'X$ through point Y' ; then $[OZ] = [OX] \cdot [OY]$. It can be easily verified that \cdot is indeed an operation on Σ ; that for every dlm $[OX]$, $[OX] \cdot [OE] = [OE] \cdot [OX] = [OX]$; and that every non-null dlm $[OX]$ has a reciprocal dlm $[OX]^{-1}$ such that $[OX] \cdot [OX]^{-1} = [OX]^{-1} \cdot [OX] = [OE]$. The reciprocal of $[OX]$ can be constructed thus: Let Z' be the point where m' meets the parallel to $E'X$ through E ; then $[OZ'] = [OX]^{-1}$. The reader should verify that the operation \cdot is commutative and associative, so that $\langle \Sigma, +, \cdot \rangle$ is indeed a field, with zero element $[OO]$ and unity $[OE]$. $\langle \Sigma, +, \cdot \rangle$ is ordered by the relation $<$. It can be shown moreover that, if line m has the property (D), $\langle \Sigma, +, \cdot \rangle$ is complete. Since all complete ordered fields are structurally equivalent, they are indistinguishable from a mathematical point of view. Any such field is usually called *the real number field* and is designated by the symbol \mathbf{R} . Its elements, *qua* elements of \mathbf{R} , are called real numbers.⁹⁷

The set of all ordered triples of elements of \mathbf{R} is denoted by \mathbf{R}^3 . We shall show how to label each point of Euclidean space with an element of \mathbf{R}^3 . In other words, we shall define a bijective mapping of

the set of Euclidean points onto \mathbf{R}^3 . To do this, we first define the *directed distance* from a plane to a point. Let π be a plane. π has two sides, which we conventionally label the *positive* and the *negative* side, respectively. Let P be a point not on π , and Q its perpendicular projection on π (i.e. the point where a line through P meets π at right angles). There is one and only one positive dlm $[OX]$, such that $PQ/OX = OX/PQ$. The directed distance from π to P is $[OX]$ if P lies on the positive side of π ; $-[OX]$ if P lies on the negative side of π . If P lies on π , its directed distance from π is $[OO]$. Now, let π_1, π_2, π_3 be three mutually perpendicular planes. Let $f^i(P)$ be the directed distance from π_i to point P ($i = 1, 2, 3$). We assign to P the ordered triple $(f^1(P), f^2(P), f^3(P))$. It will be easily seen that this rule defines a bijection of the set of all Euclidean points onto \mathbf{R}^3 . For the points at directed distance $f^i(P)$ from π_i lie all on a plane on a definite side of π_i and at a definite distance from that plane; our ordered triple determines therefore three mutually perpendicular planes which meet only at P . On the other hand, for every ordered triple of dlm's ($[OX_1], [OX_2], [OX_3]$) there is a point X whose directed distance from π_i is $[OX_i]$.

We shall hereafter use the following terminology: A bijective mapping $P \mapsto (f^1(P), f^2(P), f^3(P))$ of Euclidean space onto \mathbf{R}^3 , constructed according to the above directions, will be called a Cartesian mapping. Please observe that the definition of a Cartesian mapping involves the arbitrary choice of an ordered triple of planes, of the positive side of each of these planes and of a segment OE as a gauge for distances. The ordered triple of planes that enter into the definition of a Cartesian mapping is the *frame*, their point of intersection, the *origin* of the mapping. The frame $\langle \pi_1, \pi_2, \pi_3 \rangle$ of a Cartesian mapping is said to be right-handed if the following condition is fulfilled: If I place my right hand at the origin, with the thumb pointing toward the positive side of π_1 , and the index finger pointing toward the positive side of π_2 , I can bend the middle finger so that it points toward the positive side of π_3 . The frame is left-handed if the foregoing condition is fulfilled with the word 'left' substituted for 'right'. All frames of Cartesian mappings are either left-handed or right-handed. In this book, unless otherwise stated, we assume that they are right-handed. The three real numbers assigned to a point P by a Cartesian mapping are called the coordinates of P by this mapping. If f and g are two Cartesian mappings, the composite

mapping $g \cdot f^{-1}$ is a (Cartesian) transformation of coordinates. If P is any point of space, $g \cdot f^{-1}$ maps $f(P)$ on $g(P)$. Let $x = (x^1, x^2, x^3)$ and $y = (y^1, y^2, y^3)$ be the coordinates of points P and Q, respectively, by some Cartesian mapping. The positive square root of $(x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2$ (where we write $-y^i$ for $+(-y^i)$) is called the *distance* between points P and Q or the *length* of segment PQ and will be denoted by $|x - y|$ or by $|PQ|$. It follows from the theorem of Pythagoras that $|x - y|$ is a real number which does not depend on the particular choice of a Cartesian mapping (it is, as we shall often say, invariant under Cartesian transformations of coordinates). If Q happens to be the origin of the mapping, $y^i = [OO]$ ($i = 1, 2, 3$) and we write $|x|$ instead of $|x - y|$. Generally, if $x = (x^1, \dots, x^n)$ is any n -tuple of real numbers, we shall use the symbol $|x|$ to represent the positive square root of $\sum_{i=1}^n (x^i)^2$.

NON-EUCLIDEAN GEOMETRIES

It is unlikely that Euclid ever held his five postulates to be self-evident. Mathematicians sharing the Aristotelian conviction that only manifest truths may be admitted without proof in geometry usually did not find the fifth postulate quite so obvious as the other four. From Antiquity, many attempts were made to prove it, but the proofs proposed depended always explicitly or implicitly upon new assumptions, no less questionable than the postulate itself. In the 1820's, Janos Bolyai and Nikolai I. Lobachevsky independently of each other developed two versions of a system of geometry based at once on the denial of Postulate 5 and on the assertion of all the propositions of Euclid's system which do not depend on it. We shall call this system BL geometry. In a BL plane, for any straight line and any point outside it there are infinitely many straight lines through the latter that do not meet the former, while in a Euclidean plane, for any straight line and any point outside it there is exactly one straight line through the latter which does not meet the former. Coplanar straight lines which do not meet each other Euclid calls parallel lines. Although Postulate 5 does not mention parallels, it is applied by Euclid for the first time in the proof of an important theorem concerning them. The theory of parallels therefore provided the immediate context for the debate over Postulate 5 and the eventual development of a geometry based on its denial. We shall deal with this matter in Part 2.1.

In 1827, Carl Friedrich Gauss published his *General Disquisitions on Curved Surfaces*, doubtless the main source of inspiration for the remarkable generalization of the fundamental concepts of geometry proposed by Bernhard Riemann in 1854. From Riemann's point of view, Euclidean geometry is merely a special case among the infinitely many metrical structures that a three-dimensional continuum may possess. BL geometry, on the other hand, corresponds to a whole (infinite) family of cases. Still other, infinitely many, cases are covered by neither of these two systems. In Part 2.2, we shall refer to Gauss's work and comment on Riemann's insights.

Part 2.3 is concerned with another way of incorporating the multiplicity of geometries into a unitary system, that was proposed by Felix Klein in 1871. Riemann's conception is indeed deeper and has exerted a much stronger influence on the use of geometry in physics and on the philosophers who have reflected upon it; but Klein's idea, together with the 19th-century development of projective geometry that led to it, contributed to shape the abstract axiomatic approach that has prevailed in foundational studies since the 1890's.

2.1 PARALLELS

2.1.1 *Euclid's Fifth Postulate*

Euclid's Postulate 5 ('Axiom XI' in some manuscripts and in the older editions) has been translated thus:

If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.¹

In order to understand what this means we must assume that a straight line divides each plane on which it lies into two half-planes. A half-plane is determined unambiguously by a point on it and the limiting straight line. Thus, if P is a point on a half-plane limited by line s , we may denote the half-plane by (s, P) . Euclid speaks about two arbitrary straight lines m, n – which we must assume to be coplanar – and a third straight line, the transversal t , that intersects them, say at M and N , respectively. Interior angles are the two angles made by t and m at M in the half-plane (m, N) and the two angles made by t and n at N in the half-plane (n, M) . One interior angle at M and one at N are on one side of t , the other two are on the other. Since the four interior angles add up to four right angles or 2π , and the two at each point add up to two right angles or π , the sum of the two angles on one side of t is less than π if and only if the sum of the other two is greater than π . Euclid postulates that if one of these sums is less than π , the straight lines m and n meet at some point of the half-plane limited by t which comprises the two angles that make up the said sum. In other words, if two coplanar straight lines m and n together with a transversal t make on the same side of t interior angles whose sum is less than π , the three lines m, n and t form a

triangle, with one of its vertices on the half-plane defined by t which comprises those interior angles. In thus proclaiming the existence, under certain conditions, of a triangle and consequently the ideal possibility of constructing it, Postulate 5 follows the pattern of the first three, all of which are statements of constructibility. It follows this pattern only up to a point, however, for the constructions postulated in the former postulates are not subject to any restrictions. Postulate 5 says nothing about an alternative possibility, namely, that the interior angles made by m and n on either side of t be equal to π . In this case, the figure formed by t and the parts of m and n on one side of t is congruent with the figure formed by t and the parts of m and n on the other side of t . Therefore, if we assume that two straight lines cannot meet at more than one point, it is obvious that in this case m and n are parallel.

Euclid proves in Proposition I.28 that two (coplanar) straight lines are parallel if a transversal falling on them makes the interior angles on the same side equal to π . Neither this proposition nor any other of the twenty-seven preceding ones, depends on Postulate 5. The latter is used for the first time in the proof of I.29, which includes, among other things, the converse of the preceding statement: if two straight lines are parallel, any straight line falling on them makes interior angles on the same side equal to π . In other words, given a straight line m and a point P outside it, we can prove without using Postulate 5 that the flat pencil of straight lines through P on the same plane as m include at least *one* line parallel to m , namely, the normal to the perpendicular from P to m . By means of Postulate 5, we can prove that this is the *only* line through P which is parallel to m . The uniqueness of the parallel to a given straight line through a point outside it plays an essential role in Euclid's proof of one of the key theorems of his system, namely, Proposition I.32: "The three interior angles of a triangle are equal to two right angles."

2.1.2 Greek Commentators

Postulate 5 was probably introduced by Euclid himself or by one of his predecessors in order to solve those difficulties in the older theory of parallels to which Aristotle referred in several passages.² Its meticulously precise formulation, as compared with the bluntness of the first four postulates, is easily understandable if it is true that the

postulate was consciously designed to provide a missing link in a deductive chain: Euclid put into it exactly what he needed to prop up his proofs. The contrast in style between the long-windedness and the technicalities of Postulate 5 and the conciseness and apparent simplicity of the other four must have perplexed Euclid's readers, especially if they were wont to regard his *aitemata* or 'demands' as the self-evident principles of an Aristotelian science. Our sources indicate that some of the oldest commentators of the *Elements* questioned the wisdom of including Postulate 5 among the statements assumed without proof, and attempted to demonstrate it. Proclus had no doubts on this matter. Postulate 5, he says, "ought to be struck from the postulates altogether. For it is a theorem – one that invites many questions, which Ptolemy proposed to resolve in one of his books – and requires for its demonstration a number of definitions as well as theorems".³ Proclus adds that Euclid himself has proved the converse as a theorem (I.17). Of course, this is not a very cogent argument. More significant is Proclus' objection to some authors who maintained that Postulate 5 was self-evident. They had apparently shown that it was really equivalent to a very simple statement, which we might render, in mock-Greek no less laconic than the language of the earlier postulates, *eutheiai suneousai sumptousin*, 'convergent straight lines meet'. Proclus allows that two coplanar straight lines that make internal angles less than π on one side of a transversal, do indeed converge on that side, i.e. do indefinitely approach each other. But he observes that it is not at all evident that convergent straight lines should eventually meet, for it is a well-established fact that there are lines – e.g. hyperbolae and their asymptotes – which approach each other indefinitely but never meet. "May not this, then, be possible for straight lines, as for those other lines? Until we have firmly demonstrated that they meet, the facts shown about other lines strip our imagination of its plausibility. And although the arguments against the intersection of these lines may contain much that surprises us, should we not all the more refuse to admit into our tradition this unreasoned appeal to probability?"⁴ Further on, in his commentary on the basic propositions of parallel theory, Proclus reproduces Ptolemy's proof of Postulate 5 and shows it is inadequate. He tries to complete it but is no more successful.⁵ Nevertheless, these efforts should not be dismissed as worthless, for they have helped to bring out the implications and equivalents of Postulate 5.

2.1.3 Wallis and Saccheri

We shall not review the history of the alleged demonstrations of Postulate 5 through the medieval and renaissance periods until 1800. We shall only refer briefly to the contribution of John Wallis (1616–1703) and, more extensively, to the work of Girolamo Saccheri (1667–1733).

John Wallis published, in the second volume of his *Mathematical Works* (1693), two lectures on our subject, which he had delivered in 1651 and 1663 from his Savilian Chair of Mathematics at Oxford University. The first lecture is just an exposition of the proof of Postulate 5 given by the Arabian mathematician Nasîr-Eddîn (1201–1274), but the second one contains an original demonstration. At both the beginning and the end of his lecture, Wallis declares that any such proof is unnecessary and that we cannot take Euclid to task for having tacitly assumed or openly postulated self-evident truths such as that “two convergent [coplanar] lines finally meet”.⁶ Nevertheless, since so many have believed that Postulate 5 needs proof, Wallis sets forth his own, hoping that it will be more persuasive than those preceding it. It is based on eight lemmata. The first seven are propositions proved by the usual methods, and under the familiar assumptions, of geometry; but the eighth is a basic principle which Wallis attempts not to prove but only to clarify so that it will appear self-evident. He states it thus: “For every figure there exists a similar figure of arbitrary magnitude”.⁷ Wallis observes that, since magnitudes may be subjected to unlimited multiplication and division, Lemma VIII follows from the very essence of quantitative relations, inasmuch as every figure, while preserving its shape, may be increased or reduced without limit. He adds that Euclid in fact assumes this principle in his Postulate 3 (“to describe a circle with any centre at any distance”), for “you may continuously increase or reduce a circle in any way you wish without altering its shape, not because of its superiority to the other figures, but because of the properties of continuous magnitudes which the other figures share with the circle”.⁸ In several passages, Wallis’s text shows that the author was well aware of the use of tacit assumptions in Euclid’s proofs. It shows also that Wallis was convinced that Postulate 5 cannot be demonstrated unless we introduce another postulate in its place. He apparently expected that his own postulate, i.e. Lemma VIII, would shine forth with greater evidence. He probably felt that it would be absurd to

deny it, since that would imply that there are no similar figures of arbitrarily different sizes – in particular, no cubes or squares, for these figures can obviously be multiplied through mere juxtaposition.

In 1733, Girolamo Saccheri, a jesuit well versed in the literature on the problem of parallels, published a treatise whose Book I deals with the subject.⁹ He proposes to prove Postulate 5 by a method not yet tried, that of indirect proof. Saccheri attempts to show that the denial of Postulate 5 is incompatible with the remaining familiar assumptions of geometry. Since he was probably aware that the proofs in Book I of the *Elements* often depend on unstated premises, he chooses to treat the first 26 propositions of that book, which, as we know, do not depend on Postulate 5, as undemonstrated principles in his argument. In fact, he also employs as implicit assumptions the Archimedean postulate – if a and b are two straight segments, there exists an integer n such that $na > b$ – and a principle of continuity that may be stated as follows: If a continuously varying magnitude is first less and then greater than a given magnitude, then at some time it must be equal to it. Saccheri considers a certain plane figure, now known as a Saccheri quadrilateral. To construct one, take a straight segment AB and draw two equal perpendiculars AD and BC ; the four-sided polygon $ABCD$ is a Saccheri quadrilateral. It is easily proved that the angles at C and at D are equal. Saccheri proposes three alternative hypotheses: that both angles are right angles, or that

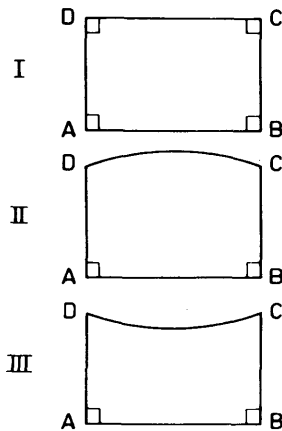


Fig. 4.

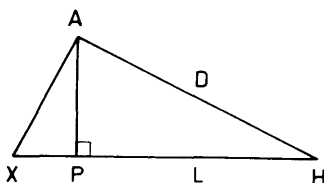


Fig. 5.

both are obtuse, or that both are acute. We shall, for brevity, speak of Hypotheses I, II and III. (Fig. 4.) Saccheri shows that if one of them is true in a single case, it is true in every case. Postulate 5 obtains under Hypotheses I and II (Proposition XIII), but Hypothesis II happens to be incompatible with some of the propositions initially admitted by Saccheri. Postulate 5 will be proved if we manage to show that Hypothesis III is also incompatible with that set of propositions. This is a long and laborious enterprise that takes up most of Saccheri's book.

The incompatibility of Hypothesis II with the set of initial assumptions is established in Proposition XIV. Its proof depends on Proposition IX, which states that, in a right triangle, the sum of the two angles adjacent to the hypotenuse is equal to, greater than or less than the remaining angle if Hypothesis I, II or III is true. Let APX be a right triangle (Fig. 5). If Hypothesis II is true, the angles at X and A are together greater than a right angle. We can find therefore an acute angle PAD such that all three angles together are equal to two right angles. Under Hypothesis II, Postulate 5 obtains; consequently PL and AD meet at a point H . Triangle XAH has two interior angles adjacent to side AX which are together equal to two right angles. This contradicts Euclid I.17. Hypothesis II is therefore incompatible with the set of initial assumptions.

In the course of his fight against Hypothesis III, Saccheri draws from it several conclusions which today are well-known propositions of BL geometry. The existence of one triangle whose three internal angles are equal to, more than or less than π , is sufficient to validate Hypothesis I, II or III, respectively (Proposition XV). If AB is a straight segment, AK a straight line normal to AB , and BD a ray on the same side of AB as K and making with AB an acute angle on the side of AK , it may, under Hypothesis III, very well happen that BD does not meet AK . (Fig. 6.) Let BR be normal to AB (with R on the

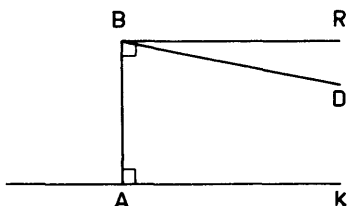


Fig. 6.

same side of AB as K). All rays BD fitting the above description fall within angle ABR. Saccheri proves that, under Hypothesis III, some of these rays meet AK while others share with it a common perpendicular. If we order the former group of rays according to the increasing size of the angle they make with AB, and the latter group according to the increasing size of the angle they make with BR, we shall find that none of these two sequences of rays possesses a last element. For reasons of continuity, Saccheri concludes that between the two sequences there exists one and only one ray which does not meet AK and does not share with it a common perpendicular. The straight line comprising this ray approaches AK indefinitely. Therefore, under Hypothesis III, there exist asymptotical straight lines as Proclus surmised.

Saccheri chooses to state these results with the help of the somewhat tricky notion of an infinitely distant point. Let T be such a point on AK, i.e. a point beyond K and beyond every other point of AK which is at a finite distance from A. The rays of one of our sequences meet AK at points increasingly distant from A; consequently, argues Saccheri, the limiting ray of that sequence meets AK at T. The rays of the other sequence are met by perpendiculars which meet AK orthogonally at points ever more distant from A; consequently, the limiting ray of this sequence must have a perpendicular which meets AK orthogonally at T. But the limiting ray of both sequences is the same ray BT. Therefore BT and AK have a common perpendicular at one and the same point. But this is absurd, for two different straight lines cannot both meet another line perpendicularly at one point – if it is true that all right angles are equal (Euclid, Postulate 4) and that two different straight lines cannot have a common segment. Saccheri does not ask himself whether everything that is true of ordinary points is necessarily true of an infinitely distant point. It would have been

safer, of course, to leave such a point entirely out of this discussion, as it was done above. But then Saccheri's first refutation of Hypothesis III would not have come about. However, he gives a second one, based this time on the notion of an infinitely short segment. We shall not go into it.

In Note II to Proposition XXI, Saccheri proposes three "physico-geometrical" experiments that might confirm Postulate 5. It is no longer necessary to travel indefinitely along two straight lines making, on the same side of a third, interior angles less than π , in order to know whether they meet or not. In the light of the theorems proved by Saccheri this question can be decided on the basis of the properties of a finite spatial configuration. The existence of one Saccheri quadrilateral with four right angles suffices to verify Postulate 5. Its truth is guaranteed also by the existence of a right triangle whose hypotenuse coincides with the diameter of a circle while its opposite vertex lies on the circumference of this circle; or of a polygonal line which is inscribed in a circle and consists of three segments, each equal to the radius of the circle, joining the extremities of one of its diameters. While Saccheri claims correctly that any one of these three figures is very easy to construct, he makes no reference at all to the fact that exact measurements are physically impossible. Yet he must have known that a piece of flat land can be divided into lots whose shape everybody would call rectangular but that nevertheless the earth is round and Postulate 5 is not applicable to the straightest lines that join points on its surface.

2.1.4 *Johann Heinrich Lambert*

Saccheri's work was not unknown to his contemporaries. Stäckel and Engel have verified its presence, since the 18th century, in several public libraries in Germany. It is mentioned in the histories of mathematics of Heilbronner (1742) and Montucla (1758). G.S. Klügel (1739–1812) studies it carefully in his doctoral dissertation on the main attempts to prove Postulate 5. Klügel concludes that Saccheri's alleged proof is not more cogent than the other thirty or so he examines. He observes that "it is possible that non-intersecting straight lines are divergent", and adds: "That this is absurd we know not by strict inferences nor by any distinct notions of the straight line and the curved line, but by experience and the judgment of our eyes".¹⁰ Indeed, our eyes would be hard put to pass judgment on the

absurdity of that statement if two straight lines diverge and hence converge very, very slowly, for then the intersection, if it occurs, will be hopelessly beyond their reach. But perhaps Klügel could not carry his sceptical remarks any further while contending for a university degree.

Klügel's dissertation is praised by the Swiss philosopher and mathematician Johann Heinrich Lambert (1728–1777) in his *Theory of Parallels*, published posthumously in 1786 but apparently written in 1766. Reading Klügel, Lambert learned about Saccheri, if he had not already had direct access to the latter's book. His own work, we shall see, may be regarded as a continuation of Saccheri's. The first section of Lambert's essay deals with methodology. It culminates in the following passage:

The difficulties concerning Euclid's 11th axiom [i.e. Postulate 5] have essentially to do only with the following question: *Can this axiom be derived correctly from Euclid's postulates and the remaining axioms? Or, if these premises are not sufficient, can we produce other postulates or axioms, no less evident than Euclid's, from which his 11th axiom can be derived?* In dealing with the first part of this question we may wholly ignore what I have called the *representation of the subject-matter* [*Vorstellung der Sache*]. Since Euclid's postulates and remaining axioms are stated in words, we can and should demand that no appeal be made anywhere in the proof to the matter itself, but that the proof be carried out – if it is at all possible – in a thoroughly symbolic fashion. In this respect, Euclid's postulates are, so to speak, like so many given algebraic equations, from which we must obtain x , y , z , etc., without ever looking back to the matter in discussion [*die Sache selbst*]. Since the postulates are not quite such formulae, we can allow the drawing of a figure as a guiding thread [*Leitfaden*] to direct the proof. On the other hand, it would be preposterous to forbid consideration and representation of the subject-matter in the second part of the question, and to require that the new postulates and axioms be found without reflecting on their subject-matter, off the cuff, so to speak.¹¹

Lambert's mathematical methodology combines a would-be total formalism in the derivation of theorems with a healthy appeal to intuition in the search for, and the statement of, postulates and axioms. Lambert apparently does not countenance the possibility that "the representation of the subject-matter" might prove insufficient or ambiguous with regard to the truth of Postulate 5.

The programme sketched in the passage quoted above will aid us in understanding some novelties in Lambert's treatment of the theory of parallels. His starting-point is a quadrilateral with three right angles. He examines three hypotheses, called by him the first, the second and

the third, which are, that the fourth angle is a right angle, that it is obtuse, or that it is acute.¹² In three separate sections, Lambert derives consequences from each of these hypotheses. In the proofs based on the second one, he studiously avoids using any of the propositions in Euclid which are incompatible with it (I.16 and its consequences): only towards the end of the section does he appeal to one of those propositions, and this just in order to carry out the refutation of the 2nd hypothesis. I think that this procedure ought to be understood in the light of Lambert's formalism which naturally leads him to explore the possibility of a consistent deductive system based on the 2nd hypothesis, no less than that of one based on the 3rd. On the other hand, Lambert the intuitionist knows of a "representation of the subject-matter" which satisfies the 2nd hypothesis, if only we agree to give an appropriate interpretation to the intrinsically meaningless terms of the corresponding formal system. "It seems remarkable to me" – he writes – "that the 2nd hypothesis should hold when we consider *spherical* triangles instead of plane triangles",¹³ in other words, when we understand by 'straight lines' the great circles on a sphere. These, as is well known, always contain the shortest path between any two points lying on them. But they are closed lines, and they intersect each other at more than one point; so they do not share those properties of ordinary straight lines used in the refutation of the 2nd hypothesis. Even more surprising are the next two remarks of Lambert: (i) The geometry of spherical triangles does not depend upon the solution of the problem of parallels, for it is equally true under any of the three hypotheses; (ii) the 3rd hypothesis, in which the fourth angle of Lambert's quadrilateral is assumed to be less than a right angle, might hold true on an imaginary sphere, i.e. on a sphere whose radius is a pure imaginary number.¹⁴ We have seen that Lambert had a formalist conception of mathematics which likened the premises of a deductive system to a set of algebraic equations whose terms may denote any object satisfying the relations expressed therein. He also discovered the modern idea of a model, that is, of an object or domain of objects which happens to fulfil precisely the conditions abstractly stated in the hypotheses of the system. Such content is supplied by the "representation of the subject-matter", which, according to Lambert, ought to guide the selection of hypotheses. Lambert's last remark shows how broadly he conceived of this kind of "representation", for an imaginary sphere is

not something we could visualize or mould in clay or in *papier maché*, but a purely intelligible entity.

Under the 3rd hypothesis the fourth angle of a Lambert quadrilateral is always acute; the bigger the quadrilateral, the more acute the angle. This makes it possible to transfer the absolute system of measurement, which is familiar in the case of angles, to the measurement of distances, areas and volumes. Indeed, it is enough to take as the absolute unit of length the base of a Lambert quadrilateral whose fourth angle has a given size and whose two sides not adjacent to that angle stand in a fixed proportion to each other. Lambert observes that "there is something alluring about this consequence which readily arouses the desire that the 3rd hypothesis be true!"¹⁵ Such advantage, however, would have to be paid for by many inconveniences, the worst of which would be the elimination of the similarity and proportionality of non-congruent figures, which Lambert believes would be ruinous to astronomy. Saccheri showed that under the 3rd hypothesis the sum of the three interior angles α , β , γ of an arbitrary triangle are less than π . Lambert shows that the 'defect', $\pi - \alpha - \beta - \gamma$, is proportional to the area of the triangle. Stäckel and Engel suggest that this result prompted Lambert's remark about the fulfilment of the 3rd hypothesis on an imaginary sphere. Indeed, the above expression for the 'defect' is obtained from the familiar formula for the area of a spherical triangle with angles α , β , γ upon a sphere of radius r , i.e. $r^2(\alpha + \beta + \gamma - \pi)$, by substituting $\sqrt{-1}$ for r . Lambert's refutation of the 3rd hypothesis is based only on the following: if it were true, two mutually perpendicular straight lines would be parallel to the same line. Lambert finds this an intolerable paradox. The 2nd hypothesis is easier to refute, for it implies that some pairs of straight lines intersect at more than one point. This consequence can be avoided if we allow for straight lines that close upon themselves, but this is, of course, just as paradoxical. Max Dehn has shown that Saccheri's and Lambert's second hypotheses – which are indeed equivalent – do not imply any of these paradoxical consequences once we strike out from our assumptions the postulate of Archimedes.¹⁶

Many treatises and memoirs on the theory of parallels were published in the late 18th and early 19th centuries. The most influential were probably those by Adrien Marie Legendre (1752–1833), which excelled more in the, often deceptive, elegance and clarity of the

proofs, than in the novelty of the results.¹⁷ The contribution of F.A. Taurinus (1794–1874), in his *Theory of Parallels* (1825) and in an appendix to his *Elements of Geometry* (1826), is more interesting. The author, who was induced to study the subject by his uncle, F.K. Schweikart (1780–1857), a professor of jurisprudence, gives unqualified assent to Euclidean geometry, but admits the possibility of developing in a purely formal way a consistent system of geometry where the three interior angles of a triangle are less than π . (This condition is equivalent to Saccheri's Hypothesis III.) Taurinus carries the analytical development of this system – which, in his opinion, “might not lack significance in mathematics”¹⁸ – much further than Saccheri or Lambert, anticipating some important results published later by Lobachevsky.

In a memorandum to Gauss of December, 1818, F.K. Schweikart had set forth the main theses of a new geometry which he called *Astralgeometrie*, probably to suggest that it might be true on an astronomical scale.¹⁹ In this geometry, the three angles of a triangle are less than π , the more so the larger the triangle. Also there exists a characteristic constant, which Schweikart defined as the upper bound of the height drawn from the hypotenuse of an isosceles right triangle. (This is, of course, equal to the distance from the vertex of a right angle to the straight line parallel to both its sides; the existence of such a parallel was the paradox which had led Lambert to reject his 3rd hypothesis.) Gauss remarked that he wholly approved of Schweikart's ideas, which seemed to him to come “from his own heart”.²⁰ Schweikart never published them, however, but persuaded his nephew Taurinus, a professional mathematician, to develop them. The remarkable results obtained by the latter are based simply on the substitution, in the formulae of spherical trigonometry, of imaginary numbers for the radius and the sides. Taurinus' success confirms Lambert's bold conjecture. Taurinus maintains that this new system is wholly unacceptable, for “it contradicts all intuition”. “It is true”, he adds, “that such a system would exhibit locally [*im Kleinen*] the same appearances as the Euclidean system; but if the representation of space may be regarded as the mere form of outer sense, the Euclidean system is indisputably the true one and we cannot assume that a limited experience could generate an illusion of the senses.”²¹ He gives seven additional reasons for the truth of Euclidean geometry, all of them about as persuasive as the first. A more novel

and interesting argument is based on the existence of the characteristic constant. There are as many different forms of the new system as admissible values of the constant. There is no reason whatsoever for preferring one of these values over the others; thus, if the new geometry were true, all its different forms would be true at the same time. Hence, Taurinus concludes, two arbitrary points would determine infinitely many straight lines, one for each value of the constant. Euclid's system, on the other hand, is univocal.

2.1.5 *The Discovery of Non-Euclidean Geometry*

The first publications in which a system of non-Euclidean geometry is presented without reservation are a paper "On the Principles of Geometry" (1829–30) by Nikolai I. Lobachevsky (1793–1856) and the "Appendix presenting the Absolutely True Science of Space" by Janos Bolyai (1802–1860).²² The former contains the essentials of a lecture delivered at the University of Kazan on February 12, 1826. The latter was printed at the end of Volume I of the *Elements of pure mathematics* (1832) by the author's father, Farkas Bolyai (1775–1856), and is a Latin translation of a paper the author had sent to his former teacher, J. W. von Eckwehr, in 1825. The system presented in each of these works is essentially the same – a consistent and uninhibited development from assumptions equivalent to Saccheri's Hypothesis III (and Schweikart's *Astralgeometrie*) – but the authors discovered it independently and put it forth in different terminology. It is the system we have agreed to call BL geometry. Before the discoveries of the Russian professor and the Hungarian captain, Carl Friedrich Gauss (1777–1855), the most illustrious mathematician of the time, had become convinced that there were no purely mathematical reasons for preferring Euclidean geometry to this non-Euclidean system and had worked successfully in the development of the latter. The posthumous edition of his papers and letters leaves no doubt about that. But Gauss never wished to publish his ideas on this matter, for fear, he confided to Bessel, of "the uproar of Boeotians".²³ By 1831, he had begun to put them in writing, so "that they not perish with me", as he told Schumacher.²⁴ But early in 1832 he received the work of Janos Bolyai, sent by his father Farkas, who had been a good friend of Gauss when they were young. Gauss thereupon realized that he could spare himself the trouble of writing out his discoveries, for his friend's son had anticipated him.²⁵

*Some writers, perhaps astonished that such a radically innovative conception could arise independently, and be accepted without qualms for the first time, outside the heartlands of European civilization, have attempted to trace the influence of Gauss upon Janos Bolyai, exerted allegedly through his father Farkas, and upon Lobachevsky, through J.M.C. Bartels, a German professor of mathematics in Kazan and an acquaintance of Gauss. But Gauss' titles to glory are so many and so great that I do not see any point in trying to place upon him the full burden of this particular discovery. Gauss himself expressly acknowledged the originality of Bolyai and Lobachevsky. On February 14, 1832, he wrote to Gerling: "A few days ago I received from Hungary a short work about non-Euclidean geometry, where I find *all my own ideas and RESULTS* developed with great elegance but in such a concentrated form that it will be hard to follow for someone to whom this matter is foreign. The author is a *very* young Austrian officer, the son of an old friend of mine, with whom I often spoke about the subject in 1798, at a time, however, when my ideas were still very far from the elaborateness and maturity they have attained through this young man's own thinking. I regard this young geometer Bolyai as a genius of the first magnitude".²⁶ Fourteen years later Gauss wrote to Schumacher: "I have recently had the occasion of once again going through Lobachevsky's booklet (*Geometrische Untersuchungen zur Theorie der Parallellinien*, Berlin 1840, bei G. Fincke. 4 sheets). It contains the elements of the geometry that must obtain and with strict consistency can obtain if Euclidean geometry is not the true one. A man called Schweikart named such a geometry astral geometry; Lobachevsky calls it imaginary geometry. You know that I have had this conviction for 54 years already (since 1792); [...] I have therefore found nothing in Lobachevsky's work that is substantially new to me, but the development follows a different road than the one I myself took, being masterfully carried out by Lobachevsky in a genuine geometric spirit. I believe I ought to draw your attention to this book which will give you thoroughly exquisite pleasure".²⁷ There is no extant document to prove that Gauss believed in the consistency of BL geometry as far back as 1792. There is a letter to Farkas Bolyai, dated December 16, 1799, in which Gauss declares that he has come to doubt the truth of geometry and that, although he knows of several apparently obvious premises from which Postulate 5 readily follows,

he is not willing to take any of them for granted.²⁸ Emphatic statements on the matter are made by him only much later, e.g. in his letter to Gerling of April 11, 1816, where he writes that he finds “nothing absurd” in the consequences of denying Postulate 5, such as that no incongruent figures can be similar to each other or that the size of the angles of an equilateral triangle must vary with the size of the sides. He adds that the existence of an absolute unit of length may seem somewhat paradoxical, but that he fails to find anything contradictory about it and that it even seems desirable.²⁹ On April 28, 1817, he writes to Olbers: “I am ever more convinced that the necessity of our geometry cannot be proved, at least not *by*, and not *for*, our HUMAN understanding. Maybe in another life we shall attain insights into the essence of space which are now beyond our reach. Until then we should class geometry not with arithmetic, which stands purely a priori, but, say, with mechanics”.³⁰ On March 16, 1819, after receiving Schweikart’s memorandum, he wrote to Gerling: “I have myself developed astral geometry to the point where I can solve all its problems completely if the [characteristic] constant C is given”.³¹ A very clear and eloquent statement of Gauss’ views is given in a letter of November 8, 1824, to Taurinus, where he bids him to keep them to himself.³²

2.1.6 Some Results of Bolyai–Lobachevsky Geometry

The founders of BL geometry took all the explicit and implicit assumptions of Euclid for granted, except Postulate 5. In Section 2.1.7, we shall have something to say about the epistemological significance of this attitude, but for the time being we, too, shall assume it when sketching a proof of some of their results. Other results, we shall quote without proof.

Let P be a point and m a straight line not through P (Fig. 7). Line m has two senses or directions which we agree to call the *plus* direction and the *minus* direction. Consider the flat pencil of straight lines through P on the same plane as m . Let us call it (P, m) . One and only one line in (P, m) is perpendicular to m ; call it t and let it meet m at Q . t has two sides which we shall label as the plus side and the minus side according to the following rule: if we change sides by moving along m in the plus direction, then we go over from the minus side of t to the plus side. The remaining lines of (P, m) belong to two sets: the set of all lines making an interior angle – i.e. an angle toward

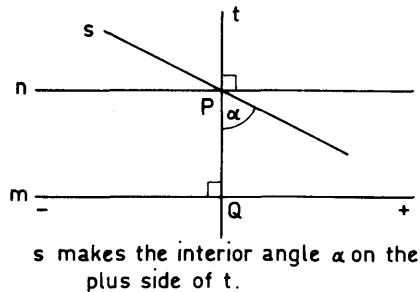


Fig. 7.

m – on the plus side of t equal to or less than one right angle (the plus set) and the set of all lines making such an angle on the minus side of t (the minus set). There is one and only one line common to both sets, namely the perpendicular to t ; let us call it n . Henceforth, we shall consider only the plus set; whatever we learn about it applies by symmetry, *mutatis mutandis*, to the minus set. For every point of m on the plus side of t there is a line of the plus set meeting m at that point. There is at least one line of the plus set which does not meet m , for n is such a line. This justifies the following definition: A line s of pencil (P, m) , making an interior angle σ on the plus side of t , is *the parallel to m in the plus direction* if and only if s does not meet m but m meets every line in (P, m) which makes an interior angle smaller than σ on the plus side of t .

Definitions of parallelism essentially identical to this were adopted independently by Gauss, Bolyai and Lobachevsky.³³ The reason for abandoning Euclid's definition is this: If Postulate 5 is true the new and the old definitions are equivalent; if Postulate 5 is false, there are two kinds of lines in (P, m) which do not meet m , namely the two parallels, one in each direction, and an infinite set of lines between them. These lines, called hyperparallels by some, have important properties not shared by the two parallels; e.g. each of them has a perpendicular which is also normal to m . Euclid's definition, however, makes no distinction between these two kinds of lines, for according to it all of them are 'parallels'.

It is clear that there is one and only one line through P which is parallel to m in the plus direction. It can be shown that, if s is that line, and P' is any point on s , then s is the one line through P' that is

parallel to m in the plus direction. In other words, parallelism in the plus direction is not relative to a particular point or pencil of lines. Moreover parallelism in the plus direction is a symmetric and transitive relation. The parallel to m in the plus direction makes at point P at a distance PQ from m an interior angle σ on the plus side of the perpendicular t . We call σ the *angle of parallelism of segment PQ* , for its size depends only on the length of this segment. In particular, it is equal to the interior angle made at P on the minus side of t by the parallel to m in the minus direction.

The last two statements are easily proved. Let P, Q, m, s and σ be as above; suppose s' is a straight line through a point P' , parallel in some direction κ to a line m' ; let the perpendicular to m' through P' meet m' at Q' , with $P'Q' = PQ$; s' makes on the κ side of $P'Q'$ an interior angle σ' . If $\sigma' > \sigma$, there is a straight line through P' making on the κ side of $P'Q'$ an internal angle equal to σ and meeting m' at H' . There exists then a triangle PQH congruent to triangle $P'Q'H'$, such that PH lies on s and QH lies on m ; but then s meets m at H and is not parallel to m , contrary to our assumption. A similar contradiction follows if $\sigma' < \sigma$. Therefore, if $P'Q' = PQ$, $\sigma' = \sigma$. The last statement preceding this paragraph follows immediately if we make $P' = P$ and $m' = m$ and let κ be the minus direction.

According to our definitions, the angle of parallelism of PQ may be equal to or less than $\pi/2$. If it is equal to $\pi/2$, the parallel to m in the plus direction is identical to the parallel to m in the minus direction: it is line n , the perpendicular to PQ through P . It can be proved that if the angle of parallelism of any given segment equals $\pi/2$ then the angle of parallelism of every segment has the same value. In that case, through each point P outside a line m there is one and only one parallel to m , the same in both directions. This is equivalent to Postulate 5. Conversely, if the angle of parallelism of any segment is less than $\pi/2$, it is less than $\pi/2$ for every segment. Consequently, all parallel lines in a given direction converge asymptotically and no two parallel lines have a common perpendicular. From this last statement it follows that if any angle of parallelism is less than $\pi/2$, the angle of parallelism of a segment of length x decreases as x increases. Following Lobachevsky, we shall hereafter designate by $\Pi(x)$ the angle of parallelism of a segment of length x . We regard Π as a real-valued function on lengths. Lobachevsky proved that unless Postulate 5 is true, Π is a monotonically decreasing continuous

function that takes all values between $\pi/2$ and 0 as x goes from 0 to ∞ .³⁴ Of course, if Postulate 5 is true, $\Pi(x) = \text{constant} = \pi/2$. In the discussion to follow, we shall disregard this case.

We shall not prove Lobachevsky's full statement but shall show that, if $\Pi(x) < \pi/2$, $0 < x < x'$ implies that $\Pi(x) > \Pi(x')$. Consider a point Q on a line m and a line PQ perpendicular to m . Let $|PQ| = x$. Produce PQ beyond P to P' and let $|P'Q| = x'$. Let s and s' be the parallels to m in a given direction κ that go, respectively, through P and P'. Since s and s' are parallel to one another, they cannot have a common perpendicular. Consequently, $\Pi(x) \neq \Pi(x')$. (*Proof*: Let H be the midpoint of PP'; the perpendicular to s through H meets s at R, s' at R'; if $\Pi(x) = \Pi(x')$, triangle PHR is congruent with triangle P'HR' and RR' is also perpendicular to s' .) If $\Pi(x) < \Pi(x')$ there is a line through P' that makes an interior angle equal to $\Pi(x)$ on side κ of PQ and meets m at a point G. P'G and s have a common perpendicular (*Proof*: By the above construction) and P'G meets s between P' and G. But this is impossible. Consequently, $\Pi(x) > \Pi(x')$. Q.E.D.

Π is an injective or 'one-one' mapping of the positive real numbers onto the open interval $(0, \pi/2)$. The inverse mapping Π^{-1} is therefore defined on $(0, \pi/2)$ and enables us to express length in terms of angular measure. This is the surprising feature of BL geometry which even Gauss and Lobachevsky judged paradoxical.³⁵ Angular measure is absolute and involves a natural unit; while length, we are wont to believe, is essentially relative, so that a sudden duplication of all distances in the universe would make no difference whatsoever to the geometrical aspect of things. This is not true of a BL world as the following example will show. Let C be a positive real number such that $\Pi(C) = \pi/4$. Take a point Q on a straight line m and let C be the length of a segment PQ that meets m orthogonally at Q. Each of the parallels to m through P makes an interior angle equal to $\pi/4$ on the corresponding side of PQ (Fig. 8). Consequently, the two parallels are

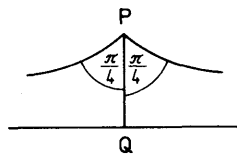


Fig. 8.

orthogonal and m is parallel to two mutually perpendicular straight lines. C is therefore the distance – the same everywhere in BL space – between the vertex of a right angle and the line parallel to its two sides.³⁶ Now, if we can ascribe a precise physical denotation to the terms ‘right angle’ and ‘straight line’, and if, under this ascription of meanings, BL geometry happens to be true of the physical world, C will be a physical distance, say, so many times the average distance from the sun to the earth. Under such conditions, the duplication of all distances would make a difference in the geometrical aspect of things, unless it were accompanied by an appropriate change in the function Π .

One of the main problems of BL geometry is the analytical determination of Π . Its elegant solution by Lobachevsky was, no doubt, a powerful inducement to accept BL geometry as a respectable branch of mathematics. Π involves an arbitrary constant. We can make this constant equal to 1 by agreeing that $C = \Pi^{-1}(\pi/4) = \ln(1 + \sqrt{2})$. Then $\Pi(x) = 2 \operatorname{arc} \cot e^x$ and $x = \ln \cot(\frac{1}{2}\Pi(x))$. These results are used in the derivation of the formulae of BL trigonometry. As anticipated by Lambert and confirmed by Taurinus, the latter are identical to the formulae of spherical trigonometry, with an imaginary number substituted for radius r (pp.66f.). Another theorem of BL geometry which deserves mention was also anticipated by Lambert. In BL geometry the three angles of a triangle equal $\pi - \delta$, where δ , the ‘defect’ of the triangle, is a positive real number. It is easily shown that if a triangle A is partitioned into triangles B_1, \dots, B_n , the defect of A is equal to the sum of the defects of B_1, \dots, B_n . This implies that the defect of a triangle increases with its area. It can be proved that the area is strictly proportional to the defect. Since the defect cannot exceed π , the area of a BL triangle has an upper bound equal to $k\pi$, where k is the coefficient of proportionality.

We may view C as the characteristic constant of BL geometry. It is quite obvious that if we let C increase beyond all bounds, BL geometry approaches Euclidean geometry as a limit. This connection between the two geometries can be shown also from another, most instructive perspective. Let m be a straight line, κ one of the directions of m , m' a line parallel to m in direction κ . Given a point Q on m , there is exactly one point Q' on m' such that the perpendicular bisector of QQ' is parallel to m (and hence to m') in direction κ . We shall say that Q and Q' are *corresponding* points. Correspondence is a

symmetric and transitive relation. Consider now the set \mathcal{M} of all lines in space that are parallel to m in direction κ . We call \mathcal{M} (including m) a family of parallels in space. The locus of the points corresponding to Q on each of these lines is a smooth surface \mathcal{H} which we call a *horosphere*. We say that m is an *axis* of \mathcal{H} . The name *horosphere* is meant to remind us of the following: if horosphere \mathcal{H} cuts axis m at Q and if P is a point of m on the concave side of \mathcal{H} , a sphere with centre P and radius PQ touches \mathcal{H} but does not intersect it; as PQ increases beyond all bounds, the sphere indefinitely approaches \mathcal{H} – in other words, a horosphere is the limit (*horos*) towards which a sphere tends as its radius tends towards infinity. Let us now consider any line m' belonging to set \mathcal{M} , i.e. any line parallel to m in direction κ ; if m' intersects \mathcal{H} at Q' , \mathcal{H} is clearly the locus of the points corresponding to Q' on every line of \mathcal{M} . Each line of \mathcal{M} is therefore an axis of \mathcal{H} . Every axis of \mathcal{H} is normal to \mathcal{H} . All horospheres are congruent. A plane ζ through an axis of a horosphere \mathcal{H} intersects \mathcal{H} on a curve h which we call a *horocycle*; h is clearly a locus of corresponding points on the parallels to the given axis that lie on plane ζ . On the other hand, if no axis of \mathcal{H} lies on ζ and ζ meets \mathcal{H} , their intersection is a circle. A horocycle is an open curve; a horocycle lying on a horosphere divides it into two parts; all horocycles are congruent; moreover, two arcs of horocycle are congruent if they subtend equal chords. Horocycles, as we see, share some of the properties of straight lines. Consider two mutually intersecting horocycles h, k on a horosphere \mathcal{H} , determined by two mutually intersecting planes ζ, η . Let w be the intersection of ζ and η . We agree to measure the curvilinear angle formed by h and k by the rectilinear angle made by two straight lines lying on ζ and η , respectively, and meeting w perpendicularly at the same point. Now let a, b, c be horocycles on a horosphere \mathcal{H} such that c cuts a and b making interior angles on the same side of c less than two right angles; then, as both Bolyai and Lobachevsky proved, a and b meet. Bolyai concluded without more ado: “From this it is evident that Euclid’s Axiom XI [i.e. Postulate 5] and all things which are claimed in geometry and trigonometry hold good absolutely in [horosphere \mathcal{H}], L-form lines [i.e. horocycles] being substituted in place of straights”.³⁷ Lobachevsky shows this in detail. He proves in particular that the three interior angles of a horospherical triangle – i.e. a figure limited by three horocycles on a horosphere – are equal to π and that

figures traced on a horosphere may be similar without being congruent.³⁸ We saw above that a horosphere is, so to speak, a sphere with infinite radius. It is well known that in Euclidean geometry a sphere with infinite radius is a plane. But of course, if Postulate 5 is true, a horosphere, i.e. a surface normal to a family of parallels in space, is a plane. No wonder then that, under Postulate 5, horospherical geometry should be identical with plane geometry. Now, if Postulate 5 is true, the radii of the sphere which, in the finite case, jointly converge to a point, tend to become, as they grow beyond all bounds, a family of Euclidean parallels, i.e. a set of non-convergent lines running equidistantly along each other. On the other hand, if Postulate 5 is false, the radii become at infinity a family of BL parallels, so that their convergence persists, though it is now asymptotic. This led Lobachevsky to remark that the transition to infinity is carried out 'better' in BL geometry than in Euclidean geometry. He appears to have thought that such smoothness of transition, occurring even where continuity is broken, was typical of the general or normal case, while the abruptness exemplified by Euclidean geometry was distinctive of a singular, unnatural case, laden with arbitrary, artificial assumptions (the degenerate case where horospheres collapse into planes). Mathematicians have since learned not to place too much trust in such intuitive considerations. But Lobachevsky's remark can still teach something to philosophers who believe that 'intuition' unconditionally favours Euclidean geometry.

2.1.7 The Philosophical Outlook of the Founders of Non-Euclidean Geometry

Euclidean and BL geometry are often described by philosophical writers as two abstract axiomatic theories which agree in all their axioms except one, Postulate 5, which is asserted by Euclidean geometry and denied by BL geometry. Both are equally consistent and hence equally admissible from a logical point of view. The question of their truth in a 'real', 'material' or transcendent sense, cannot be decided within the theories, i.e. by an examination and comparison of their contents. Although the advent of BL geometry eventually contributed to the development and popularization of the formalist philosophy of mathematics leading to the above description, its creators never viewed it in that way. Although that philosophy had been anticipated, up to a certain point, by Lambert and other 18th-

century writers, it did not guide the efforts of Gauss or Lobachevsky toward the formulation of a new system of geometry. Abstract axiomatic theories consist of the logical consequences of arbitrarily posited unproved premises – the axioms – containing diversely interpretable undefined terms – the primitives. The axioms are all equally groundless, and any one of them may be negated – unless it happens to be a consequence of the others – to obtain a different theory. The primitives are semantically neutral, the spectrum of their admissible meanings being restricted only by the net of mutual relations into which they are knit by the axioms. But the founders of BL geometry had little use for such equality and neutrality. They made a neat distinction between Postulate 5, on which they suspended judgment, and the remaining assumptions of geometry, whose truth they never questioned, and which they regarded as the unshakeable basis of what Bolyai called “the absolutely true science of space”. We might even say that, had they succeeded in their youthful attempts to derive Postulate 5 from those other assumptions, they would have sat content in the belief that the old Euclidean *Elements*, their flaw removed, furnished that definite and final knowledge of the laws of space upon which alone a fruitful and reliable physical science could be built. After their attempts failed and they became persuaded that such failure was inevitable, Lobachevsky and Bolyai directed part of their efforts towards absorbing the old geometry within a more general system, that was well defined only up to a constant. This reveals a strong penchant to preserve the unity of geometry, which may help explain why they did not pay any attention to the comfortable notion of semantic neutrality. The new ideas, indeed, would have been readily accepted, had their proponents acknowledged that the basic terms need not mean the same in the new geometry as in the old; that BL straights, for example, might after all really not be straight, at least not in the ordinary sense of the word. This admission would certainly have assuaged the antipathy aroused by the seemingly incomprehensible asymptotic convergence of BL parallels; if they were not genuinely straight, and hence not really parallel, nobody would have disputed the possibility that they converge asymptotically. But that wilful cossack, Lobachevsky, would have none of it. On the one occasion when he tried to set forth all his assumptions clearly – at the beginning of his German booklet on the theory of parallels – the first three and the fifth were devoted to

expressing the essential properties of the straight line, and were designed to leave no doubt as to the straightness of Lobachevsky's straights.

(1) A straight line fits upon itself in all its positions. By this I mean that during the revolution of the surface containing it the straight line does not change its place if it goes through two unmoving points in the surface.

(2) Two straight lines cannot intersect in two points.

(3) A straight line sufficiently produced both ways must go out beyond all bounds, and in such way cuts a bounded plane into two parts.

(5) A straight line always cuts another in going from one side of it over to the other side.³⁹

Horocycles satisfy the last three statements, but not the first; this suffices to show where lies the truth about genuine straight lines if Postulate 5 is false. The decision to use the word 'straight' in the sense set by the four postulates above is of course conventional, and if it turns out that in the world there are no straight lines in this sense, we will probably give up some of those postulates rather than do without such a familiar word. (This is indeed what pilots do when they speak of flying from Sydney straight to Vancouver.) But the point here is that Lobachevsky's usage does not constitute a departure from the established conventions governing this word, conventions which, as Proclus and most probably Euclid himself knew, do not require that only such lines as fulfil Postulate 5 be called straight – quite the contrary: it may very well happen that the lines fulfilling this postulate are horocycles, which, by those conventions, ought not to be called straight.

The indeterminateness of the characteristic constant of BL geometry prompted attempts at determining it experimentally, on the analogy of other physical constants familiar to 19th-century scientists. The successful application of Euclidean geometry in science and in everyday life indicated that the constant, if finite, must be very large. Schweikart wrote that if the constant was equal to the radius of the earth it would be practically infinite in comparison to the magnitudes we deal with in everyday life (thus vindicating the use of Euclidean geometry for ordinary purposes). Gauss commented that "in the light of our astronomical experience, the constant must be enormously larger (*unermesslich grösser*) than the radius of the earth".⁴⁰ Lobachevsky tried to evaluate the constant by using astronomical data. Rather than look for an actual example of a line parallel to two

perpendicular straight lines and measure its distance to their intersection, he set out to calculate the constant indirectly, by measuring the defect of a very large triangle. He found that the defect of the triangle formed by Sirius, Rigel and Star No.29 of Eridanus was equal to 3.727×10^{-6} seconds of arc, a magnitude too small to be significant given the range of observational error. He concluded that "astronomical observations persuade us that all lines subject to our measurements, even the distances between heavenly bodies, are too small in comparison with the line which plays the role of a unit in our theory, so that the usual equations of plane trigonometry must still be viewed as correct, having no noticeable error".⁴¹

There is another side to Lobachevsky's empiricism which ought to be mentioned here. He believed that the basic concepts of any science – which, he said, should be clear and very few in number – are acquired through our senses.⁴² Geometry is built upon the concepts of *body* and *bodily contact*, the latter being the only 'property' common to all bodies that we ought to call geometrical. Lobachevsky arrives at the familiar concepts of depthless surface, widthless line and dimensionless point by considering different possible forms of bodily contact and ignoring, *per abstractionem*, everything except the contact itself. But then these "surfaces, lines and points, as defined in geometry, exist only in our representation; whereas we actually measure surfaces and lines by means of bodies".⁴³ For "in nature there are neither straight nor curved lines, neither plane nor curved surfaces; we find in it only bodies, so that all the rest is created by our imagination and exists just in the realm of theory".⁴⁴ "In fact we know nothing in nature but movement, without which sense impressions are impossible. Consequently all other concepts, e.g. geometrical concepts, are generated artificially by our understanding, which derives them from the properties of movement; this is why space in itself and by itself does not exist for us."⁴⁵ This leads Lobachevsky to a most remarkable piece of speculation: since our geometry is not based on a perception of space, but constructs a concept of space from an experience of bodily movement produced by physical forces, why could there not be a place in science for two or more geometries, governing different kinds of natural forces?

To explain this idea, we assume that [...] attractive forces decrease because their effect is diffused upon a spherical surface. In ordinary geometry the area of a spherical surface of radius r is equal to $4\pi r^2$, so that the force must be inversely proportional to

the square of the distance. I have found that in imaginary geometry the surface of a sphere is equal to $\pi(e^r - e^{-r})^2$; such a geometry could possibly govern molecular forces, whose variations would then entirely depend on the very large number e .⁴⁵

This is, of course, a mere supposition and stands in need of better proof, "but this much at least is certain: that forces, and forces alone, generate everything: movement, velocity, time, mass, even distances and angles".⁴⁶ Here we see Kant's youthful fantasy of 1746 (p.29) making a new, much bolder, appearance. If forces, as Kant surmised, determine (physical) geometry, we cannot expect the same geometry to be everywhere applicable, for geometry must reflect the behaviour of the forces prevailing at each level of reality.

There is one further question we must examine before leaving the subject of the theory of parallels. What made Gauss, Bolyai and Lobachevsky so certain that BL geometry contained no inconsistency? The fact that no contradiction had been inferred despite their efforts did not prove that none could ever arise. Lack of familiarity with formalized deduction and deductive systems may have kept these eminent mathematicians unaware of the pitfalls concealed even in full-fledged axiomatic theories, whose assumptions have been made wholly explicit. BL geometry was a fairly complex system, where seemingly disparate lines of reasoning led to surprisingly harmonious conclusions – a trait that normally inspires trust and arouses zeal in mathematicians. Lobachevsky had also a more specific reason for believing in the consistency of BL geometry, one that may also have been known to Gauss and Bolyai, for they were familiar with the mathematical fact on which it is based. In the conclusion to his first published paper on the subject, Lobachevsky points out that after deriving a set of equations labelled (17), which express the mutual dependence of the sides and the angles of a BL triangle, he gave general formulae for the elements of distance, area and volume,

so that, from now on, everything else in geometry will be analysis, wherein calculations will necessarily agree and where nothing will be able to disclose to us something new, i.e. something that was not contained in those first equations from which all relations between geometrical magnitudes must be derived. Therefore, if somebody unflinchingly maintains that a subsequently emerging contradiction will force us to reject the principles we have assumed in this new geometry, such a contradiction must already be contained in equations (17). Let us observe however that these equations become equations (16) of spherical trigonometry as soon as we substitute $a\sqrt{-1}$, $b\sqrt{-1}$, $c\sqrt{-1}$ for sides a , b , c . But in ordinary geometry and in spherical trigonometry we only encounter relations between lines; consequently, ordinary geometry, trigonometry and this new geometry will always stand in mutual agreement.⁴⁶

In other words, the new geometry is at least as consistent as the old. This argument for the relative consistency of BL geometry does not involve the construction of a so-called Euclidean model of it, i.e. it does not require us to understand its terms in some unnatural, originally unintended sense – say, ‘straight lines’ as half-circles, ‘planes’ as some kind of curved surfaces, etc. For the argument depends upon a purely formal agreement between two sets of equations, one of which is derived within BL geometry. Moreover, the other set of equations does not belong specifically to Euclidean geometry, any more than, say, the laws of arithmetic belong to it. They are the basic equations of spherical trigonometry, which, as Lobachevsky (and Bolyai) made a point of showing, does not depend on Postulate 5. The geometry of great circles upon a sphere is certainly true within Euclidean geometry, but it is equally true in BL geometry, for it is part of that *scientia spatii absolute vera* that is built upon the assumptions common to both geometries. Indeed, it seems to me quite typical of Lobachevsky’s posture that, when he needed a formal argument to uphold the viability of his new geometry before the partisans of the exclusive validity of the old, he should have looked for it precisely in that part of geometry which was acceptable to both sides. On the other hand, although our idea of a model was not wholly foreign to him, he does not appear to have thought that one could make BL geometry respectable by providing it with a Euclidean model. His use of models aims, so to speak, in the opposite direction, namely, at making Euclidean plane geometry plausible and its early discovery and continued predominance understandable, by showing how it is realized within the new geometry, although only as a particular, extreme degenerate case.

*Formulae (17) of BL trigonometry, referred to above, are as follows. In a triangle with sides a , b , c , opposite to angles A , B , C , respectively:

$$\begin{aligned} \tan \Pi(a) \sin A &= \tan \Pi(b) \sin B, \\ \cos A \cos \Pi(b) \cos \Pi(c) &= 1 - \frac{\sin \Pi(b) \sin \Pi(c)}{\sin \Pi(a)}, \\ \cot A \sin B \sin \Pi(c) &= \cos B - \frac{\cos \Pi(c)}{\cos \Pi(a)}, \\ \cos C + \cos A \cos B &= \frac{\sin A \sin B}{\sin \Pi(c)}. \end{aligned}$$

The corresponding formulae of spherical trigonometry, equations (16) in Lobachevsky's book, are the following:

$$\begin{aligned}\sin a \sin B &= \sin b \sin A, \\ \cos A \sin b \sin c &= \cos A - \cos b \cos c, \\ \cot A \sin C &= \cos C \cos b - \cot a \sin b, \\ \cos a \sin B \sin C &= \cos A - \cos B \cos C.\end{aligned}$$

The substitutions prescribed by Lobachevsky yield the expected result because (writing i for $\sqrt{(-1)}$)

$$\begin{aligned}\sin \Pi(ia) &= \frac{1}{\cosh ia} = \frac{1}{\cos a}, \\ \cos \Pi(ia) &= \tanh ia = i \tan a, \\ \tan \Pi(ia) &= \frac{1}{\sinh ia} = \frac{1}{i \sin a}.\end{aligned}$$

2.2 MANIFOLDS

2.2.1 Introduction

By 1840, a full statement of Lobachevsky's theory had been made available in French and in German. Contrary to Gauss' expectation, no uproar was heard. Most mathematicians ignored the extravagant Russian, but some took a deep interest in the new geometry. Postulate 5 had long been sensed by many as a mildly painful thorn in the "supremely beautiful body of geometry" (to borrow Henry Savile's words).¹ We thus find Bessel, in his reply to the letter where Gauss expressed his fear of Boeotians, not unresponsive to the new ideas.

What Lambert has written and what Schweikart said have made it clear to me that our geometry is incomplete and should be given a hypothetical correction, which vanishes if the sum of the angles of a plane triangle equals 180 degrees. That would be the *true* geometry, while Euclidean geometry would be the *practical* one, at least for figures on the earth.²

A lively mathematical and philosophical discussion of the new geometrical conceptions did not begin until the 1860's however, when the fact that Gauss had been recommending them became generally known through the publication of his correspondence with Schumacher. Interest in non-Euclidean geometry was on the rise when Riemann's lecture of 1854 "On the Hypotheses which Lie at the Basis of Geometry" was finally printed in 1867. This work marks the

beginning of the modern philosophy of geometry and is the source of some of its most characteristic ideas. We must therefore analyze it with some care. To this end, we shall examine first some innovations due to Gauss which led up to it.

In the next three sections, we shall speak about smooth curves and surfaces in space which have no cusps or other singularities. We may restrict our treatment to them because we shall be concerned with general local properties of curves and surfaces, i.e. properties true of a neighbourhood of an average point on them. We take space to be the infinite three-dimensional continuum of classical geometry, in which all Euclid's theorems are valid.

2.2.2 Curves and their Curvature

The theory of plane curves, initiated in a piecemeal fashion by the Greeks, was developed in the 17th and 18th centuries with the full generality allowed by the newly-introduced method of coordinate geometry (Section 1.0.4). The study of direction, the most conspicuous local property of a curve, played a major role in the discovery of the calculus. The resources of this new mathematical discipline were used for defining the length of a curve and for conceiving in an exact quantitative fashion another important local property of plane curves, namely curvature, which we may intuitively describe as the degree to which a curve is bent at each point. In the 18th century, the new methods were applied to the study of curves in space and, eventually, to the study of surfaces. Following a pattern quite familiar in mathematics, the concept of curvature, which had originally been defined for plane curves on strongly intuitive grounds, was extended analogically to space curves and surfaces, losing in this process most of its intuitive feel.

It will be useful to introduce some technical terms. A *path* in space is a mapping c of an interval of \mathbb{R} into space, such that, for any Cartesian mapping x , the composite mapping $x \cdot c$ is everywhere differentiable. We shall usually consider *injective* paths c , such that, for any Cartesian mapping x , $x \cdot c$ possesses everywhere derivatives of all orders.³ The range of such a path always corresponds to our intuitive idea of a smooth curve; on the other hand, curves that are smooth in the intuitive sense, but which are not the range of any such path – e.g. closed curves, or curves with double points such as the figure 8 – can always be viewed as the union of the possibly overlapping ranges of several paths of the kind

described. A single curve K can be the range of many paths, defined on the same or on different intervals of R . If K is the range of two paths c and \hat{c} , related by the equation $c = \hat{c} \cdot \gamma$, γ is said to reparametrize the curve K , 'parameter' being a term traditionally used to designate the 'variable' argument of a path. c and \hat{c} are two 'parametrical representations' of K . Let c be a path defined on a closed interval $[a, b]$, and let x be a Cartesian mapping. We define the length of the curve $c([a, b])$ by

$$\lambda(a, b) = \int_a^b \left| \frac{d}{du} (x \cdot c(u)) \right| du. \quad (1)$$

The integrand is, of course, the limit approached by the length of a chord drawn from $c(u)$ to a neighbouring point $c(u+h)$ as h tends to zero. The length $\lambda(a, b)$ is equal therefore to the limit of the sequence (λ_i) of the lengths of any sequence (p_i) of polygonal lines inscribed in $c([a, b])$ between $c(a)$ and $c(b)$, whose sides grow shorter than any arbitrary segment as i increases beyond all bounds. Under the conditions imposed on $x \cdot c$, this limit can be shown to exist. Since the integrand is invariant under coordinate transformations and a reparametrization is equivalent to a substitution of variables, the above integral has a fixed value for a given curve, no matter how we choose the mapping $x \cdot c$ that represents it. The length of the arc joining $c(a)$ to an arbitrary point $c(u)$ in $c([a, b])$ is given by

$$\lambda(u) = \int_a^u \left| \frac{d}{du} (x \cdot c(u)) \right| du. \quad (2)$$

The function $u \mapsto \lambda(u)$ reparametrizes the curve $c([a, b])$. Let $c = \hat{c} \cdot \lambda$. Path \hat{c} is said to represent our curve as 'parametrized by arc length'. The parameter, in this case, is usually denoted by s . Obviously, $|d(x \cdot \hat{c}(s))/ds| = 1$. Hence

$$\lambda(s) = \int_0^s du = s, \quad (3)$$

as it ought to be.

Let c be an injective path defined on $[0, k]$, with arc length as

parameter; let x be a Cartesian mapping. We assume that the range of c lies entirely on a given plane, i.e. that it is a plane curve. As s takes all the values between 0 and k , the derivative $d(x \cdot c)/ds$ takes its values in \mathbb{R}^3 . Therefore, $x^{-1}(d(x \cdot c)/ds)$ is a point in space. The mapping $c' = x^{-1}(d(x \cdot c)/ds)$ is a path, though not necessarily an injective one. Since $c([0, k])$ is parametrized by arc length, the range of c' lies entirely on a circle of unit radius. If X is the origin of the mapping x and $P = c'(s)$, the directed segment XP is parallel to, and has the same sense as, the tangent to $c([0, k])$ at $c(s)$. For this reason, we call c' , $c'(s)$ and $c'([0, k])$ the *tangential images* of c , $c(s)$ and $c([0, k])$, respectively. We will illustrate the significance of the tangential image of a curve with the aid of a story. Suppose H is an object moving at constant speed along the curve $c([0, k])$, passing through point $c(s)$ at time s . Let H' move simultaneously on the range of c' , so that at time s , H' is at $c'(s)$. H' , of course, need not move at a constant speed; indeed, at times it may not move at all. But its movements depend at every moment on the simultaneous movements of H , through the equation relating c' to c . As the direction in which H moves changes, the position of H' changes; if the direction of H changes faster, H' moves faster. In other words, the speed of H' at time s measures the degree to which curve $c([0, k])$ is bent at the point $c(s)$. Consequently, the speed with which H' moves along the range of c' measures what we may reasonably call the *curvature* of $c([0, k])$ at each of its points. The speed of H' is given by $|d(x \cdot c')/ds|$. But this is equal to $|d^2(x \cdot c)/ds^2|$. The value of this derivative at s does not depend on the mapping x . We take it as a measure of the curvature or local 'bendedness' of our curve $c([0, k])$.

In the above discussion, the restriction to plane curves plays no role, except that of motivating our choice of the name 'curvature' for the property measured by $|d^2(x \cdot c)/ds^2|$. We may lift the restriction and preserve the name, as 18th-century mathematicians did. The range of c' is then no longer confined to a circle, but lies on a *sphere* of unit radius. We define therefore the curvature κ at a point $c(s)$ of a curve $c([0, k])$ parameterized by arc length, by

$$\kappa(s) = |d^2(x \cdot c)/ds^2|_s, \quad (4)$$

(where x is any Cartesian mapping). This definition immediately suggests a concept of *total curvature* κ_T , measuring the total change of direction of curve $c([0, k])$ as we go from point $c(s_1)$ to point $c(s_2)$;

namely

$$\kappa_T(s_1, s_2) = \int_{s_1}^{s_2} \kappa(s) ds. \quad (5)$$

But $\kappa(s)$ is equal to $|d(x \cdot c')/ds|_s|$, so that

$$\kappa_T(s_1, s_2) = \int_{s_1}^{s_2} |d(x \cdot c')/ds|_s| ds. \quad (6)$$

The total curvature of $c([s_1, s_2])$ is therefore equal to the *length* of the tangential image $c'([s_1, s_2])$.

We have defined the curvature of a curve at a point as the magnitude of an element of \mathbf{R}^3 , i.e. as a non-negative real number. In the case of plane curves, it is possible to define a signed curvature. Mathematicians have not failed to use this possibility in order to convey, through the value of the curvature, one more item of information about the curve. Orientation conventions establish a *positive* and a *negative* sense of rotation about a point in the plane. The signed curvature $\bar{\kappa}(s)$ at the point $c(s)$ is equal to $\kappa(s)$ if the tangent to the curve at $c(s)$ is constant or rotates about $c(s)$ in a positive sense; $\bar{\kappa}(s) = -\kappa(s)$ if the tangent at $c(s)$ rotates about this point in a negative sense.

2.2.3 Gaussian Curvature of Surfaces

As we did with smooth curves, we shall make our notion of a smooth surface more precise by imposing certain conditions on the admissible analytical representations of such surfaces. Let y^i and z^i denote the i th projection functions on \mathbf{R}^2 and \mathbf{R}^3 , respectively.⁴ Let ζ be a connected, open or closed region of \mathbf{R}^2 and $f: \zeta \rightarrow \mathbf{R}^3$ a differentiable function such that the matrix $[\partial(z^i \cdot f)/\partial y^j]$ ($i = 1, 2, 3; j = 1, 2$) everywhere has rank 2. If x is any Cartesian mapping, $x^{-1} \cdot f = \Phi$ maps ζ into space. If Φ is injective and, for any Cartesian mapping x , $x \cdot \Phi$ everywhere possesses partial derivatives of all orders, $\Phi(\zeta)$ is what we would normally call a smooth surface. Although not every smooth surface can be wholly represented as the range of some injective mapping of this kind, any point on such a surface has a neighbourhood which is thus representable. The full surface can then be pieced together from

the ranges of several such mappings. Our discussion will be restricted to surfaces or pieces of surfaces which are, in each case, the range of a mapping Φ defined on an open region $\zeta \subset \mathbb{R}^2$ and subject to the stated conditions. Results obtained under this restriction are not always true of a surface composed of several such pieces. Our discussion pertains therefore to the *local* geometry of surfaces and not to their *global* geometry.

It can be shown that, if $S = \Phi(\zeta)$ is a smooth surface and $P = \Phi(a, b)$ is a point on it, there exists a plane S_P which contains the tangents at P to all curves on S through that point. S_P can be naturally viewed as a 2-dimensional vector space with P for its zero vector. It is then called the *tangent plane* of S at P . If π is a plane through P normal to S_P , the intersection of π and S is a plane curve called a *normal section* of S at P . Every normal section of S at P possesses a signed curvature at P . The set of these curvatures is a bounded set of real numbers. In 1760, Leonhard Euler (1707–1783) proved that this set has a maximum $\bar{\kappa}_{\max}$ and a minimum $\bar{\kappa}_{\min}$, and that $\bar{\kappa}_{\max}$ and $\bar{\kappa}_{\min}$ are the signed curvatures at P of *two mutually perpendicular* normal sections. The tangents of these curves at P are called the *principal directions* of surface S at P . By a mild abuse of language, $\bar{\kappa}_{\max}$ and $\bar{\kappa}_{\min}$ are called the *principal curvatures* of surface S at P . Euler proved also that if Σ is a normal section of S at P , whose tangent at P makes an angle φ with the principal direction associated with $\bar{\kappa}_{\max}$, the signed curvature $\bar{\kappa}$ of Σ is given by

$$\bar{\kappa} = \bar{\kappa}_{\max} \cos^2 \varphi + \bar{\kappa}_{\min} \sin^2 \varphi. \quad (1)$$

Consider now a Cartesian mapping x with origin O . The sphere with centre O and unit radius has a unique diameter Q_1Q_2 normal to S_P . Orientation conventions enable us to choose one of the points Q_i ($i = 1, 2$) as a unique representative of S_P .⁵ We call the chosen point the *normal image* of surface S at $P = \Phi(a, b)$, and denote it by $n(a, b)$. It is clear that $(u, v) \mapsto n(u, v)$ maps ζ onto a connected subset of the unit sphere with centre O . We call $n(\zeta)$ the *normal image* of surface S . Gauss defined the *total curvature* of a surface $S = \Phi(\zeta)$ as the *area* of its normal image $n(\zeta)$. This definition is a rather natural analogical extension of our definition of the total curvature of a curve as the *length* of its tangential image. We arrived at that definition by integrating (local) curvature, eqn. (6) of Section 2.2.2. Gauss proposed a concept of curvature of a surface at a point, which bears a similar relation to his concept of total curvature of a surface. He writes:

To each part of a curved surface enclosed within definite limits we assign a *total* or *integral curvature*, which is represented by the area of the figure on the sphere corresponding to it.⁶ From this integral curvature must be distinguished the somewhat more specific curvature which we shall call the *measure of curvature*. The latter refers to a *point* of the surface, and shall denote the quotient obtained when the integral curvature of the surface element about a point is divided by the area of the element itself; and hence it denotes the ratio of the infinitely small areas which correspond to one another on the curved surface and on the sphere.⁷

Gauss may seem to be defining the 'measure of curvature' k of $S = \Phi(\zeta)$ at $\Phi(a, b)$ by

$$k(a, b) = \lim_{\zeta \rightarrow (a, b)} \frac{\text{area of } n(\zeta)}{\text{area of } \Phi(\zeta)}. \quad (2)$$

This expression is meaningless unless we have a satisfactory definition of the area of a curved surface. We may grant that Gauss possessed such a definition *in pectore*, given that in §17 of the *Disquisitiones generales circa superficies curvas* he provided the basis for the classical theory of surface area. But even if we grant this, and assume all the conceptual refinements required to make that definition truly unimpeachable, it is not obvious that the above limit exists or that it is independent of the way how ζ contracts to (a, b) . But Gauss' text does not really speak of such a limit. It does not refer to a sequence of ratios of functionally related areas allegedly converging to a 'measure of curvature'. Gauss defines the 'measure of curvature' simply as the ratio between 'elements of surface' of $n(\zeta)$ and $\Phi(\zeta)$, respectively; that is, as the ratio of *integrands*, the integral of the first of which, taken over all ζ , is equal to the area of $n(\zeta)$, and the integral of the second of which, taken over all ζ , is equal to the area of $\Phi(\zeta)$. This may sound even more perplexing than our earlier interpretation, for it amounts to – *horribile dictu* – dividing one infinitesimally small quantity by another. But Gauss, trusting in his own instinct and in the intelligence of his successors, leaves it at that and immediately proceeds to contrive a method for calculating the said ratio.⁸ It is based on the following: the tangent plane S_p of S at $P = \Phi(a, b)$ is parallel to the plane tangent to the unit sphere at $Q = n(a, b)$, for the radius OQ is, by definition, perpendicular to S_p ; therefore, reasons Gauss, given a Cartesian mapping x , with frame (π_1, π_2, π_3) , the ratio of the perpendicular projections on, say, π_1 , of the 'elements of surface' of $n(\zeta)$ and $\Phi(\zeta)$ must be equal to the ratio

of the 'elements of surface' themselves. The calculation of the ratio of the said perpendicular projections is, for Gauss, an easy matter.

After obtaining a formula enabling the calculation of his 'measure of curvature', or G-curvature (G for Gaussian), as we shall henceforth call it, Gauss derives a series of beautiful theorems. One of them states that the G-curvature of a surface S at a point P is always equal to the product of the two principal curvatures of S at P . Since Euler's results (mentioned on p.72) are perspicuous and fairly easy to prove, nothing could be simpler than using this result of Gauss' as a definition of G-curvature, whereby we would avoid the difficulties of the original Gaussian definition. This procedure is followed by some authors. The untutored reader often fails to understand why this particular number deserves to be called the curvature of the surface. Why not take the average of, or the difference between the principal curvatures? Why care for the principal curvatures at all? They are nothing but the curvatures of certain plane curves. Why use them to characterize surfaces? Singling out G-curvature among the local features of surfaces is justified *ex post facto* by the stupendous fruitfulness of that concept. Yet, while Gauss' train of thought gives us reason enough to expect this (for he defines G-curvature as a natural extension to surfaces of an important concept of the theory of curves), when G-curvature is defined as $\bar{\kappa}_{\max}$ times $\bar{\kappa}_{\min}$ its remarkable properties appear as a piece of sheer good luck. There is of course a didactic tradition which prefers this way of doing mathematics, patterning it after the juggler's craft, not the poet's art.

A second theorem proved by Gauss has particular importance in connection with our main topic. Let A, B, C be three points on a surface $\Phi(\zeta)$, joined by arcs of shortest length.⁹ Let α, β, γ be the interior angles of the curvilinear triangle ABC formed by these arcs. The real number $(\alpha + \beta + \gamma - \pi)$ is called the *excess* of triangle ABC if it is positive, the *defect* if it is negative. Gauss proved that triangles formed by shortest arcs on surfaces of positive curvature always have an excess while those formed on a surface of negative curvature always have a defect, *the excess or defect being proportional to the area of the normal image of the triangle* (i.e. to its total curvature). This result, published in 1827, if read in the light of the contemporary discoveries by Bolyai, Lobachevsky and Gauss himself, ought to have suggested a surface of negative G-curvature as a model of plane BL geometry.¹⁰ It is all the more remarkable that such a model was not

discovered until forty years later, when it was proposed by Eugenio Beltrami (Section 2.3.7).

Three simple examples will illustrate the intuitive content of the concept of G-curvature. If $\Phi(\zeta)$ is a plane, $n(\zeta)$ is a point and G-curvature is therefore constant and equal to zero. If $\Phi(\zeta)$ is a sphere of radius r , $n(\zeta)$ is the full unit sphere; the area of $\Phi(\zeta)$ is then r^2 times that of $n(\zeta)$ and the G-curvature of $\Phi(\zeta)$ is constant and equal to $1/r^2$. These results seem quite reasonable. Let us now consider a thick roll of wallpaper with a pattern consisting of transverse stripes (parallel to the roll's axis). Let us see if we can determine the G-curvature at a point on the coiled paper surface. Along the edge of a stripe the tangent plane remains constant; so the normal image of the edge is a point. The same is true of any other transverse line on the paper, i.e. any line parallel to that edge. If a point moves on the paper otherwise than along a transverse line, it continuously goes from one transverse line to another and its normal image describes a line—a circular arc—on the unit sphere. Since the area of a line is zero, the G-curvature of the wallpaper surface is everywhere equal to zero. This will not change if we pack the paper more or less tightly or if we unroll it to paste it on a wall. Moreover, if the wall is smooth and the paper, when pasted on it, fits snugly without needing to be stretched or shrunk the curvature of the surface will remain constant and equal to zero no matter what the wall's shape. (The tangent plane may now change as we move horizontally, along the transverse stripes of the paper, but it will usually remain constant along the vertical lines; if the wall is so warped that its tangent plane varies simultaneously in both the vertical and the horizontal direction, the paper will not fit on the wall unless we stretch some parts of it and cut away others.) This result is quite surprising and certainly disqualifies G-curvature as a measure of what we would ordinarily call the curvature or 'bendedness' of a surface. The break between the mathematical and the intuitive concepts of curvature, noticeable in the case of space curves, is now complete. But this should not detract us from using the mathematical concept, for, as Gauss writes, "less depends upon the choice of words than upon this, that their introduction shall be justified by pregnant theorems".¹¹ And the theorem which our wallpaper example illustrates is pregnant indeed with portentous ideas.

2.2.4 Gauss' Theorema Egregium and the Intrinsic Geometry of Surfaces

Plane geometry is usually taught at school as if no third spatial dimension existed. Euclidean plane geometry can be developed as what we might call the 'intrinsic' geometry of the plane, which studies the plane's structure purely in terms of itself, disregarding its relations to the space outside it. This becomes evident if we examine the special kind of Cartesian mappings used in plane coordinate geometry, characterizing them in terms of the Cartesian mappings we defined in Section 1.0.4. When applying the method of coordinates to plane geometry, we consider only mappings $x = (x^1, x^2, x^3)$ such that $x^3 = \text{constant}$, i.e. such that the reference plane π_3 is parallel to the plane η on which we carry out our investigations; consequently, the third coordinate of every point may be considered irrelevant and disregarded. The first two coordinates are, for each point P on η , identical with the distance from P to the mutually perpendicular lines λ_1 and λ_2 at which η intersects π_1 and π_2 , respectively. In plane geometry, our Cartesian mappings of space onto \mathbb{R}^3 can be (and in actual geometrical practice are) replaced by mappings of the plane onto \mathbb{R}^2 , which are referred, not to a triad of mutually orthogonal planes, but to a pair of perpendicular straight lines, the *axes* of the mapping. We shall call this kind of mapping a Cartesian 2-mapping.

The full import of this approach to plane geometry will perhaps more easily be grasped if we go back to the wallpaper example we introduced toward the end of the preceding section. Suppose now that a remarkably enterprising school principal resolves to decorate some of the classrooms in his school with specially designed wallpaper displaying a course of elementary plane geometry. After the paper is pasted on the flat classroom walls, the figures illustrating the proof of each theorem will look not much different than they do in the ordinary chalk-and-blackboard course, only more neatly printed and adequately drawn. Some of the figures will perhaps be such that merely from looking at them we will find the accompanying statements obvious. While the wallpaper lies rolled up in the school's storage room, the statements printed on them do not seem so obvious. My question is: are they any the less true? Or consider Fig. 9 illustrating Euclid's proof of Pythagoras' theorem. Tear out the page and roll it in any way you wish, or make a dunce cap out of it. Does

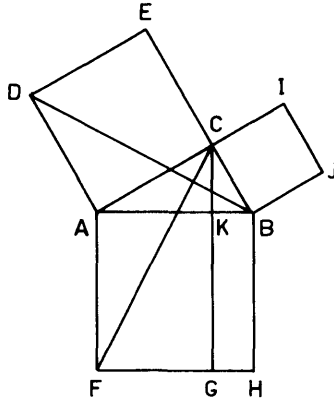


Fig. 9.

the area of figure ABHF cease being equal to the sum of the areas of ACED and BCIJ? Certainly not, provided the paper has not been stretched or shrunk. Moreover, every step in Euclid's proof retains its validity when referred to the rolled up figure, e.g. that $AB = AF$ and $CA = AD$ and even, in a sense which may initially elude us, that angle BAD is equal to angle CAF, and that *therefore* the area of AKGF is equal to that of ACED. If the reader is not persuaded let him check the proof by the method of coordinates. He may choose as axes the top and side edges of the page. On the dunce cap both will become curved lines, as will the perpendiculars drawn from them to any point of the figure. But the latter will preserve their lengths and they will continue to meet the axes orthogonally at the same points. The Cartesian 2-mapping now maps the surface of the dunce cap into \mathbb{R}^2 . But the value assigned to each point is the same as it was, so that all the equations used in the proof remain true. We can check by this method any proof on the wallpaper rolls and obtain similar results.

We now see that the G-curvature function, in assigning the same constant value 0 to the points of the plane and those of a rolled surface, does not behave in an arbitrary, geometrically irrelevant fashion. On the contrary, these two kinds of surface, like all surfaces of zero G-curvature, are so closely related, that, when viewed 'intrinsicly' as two-dimensional expanses, apart from their relations to the space outside, they must be regarded as geometrically equivalent (at least locally, i.e. on a neighbourhood of each point). Curiously

enough, G-curvature itself does not seem to be an intrinsic property of surfaces, for it is defined in terms of the varying spatial position of the tangent plane at different points of a surface. Why then is G-curvature identical in surfaces which intrinsically are indeed geometrically equivalent but which extrinsically, in terms of their relations to the rest of space, are not at all equivalent? Does this happen only to surfaces of zero curvature, whose peculiar relation to outer space measured by G-curvature is null or non-existent? Gauss' most remarkable discovery in his study of curved surfaces – *theorema egregium*, as he called it – states that this happens not exclusively to surfaces of zero curvature, but universally to all surfaces. To make this statement more precise we must first define exactly what it is, in the general case, for the intrinsic geometrical structure of two surfaces to be equivalent. Let us recall our argument illustrating the geometrical equivalence between the plane and the surface of a roll. It rested essentially on the following fact: if Φ is an injective differentiable mapping of a region ζ of \mathbb{R}^2 onto a part of the rolled surface and x is a Cartesian 2-mapping of the plane, it can happen that $\Phi \cdot x$ maps any straight line segment on the plane onto an arc of the same length on the rolled surface. An analogous relation between any two surfaces $\Psi(\eta)$, $\Psi(\zeta)$ can easily be exhibited. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a distance-preserving mapping¹² such that $f(\eta) \subset \zeta$; then, $\Theta = \Phi \cdot f \cdot \Psi^{-1}$ is an isometric mapping or an *isometry* of $\Psi(\eta)$ into $\Phi(\zeta)$ if and only if, for any arc λ on $\Psi(\eta)$, λ and $\Theta(\lambda)$ have the same length. We say that two surfaces are isometrically related or *isometric* if there exists an isometry that maps one into the other. 'Geometric equivalence' in the sense suggested above is, strictly speaking, isometric relatedness. Gauss' *theorema egregium* can now be stated thus: if two surfaces S_1 and S_2 are isometric, G_i is the G-curvature function on S_i ($i = 1, 2$), and f is an isometry of S_1 into S_2 , then $G_2 \cdot f = G_1$. We put this briefly by saying that G-curvature is invariant under isometries, or isometrically invariant. This result shows that G-curvature is a quite significant geometrical concept. In preparation for his proof of the theorem in the *Disquisitiones generales* of 1827, Gauss developed the means for studying any surface 'intrinsically', heedless of its relations to the three-dimensional space in which it is embedded. This development provided the groundwork for the generalized geometry of Riemann.

In a paper of 1825 published posthumously,¹³ Gauss proves his

theorema egregium as a consequence of the theorem (stated on p.74) which equates the excess or defect of a triangle bounded by shortest arcs on a surface to the total curvature of the region enclosed by that triangle. This discovery must have seemed paradoxical, for isometric relatedness was so obviously independent of the spatial position of the isometric surfaces, while G-curvature, as we pointed out above, was defined in terms of that position. But in his tract of 1827, Gauss showed that G-curvature could be calculated from certain isometrically invariant functions which suffice to determine what is usually called the 'intrinsic geometry' of the surface, i.e. its isometrically invariant structure.¹⁴ These functions arise when we look for a general expression for arc length on a surface. We shall consider a surface $\Phi(\zeta)$, where Φ is an injective mapping of an open region of \mathbf{R}^2 into space, fulfilling the conditions stated on p.71. An arc on $\Phi(\zeta)$ joining two points P and Q is the range of some injective path $\hat{c}([a, b])$ subject to the condition stated on p.68, such that $\hat{c}(a) = P$ and $\hat{c}(b) = Q$. Since $\hat{c}([a, b])$ lies wholly on $\Phi(\zeta)$, there exists an injective mapping $c: [a, b] \rightarrow \zeta$ with derivatives of all orders, such that $\hat{c} = \Phi \cdot c$. Given any Cartesian mapping x , the length of $c([a, b])$ is given by

$$s(a, b) = \int_a^b \left| \frac{d}{dt} (x \cdot \hat{c}) \right|_t dt = \int_a^b \left| \frac{d}{dt} (x \cdot \Phi \cdot c) \right|_t dt, \quad (1)$$

where the integrand, as we know, does not depend on the choice of x . Let $\varphi = x \cdot \Phi$; φ maps $\zeta \subset \mathbf{R}^2$ into \mathbf{R}^3 . For each $(u, v) \in \zeta$, we write $\varphi(u, v) = (\varphi_1(u, v), \varphi_2(u, v), \varphi_3(u, v))$. For each $t \in [a, b]$, we write $c(t) = (c_1(t), c_2(t))$. With this notation the integrand in eqn. (1) is given by

$$\left| \frac{d}{dt} (\varphi \cdot c) \right| = \left| \left(\sum_{i=1}^3 \left(\frac{d}{dt} (\varphi_i \cdot c) \right)^2 \right)^{1/2} \right| = \left| \left(\sum_{i=1}^3 \left(\sum_{j=1}^2 \frac{\partial \varphi_i}{\partial c_j} \frac{dc_j}{dt} \right)^2 \right)^{1/2} \right|. \quad (2)$$

Regrouping terms in the last expression and writing

$$E = \sum_{i=1}^3 \left(\frac{\partial \varphi_i}{\partial c_1} \right)^2, \quad F = \sum_{i=1}^3 \frac{\partial \varphi_i}{\partial c_1} \frac{\partial \varphi_i}{\partial c_2}, \quad G = \sum_{i=1}^3 \left(\frac{\partial \varphi_i}{\partial c_2} \right)^2, \quad (3)$$

we obtain

$$\left| \frac{d}{dt} (\varphi \cdot c) \right| = \left| \left(E \left(\frac{dc_1}{dt} \right)^2 + F \left(\frac{dc_1}{dt} \frac{dc_2}{dt} \right) + G \left(\frac{dc_2}{dt} \right)^2 \right)^{1/2} \right|. \quad (4)$$

In order to put the above into more familiar notation, we now adopt a different, otherwise revealing point of view. Since Φ is injective it has an inverse Φ^{-1} , which maps the surface $\Phi(\zeta)$ injectively onto the open region $\zeta \subset \mathbb{R}^2$. To each point P on $\Phi(\zeta)$, Φ^{-1} assigns a pair of real numbers $(\hat{u}(P), \hat{v}(P))$ which we shall call the *coordinates* of P by the *chart* Φ^{-1} . (The functions $P \mapsto \hat{u}(P)$ and $P \mapsto \hat{v}(P)$ are called the first and second coordinate functions of the chart.) Since $c = \Phi^{-1} \cdot \hat{c}$, it is obvious that $c_1 = \hat{u} \cdot \hat{c}$ and $c_2 = \hat{v} \cdot \hat{c}$. If we put $c_1 = u$ and $c_2 = v$, we obtain upon substituting in (4) the well-known expression for arc length on a surface $\Phi(\zeta)$:

$$s(a, b) = \int_a^b ds = \int_a^b \left| \left(E \left(\frac{du}{dt} \right)^2 + 2F \frac{du}{dt} \frac{dv}{dt} + G \left(\frac{dv}{dt} \right)^2 \right)^{1/2} \right|. \quad (5)$$

The integrand is called the *line element* on $\Phi(\zeta)$ and is customarily expressed thus:

$$ds = [E(du)^2 + 2F du dv + G(dv)^2]^{1/2}. \quad (6)$$

E , F and G are continuous functions on ζ , whose values at a point $c(t) = (u(t), v(t))$ do not depend on the choice of c .

Gauss' 1827 proof of his *theorema egregium* consists essentially in showing that the G -curvature function on $\Phi(\zeta)$ can be defined in terms of E , F and G . The proof is achieved by sheer force of calculation and we need not go into it. Gauss also establishes in terms of E , F and G , differential equations which must be satisfied by arcs of shortest length on $\Phi(\zeta)$.¹⁵ The result is then used to prove the theorem relating total curvature to triangular excess or defect. E , F and G also enter essentially into the expression for angular measure on $\Phi(\zeta)$. It seems that if we view E , F and G as three arbitrary (though well-behaved) functions on ζ , and forget their original definition in terms of a Cartesian mapping x , we could regard the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ as characteristic of the surface $\Phi(\zeta)$ and fully determining its intrinsic geometry. We may accept this, subject to one very important qualification: the above matrix depends essentially on the

representation of the surface as the range of a mapping Φ , or, to put it the other way around, as the domain of a chart Φ^{-1} . We can nevertheless easily establish how

$$\begin{bmatrix} E \cdot \Phi^{-1} & F \cdot \Phi^{-1} \\ F \cdot \Phi^{-1} & G \cdot \Phi^{-1} \end{bmatrix}$$

transforms when a different chart is substituted for Φ^{-1} , or, to use the technical expression, how it transforms 'under a transformation of coordinates'. We could therefore conceive the intrinsic geometry of a surface as defined by an infinite set of such matrices, transforming into one another according to definite rules. This conception is indeed clumsy, but many physicists and most engineers have managed to live with it to this day. On the other hand, the line element on the surface, although it was expressed above in chart-dependent fashion, is actually invariant under coordinate transformations. This suggests that we regard the line element as the function which characterizes the intrinsic geometry of the surface, and each matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ as a 'decomposition' of it, relative to one of the many admissible charts. But this is not quite so simple as it sounds. The line element is indeed a function, for it maps something into \mathbb{R} . But the something mapped is not the set of points on the surface. At each point the line element indicates, so to speak, the local contribution to the length of each arc passing through the point; this contribution is indeed the same for all arcs passing through the point in a certain direction, but it is usually different for arcs passing through it in different directions. The line element at each point P of a surface S is therefore a function on the set of directions through P in the tangent plane S_P . If we wish to find a mathematical entity defined on S and fully characterizing its intrinsic geometry, we shall do well to look for a mapping assigning to each point P on S a function on the said set. We shall soon learn to regard a surface S as a two-dimensional differentiable manifold (p.88). Mappings of the required sort are called covariant tensor fields on S .

Gauss' treatment of the intrinsic geometry of surfaces suggests an idea which the non-mathematical reader ought to consider carefully at this point. Let $\Phi(\zeta) = S$ be a smooth surface, as before. If x is a Cartesian 2-mapping, $x^{-1}(\zeta)$ is an open, connected region of the Euclidean plane. Call it Q . Let $\Psi = x^{-1} \cdot \Phi^{-1}$. Ψ maps S injectively

onto Q . Arcs, closed regions and angles in S are mapped by Ψ onto arcs, closed regions and (usually curvilinear) angles in Q . Let the 'length' of an arc λ in Q be equal to the length of $\Psi^{-1}(\lambda)$, the 'area' of a region A of Q be equal to the area of $\Psi^{-1}(A)$, the 'size' of an angle α in Q be the size of $\Psi^{-1}(\alpha)$, etc. These conventions establish what we may call a quasigeometry on Q . By a judicious distribution of inverted commas we can now convert every theorem of the intrinsic geometry of S into a true statement of the quasigeometry of Q . To speak more straightforwardly, we shall say that the bijective mapping Ψ induces in a region of the plane the intrinsic geometry of surface S .¹⁶ The whole procedure can be conceived in a more general way: let $\Phi(\zeta) = S$ and $\Phi'(\zeta) = S'$ be any two surfaces; obviously $\Phi' \cdot \Phi^{-1}$ induces on S' the intrinsic geometry of S . This idea can be extended to any set S endowed with an arbitrary structure G . If f maps S bijectively onto a set S' , f can be said to induce G in S' , since every true sentence $P((x_i)_{i \in I})$ concerning a family of points of S will be uniquely correlated with a true sentence ' $P'((f(x_i))_{i \in I})$ ' concerning a family of points of S' . Since the same subset of S' can be, say, a 'straight line' by virtue of one such mapping f and the 'interior of a sphere' by virtue of another mapping g , familiarity with these methods and points of view can easily lead one to think that geometry is just a matter of terminological convention.¹⁷

2.2.5 Riemann's Problem of Space and Geometry

Bernhard Riemann (1826–1866), a student of theology who converted to mathematics in Göttingen, obtained his doctoral degree in 1851 under Gauss with a dissertation on the theory of functions of a complex variable. In order to become *habilitiert*, i.e. licensed as a university instructor, he submitted a second tract, "On the possibility of representing a function by a trigonometric series" (1853). The final requirement for habilitation was to deliver a public lecture before the full faculty of philosophy. Of the three subjects proposed by Riemann, the first two were related to his former essays, but Gauss, acting on behalf of the faculty, quite unusually passed them up and opted for the last, the foundations of geometry. This choice was probably Gauss' last but not least contribution to the field. Had he instead acted strictly according to precedent, Riemann's lecture "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" would never have been written. It was read on June 10, 1854. Perhaps

it was as a concession to his non-mathematical audience that Riemann omitted all formal derivations. It is, however, unlikely that he was in a position to give them all, with the full clarity with which they can be given nowadays, after a century of efforts by noteworthy mathematicians.¹⁸ But this does not detract from the greatness of Riemann's achievement, for nearly all his unproved and *prima facie* obscure mathematical claims can be translated into intelligible and demonstrable statements.¹⁹ The same cannot be said of his epistemological conclusions, on whose very meaning all philosophers are not agreed.

Riemann begins by pointing out a feature common to all the traditional presentations of geometry: that they presuppose the concept of space and the fundamental concepts used in spatial constructions. The purely nominal definitions of these basic concepts – e.g. Euclid's definitions of point and straight line – shed no light on the supposedly essential properties and relations ascribed to these concepts in the axioms of geometry. Consequently, one fails to perceive any necessity in jointly assuming all these presuppositions; moreover, one does not even see whether their joint assertion is at all tenable. Riemann believes that in order to dispel this obscurity from the foundations of geometry we must clarify the general concept of which space is just a particular instance. That general concept he describes as the concept of a multiply extended quantity (*mehrfach ausgedehnte Grösse*). He proposes to “construct” it from “general quantitative concepts”. This construction will show that an n -fold extended quantity admits of diverse “metric relations” (*Maassverhältnisse*), “so that space constitutes only a special case of a threefold extended quantity”.²⁰

These introductory remarks deserve careful attention. Riemann obviously does not use the term ‘space’ in the very broad sense in which it is used nowadays by mathematicians. ‘Space’ is, in his words, *the space, der Raum*, a unique entity which is the site of physical bodies and the locus of physical movements. We saw in Section 1.0.3 that space, in this sense, was originally conceived as a repository of geometrical points, whose existence ensured the applicability of Euclidean geometry in natural science. If one knows of other geometries, it is reasonable to ask whether the Euclidean system is in every respect the one best suited for the description of natural phenomena. In the context of space metaphysics, this question naturally takes the form: *What geometry is true of space?* The

meaning and scope of this question will be considerably clarified if we consider space as an instance of a broader genus, each of whose species is characterized by a geometry. Such is the approach suggested by Riemann's initial remarks and outlined in the rest of his lecture. Riemann does not hesitate to describe space as a "threefold extended quantity". However it is not the only conceivable quantity of this kind, for the genus "threefold extended quantity" can be specified by several alternative "metric relations". Riemann assumes that space is characterized by a definite system of such relations which unambiguously determines the ratios between all pairs of homogeneous spatial magnitudes. This is tantamount to a geometry. Since space admits of but one such system and many more are thinkable, the true geometry of space cannot be determined by conceptual analysis alone. Riemann concludes that "those properties which distinguish space from other conceivable threefold extended quantities can be gathered only from experience".²¹

Riemann proposes next the following fundamental problem: "To find out the simplest facts from which the metric relations of space can be determined". This task has a purely conceptual side which consists in pointing out the structural features of a multiply extended quantity that are sufficient to determine its specific "metric relations". But if Riemann is right, it has an empirical side as well, to be settled by experimental research into physical phenomena. Euclid's postulates together with his tacit assumptions describe one such system of simple facts, which suffice to establish, through Pythagoras' theorem, the metric relations of space. But, in view of the foregoing, we cannot expect to deduce these facts from general quantitative concepts. This implies, according to Riemann, that they are "not necessary, but possess only empirical certainty: they are hypotheses".²² Even if we grant that their likelihood is enormous within the bounds of observation, their applicability beyond these bounds, either on the side of the very large or on that of the very small, remains an open question.

In the remaining sixteen pages of his lecture Riemann carries out a sizeable part of the programme sketched in the first two. He complains that he has obtained very little help from previous writers – only a few short hints in a paper by Gauss and "some philosophical investigations by Herbart" have guided him in his work. The lecture is divided into three parts: the first is concerned with the general concept of an n -fold extended quantity; the second explores the

mathematical problem of determining the simplest facts that govern metric relations on such quantities; the third deals with “applications to space”.

2.2.6 *The Concept of a Manifold*

Riemann conceives an n -fold extended quantity as a particular instance of a more general sort of entity which he calls a *Mannigfaltigkeit*. The English equivalent of this word is *manifold*. Both words are used in present day mathematics in a narrower sense. Confusion may arise because this technical sense of *manifold* agrees fairly well with what Riemann had in mind when he spoke of an n -fold extended quantity. A *manifold*, in Riemann’s sense, is rather like what we would nowadays call a *set* – although the empty set and sets of a single element presumably would not have counted as manifolds in his eyes. We shall put quotation marks around *manifold* when we use it in the latter sense. Riemann introduces this notion of a “manifold” in a somewhat peculiar way. Quantitative concepts he says are applicable only if a genus is given, and the latter can be specified in a variety of ways. The specifications of the genus constitute a “manifold”, which is continuous if there is a continuous transition from one specification to another, or discrete if there is not. Specifications constituting a discrete “manifold” are called the *elements* of the “manifold”; those forming a continuous “manifold” are called its *points*.²³ Although Riemann admits the possibility that space might ultimately be a discrete “manifold”, he concerns himself almost exclusively with continuous “manifolds”.

Riemann can give only two commonplace examples of continuous “manifolds”, namely, colours, and “the locations of the objects of sense” (*die Orte der Sinngegenstände*). But higher mathematics supplies a vast array of them. Riemann apparently believes that they all fall into two classes: n -fold extended quantities (for some positive integer n) and what we may call infinitely extended quantities. The latter are mentioned in passing, towards the end of Part I. While a point of an n -fold extended quantity can be referred to by an n -tuple of real numbers (*Größenbestimmungen*, i.e. “determinations of magnitude”, in Riemann’s words), you need a sequence of real numbers or even a whole continuous manifold of them to specify a point in an infinitely extended quantity. Riemann says nothing further about the latter,²⁴ so we shall ignore them and deal only with n -fold

extended quantities. Since each point of these can be referred to by an element of \mathbb{R}^n , it might seem possible to characterize an n -fold extended quantity as a set that can be mapped injectively into \mathbb{R}^n . But this characterization is doubly inadequate. On the one hand it is too broad: the injective mappings of n -fold extended quantities into \mathbb{R}^n must fulfil certain additional conditions. On the other hand, when these conditions are added, our characterization becomes unnecessarily restrictive. Riemann obviously conceives the mapping of an n -fold extended quantity S into \mathbb{R}^n , which furnishes each point of S with its own exclusive real n -tuple, as a continuous mapping, i.e. one that maps neighbouring points of S on neighbouring points in \mathbb{R}^n . Although he does not define neighbourhood in S , he makes use of such a concept when speaking of a "continuous transition" from each point of S to the others. But it is not only continuity of the mapping that is required. Riemann's discussion in Part II is based on the assumption that an n -fold extended quantity S can be mapped injectively into \mathbb{R}^n in many different ways, and that if f and g are two such mappings the composite mapping $f \cdot g^{-1}$ is everywhere differentiable to a suitably high order. Riemann probably thought that the latter condition was implied by the requirement of continuity, but we now know that it is not. As we said earlier, the characterization of n -fold extended quantities as manifolds injectable into \mathbb{R}^n , when qualified by these further requirements, is too restrictive. Thus, for example, there exists no injective mapping which assigns neighbouring pairs of real numbers to all neighbouring points of a sphere. But Riemann would certainly have considered the sphere as a twofold extended quantity – indeed, he gives it as one of his examples in Part II.5. The solution of this difficulty was suggested on p.71f.: an n -fold extended quantity may be conceived as composed of many pieces, each of which can be mapped injectively onto a part of \mathbb{R}^n , the mappings being subject to the two additional conditions mentioned above. An n -fold extended quantity, thus understood, is what we nowadays call an n -dimensional (real) differentiable manifold. This is defined as an abstract set M , associated with a collection A (an *atlas*) of injective mappings (or *charts*), each of which maps a part of M onto an open subset of \mathbb{R}^n , so that (i) each point of M is included in the domain of at least one chart of the atlas, and (ii) if f and g are two charts of the atlas, the composite mapping $f \cdot g^{-1}$ is differentiable to a suitably high order on $g(\text{dom } f \cap \text{dom } g)$.²⁵ A composite mapping such

as $f \cdot g^{-1}$ is known as a 'coordinate transformation' of the manifold $\langle M, A \rangle$. We shall hereafter require coordinate transformations to have partial derivatives of all orders. It is fairly easy to show rigorously how an $(n + 1)$ -dimensional manifold can be constructed from a one-dimensional and an n -dimensional one, which Riemann explains more or less intuitively in Part I.2; or how any n -dimensional manifold can be analyzed into submanifolds of dimensions 1 and $n - 1$, which is sketched by him in Part I.3. But it must be realized that since n -dimensional differentiable manifolds possess a rather peculiar structure, not every "manifold" (in Riemann's sense) which may be regarded as continuous in some non-trivial way will necessarily fall into one of the two classes of continuous "manifolds" Riemann acknowledged, i.e. n -fold and infinitely extended quantities.

*Two additional remarks on differentiable manifolds are in order here. In the first place, we need not assume that the abstract set M is in any sense continuous, for continuity in M comes about automatically, more or less in the following way. Given an atlas A on M , there is a unique maximal atlas A' , such that $A \subset A'$. (A' is the set of all charts f such that, for every $g \in A$, $f \cdot g^{-1}$ and $g \cdot f^{-1}$ are differentiable to the required order on $g(\text{dom } f \cap \text{dom } g)$ and on $f(\text{dom } f \cap \text{dom } g)$, respectively.) The charts of A' induce neighbourhood relations between the points of M : if P and Q belong to the domain of a chart $f \in A'$ which maps them onto neighbouring points of \mathbb{R}^n , we regard P and Q as neighbouring points of M ; if P and Q do not belong to the domain of the same chart of A' , we may look for a point R which belongs with P to the domain of a chart f_1 and with Q to the domain of a chart f_2 , and establish neighbourhood relations between P and Q through the mediation of R . (For a more exact formulation of these ideas, see Appendix, p.362.) In the second place, an abstract set M which is given the structure of an n -dimensional manifold by its association with an atlas A , can be given the structure of an m -dimensional manifold ($m \neq n$) by its association with a different atlas B .²⁶ A given atlas, however, unambiguously fixes the dimension number of the manifold. This is due to the following fact: *if $m \neq n$, there cannot be an injective, continuous and open mapping of an open subset of \mathbb{R}^n onto an open subset of \mathbb{R}^m* ; consequently, if the charts of an atlas fulfil condition (ii) stated above, their ranges must be open subsets of \mathbb{R}^n for a single positive integral value of n . The statement in italics was not proved until this century (Brouwer, 1911b). But

Riemann apparently took it for granted. In fact, it seems intuitively obvious. Faith in this sort of intuition was badly shaken however when Cantor (1878) proved that \mathbb{R}^2 can be mapped injectively – though not continuously – into \mathbb{R} and Peano (1890) proved that \mathbb{R} can be mapped continuously – though not injectively – onto \mathbb{R}^2 .

2.2.7 *The Tangent Space*

Smooth curves and surfaces in space can easily be conceived as one- and two-dimensional differentiable manifolds. Indeed, Gauss' methods for dealing with them are historically at the root of the very notion of a differentiable manifold. This notion enables us to transfer analogically the familiar concepts of the theory of surfaces to any set of arbitrary entities which has been associated with an atlas. This momentous step is nowhere elaborated upon by Riemann but is implicit in his lecture. Without spending time on definitions or conceptual analyses, he proceeds in Part II to sketch a full-fledged generalization to n -dimensional differentiable manifolds of Gauss' intrinsic geometry of surfaces. Clarity with respect to Riemann's assumptions was arrived at much later. Yet we cannot avoid making use of some of the later developments, even at the risk of its appearing anachronistic.

If P is any point of a smooth surface S , S touches at P a plane S_P , the tangent plane at P . We saw above that a consideration of tangent planes played a major role in the establishment of Gauss' theory of surfaces – a remarkable fact indeed, since the notion of a tangent plane, which lies for the most part in the space outside the surface, seems quite foreign to the project of studying the intrinsic properties of surfaces. We shall now provide each point P of an n -dimensional differentiable manifold M with the analogue of a tangent plane, namely an n -dimensional vector space $T_P(M)$, called *the tangent space* at P . $T_P(M)$ must be conceived rather abstractly, for we take M to be an arbitrary manifold. No thought of a space surrounding M should enter into the definition of $T_P(M)$. Since any smooth surface S may be treated as a two-dimensional manifold, each point P on S will have a two-dimensional vector space $T_P(S)$ attached to it. $T_P(S)$ is naturally isomorphic with the tangent plane S_P (p.72). Consequently, it can fill the latter's role in the theory of surfaces. In other words, we can identify S_P with $T_P(S)$ or, as I prefer to say, we can substitute the latter for the

former. But $T_P(S)$ belongs to S intrinsically, since we shall have defined it without making any reference to how S lies in space.

We shall outline the construction of the tangent space $T_P(M)$ at a point P of an n -dimensional manifold M . But we shall first show that the very nature of differentiable manifolds enables us to extend to them the notion of differentiability. If φ maps an m -dimensional manifold M into an m' -dimensional manifold M' , we say that φ is differentiable at $P \in M$ if, given a chart x defined at P and a chart y defined at $\varphi(P)$, the mapping $f = y \cdot \varphi \cdot x^{-1}$ possesses all partial derivatives of every order at $x(p)$. This definition makes good sense because f maps an open set of \mathbb{R}^m into $\mathbb{R}^{m'}$. The differentiability of φ does not depend on the choice of charts x, y , because all coordinate transformations of M and M' are differentiable. \mathbb{R}^n is made into a manifold by associating with it an atlas whose sole chart is the identity mapping $a \mapsto a$. We stipulate that \mathbb{R}^n (for every positive integer n) possesses this manifold structure. We can now define a *path* in a manifold M as a differentiable mapping of an open interval of \mathbb{R} into M .²⁷ Let $\mathcal{C}_P(M)$ be the set of all paths c which are defined on some open interval about zero and are such that $c(0) = P$. Let $\mathcal{F}_P(M)$ be the set of all differentiable functions that map some neighbourhood of P into \mathbb{R} . If $c \in \mathcal{C}_P(M)$ and $f \in \mathcal{F}_P(M)$, $f \cdot c$ maps an interval of \mathbb{R} into \mathbb{R} . The derivative $d(f \cdot c)/dt$ is defined at zero; its value there will be denoted by $d(f \cdot c)/dt|_0$. We assign to each $c \in \mathcal{C}_P(M)$ a function $\dot{c}_P: \mathcal{F}_P(M) \rightarrow \mathbb{R}$ defined as follows:

$$\dot{c}_P(f) = d(f \cdot c)/dt|_0. \tag{1}$$

The set of these functions \dot{c}_P is endowed with a standard linear structure (Appendix, p.364). With that structure it is called *the tangent space* of M at P and denoted by $T_P(M)$. Moreover, if c is any path such that $c(u) = P$ for some real number u , we assign to c a unique vector $\dot{c}_P \in T_P(M)$ according to the following rule. Let τ_u denote the translation $\mathbb{R} \rightarrow \mathbb{R}$; $x \mapsto x + u$. Let $\gamma = c \cdot \tau_u$. Then, $\gamma \in \mathcal{C}_P(M)$ and $\dot{\gamma}_P$ is defined by (1). We set

$$\dot{c}_P = \dot{c}_{c(u)} = \dot{\gamma}_{\gamma(0)} = \dot{\gamma}_P. \tag{2}$$

If K is the range of c , K is a submanifold of M and the canonical injection $i: K \rightarrow M$; $a \mapsto a$ is an imbedding. It can be shown that \dot{c}_P spans $T_P(K)$, the one-dimensional tangent space of K at P . The union

of the tangent spaces of an n -dimensional manifold M can be given the structure of a $2n$ -dimensional manifold, the tangent bundle TM . It is thus possible to define differentiable mappings of M into TM and vice versa.²⁸

2.2.8 Riemannian Manifolds, Metrics and Curvature

The second item of Riemann's programme is concerned with metric relations in n -dimensional manifolds and the simplest conditions under which they can be determined. Metric relations (*Maassverhältnisse*) are what enable quantitative comparisons between the parts of a "manifold" (in Riemann's sense of the word). Riemann observes that the parts of a discrete "manifold" can be quantitatively compared by *counting*, but if the "manifold" is continuous – as all manifolds in the special sense defined above are – quantitative comparison can be made only by *measurement*. "Measurement", says Riemann, "consists in a superposition of the quantities to be compared. Therefore it requires a means of transporting one quantity to be used as a standard (*Maassstab*) for the others. Otherwise one can compare two quantities only if one is a part of the other, and then only as to *more* or *less*, not as to how much."²⁹ This passage, which turns all of a sudden from the lofty musings of ontological and mathematical abstraction to down-to-earth tasks reminiscent of tailoring and bartending, has weighed heavily on the minds of philosophers of geometry for over a century. It is all the more remarkable, since measurement in the physical sciences is rarely effected by the superposition of a standard upon the object to be measured, either because the latter is too small or too large, or because it lies too far away, or even because superposition is repugnant to its very nature.³⁰ I do not see very well how one can transport (*forttragen*) a part of a manifold while the rest of it remains unmoved. But this is a question we need not discuss now (cf. pp.159f., 174f.). Our present interest in the above passage stems from its connection with Riemann's investigations of Part II. His aim there is to determine the conditions under which measurements can be performed on a manifold. To this end, he instinctively translates the passage's obscure operational prescription – free mobility of the standard of measurement inside the manifold – into a neat mathematical requirement, which is that *magnitudes be independent of their position in the manifold* (*Unabhängigkeit der Grössen vom Ort*). This requirement,

he says, can be satisfied in several ways. The first that comes to mind consists in supposing that the length of a line is independent of the way how the line lies in the manifold (*Unabhängigkeit von der Lage*), so that every line can be measured by every other line. If this obtains, the length of an arc in a manifold will be determinable as an intrinsic property, i.e. as a property belonging to the arc as a one-dimensional submanifold, no matter what its relation to the points outside it.³¹

The length of an arc in Euclidean space was traditionally conceived as the limit of a sequence of lengths of polygonal lines inscribed in the arc. This conception is extended quite naturally to \mathbb{R}^n , where 'straight' segments are easily discerned.³² The length of the straight segment joining $a = (a_1, \dots, a_n)$ to $b = (b_1, \dots, b_n)$ is defined, by an immediate generalization of the theorem of Pythagoras, as $|a - b| = |(\sum_{i=1}^n (a_i - b_i)^2)^{1/2}|$. This method of definition is not intrinsic in the above sense, and is not generally applicable to arcs in an arbitrary n -dimensional manifold, since one cannot know beforehand whether anything like a polygonal line will even exist in such a manifold. The traditional definition of arc length can be given however a different reading in the light of the concept of a tangent space developed in the foregoing section. As we saw on page 69, the length of a path c was given classically by an integral which we may note, for brevity, as $\int f \cdot c(t) dt$. The 'element of length', $f \cdot c(t)$, was interpreted as the length of an arbitrarily short straight line tangent to the path at $c(t)$. This notion is somewhat mysterious, for the length of an arbitrarily short line is not a definite number at all (unless we simply equate it to zero). The obscurity is avoided, however, if we conceive f as a function which assigns to each point $c(t)$ the 'length' of the tangent vector $\dot{c}_{c(t)}$ defined in eqn. (2) of Section 2.2.7. This will make sense only if that vector has been given a 'length'. This is usually done by defining a 'norm' on the vector space to which it belongs.³³ Since the concepts of tangent vector and tangent space are intrinsic, the reinterpreted definition of arc length satisfies Riemann's requirement and can be extended to an arbitrary manifold. We shall now see how this is done.

Let M be an n -dimensional differentiable manifold. To each point $P \in M$ there is attached an n -dimensional vector space, the tangent space $T_P(M)$. Consider any smooth arc in M .³⁴ We may regard it as the range of an injective path c . At a point $P \in c(t)$, a definite element of $T_P(M)$ is associated with path c , namely \dot{c}_P . If, in every tangent space

of M , there is a norm, each vector \dot{c}_P possesses a definite length $\|\dot{c}_P\|$ which we may regard as an index, so to speak, of the lengthening that our arc experiences as it passes through P . The length of the arc $c([a, b])$ is then given by the integral

$$\int_a^b \|\dot{c}_{c(t)}\| dt. \quad (1)$$

This definition is indeed intrinsic, for $\dot{c}_{c(t)}$ spans the tangent space $T_{c(t)}c([a, b])$.³⁵ Since we are dealing with an arbitrary manifold M , the norm in its tangent spaces must be conceived quite broadly. It need not even be defined in all of them in the same way. It is required only that the norm of $T_P(M)$ does not change abruptly as P ranges over M . This demand leaves enormous latitude of choice. Riemann imposes two further restrictions. The first is that the norm in each tangent space must be a positive homogeneous function of the first degree; i.e. that for any vector v and any real number α , $\|\alpha v\| = |\alpha| \|v\|$. This requirement agrees well with our intuitive idea of length and is ordinarily included in the general definition of a norm on vector spaces. It ensures that the length of the arc $c([a, b])$ will not depend on the choice of its parametrical representation c . Riemann's second restriction sounds less natural. It amounts to demanding that the manifold M be what we now call a *Riemannian manifold*. Riemann is well aware that this is not really necessary for a reasonable solution of his problem. He observes, however, that the discussion of a more general case would involve no essentially different principles, but would be rather time-consuming and throw comparatively little new light on the study of space. (Riemann, H, p.14).

Let us say what we mean by a Riemannian manifold. A given vector space V determines a vector space $\mathcal{F}_2(V)$ of bilinear functions on $V \times V$. Let $\mathcal{F}_2(M)$ denote the union of the spaces $\mathcal{F}_2(T_P(M))$ determined by each tangent space $T_P(M)$ of a manifold M . $\mathcal{F}_2(M)$ is endowed with a standard differentiable structure (compare p.366). It therefore makes sense to speak of a differentiable mapping of M into $\mathcal{F}_2(M)$. A *Riemannian manifold* or *R-manifold* is a pair $\langle M, \mu \rangle$ where M is a differentiable manifold and μ is a differentiable mapping of M into $\mathcal{F}_2(M)$ which assigns to each $P \in M$ a bilinear function $\mu_P: T_P(M) \times T_P(M) \rightarrow \mathbf{R}$, and fulfils the following three conditions:

(i) μ is symmetric, i.e. for every $P \in M$, and every $v, w \in T_P(M)$, $\mu_P(v, w) = \mu_P(w, v)$;

(ii) μ is non-degenerate, i.e. for every $P \in M$, if $v \in T_P(M)$, $\mu_P(v, w) = 0$ for every $w \in T_P(M)$ if and only if $v = 0$;

(iii) μ is positive definite, i.e. for every $P \in M$, and every $v \in T_P(M)$, $\mu_P(v, v) \geq 0$, equality obtaining if and only if $v = 0$.

(iii) clearly implies (ii); (ii), in its turn, implies that for every $P \in M$, if (e_i) is a basis of $T_P(M)$, the matrix $[\mu_P(e_i, e_j)]$ is non-singular (i.e. its determinant is not equal to zero).

If $\langle M, \mu \rangle$ is an R -manifold, μ is called an R -metric on M . If μ fulfils (i) and (ii), but $\mu_P(v, v)$ takes values less than, equal to or greater than 0 (depending on the argument v), we say that μ is an *indefinite* metric on M . A pair $\langle M, \mu \rangle$, where M is a differentiable manifold and μ is either an R -metric or an indefinite metric on M , is called a *semi-Riemannian manifold*. The study of semi-Riemannian manifolds has become important due to their use in the theory of relativity. Riemann concerned himself exclusively with R -manifolds. An R -manifold structure can be defined on a wide variety of differentiable manifolds.

If μ is an R -metric on a manifold M and P is any point of M , $v \mapsto |(\mu_P(v, v))^{1/2}|$ is a norm in $T_P(M)$. If we substitute this norm in expression (1) we obtain the standard definition of arc length on R -manifolds. An R -manifold M is made into a metric space, in the usual sense,³⁶ if a distance function $d: M \times M \rightarrow \mathbf{R}$ is defined as follows: If $P, Q \in M$, we let $L(P, Q)$ denote the set of real numbers $\{\lambda | \lambda$ is the length of an arc in M , joining P and $Q\}$; then $d(P, Q) = \inf L(P, Q)$. In other words, the distance between two points P, Q of an R -manifold M is the infimum or greatest lower bound of the lengths of the arcs joining P to Q . This is the standard metric structure of R -manifolds. Since it is ultimately determined by the mapping μ that characterizes each such manifold, μ is customarily called the metric of the manifold. (This terminology has given rise to some philosophical misunderstandings due to the fact that indefinite metrics do not make their respective semi-Riemannian manifolds into metric spaces.)

Consider an n -dimensional manifold M , endowed with an R -metric μ . Let $U \subset M$ be the domain of a chart x . x maps an arbitrary point $P \in U$ on the real number n -tuple $(x^1(P), \dots, x^n(P))$. For each coordinate function x^i there is a unique path c^i through P , defined on some open neighbourhood of 0, such that if $Q \in U$ and $Q = c^i(t)$ for

some real number t in the domain of c^i , then the i -th coordinate of Q , that is, $x^i(Q)$, equals $x^i(P) + t$, while all the remaining coordinates of Q are equal to the respective coordinates of P . (In other words, all coordinate functions except x^i are constant on the image of c^i .) The tangent vector \dot{c}_P^i is denoted by $\partial/\partial x^i|_P$. The set of vectors $(\partial/\partial x^i|_P)$ ($1 \leq i \leq n$) is a basis of $T_P(M)$.³⁷ We define a set of n^2 functions g_{ij} on U :

$$g_{ij}(P) = \mu_P \left(\frac{\partial}{\partial x^i} \Big|_P, \frac{\partial}{\partial x^j} \Big|_P \right) \quad (1 \leq i, j \leq n). \quad (2)$$

These functions can be shown to be differentiable, as they are composed of differentiable mappings.³⁸ Let g be the determinant of the matrix $[g_{ij}]$ and let G_{ij} be the cofactor of g_{ij} in this matrix. Since μ is non-degenerate, $g \neq 0$. A second set of n^2 differentiable functions g^{ij} is defined on U by

$$g^{ij} = (1/g)G_{ij}. \quad (3)$$

Clearly

$$\sum_{i=1}^n g_{ik}g^{ij} = \delta_k^j \quad (\text{i.e. } 1 \text{ if } j = k \text{ or } 0 \text{ if } j \neq k). \quad (4)$$

Since μ is symmetric, $g_{ij} = g_{ji}$ and $g^{ij} = g^{ji}$, so that each set comprises at most $n(n+1)/2$ different functions. We define two further sets of functions on U , for use later:³⁹

$$\begin{aligned} [ij, k] &= \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right), \\ \{h\}_{ij} &= \sum_{k=1}^n g^{hk} [ij, k]. \end{aligned} \quad (1 \leq i, j, k, h \leq n) \quad (5)$$

We shall now express arc length in U by means of the functions g_{ij} . Consider any smooth arc in U . As on p.91 we regard it as the range of an injective path c . The integral (1) gives then the length of our arc between points $c(a)$ and $c(b)$. On the R -manifold $\langle M, \mu \rangle$ the integrand of (1) is

$$\|\dot{c}_{c(t)}\|_{c(t)} = |(\mu_{c(t)}(\dot{c}_{c(t)}, \dot{c}_{c(t)}))^{1/2}|. \quad (6)$$

Since $c(t) \in U$, we can express $\dot{c}_{c(t)}$ as a linear combination of the basis vectors $(\partial/\partial x^i|_{c(t)})$:

$$\dot{c}_{c(t)} = \sum_{i=1}^n \dot{c}_{c(t)}(x^i) \frac{\partial}{\partial x^i} \Big|_{c(t)} = \sum_{i=1}^n \frac{dx^i \cdot c}{dt} \Big|_t \frac{\partial}{\partial x^i} \Big|_{c(t)} \quad (7)$$

Since μ_P is bilinear, we have that

$$\begin{aligned} \|\dot{c}_{c(t)}\|_{c(t)}^2 &= \sum_{ij} \mu_{c(t)} \left(\frac{\partial}{\partial x^i} \Big|_{c(t)}, \frac{\partial}{\partial x^j} \Big|_{c(t)} \right) \frac{dx^i \cdot c}{dt} \Big|_t \frac{dx^j \cdot c}{dt} \Big|_t \\ &= \sum_{ij} g_{ij}(c(t)) \frac{dx^i \cdot c}{dt} \Big|_t \frac{dx^j \cdot c}{dt} \Big|_t. \end{aligned} \tag{8}$$

The integrand $\|\dot{c}_{c(t)}\|$ appears thus, on the domain U of a given chart x of our R -manifold (M, μ) , as a reasonable generalization of Gauss' line element.⁴⁰ A short calculation shows that if v is a tangent vector at a point $P \in U$, its squared norm $\|v\|^2$ is equal to the value at (v, v) of the function $\sum_{ij} g_{ij}(P) dx^i(P) \otimes dx^j(P)$. The metric μ can therefore be expressed on U as $\sum_{ij} g_{ij} dx^i \otimes dx^j (1 \leq i, j \leq n)$.⁴¹ In particular, the standard metric of Euclidean space can be expressed in terms of any Cartesian mapping x as $dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3$. By analogy, any n -dimensional R -manifold whose metric can be put into the form $\sum_{i=1}^n dx^i \otimes dx^i$ relatively to some global chart x is called an n -dimensional Euclidean space (e.g. R^n with its standard metric).

Let U be, as above, the domain of chart x in R -manifold M . A path γ on U is called a *geodesic* if it is a solution of the differential equations:⁴²

$$\frac{d^2 x^i}{dt^2} + \sum_{j,k=1}^n \left\{ \begin{matrix} i \\ j \ k \end{matrix} \right\} \cdot \gamma \frac{dx^j \cdot \gamma}{dt} \frac{dx^k \cdot \gamma}{dt} = 0 \quad (0 \leq i \leq n). \tag{9}$$

The range of γ is a geodetic arc. It can be proved that if $P \in U$, there is an open neighbourhood of P , $V \subset U$, such that every point $Q \in V$ is joined to P by a geodetic arc, which is the shortest arc joining P and Q . In V , each geodesic γ such that $\gamma(0) = P$ is fully determined if we are given $\dot{\gamma}_P$. Consider the mapping $\text{Exp}_P: \dot{\gamma}_P \rightarrow \gamma(1)$ defined on the set of vectors $\{\dot{\gamma}_P | \gamma(0) = P\}$. It can be proved that Exp_P maps a neighbourhood of 0 in $T_P(M)$ diffeomorphically onto a neighbourhood of P contained in V .⁴³ We shall see that an essential step in Riemann's investigations rests on these results, which he, with his incredible flair for mathematical truth, assumed without proof.

We have seen that the integrand of (1) can on the domain of each chart of an R -manifold be equated to the square root of a chart-related quadratic expression (6). Riemann rightly maintains that the value of this expression does not depend on the choice of the chart, being (as we shall say) invariant under coordinate transformations. This observation appears trivial indeed if the matter is approached in

the above manner. We shall call the integrand of (1) the *line element* of the manifold. The quadratic form taken by the line element on the manifolds given his name is used by Riemann to characterize them. He is well aware that they are just a special kind of manifold, what he calls the "simplest cases". The "next simple case", he says, would consist of manifolds whose line element can be expressed as the fourth root of an expression of fourth degree. "Investigation of this more general class", he adds, "would indeed involve essentially the same principles, but would be rather time consuming and would throw comparatively little new light on the study of space."⁴⁴ That is why he restricts his research to what we call *R*-manifolds. He observes that the chart-related expression of the line element depends on $n(n+1)/2$ arbitrary functions (g_{ij}), whereas coordinate transformations are given by n equations. There remain therefore $n(n-1)/2$ functional relations which do not depend on the choice of chart but must be characteristic of the manifold. They should suffice to determine metrical relations on an n -dimensional *R*-manifold M . In his lecture, Riemann approaches this question *locally*, showing how to find $n(n-1)/2$ quantities at an arbitrary point $P \in M$ which, according to him, determine metrical relations in a neighbourhood of P . But before setting out to show this, he makes an important philosophical point. The line element on M takes at each point $P \in M$ the Euclidean form $(\sum_{i=1}^n dx^i(P) \otimes dx^i(P))(\dot{c}_P, \dot{c}_P)^{1/2}$, for a suitable chart x defined on all M , if and only if the functions g_{ij} determined by x satisfy

$$g_{ij} = \delta_j^i \quad (1 \leq i, j \leq n). \quad (10)$$

This presupposes a very special choice of the $n(n-1)/2$ arbitrary conditions that according to Riemann govern metrical relations on M . Consequently, the concept of Euclidean space is very far from being coextensive with that of a three-dimensional *R*-manifold, and far less with that of a three-dimensional manifold *überhaupt*. Just as Riemann had announced at the beginning of his lecture, the general notion of a threefold extended quantity does not, in any way, prescribe a Euclidean character to space.

Let P be any point in an *R*-manifold M . In order to find the $n(n-1)/2$ quantities which supposedly determine metrical relations near P , Riemann chooses a very particular chart at P . Let Exp_P map a neighbourhood of $0 \in T_P(M)$ diffeomorphically onto a neighbourhood W of P . Choose a basis (Y_i) on $T_P(M)$ such that $\mu_P(Y_i, Y_j) = \delta_j^i (1 \leq i,$

$j \leq n$), that is, a so-called orthonormal basis. Let $k: T_P(M) \rightarrow \mathbb{R}^n$ be given by $k(\sum_{i=1}^n a_i Y_i) = (a_1, \dots, a_n)$. The chart chosen by Riemann is defined on W as

$$x = k \cdot \text{Exp}_P^{-1}. \tag{11}$$

We call it a Riemannian normal chart. It can be shown that in terms of it

$$g_{ij}(P) = \delta_j^i, \quad \left. \frac{\partial g_{ij}}{\partial x^k} \right|_P = 0, \quad (0 \leq i, j, k \leq n). \tag{12}$$

This has an important implication that fully justifies the choice of the peculiar chart. Consider the Taylor expansion of the g_{ij} about P :

$$g_{ij} = g_{ij}(P) + \sum_k \left. \frac{\partial g_{ij}}{\partial x^k} \right|_P x^k + \frac{1}{2} \sum_{k,h} \left. \frac{\partial^2 g_{ij}}{\partial x^k \partial x^h} \right|_P x^k x^h + o(|x|^2), \tag{13}$$

where $o(|x|^2)$ denotes a function $f: M \rightarrow \mathbb{R}$ such that

$$\lim_{Q \rightarrow P} \frac{f(Q)}{|x(Q)|^2} = 0. \tag{14}$$

Since the first derivatives of the g_{ij} vanish at P , the deviation of the g_{ij} from the Euclidean value they attain at P is measured, in a suitable neighbourhood of P , by the third term of the above expansion. Let us write

$$\frac{1}{2} \frac{\partial g_{ij}}{\partial x^k \partial x^h} = C_{ij, kh} \quad (1 \leq i, j, k, h \leq n). \tag{15}$$

The Taylor expansion of the squared norm in tangent spaces near P can now be expressed in terms of our Riemannian normal chart:

$$\begin{aligned} \| \cdot \|^2 &= \sum_{ij} g_{ij} dx^i dx^j \\ &= \sum_i dx^i dx^i + \sum_{i,j,k,h} C_{ij, kh} x^k x^h dx^i dx^j + o(|x|) \\ &= \Delta_1 + \Delta_2 + o. \end{aligned} \tag{16}$$

This means that, if v is a sufficiently small vector at a point Q near P , $\sum C_{ij, kh} x^k(Q) x^h(Q) dx_Q^i(v) dx_Q^j(v)$ (summation implied over all four indices) is the *correction* that must be added to the Euclidean value $\sum dx_Q^i(v) dx_Q^j(v)$ to obtain the squared length of v . Since P is arbitrary, M can be covered by a collection of Riemannian normal charts. Thus, it appears that the key to metrical relations on M could be found

through the study of the second term Δ_2 of expansion (16). Riemann does not write the latter out as we do, but simply states that it is given by a quadratic expression in the $(x^i dx^j - x^j dx^i)(1 \leq i, j \leq n)$. This implies that there exist numbers $R_{ij, kh}$ such that

$$\begin{aligned} \Delta_2 &= \sum_{i,j,k,h} C_{ij, kh} x^i x^j dx^k dx^h \\ &= \sum_{i,j,k,h} R_{ij, kh} (x^i dx^j - x^j dx^i)(x^k dx^h - x^h dx^k). \end{aligned} \quad (17)$$

Riemann conceives the differentials dx^i as infinitesimals, i.e. as the coordinates of a point P' 'infinitely near' P . The x^i are, of course, the coordinates of an arbitrary point Q in W . Viewed in this way, Δ_2 is an infinitesimal quantity of the fourth order, which, Riemann says, when divided by the area A of the infinitesimal geodetic triangle PQP' , equals a finite quantity Δ_2/A . Riemann claims that this quantity, multiplied by $-3/4$, equals the G -curvature at P of the two-dimensional submanifold of M on which the triangle PQP' lies. This implies that Δ_2/A does not depend on the chart x and has exactly the same value for every two points P', Q in V which are such that the geodetic arcs joining them to P lie on the same two-dimensional submanifold of M . Riemann adds:

We found that $n(n-1)/2$ functions of position were necessary for determining the metric relations of an [n -dimensional R -manifold]; hence, if the [G -curvature] is given in $n(n-1)/2$ surface directions at each point, the metric relations of the manifold can be determined, provided only that there are no identities among these values, and indeed this does not, in general, occur. The metric relations of these manifolds, in which the line element can be represented as the square root of a differential expression of the second degree, can thus be expressed in a way completely independent of the choice of coordinates.⁴⁵

We cannot stop here to prove or disprove these portentous claims, but a few more observations might help to clarify them. The obscurest point lies perhaps in the treatment of the dx^i as infinitesimals. This is readily justified in the context of formal differential geometry (see Note 8, *ad finem*), but I shall abide by the now standard approach to the subject and view them as covector fields defined on a neighbourhood of P .⁴⁶ Since we consider only their value at P we write dx^i for dx^i_P .⁴⁷ We define a quadratic function $F: T_P(M) \times T_P(M) \rightarrow \mathbf{R}$ by giving its value at an arbitrary pair (X, Y) of

vectors in $T_P(M)$:

$$F(X, Y) = \sum_{i,j,k,h} C_{ij, kh} dx^i(X) dx^j(X) dx^k(Y) dx^h(Y). \tag{18}$$

It can be shown that the numbers $C_{ij, kh}$, defined in (15), fulfil the conditions⁴⁸

$$C_{ij, kh} + C_{ih, jk} + C_{ik, hj} = 0, \quad C_{ij, kh} = C_{kh, ij}, \quad (1 \leq i, j, k, h \leq n). \tag{19}$$

It is then purely a matter of tedious calculation to show that F can be expressed in terms of the forms $(dx^i \wedge dx^j)$ as follows:

$$F(X, Y) = \frac{1}{3} \sum_{i,j,k,h} C_{ij, kh} (dx^i \wedge dx^k)(dx^j \wedge dx^h)(X, Y). \tag{20}$$

If X and Y span a two-dimensional subspace α in $T_P(M)$, we can assign to α a number $k(\alpha)$, invariant under coordinate transformations and independent of the choice of the generators X and Y .⁴⁹

$$k(\alpha) = - \frac{3F(X, Y)}{\mu_P(X, Y)}. \tag{21}$$

Riemann's claims can now be stated as follows: If $n = 2$, so that $\alpha = T_P(M)$, $k(\alpha)$ is the G -curvature of M at P . If $n > 2$ and β is a neighbourhood of 0 in α , such that Exp_P is a diffeomorphism on β , $k(\alpha)$ is the G -curvature at P of the two-dimensional submanifold $M' = \text{Exp}_P(\beta)$ (regarded as an R -manifold with metric $\mu \cdot i$, where μ is the R -metric on M and $i: M' \rightarrow M$ is the canonical injection.) M' coincides on a neighbourhood of P with the locus of all geodesics through P whose tangent vector at P belongs to α . Let (Y_1, \dots, Y_n) be a basis of $T_P(M)$; then there are $n(n - 1)/2$ two-dimensional subspaces α_{ij} , spanned by the vector pairs (Y_i, Y_j) ($1 \leq i < j \leq n$). Riemann's chief claim in the passage quoted is that metrical relations on M are fully determined if we are given the values $k(\alpha_{ij})$ for every one of these subspaces α_{ij} at each point $P \in M$.⁵⁰

Riemann pays special attention to two kinds of manifolds. A manifold M belongs to the first of them when at every point $P \in M$, $k(\alpha) = 0$ for every two-dimensional subspace $\alpha \subset T_P(M)$. There can then be defined on a neighbourhood of each $P \in M$ a chart x such that, on its domain, $\mu(\partial/\partial x^i, \partial/\partial x^j)$ equals 1 if $i = j$ and equals 0 otherwise ($1 \leq i, j \leq n$). The domain of x can evidently be mapped isometrically

into \mathbb{R}^n (with its standard metric; see p.95). M is what Riemann calls a *flat* manifold. The second kind of manifolds considered by Riemann includes flat manifolds as a subclass. He calls them manifolds of *constant curvature*. If M is such a manifold $k(\alpha)$ equals the same real number for every two-dimensional subspace $\alpha \subset T_P(M)$ at every point $P \in M$. Schur (1886a) subsequently proved that M is a manifold of constant curvature if the preceding condition is fulfilled at any point $P \in M$. In Part III.1, Riemann observes that only if space is a manifold of constant curvature one may maintain that “the existence of *bodies*”, and not just that of widthless *lines*, does not depend on how they lie in space. In other words, only if space has a constant curvature does it make sense to speak of rigid bodies. We shall see in Sections 3.1.1–3.1.3 that Helmholtz regarded the existence of rigid bodies as a *conditio sine qua non* for the measurement of distance in physical space. If he is right, then physical geometry must rest on the assumption that space is a manifold of constant curvature. This restricts the spectrum of viable physical geometries considerably. Helmholtz’ ‘problem of space’ consists of determining that spectrum, under his just-mentioned assumption, by purely mathematical means. In order to solve this problem one must give a clear mathematical formulation to the idea that the existence of bodies is independent of how they lie in space. This can reasonably be understood to mean that any geometrical body placed in an arbitrary position can be copied isometrically about any point and in any direction. Now, one can only speak of isometrical copying with regard to a manifold in which metrical relations are determined. If, as Riemann contends, the latter depend wholly on the values of k (see however, Note 50), the required copies can certainly be made in a manifold of constant curvature, for metrical relations in such a manifold are “exactly the same in all the directions around any one point, as in the directions around any other, and thus the same constructions can be effected starting from either”.⁵¹ On the other hand, if P is a point of a manifold M and α and α' are two-dimensional subspaces of the tangent space $T_P(M)$ such that $k(\alpha) \neq k(\alpha')$, let β and β' be the neighbourhoods of 0 in α and α' , respectively, which are diffeomorphically mapped into M by Exp_P . $\text{Exp}_P(\beta)$ is then a two-dimensional submanifold of M whose tangent space at P is α . But it is impossible to construct an isometrical copy of $\text{Exp}_P(\beta)$ with tangent space α' at P . This can be seen as follows: $\text{Exp}_P(\beta)$ is

covered by all geodesics through P whose tangent vector belongs to α . The isometric copy of a geodesic is a geodesic. But all geodesics through P with tangent vector in α' lie on $\text{Exp}_P(\beta')$. Hence an isometric copy of $\text{Exp}_P(\beta)$ can have α' for its tangent space at P only if it coincides with $\text{Exp}_P(\beta')$ on a neighbourhood of P . This however is impossible, since $\text{Exp}_P(\beta)$ and $\text{Exp}_P(\beta')$ are surfaces whose G -curvatures differ at P . We may conclude, therefore, that unless M is a manifold of constant curvature not even surfaces – let alone bodies – are independent of how they lie in space. Riemann gives a general formula for the line element of a manifold of constant curvature K :

$$ds = \frac{1}{1 + \frac{K}{4} \sum x^i x^i} \sqrt{\sum dx^i dx^i}. \quad (22)$$

Riemann illustrates these ideas in a brief discussion of surfaces of constant curvature. If the curvature is $K > 0$, the surface can be mapped isometrically into a sphere of radius $1/\sqrt{K}$. If $K = 0$, it can be mapped isometrically into a Euclidean plane.

In his cursory reference to surfaces of constant curvature $K < 0$, Riemann does not mention the fact that they can be mapped isometrically into a BL plane, but I am convinced that he was aware of it. After all, BL-space geometry was at that time the only known example of a three-dimensional manifold with a non-Euclidean metric, and it is more than likely that concern with its viability and significance – which surely was not lacking in Gauss' entourage – prompted Riemann's own revolutionary approach to the question. His entire exposition is designed to bring out the fact that Euclidean manifolds, i.e. manifolds of constant zero curvature, constitute only a very peculiar species of a vast genus.⁵²

*Riemann extended the Gaussian concept of curvature to an arbitrary n -dimensional R -manifold M by using what we may call sectional curvatures, i.e. the G -curvatures of two-dimensional submanifolds of M . The value of these sectional curvatures at a point $P \in M$ is given by the function F defined on $T_P(M) \times T_P(M)$, (20). We would possess a general conception of the curvature of M if we could determine, once and for all, the F function attached to each point of M . This job is performed by the celebrated Riemann tensor (in its covariant form). Riemann himself went a long way towards its definition in his prize-essay of 1861.⁵³ His work was completed by

Christoffel (1869), who gave a definition of the tensor in terms of its components in an arbitrary chart. That the tensor had, so to speak, geometric substance, and was not an ephemeral chart-dependent appearance, was proved in classical mathematics by showing that in a coordinate transformation the components transform according to fixed rules.

*A deeper insight into the geometric meaning of curvature was gained through Levi-Civita's work on parallel transport (1917). As explained subsequently by Weyl a differentiable manifold M can be endowed with an *affine structure*, which determines, for each point $P \in M$ and each path k through P , a linear bijection of the tangent space $T_P(M)$ onto each tangent space attached to a point on the range of k . If Q is such a point, we denote by τ_{PQ}^k the mapping of $T_P(M)$ onto $T_Q(M)$ determined, for path k , by the affine structure of M . The mappings τ fulfil the following requirements: τ_{QP}^k is the inverse of τ_{PQ}^k ; also, if k , P and $v \in T_P(M)$ are fixed, $\tau_{PX}^k(v)$ describes a smooth curve in the tangent bundle TM as X varies over the range of k . We may therefore view the vector v as being carried 'parallel to itself' along the range of k , from P to Q , as X goes from the former point to the latter. $\tau_{PQ}^k(v)$ is said to be the image of v by *parallel transport* from P to Q , along the path k ; v and $\tau_{PQ}^k(v)$ are *parallel* vectors relative to k . Two vectors belonging, respectively, to $T_P(M)$ and $T_Q(M)$ which are parallel relative to k are not generally parallel relative to a different path k' joining P and Q . An affine structure Λ on M determines a collection of paths called the (affine) geodesics of $\langle M, \Lambda \rangle$. They can be characterized as follows: if k is a geodesic through P and Q in M and v is a vector tangent to k at P , then $\tau_{PQ}^k(v)$ is a vector tangent to k at Q . In other words, all vectors tangent to a geodesic are parallel relative to it. If μ is an R -metric on M , there is a unique affine structure Λ_μ such that the affine geodesics of $\langle M, \Lambda_\mu \rangle$ are the metric geodesics of $\langle M, \mu \rangle$, i.e. the paths which satisfy equations (9). This means that if an arc κ , joining P and Q , is the range of a geodesic of $\langle M, \Lambda_\mu \rangle$, κ is an extremal, i.e. κ is either longer or shorter than all other nearby arcs joining P and Q . Suppose now that M is endowed with a metric μ and that the affine structure Λ_μ defines the mappings τ described above. Let $c: [a, b] \rightarrow M$ be a path such that $c(a) = c(b) = P$. Denote the mapping by parallel transport of $T_{c(a)}(M)$ onto $T_{c(b)}(M)$ by τ_{PP}^c . Then, for every non-zero vector $v \in T_P(M)$, $\tau_{PP}^c(v)$ will normally differ from v . If the range of c is a small closed circuit, the

said difference is measured by the components of the Riemann tensor.

*In the Appendix, the affine structure of a manifold is introduced by means of Koszul's concept of a linear connection. This provides also a straightforward definition of the Riemann tensor. Let M be an R -manifold with metric μ and let ∇ be the linear connection which determines the unique affine structure Λ_μ . For any vector fields X, Y, Z, W on M , let

$$\begin{aligned} \bar{R}((X, Y), Z) &= \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z \\ R(X, Y, Z, W) &= \mu(\bar{R}((Z, W), Y), X) \end{aligned} \quad (23)$$

The mapping $(X, Y, Z, W) \mapsto R(X, Y, Z, W)$ is a covariant tensor field of order 4. We call it the covariant Riemann curvature tensor.⁵⁴ The name is justified because, if $P \in M$ and F is the quadratic function defined in (18)

$$R_P(X_P, Y_P, X_P, Y_P) = -3F(X_P, Y_P) \quad (24)$$

(where R_P, X_P and Y_P denote, respectively, the values of R, X and Y at P). Since ∇ is determined by μ , (23) and (24) imply that metric determines curvature, i.e. that the generalized version of Gauss' *theorem egregium* holds in every R -manifold. (Concerning Riemann's claim that, conversely, curvature determines metric, see Note 50.)

2.2.9 Riemann's Speculations about Physical Space

Part III of Riemann's lecture concerns the 'application' of the foregoing to space. It rests on the assumption that space is an extended quantity and, consequently, a "manifold", i.e. the set of "specifications" (*Bestimmungsweisen*) of a genus (*allgemeiner Begriff*). Since space is probably a continuous "manifold" – or, at any rate, is generally treated as if it were one – its elements are called "points" (p.85). Indeed, Riemann's choice of the word "point" to designate the elements of a continuous "manifold" is doubtless motivated by the ordinary use of the word when speaking of space. Riemann therefore openly treats space as the (structured) aggregate of its points. For a mathematician, this view is reasonable enough, since every proposition belonging to a geometric theory can be formulated as a statement concerning the structured set of points postulated by the theory. The view should also satisfy a physicist, for

'space' can only be to him the structured point-set where bodies or their theoretical representations are located by the mathematical theory he is working with, or – if you prefer a metaphysical manner of speech – that entity, whatever it may be, which the said point-set is supposed to represent. Even if the representation of such an entity by any particular theory is admittedly inadequate, the general description of space as a structured point-set should not be objectionable to the physicist, since a better representation can only consist – as long as physics is mathematical and mathematics is not expelled from Cantor's paradise – in a differently structured point-set.

According to Riemann, all we can say about space without resorting to experience is that it is one among many possible kinds of "manifold". It may even be a discrete "manifold". However, Riemann considers at length only a smaller range of alternatives, namely, finite-dimensional extended quantities, i.e. finite-dimensional differentiable manifolds. This tentative limitation of admissible alternatives, like the further restriction to R -manifolds, is clearly founded, to Riemann's mind, upon an empirical consideration, *viz.* the success of Euclidean geometry within "the limits of observation". Riemann distinguishes between two kinds of properties of manifolds: "extensive or regional relations" (*Ausdehnungs- oder Gebietsverhältnisse*) and "metric relations" (*Maassverhältnisse*). We have already spoken about the latter. The former I take to be the relations determined by the differentiable structure of the manifold. They include the topology of the manifold and all so-called topological properties (i.e. properties preserved by homeomorphisms), but that is not all what they include. Riemann points out an important difference between these two kinds of properties: while the variety of "extensive relations" is discrete, that of "metric relations" is continuous. Consequently, empirical statements concerning the former, though hypothetical, are apt to be exact. Thus, we usually assume that space has three dimensions and, if this turns out to be wrong, space will have four, five or another integral number of dimensions. By contrast, empirically verifiable hypotheses concerning the metric relations of space are necessarily imprecise, and they can hold only within a certain range of experimental error. Thus, the statement that space is Euclidean, that is, that its curvature is everywhere exactly zero, is not admissible as a scientific conjecture: we can hypothesize at best that

the curvature of space lies within the interval $(-\varepsilon, \varepsilon)$, for some real number $\varepsilon > 0$. This conclusion, unstated by Riemann but clearly implied by his remarks, has considerable importance, for the geometry of a manifold is non-Euclidean—either spherical or BL—once its constant curvature deviates ever so slightly from zero. If hypotheses concerning space curvature can only assign it intervals, not fixed values, even the supposition that space curvature must be constant appears to be ruled out. If there is no empirical means of telling which value, within a given interval, space curvature does, in fact, take on, the latter may just as well vary gradually within that interval from place to place or from time to time. Towards the end of his lecture, Riemann advances an even bolder conjecture, namely, that space curvature may vary quite wildly within very small distances, provided the *total* curvature over intervals of a suitable size is approximately zero. The celebrated hypothesis on the “space-theory of matter” put forward by W.K. Clifford (1845–1879) in 1870 is little more than a restatement of this conjecture of Riemann’s. Clifford wrote:

I hold in fact

(1) That small portions of space are in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.

(2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

(3) That this variation of the curvature of space is what really happens in the phenomenon which we call the motion of matter, whether ponderable or ethereal.

(4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity.⁵⁵

These conjectures concerning the microphysical variability of space curvature are often said to anticipate the conception, propounded in Einstein’s theory of gravitation, of a four-dimensional *space-time* manifold, whose curvature changes from point to point at the macro-physical level.

Another remark by Riemann does unquestionably anticipate Einstein. He notes that a manifold may—indeed, must—be unlimited (*unbegrenzt*), even if it is not infinite (*unendlich*). Lack of boundaries or limits is an “extensive” property belonging to the manifold as such, while infinitude depends on the metric.⁵⁶ “That space is an unlimited triply extended manifold”, says Riemann, “is an assumption involved

in every conception of the external world. At every moment, we complete the domain of actual perceptions and construct the possible place of sought-for objects in accordance with the said assumption, which is being continually confirmed by means of these applications. The unlimitedness of space therefore carries greater empirical certainty than any other external experience. But its infinitude does not in any way follow from this; for, assuming that bodies are independent of position and that space is therefore of constant curvature, space would be finite if that curvature had ever so small a positive value. By prolonging into shortest lines the initial directions on a surface element, one would obtain an unlimited surface of positive constant curvature, that is, a surface which in a triply extended flat manifold would take the form of a sphere, and which consequently is finite."⁵⁷

These remarks open the way to further speculations about the global properties of space, analogous to those made by 20th-century cosmologists in the wake of Einstein. But Riemann cuts short the flight of scientific imagination. "Questions about the very large", he observes, "are idle questions for the explanation of nature." But such is not the case with questions about the very small. They are of paramount importance to natural science, for "our knowledge of the causal connection of phenomena rests essentially upon the exactness with which we pursue such matters down to the very small".⁵⁸ Questions concerning the metric relations of space in the very small are therefore not idle. If the size and shape of bodies is independent of their position, space curvature is constant and its value can be conjectured on the basis of astronomical observations. They show, Riemann says, that it can differ only insignificantly from zero.

But if such an independence of bodies from position is not the case, no conclusions about metrical relations in the infinitely small can be drawn from those prevailing in the large; at every point the curvature in three directions can have arbitrary values provided only that the total curvature of every measurable portion of space is not noticeably different from zero. Still more complicated relations can occur if the line element cannot be represented, as was assumed, by the square root of a differential expression of the second degree. Now it seems that the empirical concepts on which the metric determinations of space are founded, namely, the concept of a rigid body and that of a light ray, are not applicable in the infinitely small; it is therefore quite conceivable that the metrical relations of space in the infinitely small do not agree with the assumptions of geometry; and indeed we ought to hold that this is so if phenomena can thereby be explained in a simpler fashion.⁵⁹

The intellectual freedom displayed by the young Riemann in the preceding lines must have overwhelmed his audience. His last suggestions reach well beyond Einstein's theories to some recent speculations concerning the breakdown of space concepts in particle physics.

2.2.10 *Riemann and Herbart. Grassmann*

Riemann names Gauss and Herbart as his only authorities. His relations to Gauss ought to be plain by now. Let us dwell a little further upon his relation to Herbart. In a posthumously published note Riemann declared:

The author is Herbartian in psychology and in the theory of knowledge [...] but on the whole he does not subscribe to Herbart's philosophy of nature and the philosophical disciplines related to it (ontology and synechology).⁶⁰

Stimulated by this statement of philosophical allegiance, some writers have sought to determine the specific influence of Herbart on Riemann's philosophy of space and geometry. Bertrand Russell lists five items in Herbart's writings which "gave rise to many of Riemann's epoch-making speculations", namely, the psychological theory of space, the construction of extension out of series of points, the comparison of space with the tone and colour series, Herbart's general preference for the discrete above the continuous and his belief in the great importance of classifying space with other "manifolds" (called by him *Reihenformen*).⁶¹ Of these items, the third is strictly Herbartian and has probably led to Riemann's general description of a "manifold" as the set of specifications of a genus, a description that better suits the manifold of colours and colour-hues than it does the points of space. Classifying space with time is commonplace in modern philosophy, and the word *manifold* (*Mannigfaltigkeit*) had been employed by Kant to name the class to which they both belong.⁶² Herbart's psychological theory of space belongs to that part of Herbart's philosophy to which Riemann professed allegiance, but I fail to perceive its influence on the lecture of 1854, except insofar as it may have inspired its strong empiricist bias. But then empiricism was rampant in Germany in the 1850's – due in part to Herbart's lifelong work. Herbart's psychology purported to show how our representation of space can be reconstructed from

empirical beginnings; but psychogenesis has no place in Riemann's lecture, the empiricism in which bears a logical stamp.⁶³ (Riemann does not speak of the origin of representations, but of hypotheses lying at the foundation of a deductive science, which must be accepted or rejected in accordance with the success and simplicity of the explanation they give of phenomena.) I do not know what Russell had in mind when he spoke of Herbart's "general preference for the discrete above the continuous", so that I cannot judge wherein such preference shows up in Riemann's writings. As for the second item, the construction of extension out of series of points, it presumably refers to the construction of the line out of a pair of points in Herbart's theory of the continuum or synechology (i.e. in one of the parts of Herbart's system to which Riemann did not subscribe). The first step in the construction is the following "extremely simple" thought: "Two simple entities, which we denote by A and B, can be, but at the same time cannot be, together."⁶⁴ This simple but paradoxical thought generates a third entity between A and B which poses the same paradox. Endless iteration of the paradox generates the line. Providing that Herbart's construction works and is not just sheer nonsense, we may conclude that it yields a dense point-set (like the set of points assigned rational coordinates by a Cartesian mapping of a Euclidean line), but not a linear continuum (i.e. a point-set structurally equivalent to \mathbb{R}). Herbart somehow acknowledges this limitation of the proposed scheme when he declares that the line generated by his construction is a "rigid" line, not a "continuous" one.⁶⁵ According to him, a true continuum "does not consist of points, even if it *arises* from them", and is therefore not a point-set.⁶⁶ Riemann, on the other hand, resolutely conceived of space as a *continuous point-set*, endowed with a differentiable structure which he must have known that a merely dense set cannot possess. A continuous point-set cannot be constructed from its elements (in fact, this is the main objection of the intuitionist school of Brouwer, Weyl, etc., to the set-theoretical approach to continua). Riemann, however, is not content to have the points of his extended quantities simply stand as given in certain mutual relations. He outlines a so-called "construction" of an n -dimensional manifold by successive or serial transition, in certain well-regulated ways, from one of its points to the others. But his construction is very different from Herbart's. Curiously enough, the same construction is formulated, almost in

Riemann's terms, in the context of an earlier proposal for the generalization of geometry, the *Theory of Extension* published in 1844 by Hermann Grassmann (1809–1877).⁶⁷

There is no evidence that Riemann ever read Grassmann. In fact the latter's book was generally ignored by mathematicians until many of his findings were rediscovered by others, and Hermann Hankel (in 1867) and Alfred Clebsch (in 1872) drew everyone's attention to his pioneering work. However, since Grassmann's programme has so much in common with Riemann's, a brief comparison would not be out of order here. In a summary published in 1845 in Grunert's *Archiv*, Grassmann describes his "theory of extension" as "the abstract foundation of the theory of space (geometry)". "I.e. it is the pure mathematical science freed from every spatial intuition, whose special application to space is geometry."⁶⁸ The latter, "since it refers to something given in nature, namely space, is not a branch of pure mathematics, but an application of it to nature; however, it is not merely an application of algebra, [...] for algebra lacks the concept of a variety of dimensions, which is peculiar to geometry. What is needed therefore is a branch of mathematics whose concept of a continuously variable quantity incorporates the notion of differences corresponding to the dimensions of space. Such a branch is my theory of extension."⁶⁹ This theory overcomes the restriction to three dimensions imposed on geometry by its physical referent. But not only does Grassmann anticipate Riemann in his attempt at a general treatment of "extended quantities"; in a specific methodological area he appears more modern than his younger contemporary: he sets out and develops a coordinate-free geometrical calculus—a "truly geometrical analysis", as he calls it⁷⁰—which directly subjects points, lines, etc., to algebraic operations. Nevertheless, Grassmann's theory of extension is not a general theory of manifolds, but only a theory of n -dimensional vector spaces with the usual Euclidean norm. Compared to Riemann's theory, it is a rather restricted generalization of geometry. Its limitation is probably due to the fact that in developing his general theory Grassmann simply took it for granted that ordinary geometry was correct in its special (physical) field of application. Riemann, on the other hand, educated at Gauss' Göttingen, questioned this assumption from the outset, and this no doubt guided the formation of his thoughts. By probing deeper he was able to give his theory a broader scope.⁷¹

2.3 PROJECTIVE GEOMETRY AND PROJECTIVE METRICS

2.3.1 *Introduction*

The development of non-Euclidean geometry in Central and Eastern Europe was half-hidden from the public owing to the obscurity of two of its creators and the shyness of the third. In almost the same period, the work of Jean-Victor Poncelet (1788–1867), who, in the limelight of Paris, was laying the foundations of projective geometry, received more attention. Partly because of its simplicity and beauty, and partly, no doubt, because of its deceptive appearance of Euclidean orthodoxy, the new discipline was in a short time well-known, accepted and taught in the universities, often under the alluring name of 'modern geometry'. Since there was no provocative negation expressed in its name and since its radicalism was hidden beneath seductive appeals to intuition, no philosopher ever raised his voice against it. Yet, at bottom, projective geometry is much more 'unnatural' than, say, BL geometry, which only negates a Euclidean postulate whose intuitive evidence had been questioned for centuries, while, in projective geometry, the basic relations of linear order and neighbourhood between the points of space are upset. Projective geometry ignores distances and sizes, and thus may be regarded as essentially non-metric.¹ Nevertheless, in 1871, Felix Klein (1849–1925), following the lead of Arthur Cayley (1821–1895), showed how to define metric relations in projective space. Making conventional and, from the projective point of view, seemingly inessential variations in the definition of those relations, one obtained a metric geometry satisfying the requirements of Euclid, or one satisfying those of Bolyai and Lobachevsky, or, finally, a geometry where triangles had an excess, as in spherical geometry, but where straight lines would not meet at more than one point. In order to make these results understandable to readers with no previous knowledge of projective geometry, we shall present the main ideas of this geometry in a more or less intuitive fashion in Section 2.3.2 which follows. A rigorous analytical presentation of them will be given in Section 2.3.3. Although an axiomatic characterization of projective space would provide the best approach to such a thoroughly unintuitive entity, we shall not give one because neither Klein nor his predecessors judged it necessary or even useful and we wish to look at the subject as much as possible from their point of view.

2.3.2 Projective Geometry: An Intuitive Approach

The origins of projective geometry can be traced to the study of perspective by Renaissance painters and architects. It was assumed that one could obtain a faithful representation of any earthly sight upon a flat surface S by placing S between the observer and the objects seen and 'projecting' the latter onto S from a single point P located inside the observer's head. The *projection* of a point Q in space from P onto S is simply the point where S meets the straight line joining P to Q . The study of projections suggests, as we shall see, a seemingly innocent device, which makes for greater simplicity and uniformity. This consists in adding to every straight line an ideal point or 'point at infinity'. The first modern mathematician to do this was Johannes Kepler in 1604. Projective methods involving the use of ideal points were successfully used in the solution of geometrical problems by Girard Desargues (1591–1661), followed by Blaise Pascal (1623–1662) and Philippe de la Hire (1640–1718). In the 18th century, the value of these methods was eclipsed by the tremendous success of analytical methods. The revival of projective methods in France in the early 19th century owed much to the influence of Gaspard Monge (1746–1818), one of whose pupils was Poncelet.

To explain the meaning and use of ideal points, we shall consider the projection of one line on another from a point outside both. We initially assume that Euclid's geometry is valid. Now, let m, n be two straight lines meeting at P and let O be a point of the plane (m, n) , neither on m nor on n . Let $\lambda(O)$ be the flat pencil of lines through O on the plane (m, n) . $\lambda(O)$ includes a line m' which does not meet m and a line n' which does not meet n (Fig. 10). All other lines in $\lambda(O)$

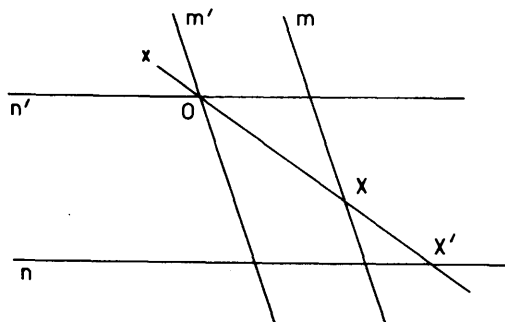


Fig. 10.

meet both m and n . If $X \in m$, there is a single line in $\lambda(O)$ which meets m at X . Let us denote this line by x . The *projection* of m on n from O is the mapping which assigns to a point X in m the point $x \cap n$ where x meets n . This mapping is injective. It is defined on $m - \{m \cap n'\}$. Its range is $n - \{n \cap m'\}$. Points near $m \cap n'$ but on different sides of it are mapped very far from $n \cap m'$, on the opposite extremes of n ; while points lying very far from $m \cap n'$, on the opposite extremes of m , are mapped near $n \cap m'$ but on different sides of it. The domain of the projection is cut into two parts by $m \cap n'$ and within each part the projection is continuous (mapping neighbouring points onto neighbouring points). The parallel lines m, m' (n, n') do not share a *point* but they have the same *direction*. Let us use the term 'a meet' to denote either a point or a direction. Instead of saying that two lines p, q have a meet, Y , in common, we may say that they *meet at* Y . Neighbourhood relations between the meets of a line q can be defined very easily. Let P be a point outside q ; then every meet of q belongs to a line through P . A neighbourhood of a line m through P is any angle with vertex at P containing points of m in its interior. Let us say that m *belongs* to such an angle. (Obviously, m also belongs to the vertically opposite angle.) Let X be the meet of m and q . Then the meets of q with all lines belonging to a given neighbourhood of m constitute a neighbourhood of X . It can be easily seen that this definition of neighbourhoods on q does not depend on the choice of point P .² According to our stipulations, every neighbourhood of the direction of a line q includes points on either extremity of q . We now redefine the projection of m on n from O as the mapping which assigns to every meet X of m a meet X' of n , so that X and X' belong to the same line through O . The projection thus defined is a bijection defined on the set of meets of m ; its range is the set of meets of n . The projection is a continuous mapping.³ Henceforth, and in accordance with established mathematical usage, we shall call every meet a *point*. A meet which is not a point in the ordinary sense of the word is what we call an *ideal point* or a point *at infinity*. A straight line m regarded as the set of its meets and endowed with the neighbourhood structure induced on it by a flat pencil through a point outside it, is called a *projective line*. As in ordinary geometry, we use the term *open segment* to denote an open connected (proper) part of a projective line.⁴ If A and B are any two points on a projective line m , $m - \{A, B\}$ consists of two open

segments which join A to B. One of these segments includes the ideal point (unless A or B is itself that point). Consequently, given three points A, B, C on m , any two of them are joined by a segment which does not include the other; it makes no sense, therefore, to say that one of them lies between the other two. Four points on m , — A, B, C, D — can always be grouped in two pairs, say (A, C), (B, D), such that each segment joining the points in one couple includes one of the points in the other; we say then that the points in each couple *separate* the points in the other. On a projective line we cannot define a linear order but we can define a cyclic order. This is only natural, since neighbourhood relations on the projective line are based on the neighbourhood relations of a flat pencil of lines through a point.

Let us consider now the projection of a plane on another plane from a point outside both. To avoid repetition let us regard m, n in Fig. 10 as the intersection of planes α, β with the plane $(O, m \cap n', n \cap m')$. The projection of α on β from O is determined by the intersections of α and β with all the straight lines through O. Let us denote this bundle of lines by $\sigma(O)$. We regard every straight line on α and β as a projective line and we let $\sigma(O)$ induce neighbourhood relations on both planes. The projection is plainly a continuous bijective mapping of α (including its ideal points) onto β (including its ideal points). We agree to regard the set of ideal points on each plane as an ideal straight line (where it meets every plane parallel to it). Clearly, the projection maps straight lines onto straight lines and preserves incidence relations between straight lines and points. (If m meets m' at Q, the projection of m meets the projection of m' at the projection of Q, etc.)

A plane endowed with an ideal line and with the neighbourhood structure induced by a bundle of straight lines through a point outside it is called a projective plane. This is a very peculiar sort of entity, as we shall now see. Consider three points, A, B, C, on a projective plane π . The lines AB, AC, BC divide π into four regions (Fig. 11). Any two points within one of these regions are joined by a segment wholly within the region. Two points belonging to two different regions are joined by segments that cut at least one of those three lines. We shall now assign a sense to the perimeter of each region, namely, the sense ABCA. This is counterclockwise on Region I of the figure, that is, on the region which has no ideal points. But on the other three regions, which meet the ideal line, the sense prescribed

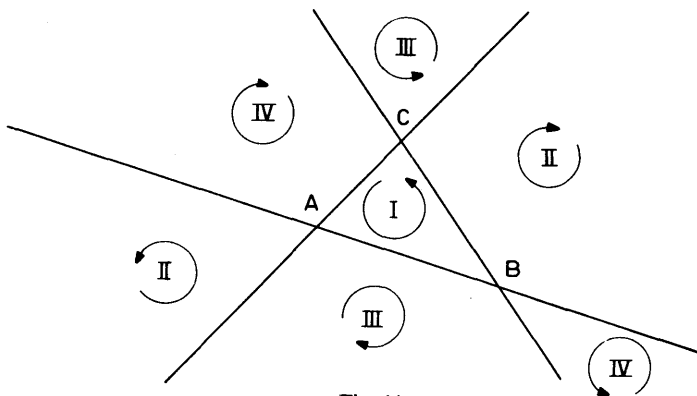


Fig. 11.

appears to be clockwise on one side of that line and counterclockwise on the other side. We therefore cannot assign a sense unambiguously to every closed polygonal line on π . The projective plane is *non-orientable*. It is also a *one-sided* surface, as we shall try to show. The reader is presumably acquainted with the one-sided Möbius strip. We shall show that it can be regarded as a strip cut out of the projective plane. To do this, we shall construct several homeomorphisms which will be useful later on.⁵ The projective plane can be mapped homeomorphically onto the pencil $\sigma(O)$ of straight lines through a point O not on that plane. Take a sphere S centred at O . We shall say that $x, y \in S$ stand on the relation E if a line of $\sigma(O)$ goes through x and y (i.e. if x and y are either identical or antipodal). E is an equivalence. The pencil can be mapped homeomorphically onto the quotient set S/E . Take one half of S , including the equator that divides it from the other half. If we agree to regard each pair of antipodal points on the equator as a single point, we obtain a structure homeomorphic to S/E . The perpendicular projection of this figure on its equatorial plane maps it homeomorphically onto a circular disk whose peripheral points are regarded as identical whenever they lie on the same diameter. Two parallel chords equidistant from the centre of the disk define a strip on it which is plainly homeomorphic to a rectangle two opposite sides of which have been identified in reverse order (Fig. 12). Such a rectangle is a Möbius strip. Through the inverses of the homeomorphisms we have described, the Möbius strip is mapped homeomorphically onto a strip of the projective

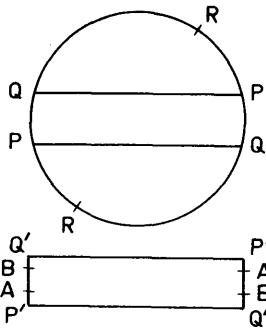


Fig. 12.

plane. This does not prove, but somehow makes plausible that the latter is also a one-sided surface.

If every plane in Euclidean space has been turned into a projective plane, we can naturally regard the set of these planes as a new kind of space, namely projective space. This space includes an *ideal plane*, formed by all the ideal points that have been added to every ordinary plane.

The ideal plane is determined by the pair of ideal lines where it meets any two non-parallel ordinary planes. We cannot establish neighbourhood relations in projective space by appealing to some intuitively representable topological structure outside it (such as the pencil $\sigma(O)$ we used in the case of the projective plane), because every intuitive spatial configuration is comprised in it. We might, however, attempt to define its neighbourhood structure from within.⁶ But we shall go no further in the consideration of projective space until we have made the notion of a projective plane clearer and less problematic.

2.3.3 Projective Geometry: A Numerical Interpretation

We have introduced projective space insidiously, by a series of natural, apparently intuitive steps. The result arrived at is, however, ostensibly counterintuitive and we cannot be sure that it is truly viable. A contemporary mathematician would dispel our doubts by producing an axiom system that unambiguously determines the structure we expect projective space to have and then proving its consistency. But before the publication of Pasch's *Lectures on Modern Geometry* (1882), even the most distinguished mathematicians had a

rather poor grasp of axiom systems. In the matter of the viability of projective planes and of projective space, they simply trusted their instinct; or else, they constructed a real number structure and identified it with the projective plane (or space). Though the latter procedure may seem artificial to philosophical readers, it provides the shortest way to understanding Klein's work on non-Euclidean geometries. In presenting a numerical model of the projective plane, we shall try to dispel any appearance of arbitrariness by introducing it through a short motivating discussion instead of presenting it ready-made like a rabbit out of a mathematician's hat.

Let \mathcal{E}^2 denote the Euclidean plane and let x be a Cartesian 2-mapping. Let $\langle x_1, x_2 \rangle$ denote the point $P \in \mathcal{E}^2$ such that $x^1(P) = x_1$, $x^2(P) = x_2$. A straight line m on \mathcal{E}^2 is a set

$$m = \{ \langle x_1, x_2 \rangle \mid u_1x_1 + u_2x_2 + u_3 = 0; u_i \in \mathbf{R}; \\ (u_1, u_2, u_3) \neq (0, 0, u_3) \}$$

We obtain the same line m if we multiply both members of the equation $u_1x_1 + u_2x_2 + u_3 = 0$ by an arbitrary real number $k \neq 0$. A straight line m on \mathcal{E}^2 is determined, therefore, by a set of linearly dependent⁷ elements of \mathbf{R}^3 , $\{ (ku_1, ku_2, ku_3) \mid k \neq 0; (u_1, u_2, u_3) \neq (0, 0, u_3) \}$. Let $(u_1, u_2, u_3), (v_1, v_2, v_3)$ be two linearly independent elements of \mathbf{R}^3 . Then (u_i) and (v_i) represent two different lines m, n on \mathcal{E}^2 . If m, n are not parallel, they meet at a point whose coordinates are the solution of the following system:

$$\begin{aligned} u_1x_1 + u_2x_2 + u_3 &= 0, \\ v_1x_1 + v_2x_2 + v_3 &= 0. \end{aligned} \tag{1}$$

Let us multiply both sides of these equations by a real number $p_3 \neq 0$. If we set $p_3x_1 = p_1$ and $p_3x_2 = p_2$, we obtain the system:

$$\begin{aligned} u_1p_1 + u_2p_2 + u_3p_3 &= 0, \\ v_1p_1 + v_2p_2 + v_3p_3 &= 0. \end{aligned} \tag{2}$$

This system has infinitely many linearly dependent solutions (kp_1, kp_2, kp_3) , with $k \neq 0, p_3 \neq 0$, each one of which determines the same point on \mathcal{E}^2 . The foregoing method furnishes a remarkably symmetric representation of the points and the lines of \mathcal{E}^2 by real number triples. One asymmetry remains however: *one* of the first two terms of the representative triple must be different from zero for lines, while the *third* term must be different from zero for points. Let us now

consider a pair of parallel lines. They are represented by the linearly independent triples (u_1, u_2, u_3) , (v_1, v_2, v_3) if and only if eqns. (1) have no solution, that is, if and only if (u_1, u_2) , (v_1, v_2) are linearly dependent. Suppose $u_i = kv_i$ ($i = 1, 2$). We can now write eqns. (2) as follows:

$$\begin{aligned} u_1p_1 + u_2p_2 + u_3p_3 &= 0, \\ u_1p_1 + u_2p_2 + kv_3p_3 &= 0. \end{aligned} \quad (3)$$

Subtracting the second equation from the first, we obtain

$$(u_3 - kv_3)p_3 = 0. \quad (4)$$

But $u_3 \neq kv_3$, since the triples (u_1, u_2, u_3) and (v_1, v_2, v_3) are supposed to represent different lines. Consequently

$$p_3 = 0. \quad (5)$$

System (3) has indeed a solution but this solution, being of the form $(x, y, 0)$, does not represent a point of \mathcal{E}^2 . This is as it should be, for parallel lines do not meet on \mathcal{E}^2 . Let us now endow each line on \mathcal{E}^2 with an additional 'point' where it 'meets' all the lines that are parallel to it. We know at once how to represent these points, namely, by a solution of a system of type (3), i.e. by a triple of the form $(p_1, p_2, 0) \neq (0, 0, 0)$. Two of these 'points' should determine a 'line', namely, the ideal line of the enriched plane. Can we represent that 'line' by a real number triple? We ought to be able to determine it by solving the following system

$$\begin{aligned} u_1p_1 + u_2p_2 + u_3p_3 &= 0, \\ u_1q_1 + u_2q_2 + u_3q_3 &= 0. \end{aligned} \quad (6)$$

where the triples (p_i) and (q_i) represent two different ideal points, so that $p_3 = q_3 = 0$ and consequently (p_1, p_2) and (q_1, q_2) are linearly independent. This implies that $u_1 = u_2 = 0$. Since u_3 is arbitrary, the ideal line is represented by any triple of the form $(0, 0, u_3)$ with $u_3 \neq 0$. We must exclude the case $u_3 = 0$, because $(0, 0, 0)$ is linearly dependent on every other element of \mathbf{R}^3 , and therefore cannot represent a specific line.

For those to whom the projective plane, as it was introduced in Section 2.3.2, is clearly conceivable, the method sketched above enables a numerical representation of its lines and points. For those to whom, as we assume at the beginning of this section, the notion of the

projective plane presented in Section 2.3.2 is not clear, it may be defined and have a sense bestowed upon it by means of the numerical representation. One proceeds as follows. Let $\hat{\mathbf{R}}^3$ denote $\mathbf{R}^3 - (0, 0, 0)$. Let E denote the relation of linear dependence between pairs of elements of $\hat{\mathbf{R}}^3$. E is an equivalence. Let \mathcal{P}^2 denote the quotient set $\hat{\mathbf{R}}^3/E$. We call \mathcal{P}^2 the *projective plane*. If $(x_i) = (x_1, x_2, x_3)$ belongs to $\hat{\mathbf{R}}^3$, we denote its equivalence class by $[x_i]$. \mathcal{P}^2 is therefore the set $\{[x_i] | (x_i) \in \hat{\mathbf{R}}^3; i = 1, 2, 3\}$. \mathcal{P}^2 is endowed with the strongest topology which makes $(x_i) \rightarrow [x_i]$ a continuous mapping. In this topology, those and only those subsets of \mathcal{P}^2 are open whose inverse image by the said mapping is open in \mathbf{R}^3 . We call $[x_i]$ a *point* of \mathcal{P}^2 ; the triple $(x_i) \in \hat{\mathbf{R}}^3$ provides a set of *homogeneous coordinates* representing the point $[x_i]$. Hereafter, we shall usually denote each point of \mathcal{P}^2 by a set of homogeneous coordinates representing it. Given a triple of real numbers $(u_1, u_2, u_3) \neq (0, 0, 0)$, the set of points in \mathcal{P}^2 denoted by the solutions of the equation

$$\sum_{i=1}^3 u_i x_i = 0 \quad (7)$$

is a *line* in \mathcal{P}^2 . This line can naturally be denoted by the set $(u_i) \in \hat{\mathbf{R}}^3$. Since the solutions of (7) are also solutions of

$$\sum_{i=1}^3 k u_i x_i = 0 \quad (8)$$

for any real number $k \neq 0$, the line in question can be denoted by any member of the equivalence class $[u_i]$. A line (u_i) is *incident on* (or *passes through*) a point (x_i) – which is then said to be incident on or to lie on (u_i) – if and only if $\sum_{i=1}^3 u_i x_i = 0$. Two or more points are *collinear* if they all lie on one line; this line is their *join*. Two or more lines are concurrent if they all pass through one point; this point is their *meet*. It is merely a matter of algebra to prove that any two points in \mathcal{P}^2 have one and only one join and that any two lines in \mathcal{P}^2 have one and only one meet. If plane projective geometry concerns the properties and relations of the points and lines of \mathcal{P}^2 , the proofs of its theorems will consist of equations where different points and lines are denoted by different elements of $\hat{\mathbf{R}}^3$. Now, if we choose, say, the symbols $(p_i), (q_i), (r_i), \dots$ to denote points, while $(u_i), (v_i),$

$(w_i), \dots$ denote lines, the equations in which these symbols occur will still hold if we let $(p_i), (q_i), (r_i), \dots$ denote lines, while $(u_i), (v_i), (w_i), \dots$ denote points. Consequently, any true statement of plane projective geometry gives rise to a 'dual', that is, another true statement obtained from the former by substituting *point* for *line*, *collinear* for *concurrent*, *meet* for *join*, and vice versa, wherever these words occur in the former statement. This is called the *principle of duality*. In our numerical interpretation of projective geometry the principle is trivial. But Gergonne (1771–1859), who formulated it as a general principle in 1825, did not have this interpretation at his disposal. He discovered duality by noticing pairs of complementary or 'dual' theorems, proved under the usual intuitive (or pseudo-intuitive) conception of the projective plane.

Our numerical interpretation of the projective plane shows that projective geometry is at least as consistent as the theory of real numbers. Since every theorem can be stated as a relation between number triples and every proof can be carried out through a sequence of ordinary algebraic calculations, any contradiction arising in projective geometry would show up as a contradiction in elementary algebra. Though this result should remove the doubts expressed at the beginning of this section, we shall now give a fully intuitive representation of the projective plane for the benefit of readers who stand in awe of numbers. Let P be a point in Euclidean space \mathcal{E}^3 and let $\hat{\mathcal{E}}^3$ denote $\mathcal{E}^3 - \{P\}$. We define an equivalence F on $\hat{\mathcal{E}}^3$ as follows: xFy if and only if x and y lie on the same line through P . Consider the quotient set $\hat{\mathcal{E}}^3/F$. The 'points' of $\hat{\mathcal{E}}^3/F$ are the lines through P . Two lines through P define a plane through P , which we shall call their *join*. Two planes through P determine a line through P , which we shall call their *meet*. With these stipulations the pencil of lines and the bundle of planes through P furnish an adequate representation of the projective plane. They can, in fact, be identified with \mathcal{P}^2 , our numerical representation. If x is a Cartesian mapping with its origin at P , we can represent a line m through P by the x -coordinates of any one of the points of $\hat{\mathcal{E}}^3$ that lie on m . In this way, we assign a full equivalence class of homogeneous coordinates in $\hat{\mathcal{R}}^3$ to each meet of $\hat{\mathcal{E}}^3/F$, thereby mapping $\hat{\mathcal{R}}^3/E$ onto $\hat{\mathcal{E}}^3/F$. We do likewise with the joins of $\hat{\mathcal{E}}^3/F$, i.e. the planes through P . Call this mapping f . It is not difficult to see that if A is the join or meet of B and C in $\hat{\mathcal{R}}^3/E$, $f(A)$ is the join or meet of $f(B)$ and $f(C)$ in $\hat{\mathcal{E}}^3/F$. The existence of f shows that $\hat{\mathcal{R}}^3/E$

and \mathcal{E}^3/F are isomorphic, i.e. that they both possess the same projective structure. Our intuitive representation of the projective plane makes an important result immediately obvious. All planes through P have the same status. We cannot select one among them to play the rôle of the ideal line, except by an arbitrary stipulation. Consequently, from a purely projective point of view there is no essential difference between the ideal line and every other line. This is not so clear in the numerical representation because the ideal line has a seemingly peculiar equation (namely $p_3 = 0$). But it could be inferred from the principle of duality: there is no such thing as a privileged line in \mathcal{P}^2 , formed by a distinguished class of 'ideal' points, because there is no such thing as a privileged pencil, formed by a distinguished class of lines.

The numerical representation of the projective plane suggests a generalization which is assumed by Klein in his work on Non-Euclidean geometry. Let C denote the field of complex numbers. If $\hat{C}^3 = C^3 - (0, 0, 0)$ and if E denotes the relation of linear dependence between two elements of \hat{C}^3 , we denote the quotient set \hat{C}^3/E by \mathcal{P}_C^2 . We call this the *complex projective plane*.⁸ Points and lines in \mathcal{P}_C^2 are defined in the same terms as in \mathcal{P}^2 . \mathcal{P}^2 may be regarded as a proper subset of \mathcal{P}_C^2 , formed by the equivalence classes $[p_1, p_2, p_3]$ one of whose representative triples consists exclusively of complex numbers whose imaginary part is zero. We call these points the real points of \mathcal{P}_C^2 . By eliminating from \mathcal{P}^2 the line $u_3 = 0$ we obtain the so-called affine plane, which is simply the Euclidean plane regarded as a proper subset of the projective plane (and deprived, as such, of the Euclidean metric structure). We shall denote the affine plane by \mathcal{E}^2 . For the sake of completeness, we may mention that if $\hat{R}^{n+1} = R^{n+1} - \{0\}$, $\hat{C}^{n+1} = C^{n+1} - \{0\}$ and E denotes linear dependence in one or the other of these sets, $\mathcal{P}^n = \hat{R}^{n+1}/E$ and $\mathcal{P}_C^n = \hat{C}^{n+1}/E$ are called, respectively, the *real* and the *complex n -dimensional projective space*.

2.3.4 Projective Transformations

We shall consider two kinds of mappings defined on \mathcal{P}^2 . A *collineation* is a continuous injective mapping of \mathcal{P}^2 onto itself that matches points with points and lines with lines, preserving incidence relations between lines and points. Let (p_i) , (q_i) denote points while (u_i) , (v_i) denote lines. The general analytic expression of the collineation $(p_i) \mapsto (q_i)$, $(u_i) \mapsto (v_i)$ is

$$\begin{aligned} kq_i &= \sum_{j=1}^3 a_{ij}p_j, \\ ku_i &= \sum_{j=1}^3 a_{ji}v_j, \end{aligned} \quad (|a_{ij}| \neq 0; i = 1, 2, 3; k \neq 0). \quad (1)$$

This mapping preserves incidence between lines and points since

$$\sum_{i=1}^3 v_i q_i = \frac{1}{k} \sum_{i,j=1}^3 v_i a_{ij} p_j = \sum_{j=1}^3 u_j p_j. \quad (2)$$

Let A_{ij} denote the cofactor of a_{ij} in the matrix $[a_{ij}]$. The inverse of (1) is then given by

$$\begin{aligned} k'v_i &= \sum_{j=1}^3 A_{ij}u_j, \\ k'p_i &= \sum_{j=1}^3 A_{ji}q_j, \end{aligned} \quad (i = 1, 2, 3). \quad (3)$$

A *correlation* is a continuous injective mapping which assigns a point to each line and a line to each point in \mathcal{P}^2 , so that collinear points are mapped on concurrent lines, and vice versa. We obtain the general expression of the correlation $(p_i) \mapsto (v_i)$, $(u_i) \mapsto (q_i)$ by simply interchanging (v_i) and (q_i) in (1):

$$\begin{aligned} kv_i &= \sum_{j=1}^3 a_{ij}p_j, \\ ku_i &= \sum_{j=1}^3 a_{ji}q_j, \end{aligned} \quad (|a_{ij}| \neq 0; i = 1, 2, 3; k \neq 0). \quad (4)$$

Let φ be a correlation. If P is a point in \mathcal{P}^2 , $\varphi(P)$ is a line m in \mathcal{P}^2 , $\varphi(m)$ is another point Q . If $\varphi(m) = \varphi(\varphi(P)) = P$, for every P in \mathcal{P}^2 , the correlation is called a *polarity*. A polarity is therefore an involutory correlation, a correlation which is its own inverse. It is easily seen that if φ is a polarity which maps a point P on a line m , $\varphi(\varphi(m)) = \varphi(P) = m$. It can be shown that equations (4) define a polarity if and only if $a_{ij} = a_{ji}$. The image of a point under a polarity is called its *polar*; the image of a line, its *pole*. Two points P, Q are said to be conjugate with respect to a polarity φ if Q lies on $\varphi(P)$, that is, on the polar of P ; since φ preserves incidence and is involutory P must lie on $\varphi(Q)$. Two lines m, n are conjugate with respect to φ if m passes through $\varphi(n)$; n , of course, passes through $\varphi(m)$. Let $\varphi: (p_i) \mapsto (v_i)$ be

given by

$$kv_i = \sum_{j=1}^3 a_{ij}p_j, \quad (|a_{ij}| \neq 0, a_{ij} = a_{ji}, k \neq 0, i = 1, 2, 3). \quad (5)$$

If (p_i) and (q_i) are conjugate points, (q_i) lies on (v_i) ; hence

$$\sum_{i=1}^3 v_i q_i = \sum_{i,j=1}^3 a_{ij} q_i p_j = 0. \quad (6)$$

A similar equation expresses the condition that must be fulfilled by two conjugate lines. A point lying on its own polar and a line passing through its own pole are called self-conjugate. A polarity is called elliptic if it has no self-conjugate points (or lines) or hyperbolic if it has at least one. It can be proved that a hyperbolic polarity has infinitely many different self-conjugate points and lines. Take the polarity given by (5). The condition for a point (p_i) to be self-conjugate follows immediately from (6):

$$\sum_{i,j=1}^3 a_{ij} p_i p_j = 0. \quad (7)$$

This is a quadratic equation whose solutions, if they exist (i.e. if the polarity is hyperbolic), are the points of a conic.^{8a} The tangent to the conic at a point P is the polar of P. The condition for a line (u_i) to be self-conjugate is of course

$$\sum_{i,j=1}^3 a_{ij} u_i u_j = 0. \quad (8)$$

If (7) has solutions, (8) has solutions as well. They are precisely the tangents to the conic defined by (7). The pole of each tangent m is its point of tangency. A conic may be regarded as a set of points, or, dually, as a set of lines, namely, the tangents that envelop it. If we regard it both ways, (7) and (8) represent the same conic which may be said to be its own image under polarity (5). This property of being a locus of self-conjugate points and lines under a fixed (hyperbolic) polarity is normally used to *define* conics in (real) plane projective geometry. The definition does not depend on the numerical interpretation we have made the basis of our discussion.⁹ If, in eqns. (7) and (8), the matrix of the coefficients a_{ij} happens to be singular ($|a_{ij}| = 0$), of rank 2 or rank 1, those equations are said to determine a

degenerate conic; the points of such a conic lie on two lines or on a single line, respectively, according to the rank of the matrix.

All these concepts can be defined analogously on the complex projective plane \mathcal{P}_C^2 . Polarities of the form (5) are called *projective*. Since every quadratic equation of the form (7) has complex solutions, every projective polarity in \mathcal{P}_C^2 defines a conic which is the locus of its self-conjugate points. A projective polarity is called hyperbolic if the conic defined by it includes real points and elliptic if all its points are imaginary.¹⁰ Let \bar{a} denote the complex conjugate of $a \in \mathbb{C}$. An injective, incidence-preserving, continuous involutory mapping $(u_i) \mapsto (\bar{p}_i)$, $(p_i) \mapsto (\bar{u}_i)$ on \mathcal{P}_C^2 is called an *anti-projective* polarity. Its general expression is

$$ku_i = \sum_{j=1}^3 a_{ij} \bar{p}_j, \quad (|a_{ij}| \neq 0, a_{ji} = \bar{a}_{ij}, k \neq 0). \quad (9)$$

Its self-conjugate points form an *anti-conic* given by

$$\sum_{i,j=1}^3 a_{ij} p_i \bar{p}_j = 0. \quad (10)$$

Two further remarks concerning \mathcal{P}_C^2 will be useful later. Firstly, a system formed by a quadratic equation $\sum_{i,j=1}^3 a_{ij} p_i p_j = 0$ and a linear equation $\sum_{i=1}^3 u_i p_i = 0$ regularly has two solutions in $\hat{\mathbb{C}}^3$. This means that every conic in \mathcal{P}_C^2 regularly meets every straight line at two points.¹¹ The second remark concerns a particular kind of conics we shall call *circles*, because of the formal analogy between their characteristic equation and that of an ordinary Euclidean circle.¹² They are the conics defined by polarities whose matrix $[a_{ij}]$ has the form

$$\begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & a^2 + b^2 + c^2 \end{bmatrix}.$$

The equation of a circle is given therefore by

$$(p_1 - ap_3)^2 + (p_2 - bp_3)^2 + c^2 p_3^2 = 0. \quad (11)$$

Where does a circle meet the ideal line? According to our first remark, at two points. We can calculate their coordinates by substituting in (11) the value $p_3 = 0$ which characterizes all ideal points. We obtain the equations

$$p_1^2 + p_2^2 = 0, \quad p_3 = 0. \quad (12)$$

Two points of \mathcal{P}_C^2 satisfy these equations, namely $(1, i, 0)$ and $(1, -i, 0)$. They are both imaginary. Clearly, they do not depend on the parameters a, b, c which define a given circle. Therefore every circle meets the ideal line at these two points which are called the *circular points* of \mathcal{P}_C^2 .

2.3.5 Cross-ratio

We call the set of all collineations and correlations defined on \mathcal{P}^2 the *projective transformations* of the (real) plane. (Real) plane projective geometry will determine the properties and relations which are preserved by (real) projective transformations. Some of them were specified in the very definition of collineations, namely incidence between points and lines, collinearity of points, concurrence of lines. Correlations, on the other hand, map concurrent lines on collinear points and collinear points on concurrent lines. If a line m passes through a point P and if φ is a correlation, point $\varphi(m)$ will lie on line $\varphi(P)$. Consider now a real-valued function f defined on $(\mathcal{P}^2)^n$. We say that f is an *n-point projective invariant* if, given any projective transformation φ , $f(Q_1, \dots, Q_n) = f(\varphi(Q_1), \dots, \varphi(Q_n))$, for every set of n points (or lines) $\{Q_1, \dots, Q_n\}$ in \mathcal{P}^2 . Sophus Lie showed that there are no such invariants for $n \leq 3$.¹³ This means, in particular, that given a collineation φ and a function $f: \mathcal{P}^2 \times \mathcal{P}^2 \rightarrow \mathbb{R}$, we cannot have $f(\varphi(P), \varphi(Q)) = f(P, Q)$ for every pair of points P, Q in \mathcal{P}^2 . It would seem, therefore, that the concept of distance can have no place at all in projective geometry. We shall see, however, that it can be introduced in a roundabout way.

Lie shows that every 4-point (4-line) projective invariant is reducible to the so-called *cross-ratio* between four collinear points (four concurrent lines).¹⁴ If (p_i) and (q_i) are two different points of \mathcal{P}^2 , every point (x_i) on their join will satisfy the equation

$$\begin{vmatrix} x_1 & p_1 & q_1 \\ x_2 & p_2 & q_2 \\ x_3 & p_3 & q_3 \end{vmatrix} = 0. \quad (1)$$

It is easily seen that every solution of (1) has the form

$$x_i = kp_i + mq_i \quad ((k, m) \neq (0, 0), i = 1, 2, 3). \quad (2)$$

$(kp_i + mq_i)$ and $(k'p_i + m'q_i)$ denote the same point if and only if $\begin{vmatrix} k & k' \\ m & m' \end{vmatrix} = 0$. Let P_1, P_2, P_3, P_4 be four collinear points such that

$P_1 \neq P_4$ and $P_2 \neq P_3$. Let (p_i) , (q_i) denote two points of the line on which the points P_r lie. Each point P_r ($r = 1, 2, 3, 4$) is then denoted by $(k_r p_i + m_r q_i)$ for some pair of real numbers (k_r, m_r) , not both zero. The cross-ratio of P_1, P_2, P_3, P_4 (in that order) is then defined by the following equation:

$$(P_1, P_2; P_3, P_4) = \frac{\begin{vmatrix} k_1 & k_3 \\ m_1 & m_3 \end{vmatrix} \cdot \begin{vmatrix} k_2 & k_4 \\ m_2 & m_4 \end{vmatrix}}{\begin{vmatrix} k_2 & k_3 \\ m_2 & m_3 \end{vmatrix} \cdot \begin{vmatrix} k_1 & k_4 \\ m_1 & m_4 \end{vmatrix}}. \quad (3)$$

If P_3 is (p_i) and P_4 is (q_i) , $(k_3, m_3) = (1, 0)$ and $(k_4, m_4) = (0, 1)$. In that case

$$(P_1, P_2; P_3, P_4) = \frac{m_1 k_2}{k_1 m_2}. \quad (4)$$

Since we are always free to make that assumption, it is clear that, given three arbitrary collinear points A, B, C ,

$$(A, A; B, C) = 1. \quad (5)$$

The cross-ratio of four collinear points depends not on the choice of the homogeneous coordinates that represent them, but only, as we gather from eqn. (5), on the parameters which determine the relative positions of two of the points with respect to the other two, on the line to which all four belong. The cross-ratio of four concurrent lines is defined analogously. It is a matter of mere calculation to show that the cross-ratio is preserved by projective transformations, i.e. that, if φ is a projective transformation, then, for every four collinear points or concurrent lines M_1, M_2, M_3, M_4 ,

$$(M_1, M_2; M_3, M_4) = (\varphi(M_1), \varphi(M_2); \varphi(M_3), \varphi(M_4)). \quad (6)$$

If $(M_1, M_2; M_3, M_4) = -1$ the four points or lines are said to be harmonic; M_4 is called the fourth harmonic to M_1, M_2, M_3 .

2.3.6 Projective Metrics

We are now ready to present Klein's interpretation of plane non-Euclidean geometries. We shall see that it rests upon the introduction of a metric, or rather, of a variety of metrics, in the complex projective plane $\mathcal{P}^2_{\mathbb{C}}$. All that we have said in Section 2.3.5 concerning projective transformations and the cross-ratio applies, *mutatis*

mutandis, to \mathcal{P}_C^2 . Let ζ be a conic in \mathcal{P}_C^2 . Let K_ζ denote the set of all collineations which map ζ onto itself. The join of two points P, Q meets ζ at two points which we shall denote by $(PQ/\zeta)_1$ and $(PQ/\zeta)_2$. If $\varphi \in K_\zeta$, each of these points is mapped on one of the points

$$(\varphi(P)\varphi(Q)/\zeta)_i = \varphi((PQ/\zeta)_i), \quad (i = 1, 2). \quad (1)$$

Since the cross-ratio is a projective invariant, it follows immediately from eqns. (3) and (6) of Section 2.3.5 that

$$\begin{aligned} (P, Q; (PQ/\zeta)_1, (PQ/\zeta)_2) \\ = (\varphi(P), \varphi(Q); (\varphi(P)\varphi(Q)/\zeta)_1, (\varphi(P)\varphi(Q)/\zeta)_2). \end{aligned} \quad (2)$$

Let f_ζ denote the complex-valued function $(P, Q) \rightarrow (P, Q; (PQ/\zeta)_1, (PQ/\zeta)_2)$, defined on $\mathcal{P}_C^2 \times \mathcal{P}_C^2$; ¹⁵ f_ζ is preserved by every collineation of K_ζ , since (2) implies that, for every $\varphi \in K_\zeta$

$$f_\zeta(P, Q) = f_\zeta(\varphi(P), \varphi(Q)). \quad (3)$$

We now define a function d_ζ on point-pairs of the complex projective plane:

$$d_\zeta(P, Q) = c \cdot \log f_\zeta(P, Q) \quad (4)$$

(where c is an arbitrary non-zero constant and $\log x$ denotes the principal value of the natural logarithm of x). The function d_ζ has some properties that make it a good choice for a (signed) distance function on \mathcal{P}_C^2 . In the first place, $d_\zeta(P, P) = 0$ for every point P not on ζ . (See eqn. (5) of Section 2.3.5.) In the second place, if P_1, P_2, P_3 are collinear points not on ζ ,¹⁶

$$d_\zeta(P_1, P_2) + d_\zeta(P_2, P_3) = d_\zeta(P_1, P_3). \quad (5)$$

In the third place, if P and Q are different points not on ζ ,

$$d_\zeta(P, Q) = -d_\zeta(Q, P). \quad (6)$$

It is true that d_ζ is undefined on a point-pair if one (or both) of its members lies on ζ . But we can make sense of the statement that if Q lies on ζ then $d_\zeta(P, Q)$ is infinite for every point P not on ζ . Suppose that $Q = [kp_i + mq_i]$ lies on ζ and that $(Q_j) = ([kp_j + m_jq_j])$ ($i = 1, 2, 3$; $j = 1, 2, \dots$) is a sequence of points not on ζ (but all on the same line through Q) such that $(|k - k_j|)$ and $(|m - m_j|)$ are null sequences. Then, if P is a point not on ζ lying on QQ_1 , $|d_\zeta(PQ_j)|$ increases with j beyond all bounds. Therefore, if we persist in regarding d_ζ as a distance

function we may say that the points on ζ are infinitely distant from the remaining points of \mathcal{P}_C^2 . Still, d_ζ has a property that is rather unusual for a distance function: d_ζ is complex-valued and, whatever the value assigned to the arbitrary constant c , there will be, for every choice of ζ , point-pairs (P, Q) such that $d_\zeta(P, Q)$ has a non-zero imaginary part. One ought not to dispute about names and every mathematician should feel free to call a complex-valued function like d_ζ a 'distance function' on \mathcal{P}_C^2 . But the 'geometry' thereby defined is not what is known as a *metric* geometry in contemporary mathematics.¹⁷ However, as we shall see, d_ζ when restricted to a well-chosen region of \mathcal{P}_C^2 does define a real-valued metric function. This is the substance of Klein's discovery.

The development leading to the definition of d_ζ can be dualized by substituting any pair of lines m, n for the points P, Q. Then $(mn/\zeta)_1$ and $(mn/\zeta)_2$ will denote, of course, the two tangents to the conic ζ that pass through the meet of m and n . As a function on line-pairs, d_ζ seems to be a good choice for an angle-function, i.e. a function whose value measures the size of the angle formed by its two arguments. The choice is strongly recommended on account of the following result due to Laguerre.¹⁸ Let m, n be two lines on the affine plane $\mathcal{E}^2 \subset \mathcal{P}_C^2$, which meet at a point P in \mathcal{E}^2 . We shall denote by \bar{m} and \bar{n} the extension of m and n to \mathcal{P}_C^2 , i.e. the sets of points in \mathcal{P}_C^2 that satisfy the equations characteristic of m and n . There are two lines r, r' that join P to the two circular points of \mathcal{P}_C^2 (r, r' have each only one real point, namely P). As Laguerre showed, the ordinary Euclidean value of the angle made at P by m and n is equal to $1/2i$ times the natural logarithm of the cross-ratio $(\bar{m}, \bar{n}; r, r')$. The circular points can be regarded as a degenerate line conic.¹⁹ If we let ζ denote this conic and if we take $c = 1/2i$ our function d_ζ as defined on line-pairs measures the size of Euclidean angles. This interpretation of Laguerre's result was given by Cayley in 1859.²⁰ By duality he obtained a distance function defined on point-pairs of the affine plane. Remarkably enough, this distance function is none other than the ordinary Euclidean metric function. Cayley's discovery linked angle size to segment length – a welcome achievement at a time when projective geometry was regarded as a natural extension of ordinary Euclidean geometry (and not as something utterly different from it, as we regard it here). Indeed angles are the duals of segments, so that the measure of the latter should be the dual of the measure of the

former – a *prima facie* paradoxical requirement, given the notorious differences between the two kinds of measure.²¹

Cayley defined d_ζ quite generally, relatively to an arbitrary conic ζ , which he called the Absolute.²² He writes: “The metrical properties of a figure are not the properties of a figure considered *per se*, apart from everything else, but its properties when considered in connection with another figure, *viz.* the conic termed the Absolute”.²³ Cayley considers two cases. When the Absolute is an ordinary (imaginary) conic, we obtain the metrical properties characteristic of spherical geometry; when the Absolute degenerates into the pair of circular points at infinity, we obtain the metrical properties of ordinary plane geometry. However, he disregards what seems to be the most natural case, *viz.*, when the Absolute is an ordinary real conic, such as an ordinary circle.²⁴ It was Klein who first considered this case and pointed out its relation to BL-geometry. Klein showed that d_ζ , judiciously restricted to a subset of \mathcal{P}_ζ^2 in accordance with the choice of ζ , constitutes a metric function on the point-pairs and line-pairs (i.e. on the segments and angles) comprised in that subset. Klein considered three cases:²⁵

(i) ζ is a real conic. Let I_ζ denote its interior, i.e. the set of real points from which no real tangent to ζ can be drawn. d_ζ restricted to I_ζ is a metric function. If $\varphi \in K_\zeta$ (i.e. if φ is a collineation that maps ζ onto itself), $\varphi|I_\zeta$ preserves $d_\zeta|I_\zeta$, and is therefore an isometry. The metric geometry thus defined on I_ζ Klein calls *hyperbolic geometry*. I_ζ with this metric structure can be mapped isometrically onto the BL plane. Consequently, hyperbolic geometry is essentially identical with BL geometry.

(ii) ζ is a purely imaginary conic. d_ζ restricted to the real projective plane \mathcal{P}^2 is also a metric function. Klein calls the metric geometry thus obtained *elliptic geometry*. In it, as in spherical geometry, triangles have an excess, but two straight lines meet at one and only one point. If $\varphi \in K_\zeta$, $\varphi|\mathcal{P}^2$ is an isometry. Elliptic geometry satisfies Saccheri’s hypothesis of the obtuse angle. This does not conflict with Saccheri’s refutation of that hypothesis, because straight lines in elliptic geometry possess the neighbourhood structure of real projective lines, so that their points are ordered cyclically, not linearly – a possibility which was of course excluded by the Euclidean premises of Saccheri’s argument.

(iii) ζ is a degenerate conic. There are five different kinds of

degenerate conics on \mathcal{P}_C^2 ,²⁶ but Klein (1871) considers only one of them, *viz.* ζ regarded as a locus of points consists of the ideal line taken twice, while as an envelope of lines it consists of the two imaginary pencils through the two circular points. In this case d_ζ restricted to the affine plane defines the ordinary Euclidean metric. Klein calls this geometry *parabolic*. A special difficulty arises in this case in connection with the definition of d_ζ as a distance function on point-pairs. The join of two points P, Q in \mathcal{E}^2 meets the degenerate conic ζ at just one point taken twice. In other words $(PQ/\zeta)_1$ is identical with $(PQ/\zeta)_2$, so that $f_\zeta(P, Q) = 1$ and $d_\zeta(P, Q) = 0$. Klein avoids this difficulty by means of a limit operation in the course of which he approaches the parabolic case from either the elliptic or the hyperbolic cases.²⁷ If $\varphi \in K_\zeta$, $\varphi|\mathcal{E}^2$ is not always an isometry but it belongs to what Klein calls the *principal group of transformations* of Euclidean space, formed by the Euclidean *isometries* (translations, rotations, reflections) and *similarities* (bijective mappings of space onto itself which preserve shape but multiply areas by a constant factor). In his posthumous *Lectures on Non-Euclidean Geometry* (1926) Klein briefly examines the other four degenerate cases. He does not pay much attention to the resulting geometries because angle-measure in them is not periodic – a fact that, in Klein’s opinion, makes them inapplicable to the real world, since “experience shows us that a finite sequence of rotations [about the axis of a bundle of planes] finally takes us back to our starting point”.²⁸

Klein’s results are at first sight quite impressive. The difference between Euclidean geometry and the two classical non-Euclidean geometries (BL or acute-angle geometry and obtuse-angle geometry) seems to depend merely on the choice of a particular kind of conic. Now, from a purely projective point of view all conics are equivalent, since they can be carried onto one another by projective transformations. Thus the difference between these geometries would appear to be inessential. The appearance is deceiving, however, for the restricted domains of hyperbolic, elliptic and parabolic geometry within \mathcal{P}_C^2 are not projectively equivalent. Thus, if ζ is a real conic and ζ' a purely imaginary one, and if φ is a collineation which maps ζ onto ζ' , $\varphi(I_\zeta)$ must include some purely imaginary points. In other words the φ -image of I_ζ , the hyperbolic plane, includes points not comprised in \mathcal{P}^2 , the elliptic plane. Analogous results occur with respect to \mathcal{E}^2 , the parabolic plane. In his *Lectures* Klein sometimes

uses the terms “hyperbolic” and “elliptic” as names for the geometries defined by d_ζ on the whole of $\mathcal{P}_\mathbb{C}^2$ (strictly speaking, on $\mathcal{P}_\mathbb{C}^2 - \zeta$) when ζ is a real conic or a purely imaginary one.²⁹ Let us call these geometries e -hyperbolic and e -elliptic (e for *extended*). We may add e -parabolic geometry. These three geometries are indeed projectively equivalent. But they are not metric geometries in the ordinary sense of the expression because d_ζ is not a real-valued function on $(\mathcal{P}_\mathbb{C}^2 - \zeta)^2$. And, of course, e -hyperbolic geometry is not identical with BL geometry nor is e -parabolic geometry identical with Euclidean geometry.

Klein did not limit his consideration to the two-dimensional case, as we have, but defined projective metrics for the three-dimensional case too. As we know, the complex projective space $\mathcal{P}_\mathbb{C}^3$ can be identified with \hat{C}^4/E , where \hat{C}^4 is $C^4 - \{(0, 0, 0, 0)\}$ and E denotes the relation of linear dependence in \hat{C}^4 . Our discussion applies without much change to $\mathcal{P}_\mathbb{C}^3$ if we take ζ to be a quadric surface. If ζ is a real quadric and I_ζ is its interior (i.e. the set of real points from which no real tangent to ζ can be drawn), $d_\zeta|_{I_\zeta}$ defines the hyperbolic metric on I_ζ and we have an equivalent of BL-space geometry. If ζ is a purely imaginary quadric, $d_\zeta|_{\mathcal{P}^3}$ defines the elliptic metric on the real projective space. Finally, we obtain the parabolic (or Euclidean) metric on the affine space \mathcal{E}^3 (i.e. \mathcal{P}^3 minus the ideal plane $x_4 = 0$) by restricting d_ζ to \mathcal{E}^3 when ζ is the degenerate quadric formed by the imaginary circle where every sphere meets the ideal plane $x_4 = 0$. (Spheres are defined analogously with circles; see p.123.) Parabolic geometry occurs in only one of the possible degenerate cases which in three dimensions number fifteen.

Cayley accepts the numerical interpretation of $\mathcal{P}_\mathbb{C}^n$ without reservation. This was, indeed, the only reasonable attitude before the advent of axiomatics.³⁰ Klein, on the other hand, believes that the numerical manifold must be somehow grounded on intuition. The snag is that the classical intuitive – or pseudointuitive – construction of projective space depends essentially on Euclid’s Postulate 5. Thus, our method of projecting a line m on a line n from a point O on plane mn but outside both m and n (pp.111f.) presupposes that there is a unique line m' through O which does not meet m and a unique line n' through O which does not meet n . But, if projective geometry rests on Postulate 5, Klein’s projective foundation of non-Euclidean geometries can hardly be consistent. We shall see in Section 2.3.9

how Klein finally succeeded in establishing projective geometry on what he judged was an intuitive basis, without resorting to Postulate 5.

Before studying the two- and three-dimensional cases, Klein considers linear transformations on a (complex) projective line. They are of two kinds: those that leave one point invariant (parabolic transformations) and those that leave two points fixed. The latter fall into two subclasses: hyperbolic transformations, in which the two fixed points are real, and elliptic transformations in which the fixed points are conjugate imaginary. The reader can satisfy himself that if ζ is a conic (or a quadric) like those we have considered, a collineation which maps ζ onto itself will induce an elliptic, hyperbolic or parabolic transformation in a fixed line if ζ is, respectively, imaginary, real or equal to the two circular points (or to the imaginary circle where every sphere meets the ideal plane). This terminology, due to Steiner, is thus clearly the source of Klein's nomenclature. I have not been able to verify the reason for Steiner's choice of words, but it is easily guessed. Every linear transformation that maps a line onto itself can be associated with a characteristic quadratic equation. The transformation is elliptic, parabolic or hyperbolic, in the above sense, if the discriminant of this equation is less than, equal to or greater than 0, i.e. if the conic represented by this equation is an ellipse, a parabola or a hyperbola.

Trained mathematicians who read Klein cannot have failed to appreciate the point of his use of *parabolic* as the new, scientifically grounded name of Euclidean geometry. The parabola is the exceptional conic, while the ellipse and the hyperbola must be viewed as typical. Furthermore, Klein recalls that *parabolic mappings* of a line onto itself are the special case, as opposed to the general one with two real or two imaginary fixed points. "Correspondingly", he adds, "there will be just two essentially different kinds of projective metrics on fundamental figures of level one [i.e. lines and flat pencils]: a general one which uses transformations of the first kind [i.e. non-parabolic], and a special one which uses transformations of the second kind [i.e. parabolic]. The ordinary metric on a flat pencil [i.e. the familiar system for measuring the size of angles] belongs to the first kind because in a rotation of the pencil about its centre two distinct lines remain fixed, namely, the lines that go through the infinitely distant imaginary circular points. On the other hand, the

ordinary metric on the straight line [i.e. the familiar system for measuring the length of segments] belongs to the second kind because a displacement of the straight line along itself, under the assumptions of ordinary parabolic geometry, leaves just one point unchanged, namely, the infinitely distant point."³¹ This difference between the two fundamental metrical systems of geometry disappears in the elliptic and the hyperbolic cases. This is as should be expected, if the latter indeed are more general and consequently more natural.

2.3.7 Models

Klein's work is often linked to the construction of so-called Euclidean models of non-Euclidean geometry. Thus, Borsuk and Szmielew, in their well-known *Foundations of Geometry*, describe a Beltrami-Klein model of BL-geometry.³² We shall presently see to what extent such a characterization of Klein's work is justified. Strictly speaking, a model can be conceived only in relation to an abstract axiomatic theory. If you are given a set of sentences S which contain undefined terms t_1, \dots, t_n , you can look for a model of S , that is, a domain of entities where, through an arbitrary but consistent interpretation of terms t_1, \dots, t_n , the sentences of S come true. In this strict sense, we cannot ascribe a model-building intention to Klein who, in 1871, did not have the notion of an abstract axiom system. But we also speak of models in a looser sense whenever a structured collection of objects is seen to satisfy a set of mathematical statements, given a suitable, though usually unfamiliar, reading of its key words. We thus say that the pencil of straight lines through a point P in space provides a model of the projective plane if we accept the following semantic equivalences: 'a point' = a line through P ; 'a line' = a plane through P ; 'point Q is the meet of lines m and m' ' = line Q is the intersection of planes m and m' ; 'line m is the join of points Q and Q' ' = plane m is spanned by lines Q and Q' (p.119). In this looser sense, Klein's theory does indeed supply models for Euclidean and non-Euclidean geometry, but his models are projective and therefore not Euclidean (because, as we have repeatedly observed, projective space is not Euclidean space, Postulate 5 is false in projective geometry, etc.). Thus parabolic plane geometry on the affine plane $\mathcal{E}^2 \subset \mathcal{P}_C^2$ provides a somewhat peculiar model of ordinary Euclidean plane geometry: points are points and straight lines are straight lines, but distances between pairs of points and angles between pairs of

lines are defined with respect to a fixed entity *located outside the affine plane itself*. On the other hand, hyperbolic plane geometry on the interior of an ellipse may be viewed as a Euclidean model of BL plane geometry if we no longer consider its domain of definition to be a subset of \mathcal{P}_C^2 and regard it as a region of the Euclidean plane. Thus, we may define hyperbolic geometry in the interior of a circle (O, r) with centre O and radius r . A BL point is any ordinary point inside this circle; a BL line is any chord (not including the points where it meets the circumference of the circle). Let P be a BL point and m a BL line not through P meeting the circumference of (O, r) at A and B . There are two parallels to m through P , namely the two chords that join P to A and to B (Fig. 13). These parallels divide the chords through P into two groups: those that meet m and those that do not meet m (*scil.* those that do not meet the chord m in the interior of (O, r) .) In order to complete the model we must introduce projective concepts. If Q and R are two points on m (inside (O, r)), the (undirected) distance between Q and R is taken to be equal to $\frac{1}{2}|\log(Q, R; A, B)|$.³³ This value is preserved by all linear transformations (of the entire projective plane) that map circle (O, r) onto itself. The restrictions of these transformations to the interior of (O, r) play the role of BL isometries (motions and reflections). This is, in essence, the “Beltrami–Klein” model given by Borsuk and Szmielew. I leave it to the reader to decide whether it is a genuine Euclidean model.

This model was found by Eugenio Beltrami (1835–1900) some time before the publication of Klein’s paper. In his “Saggio di interpretazione della geometria non euclidea” (1868), Beltrami sets out to find a Euclidean realization of BL plane geometry and discovers it in a surface of negative curvature. The flat model we have just described is only used as an aid in Beltrami’s investigations. Beltrami is aware that the new geometrical conceptions are bound to bring about deep changes throughout classical geometry. But he is persuaded that the introduction of new concepts in mathematics cannot

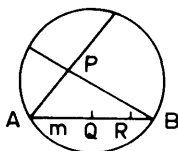


Fig. 13.

upset acquired truths; it can only modify their place in the system or their logical foundations and thereby increase or decrease their value and utility. With this understanding, Beltrami has tried to justify to himself (“dar ragione a noi stessi”) the results of Lobachevsky’s theory. Following a method he believes to be “in agreement with the best traditions of scientific research”, he has attempted “to find a real substrate for this theory before admitting the need for a new order of entities and concepts to support it”.³⁴ To Beltrami’s mind, a “real substrate” is perforce a Euclidean model. He thinks he has succeeded in his attempt as far as BL plane geometry is concerned, but he believes it impossible to do likewise in the case of BL space geometry.³⁵

Beltrami reasons thus: A “real substrate” for the BL plane must be found in a curved surface in Euclidean space, since a Euclidean plane can provide a model only of itself, unless we tamper with the ordinary meaning of distance, and this he seems unwilling to do. It must be a surface of constant G-curvature, for only on such surfaces can we apply the “fundamental criterion of proof of elementary geometry”, namely, the superposability of congruent figures (*la sovrapponibilità delle figure eguali*). The most essential ingredient of a geometric construction is the straight line. Its analogue on a surface of constant curvature is the geodetic arc. The analogy breaks down on surfaces of constant positive curvature, for there exist on them point-pairs which do not determine a unique geodetic arc. How about surfaces of negative curvature, or “pseudospheres” as Beltrami calls them? To prove that every pair of points on a pseudosphere is joined by one and only one geodetic arc, Beltrami sets up a special chart with coordinate functions u, v . Relative to this chart, the element of length on a pseudosphere with constant curvature equal to $-1/R^2$ is given by

$$ds^2 = R^2 \frac{(a^2 - v^2) du^2 + 2uv du dv + (a^2 - u^2) dv^2}{(a^2 - u^2 - v^2)^2}. \quad (1)$$

The main advantage of this chart is that every linear equation in u and v represents a geodetic line and every geodetic line is represented by a linear equation in u and v . In particular, the lines $u = \text{constant}$ and $v = \text{constant}$ are geodetic. The angle θ formed by the lines $u = \text{constant}$ and $v = \text{constant}$ at (u, v) is given by

$$\begin{aligned} \cos \theta &= \frac{uv}{((a^2 - u^2)(a^2 - v^2))^{1/2}}, \\ \sin \theta &= \frac{a(a^2 - u^2 - v^2)^{1/2}}{((a^2 - u^2)(a^2 - v^2))^{1/2}}. \end{aligned} \quad (2)$$

Consequently, if either $u = 0$ or $v = 0$, $\theta = \pi/2$, so that all the lines $u = \text{constant}$ are orthogonal to $v = 0$ and all the lines $v = \text{constant}$ are orthogonal to $u = 0$. The geodetic lines $u = 0$, $v = 0$ are called fundamental. Formulae (2) show that the admissible values of u , v are limited by the condition

$$u^2 + v^2 \leq a^2. \quad (3)$$

Following the procedure sketched on pp.81f., we can represent the relevant region of the pseudosphere on a plane. Just let x be a Cartesian 2-mapping and take the point $(x_1^{-1}(u), x_2^{-1}(v))$ as the representative of the point with coordinates (u, v) . The region of the pseudosphere covered by our chart is then represented by the interior of a circle with radius a , whose centre lies at the origin of the Cartesian 2-mapping x . Beltrami calls this circle "the limit circle" (*il cerchio limite*). The geodetic lines of the pseudosphere are represented by the chords of the limit circle. In particular, the geodetic lines $u = \text{constant}$, $v = \text{constant}$ are represented by chords parallel to the coordinate axes $x_1 = 0$, $x_2 = 0$. The interior of the limit circle is, of course, none other than the Beltrami-Klein model of the BL plane we met above. Beltrami only uses it to prove that a geodetic line on the pseudosphere is uniquely determined by two of its points. In the rest of his paper, he proves in detail that Lobachevsky's geometry is satisfied on the pseudosphere if we identify BL straights with geodetic lines. Any pair of geodetic lines can be chosen as fundamental. The BL distance between two points is equal to the ordinary Euclidean length of the geodetic arc that joins them. Beltrami does not mention one important fact however, namely that his pseudosphere possesses singularities on a part outside the region onto which the BL plane can be mapped isometrically. For a pseudosphere (Fig. 14) is a surface of revolution generated by a tractrix

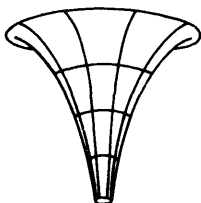


Fig. 14.

which is a curve with a cusp.³⁶ The singularities of the pseudosphere are on the circle described by the cusp. We may ask if there exists a surface in Euclidean space with no such singularities, onto which the BL plane could be mapped isometrically. The question was answered *negatively* by David Hilbert in 1901.³⁷ In his paper, Beltrami suggests, but does not prove, another very important negative conclusion: no *isometric* model of BL 3-space can be constructed in Euclidean 3-space. The auxiliary representation of the BL plane as the interior of a Euclidean circle can, of course, be generalized to any number of dimensions. BL 3-space can thus be mapped homeomorphically onto the interior of a Euclidean sphere.

As an immediate consequence of Beltrami's researches, we conclude that the interior of the limit circle is indeed, as we have stated, a model of the BL plane, with its chords representing BL straights. This model can be used in constructing two more flat models of the BL plane, which were discovered by Henri Poincaré. We shall call them the Poincaré disk and the Poincaré half-plane.³⁸ We obtain them as follows. Consider a (Euclidean) sphere with its centre at point $(0, 0, 0)$ and its north pole at point $(0, 0, 1)$. Let the Beltrami-Klein model be given on the equatorial plane of this sphere, the equator being the limit circle. We project the equatorial plane perpendicularly onto the southern hemisphere: the limit circle goes onto itself and the chords go over onto half-circles which are normal sections of the southern hemisphere. These half-circles now represent the BL straights. Let us now map the southern hemisphere stereographically from the north pole into the tangent plane through the south pole.³⁹ We thus obtain the Poincaré disk, which is a circle whose circumference lies on the image of the equator (Fig. 15). The interior of the Poincaré disk represents the entire BL plane. Since the stereographic projection preserves circles and angles, BL lines are represented by circular arcs orthogonal to the circumference of the Poincaré disk. The Poincaré half-plane is obtained by a slightly different procedure, mapping the southern hemisphere stereographically from the point $(0, -1, 0)$ into the tangent plane through the point $(0, 1, 0)$. The equator goes over onto a straight line which we call the horizon. The southern hemisphere is mapped onto one of the two half-planes determined by the horizon. This is the Poincaré half-plane. BL straights are represented on it by the semicircles and the straight rays orthogonal to the horizon. The straight rays are the

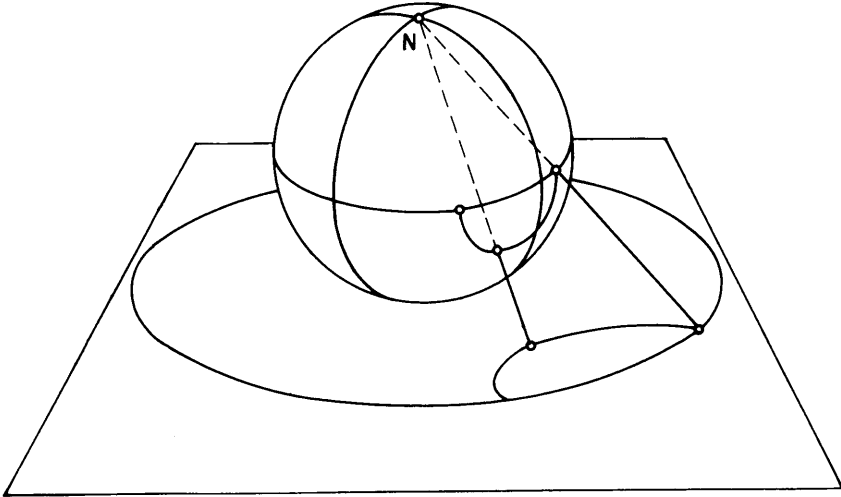


Fig. 15.

images of the semicircles orthogonal to the equatorial plane that pass through the point $(0, -1, 0)$. The distance between two points P and P' on the Poincaré disk or on the Poincaré half-plane can be calculated as follows. Let (PP') denote the circle through P and P' whose centre lies on the circumference of the Poincaré disk or on the horizon of the Poincaré half-plane; let (PP') meet that circumference or horizon at Q and Q' ; then, if $(P, P'; Q, Q')$ denotes the cross-ratio of the radii of circle (PP') which pass respectively through P, P', Q and Q' , the distance between P and P' is equal to $\frac{1}{2} \log (P, P'; Q, Q')$.

2.3.8 Transformation Groups and Klein's Erlangen Programme

Our exposition of Klein's theory was based mainly on his first paper entitled "On the so-called non-Euclidean geometry" (1871). In this and the next section, we shall deal with some additional points brought up in the second paper he published under that title (Klein, 1873). It is divided into two unconnected parts.⁴⁰ The main purpose of the first part is to prove that his projective metric geometries (elliptic, parabolic, or hyperbolic) are the same thing as Riemann's geometries on a manifold of constant curvature (greater than, equal to, or less than 0). In order to show this, Klein states what he understands by an n -dimensional manifold:

If n variables x_1, x_2, \dots, x_n are given, the infinity to the n th value-systems we obtain if we let the variables x independently take the real values from $-\infty$ to $+\infty$, constitute what we shall call, in agreement with usual terminology, a manifold of n dimensions. Each particular value-system (x_1, \dots, x_n) is called an *element* of the manifold.⁴¹

It is not clear whether "the real values from $-\infty$ to $+\infty$ " are just the values between these two extremes, i.e. all the real numbers, or include $-\infty$ and $+\infty$. If they exclude the latter, an n -dimensional manifold in Klein's sense is simply \mathbf{R}^n . Now \mathbf{R}^n is not the same as an n -dimensional manifold in Riemann's sense (pp.86ff.), but if we endow it with the usual differentiable structure, it is diffeomorphic to any 'coordinate patch' (the domain of a chart) of such a manifold. On the other hand, if Klein's variables may take the values $-\infty$ and $+\infty$, an n -dimensional manifold in his sense is a very peculiar entity whose topology would require some further specification. In the light of Klein's usage in the paper we are discussing, I conclude that the truth lies somewhere between the two alternatives: a manifold composed of "real"-valued n -tuples turns out to be identical with \mathcal{P}^n . As we know, this is not homeomorphic to \mathbf{R}^n , let alone to any arbitrary manifold in Riemann's sense. But it is not the same as the set $\{(x_1, \dots, x_n) | -\infty \leq x_i \leq +\infty; 1 \leq i \leq n\}$. Klein adds that in the course of his arguments he will let the variables x_1, \dots, x_n take arbitrary complex values as well. This implies, in my opinion, that "an n -dimensional manifold" in Klein's paper (1873) is but another name for the complex n -dimensional projective space $\mathcal{P}_{\mathbb{C}}^n$. This may readily be conceived as an n -dimensional complex differentiable manifold (i.e. one with complex-valued charts).⁴² But it is not diffeomorphic to every complex n -dimensional manifold. Nor does it play a special role among them, like that of \mathbf{C}^n (to which every complex manifold is diffeomorphic locally).

I have dwelt at length on such scholastic niceties, as a prelude to the following remark. Klein will show that Riemann's geometries of constant curvature can be regarded as the theories of certain structures defined on (or in?) $\mathcal{P}_{\mathbb{C}}^n$. However, for Riemann, those geometries are but peculiar members of the vast family of Riemannian geometries, dealing with R -manifolds of arbitrary curvature. Within this family the geometries of constant curvature are, so to speak, degenerate cases. Hence by confining his discussion to structures definable on a particular n -dimensional manifold, Klein loses sight of the full scope of Riemann's conception. Geometries of constant

curvature are taken from the context in which they were originally defined, and granted a privileged status.

However this does not mean that Klein deals with them in isolation. They have a well-defined position in a different system which, although the ordinary Riemannian geometries are excluded from it, can be extended to cover many new geometries. After his description of n -dimensional manifolds, Klein sketches the main ideas of this system. A more detailed exposition is given in the "Programme" he submitted to the Faculty at Erlangen at the time of joining it.⁴³ The driving force behind it appears to be his desire to find a unifying concept by means of which to comprehend and organize the wealth of disparate discoveries in 19th-century geometry. He found it in the concept of a group of transformations.⁴⁴ We may characterize it as follows. For any set S , a bijective mapping $f: S \rightarrow S$ is called a *transformation* of S (into itself). Let T be the set of all transformations of S . T has the following properties: (i) if f and g belong to T , the composite mapping $f \cdot g$ belongs to T ; (ii) if f belongs to T , the inverse mapping f^{-1} belongs to T . Given that, for every $f, g, h \in T$, $f \cdot (g \cdot h) = (f \cdot g) \cdot h$, and $f \cdot f^{-1}$ is equal to the identity transformation $x \mapsto x$ (which belongs to T), T is a group, with group product \cdot (composition of mappings). Let G be a subgroup of T ; G is a *transformation group* of S . If, for every $x \in S$ and every $f \in G$, whenever x has the property Q , $f(x)$ has Q , we say that the group G *preserves* Q . We may say likewise that G preserves a relation or a function defined on S^n . Any property, relation, etc., preserved by G is said to be *invariant* under G , or *G-invariant*.

Klein uses these ideas to define and classify geometries. Let S be an n -dimensional manifold and let G be a group of transformations of S . By adjoining G to S (as Klein says) we define a geometry on S , which consists in the theory of G -invariants. If H is a subgroup of G , the theory of H -invariants is another geometry, subsumed under the former. The most general group of transformations of an n -dimensional manifold mentioned by Klein is the group of homeomorphisms (continuous bijective mappings whose inverses are continuous also). The manifold \mathcal{P}_c^n is endowed with the usual topology. Homeomorphisms form a group since the inverse of a homeomorphism and the product of two homeomorphisms are homeomorphisms. The invariants of this group are studied by *analysis situs* (known today as topology). The hierarchy of subgroups of this group determines the

hierarchy of geometries. Klein's conception does indeed provide a common framework within which can be situated many different tendencies in the geometry of his day. The reader will easily understand how the Cayley–Klein theory of projective metrics falls into this scheme. The set of all collineations is a subgroup of the homeomorphisms of $\mathcal{P}_{\mathbb{C}}^n$. The set of all collineations that map a given hypersurface ζ of second degree onto itself is a subgroup of that group. The function d_{ζ} suitably defined on $\mathcal{P}_{\mathbb{C}}^n \times \mathcal{P}_{\mathbb{C}}^n$ is invariant under this subgroup.⁴⁵ Klein shows in the Programme how other, newly-developed branches of geometry can be better understood in this way. An important one which he does not mention is affine geometry. This is defined on \mathcal{P}^n by the group of projective transformations that map a given hyperplane onto itself. If we excise this hyperplane from \mathcal{P}^n we obtain affine space.

It seems reasonable to regard two figures as equal, in a given geometry, if one is the image of the other under a transformation belonging to the characteristic group. Thus, in topology, a sphere is equal to a cube (but not to an anchor-ring); in projective geometry, a circle is equal to a hyperbola; in BL geometry only congruent figures are equal. With the aid of the group concept we can establish equivalences also between apparently different geometries. Let M be a manifold on which a geometry is defined by a group G . Let f map M bijectively onto an arbitrary set M' . The mapping $g' = f \cdot g \cdot f^{-1}$ ($g \in G$) is a transformation of M' . The set $G' = \{g' | g \in G\}$ is a transformation group of M' , which defines on this set what we may reasonably call 'the same geometry' that G defines on M .⁴⁶ Two examples will show this: Let G preserve the property of being a straight line on M . We shall say that $f(a)$ is a 'straight' on M' whenever a is a straight line on M . Obviously G' preserves the property of being a 'straight' on M' . Likewise, if a distance function d on M is G -invariant, the function $d': (x, y) \mapsto d(f^{-1}(x), f^{-1}(y))$, which is a distance function on M' , is G' -invariant.

When introducing the concept of equivalence between geometries, Klein is on the verge of abandoning the narrow notion of a manifold used in the paper of 1873. At times, it seems as though a manifold is for him simply a structured set, its structure being determined by the adjoined group. A geometry is determined not by the particular nature of the elements of the manifold on which it is defined but by the structure of the group of transformations that defines it. One and the

same geometry will be defined on completely different manifolds by structurally identical (isomorphic) groups of transformations. The readiness to identify, say, straight lines with circles, planes with points, if we can but set up among the former a structure equivalent to one found among the latter, stems from the newly-acquired awareness that structure (relational nets) is all that geometers really care for. It is not the nature of points and lines (which nobody has ever been able to explain) but how they stand to one another in a system of relations of incidence and order which is the concern of projective geometry, and this is sufficiently known once we know the group which preserves this system. Klein's group-theoretical approach to geometry is a principal antecedent of the modern axiomatic method, as developed in the late 19th century by Peano and his school and by David Hilbert (Part 3.2). This method is based on the assumption that the objects of a mathematical theory need not be ascribed more than what is strictly necessary for them to sustain the relations we require them to have to one another. The basic objects of such a theory are determined just by its basic propositions, the *axioms* that lay out the relational net into which those objects are inserted. Such a determination is as much as a mathematical theory requires.

Klein's conception is, of course, narrower than the general structural viewpoint just expressed. Thus, Riemann's geometry of manifolds will not fit into it. If M is an R -manifold of non-constant curvature, it may happen that arc-length is preserved by no group of transformations of M other than the trivial one which consists of the identity alone. But this trivial group cannot be said to characterize anything, let alone the Riemannian geometry of M . Klein just shows how his scheme can be extended to cover the geometries of constant curvature. That these are equivalent to Klein's projective metric geometries is doubtless true (subject to the qualifications discussed above). But Klein's argument for this equivalence (Klein, 1873, §6) follows an un-Riemannian line. He writes: "When we ascribe a definite, constant non-vanishing curvature to a manifold, we are specifying the mere concept of an n -fold extended manifold by adding to it [...], as a further determination, a transformation group which is constructed in well-known fashion by requiring free mobility of rigid bodies".⁴⁷ Beltrami has shown "that in a manifold of constant curvature the coordinates can be chosen so that geodesic lines are represented by

linear equations".⁴⁸ From Klein's point of view, this is stated as follows: "The transformation group adjoined to a manifold when we ascribe to it a constant curvature is contained, for a suitable choice of coordinates, in the group of linear transformations." In the light of his own study of projective metrics Klein concludes: "The transformation group which preserves the metric on a manifold of constant curvature consists, for a suitable choice of coordinates, in the group of linear transformations that preserve a given quadratic equation".⁴⁹ Whereas Riemann held free mobility of figures – without dilation or contraction – to be a consequence of the metric structure of manifolds of constant curvature and of their characteristic symmetries, Klein conceives it as a result of the invariance of certain properties and relations under a given group. This is the primary fact. A suitable choice of coordinates enables him to find an elegant analytic representation of this group, from which the curvature and the remaining properties of the manifold can be computed.

Interest in Riemannian geometry increased considerably after Ricci and Levi-Civita (1901) created the tensor calculus and Einstein (1916) used a four-dimensional semi-Riemannian manifold of non-constant curvature to represent physical space-time.⁵⁰ Some attempts were made to incorporate Riemannian geometry in Klein's scheme. Schouten suggested the following use of Klein's concept of adjunction: A Riemannian structure is defined on a differentiable manifold by "adjoining" a given quadratic differential expression to the group of diffeomorphic transformations of the manifold.⁵¹ Elie Cartan objects that this deprives Klein's concept of all meaning. "En poussant jusqu'au bout l'extension abusive faite du principe d'adjonction, on pourrait dire que tout problème mathématique rentre dans le cadre du programme d'Erlangen; il suffit d'adjoindre au groupe de toutes les transformations possibles les données du problème à résoudre." (E. Cartan, 1927, p.203). Cartan's own approach is much subtler. A description of it lies beyond the scope of this book. Cartan's ideas have led to the very fruitful application of group theory to modern differential geometry. But they go beyond the bounds of the Erlangen programme. Recent writers neatly distinguish Klein geometry, which deals with structures governed by the Erlangen scheme, from differential geometry, the general theory of differentiable manifolds. (See Jasinska and Kuchrzewski, (1974).)

2.3.9 Projective Coordinates for Intuitive Space

The second part of Klein (1873) studies an important matter we mentioned briefly on p.131 namely “the possibility of constructing projective geometry [. . .] without assuming the axiom of parallels”.⁵² To “construct projective geometry” apparently means here to put the numerical manifold studied in the first part of the paper in connection with the intuitive space which Klein believed was the proper subject-matter of geometry. Klein’s proof of possibility consists in showing that any intuitively accessible spatial region can be mapped bijectively onto an open subset of the real projective manifold \mathcal{P}^3 in such a way that the intuitive relations of neighbourhood and order are preserved by the mapping. Each point of the region is thereby assigned a unique point of \mathcal{P}^3 , that is, an equivalence class of real homogeneous coordinates. Any intuitively given space can, in this sense, be embedded in \mathcal{P}^3 – and consequently in $\mathcal{P}_\mathbb{C}^3$ also – and be identified with a part of it.

A method for assigning homogeneous coordinates to the points of space had been developed by von Staudt. In his paper of 1871, Klein asserts, without proof, that von Staudt’s method does not depend on the axiom of parallels.⁵³ In 1873, he sets out to prove this assertion. The proof presupposes only that space can be analyzed into points, straight lines and planes in the familiar fashion, and that it is continuous in the sense that we shall define below. The assumption of continuity was formulated in Klein (1874).

Von Staudt had shown how to associate a unique point to three given collinear points. For simplicity’s sake, we restrict our discussion to the plane, but von Staudt’s construction can be easily extended to three-dimensional space. Let A , B and C be three collinear points (Fig. 16). Choose three lines, p , q , r , such that p and q go through A and not through B while r goes through B and not through A . r meets p at P and q at Q . Denote the join of P and C by s . s meets q at S . Denote the join of B and S by t . t meets p at T . The join of T and Q meets line AB at D . D is determined by A , B and C and does not depend on the choice of p , q and r . Moreover, if we exchange A and B , we obtain the same point D . The construction can be dualized to obtain a unique line d associated with three concurrent (coplanar) lines a , b and c .

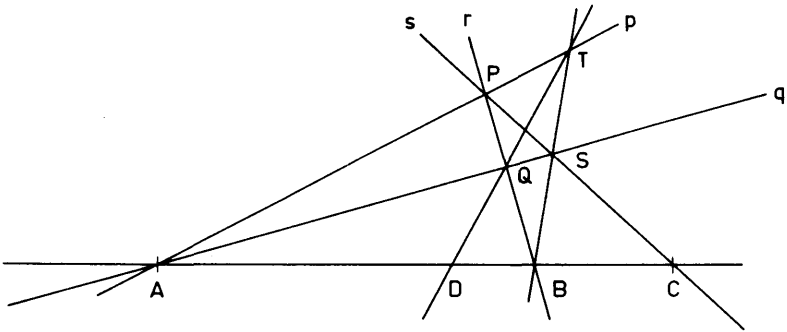


Fig. 16.

We have made no stipulations regarding the relative distances of points A , B , C . Indeed, we need not even assume that the concept of distance can be meaningfully applied to them. However, if A and B lie on a Euclidean plane α and C happens to be the midpoint of segment AB , we can easily verify that the join of T and Q is parallel to AB , so that point D does not exist (unless we place it 'at infinity'). In the dual construction, of course, line d will always be found to exist. If c happens to form equal angles with a and b , d is perpendicular to c . Define now a Cartesian 2-mapping $x: \alpha \rightarrow \mathbb{R}^2$. If $P \in \alpha$, denote by \bar{P} the number triple $(x^1(P), x^2(P), 1)$. The mapping $P \mapsto \bar{P}$ assigns a set of homogeneous coordinates to each point $P \in \alpha$. If D is, as above, the point associated by von Staudt's construction with three collinear points A , B and C on α , it can be shown that the cross-ratio $(\bar{A}, \bar{B}; \bar{C}, \bar{D}) = -1$. In other words, \bar{D} is the fourth harmonic to \bar{A} , \bar{B} and \bar{C} . Hence, it is not unnatural to describe von Staudt's construction as a method for finding the 'fourth harmonic' to three given collinear points (or to three concurrent coplanar lines) in Euclidean space.

Hereafter, we use this terminology regardless of its Euclidean motivation. We simply call a line (or a point) the fourth harmonic to three coplanar concurrent lines (collinear points) if it can be associated with them by von Staudt's construction. Given three coplanar concurrent lines u , v and w , we say that a line m belongs to the *harmonic net* (uvw) if $m = u$ or $m = v$ or $m = w$ or m is the fourth harmonic to three lines belonging to (uvw) . A harmonic net of collinear points is defined analogously.

As I said above, we shall postulate that space is continuous in the following sense: If X is a flat pencil of lines, partitioned into two subsets X_1 and X_2 such that no pair of lines of X_1 is separated by a pair of lines of X_2 , there exists a pair of lines a, b in X which separates every line in $X_1 - \{a, b\}$ from every line in $X_2 - \{a, b\}$. Zeuthen proved that if this is assumed, then, for every flat pencil X and every harmonic net Y contained in X , each pair of lines belonging to X is separated by a pair of lines belonging to Y . This means that Y is everywhere dense in X . We shall refer to the foregoing assertion as *Zeuthen's lemma*.⁵⁴

Let us add an object ∞ to the field \mathbf{Q} of rational numbers, postulating that $\infty + \infty = \infty$, $\infty - \infty = 0$, $\infty/\infty = 1$; that for every $a \in \mathbf{Q}$, $\infty > a$, $\infty + a = \infty$, $\infty/a = \infty$, $a/\infty = 0$, and that if $a \neq 0$, $a/0 = \infty$. Any harmonic net (uvw) contained in a flat pencil X can be mapped injectively into $\mathbf{Q} \cup \{\infty\}$, in the following standard fashion. We assign the numbers 0, 1 and ∞ to u, v and w , respectively. We agree that if a, b and c are the numbers assigned to three lines of the net, their fourth harmonic be assigned the number x determined by equation

$$\frac{(x-b)(c-a)}{(x-a)(c-b)} = -1. \quad (1)$$

In particular, the fourth harmonic to u, v and w will be assigned the number $1/2$; the fourth harmonic to u, w and v , the number -1 . The reader should satisfy himself by studying von Staudt's construction that this mapping preserves cyclic order: if $a < b$, the lines numbered a, b separate the lines numbered c, ∞ if, and only if, $a < c < b$. Zeuthen's lemma implies that the image of the harmonic net (uvw) by this mapping is everywhere dense in $\mathbf{R} \cup \{\infty\}$. It is clear, on the other hand, that not every line in the pencil X can belong to the net (uvw) .⁵⁵ We shall nevertheless define an extended harmonic net $(uvw)'$ which includes every line in X . Let a, a_1, a_2, \dots and b, b_1, b_2, \dots be two monotonic sequences of rational numbers which belong to the image of (uvw) by the above mapping and converge to the same real number c ($a < c < b$). Since the image of (uvw) is everywhere dense in \mathbf{R} , such sequences exist for every $c \in \mathbf{R}$. Continuity implies that there is a unique line m in X such that m and w separate every line in the first sequence from every line in the second. We assign to m the real number c . The extended net $(uvw)'$ is formed by every line which belongs to (uvw) or is assigned a real number by the foregoing rule.

Obviously $(uvw)' = X$. Moreover, our rules for assigning numbers to the lines of $(uvw)'$ determine a bijective mapping of X onto $\mathbb{R} \cup \{\infty\}$, which preserves cyclic order in the way explained above.

We shall now show how to assign homogeneous coordinates to every point of a finite region of the plane using von Staudt's construction. To avoid unnecessary complications, we consider a convex plane region S (i.e. a region such that if points A and B lie on it, the entire segment AB is contained in it). Choose two straight lines p and q which meet at a point O in S . Pick three more points in S , one on p one on q , one outside both lines. (The reader is advised to draw a diagram.) We denote these points by P , Q and E , respectively. Consider the flat pencil through P . We assign the numbers 0 , 1 and ∞ to lines PO (that is p), PE and PQ , respectively. This determines, as we know, a mapping of the entire pencil through P onto $\mathbb{R} \cup \{\infty\}$. We do the same with the pencil through Q , assigning 0 to QO , 1 to QE and ∞ , once more, to QP . Let X be any point of S . Then, unless X lies on line PQ , X is the meet of a line through P and a line through Q . Let u and v be the numbers assigned, respectively, to those two lines by the above mappings. We assign to X the class of homogeneous coordinates $[u, v, 1]$. If X lies on PQ , the segment joining X to O is entirely contained in S . Let Y be a point on this segment, distinct from X and O . Y is, of course, the meet of a line through P , numbered, say, s , and a line through Q , numbered t . We assign to X the class $[s, t, 0]$. It is readily seen that this assignment does not depend on the choice of Y . In particular, according to this rule P is assigned the class $[0, 1, 0]$ and Q , the class $[1, 0, 0]$. As we know, each equivalence class of real homogeneous coordinates of the form $[x_1, x_2, x_3]$ ($x_i \neq 0$ for some value of i) is a point of \mathcal{P}^2 . We have therefore defined a mapping of S into \mathcal{P}^2 . The mapping is obviously injective. It maps collinear points of S on collinear points of \mathcal{P}^2 .⁵⁶ If \bar{P} is the image of a point $P \in S$ by this mapping, then every neighbourhood of \bar{P} (in the topology defined on \mathcal{P}^2) contains the image of some neighbourhood of P (in the intuitive sense of the word 'neighbourhood'). We can easily define a similar mapping of any convex spatial region V into \mathcal{P}^3 . This mapping can be extended to any convex region V' which contains V . In this way, intuitive space can be identified with a part of \mathcal{P}^3 and projective geometry can be said to be grounded on spatial intuition.

2.3.10 Klein's View of Intuition and the Problem of Space-Forms

The preceding discussion involves a view of geometric intuition and its role in science which is expressed by Klein in several of his writings.⁵⁷ He apparently believed that every normal human adult has the ability to form geometrical images according to a fixed pattern. This faculty or its exercise he called *intuition* (*Anschauung*). Intuition lies at the root of scientific geometry and is an indispensable aid in geometric discovery. Klein at one point states that geometric intuition is an inborn talent. He holds, however, that it is developed by experience. "Mechanical experiences, such as we have in the manipulation of solid bodies, contribute to forming our ordinary metric intuitions, while optical experiences with light-rays and shadows are responsible for the development of a 'projective' intuition".⁵⁸ However, geometric intuition is insufficient for unambiguously determining geometric notions and for deciding between certain incompatible geometrical propositions. Klein proposes the following case in point. After choosing a straight line m within one's grasp, one imagines a point P in Sirius. Either there is but one line through P which is parallel to m or there are many such lines, lying very nearly at right angles with the perpendicular from P to m . Which is the case? We must acknowledge the impotency of intuition to decide the issue. Either alternative is compatible with it. Either one involves an 'idealization' of intuitive data, i.e. the introduction, by intellectual *fiat*, of precision not possessed by the data. The tendency to idealize is strong in ordinary perception: we *see* surfaces as smooth and flat which, under careful observation, exhibit minute irregularities.⁵⁹ Scientific geometry carries idealization to a limit: widthless lines and dimensionless points replace the strips and dots of intuition. Familiarity with these idealized objects develops what Klein calls a "refined intuition" which should not be confused with the "naïve intuition" we have been speaking about. Such "refined intuition" is required for following many of the proofs in Euclid. But then, "refined intuition is not properly an intuition at all, but arises through the logical development from axioms considered as perfectly exact".⁶⁰

Klein advances the idea that naïve geometric intuition has a threshold of exactness which does not meet the requirements of

traditional geometric thought. This idea was suggested to him by the psychological notion of a threshold of sensation, below which stimuli fail to arouse consciousness. The idea was first introduced by Klein in a lecture (1873b) intended to make Weierstrass' nowhere-differentiable continuous functions more palatable to scientists. Given a Cartesian 2-mapping z , the graph of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ can be represented on the plane by the set $F = \{z^{-1}(x, f(x)) \mid x \in \mathbb{R}\}$. If f is continuous (in the technical sense) and injective, F must be a widthless, gapless curve (in the intuitive sense). If f is nowhere differentiable, this curve nowhere has a definite direction: if P is any point of F , there is no tangent to F at P . This is generally held to be counter-intuitive. But, Klein observes, a departure from intuition is already involved in the notion of a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$. Such a function assigns a real number to every real number, or, if you wish, a point on a widthless, gapless line to every point on a widthless, gapless line. But intuitive lines are actually narrow strips. They are, of course, gapless because between any two non-overlapping dots in them we can always mark a third dot (which possibly overlaps with each of the former). The idealizing move that takes us from narrow strips to widthless lines, which are continuous sets of dimensionless points, is not blatantly counterintuitive; it can even be said to be suggested by intuition. But it, in fact, goes beyond intuition, and we must not be surprised if it eventually leads to the notion of a directionless curve, which is unimaginable. We find an analogous development in the foundations of projective geometry: projective intuition (if such a thing exists) suggests the existence of a 'point at infinity' which comes after every finitely distant point on both extremities of a straight line.⁶¹ If we adopt it, we are led inevitably to postulating a line at infinity on every plane, and a plane at infinity in three-dimensional space. But we cannot, by any stretch of the imagination, attach an intuitive content to the latter plane.

In the wake of these remarks it should come as a surprise to learn that Klein rejected Pasch's use of the axiomatic method. Pasch demanded that the full intuitive content of geometry should be expressed in axioms, from which the remaining geometrical truths would be derived by strict deductive inference (Section 3.2.5). Thus, geometrical questions could be settled by an appeal to the axioms, without our having ever to bring in intuition. Klein objects: "I find it impossible to develop geometrical considerations unless I have

constantly before me the figure to which they refer [. . .]. A purely calculating analytic geometry, which does away with figures, cannot be regarded as genuine geometry [. . .]. An axiom is a *demand* that compels me to make *exact statements* out of inexact intuition." ("die *Forderung*, vermöge deren ich in die ungenaue Anschauung *genaue Aussagen hineinlege*", – Klein (1890), p.571.) I, for my part, fail to see how an admittedly imprecise image can be of any help in the actual proof of statements concerning the unambiguous ideal entities determined by the axioms. (Compare Klein, *Elementarmathematik*, Vol. III, p.8.)

The limitations of geometric intuition give rise to an interesting problem to which Klein devoted some attention in his later writings on non-Euclidean geometry. All that we can represent to ourselves in our imagination lies in a finite region of space; neither our inborn geometric intuition (if we have one) nor the increased intuitive abilities we acquire through mechanical and optical experiences can help us in any way to visualize the whole of space. Geometry determines through idealization the exact (topological, projective, metric) structure that we ascribe to the spatial region which we are able to imagine. Does this postulated exact structure determine the global structure of space as well? Not at all. We know already that a region homeomorphic to a connected subset of Euclidean 3-space can belong to many very different topological spaces. The definition of a metric on the intuitively accessible spatial region restricts the set of globally different spaces to which this region can belong, but even then their diversity is astonishing. Klein arrived at this remarkable conclusion guided by W.K. Clifford's discovery of a class of surfaces in elliptic 3-space which are locally isometric to the Euclidean plane.⁶²

Euclidean parallels are coplanar non-intersecting everywhere equidistant straight lines. There are congruence-preserving transformations of Euclidean space – parallel *translations* – which map each member of a given family of parallel lines onto itself. Euclidean parallels exist only in Euclidean space. In BL space there are coplanar non-intersecting straight lines. Denote a set of such lines by Γ . Since no pair of members of Γ are everywhere equidistant there is no congruence-preserving transformation of BL space (except the identity) which maps each member of Γ onto itself. Non-intersecting coplanar straight lines of BL space lack therefore the most interesting property of Euclidean parallels. A better analogy to Euclidean

parallels might be provided by everywhere equidistant skew (i.e. non-coplanar and consequently non-intersecting) lines. Do such lines exist? Let us place ourselves in a three-dimensional space of arbitrary constant curvature. Suppose m and n are two everywhere-equidistant skew lines. Choose two points A, B on m . The perpendiculars from m to n at A and B meet n at A' and B' , respectively. $AA' = BB'$. $AA'BB'$ is a skew rectangle. It can be shown that $AB = A'B'$ and that the right triangle ABB' has an excess. (Bonola, NEG, p.201.) Consequently, everywhere-equidistant skew lines cannot exist in Euclidean or in BL space. Clifford showed however that real lines satisfying this description do exist in three-dimensional elliptic space. We call them C-parallels (C for Clifford). To see how they can be constructed let us recall that the congruence-preserving transformations of elliptic 3-space – the elliptic motions – are the collineations of \mathcal{P}_C^3 which map a given imaginary quadric ζ onto itself. ζ has two families of (imaginary) rectilinear generators, F and F' . There are elliptic motions, which we shall call displacements of Class 1, which map every line in F' onto itself while displacing each point P on any line $m \in F$ along the line in F' that meets m at P . For any given displacement f of Class 1, there are two conjugate imaginary lines m_1, m_2 in F , such that for every $P \in m_i$ $f(P) = P$ ($i = 1, 2$). We say that f fixes m_1 and m_2 pointwise. Displacements of Class 2 are characterized by interchanging F and F' in the description of displacements of Class 1. If we regard the identity as belonging to both classes of displacements, we may say that every elliptic motion is the product of a displacement of Class 1 and a displacement of Class 2. Let f denote the displacement of Class 1 (not the identity) which fixes pointwise the lines m_1 and m_2 in F . Consider now the family of straight lines $\Gamma_f = \{m | m \text{ is real and } m \text{ meets } \zeta \text{ on } m_1 \text{ and } m_2\}$. If $m \in \Gamma_f$, f fixes two points on m , so that f maps m onto itself. Since f is an elliptic motion, any two lines in Γ_f are everywhere equidistant. Γ_f is therefore a family of C-parallels. Every displacement of Class 1 (except the identity) determines thus a family of C-parallels in elliptic space. Two lines belonging to such a family will be called C_1 -parallel. C_2 -parallels are defined likewise, by substituting 'Class 2' for 'Class 1' in the foregoing discussion. Let a be a real line which meets ζ at P_1 and P_2 . P_i is the meet of two generators $g_i \in F$ and $h_i \in F'$ ($i = 1, 2$). Let P be a real point not on a . There is exactly one line b through P which meets ζ on g_1 and g_2 ; there is also one line b' through P which meets ζ on h_1 and h_2 . Clearly

a and b (b') are C_1 - (C_2 -) parallel. If P lies on the polar⁶³ of a , then b and b' coincide. If three lines a, b, c are C_1 - (C_2 -) parallel, then all the straight lines that meet a, b and c are C_2 - (C_1 -) parallel. Consequently, any three C_1 - (C_2 -) parallel lines define a ruled surface in elliptic 3-space. Such a surface is called a Clifford surface. Let Σ denote a Clifford surface. It can be proved that Σ is a surface of revolution with two axes. Σ is closed and has a finite area. It is not difficult to see that Σ has a constant Gaussian curvature equal to 0: Any pair of C_1 -parallels a, b on Σ will meet a pair of C_2 -parallels r, s on Σ at right angles; a, b, c, d form a rectangular quadrilateral with equal opposite sides. Such a quadrilateral can exist only on a surface of zero G -curvature.⁶⁴ It follows that every connected proper subset of Σ can be mapped isometrically into the Euclidean plane.

The discovery of Clifford surfaces immediately suggests the following problem: To determine all the globally non-homeomorphic surfaces in elliptic, parabolic and hyperbolic 3-space which are locally isometric to the Euclidean (or to the elliptic or to the hyperbolic) plane. The problem, when generalized to hypersurfaces in elliptic, parabolic and hyperbolic space of any dimension number, is known as the Clifford-Klein problem of space-forms. The question can be approached also from an 'intrinsic' point of view, that is, without paying heed to the particular structure of the embedding space. Thus, we may ask for all the types of globally non-homeomorphic n -dimensional Riemannian manifolds of zero curvature, all of which, as we know, are locally isometric to \mathbb{R}^n (with the standard Euclidean metric). Each of these types is said to be an n -dimensional Euclidean space-form. The case $n = 3$ is philosophically interesting. It can be shown that there are seventeen distinct families plus a one-parameter family (i.e. a family of families, indexed by \mathbb{R}) of Riemannian manifolds such that (i) any member of one of the families is locally isometric with Euclidean 3-space; (ii) two members of the same family are globally diffeomorphic; but (iii) no member of one family can be mapped diffeomorphically onto a member of another family. Ten of the families are topologically compact, so that the spaces belonging to them can be said to have a finite volume.⁶⁵ Let us assume for a moment that men actually have – as Kant taught in 1770 – an a priori intuition of space which requires them to ascribe to it a Euclidean metric. Since we cannot visualize the whole of space, our a priori intuition would not enable us to decide whether it is actually an

infinite space or whether it belongs to one of the ten families of compact spaces that are locally isometric to \mathbb{R}^3 . This indeterminacy is, in a sense, built into the concept of an a priori intuition of Euclidean space. Analogous remarks apply to empirically based spatial intuitions. Commenting on such matters, Klein wrote in 1897:

Our empirical measurements have an upper bound, given by the size of the objects which are accessible to us or which fall under our observation. What do we know about spatial relations in the very large (*im Unmessbar-Grossen*)? Absolutely nothing, to begin with. We can only resort to postulates. Hence I regard all the topologically different space-forms as equally compatible with experience.⁶⁶

The choice between these forms, Klein adds, should be guided by the principle of economy of thought. In his lectures on non-Euclidean geometry, published posthumously thirty years later, he remarks:

Let us assume that the space about us exhibits a Euclidean or a hyperbolic structure. We can by no means infer from this that space has an infinite extent. Because, for instance, Euclidean geometry is entirely compatible with the hypothesis that space is finite, a fact that has been formerly overlooked. The possibility of ascribing a finite content to space whatever its geometrical structure, is particularly welcome because the idea of an infinite expanse, which was originally looked upon as a substantial progress of the human mind, gives rise to many difficulties, e.g. in connexion with the problem of mass-distribution.⁶⁷

We see thus that towards the end of his life, probably after studying Einstein's writings on gravitation and cosmology, Klein came to think that cosmological considerations can furnish empirical criteria for choosing between globally non-homeomorphic though locally isometric space-forms.

FOUNDATIONS

Plato held that specialized knowledge becomes true science only if one is aware of its foundations. To inquire into these, however, was not a task for the specialist but for the dialectician, whom we would call the philosopher. Yet philosophers from Plato to Kant have not contributed much to our awareness of the foundations of geometry. Some of them did discuss the nature of geometrical objects and the source of geometrical knowledge, but they were content to accept the principles of geometry proposed by Euclid, and they rarely went into details concerning, say, the justification of this or that particular principle or the relationship between the principles and the body of geometrical propositions. Shortly after 1800, G.W.F. Hegel claimed that mathematical axioms, insofar as they are not mere tautologies, ought to be proved in a philosophical science, prior to mathematics. Euclid, said Hegel, was right not to attempt a demonstration of Postulate 5, for such a demonstration can only be based on the *concept* of parallel lines and therefore pertains to philosophy, not to geometry.¹ However Hegel himself did not provide the demonstration.

In 1851 Friedrich Ueberweg, a young German philosopher, published an essay on *The principles of geometry, scientifically expounded*. The purpose of this work is to show how the propositions of geometry can be derived by logical means from a few empirically obvious truths. With this aim in mind, Ueberweg tries to build Euclid's system on a novel set of axioms, centred upon the concept of rigid motion. Another philosopher, the Belgian, J. Delboeuf, a pupil and friend of Ueberweg, published nine years later a different reconstruction of Euclidean geometry, based on the principle that shape cannot depend on size. Delboeuf was probably the earliest philosophical writer who had first-hand acquaintance with the works of Lobachevsky. He describes BL geometry as "une science enchainée quoique fausse", and he mentions it to illustrate the fact that false premises need not imply inconsistent conclusions.²

The first works on the foundations of geometry responsive to the

full impact of the new geometries did not appear until the late 1860's, the most important being Riemann's lecture of 1854 (published in 1867) and three papers of Hermann von Helmholtz (1866, 1868, 1870), who was not a mathematician, nor a professional philosopher, but a physician – and a great physicist – with a philosophical turn of mind. These works may be regarded as the starting-point of a series of investigations of an increasingly mathematical character that eventually culminated in David Hilbert's *Foundations of Geometry* (1899). Among the publications on the subject which appeared between Helmholtz's and Hilbert's the most remarkable are perhaps two papers by Sophus Lie, "On the foundations of geometry" (1890), reworked and expanded in Part V of the third volume of his *Theory of Transformation Groups* (1893). Lie's approach stems directly from Helmholtz's. His aim is to give an exact solution of the latter's "problem of space", originally conceived as a typical epistemological question which may be stated thus: *Which among the infinitely many geometries whose mathematical viability has been shown by Riemann's theory of manifolds are compatible with the general conditions of possibility of physical measurement?* Implicit in the question is the operationist belief that only such geometries as are compatible with the latter conditions are suitable for the role of a physical geometry. We shall discuss this problem in Part 3.1.

We shall see that Lie, while correcting and improving Helmholtz's mathematical treatment of the problem of space, lost sight of its epistemological significance and understood it as a problem in pure mathematics, *viz.* *What properties are necessary and sufficient to define Euclidean geometry and its near relatives, the geometries of constant non-zero curvature?* Lie's answer to this modified question anticipates Hilbert's achievement, insofar as it gives an exact axiomatic characterization of Euclidean geometry. But Hilbert's approach is very different from Lie's. Like Riemann and Helmholtz, Lie conceives of space as a differentiable manifold, whose local topology depends on the familiar, tacitly assumed properties of the real number continuum (its global topology, he simply ignores). Hilbert, on the other hand, makes no prior assumptions concerning the relation between geometry and analysis. As a matter of fact, the first version of his axiom system does not even ensure that space can be endowed with a differentiable structure.³ And he makes a point of showing how unexpectedly far one can proceed in the construction of Euclidean

geometry without using the Archimedean postulate that every segment is less than a multiple of a given segment, which is apparently a prerequisite of analytic geometry. Hilbert, whose style recalls Greek geometry at its best, disentangles the simplest conditions that determine the structure of Euclidean space and shows how the various aspects of that structure depend on one or the other of those conditions. Hilbert's book was preceded by Pasch's *Lectures on Modern Geometry* (1882) and the axiomatizations of Peano (1889, 1894) and other mathematicians of the Italian school. Its impact upon the methodology of mathematics and the philosophical interpretation of the nature of mathematical knowledge has been immense. The background and significance of Hilbert's book are the subject of Part 3.2.

3.1 HELMHOLTZ'S PROBLEM OF SPACE

3.1.1 *Helmholtz and Riemann*

The writings of Hermann von Helmholtz (1821–1894) on the foundations of geometry are few and short, but they have exerted an enormous influence. His solution of the Helmholtz problem of space was reported in a paper "On the factual foundations of geometry" (1866) and presented with proofs in a second paper "On the facts which lie at the foundation of geometry" (1868). His conclusions are marred by the fact that he ignored BL geometry. Apparently, he learnt about it from Beltrami's "Saggio" (1868) and "Teoria fondamentale degli spazii di curvatura costante" (1868/69). A note correcting his omission appeared in 1869 in the same journal that published the paper of 1866. His well-known lecture "On the origin and significance of geometrical axioms" (1870) states his final solution of the space problem and draws philosophical conclusions from it. We shall see that this lecture not only provides almost all the material for the philosophical debate on geometry during the last third of the 19th century, but contains the germ of several important epistemological ideas which have been very influential in the 20th century.

Helmholtz says that his investigations concerning the localization of perceived objects in the visual field led him to study the origin and nature of our common intuitive representations of space. In this connection, he met one question whose answer, he believes, pertains

also to exact science, namely “which propositions of geometry express truths of factual significance and which are merely definitions or a consequence of definitions and of the chosen mode of expression?”¹ Helmholtz’s question asks, essentially, for the foundations of geometry, regarded as a science of physical space. He says that in his investigation of the problem he followed a path not too distant from the one pursued by Riemann, with whose lecture “Ueber die Hypothesen” he became acquainted later. There is however an essential difference between him and Riemann, which Helmholtz emphasized in the very title of his second paper: “On the *facts* [instead of the *hypotheses*] which lie at the foundation of geometry”. In Riemann’s theory only the most general principle of physical geometry, stating that space is what Riemann called a “manifold”, passes for an a priori principle that follows directly from the concept of extendedness or spatiality. The specification of this “manifold” as a continuous one and the remaining principles of geometry are hypotheses, which Riemann also describes as “facts” (*Tatsachen*),² because they can be confirmed or refuted by experience, but which, like all scientific hypotheses, never can attain full precision and certainty. There is one exception: because of its peculiar nature, the number of dimensions of physical space, though factual, is known exactly with almost unimpeachable certainty, through simple, familiar, prescientific experiences.³ But all the other fundamental principles of geometry are valid only approximately, within the limits of observation, and are subject to revision. Moreover they share with the newest hypotheses of mid-19th-century physics one rather disquieting trait: they involve highly complex and seemingly abstruse conceptual constructions, the adequacy of which can only be determined indirectly, through the empirical test of particular, often remote consequences. Thus, Euclidean geometry is characterized by three hypotheses, which, together with the two principles we have already mentioned, may be regarded as Riemann’s axioms for this geometry:

(R1) Space is a differentiable manifold.

(R2) Space has three dimensions.

(R3) The element of length is given by the square root of a quadratic differential expression $\sum g_{mn} dx^m dx^n$, where the g_{mn} denote differentiable functions of the coordinates x^1, x^2, x^3 , with non-singular matrix $[g_{mn}]$.

(R4) The curvature of space is constant.

(R5) Space is flat, i.e. its curvature is everywhere equal to zero.⁴

Helmholtz readily accepted this characterization, but he apparently felt that such a fundamental science as geometry, which lies at the very basis of mechanics and the other physical sciences, ought not to rest upon an uncertain hypothetical foundation which is indirectly and only approximately verifiable. He was therefore very happy to find out that R3 and R4 are logical consequences of a "fact of observation", as familiar and indisputable as, say, the fact that space is three-dimensional: "the observed fact that the movement of rigid figures is possible in our space, with the degree of freedom that we know",⁵ so that the congruence of bodily figures is independent "of place, of the direction of the coincident figures, and of the path over which they have been brought into coincidence".⁶ As to Axiom R5, Helmholtz mistakenly believed, until Beltrami disabused him, that it follows from another, not quite so obvious fact, which was generally accepted at that time; namely, that space is infinite.^{6a}

Riemann himself had pointed out that R4 follows from the assumption that the shape and size of bodies do not depend upon their position or their movements in space. He backed this statement, not with a strict mathematical proof, but with good plausible arguments.⁷ Helmholtz was content to accept them. His contribution consisted in showing that the more general Riemannian hypothesis R3 can also be inferred from this assumption. He believes that the epistemological significance of this result is considerable because the very possibility of physical geometry rests upon the existence of rigid bodies that can be transported everywhere, undeformed. If this condition were not fulfilled, no spatial measurement could be performed, for the possibility of actual physical measurement does not only require—as Riemann thought—that the length of *lines* be independent of how they lie in space, but also that the size and shape of *bodies* remain unaltered, while they rotate or travel in any way whatsoever.⁸ A mathematical theory of space which does not make allowance for the possibility of measurement does not deserve the name of *geometry*, since no *metrein*, no measuring, can be performed within its framework. If Helmholtz is right, there are no more geometries than those which agree with Axioms R1–R4, that is, the Riemannian geometries of constant curvature. The vast array of manifolds, both Riemannian

and non-Riemannian, which Riemann's lecture had opened to exploration, are not then a proper subject for geometry, but merely the field of some abstruse *jeu d'esprit*, or at best, perhaps, an auxiliary branch of mathematical analysis.

3.1.2 *The Facts which Lie at the Foundation of Geometry*

In order to demonstrate R3, Helmholtz analyses the fundamental fact of the existence of rigid bodies and states its necessary conditions in the form of axioms. They are preceded by a restatement of R1, which provides the groundwork for the whole argument. This is carried out for the 3-dimensional case only. Helmholtz's axioms read approximately as follows:

(H1) Space is an n -fold extended manifold, that is, each point in space can be determined by measurement of n arbitrary, continuously and independently variable quantities (coordinates).

(H2) There exist in space movable point-systems, called rigid bodies, which fulfil the following condition: the $2n$ coordinates of every point-pair in the system are related by an equation which is independent of the movement of the system and is the same for all congruent point-pairs. (Two point-pairs are congruent if they can be made to coincide, simultaneously or successively, with the same pair of points in space).

(H3) (a) Rigid bodies can move freely, that is, any point in such a body can be carried continuously to the position of any other point in space, provided that this movement is compatible with the equations that relate its coordinates to those of the other points of the body. (b) Any point in a rigid body may be regarded as absolutely movable; if this point is fixed, i.e. if its coordinates do not change, any other point is tied to it by an equation and one of the coordinates of the latter becomes a function of the remaining $n - 1$ coordinates; if two points are fixed, any third point is tied by two equations, etc. Generally speaking, therefore, the position of a rigid body depends on $n(n + 1)/2$ quantities.

(H4) When a solid body rotates about $n - 1$ points belonging to it, and these are chosen so that the position of the body depends only on a single independent variable, rotation without reversal eventually carries the body back to its initial position. ('Rotation of a body about k points belonging to it' means movement of the body while those k points remain fixed; a movement is said to be reversed if the

coordinates of every point retake successively, in reverse order, the same values they had taken before.)

H1 implies that the movement of a point is attended by a continuous change of one or more of its coordinates. Helmholtz remarks that continuity, in this case, does not mean only that the coordinates will take all values between two extremes, but also that the quotient of the correlative variations of two coordinates changing together will approach a limit as those variations decrease. Helmholtz is apparently aware of Weierstrass' discovery of a nowhere differentiable continuous function, and he reacts by explicitly narrowing down the meaning of continuity so that it implies differentiability. It seems clear that by a value of a variable quantity we must here understand a real number. Since Helmholtz considers spherical geometry as one of the geometries compatible with his first axiom (indeed, with all four), this axiom cannot mean that a single coordinate system (a single chart, as we would say now) covers the whole of space. I take it therefore that Helmholtz's H1 (like Riemann's R1) really says that space is an n -dimensional differentiable manifold.

H2 postulates the existence of rigid bodies. These are defined as movable systems of spatial points which fulfil a specified condition in pairs. Two point-pairs fulfilling the same condition are said to be *congruent*. I understand that these point-systems consist of any number of points of *general* position in space (i.e. such that their respective coordinates do not all satisfy a given system of n linear equations, or some other restrictive condition of this kind). This interpretation of rigid bodies as point-systems may be thought to aim at dephysicalizing the concept of a body, and thereby adapting it to its new role as a fundamental concept in pure geometry. But then we should dephysicalize the concept of movement as well. It does not make much sense to speak of transporting an immaterial point from one place to another. One usually grapples with this difficulty by resorting to the concept of a mapping. An injective mapping of a region of space onto another does not actually move the points of the former into coincidence with the latter, but it defines a one-to-one correspondence between points which bears an analogy to the relation between the initial and the final positions of a moving body. If we represent bodies by immaterial point-systems, we may as well represent movements by injective mappings. Let K be a point-system representing a body k at rest in space S ; if $f: K \rightarrow S$ is an injective

mapping which preserves congruence, then $f(K)$ represents k in the position attained after a movement represented by f . Of course, f represents equally well any movement which carries k from K to $f(K)$. But then, all these movements are equivalent for the geometer, who only heeds the size and shape of the body before and after the movement. The condition on point-pairs stated in H2 can now be read as a restriction imposed on injective mappings. An injection $f: K \rightarrow S$ represents a movement of k from position K if, and only if, f preserves a given function g defined on $S \times S$, that is, if for every P, Q in K , $g(P, Q) = g(f(P), f(Q))$. If $f_1: K \rightarrow S$ and $f_2: f_1(K) \rightarrow S$ are injections which preserve g , $f_2 \circ f_1$ represents a movement of k from K to $f_2(f_1(K))$. This suggests a bolder approach to the representation of movements by means of mappings. Let t be a transformation of S (Section 2.3.8). If t preserves g , the restriction of t to K represents a movement of k from K . Since K is arbitrary, we may forget about it, and consider t itself as the representative of every movement of an arbitrary body k , from any given position K to $t(K)$. We call such a transformation a motion of S . It is readily seen that the set of motions of S is a transformation group. This conception is the basis of Lie's treatment of Helmholtz's problem (Section 3.1.4). Helmholtz's own proof of R3 suggests that the said conception was not altogether foreign to him, but we cannot be sure that he actually had thought of it. In fact, I do not believe that the dephysicalization apparently aimed at by H2 was ever seriously meant by Helmholtz. He regarded geometry as essentially oriented towards its physical and technical applications. This determines the fundamental conditions H2–H4 which he required every geometry to fulfil. He wrote that "the axioms of geometry do not speak about spatial relations only, but also at the same time about the mechanical behaviour of our most rigid bodies in motion".⁹ We might not be too far off the mark, therefore, if we say that Helmholtz did not expect his movable rigid point-systems to be altogether immaterial, but that he conceived them as entities of an unspecified materiality, like the mass-points mentioned in mechanical treatises.

We have divided H3 into two parts, marked (a) and (b). Helmholtz seems to consider (b) merely as a consequence or explanation of (a), for he does not italicize it, as he does (a), and he uses in the first sentence of (b) the word *therefore* (in German: *also*). We shall see later that (b) adds, in fact, a new and significant condition to H3(a).

This will be better understood when we come to speak of Lie's criticism of Helmholtz.¹⁰

H4 is called the axiom of monodromy. It says that the rotation of an n -dimensional rigid body about $n - 1$ fixed points belonging to it, if continued long enough in the same sense, takes every point in the body back to its initial position. We normally regard it as intuitively obvious in the case of a body in three-dimensional Euclidean space rotating about two fixed points. The intuitive idea of continuous movement is given a strict mathematical interpretation within the framework of Lie's theory. But Lie shows, on the other hand, that the axiom of monodromy is superfluous: R3 and R4 can be derived from H1–H3 alone, on a suitable interpretation of axioms H2 and H3; but they cannot be derived from H1–H4 if H2 and H3 are not given this interpretation. In Helmholtz's reasoning, however, H4 plays an essential role.

Helmholtz's argument, as we said above, depends also on a fifth axiom:

(H5) Space has three dimensions.

This is mentioned explicitly at the outset.¹¹ Helmholtz goes on to say that, since his proof aims at establishing Riemann's Axiom R3, which is concerned with the differentials of the coordinates, he will apply H2–H4 only to points whose respective coordinates differ infinitesimally. He apparently thinks that he will thereby weaken his assumptions, since he need not extend them to bodies of any arbitrary size.¹² But Lie will show that he has radically changed them, since the assumption that H2–H4 apply to infinitesimal displacements neither implies nor is implied by the statement that these axioms apply to finite movements. From a logical point of view, this does not really matter, for we may take Helmholtz's remark as actually fixing the intended scope of his axioms, and there is then nothing essentially wrong about his proof. However, the matter is not epistemologically indifferent, because the validity of the axioms in the infinitesimal cannot be established by empirical observation, except indirectly, through the verification of their consequences. Hence, if we must *assume* that they are true of infinitesimal displacements (instead of *inferring* it from the fact that they are true of finite displacements), the axioms will not provide a factual foundation of geometry, in Helmholtz's sense, but will behave as mere hypotheses like Riemann's axioms.

We need not go into the details of Helmholtz's argument. Axiom H1 ensures the applicability of mathematical analysis. H3 is quite essential, for the proof depends mainly on the consideration of infinitesimal rotations about three concurrent axes (or rather, about three linearly independent tangent vectors at a point). H4 is used for proving that the solution of a certain system of differential equations must be a periodic function (and that, consequently, a parameter which appears in that solution can only take imaginary values). The conclusion is that a certain quadratic expression in the coordinate differentials remains unchanged "in all rotations of the system" about a given point.¹³ "This quantity – says Helmholtz – can therefore be used as a measure, independent of the rotational movements, of the spatial difference between the points (r, s, t) and $(r + dr, s + ds, t + dt)$."¹⁴

After thus inferring R3 from his axioms, Helmholtz accepts, on the strength of Riemann's plausible reasons, that R4 also follows from them. Helmholtz concludes in 1866 and 1868 that the only possible geometries are the spherical geometry of positive constant curvature and the Euclidean geometry of zero curvature. Consequently, "if we postulate the infinite extension of space, no geometry is possible except the one Euclid taught".¹⁵ Helmholtz either overlooks the possibility of a space with constant negative curvature (which Riemann had mentioned in passing), or mistakenly assumes that such a space must be finite. This error was corrected by him in a short note, published on April 30, 1869, which, as we mentioned above, refers to two papers by Beltrami. In their original formulation, Helmholtz's papers of 1866 and 1868 must have sounded to non-mathematicians as a proof that Euclidean geometry is solidly founded on facts, for the infinity of space was a commonplace of contemporary astronomy. This conclusion, however, could have been questioned even in terms of those two papers alone, without invoking BL geometry. Because if we admit, as Helmholtz does, the possibility of a three-dimensional spherical geometry, space can be unlimited without being infinite (Riemann, as we may recall, had made this plain). But if this is so, there is really no reason for maintaining that space is infinite – unless, of course, we know on other grounds that it is Euclidean (or BL).

3.1.3 Helmholtz's Philosophy of Geometry

Helmholtz's final solution of his problem of space may be stated thus: *Space is a three-dimensional R-manifold with constant curvature.* The solution rests on three premises: (a) Space is an n -dimensional

differentiable manifold; (b) $n = 3$; (c) rigid bodies exist. Premise (a) is apparently regarded by Helmholtz as analytic, i.e. as an explanation of the meaning of the word *space*. It does not lack factual contents, however, insofar as it states (or implies) that such a space exists. Premise (b) is treated by Helmholtz as stating a universal trait of human experience, a sort of factual a priori. Premise (c) is repeatedly described by him as a fact, though, as we shall see, it would be a fact of a rather peculiar sort. The stated solution opens three main alternatives: Space is either (i) a spherical space or (ii) a Euclidean space or (iii) a BL space. Alternatives (i) and (iii) comprise, in fact, a continuous spectrum of possibilities, depending on the exact value of space curvature; but Helmholtz does not discuss this side of the matter. A decision between alternative (i) and the other two could be empirically reached if we could test the statement that space has a finite extension. But it does not seem possible to choose, on purely geometric grounds, between alternatives (ii) and (iii).

We gather these results from Helmholtz's lecture of 1870, "On the origin and significance of geometric axioms". This is mainly intended to present the discoveries of Bolyai, Lobachevsky, Gauss, Riemann and of Helmholtz himself, as a scientific basis for an empiricist philosophy of geometry, directly opposed to Kant's apriorism. But it also contains what is perhaps the first statement of a conventionalist position in this field (restricted, however, to a choice of two geometries). Finally, insofar as the factual foundation of geometry, according to the empiricist philosophy of Helmholtz, is, as we said, a peculiar fact, which is viewed as a condition of the very possibility of geometrical knowledge, Helmholtz can be regarded as paving the way for a new brand of apriorism, developed in our century by Hugo Dingler (1881–1954).

Helmholtz's researches on auditive and visual perception persuaded him that sensory stimuli only supply *signs* of the presence of the objects surrounding us, but do not give us a passably adequate idea of such objects. Such signs, in themselves quite devoid of sense, acquire a meaning by virtue of which they become a vehicle of knowledge, through a long process of association and comparison, beginning in the earliest days of childhood. This constitutes the foundation of inductive inferences, which eventually become so habitual, that they are automatically and instantly performed. Helmholtz's conception of perceptual knowledge is not too different from Kant's, who spoke of "spelling out sensory appearances, in order to read them as

experience".¹⁶ But Kant thought that geometrical knowledge, i.e. knowledge of the spatial structure to which all things around us must conform, is not acquired in this fashion, but is based on an immediate awareness of space which accompanies all our perceptions of spatial things but is not determined by them. Kant described this awareness as a kind of self-knowledge, *viz.* our intuitive (i.e. non-mediated and non-generic) knowledge of the 'form' of outer sense. This Helmholtz rejects. The science of geometry contains a vast – and ever growing – array of truths which can be inferred by purely logical means from a few principles or axioms. These axioms, in their turn, do not express the content of a non-empirical awareness of the structure of space, but like everything else we know about the material world in which we live, they are learnt through the processes of manipulation and observation, guided by intelligent comparison and inference, which Helmholtz, like Kant, calls experience. The existence of alternative consistent systems of geometrical axioms has probably contributed to suggest this empiricist thesis. A professional natural scientist like Helmholtz, if faced by several equally rational theories that purportedly concern the same subject-matter, will feel inclined to judge that only experience can decide between them. But such feelings are not a rational ground of belief. A truly powerful argument for Helmholtz's geometrical empiricism is provided by his own discovery that the existence of rigid bodies, reputedly "a fact of observation", goes a long way to determine the structure of space.

Like most scholars of his time, Helmholtz believed that Kant's claim that geometrical knowledge is non-empirical rests on the alleged fact that we can only visualize (*anschaulich vorstellen*) spatial relations which agree with Euclid's geometry. By visualization, we must understand here that kind of imaginative representation of spatial figures which we all have had while attempting to solve a problem in elementary geometry with closed eyes. Helmholtz attacks this supposedly Kantian position from two sides. In the first place, in order to establish the unavailability of the Euclidean axioms, the inner visualization ought to be absolutely exact. "Otherwise we could not say whether two straight lines prolonged to infinity will intersect once only or twice, or whether every straight line that cuts one of two parallels must also cut the other lying in the same plane. Imperfect ocular estimates cannot pass for the transcendental intuition, since the latter demands absolute precision (*man muss nicht das so un-*

vollkommene Augenmass für die transcendente Anschauung unterschieden wollen, welche letztere absolute Genauigkeit fordert).¹⁷ It goes without saying that our images of geometrical objects lack such precision, especially with regard to their metrical properties. In the second place, we are actually able to visualize the state of affairs in a non-Euclidean space. This is not easy, but it is not very much harder than visualizing, say, all the loops of a thread tied in a difficult knot, or the plan of a labyrinthic building which we have just finished visiting. The important thing is to understand rightly what it means to visualize a state of affairs we have never met in actual experience. "By the much abused expression 'to visualize' (*sich vorstellen*) or 'to be able to figure out how something happens', I understand – and I do not see how anything else can be understood without it losing all meaning – the power of imagining the whole series of sensible impressions that would be had in such a case."¹⁸ Helmholtz proposes several examples of visualization of non-Euclidean situations which were probably suggested by his experiments with distorting eye-glasses. One of them anticipates Lewis Carroll's *Through the Looking-glass*. A convex mirror maps an open region of ordinary space into an imaginary space where bodies behave in a most remarkable fashion. The mapping is injective, and every straight line and every plane in the outer world is represented by a line and a surface in the image.

The image of a man measuring with a rule a straight line from the mirror would contract more and more the farther he went, but with his shrunken rule the man in the image would count out exactly the same number of centimetres as the real man. And, in general, all geometrical measurements of lines or angles made with regularly varying images of real instruments would yield exactly the same results as in the outer world, all congruent bodies would coincide on being applied to one another in the mirror as in the outer world, all lines of sight in the outer world would be represented by straight lines of sight in the mirror. In short, I do not see how men in the mirror are to discover that their bodies are not rigid solids and their experiences good examples of the correctness of Euclid's axioms. But if they could look out upon our world as we can look into theirs, without overstepping the boundary, they must declare it to be a picture in a spherical mirror, and would speak of us just as we speak of them; and if two inhabitants of the different worlds could communicate with one another, neither, so far as I can see, would be able to convince the other that he had the true, the other the distorted, relations.¹⁹

A second example shows that something similar may be said of BL space (which Helmholtz calls pseudospherical space), as represented in the interior of a Euclidean sphere (Section 2.3.7). Beltrami's mathematical description of this model enables us to predict exactly

what an observer placed in the centre of it would see. In agreement with the above definition of 'to visualize', Helmholtz concludes: "We can picture to ourselves the look of a pseudospherical world in all directions, just as we can develop the concept of it".²⁰

But the fact that the axioms of geometry are not known a priori does not imply, in Helmholtz's opinion, that we do not have an intuitive, non-empirical awareness of spatiality as such. Indeed, "Kant's theory of the forms of intuition given a priori is a very clear and happy expression of the state of affairs; but these forms must actually be so empty and free (*inhaltsleer und frei*) that they might receive every contents which can make its appearance in the corresponding form of perception".²¹ How does Helmholtz conceive space as an a priori 'form of sense'? To make himself clear, he proposes an analogy: it lies in the nature of our visual faculty that we must see everything in the guise of colours spread out in space. "This is the innate form of our visual perceptions."²² But this does not in any way determine how the colours that we actually see lie beside one another or how they succeed each other. Likewise, the representation of all external objects in spatial relations might be the form given a priori in which alone we can represent such objects; but this does not imply that certain spatial perceptions must go together, e.g. that if a triangle is equilateral its angles must be equal to $\pi/3$. Helmholtz emphasizes that the general form of extendedness that we may regard as given a priori must be quite indeterminate. This cannot mean, however, that it has no determinations at all. Shall we maintain, as suggested by Schlick, that its determinations are indescribable, like, say, the difference between sweet and bitter?²³ There is one passage which clearly implies that at least the dimension number is a definite property of the general form of our outer sense (as understood by Helmholtz).²⁴ Since the number of dimensions of space is conceived by him in connection with its manifold structure, consistency requires that we also regard this structure as included in the general form of extendedness.²⁵ This does not mean, of course, that the mathematical notion of a differentiable manifold is known to infants. But it must mean, if it means anything, that the said notion arises from our attempt at intellectually dissecting and recomposing a natural idea of space we have always been familiar with. If this is right, the properties of pure space can be stated in axioms; indeed they have been so stated by Helmholtz himself (in H1). But they are

not mentioned in the traditional axioms of geometry, which are what Helmholtz has in mind when he says that the axioms are empirical. Such axioms, he says, do not belong to the pure theory of space for they speak of quantities. But “we can speak of quantities only if we know of some way by which we can compare, divide and measure them. All space-measurements, and, therefore, in general, all quantitative concepts applied to space, assume the possibility of figures moving without change of form or size”.²⁶ We have experienced the existence of such figures since our earliest youth. But it does not follow from the pure idea of space. Helmholtz recalls that Riemann had shown that such figures can only exist in an R -manifold of constant curvature.²⁷ His own mathematical researches, as we saw in Section 3.1.2, have allegedly shown that the existence of rigid figures suffices to determine the geometry of a manifold up to a parameter (the constant Riemannian curvature). In this sense, and if we grant his operationist premise, Helmholtz may be right in claiming that geometry rests on a factual foundation. But the fact upon which it is said to rest is a very special fact. Strictly speaking, there are no absolutely rigid bodies. Every solid piece of matter is liable to suffer deformations under the influence of heat, gravitation, etc. Two congruent bodies moved about for some time along different paths will no longer fit exactly into the same mould. How can we analyze the physical causes acting on our bodies so that we may conclude that the deformation of the latter is not caused just by their displacement? Helmholtz was well aware of this difficulty. After introducing his basic question “What propositions of geometry express factually significant truths?” he adds:

It is not easy to answer this question [...] because the spatial figures of geometry are ideals to which the bodily figures of the real world can only approximate, without ever satisfying all the requirements of the concept, and because we must judge the permanence of shape, the flatness of the planes and the straightness of the lines we find in a solid body, precisely by means of the same geometrical propositions we wished to prove factually in this particular case.²⁸

The fact that (approximately) rigid bodies exist can only be known, therefore, if we possess the idea of a (perfectly) rigid body. Bearing this in mind, Helmholtz acknowledges that “the notion of a rigid geometrical figure may be conceived indeed as a transcendental notion, which has been formed independently of actual experiences, and which will not necessarily correspond with them”.²⁹ He adds:

Taking the notion of rigidity thus as a mere ideal, a strict Kantian might certainly look upon the geometrical axioms as propositions given a priori by transcendental intuition, which no experience could either confirm or refute, because it must first be decided by them whether any natural bodies can be considered as rigid. But then we should have to maintain that the axioms of geometry are not synthetic propositions in Kant's sense, because they would state only what follows analytically from the notion of a rigid geometrical figure, as it is required for measurement. Only such figures as satisfy those axioms could be acknowledged as rigid figures.³⁰

The reference to a transcendental intuition is probably ironic,³¹ but Helmholtz's claim in this passage is perfectly serious and very important. It clearly anticipates the familiar epistemological conception of scientific notions and theories as free creations of the human mind, which are not originated in experience but must be tested by it. But the notion we are concerned with here is not an ordinary scientific notion. According to Helmholtz, if the facts fail to satisfy it, spatial measurement will turn out to be impossible. Consequently, a whole field of experience, which provides the basis for physical science, will fail to exist. The notion of a rigid body must therefore be regarded, if Helmholtz is right, as a concept constitutive of physical experience, that is, as a transcendental concept in the proper Kantian sense. And experience, at least objective, scientific, measurement-controlled experience, cannot but conform to it. Helmholtz stands therefore nearer to Kant than it seems at first sight. There is one big novelty, however. The role of the concept of a rigid body in the constitution of scientific experience does not consist in presiding, like a Kantian category, a purely mental process of organization of sense-data; but in regulating the manufacture and use of material instruments of measurement. This idea will be taken up and worked out by Hugo Dingler.

A space which contains perfectly rigid bodies is an *R*-manifold of constant curvature. Will experience allow us to decide between the different alternatives contained in this notion? A positive curvature is excluded if space is not finite. But even granting that it is infinite, we still have the choice between Euclidean and BL geometry. Let us hear what Helmholtz has to say about this:

We have no other mark of rigidity of bodies or figures but the congruences they continue to show whenever they are applied to one another at any time or place and after any rotation. We cannot however decide by pure geometry [...] whether the coinciding bodies may not both have varied in the same sense. If we judged it useful for any

purpose we might with perfect consistency look upon the space in which we live as the apparent space behind a convex mirror [. . .]; or we might consider a bounded sphere of our space, beyond the limits of which we perceive nothing further, as infinite pseudo-spherical space. Only then we should have to ascribe to the bodies which appear to us to be rigid, and to our own body, corresponding dilatations and contractions and we should have to change our system of mechanical principles entirely.³²

On the face of it, this passage says that the choice between Euclidean and BL geometry is a matter of convenience, so that, at least within this limited spectrum of alternatives, geometry is conventional. In terms of Helmholtz's original question, this conclusion can be stated saying that Euclid's fifth postulate is not a "truth of factual significance" but a consequence of the chosen mode of expression. This position was held later by Henri Poincaré (1854–1912).³³ Did Helmholtz anticipate him? I would say yes, insofar as he did publish the passage quoted above, which clearly suggests the conventionalist thesis. But it is apparent that Helmholtz did not understand his words quite in that sense. He points out that a change in the geometry would impose a change in the laws of mechanics. And he is not willing to grant that the latter are, up to a point, no less conventional than the former. In his opinion, the reference to mechanics settles the question. This implies of course that a decision concerning the truth of Postulate 5 cannot be reached by geometrical experiments alone. But that was to be expected after Helmholtz's earlier assertion that the axioms of geometry do not belong to the pure theory of space. He now adds: "Geometric axioms do not speak about spatial relations only, but also at the same time about the mechanical behaviour of our most rigid bodies in motion".³⁴ As a consequence of it, we must conclude that geometry – that is, physical geometry – does not provide a groundwork for mechanics but must be built jointly with it. It might even seem that Helmholtz expects the more elementary mechanical principles to provide a foundation for geometry. At any rate, he does not ask how much in *these* principles has a factual import, and how much is merely a matter of linguistic preference. One thing is clear: *pure* physical geometry is indeterminate; "but if to the geometrical axioms we add propositions relating to the mechanical properties of natural bodies, were it only the principle of inertia, or the proposition that the mechanical and physical properties of bodies are, under otherwise identical circumstances, independent of place, such a system of propositions has a real import which can be confirmed or

refuted by experience, but just for the same reason can also be gained by experience".³⁵ This is a very powerful argument against the Kantian philosophy of geometry and is perhaps the main reason why the latter could not survive the discovery of non-Euclidean geometries: a priori knowledge of physical space, devoid of physical contents, is unable to determine its metrical structure with the precision required for physical applications.

In his reply to Land (1877), Helmholtz upholds an unmitigated empiricism. He proposes a simple experiment in order to decide the issue between the three geometries of constant curvature. I paraphrase: As soon as we have a method to determine whether the distances between two-pairs are equal ("i.e. physically equivalent") we can also determine if three points A, B, C are so placed that no other point $D \neq B$ satisfies the equations $d(D, A) = d(B, A)$ and $d(D, C) = d(B, C)$ (where d stands for *distance*). We say then that A, B, C lie in a straight line. Let us choose three points P, Q, R not in a straight line, such that $d(P, Q) = d(Q, R) = d(R, P)$, and two further points A, B, such that $d(A, P) = d(B, P)$, and P, Q, A on the one hand, and P, R, B on the other, lie in a straight line (Fig. 17). Then, if $d(A, P) = d(A, B)$, the Euclidean geometry is true; but BL geometry is true if $d(A, B) < d(P, A)$ whenever $d(P, A) < d(P, Q)$ and spherical geometry is true if $d(A, B) > d(P, A)$ whenever $d(P, A) < d(P, Q)$. (Helmholtz, G, p. 70). Helmholtz is right indeed, *if we have a method to determine whether the distances between two point-pairs are equal, that is, physically equivalent*. The whole issue turns therefore about this notion, which Helmholtz defines as follows: "Physisch gleichwertig nenne ich Raumgrößen, in denen unter gleichen Bedingungen und in gleichen Zeitabschnitten die gleichen physikalischen Vorgänge

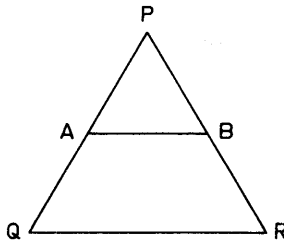


Fig. 17.

bestehen und ablaufen können." ("I call such spatial magnitudes physically equivalent in which equal physical processes can occur and develop under equal conditions and in equal times." – Helmholtz, G, p.69; my translation; the passage does not occur in the English text published in *Mind* in 1878.)

*Schlick objects that we cannot measure time without measuring distances in space, so that we cannot determine the physical equivalence of two magnitudes unless we know beforehand how to determine the equality of distances (Schlick in Helmholtz, SE, p.143). Schlick is wrong; our most exact clocks measure time without having to measure space. But it does not seem possible to establish, with a reasonable degree of accuracy, the physical equivalence of two spatial magnitudes, in the above sense, unless we employ methods of observation and control which involve the measurement of distances.

3.1.4 *Lie Groups*

There is a story that, just as the monarchs of Portugal and Castile partitioned the New World among themselves by the treaty of Tordesillas of 1494, so Felix Klein and Sophus Lie (1842–1899), while studying in Paris in the late 1860's, divided the emerging realm of group theory: Klein would take up discontinuous groups, letting Lie concentrate upon the continuous ones. The results of Lie's explorations are contained in the monumental *Theory of Transformation Groups*, edited with the assistance of Friedrich Engel. Part V of the third volume is devoted to a detailed study of the Helmholtz problem of space. Lie says that this problem was brought to his attention by his friend Klein, who told him that many mathematicians would not accept Helmholtz's reasoning, and suggested that the problem might be attacked successfully with the resources of Lie's group theory.³⁶ Lie reported his results on this matter in 1886, and published them with proofs in 1890. The content of his two papers of 1890 has been inserted in the considerably expanded treatment of Helmholtz's problem contained in Lie's big book.

Lie's approach to the problem is wholly foreign to the philosophy of physics. He treats it as a problem concerning the axiomatic foundations of geometry, regarded as a branch of pure mathematics. If Helmholtz's reasonings were correct, his axioms H1–H4 would provide a very concise characterization of Euclidean geometry and the classical non-Euclidean geometries, that is of the geometries

regarded as respectable since Klein's paper of 1871. Lie rejects Helmholtz's argument and he is not very happy about his axioms, but he translates the latter into several alternative sets of statements, which do provide an adequate characterization of those geometries. Lie's reformulations of Helmholtz's axioms are all based on the idea we have already mentioned in Section 3.1.2. We said there that the movements of a rigid figure in space – in terms of which Helmholtz developed his own version of the axioms – can be represented by a group of transformations of space. Lie takes this for granted and studies such transformations in the context of his theory of continuous groups. Before sketching Lie's treatment of Helmholtz's problem we must say a few words about this theory and show how the Helmholtzian concept of rigid movement can be made to fit into it.

Lie's theory is concerned with what he calls finite continuous groups of transformations acting on a manifold. A finite continuous group in Lie's parlance is what we now would call an n -dimensional connected Lie group. An n -dimensional Lie group is a set G which has the structure of a group and also that of an n -dimensional differentiable manifold. Between both structures there is the following relation: the group product $(g, h) \mapsto gh$ (which assigns to every pair (g, h) of elements of the group the product of h by g) is a differentiable mapping. A Lie group is connected if it is not the union of two disjoint open non-empty subsets.³⁷ The groups studied by Lie are usually complex manifolds (i.e. manifolds charted onto open subsets of C^n). They always are analytic manifolds (i.e. the coordinate transformations and the mapping $(g, h) \mapsto gh$ can be developed into convergent power series in a neighbourhood of each point in their domains). We say that Lie group G acts on a differentiable manifold M if there is a surjective differentiable mapping $f: G \times M \rightarrow M$, such that for every h, g in M and every m in M ,

$$f(hg, m) = f(h, f(g, m)), \quad f(e, m) = m, \quad (1)$$

(where e denotes the neutral element of G). We call f the action of G on M . To each g in G we associate the mapping $L_g: m \mapsto f(g, m)$, defined on M . This is indeed a transformation of M .³⁸ The set $\{L_g \mid g \in G\}$ is a transformation group homomorphic to G . It is isomorphic to G if, and only if, e is the only element of G such that $f(e, m) = m$ for every m in M . If this condition is fulfilled, we say that G acts *effectively* on M . Given a group G acting on a manifold M we

can easily define a group H that acts effectively on M . Let f denote the action of G on M . The set $K = \{g \mid g \in G \text{ and } f(g, m) = m \text{ for every } m \text{ in } M\}$ is a normal subgroup of G . The quotient group G/K acts effectively on M . It is not unreasonable, therefore, to restrict our discussion to effective groups. Since such a group is isomorphic to its associated transformation group, we need not distinguish it from the latter. This entitles us to write $g(m)$ or simply gm to denote both $f(g, m)$ and $L_g(m)$ (where g belongs to a group G acting through f on a manifold M which includes m). A group G acts *transitively* on a manifold M if for every pair x, y in M there is a g in G such that $gx = y$.³⁹

Lie studies m -dimensional Lie groups acting on an n -dimensional analytic manifold called R_n or "the space (x_1, x_2, \dots, x_n) ". The value of n is sometimes specified. Lie's researches are local in a twofold sense: (i) they are concerned with a neighbourhood of the identity element of the group or, at most, with the subgroup generated by such a neighbourhood; (ii) they consider the action of the group only on an open subset of R_n on which a chart is defined.

We must be careful not to confuse Lie's R_n with our R^n (the n th Cartesian power of the set of real numbers, endowed with the standard differentiable structure). Lie – or is it his editor Engel? – invites this confusion when, speaking of R_3 , he calls it "ordinary space" (*der gewöhnliche Raum*). But in actual usage, R_n denotes a complex manifold (also if $n = 3$). Reading Volume I of Lie's *Theorie der Transformationsgruppen* one has, at times, the feeling that "the space (x_1, \dots, x_n) " denotes *any* analytic complex manifold, or perhaps only the domain of a chart of such a manifold. But when Lie comes to determine all groups of this or that kind acting on R_n it becomes evident that R_n denotes a definite complex manifold. Contrary to what could be expected, this manifold is not homeomorphic to C^n . R_n is said to include an $(n - 1)$ -dimensional hyperplane "at infinity".⁴⁰ We conclude, therefore, that Lie's R_n is none other than our \mathcal{P}_C^n , that is, complex n -dimensional projective space. In some passages, Lie deals with what he calls real groups; these are m -dimensional real manifolds (the chart ranges are open subsets of R^m) and they are allowed to act upon a real manifold. The latter is also called R_n ; I take it that in this case R_n denotes \mathcal{P}^n , real n -dimensional projective space.⁴¹

We shall now explain briefly how the idea of rigid movement can be inserted in the framework of Lie's theory. We take the matter up

where we left it on p.160. The movements of a rigid body in Euclidean 3-space \mathcal{E}^3 are represented by a group of transformations of \mathcal{E}^3 , the group of Euclidean motions. According to Helmholtz's axiom H2 this group preserves a function on $\mathcal{E}^3 \times \mathcal{E}^3$ whose value is the same for all congruent point-pairs. However, not every transformation of \mathcal{E}^3 which preserves congruence between point-pairs will preserve congruence between spatial figures. The group of Euclidean motions is therefore a subset of the group of transformations of \mathcal{E}^3 which preserve congruence between point-pairs. Let x denote a Cartesian mapping and let P_0, P_1, P_2, P_3 be four points in \mathcal{E}^3 such that $x(P_0) = (0, 0, 0)$, $x(P_1) = (1, 0, 0)$, $x(P_2) = (0, 1, 0)$ and $x(P_3) = (0, 0, 1)$. An isometry $g: \mathcal{E}^3 \rightarrow \mathcal{E}^3$ is completely determined if we know the four values $g(P_i)$ ($0 \leq i \leq 3$); in other words, the 12-tuple formed by the coordinates of these four points defines a unique isometry g . But these coordinates cannot be chosen arbitrarily. Indeed, if we fix $g(P_0)$, the images of the other three points by g must lie on the unit sphere centred at $g(P_0)$. Thus, if we know the three coordinates of $g(P_0)$, the position of $g(P_1)$ depends only on two additional arbitrary real numbers, e.g. the two angles which the directed line from $g(P_0)$ to $g(P_1)$ makes with the planes $\{P \mid x^1(P) = 0\}$ and $\{P \mid x^2(P) = 0\}$. If we fix both $g(P_0)$ and $g(P_1)$, then $g(P_2)$ and $g(P_3)$ must lie at right angles on the unit circle centred at $g(P_0)$ on the plane perpendicular to the line $(g(P_0), g(P_1))$. The choice of a single real number will therefore suffice to fix $g(P_2)$. If $g(P_0)$, $g(P_1)$ and $g(P_2)$ are known, there are only two positions which $g(P_3)$ can take, namely, the two points at unit distance from $g(P_0)$ on the perpendicular through this point to the plane on which $g(P_0)$, $g(P_1)$ and $g(P_2)$ lie. If $g(P_3)$ is one of these two points, the tetrahedron K whose four vertices lie at the four points P_i ($0 \leq i \leq 3$) is congruent with the tetrahedron $g(K)$ whose vertices lie at the points $g(P_i)$; if $g(P_3)$ is the other point $g(K)$ is a mirror image of K . The transformations of \mathcal{E}^3 that preserve congruence between point-pairs fall into two classes: the class of those which map K onto a congruent tetrahedron and the class of those which map K onto a mirror image. Only a transformation of the first class is a Euclidean motion. Our analysis shows that six arbitrary real numbers suffice to define it uniquely. There are many ways of choosing those six numbers, but if we settle upon one, we define thereby an injective mapping of the set \mathcal{M} of the Euclidean motions into \mathbb{R}^6 . By suitably modifying and restricting this mapping we can obtain an atlas which

bestows on \mathcal{M} the structure of a 6-dimensional analytic manifold. \mathcal{M} is clearly a group. To show that it is a Lie group acting on \mathcal{E}^3 we would also have to show that the action and the group product $(g, h) \mapsto gh$ are analytic mappings.

*Assuming that they are, we shall discuss briefly Lie's method of representing the action of \mathcal{M} on \mathcal{E}^3 . Let U be the domain of a chart t defined at the identity e in \mathcal{M} , with coordinate functions t^1, \dots, t^6 . Let y be a Cartesian mapping of \mathcal{E}^3 , with coordinate functions y^1, y^2, y^3 . Then (t, x) is a chart of $\mathcal{M} \times \mathcal{E}^3$ defined on $U \times \mathcal{E}^3$. Let f denote the action of \mathcal{M} on \mathcal{E}^3 . The restriction of f to $U \times \mathcal{E}^3$ can be represented by three functions $f_i: t(U) \times y(\mathcal{E}^3) \rightarrow \mathbf{R}$, defined as follows:

$$f_i(t^1(g), \dots, t^6(g), y^1(P), \dots, y^3(P)) = y^i(f(g, P)),$$

$$(g \in U, P \in \mathcal{E}^3, i = 1, 2, 3). \tag{2}$$

Lie writes t_j for $t^j(g)$, y_k for $y^k(P)$, y'_i for $y^i(f(g, P))$. The above representation is then given as follows:

$$y'_i = f_i(t_1, \dots, t_6, y_1, \dots, y_3) \quad (i = 1, 2, 3). \tag{3}$$

Since $f(g, P)$ is the point $g(P)$ on which $P \in \mathcal{E}^3$ is mapped by the motion g , the functions f_i give us, for a fixed g , the coordinates of $g(P)$ in terms of the coordinates of P . On our assumption that \mathcal{M} is indeed a Lie group acting on \mathcal{E}^3 , the representative functions f_i are analytic. If $g, h \in U, P \in \mathcal{E}^3$, we have that

$$y^i(f(gh, P)) = y^i(f(g, f(h, P))) = f_i(t(g), y(f(h, P)))$$

$$= f_i(t(g), f_1(t(h), y(P)), f_2(t(h), y(P)),$$

$$f_3(t(h), y(P))), \tag{4}$$

(where $t(g)$ denotes the sextuple $(t^1(g), \dots, t^6(g))$, etc.). Knowledge of the functions f_i enables us therefore to compute the coordinates of $f(g, P)$ for every g in \mathcal{M} which is the product of elements of U . Lie usually represents the action of a Lie group G on the space \mathbf{R}_n by means of analytic functions defined like the f_i above. Let us call this the standard representation of group action. The representation is local. However, though it is originally defined only on a neighbourhood U of $e \in G$, it can be extended, in the fashion we have explained, to the full subgroup of G generated by U . This subgroup is

equal to G if G is connected and U includes the inverse of each of its elements. On the other hand, no representative functions f_i can be defined on an arbitrary pair (g, x) ($g \in G, x \in \mathbb{R}_n$) unless there is a chart defined at both x and $f(g, x)$. Since $\mathbb{R}_n (= \mathcal{P}_\mathbb{C}^n)$ is not wholly covered by any chart, it may well happen that the latter condition is not fulfilled.

3.1.5 Lie's Solution of Helmholtz's Problem

Lie's approach to geometry was deeply influenced by Klein's views (Part 2.3). To obtain a unified theory covering Euclidean geometry and the classical non-Euclidean systems, we imbed the Euclidean space \mathcal{E}^3 in $\mathcal{P}_\mathbb{C}^3$. Each motion f in \mathcal{M} is extended to the whole of $\mathcal{P}_\mathbb{C}^3$. The set \mathcal{M} of (extended) Euclidean motions is then a subgroup of the general group of analytic transformations of $\mathcal{P}_\mathbb{C}^3$ into itself. \mathcal{M} is, in fact, contained within another subgroup of this group, which includes all collineations of $\mathcal{P}_\mathbb{C}^3$. The said subgroup contains other important subgroups, related to the Euclidean motions, namely the groups of collineations that map a non-degenerate quadric onto itself. These fall into two families. Each group of one family maps a given real quadric onto itself, each group of the other maps a purely imaginary quadric onto itself. Lie chooses a quadric of each kind and takes the corresponding group as a representative of its family. The two groups thus defined are called by him the groups of non-Euclidean motions of \mathbb{R}_3 (that is, of $\mathcal{P}_\mathbb{C}^3$). These names and concepts are easily extended to the n -dimensional case.

These ideas provide the context for Lie's statement of Helmholtz's problem: *What properties are necessary and sufficient to characterize the group of Euclidean motions and the two groups of non-Euclidean motions of $\mathcal{P}_\mathbb{C}^n$, thus distinguishing them from all other groups of analytic transformations of $\mathcal{P}_\mathbb{C}^n$?*⁴² Let us call this the Helmholtz-Lie or HL problem. Lie gives two principal solutions of it, one of which is valid only for $n = 3$. They are preceded by two other subsidiary solutions, based on a direct reworking of Helmholtz's paper of 1868. Lie's treatment of the problem rests on a close study of the group-theoretical implications of Helmholtz's axioms H2 and H3. In order to state these implications, we introduce the concept of a group-invariant. Let G be a Lie group acting on a manifold M . An n -point invariant of G is a function $f: M^n \rightarrow \mathbb{R}$, such that for every $g \in G, x_1, \dots, x_n \in M, f(x_1, \dots, x_n) = f(g(x_1), \dots, g(x_n))$. To avoid trivial exceptions, we exclude constant functions. We say that an invariant f

depends on other invariants f_1, \dots, f_k , if the value of f at each point x in M is determined by the values of f_1, \dots, f_k at x . An n -point invariant of G is *essential* if there does not exist a set $\{f_i\}_{1 \leq i \leq k}$ of m_i -point invariants ($m_i < n$ for every i), on which it depends. Now let M denote the still indeterminate n -dimensional manifold mentioned in Helmholtz's axioms. Let us represent the movements of a rigid body in M by a Lie group G acting on M . Since H3(a) postulates free mobility, G must act transitively on M . Consequently G cannot have a one-point invariant. H2 clearly states that G has a two-point invariant (described there as "an equation" not altered by movement, between the $2n$ coordinates of each point-pair). We denote this invariant by d . H3(a) implies that G has no essential n -point invariants for $n > 2$. H3(b) implies that G has only one two-point invariant (or rather, that all two-point invariants of G depend only on one). Let us explain this. H3(a) states that the movements of a point $x \in M$ are restricted only by the equations binding its coordinates to those of every other point of M . In other words, they are restricted only by the requirement that $f(x, y) = f(g(x), g(y))$ for every $y \in M$, $g \in G$, and for every two-point invariant f of G . This means that there can be no essential n -point invariant of G for $n > 2$, because if there was one its existence would impose additional restrictions on the movement of x . We know that G has at least one two-point invariant d . H2 and H3(a) do not imply that there are no other two-point invariants of d , but such is the purport of H3(b). According to the latter, if we fix a point x in M , every other point y in M is bound in its movements by a single equation. We see now that this can only refer to the condition $d(x, y) = d(g(x), g(y))$ for every $g \in G$. But if G had another two-point invariant d' , not dependent on d , every $g \in G$ would have to fulfil the additional condition $d'(x, y) = d'(g(x), g(y))$, and this requirement would further restrict the movements of an arbitrary $y \in M$ when a given $x \in M$ is fixed.

Lie understands that the n -dimensional manifold we have been calling M in his R_n (that is, \mathcal{P}^n), or an open subset of R_n . The foregoing analysis shows that solution of the HL problem along Helmholtz's lines could be obtained by solving first the following group-theoretical problem: *To determine all the finite continuous groups of R_n which have no one-point invariant, exactly one (independent) two-point invariant and no essential k -point invariant for $k > 2$.* The problem is solved for R_3 in Lie's second paper of 1890. Lie shows that all such groups must be 6-dimensional. The powerful

machinery of his theory enables him to give an exhaustive list, both for the general case of complex groups and for the case of real groups. These lists include the three groups of Euclidean and non-Euclidean motions, but they also include several other groups. The HL problem will be solved if we can find properties of the first three groups which are not shared by the remaining groups.

We shall omit the details of Lie's alternative axiom systems,⁴³ and shall only sketch the main idea of his final solution of the n -dimensional case. This solution deals with real Lie groups, that is, Lie groups charted into \mathbb{R}^m ("groups with real parameters", in Lie's idiom). The manifold on which they act, denoted by R_n , is, as usual, the complex space $\mathcal{P}^n_{\mathbb{C}}$. Lie's characterization of the Euclidean and non-Euclidean groups of motions utilizes the concept of *free mobility in the infinitesimal*, which we shall now define. Let G be a Lie group acting on a manifold M . We say that G fixes a tangent vector v at $P \in M$ if, for every g in G , $g_{*P}(v) = v$ (this implies, by the way, that, for all $g \in G$, $g(P) = P$; we express this by saying that G fixes P).⁴⁴ Now let G be a real Lie group acting on R_n ($n \geq 3$). We say that G possesses free mobility in the infinitesimal at a real point $P \in R_n$ if, for every set of $n - 2$ linearly independent tangent vectors v_1, \dots, v_{n-2} at P , there is a proper subgroup of G which fixes v_1, \dots, v_{n-2} , but the only subgroup of G which fixes $n - 1$ linearly independent tangent vectors at P is the improper subgroup $\{e\}$, whose sole member is the identity. Lie's conclusion is stated thus: If a real finite continuous group of R_n ($n \geq 3$) possesses free mobility in the infinitesimal at every point of general position, it is a transitive $\frac{1}{2}n(n + 1)$ dimensional group which is similar (through a real point-transformation) to the group of Euclidean motions or to one of the two groups of non-Euclidean motions of R_n .⁴⁵ The Euclidean group distinguishes itself from the others because it alone possesses a proper normal subgroup (the group of translations).⁴⁶ By a point of general position I suppose that we must understand an arbitrary real point inside a given connected open set of R_n . In fact, the group of Euclidean motions does not possess free mobility in the infinitesimal at the points "at infinity".⁴⁷

Lie takes his solution of the HL problem for a conclusive proof of Riemann's claim that only on R -manifolds of *constant* curvature can a figure be freely rotated or displaced without expanding or contracting. In a way he is right. But we should not overlook an important

difference between Lie's approach and Riemann's. The latter thought he spoke about different spaces, which may have incompatible global topological properties (thus, his spherical space is compact, while Euclidean space is not). He believed that one of these spaces (at most) could provide a true representation of physical space. Lie, on the other hand, is concerned with different groups acting on one and the same manifold, complex projective space. The basic geometrical concept of congruence is *defined* on this space, or rather, on a suitable open subset of it, by the *choice* of one of those groups. Two figures inside the suitable region will be said to be congruent if one is the image of the other by a transformation of the chosen group. Did Lie take \mathcal{P}^3 for an adequate representation of physical space? This thoroughbred mathematician does not waste one word on the matter. But he was no doubt aware of the fact that every problem in 19th-century mathematical physics has to do with entities which can be represented in an open subset of \mathcal{P}^3 . Questions concerning the global structure of real space he would probably have dismissed as metaphysical.

*Lie shows in a short note that Riemann's Postulate R3 follows directly from the requirement of free mobility in the infinitesimal. The proof does not depend on the theorem that characterizes the Euclidean and the non-Euclidean groups. "Every real group (of R_n) which possesses free mobility in the infinitesimal at a real point of general position leaves a positive definite quadratic differential expression invariant [. . .]. Riemann's axiom concerning the line element can be thus derived from the axiom of free mobility in the infinitesimal, even without actually determining the groups that possess such free mobility." (Lie, TT, Vol. III, pp. 496 f.). The reader will recall that the main aim of Helmholtz's paper of 1868 was to deduce R3 from the requirement of free mobility of (finite) rigid bodies.

3.1.6 Poincaré and Killing on the Foundations of Geometry

Other mathematicians applied Lie's theory of transformation groups in their researches on the foundations of geometry at about the same time as he did. We shall comment here on two works by Henri Poincaré (1854–1912) and Wilhelm Killing (1847–1923).

On November 2, 1887, Poincaré submitted to the Société Mathématique de France a paper on the fundamental hypotheses of geometry.⁴⁸ In it, he recalls that geometry, as a demonstrative science,

must rest on undemonstrated premises. These, however, will not be found among the propositions stated under the name of *axioms* at the head of geometrical treatises, for such are either definitions or general principles of mathematical analysis. The necessary assumptions are introduced surreptitiously in the proofs of particular theorems. Not all these assumptions are necessary, however, for some of them could be deduced from the others. This leads to the following problem: *To state without redundance all the necessary assumptions of geometry.* Poincaré's paper is an attempt to solve this problem for two-dimensional or plane geometry.

He begins with a short discussion of a family of two-dimensional geometries which he calls "quadratic". These are obtained from projective space geometry, but, as two-dimensional geometries, they can be made to stand on their own feet. The general characterization of a quadratic geometry is given as follows. Let S be a quadric. Any line where S intersects a plane which passes through its diameter, we call 'straight'; every other plane section of S we call a 'circle'. If m, n are two 'straights' meeting at a point P in S , the size of the angle (m, n) is defined as follows: let p, q be the two rectilinear generators of S through P ; let k denote the cross-ratio $(m, n; p, q)$; the size of the angle (m, n) is $\log k$ if p, q are real, $(1/i) \log k$ if p, q are imaginary (this depends only on the nature of S). The length of a segment PQ on a 'straight' m is defined as follows: m is obviously a conic; let X, Y be its two points at infinity; we denote by k the cross-ratio $(P, Q; X, Y)$; the length of PQ is $\log k$ if m is a hyperbola and $(1/i) \log k$ if m is an ellipse (again this depends only on the nature of S). On the basis of ordinary projective space geometry, we can obtain infinitely many theorems about 'straights' and 'circles' and the figures formed by them on a quadric S . Now drop the quotation marks. If S is an elliptic paraboloid, the theorems will read exactly like the theorems of Euclidean plane geometry. If S is a two-sheet hyperboloid, the theorems read like those of BL plane geometry. If S is an ellipsoid, they agree with the theorems of spherical geometry. These are the three two-dimensional geometries familiar to Poincaré (Klein's elliptic geometry has apparently escaped him). But there are still other quadratic geometries, which arise if S is a non-degenerate one-sheet hyperboloid, or one of its degenerate forms. Poincaré's first aim is to furnish all quadratic geometries (regarded as plane geometries, i.e.

independently of their original definition by means of projective space geometry) with a common axiomatic foundation.

Poincaré states two axioms common to every two-dimensional geometry:

(A) The plane has two dimensions.

(B) The position of a figure in the plane is determined by three conditions.

He considers that these apparently simple axioms entitle him to use Lie's group theory in the further specification of viable two-dimensional geometries. This implies that he, in fact, understands Axioms A and B as follows:

(A') The plane is a two-dimensional differentiable manifold.

(B') The motions of the plane constitute a three-dimensional Lie group acting on the plane.

Here, I mean by *motion* a transformation of the plane which maps every plane figure onto a figure regarded as equal to it (the *same* figure in a possibly different *position*, to use the words of Axiom B). Using B' we can characterize a two-dimensional geometry by choosing as its group of motions a three-dimensional group acting on R^2 . This choice determines what figures are regarded as equal (or *the same*) in that geometry. Lie had determined all three-dimensional groups of R^2 . Two of them are excluded by the following axiom:

(C) If a plane figure is not allowed to leave the plane and if two of its points are fixed, then the whole figure is fixed.

(This is equivalent to the following: C'. The group of motions of the plane does not contain a one-dimensional subgroup which fixes two points of the plane.) The remaining three-dimensional groups are the groups of motions of the quadratic geometries. These include Euclidean, BL and spherical geometry, and also, as we said, the geometry defined by a one-sheet hyperboloid. Poincaré takes pleasure in describing the paradoxical features of this geometry: (a) The length of the segment joining two points on the same rectilinear generator of the hyperboloid is equal to zero. (b) We recall that 'straights' are plane sections determined by a plane passing through a diameter of the hyperboloid; there are two kinds of them, namely ellipses and hyperbolae; no real motion of this geometry can map a 'straight' of one kind into one of the other kind. (c) No motion except the identity will fix a point P on a 'straight' m while mapping m onto itself. This

geometry will therefore be excluded by adding one of the equivalent axioms D or E.

(D) The distance between points P and Q is equal to 0 only if $P = Q$.⁴⁹

(E) If m and n are two straight lines meeting at a point P, m can be rotated about P until it coincides with n .

The axioms A, B, C and D or E are therefore sufficient to characterize the three classical geometries. Spherical geometry is excluded by:

(F) Two straight lines cannot meet at more than one point.

BL geometry is excluded by:

(G) The sum of the three angles of a triangle is constant.

Actually, A, B, C, G suffice to characterize Euclidean plane geometry, because D, E, F can be inferred from them. At the end of his paper, Poincaré makes a few important epistemological remarks which we shall examine when we discuss his philosophy of geometry (Part 4.4).

Killing's long paper "Ueber die Grundlagen der Geometrie" (1892) is more ambitious but less successful than Poincaré's. He deals from the outset with n -dimensional geometry. He sets up a system of eight axioms stated in familiar, intuitively appealing terms. But the reasonings based upon them do not seem to follow from them in a clear-cut way. Killing points out that a demonstrative science does not only require a set of undemonstrated premises but also a set of undefined concepts. His choice of primitive concepts for geometry is somewhat surprising. They are: *solid body, part of a body, space, part of space, to occupy a space (to cover), time, rest, movement*. I do not dispute that any set of terms can be chosen as primitive if they are appropriately combined in axioms in which no other non-logical terms occur. But Killing's use of his primitive concepts is not so neat. Thus, apparently because *time* is one of them, he feels entitled to introduce in the axioms such expressions as *simultaneously, earlier than, as soon as*, whose meaning is not explicitly defined, nor, it seems, sufficiently determined by the axioms in which they occur. To give an idea of Killing's style, let us quote Axiom V:

A body which before a movement has no part in common with a space and lies entirely within that space after the movement, reaches in the course of the movement a position in which only a part of it lies within that space.⁵⁰

Commenting on this axiom, Killing says that it asserts that movement

is continuous. Since not a word about continuity is said in the remaining axioms, we must understand that the fundamental concepts of *space*, *time* and *body* tacitly include all the properties traditionally associated to that term. When such a wealth of assumptions is hidden in the intended meaning of the primitive terms, the deductive method becomes a royal road to truth, as readers of metaphysical literature well know.

Although Killing's work is instructive in more than one way, we shall make only a few remarks about it.

(i) Killing says that his first seven axioms define a theory equivalent to the general theory of finite continuous transitive groups of transformations.⁵¹ The theory of intransitive groups, he says, can be obtained from the former through the study of subgroups. This theory he proposes to call *generalized geometry*, the science of "space-forms in a general sense". "Space-forms properly so-called" are defined however by adding Axiom VIII, which "supplies the concepts of size and shape".⁵² The purport of this axiom is approximately the following: Let A be a body consisting of two disjoint but connected parts B and C; let B occupy a space S at the beginning of a movement *m*. If, at no time during *m*, $B \cap S = \emptyset$, there exists a space $S' (\neq \emptyset)$ such that at all times during *m*, $C \cap S' = \emptyset$. C is therefore confined during *m* to a spatial region S'' which is contained in the complement of S' . Now, if a body K lies partly within S' at the beginning of a movement *m'* and partly within S at the end of *m'*, there is a time during *m'* when K lies partly in S'' (end of the axiom).

(ii) Killing tries to give a topological definition of the number of dimensions of a space. His words are far from clear and I am not sure that I have understood them rightly. His definition may be translated thus:

If $n + 1$ parts of space are mutually connected, each to each, and this connection persists after we remove from every part those regions which are not connected with another part, the highest number *n* which can be thus obtained is the number of dimensions.⁵³

I take it that space is endowed with a topology and that a part of space is the closure of an open set. Two parts of space are connected if they share a boundary point. "The regions which are not connected with another part" are those which lie outside an arbitrarily small neighbourhood of the common boundary. On this interpretation,

however, a plane will be seen to have infinite dimensions, since we can find, for every number n , $n + 1$ triangular parts which meet at a common vertex. We avoid this counterexample if we understand that two parts A and B are connected, in Killing's sense, only if there is a point P on their common boundary which does not lie on the boundary of any other part. (Killing himself, however, suggests nothing of the sort.) But even with this proviso, the plane would have three dimensions, not two, according to Killing's definition, as one can gather from Fig. 18. Killing's definition shows anyhow that as far back as 1892 there was a mathematician who was no longer willing to go on repeating that an n -dimensional space is a space that can be charted by means of n coordinate functions.

(iii) Killing's paper contains one contribution of permanent value: the concept of what we now call a Killing vector field. Let M be an n -dimensional R -manifold. Let the metric be defined in terms of a chart x by $g_{ij} = \mu(\partial/\partial x^i, \partial/\partial x^j)$. Let X be a vector field on M such that, in terms of x , $X = \sum_i \xi^i \partial/\partial x^i$. Then X is a Killing vector field if the ξ^i are solutions of Killing's system of differential equations:⁵⁴

$$\sum_{k=1}^n \left(\frac{\partial g_{ij}}{\partial x^k} \xi^k + g_{ik} \frac{\partial \xi^k}{\partial x^j} + g_{jk} \frac{\partial \xi^k}{\partial x^i} \right) = 0, \quad (1 \leq i, j \leq n). \quad (1)$$

It can be shown that a one-parameter group of transformations acting on an R -manifold preserves congruence if, and only if, it is generated by a Killing vector field.⁵⁵ Lie's results (p.178) imply that an n -dimensional manifold M admits at most $n(n+1)/2$ independent Killing vector fields, which generate its $n(n+1)/2$ -parameter transitive group of motions. If M admits the maximum number of Killing vectors it is said to be maximally symmetric. As we know, an R -manifold is maximally symmetric only if it has constant curvature. We shall illustrate the concept of a Killing vector field with an example. Let S be a surface of revolution with only one axis of

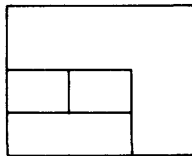


Fig. 18.

symmetry. A figure F on S cannot generally be transported over S in an arbitrary direction, without losing its original shape. However, any rotation about the axis of S maps F onto a congruent figure. Under such a rotation each point of F describes a curve which lies wholly on a plane normal to the axis of S . Every curve fulfilling this condition is the range of an integral path of a Killing vector field on S . On the other hand, the integral paths of every Killing vector field on S have their ranges along such curves. The notion of a Killing vector field suggests the convenience of (and provides an instrument for) studying the geometry of R -manifolds with intransitive groups of motions, that is, manifolds where congruence-preserving transformations which map a given point on any arbitrary point are not generally available. The study of such manifolds liberates us from Helmholtz's dogma of complete free mobility and the consequent recognition of only three 'established' geometries. We go, thus, a long way back to Riemann's broadminded conception of geometry.

3.1.7 Hilbert's Group-Theoretical Characterization of the Euclidean Plane

A drastic change in the approach to the HL problem was brought about by David Hilbert in his article "Ueber die Grundlagen der Geometrie" (1902). Three years earlier, he had published his celebrated *Grundlagen der Geometrie*, in which, as we shall see in Part 3.2, he endeavoured to analyze the presuppositions of Euclidean geometry into a number of simple conditions, instead of expressing their full import in a few powerful premises. The concept of a transformation group plays no role in that book. In the paper of 1902, however, as in the writings of Lie and Poincaré, the group of motions defines the geometry, and this makes for a great conciseness in the statement of its basic principles. But Hilbert finds that the assumptions of his predecessors were unnecessarily strong and shows how to characterize either Euclidean or BL geometry by means of a surprisingly frugal set of axioms. The paper is a masterpiece of careful, patient mathematical reasoning, making very few demands on the reader's specialized knowledge. We shall state and explain his axioms, so as to show wherein lies the novelty of his approach.

We saw that Lie characterized the classical metric geometries by means of a Lie group, i.e. a group which is a differentiable manifold, with differentiable, indeed analytic group mappings $(g, h) \mapsto gh$ and

$g \mapsto g^{-1}$. According to Hilbert, these conditions can be expressed in purely geometric terms only in a very unnatural and complicated way. The axiom system propounded by him, though based on the group concept, contains only very simple geometric requirements. The paper studies solely the foundations of plane geometry, but Hilbert believes that a similar axiom system can be set up for higher-dimensional geometries.

Hilbert first defines the plane. His definition is long, but translated into present-day terminology it is tantamount to the following: The plane is a topological space homeomorphic to \mathbb{R}^2 .

Hilbert did not possess, in 1902, the modern concept of a topological space, but his definition of the plane was a significant step in the development of that concept. He employs the notion of a Jordan domain. Let us explain what this means. Let f be a continuous mapping of a closed interval $[a, b] \subset \mathbb{R}$ into \mathbb{R}^2 . If f is injective on the open interval (a, b) , f is called a (plane) Jordan curve. If $f(a) = f(b)$, f is closed. Camille Jordan proved in a paper which set new standards of mathematical rigour that a closed Jordan curve f divides \mathbb{R}^2 into two regions, the interior and the exterior of f , so that, if $g: [0, 1] \rightarrow \mathbb{R}^2$ is another Jordan curve such that $g(0)$ lies on the interior of f and $g(1)$ lies on its exterior, then $g(t)$ lies on the range of f for some t ($0 < t < 1$). The interior of a Jordan curve is called by Hilbert a Jordan domain (*Gebiet*). If we endow \mathbb{R}^2 with the standard topology, every Jordan domain is indeed a connected open set. Let us paraphrase Hilbert's definition of the plane: A plane is a set π of objects called points, such that (i) there exists an injective mapping $x: \pi \rightarrow \mathbb{R}^2$; (ii) let $P \in \pi$; a neighbourhood (*Umgebung*) of P is a Jordan domain U_P such that $x(P) \in U_P \subset x(\pi)$; there exists a neighbourhood of P ; (iii) if U_P is a neighbourhood of P and J is a Jordan domain such that $x(P) \in J \subset U_P$, then J is a neighbourhood of P ; (iv) if $P, Q \in \pi$, U_P is a neighbourhood of P and $x(Q) \in U_P$, then U_P is a neighbourhood of Q ; (v) if $P, Q \in \pi$, there is a neighbourhood U_P of P such that $x(Q) \in U_P$. This definition implies that π is homeomorphic to \mathbb{R}^2 , if we allow x to induce a topology on π in the following obvious way: if J is a Jordan domain contained in $x(\pi)$, $x^{-1}(J)$ is an open set of π ; all open sets of π are such by virtue of this stipulation and the topological axioms. This is the weakest topology which makes x into a continuous mapping. Relatively to it, x is evidently a homeomorphism of π onto $x(\pi)$. We shall show that $x(\pi)$ is a connected open set of \mathbb{R}^2 . Choose

any point $p \in x(\pi)$; by (ii) there exists a closed Jordan curve whose interior contains p and is contained in $x(\pi)$; consequently $x(\pi)$ is open. Choose any two points p, q in $x(\pi)$; by (v) there exists a closed Jordan curve whose interior contains p and q and is contained in $x(\pi)$; consequently $x(\pi)$ is connected. Therefore, $x(\pi)$ is homeomorphic to \mathbb{R}^2 . Hence, π is homeomorphic to \mathbb{R}^2 . On the other hand, if we assume this, conditions (i)–(v) will follow.

Let x map π homeomorphically onto \mathbb{R}^2 . We paraphrase Hilbert's definition of motion: a motion of π is a bijective continuous mapping $g: \pi \rightarrow \pi$, such that, if $f: [a, b] \rightarrow \mathbb{R}^2$ is a closed Jordan curve, $x \cdot g \cdot x^{-1} \cdot f$ is a closed Jordan curve with the same (clockwise or counterclockwise) sense as f . Let gP denote the value of a motion g at $P \in \pi$. We consider now a set G of motions of π . If $P \in \pi, g \in G$ and $gP = P$, we call g a rotation about P . Let G_P denote the set of rotations about P . If $Q \in \pi$, the set $\{gQ \mid g \in G_P\}$ is called the *true circle* through Q , centred at P . If $(A, B, C), (A', B', C')$ are two point-triplets and there exists a motion g such that $gA = A', gB = B'$ and $gC = C'$, we say that the two triplets are congruent ($ABC \cong A'B'C'$). We can now state Hilbert's axioms:

(I) If $g \in G, h \in G$, then $g \cdot h \in G$.

(II) A true circle is an infinite set.

(III) Let $A_1, A_2, A_3, A'_1, A'_2, A'_3$ be points of π . If, for every $\varepsilon > 0$ there is a $g \in G$ such that $|x(A'_i) - x(gA_i)| < \varepsilon$ ($i = 1, 2, 3$), then $A_1A_2A_3 \cong A'_1A'_2A'_3$.⁵⁶

Hilbert proves that, if G fulfils these three axioms, then G is either the group of Euclidean motions or the group of BL motions of the plane.⁵⁷ Congruence, as defined above, is therefore synonymous either with Euclidean congruence or with BL congruence. Thus, the strong but quite familiar assumptions expressed in the definitions of *plane* and *motion*, plus Axioms I–III are altogether sufficient to characterize Bolyai's absolute geometry of the plane. We obtain Euclidean or BL plane geometry by merely adding the axiom of parallels or its negation.

Towards the end of his paper, Hilbert draws our attention to the difference between this axiomatic foundation of geometry and the one given in his *Grundlagen*. In the earlier book, continuity is postulated last. This naturally leads to ask which of the familiar propositions and proofs of geometry do not depend on this assumption. The answer to this question is quite surprising, as we shall see. In the paper of 1902,

continuity is assumed from the outset in the definition of the plane and of its motions, and Hilbert takes full advantage of it in his proofs. The main task consists now in finding the minimal conditions that must be added to continuity in order to define the basic geometrical notions of the circle and the straight line and to ascribe them the properties required for the construction of geometry. Such conditions, as we saw, are very modest indeed.

Elliptic geometry had been excluded from the very beginning by the assumption that the plane is homeomorphic to \mathbb{R}^2 . Hilbert observes in a footnote that there should be no difficulty in including it if we suitably modify his concepts and reasonings.⁵⁸ Such a modification would involve, of course, a change in the global topological properties of the plane. Hilbert does not employ this terminology. But his approach shows a new awareness of the significance of global properties, which had been so frightfully neglected in Lie-Engel's book, and in Poincaré's paper of 1887.

In the light of Hilbert (1902) we can restate the HL problem thus: Let S be a topological space and G a group of transformations of S ; what additional requirements must be met by S and G in order that G be characterized as one of the classical – i.e. Euclidean, hyperbolic, elliptical or spherical – groups of motions? J. Tits solved this problem in 1952. An improved solution was given by H. Freudenthal (1956), who has summarized his results as follows. Let S be a locally compact connected metric space. Let there exist, for any two sufficiently small congruent triangles in S , an isometry of S that maps one of the triangles onto the other. Then S is a real Euclidean, hyperbolic, elliptic or spherical space. (Freudenthal in Behnke *et al.*, FM, Vol. II, p. 532; for details, see Freudenthal's survey article in English, "Lie Groups in the Foundations of Geometry" (1965)). Another, very elegant solution of the HL problem which does not depend on differentiability assumptions was given in 1955 by Herbert Busemann in the context of his theory of G -spaces. (Busemann, GG, p. 336; I thank Professor H. Schwerdtfeger for drawing my attention to Busemann's work.)

3.2 AXIOMATICS

3.2.1 *The Beginnings of Modern Geometrical Axiomatics*

The geometer's ability to derive by sheer force of reasoning a multitude of complex and abstruse propositions from a few simple

and apparently obvious truths has always aroused the admiration of learned men and was probably the main reason why Euclid's *Elements* were given a privileged position in Western education. The deductive structure of the *Elements* was imitated in the two greatest scientific works of the 17th century, the Fourth Day of Galilei's *Discorsi* and Newton's *Mathematical Principles of Natural Philosophy*. It was also regarded by most philosophers of that time as an example to be followed in their writings, though only Spinoza had the courage to do so ostensibly.¹ Careful students of the *Elements* were by then aware that the book did not always live up to the standards of logical rigour for which it was praised and which it certainly observed in many of its proofs. We noted on p.44 that John Wallis knew that many demonstrations in Euclid depend on unstated assumptions. In his *Eléments de géométrie* (1685) Father B. Lamy (1640–1715) made a point of formulating several propositions “contained in the idea of a straight line”, which “geometers assume [...] without saying so”.² A similar tendency to make explicit that which is tacitly understood in the *Elements* is noticeable in some 18th-century German textbooks, such as Andreas Segner's *Elementa arithmeticae et geometriae* (1739) and A.G. Kästner's *Anfangsgründe der Arithmetik, Geometrie, Trigonometrie und Perspektive* (1758).³ Surprisingly, however, no attempt at bringing out every presupposition of Euclid and filling all the gaps in his proofs was carried out in earnest until the end of the 19th century. In his *Lectures on Modern Geometry* (1882), Moritz Pasch gave a rigorous axiomatic reconstruction of projective geometry. Further contributions to geometrical axiomatics were made by the Italians Peano (1889, 1894), Veronese (1891), Enriques (1898), Pieri (1899a, b). Hilbert published his *Foundations of Geometry* in 1899.

One might feel inclined to think that the rise of non-Euclidean geometry – which could not resort to genuine or apparent intuition in its proofs – must have powerfully contributed to stimulate the interest of geometers in unimpeachable logical deduction. In fact, Bolyai's monograph and the better parts of Saccheri's book are models of careful reasoning, and J.H. Lambert, the remarkable forerunner of Bolyai and Lobachevsky, was one of the first to see clearly that geometrical proofs should not depend on a “representation of their subject-matter”.⁴ It is probably no accident that J. Hoüel, who in the 1860's published French translations of Bolyai's *Absolute Science of Space* and Lobachevsky's *Theory of Parallels*, of Gauss'

correspondence with Schumacher and the basic papers by Riemann, Helmholtz and Beltrami, should have worked at the same time on a new axiom system for Euclidean geometry (Section 3.2.4). Nevertheless, by today's standards, none of these writers really went much further than Euclid himself in making explicit the premises of geometry. A statement such as Pasch's axiom, which says that a straight line running into the interior of a triangle eventually comes out of it, was, so to speak, too transparent for our authors to see it, present and active, in the most basic proofs of geometry.

Ernest Nagel (1939) is probably right in stressing the importance of projective geometry in the development of the new axiomatics. This branch of geometry was fully axiomatized by Pasch two decades before Hilbert's axiomatization of Euclidean geometry. Indeed, as we suggested on p.110, the counterintuitive features of the projective plane and of projective space made their axiomatic characterization almost imperative. Moreover, as Nagel rightly observed, duality, correlations and the free choice of the fundamental elements of space were certainly instrumental in making 19th-century geometers aware that their true concern was with abstract structures, not with particular things. The organization of geometry as a strictly deductive science, a collection of gapless axiomatic theories, was, of course, the natural way to deal with structures, because axiomatic theories are constitutively abstract. The unavoidably abstract nature of axiomatic theories will be explained in the next section. But one can approximately see what it means by recalling the well-known thesis of Hilbert, that the planes, lines and points of his *Grundlagen* may be taken to be *any* threefold collection of things – Hilbert once proposed chairs, tables and beer-mugs – which, given a suitable interpretation of the undefined properties of incidence, betweenness and congruence, happen to stand in the relations characterized by his axioms. The true subject-matter of the axioms and the theorems inferred from them is the net of relations in which points, lines and planes are caught, not the individual nature of the points, lines and planes themselves. What matters is the type of those relations as such, not those idiosyncratic traits they might derive from the peculiarities of the objects holding them. Now, the discovery of projective duality was certainly apt to suggest such a view of geometry, and of mathematics generally. Duality implies that the theorems, say, of plane projective geometry are true of the plane whether we regard it

as a set of points grouped in lines or as a set of lines grouped in points (pencils) (p.119). Correlations, which assign a point to each line and a line to each point, map the plane in the former acceptation onto it in the latter (and vice versa). Correlations are 'structure-preserving' in the following sense: if f is a correlation and P, Q are two points on line r , which meets line s on point X , then $f(r)$ and $f(s)$ are two points on line $f(X)$, and $f(P), f(Q)$ are two lines through $f(r)$. The suggestion lies near at hand that the substance of geometrical statements about collinear points and concurrent lines consists in what they say concerning the net of incidence relations in which points and lines are enmeshed, not in any information they might contain regarding the intrinsic nature of points and lines as such. This standpoint was strengthened when Plücker showed that space need not be viewed as composed of points and that one could also choose different kinds of curves or surfaces for its ultimate constituents.⁵ Depending on our choice of fundamental elements, space will exhibit a different structure. What matters geometrically are these several structures, not their embodiment in that unique entity, space. As we saw on pp.139ff., a structuralist view of geometry was quite clearly put forward in Klein's Erlangen Programme. Though Klein himself was wary of axiomatics (p.148), its development was certainly favoured by the increasing popularity of his views, for, as we shall now see, an axiomatic theory is most naturally suited to characterize an abstract structure.

3.2.2 Why Are Axiomatic Theories Naturally Abstract?

By an indicative sentence I mean a sentence liable to be *asserted*, i.e. used for stating a truth or a falsehood. Thus "Peter is five years old", and "If Peter were older, I should be happy to let him drive my car", are indicative sentences, while "Peter, for heaven's sake, will you stop mixing your Molotov cocktails on my desk!" is not. In the rest of this section, a sentence means always an indicative sentence. An axiomatic theory is determined by a set of sentences, the *axioms* of the theory. This set can be finite or infinite, but one must at any rate be able, in principle, to tell whether a given sentence belongs to it or not.⁶ The theory comprises all the sentences which are logical consequences of its set of axioms. These are aptly called the *theorems* of the theory. (Note that according to this definition every axiom is a theorem.) I shall now try to show that the very fact that axiomatic

theories are held together, so to speak, by the bonds of logical consequence, implies that they are essentially abstract, in the sense roughly sketched in the foregoing section and which I shall make more precise below.

Instead of saying that axiomatic theories are *abstract*, one often says that they are *formal*, because they are concerned with form, not with matter or content. But one must not confuse this meaning of *formal*, with that which opposes this word to *informal*. Axiomatic theories can be *formalized*, i.e. they can be expressed in a formal style, usually in an artificial language, in which word formation and sentence construction are subject to strict rules. The set of words and the set of sentences of such a language must be computable (see Note 6). Artificial languages employed in the formalization of axiomatics normally contain also a computable set of finite sequences of sentences, called proofs, which, if the formalization is sound, are so built that the last sentence or *conclusion* of a proof P is always a logical consequence of a computable subset of the sentences in P , called the *premises* of the proof. If all the premises of a proof P belong to a set Σ , P is said to be a proof from Σ ; its conclusion is then *provable from* Σ . If a sentence S is provable from the formalized version of the axioms of a theory, S obviously expresses a theorem of the theory. The set of sentences provable in a given artificial language from a given set of axioms is generally not computable. In a sound formalization of a theory a sentence will be provable from its axioms only if it is a theorem. On the other hand, one cannot expect, as a rule, that all theorems will be thus provable. Formality, as opposed to informality, is thus only an additional convenience, while formality in the sense of abstractness is, I contend, an essential feature of mathematical theories. Practically all contemporary mathematical writings are formal or abstract but, thank God, very few are formalized.

To prove my contention, I must elucidate the relation of logical consequence. This is a relation between a sentence and a set of sentences from which the former is said to follow. I do not know of any satisfactory explication of logical consequence applicable to the full range of sentences of a natural language. But here it will suffice to consider a fragment of English (or of any other civilized language) which contains all that is necessary for the statement of mathematical propositions. The smallest such fragment, if we give up all

embellishments, turns out to be very poor indeed. We shall call such a fragment *m*-English. Research done in the last hundred years has given us a pretty good idea of what *m*-English must look like. To avoid raising questions which would be out of place here, I shall give a rather crude sketch of its main features. In order to rid mathematical discourse of cumbersome circumlocutions and ambiguities, the meagre provision of ordinary English pronouns is supplemented or replaced in *m*-English by a computable set of symbols, known as *variables* (usually letters with or without numerical indexes, such as we use in mathematical statements throughout this book). All *m*-English sentences are in the present indicative, unqualified by so-called modalities. Sentences fall into two easily distinguished classes, which we shall call *basic* and *non-basic*. There are also two classes of basic sentences. A *basic sentence of the first class*, when asserted, ascribes a property to an entity or a relation to an ordered *n*-tuple of entities. A basic sentence of this class includes only two kinds of expressions: *n*-place *predicators* ($n \geq 1$), which, when the sentence is asserted, signify the ascribed property or relation, and *designators*, which when the sentence is asserted, denote the entity or entities to which the ascription is made. Predicators and designators will hereafter be called *interpretable words*. These include all variables. All other interpretable words are called *constants*. We must distinguish between *object* variables and constants, which behave as designators, and *predicate* variables and constants, which behave as predicators. Object variables are usually all of a kind, but in some contexts they can fall into several distinct classes. (Thus, in Hilbert's *Grundlagen* we find point variables A, B, C, \dots , line variables a, b, c, \dots and plane variables $\alpha, \beta, \gamma, \dots$) Predicate variables and constants can be classified from two points of view: (i) first-order predicate variables and constants stand for properties and relations of objects; second-order predicate variables and constants stand for properties and relations of properties or relations of objects, etc.; (ii) first-degree predicate variables and constants (of each order) stand for properties, *n*th degree predicate variables and constants ($n > 1$) for *n*-ary relations. Variables of each kind are ordered by numbering or by any other appropriate method (alphabetical order, etc.). A basic sentence of the second class consists of two designators, separated by the symbol '=' or one of its verbal equivalents ('is equal to', etc.). Such a sentence, when asserted, says that the entities denoted by

either designator are one and the same. The reader will note that '=' is not an interpretable word. Non-interpretable words in *m*-English are sometimes called logical words. Non-basic sentences are of two kinds: truth-functional sentences and existential sentences. A sentence *S* is truth-functional if its truth-value (*true* or *false*) is univocally determined, according to a fixed rule, by the truth-value of one or more sentences S_1, \dots, S_n , distinct from *S*, called its *components*. A sentence *S* is existential if it can be obtained from some other sentence *S'* by (i) substituting a suitable variable *x*, which does not occur in *S'*, for every occurrence of a given interpretable word in *S'*; (ii) prefixing the phrase 'there is an *x* such that' (which we shall abbreviate (*Ex*)). This phrase is called an *existential quantifier* and is said to *bind* the variable *x*. The bound variable *x* evidently behaves in the modified text of *S'* as a relative pronoun referred to (*Ex*). Every sentence *S* stands in a definite relation to a finite set of basic sentences which we call its base. If *S* is basic, its base is {*S*}. If *S* is truth-functional, its base is the union of the bases of its components. If *S* is existential, its base is the base of the sentence *S'* obtained from it by (i) dropping the first existential quantifier of *S* and (ii) replacing all occurrences of the variable bound by this quantifier by the first variable of the same kind which does not occur in *S*.

The fragment of *m*-English obtained by eliminating all expressions in which *k*th- or higher-order predicate variables occur (for some fixed positive integer *k*) will be called *k*th-order English. A *k*th-order axiomatic theory is the set of *k*th-order English logical consequences of a computable set of *k*th-order English sentences. Much progress has been made in the study of first-order theories.

Interpretable words are generally ambiguous. In order to make a definite statement by asserting a sentence *S*, one must fix the entity denoted by each designator in *S*, the property or relation ascribed by each predicator in *S* and the domain of entities over which all bound object variables are allowed to range. We take this domain to be non-empty. If there are several types of object variables, a non-empty domain of entities must be assigned to each type. Each such domain must include the denotata of the object constants of the corresponding type. This assignment fixes the range of every predicate variable. Thus, if all object variables range over a domain *D*, first-order predicate variables of *n*th-degree range over all *n*-ary relations

between elements of D , etc. Such an assignment of meanings to the interpretable words occurring in a set of sentences K will be called an *interpretation* of K .⁷ Interpretations must be *viable* – that is, they must assign to each word a meaning suited to its nature (third degree predicators should signify ternary relations, etc.) – and *consistent* – that is, each interpretable word must be assigned the same meaning wherever it occurs in K . The study of these matters is greatly eased by the assumption that every interpretation assigns a fixed denotation in the appropriate domain to each m -English variable. Hereafter, we assume that every interpretation fulfils these requirements. Let I be an interpretation of a set of sentences K . Let D_I be the non-empty domain assigned by I to the object variables of K (D_I can be partitioned into several domains, one for each type of object variable, as we observed above). If, on this interpretation, every sentence of K is true, we say that I *satisfies* K or is a *modelling* of K , and we call D_I a *model* of K . The reader should bear in mind, though, that a given domain D_I is a model of a set of sentences K through its association with a modelling I . The same domain might afford a different model when associated with another modelling.

We can now characterize logical consequence in m -English. A sentence S is a logical consequence of a set of sentences K if, and only if, every interpretation of $K \cup \{S\}$ which satisfies K also satisfies $\{S\}$. We use the abbreviation ' $K \models S$ ' for 'sentence S is a logical consequence of the set of sentences K '. It is clear that if $K \models S$ and K and S are consistently interpreted in any viable manner, S cannot be false if every sentence in K is true.

We see at once why axiomatic theories are essentially abstract or formal. Let K be the axioms of a theory T . Then S is a theorem of T if and only if $K \models S$. This relation does not depend on a particular interpretation of K and S . Indeed, we can replace all interpretable words in K and S by meaningless letters – as Aristotle did in his *Prior Analytics*, the earliest extant study of logical consequence – and it will still make sense to say that $K \models S$. The mathematician who studies an axiomatic theory need not worry about the referents of its sentences, though he will probably find that a model of its axioms can be a good guide in the search for theorems. The important thing is that, for every conceivable interpretation of the theory, if the axioms are true, then the theorems are also true.

Since the relation $K \models S$ holds independently of any particular

interpretation of K and S , we may say that it holds for the uninterpreted sentences of $K \cup S$, and that the study of axiomatic theories is a study of uninterpreted sentences. But we must be very careful not to confuse the uninterpreted sentences of a meaningful language, such as m -English or an artificial language into which m -English sentences might be translated, with the meaningless strings of symbols of a so-called uninterpreted calculus. A calculus is simply one of those artificial languages we mentioned on p.192, which have a computable set of words and a computable set of sentences. A calculus is said to be uninterpreted if no rules have been established for ascertaining the meaning of its words or the truth value of its sentences. Words and sentences are then nothing but strings of marks, potentially significant only in the loosest sense. If an uninterpreted calculus has a computable set of proofs, the conclusions of such proofs cannot be said to be logical consequences of their premises. They might or might not be, depending on the rules eventually agreed upon for determining the truth-value of sentences. On the other hand, an m -English sentence is not wholly devoid of sense, even if its interpretable words have not been given an unambiguous meaning or have been replaced by variables; just as a blank cheque signed by me is not a meaningless piece of paper, even if I have not named the beneficiary and have not written in the amount. Indeed, if the theorems of an abstract axiomatic theory were nothing but the provable strings of symbols of an uninterpreted calculus, mathematicians would be a sad lot. Not only would they, according to this view, spend the best time of their lives, that is, the time when they actually work on formalized theories, scribbling meaningless inkmarks according to fixed rules, but in their everyday professional work, in which they reason informally yet rigorously from ordinary language premises, they would be no better than a pack of fools who push pieces of ordnance around trusting that 'in principle' some wise man might understand their doings as moves in a strictly regulated game of strategy.

Because the uninterpreted sentences of an axiomatic theory are not meaningless, the variety of situations which they can describe when interpreted is not unlimited. The following example ought to make this clear. Let T be a first-order predicator of the third degree and let small italics be object variables of the same type. $Txyz$ says that x, y, z stand in relation T . We use the following abbreviations: if S is a

sentence, $\neg S$ is the negation of S , i.e. a sentence which is true if, and only if, S is false; 'for all x ' amounts to 'it is not the case that there is an x such that it is not the case that' (i.e. $\neg(Ex)\neg$). We characterize T by means of the following m -English sentences:

(i) For all x, y, z , $Txyz$ only if $x \neq y \neq z \neq x$.

(ii) For all x, y, z , $Txyz$ only if $\neg Tyzx$.

(iii) For all x, y, z, w , if either $Txyz$ or $Tzxy$ or $Tyzx$, and either $Txwz$ or $Tzxw$ or $Twzx$, and $y \neq w$, then either $Tyxw$ or $Twyx$ or $Txwy$.

(iv) For all x, y , if $x \neq y$, there is a z such that $Txzy$.

(v) For all x, y, z, u, v , if $x \neq y \neq z \neq x$, and $\neg Txyz$ and $\neg Tzxy$ and $\neg Tyzx$ and $Tyzu$ and $Tzvx$, there is a w such that $Txwy$ and either $Tuvw$ or $Twuv$ or $Tvwu$.

(vi) There is an x and a y and a z such that $x \neq y \neq z \neq x$ and $\neg Txyz$ and $\neg Tzxy$ and $\neg Tyzx$.⁸

Sentences (i)–(iii) come true if you let the object variables range over the set of integers and interpret $Txyz$ to mean " $x < y < z$ ". But on this interpretation, (iv) and (vi) are false ((v) on the other hand is trivially true, precisely because the interpretation does not satisfy (vi)). The following interpretations satisfy sentences (i)–(v): (a) object variables range over rational numbers, $Txyz$ means that $x < y < z$; (b) object variables range over instants, $Txyz$ means that x precedes y and y precedes z ; (c) object variables range over the points of a Euclidean plane, $Txyz$ means that x, y, z are collinear and y lies between x and z . Interpretations (a) and (b) fail to satisfy (vi). On the other hand, (c) is a modelling of the full set (i)–(vi). We obtain another modelling (c') if, in (c), we substitute 'BL' for 'Euclidean'.

Our example shows that by adding new axioms, which are not a logical consequence of the others, we can narrow down the range of interpretations which satisfy a theory. If this process were to lead us eventually to a theory which had one and only one modelling, such a theory would not be abstract, for it would characterize a unique domain of objects. We shall see, however, that this requirement cannot be satisfied. For greater precision, we restrict our discussion to first-order theories. The results we are about to state do not apply only to first-order axiomatic theories, as defined on p.194, but to any set of first-order sentences that includes all its first-order logical consequences. We call such a set a first-order theory in the extended sense. Let T be such a theory. Let C_T be the set of all constants

occurring in the sentences of T . Denote by T^* the set of all first-order English sentences whose constants belong to C_T . Plainly $T \subset T^*$. Let I_1 and I_2 be two modellings of T ; D_1 and D_2 , the corresponding models. I_1 and I_2 are said to be *structurally equivalent* if there exists a bijective mapping $f: D_1 \rightarrow D_2$ such that a sentence of T^* is true in I_1 of a collection of objects of D_1 if, and only if, it is true in I_2 of their respective images by f .⁹ T is a categorical theory if any two modellings of T are structurally equivalent. If T is axiomatic and categorical, we say that its axioms form a categorical system. Obviously, all the models of a categorical theory T are exactly alike with regard to the properties and relations characterized by T . Nevertheless, two models of T can stand in sharp contrast because of other properties and relations, which their respective objects exhibit, but which T , as applied to these models, does not even mention. A categorical axiom system will therefore specify a unique abstract structure of properties and relations, but not a unique set of things in which that structure is embodied. Obviously, non-categorical systems determine their modellings even more loosely. As a matter of fact, all the more important first-order theories of mathematics – namely, all those that have an infinite model – are not categorical.

There is another sense of the word *categorical*, in which Peano's axioms of arithmetic and Hilbert's axioms of geometry are indeed categorical systems, as it is often said (see e.g. Kline, MT, p.1014; however Kline's definition of *categorical* agrees better with *our* sense of the word). We call this the *classical* or *c-sense*. An early characterization of it will be found on pp.240f. In our own terms, we may informally define it as follows: A first-order theory T is *c-categorical* if (i) T is a specification of set theory and (ii) all modellings of T , in which the predicates 'is a set' and 'is a member of' are assigned their ordinary English meanings, are structurally equivalent.¹⁰ (i) means that T includes set theory and characterizes a specific type of sets (sets endowed with a specific 'structure'). (ii) implies that in all relevant cases, 'set' and 'set-membership' must be understood in their natural, naïve meaning. (ii) is, in fact, tantamount to treating the two basic set-theoretical predicates in question as non-interpretable words. Such was indeed at the turn of the century the favourite approach to those predicates,¹¹ but it was subsequently abandoned by most mathematicians when the set-theoretical paradoxes created the impression that the naïve understanding of set and set-membership

was not sufficiently precise for mathematical use. This led to the now current practice of axiomatizing set theory, whereby the permissible interpretations of the set-theoretical predicates are characterized by means of axioms in which they occur as undefined terms. The axiomatic approach to set theory in its turn raises difficulties which have, of late, become intolerable. But we cannot deal with them here.¹²

Mathematicians do not claim that their theorems are true but that they follow from their axioms. Some authors conclude from this that every mathematical theory is hypothetical, as they say, since its truth depends on the truth of its axioms, and the latter, they contend, are not held to be true, but are put forward only as suppositions or hypotheses. To judge the merits of this opinion one should bear in mind the following remarks. Let S be a theorem of a theory with axioms K . Mathematicians will state then that $K \models S$. This statement is a good deal stronger than what we would ordinarily call a hypothetical statement. $K \models S$ does not say merely that S will come true if a situation described by K , in some familiar acceptance of these sentences, is fulfilled. $K \models S$ says that S is true in every interpretation of $K \cup \{S\}$ which satisfies K . This far-reaching claim is made by mathematicians unconditionally, when they assert that $K \models S$. On the other hand, this claim would be trifling, if K is not true in any interpretation. Consequently, though the mathematician who states that S is a theorem which follows from K need not hold K to be true in a particular interpretation, he ought to make sure, lest his statement be pointless, that there is at least one modelling of K . This requirement is fulfilled by the more important mathematical theories if (i) the set of natural numbers exists (ii) the conditional existential postulates of ordinary axiomatic set theory are true, in their familiar English meaning.¹³ Neither of these assumptions can be said to be beyond every reasonable doubt. They may be viewed as the hypotheses which lie at the foundation of mathematics. Yet it is not the *truth* of mathematical theories, but rather their *significance*, that may be said to rest on this hypothetical basis.

3.2.3 *Stewart, Grassmann, Plücker*

The thesis that mathematical truths are hypothetical was held about a century before the rise of modern axiomatics by the Scottish philosopher Dugald Stewart (1753–1828). Stewart's position was

motivated partly by the cogency of mathematical demonstrations, partly by the fact that the theorems of geometry cannot be really true, since the dimensionless points, widthless lines, etc., to which they refer, are not actually found in nature.¹⁴ The theorems would be true, however, *if* these entities existed.

In mathematics – he writes – the propositions which we demonstrate only assert a connection between certain suppositions and certain consequences. Our reasonings, therefore, in mathematics, are directed to an object essentially different from what we have in view, in any other employment of our intellectual faculties – not to ascertain *truths* with respect to actual existences, but to trace the logical filiation of consequences which follow from an assumed *hypothesis*. If from this *hypothesis* we reason with correctness, nothing, it is manifest, can be wanting to complete the evidence of the result; as this result only asserts a necessary connection between the supposition and the conclusion.¹⁵

Stewart was one of the first writers to make this point so clearly.¹⁶ On the other hand, his discussion of this matter does not show any awareness that mathematics, thus conceived, will per force be abstract or formal.

In a tedious discussion of “mathematical axioms”, Stewart denies that these are the “foundation on which the science rests”. This is because he understands by axioms such generalities as Euclid proposed under the name of *common notions*. “From these axioms – says Stewart – it is impossible for human ingenuity to deduce a single inference.” He contrasts them with such genuine principles as “All right angles are equal to one another”, or Postulate 5, “which bear no analogy to such barren *truisms* as these: – ‘things that are equal to one and the same thing are equal to one another’; – etc.”¹⁷ In Stewart’s opinion, the principles of geometry are not the axioms, but the definitions. These he understands as hypotheses, which involve the assumption that the defined entities exist.

A conception of mathematics as the study of abstract structures or ‘forms’ freely conceived by the human intellect and devoid of intuitive contents was resolutely put forward by Hermann Grassmann (1809–1877) in his *Ausdehnungslehre* (1844). Since geometry refers to a given natural object, namely space, it does not belong to mathematics. Nevertheless, there must be a branch of mathematics “which in a purely abstract fashion generates laws similar to those which, in geometry, are bound to space”.¹⁸ That branch is the theory of

extension developed in the book. This should provide a foundation for geometry.

The specific principles of geometry must be based on our intuition of space. These principles are correctly conceived if they jointly express "the complete intuition of space" and if everyone among them contributes something to his purpose. Earlier presentations of geometry are defective, in part because they include principles which do not express any fundamental intuition of space; in part because they omit principles which do express such intuitions, and "which, later on, when it becomes necessary to use them, must be tacitly taken for granted".¹⁹ Grassmann maintains that the following two principles provide all that is required:

(I) Space is equally constituted in all places and in all directions, so that equal constructions can be carried out in all places and in all directions.

(II) Space is a system of the third level.

Principle II uses a technical term of the theory of extension and in this way subordinates geometry to that theory. A system of the third level (*System dritter Stufe*) is an instance of what Riemann called a three-dimensional continuous extended quantity.²⁰ But Grassmann assumes throughout that such a system is naturally endowed with the structure described in his book, which is that of a 3-dimensional real vector space, with the standard scalar product and the norm defined thereby.²¹ If we understand third level systems in their full Grassmannian sense, Principles I and II can only be satisfied by Euclidean-space geometry, which was probably the only three-dimensional geometry which Grassmann had ever heard of in 1844. However, Grassmann's contention that these two principles actually do provide a sufficient basis for geometry is very nearly true. There are, of course, obscure points in the foundations of the theory of extension itself. Thus, Veronese objects that it rests on an imprecise concept of continuity.²² There is also the difficulty of explaining how the structure of a third level system is embodied in space. Euclidean space can be given the structure of a three-dimensional normed real vector space by picking any point P to be the zero vector and choosing the tips of three mutually perpendicular congruent segments drawn from P for defining an orthonormal basis.²³ But this proposal makes sense only if we know how to recognize straight, congruent, perpendicular segments. Other mathematicians, working on the axiomatic foundation of geometry, will

devote considerable efforts to the exact characterization of such elementary concepts. Indeed, one might say that the major interest of the axiomatic systems of Pasch, Hilbert, etc., lies precisely in this.

As far as I can see, Grassmann had no thought of associating the formal or abstract nature of mathematics with the mathematician's search for logical consequences of the principles assumed by him. His contemporary, Julius Plücker (1801–1868), saw, at any rate, a connection between the scope of mathematical statements and the methods of mathematical proof. It is not clear, however, whether he considered this connection as a happy accident, an unexpected bonus, so to speak, of the methods employed, or whether he understood that abstractness and generality were of the very nature of the relations between sentences which such methods were designed to prove. Plücker writes:

If we carry through the proof of a theorem concerning straight lines (using the letters a, b, c, \dots to designate linear forms in two variables for representing such lines), we have, in fact, demonstrated an untold number of theorems. For if by the letters a, b, c, \dots , we no longer designate linear expressions but any general function in two variables, provided they are of the same degree, the conditional equations $F(a, b, \dots, m, n, \dots) = 0$ [which formulate the relations which hold between straight lines in the initial hypothesis], as well as all the equations derived from them, retain their meaning. [...] If we have such a proof-schema we may relate it to lines of any arbitrary order. [...] We may therefore carry over every theorem in projective geometry to curves of any arbitrary degree.²⁴

Every geometrical relation is to be viewed as the pictorial representation of an analytic relation, which, irrespective of every interpretation, has its independent validity.²⁵

3.2.4 Geometrical Axiomatics before Pasch

The novelty of Pasch's approach to the axiomatic foundation of geometry will be appreciated best by comparison with earlier efforts in this direction. We shall consider a few examples in this section.

The most popular text-book of geometry in the 19th century and perhaps the most successful mathematical best-seller ever was Legendre's *Eléments de géométrie* (1794), whose 37th French edition appeared in 1854. Legendre simplified Euclid's list of principles considerably. The earlier editions give definitions of *geometry*, *extension*, *line*, *point*, and *straight line*, and five axioms, mostly of the kind that Dugald Stewart said would never yield a single conclusion. On this slender basis, geometry can be built only with the aid of surreptitious assumptions. In

fact, Legendre's work can be profitably used by teachers of logic as a source-book of elegant, subtly fallacious arguments. Its showpiece is, of course, the demonstration of the parallel postulate.²⁶

Bernard Bolzano (1781–1848), the great Czech philosopher and mathematician, published in 1804 a booklet on the foundations of geometry, entitled *Betrachtungen über einige Gegenstände der Elementargeometrie* (Considerations on some objects of elementary geometry). Bolzano's attitude is a far cry from Legendre's complacency. In the preface, he states his conviction that no allegation of self-evidence can cancel the obligation of demonstrating a proposition, unless it is perfectly clear that no such demonstration is necessary and why it is not necessary (pp.IIff.). The book is divided into two parts. In the first, he claims to prove the main propositions about triangles and parallels while presupposing the "theory of the straight line". The second part is an avowedly provisional and incomplete presentation of the latter theory, which Bolzano considers "the hardest subject in geometry" (p.X). His own treatment of it rests on the following:

Principle. We do not have an idea a priori of any definite spatial thing. Consequently several entirely equal spatial things must be possible, of which exactly the same predicates are true. Therefore, if any spatial thing *A* is possible at a point *a*, a spatial thing *B*, equal to *A* ($B = A$), must be possible at any other point *b*.²⁷

The sentence I have italicized may be taken for a statement of the principle of homogeneity which, as we know, characterizes the maximally symmetric spaces that many late 19th-century mathematicians regarded as the proper subject-matter of geometry. (See p.184). Bolzano's choice of this principle as the foundation of the theory of the straight line and, hence, of all geometry bespeaks his sure grasp of essentials. The theory he builds on it is less remarkable for the cogency of its proofs than for the meticulous precision of its statements. Today, we would allow many of these statements to stand unproved, but Bolzano's contemporaries did not even take the trouble of formulating them. The relation between two points *a* and *b* is analyzed into two factors: the *distance* ab from *a* to *b* and the *direction* $D(a, b)$ from *a* toward *b* (§6). Bolzano demands a proof of the fact the "the distance from *a* to *b* is equal to the distance from *b* to *a*", but he confesses that he is as yet unable to supply one (§11). On the other hand, he claims to have proved that for any point *a*

there is one and only one point b which lies in a given direction and at a given distance from a (§10). This implies that a three-point system or *triangle* is uniquely determined by a point a , two directions from a and two distances marked, respectively, along each of those directions (§18). The relation between two directions $D(a, x)$ and $D(a, y)$ from the same point a is also analyzed into two factors: the *angle* between $D(a, x)$ and $D(a, y)$ and the *half-plane*, determined by $D(a, x)$ on which $D(a, y)$ lies (§13). These two factors are seen to correspond, respectively, to the factors of distance and direction that determine the relation between two points. Bolzano assumes without proof that the angle between two directions does not depend on the order in which they are taken (§14). Let $D(a, x)$ be a direction and let $D(a, y)$ ($\neq D(a, x)$) be the only direction stemming from the same point a which forms a given angle with $D(a, x)$. $D(a, x)$ and $D(a, y)$ are then said to be *opposite* directions (§15). This definition does not imply that the direction opposite to a given direction is unique, for there might be many different angles such that $D(a, x)$ makes each with one and only one direction (§16). Moreover, as Bolzano boldly points out, it does not even imply that opposite directions exist, for it is conceivable that every direction makes every given angle with several directions at a time (§24). However, according to him, the concept of opposite directions furnishes the basis for a satisfactory definition of the straight line if we grant one more assumption. This can be paraphrased as follows: If a , b and c are three points and $D(a, b)$ is the same as $D(a, c)$, then either $D(b, a)$ is the same as $D(b, c)$ and $D(c, a)$ is opposite to $D(c, b)$, or $D(b, a)$ is opposite to $D(b, c)$ and $D(c, a)$ is the same as $D(c, b)$; but if $D(a, b)$ is opposite to $D(a, c)$, $D(b, a)$ is the same as $D(b, c)$ and $D(c, a)$ is the same as $D(c, b)$ (§24). Bolzano contends that this assumption, like the two we mentioned earlier (§§11, 14), can be proved without using the concept of straight line.²⁸ He defines: a point m lies *between* points a and b if $D(m, a)$ is opposite $D(m, b)$; a *straight line* between two points a and b is an object that contains all the points lying between a and b and no other points (§26). It follows at once that any two points will determine a straight line between them. Bolzano 'proves' that if a point c lies between two points a and b , the straight line between a and c together with that between c and b form the straight line between a and b (§31). He fails to mention that c must be added to the former two lines to complete the latter.

Part I of Bolzano's work, which he considered more perfect, is not so interesting as Part II. It begins with definitions of equality (*Gleichheit*) and similarity (*Aehnlichkeit*). Two spatial things are equal if their determining elements are equal (§6). This is, of course, the kind of equality that we usually call 'congruence'. It follows from Part II, §18, that two triangles abc and $a'b'c'$ are equal in Bolzano's sense if sides ab and ac are equal to sides $a'b'$ and $a'c'$, respectively, and angle bac is equal to angle $b'a'c'$ (§14). Two spatial things are similar if all predicates that can be attributed to one of them by comparing its parts with one another, can also be attributed to the other (§16). Bolzano argues that two things are similar if their determining elements are similar (§17). He introduces a principle that we may call the principle of the relativity of distance: "No particular idea of any *definite distance*, i.e. of the definite way how two points lie outside each other, is given to us a priori" (§19). This principle is certainly not a consequence of the principle of homogeneity that we quoted on p.203, but it does look like a specification of the general epistemological statement that Bolzano inserted in his formulation of the latter principle: "We do not have an idea a priori of any definite spatial thing". If Bolzano understood the relativity of distance as a logical consequence of this statement he ought to have concluded also that we have no particular idea of a definite angle, such as the angle between two opposite directions, and his theory of the straight line would have crumbled down. On the other hand, if the relativity of distance is admitted as an independent principle, his theory of triangles and parallels presupposes more than just the theory of the straight line.

The relativity of distance is used by Bolzano to prove that two triangles abc , $a'b'c'$ are similar if angle bac equals angle $b'a'c'$ and $ab/ac = a'b'/a'c'$ (§21), and that in two similar triangles the angles opposite to proportional sides are equal (§23). He also proves (using §14) that if m is a straight line and a is a point outside it there is one and only one line through a that is perpendicular to m (§32). §§21, 23 and 32 are all that is required for proving the theorem of Pythagoras (§37), which, as we know, is the keystone of plane Euclidean geometry. Bolzano's proofs of the said three premises are defective but they could be improved with the resources at his disposal. This cannot surprise us, for the relativity of distance is essentially the same principle that John Wallis had used for proving Postulate 5 (p.44).

Staudt's *Geometrie der Lage* (1847) is often regarded as an important step towards a rigorous axiomatization of geometry. Though the book is not an axiomatic treatise, von Staudt, who was intent on making projective geometry into an autonomous science, independent of measurement, carefully states a long list of spatial properties and relations that he takes for granted, presumably because he thinks that they are intuitively obvious. The essential topological assumptions uncovered by Klein (1872b, 1874) remain unstated.²⁹

In his *Prinzipien der Geometrie* (1851), Friedrich Ueberweg (1826–1871) breaks new ground by proposing to base Euclidean geometry on the idea of rigid motion. This, as we saw in Section 3.1.2, is the keystone of Helmholtz's foundational work. A similar standpoint was adopted by Hoüel and Méray and it ultimately underlies Peano's treatment of congruence and Pieri's exact characterization of the common groundwork of Euclidean and BL geometry. We shall examine Ueberweg's axiom system in connection with his philosophical views in Section 4.1.2. I wish to note here, however, that Ueberweg thought that Euclidean space was the only conceivable three-dimensional manifold in which a figure can be moved rigidly, that is, undeformed, from any place and in any direction. Helmholtz, after reading Riemann and Beltrami, concluded that this feature is shared also by the spaces of constant positive and negative Riemannian curvature. Lie rigorously proved in the 1880's that no other three-dimensional Riemannian manifolds possess this property.

Ueberweg's friend and pupil, the Belgian philosopher J. Delboeuf (1831–1896) had rejected Ueberweg's characterization of Euclidean space at an earlier date, in his *Prolégomènes philosophiques à la géométrie* (1860). Geometry, he says, like every other science, must be grounded on *postulates* or *hypotheses*, i.e. first truths, regarded as objective, which state the fundamental qualities of its object.³⁰ "The objects of geometry are the *determinations of space*. We must therefore carefully analyse the contents of the notion of *determination* and of the notion of *space*. The results of our analysis will be the premises we are looking for."³¹ The scientific concept of space, he adds, is that of "an homogeneous receptacle", all of whose parts are endowed with the same properties.³² The homogeneity of space has two aspects: (i) "a definite portion of space can be carried anywhere in space"; (ii) "the general properties of such a portion are independent of its magnitude." Ueberweg's axioms determine property (i) only. A

manifold characterized by them, Delboeuf calls *isogeneous*. For it to be homogeneous it must also possess property (ii). "It follows that every determination of space, i.e. every *figure*, possesses two sorts of properties: some, which are independent of the *size* (*grandeur*) of the figure, belong properly to its *shape* (*forme*); [...] the others depend only on its size and are common to it and every other quantity. [...] *The mutual independence of shape and size is the first postulate of geometry.*"³³ This is, in fact, the assumption that John Wallis substituted for Euclid's Postulate 5 (p.44). We have just seen that Bolzano used an equivalent assumption for proving Pythagoras' theorem without using that postulate (p.205). However, neither Delboeuf nor his contemporaries were acquainted with the writings of Wallis and Bolzano. We may, therefore, credit Delboeuf with the independent discovery of the aforesaid remarkable characteristic of Euclidean space. His use of it in the deductive construction of geometry is unfortunately somewhat disappointing. He conceives of surfaces as boundaries of spaces, lines as boundaries of surfaces and points as boundaries of lines. A straight line is a homogeneous line. A plane is a homogeneous surface. Such lines and surfaces are given together with homogeneous space.³⁴ Delboeuf makes no further assumptions. Those we have mentioned are perhaps strong enough, but one would have wished that he had analyzed them somewhat more fully before attempting to deduce from them the fundamental propositions of geometry.

J. Hoüel (1823–1886), a French mathematician who devoted much time to the translation of the sources of non-Euclidean geometry into his language, wrote his *Essai critique sur les principes fondamentaux de la géométrie élémentaire* (1867) "to show the superiority of Euclid over most contemporary authors, in the exposition of the first principles of geometry".³⁵ The work consists of an annotated translation of Book I of Euclid's *Elements* and an "Exposition of the first principles of elementary geometry", which proposes a new axiom system.³⁶ Nine notes follow, some of them quite interesting. Though Hoüel does not say so explicitly, it is clear that, to his mind, Euclid's superiority over 19th-century writers lies mainly in the fact that he counted Postulate 5 among the indemonstrable principles of geometry. On this essential point, Euclid obviously sided with Hoüel's favourite non-Euclidean authors, against Legendre and his school.

Geometry, says Hoüel, is the study of a concrete magnitude, namely extension (*l'étendue*), which affects our senses. The latter reveal to us the fundamental properties of that particular kind of magnitude. Among the many properties thus disclosed, some are so simple, so easily verified, that people assimilate them to the abstract truths of arithmetic, the general science of magnitude. From such properties, stated in *axioms*, one can infer others, some of them no less evident than the first, others more recondite, which can only be brought to our attention by reasoning. These other properties are stated in *theorems*. The division between axioms and theorems is, up to a point, arbitrary. The number of axioms can also vary. The geometri-
cian should reduce them to a minimum and determine precisely how each theorem depends on them. Hoüel proposes four axioms. The first three amount, I should say, to a precise statement of Ueberweg's characterization of space (p.262). The fourth is equivalent to Euclid's fifth postulate.

"Geometry – writes Hoüel – is founded on the undefinable experimental notion of *solidity* or *invariability* of figures" (Hoüel, PFGE, p.41). A surface is the limit or boundary of two portions of space; the boundary of two portions of surface is a line; the boundary of two portions of line is a point. The object of geometry is the study of lines and surfaces. A figure is any set of points, lines or surfaces considered as invariable as to shape. (Hoüel, PFGE, p.42). Hoüel's four axioms are:

(I) Three points suffice, in general, to fix the position of a figure in space.

(II) There exists a line, called a *straight line*, whose position in space is fixed completely by the position of any two of its points, and which is such that every portion of this line is applied exactly on any other portion as soon as the two portions have two points in common.

(III) There exists a surface such that a straight line which passes through two of its points is entirely contained in it, and such that any portion of this surface can be applied exactly on the surface itself, either directly, or after inverting it by means of a half-rotation about two of its points. This surface is the plane. (Two straight lines, on the same plane, which do not meet even if indefinitely prolonged, are said to be *parallel*.)

(IV) Through a given point, one can draw only one parallel to a given straight line.

A few explanations clarify the language of Axiom III. But no further assumptions are made. In a beautiful note, Hoüel shows, following Farkas Bolyai, that the concept and the existence of the straight line and the plane can be established on a simpler basis. This is provided by the concept of *equal distance* between *pairs of points*, and the properties of the sphere, i.e. of the locus of points equidistant from a given point. But no attempt is made to determine which of these properties must be accepted as intuitively obvious, which follow from them. (Hoüel, PFGE, pp.71-73). Another note deals with the idea of "geometrical movement" which underlies Axioms I-III. This disregards the time required to perform the movement and is not "more complex than the ideas of magnitude or extension" (loc. cit., p.70). Hoüel fails to note that, if "geometry is founded on the [...] invariability of figures", geometric movement is not only indifferent to time, but also to the path followed by the moving figure (See pp.159f.).

In an essay on "The role of experience in the exact sciences", appended as Note I to the second edition of his book (1883), Hoüel treats geometry as an abstract deductive science, whose axioms are satisfied to a good approximation by the standard empirical interpretation. Such sciences are concerned with transformation laws of phenomena which can be determined "exactly", that is to say, so well that the remaining uncertainty is practically negligible. They consist of two parts: one, based on observation and experience, gathers facts and inductively derives the principles which are the foundations of the science; the other "is just a branch of general logic", which combines the principles in order "to deduce the representation of the observed facts and to predict new facts". When dealing with this combination of principles, one can ignore their experimental origin and the relationship of their consequences to real facts. On the other hand, it is important to verify whether the principles are mutually compatible and whether they can be reduced to a smaller set. Hoüel defines an *operation* as the "act which transforms one phenomenon into another".³⁷

To a succession of phenomena corresponds a combination of operations. In order to apply logic to the combination of operations it is in no sense necessary to know their real meaning and how they are performed. It is enough to have determined some abstract properties of these operations, which we might call *combinatory properties*. An abstract theory of the operations can be built on the sole consideration of these properties [...]. Operations can be simple, like the fundamental operations of algebra

[...]. In other cases, they are more complex: such are the constructions of geometry.³⁸

In these rational and abstract sciences it is essential to distinguish the hypotheses, considered in themselves, which are a priori essentially arbitrary and are subject only to the condition that they do not contradict each other; and the value of these hypotheses, regarded with a view to applications. Every abstract science, founded upon non-contradictory hypotheses and developed according to the rules of logic, is, in itself, *absolutely true*.³⁹

Paul Rossier, in his valuable survey of the history of geometrical axioms, extols the “revolutionary character”⁴⁰ of Méray’s *Nouveaux éléments de géométrie* (1874). Charles Méray (1835–1911) was a distinguished French mathematician, whose construction of the real number system, published earlier than Weierstrass’ and before Dedekind and Cantor developed theirs, deserves indeed to be better known.⁴¹ His textbook of geometry, however, seems to me an elaborate exposition of Hoüel’s ideas, which shows some improvements, but does not break substantially new ground.

3.2.5 Moritz Pasch

The *Lectures on Modern Geometry* published by Moritz Pasch (1843–1930) in 1882 are based on a course he taught from 1873.⁴² Pasch regards geometry as “a part of natural science”⁴³, whose successful application in other parts of science and in practical life rests “exclusively on the fact that geometrical concepts originally agreed exactly with empirical objects”.⁴⁴ It distinguishes itself from other parts of natural science because it obtains only very few concepts and laws directly from experience. It aims at deriving from these by purely deductive means, the laws of more complex phenomena. The empirical foundation of geometry is described in the second edition of Pasch’s book (1926) as a nucleus (*Kern*) of concepts and propositions. The nuclear concepts (*Kernbegriffe*) refer to the shape, size and reciprocal position of bodies.⁴⁵ These concepts are not defined, since no definition could replace the exhibition of appropriate natural objects (*der Hinweis auf geeignete Naturgegenstände*), which is the only road to understanding such simple, irreducible notions.⁴⁶ All other geometrical concepts must be defined in terms of the nuclear concepts or of previously defined concepts. The application of geometrical concepts is liable to some uncertainty (*Unsicherheit*), “as it happens with almost all the concepts which we have developed in order to grasp phenomena”.⁴⁷ The nuclear propositions (*Kernsätze*)

connect the nuclear concepts.⁴⁸ Their geometrical contents “cannot be grasped apart from the corresponding diagrams (*Figuren*). They state what has been observed in certain very simple diagrams”.⁴⁹ Instead of *nuclear propositions*, we shall, hereafter, say *axioms*. All other geometrical propositions must be proved by the strictest deductive methods.⁵⁰ Only those proofs are admissible in which every single step is grounded upon previously established propositions and definitions.⁵¹ All premises, without exception, must be stated explicitly, even if they look trifling (*unscheinbar*).⁵² Proved propositions are called theorems (*Lehrsätze*). “Everything that is needed to prove the theorems must be recorded, without exception, in the axioms.”⁵³ These must embody, therefore, the whole empirical material elaborated by geometry, so that “after they are established it is no longer necessary to resort to sense perceptions”.⁵⁴ “Theorems are not founded (*begründet*) on observations, but proved (*bewiesen*). Every conclusion which occurs in a proof must find its confirmation in the diagram, but it is not justified by the diagram, but by a definite earlier proposition (or definition).”⁵⁵ Pasch clearly understands the implications of these methodological demands:

If geometry is to be truly deductive, the process of inference must be independent in all its parts from the *meaning* of the geometrical concepts, just as it must be independent from the diagrams. All that need be considered are the *relations* between the geometrical concepts, recorded in the propositions and definitions. In the course of deduction it is both permitted and useful to bear in mind the meaning of the geometrical concepts which occur in it, but *it is not at all necessary*. Indeed, when it actually becomes necessary, this shows that there is a gap in the proof, and (if the gap cannot be eliminated by modifying the argument) that the premises are too weak to support it.⁵⁶

The empirically-grounded geometry deductively built by Pasch can therefore become the prototype of an abstract science, which ignores the origin of its principles and does not care about the applicability of its conclusions. In a paper of 1917, Pasch calls this science *hypothetical geometry*, because it rests on “hypothetical propositions”, which combine “hypothetical concepts”.⁵⁷

The *Lectures* are concerned with the projective properties of spatial figures. Undefined concepts are *point*, straight *segment*, *flat surface*. A point is a body which cannot be divided within the limits of observation.⁵⁸ Two points are *joined* by a segment, that is, a straight path between them, which includes many other points within it. A flat surface is a limited surface, which contains many points and

segments (though not necessarily every segment joining two of its points: a flat surface need not be convex). These concepts are characterized by two sets of axioms. The straight *line* (*Gerade*) and the *plane* (*Ebene*) are defined in terms of them. Pasch says that in order not to impair the (empiricist) standpoint adopted by him he has had to resort to the undefined concept of *congruence* in the definition of coordinates.⁵⁹ This is a relation between *figures*, that is, rigid configurations of two or more points.

The fundamental relations between points and segments are governed by the following nine axioms:⁶⁰

(S I) Two points can always be joined by a unique segment. (The segment joining points A and B is denoted by AB; A and B are its endpoints).

(S II) Given a segment, one can always indicate a point which lies within it.

(S III) If point C lies within segment AB, point A lies outside segment BC.

(S IV) If point C lies within segment AB, every point of segment AC is a point of segment AB.

(S V) If points C and D lie within segment AB and D lies outside segment AC, D lies within segment BC.

(S VI) Given two points A and B, one can always choose a point C, such that B lies within segment AC.

(S VII) If point B lies within segments AC and AD, then either point C lies within segment AD or point D lies within segment AC.

(S VIII) If point B lies within segment AC and point A lies within segment BD and if points C and D are joined by a segment, then A and B lie within segment CD.

(S IX) Given two points A, B, one can always choose a third point C such that none of the three points lies within the segment joining the other two.

If points A, B, C are such that one of them lies within the segment joining the other two, A, B, C are said to be *collinear*. If A, B, C are collinear, C is said to lie on the *line* AB, which is then said to go through C. Two lines are said to meet if there is a point which lies on both.

The fundamental relation between points, segments and flat surfaces are stated in the following four axioms.⁶¹

(E I) A flat surface can be laid through any three given points. (The

points are then said to be contained in the flat surface. Points contained in a flat surface P are called points of P .)

(E II) If two points of a flat surface are joined by a segment, there exists (*existirt*) a flat surface which contains every point of the foregoing, and also contains this segment.

(E III) If two flat surfaces P, P' have a point in common, one can indicate another point which is contained in a flat surface together with every point of P and in another flat surface together with every point of P' .

(E IV) If A, B, C, D are points of a flat surface, and point F lies within the segment AB , the line DF goes through a point of the segment AC or through a point of the segment BC . (Though Pasch does not say so, we must assume that A, B and C in E IV are non-collinear points.)

If four points A, B, C, D are contained in a flat surface and A, B and C are not collinear, D is said to lie on the plane ABC , which is called a plane through D .

From a philosophical point of view, Pasch's most remarkable feat is the introduction of the ideal elements of projective geometry using only the ostensive concepts of point, segment and flat surface and the empirically justifiable axioms S and E . We cannot consider this in detail, but I shall sketch Pasch's method.

Pasch proves the following theorem: Given four lines p, q, r, s , if the pairs $(p, q), (p, r), (p, s), (q, r), (q, s)$ are coplanar, but neither r nor s lie on plane pq , then the pair (r, s) is also coplanar.⁶² Let (a, b) be a pair of coplanar lines. A line c will be said to belong to the bundle ab if c does not lie on plane ab but (a, b) and (b, c) are coplanar, or if c does lie on plane ab and there is a fourth line d , not on plane ab , such that $(a, d), (b, d)$ and (c, d) are coplanar. The foregoing theorem implies that if two lines g, h belong to bundle ab , the pair (g, h) is coplanar, and the lines a, b belong to the bundle gh . Consequently a bundle is determined by any pair of lines belonging to it. If two lines of a bundle meet at a point A , all the lines in the bundle meet at A . Moreover, every straight line through A belongs to that bundle. We shall let A denote the bundle whose lines meet at point A . There are bundles, however, whose lines do not meet. These will be also denoted by capital letters, which, in this case, of course, do not at the same time denote points. Pasch stipulates that the sentence "point S lies on line g " will be understood *to mean the same as the*

sentence “line g belongs to bundle S ”.⁶³ Then, if S does not denote a point in the proper sense of the word, it is said to denote an *improper* point (*uneigentlicher Punkt*). A point S in this wider sense lies on a plane P (and P goes through S), if S lies on a line which is contained in P . Let A, B be two distinct points. Let AB denote the family or ‘pencil’ of planes through both A and B . C is said to be a point of pencil AB if C is a point which lies on every plane of the pencil. AB denotes also the line through A and B , if such a line exists. In that case, the line AB is the intersection of all planes of pencil AB and every plane in which line AB is contained belongs to this pencil. There are, of course, pencils of planes which do not have a line in common. Pasch stipulates that the sentence “point S lies on line AB ” will be understood to mean the same as “ S is a point of pencil AB ”.⁶⁴ Then, if AB does not denote a line in the proper sense, it is said to denote an improper line. Obviously, any pair of proper or improper points determines a line in this wider sense. A line is proper if one point on it is proper. Let a, b, c, d be proper lines through a proper point X . Using axioms S and E , Pasch is able to define the familiar relation ‘lines a and b are separated by lines c and d ’ (p.390). He proves that any four proper lines through a proper point can be grouped in two pairs, one of which is separated by the other. Consider now any set of four points A, B, C, D , on a (proper or improper) line m . Let X be a proper point not on m . We say that points A and B are separated by points C and D if the lines AX, BX are separated by the lines CX, DX . It can be shown that this relation does not depend on the choice of X . Pasch proves the following theorem: If A, B, C, D are four points such that the lines BC and AD meet, then the lines AC and BD meet and the lines AB and CD also meet.⁶⁵ Since the lines and points concerned need not all be proper, A, B, C, D might not be coplanar. Pasch stipulates, however, that the sentence “point D belongs to plane ABC ” will be understood to mean the same as “lines AD and BC meet”.⁶⁶ If AD and BC are not actually coplanar, ABC is said to denote an improper plane. It can be readily shown that, if words are used in their new, extended sense, two coplanar lines always meet. Also, every line meets every plane. Two planes always have a common line; three planes, a common point. The improper elements introduced by Pasch play exactly the same role as the elements ‘at infinity’ of classical projective geometry.

We turn now to Pasch’s concept of congruence. Let a, b, c, \dots

denote proper points. Two pairs of proper points, ab , $a'b'$, each marked on a rigid body, are said to be *congruent* if we can place a on a' so that b falls on b' ; also if there is a point-pair $a''b''$, marked on a rigid body, which is congruent with both ab and $a'b'$. This intuitive notion can be extended in an obvious way to figures of more than two points. According to Pasch, the following statements are evidently true of configurations of proper points marked on one or more rigid bodies, when congruence is understood in the foregoing sense. They are adopted as axioms of congruence.⁶⁷

(K I) Figure ab is congruent with figure ba .

(K II) Given a figure abc , there is one, and only one, proper point b' , distinct from a , b and c , such that ab is congruent with ab' and b' lies within segment ac or c lies within segment ab' .

(K III) If c lies within segment ab and if figure abc is congruent with figure $a'b'c'$, then c' lies within segment $a'b'$.

(K IV) If c_1 lies within segment ab , there is an integer $n \geq 1$ and n points c_2, \dots, c_{n+1} on line ab , such that segment ac_1 is congruent with segment $c_i c_{i+1}$ ($1 \leq i \leq n$) and b lies within segment ac_{n+1} . (*Axiom of Archimedes*).

(K V) If segment ac is congruent with segment bc , figure abc is congruent with figure bac .

(K VI) If two figures are congruent, their homologous parts are congruent.

(K VII) If two figures are congruent with a third figure, they are congruent with each other.

(K VIII) Given two congruent figures, if a point is added to one, one can always add a point to the other in such a way that the enlarged figures are congruent.

(K IX) Given two figures ab and fgh , such that ab is congruent with fg and h does not lie on line fg , if F is any flat surface with contains a and b , there is a flat surface G which contains F and exactly two points c and d , such that the figures abc and abd are congruent with fgh . There is, moreover, a point within segment cd which lies on line ab .

(K X) Two figures $abcd$ and $abce$ are not congruent unless all their points are contained on the same flat surface.

Axiom K VI introduces the new undefined term *homologous parts*. Its meaning is elucidated intuitively by Pasch: they are the parts which cover one another when two congruent figures are superposed. But

this elucidation is of no avail when drawing inferences from the axioms. Our conclusions should depend only on what the axioms themselves say. Now K VI, the only axiom where the term *homologous parts* occurs, does not really tell us much about them. It merely says that, if two congruent figures do contain such parts (God knows which!) as go by the name of “homologous parts”, these parts are congruent. Obviously, this will not do. Perhaps the following axiom would serve Pasch’s purpose better:

(K VI’) If figure F is congruent with figure F' , there is a bijective mapping $g: F \rightarrow F'$ such that every figure contained in F is congruent with its image by g .

Pasch’s axioms of congruence were a useful contribution to the analysis of congruence in Euclidean geometry, but their need in a system of projective geometry is far from being obvious. Pasch says that they enable him to introduce coordinates in a manner which does not prejudice his empiricist standpoint. But I am afraid that empiricism is inconsistent with the congruence axioms themselves, at least with K VII. This implies that congruence is a transitive relation. But one can easily produce a finite sequence of figures $a_1b_1, a_2b_2, \dots, a_nb_n$, such that a_ib_i can be made to coincide with $a_{i+1}b_{i+1}$ ($1 \leq i < n$), *within the limits of observation*, though a_1b_1 cannot be made to coincide with a_nb_n .

Axioms S and E provide a foundation for von Staudt’s construction of the fourth harmonic to three given collinear points.⁶⁸ Axioms K are invoked to justify the assignment of homogeneous coordinates to points of space after the manner of von Staudt and Klein (Section 2.3.9). The use of congruence axioms for this task is not quite consonant with von Staudt’s idea of projective geometry as a measurement-free science. Pasch’s argument is, on the other hand, the first truly rigorous proof of Klein’s contention that the assignment of homogeneous coordinates does not depend on Euclid’s parallel postulate. Axioms K, and in particular the Archimedean axiom K IV, enter essentially into the proof of a theorem on harmonic nets which replaces Zeuthen’s lemma (p.145) in Pasch’s construction of projective coordinates.⁶⁹ We defined harmonic nets on p.144. Three (proper or improper) collinear points A, B_0, B_1 , determine the net (AB_0B_1) . We call B_0 the zeroth and B_1 the first element of this net. The n th element of (AB_0B_1) is defined as the fourth harmonic to $AB_{n-1}B_{n-2}$ ($n > 1$). The theorem proved by Pasch can be stated thus:

Let A, B_0, B_1, P be collinear points. If A and B_1 are separated by B_0 and P , there is a positive integer n , such that the n th point of the harmonic set (AB_0B_1) is identical with P or is separated by A and P from the $(n+1)$ th point of the net B_{n+1} . In this last case, B_0 and B_{n+1} are also separated by A and P .⁷⁰

As we noted in p.145 Zeuthen's lemma follows from a postulate of continuity. The same can be said of the above theorem. Pasch points out that it is a consequence of the following axiom P, which can therefore be substituted in his system for axioms K:

(P) Let A_0, B be two distinct points. There exists then (i) a sequence of points A_1, A_2, A_3, \dots within segment A_0B , such that, for every positive integer i , A_i lies between A_{i-1} and B ; (ii) a point C of segment A_0B (possibly identical with B), such that no point of the sequence A_1, A_2, A_3, \dots lies between C and B , and that, given any point D within segment A_0C , not every point of the sequence A_1, A_2, A_3, \dots lies between A and D .⁷¹

Pasch believes however that Axiom P cannot be justified; firstly, because no empirical observation can refer to an infinite collection of things, and, secondly, because we cannot assume that a segment includes an infinite number of points, unless we broaden again the meaning of *point*, making it even more remote from its original intuitive sense.⁷²

Pasch's empiricist standpoint has another interesting consequence. Rational homogeneous coordinates provide numerical labels ("point-formulae", Pasch calls them) for every point of space. Moreover, a given assignment of such coordinates will label each point with more than one equivalence class of rational number quadruples. This is due to the fact that lines are not indefinitely divisible. There is a threshold below which one cannot distinguish points on a line. This can be stated more precisely thus: Let Φ denote a particular assignment of rational homogeneous coordinates to the points of a line m (according to the method of von Staudt-Klein-Pasch). If Φ assigns to point P on m the pair of rational numbers (x, y) —or, as we shall say for brevity, if (x, y) are Φ -coordinates of P ,—there is a rational number $\varepsilon > 0$ (dependent on Φ and P) such that, if x' is any rational number larger than $x - \varepsilon$ and smaller than $x + \varepsilon$, (x', y) are Φ -coordinates of P . Pasch acknowledges that these ideas are foreign to the usual conception of geometry. It is essential, he says, to show how the usual theory can be built upon the "empiricist infrastructure" developed by him. Consider again the foregoing example. Let (x_1, x_2) be Φ -coordinates of a point P on m . If $x_1/x_2 < g_1/g_2$ and (g_1, g_2) are

Φ -coordinates of P , then (h_1, h_2) are also Φ -coordinates of P whenever $x_1/x_2 < h_1/h_2 < g_1/g_2$. Pasch proposes the following stipulation: if h_1, h_2 are any real numbers such that $x_1/x_2 < h_1/h_2 < g_1/g_2$, we shall regard (h_1, h_2) as Φ -coordinates of a point $P' \neq P$, which approximately represents P . "We obtain thus a set of points which is not only everywhere dense, but also continuous. We thus attain a view of the straight line and its points which in the usual theory, i.e. in *mathematical geometry*, is given, from the outset, as something ready-made. While *physical geometry* need not discriminate between certain point-formulae, such as (x_1, x_2) and (h_1, h_2) in the example we have just given, in *mathematical geometry* these are unconditionally distinguished as so many 'mathematical' points."⁷³

3.2.6 Giuseppe Peano

Giuseppe Peano (1858–1932) is known chiefly for the five axioms which bear his name, and which provide the necessary and sufficient foundation of the elementary theory of natural numbers. They were published in 1889 in the artificial, canonical language invented by Peano for the communication of mathematical ideas. About the same time, he began working on the axiomatics of geometry. His contributions are contained in the pamphlet *I principii di geometria logicamente esposti* (1889) and in a long paper "Sui fondamenti della geometria" (1894). The former expresses in Peano's artificial language a set of axioms directly inspired by Pasch's axioms S and E. They constitute the groundwork of what Peano terms – borrowing Staudt's phrase – *geometria di posizione*. Today we would call them axioms of incidence and order. Peano derives some theorems, also in the artificial language, and adds sixteen pages of explanations and comments in Italian. In the paper of 1894, Peano reproduces the axioms of 1889 in Italian translation and adds a set of axioms of congruence, in fact, axioms of motion – motion being explicitly conceived by Peano as a transformation of the set of all points.

Geometrical discourse – says Peano – includes two kinds of words: geometrical words and words belonging to logic.⁷⁴ Geometrical words should be for the most part introduced through definitions, but it is, of course, inevitable to leave some undefined. After listing these, one should never use a geometrical word which has not been defined, directly or indirectly, in terms of them. Logical words are innumerable

in ordinary language, but Peano claims to have shown that they can be reduced to very few. The chief advantage of his artificial language is that it restricts the indispensable logical ingredient of discourse to a very small set of unambiguous words and constructions. It also enables us to codify the rules of inference, but this side of the matter, though duly exploited by Peano, is not emphasized by him in these works.

Peano agrees that the undefined terms of geometry must signify some very simple ideas, common to all mankind.⁷⁵ But this ordinary meaning of the basic or, as Peano says, *primitive* concepts of geometry is actually irrelevant to geometric theory. Thus, Peano's geometric Axiom I says "Class 1 is not empty" (" $1 = \Lambda$ "). If objects a, b belong to class 1, ab denotes a subset of class 1 (" $a, b \in 1. \supset .ab \in K1$ "). Class 1 is called in ordinary language, the class of *points*; ab is called the *segment* determined by points a and b . But geometric reasoning should not be influenced by the suggestions contained in these words. It must rest entirely on the axioms which determine the properties of the undefined objects of class 1 and of the undefined relation $c \in ab$ (read " c belongs to segment ab " or " c lies between a and b "). Peano drives this point home quite resolutely:

We are given thus a category of objects (*enti*) called points. These objects are not defined. We consider a relation between three given points. This relation, noted $c \in ab$, is likewise undefined. The reader may understand by the sign 1 any category of objects whatsoever, and by $c \in ab$ any relation between three objects of that category. [...] The axioms will be satisfied or not, depending on the meaning assigned to the undefined signs 1 and $c \in ab$. If a particular group of axioms is verified, all propositions deduced from them will be true as well.⁷⁶

Peano's "geometry of position" is based on seventeen axioms. The first eleven agree essentially with Pasch's axioms S. Peano uses the logical notions of negation, conjunction, disjunction, implication, equivalence, existential generalization and identity, the set-theoretical notions of belong to a set, being part of a set, the empty set, the union and the intersection of two sets, the singleton $\{x\}$, i.e. the set whose only element is the object x , and the two undefined geometrical concepts mentioned above, namely, the class or set of all points, and the point-set ab , determined by points a and b . My English version of Axioms I–XVII follows the original text in the artificial language, rather than Peano's Italian translation.⁷⁷

(P I) The class of points is not empty.

(P II) If a is a point, there is a point x which is not identical with a .

(P III) If a is a point, segment aa is empty.

(P IV) If a and b are distinct points, segment ab is not empty.

(P V) If a and b are points and c belongs to segment ab , c belongs to segment ba .

Definition: Instead of saying that b belongs to segment ac ($b \in ac$), we say that c lies on ray $a'b$ ($c \in a'b$). "The ray $a'b$ is, so to speak, the shadow of b when illuminated from a ." (Peano (1894), p. 56).

(P VI) If a and b are points, a does not belong to segment ab .

(P VII) If a and b are distinct points, ray $a'b$ is not empty.

(P VIII) If a and d are points, $c \in ad$ and $b \in ac$, then $b \in ad$.

(P IX) If a and d are points, and $b \in ad$ and $c \in ad$, then either $b \in ac$ or $b = c$ or $b \in cd$.

(P X) If a and b are points and $c \in a'b$ and $d \in a'b$, then either $c = d$ or $c \in bd$ or $d \in bc$.

(P XI) If a, b, c, d are points and $b \in ac$ and $c \in bd$, then $c \in ad$.

Definition: If a, b are distinct points, the line (ab) is the set $b'a \cup \{a\} \cup ab \cup \{b\} \cup a'b$. Three points are said to be *collinear* if they all belong to a given line. (In other words: a point is collinear with two distinct points if it is identical with one of them or if one of the three points belongs to the segment determined by the other two.)

(P XII) If r is a line, there is a point x which does not belong to r .

(P XIII) If a, b, c are three non-collinear points, and $d \in bc$ and $e \in ad$ there is a point f such that $f \in ac$ and $e \in bf$.

(P XIV) If a, b, c are three non-collinear points and $d \in bc$ and $f \in ac$, there is a point e such that $e \in ad$ and $e \in bf$.

Definition: A set of points is called a *figure*. If a is a point and k a figure, ak denotes the set $\{x \mid x \in ay, y \in k\}$. Peano proves that if a, b, c are three non-collinear points, $a(bc) = b(ac)$. This set can therefore be denoted by abc . It is called the *triangle* abc . If a is a point and k a figure, $a'k$ denotes the set $\{x \mid x \in a'y, y \in k\}$. (If r is a line, $a'r$ is the half-plane determined by r and a , and not including a .) If b, c are points, $a'(b'c)$ is the *angle* limited by rays $a'b$ and $b'c$. Let a, b, c be three non-collinear points. *Plane* (a, b, c) is the union of segments ab, ac, bc , rays $a'b, b'a, a'c, c'a, b'c, c'b$, triangle abc , figures $a'bc, b'ca, c'ab$, and angles $a'b'c, b'c'a$ and $c'a'b$. Four points are said to be *coplanar* if they belong to the same plane. (In other words, a point is coplanar with three distinct points if it is collinear with

one of them and with a point collinear with the other two.) Peano proves that if two distinct points a, b belong to a line r and to a plane p , r is contained in p (Peano (1889), §11, p.29).

(P XV) If h is a plane, there is a point a which does not belong to h .

(P XVI) If p is a plane, a a point not belonging to p and $b \in a'p$, then, if x is any point, either $x \in p$ or the intersection of p and ax is not empty or the intersection of p and bx is not empty.

Definition: A figure k is said to be *convex* if every segment determined by a pair of points of k is contained in k .

(P XVII) If h is a convex figure, a and b are points, $a \in h$ and $b \notin h$, there is a point x such that (i) either $x = a$ or $x \in ab$ or $x = b$; (ii) ax is contained in h ; (iii) the intersection of bx and h is empty. (P XVII implies that if a and b are points and k is a non-empty set of points contained in ab , there is a point x belonging to $\{a\} \cup ab \cup \{b\}$, such that (i) $k \cap xb = \emptyset$, (ii) for every point $y \in ax$, $k \cap yb \neq \emptyset$. To show this, choose $h = ak \cup \{a\}$. P XVII postulates, therefore, the continuity of the straight line).

A set of axioms is said to be independent if none of them is a logical consequence of the others. If a set of axioms is not independent, you can eliminate one or more axioms, and obtain a smaller set, which still determines the same axiomatic theory. In his paper of 1894, after reproducing P I–P XI, Peano remarks that the “first scientific question” regarding them is whether they are independent or not. He adds:

The independence of some postulates from others can be proved by means of examples (*esempi*). The examples for proving the independence of the postulates are obtained by assigning arbitrary meanings (*dei significati affatto qualunque*) to the undefined signs. If it is found that the basic signs, in this new meaning satisfy (*soddisfino*) a group of the primitive propositions, but not all, it will follow that the latter are not necessary consequences (*conseguenze necessarie*) of the former. [...] Hence, to prove the independence of n postulates, it would be necessary to give n examples of interpretation (*esempi di interpretazione*) of the undefined signs [...], each of which satisfies $n - 1$ postulates, and not the remaining one.⁷⁸

It is clear that, in 1894, Peano already understood the nature of axiomatic theories in the manner explained in Section 3.2.2. He proposes several interpretations of the undefined concepts of *point* and *segment* which show that some of the first eleven axioms are not a consequence of the others. He does not prove, however, the independence of the whole set. Let us mention three of Peano’s

“examples”. (1) If *point* means *integer* and $c \in ab$ means $a < c < b$, all axioms P I–P XI are verified, except P IV. (2) If *point* means *a real number of the closed interval* $[0, 1]$, and $c \in ab$ means $a < c < b$, all axioms P I–P XI are verified, except P VII. (3) Pick three lines through a point P. Eliminate all points to the left of P. We obtain three half-lines originating at P. Let *point* mean a point of any of these half-lines. If $c \in ab$ means that c lies on the shortest way leading from a to b over points in the agreed sense, all axioms P I–P XI are verified, except P X.

Peano’s axiomatic treatment of congruence depends on one more set-theoretical notion, besides those listed on p.219: the concept of a mapping (*corrispondenza*). This, like all other set-theoretical ideas, is viewed by Peano as a part of logic. Peano writes fx for the value assigned by the mapping f to an object x . He introduces a class of mappings, called *affinities*, defined on the set of points characterized by P I–P XVII. Let a and b be points. If f is an affinity and $c \in ab$, then $fc \in (fa)(fb)$. P III implies then that affinities are injective. It can be easily shown that affinities map collinear points on collinear points, coplanar points on coplanar points. If f and g are affinities, the composite mapping $g \cdot f$ is an affinity. The identity mapping $x \mapsto x$ is obviously an affinity. Let ab be a segment, f an affinity. Is $f(ab)$ identical with the segment $(fa)(fb)$? Peano declares that he does not know the answer to this question. In other words, he does not know whether the inverse mapping f^{-1} is an affinity, and he cannot say whether affinities, in his sense of the word, form a group.

The idea of congruence is introduced through the axiomatic characterization of a class of affinities, called *motions*. Two figures k, k' , are said to be congruent if there is a motion f such that $k' = fk$. There are eight axioms of motion. The last four can be summarized in one, using the defined concepts of half-line and half-plane.

(M 1) The class of motions is contained in the class of affinities. (Peano remarks that M 1 is equivalent to Pasch’s Axiom K III.)

(M 2) The identity mapping is a motion.

(M 3) If f is a motion, the inverse mapping f^{-1} is a motion.

(M 4) If f and g are motions, the composite mapping $g \cdot f$ is a motion.

Definition: Given two points a, b , the half-line $Hl(a, b)$ is the set $ab \cup \{b\} \cup a'b$. Given three non-collinear points a, b, c , let $r =$

$Hl(a, b) \cup Hl(b, a)$; the half-plane $Hp(ab, c)$ is the set $\{x \mid x \in y'z, y \in r, z \in cr\}$. (Remember that $cr = \{x \mid x \in cw, w \in r\}$.)

(M 5) If a, b, c are three non-collinear points and x, y, z are three non-collinear points, there is a unique motion which maps a on x , $Hl(a, b)$ onto $Hl(x, y)$, and $Hp(ab, c)$ onto $Hp(xy, z)$.

From these axioms, Peano derives some theorems concerning axial symmetry and orthogonality, translations and rotations.

3.2.7 The Italian School. Pieri. Padoa

Peano's conception of axiomatized geometry as an abstract science was shared in Italy in the 1890's, not only by the group of mathematicians who collaborated with him in the formulation of all mathematical theories in the artificial language, but also by others who did not take part in this enterprise and even looked askance on it. H. Freudenthal credits G. Fano with the first unambiguous statement of the abstract view of geometry. In a paper of 1892 concerning the postulates of n -dimensional linear geometry, Fano declares:

As a basis for our study we posit an arbitrary *manifold* of objects of any nature whatsoever, which, for brevity, we shall call *points*, on the understanding, however, that this name is independent of their own nature.⁷⁹

As we saw above, Peano had said as much three years earlier (p.219, reference 76), and it should not be too hard to discover other statements of the same idea in contemporary Italian literature. Thus, Giuseppe Veronese (1854–1917), in the historico-critical appendix to his influential book *Fondamenti di Geometria* (1891), criticizes Pasch for paying too much attention to the intuitive meaning of undefined geometrical concepts. This forced him to distinguish quite unnecessarily between proper and improper objects, though both have the same geometrical properties, and led him to restrict the scope of his axioms, so that they did not clash with the evidence of the senses. Pasch, observes Veronese,

rightly maintains that proofs must be independent of the intuition of the figure, or rather, as he understands it, of the sense representation of the figure. This aim, however, cannot be fully attained [...] unless the axioms give us well-defined abstract properties independently of intuition.⁸⁰

Veronese demands that geometrical theories should be so conceived that, when intuition is disregarded, they become "a system of purely

abstract truths, in which the axioms play the role of well-determined definitions or abstract hypotheses".⁸¹ A similar approach underlies the *Lezioni di geometria proiettiva*, by Federico Enriques (1871–1946), which circulated in lithographed form since 1894, and were issued in print in 1898. The new view of geometry was made known to the international philosophical community at the Paris Congress of 1900 by Peano's follower Mario Pieri (1860–1913), in a paper "On geometry regarded as a purely logical system".

While Veronese and Enriques stressed the empirical origin of the undefined concepts of geometry, and even Peano wrote that an axiomatic theory deserves the name of geometry only if its postulates state "the result of the simplest and most elementary observations of physical figures",⁸² Pieri regards the connection of geometry with experience as an inessential historical accident. He compares the ordinary spatial representation of geometrical points and lines with the medieval conception of negative integers as debts.⁸³ Geometry is not more closely related to the study of bodily extension than arithmetic is related to bookkeeping.

If you maintain that the postulates of geometry are nothing but rigorous formulations of the intuitive concept of physical space (which merely impress stability and a seal of rationality on the facts of spatial intuition), you ascribe, in my opinion, too much importance to an objective representation, which you treat as a *conditio sine qua non* of the very existence of geometry, whereas the latter can, in fact, very well subsist without it. Today, geometry can exist independently of any particular interpretation of its primitive concepts, just like arithmetic.⁸⁴

Indeed, after the work of Bolyai and Lobachevsky, one can no longer expect geometrical axioms to be intuitively evident. "How could you account for the intuitive evidence of the postulates proper to so-called non-Euclidean geometries, after you have found Axiom XII on parallels evident, or vice versa?"⁸⁵ It is pointless to demand that the primitive concepts of geometry be intuitively clear, since these ("with the exception of the logical categories, which are necessary to all discourse and consequently cannot be described by words") can be given through "implicit definitions or *logical descriptions* [...] or as the *roots* of a system of simultaneous logical equations".⁸⁶

For instance: we call, respectively, *point* and *motion* every determination of classes Π and M which have the following properties: . . . (list here the premises concerning points and motions, denoted respectively, by Π and M).⁸⁷

Such a description conceals, in fact, a system of postulates. But since these, dressed as definitions, amply exhibit their nature as conditional propositions concerning the primitive concepts (i.e. their naturally arbitrary character, etc.), nobody will ask whether they are self-evident or not. The postulates, like every conditional proposition, *are neither true nor false*: they only express conditions which may or may not be verified. Thus, the equation $(x + y)^2 = x^2 + 2xy + y^2$ is *true* if x, y denote real numbers, *false* if they denote quaternions.⁸⁸

Such is geometry as an “hypothetic doctrine”, “*la science de tout ce qui est figurable*”, a “purely speculative and abstract system, whose objects are pure creations of our minds and whose postulates are simple acts of our will”.⁸⁹

Before presenting his ideas on axiomatic geometry to the Paris Congress, Pieri had shown how to carry them out, in two memoirs submitted to the Academy of Sciences in Turin: “The principles of the geometry of position, organized in a logico-deductive system” (accepted for publication on December 19, 1897) and “On elementary geometry as an hypothetico-deductive system” (accepted on May 14, 1899).⁹⁰ The former takes its cue from Staudt and Cayley, who tried to build projective geometry as a science “independent of every other mathematical or physical theory”, unaided by “measurements performed with transportable units in space”.⁹¹ Pieri does not attempt to conceal the thoroughly counterintuitive nature of this science, but proposes to establish it firmly as “an hypothetical science, altogether independent of *intuition*, not only in its method, but also in its *premises*”.⁹² Pieri assumes only two undefined concepts: the projective *point*, and the *join* of two points. These are combined in nineteen axioms. In an appendix, Pieri demonstrates what he calls the “ordinal independence” of his axioms, that is to say, that the $(n + 1)$ th axiom is not a logical consequence of the n axioms that precede it ($1 \leq n < 19$). Some of the interpretations proposed in Pieri’s independence proofs determine what are now generally known as finite geometries, i.e. finite collections of objects which satisfy some typically geometrical axioms.⁹³

Pieri’s monograph on elementary geometry proposes a system of twenty axioms, adequate to support the common groundwork of Euclidean and BL geometry (i.e. Bolyai’s *scientia spatii absolute vera*). The addition of the parallel postulate or its negation suffices to determine one or the other. Pieri defines every geometrical concept in terms of these two: *point* and *motion*. The first axioms characterize

the set of motions as a group of transformations acting transitively on the set of points. Pieri's axiomatic reconstruction of elementary geometry agrees thus with Klein's Erlangen Programme and follows the lead of Helmholtz and Lie. But instead of relying on the familiar attributes of the 'number manifold' \mathbb{R}^3 , Pieri patiently analyses the properties which must be ascribed to the class of mappings called motions and to their domain, the set of points, in order to determine fully and exactly the classical structure of geometry. Pieri points out that all his axioms can be translated into Peano's artificial language, in which, indeed, most of them were originally conceived.⁹⁴

Besides Pieri's paper on geometry as a logical system, the Proceedings of the First International Congress of Philosophy contain several other articles on axiomatics by members of Peano's group. Peano himself spoke about mathematical definitions, which, he said, "are reducible to an identity, whose first member is the name to be defined, while the other expresses its value".⁹⁵ Burali-Forti contrasted such full-fledged *nominal* definitions, which determine concepts, with "definitions by abstraction" and "definitions by postulates", which yield *intuitions*.⁹⁶ The former, he believed, are somehow superior to the latter. He proposed a nominal definition of natural number in purely set-theoretical terms, which essentially repeats, with less elegance and clarity, Frege's feat of sixteen years before,⁹⁷ a shocking instance of the lack of communication between scientists of different countries in the late 19th century. Alessandro Padoa (1868–1937) presented an axiomatic theory of integers, preceded by a short description of an "arbitrary deductive theory", which summarizes the main ideas on axiom systems which we have met up to now and advances a very important result on definability. Deductive theories, says Padoa, must start from a system of undefined symbols combined in a system of unproved propositions. We can imagine that the former are "entirely devoid of meaning" and that the latter, "far from stating *facts*, i.e. *relations* between the *ideas* represented by the undefined symbols, are nothing but *conditions* with which the undefined symbols must comply".⁹⁸

It can happen that there are many (indeed infinitely many) *interpretations* of a system of *undefined* symbols which *verify* the system of *unproved* propositions, and, consequently, every proposition of a theory. The system of undefined symbols can be considered then as the *abstraction* of all these interpretations.⁹⁹

Padoa discusses next the possibility of reducing the system of undefined symbols or the system of unproved propositions of a theory without changing the theory itself. The latter reduction can be achieved, as we know, if one of the unproved propositions is a logical consequence of the others. A system of unproved propositions is therefore irreducible in Padoa's sense if, and only if, there is, for every proposition belonging to it, "an interpretation of the system of undefined symbols which verifies all the unproved propositions, except that one".¹⁰⁰ On the analogy of this procedure (due to Peano) for proving the irreducibility or independence of axiom systems, Padoa puts forward a novel method for proving the irreducibility of a system of undefined symbols. Such a system can be reduced without modifying the theory that rests on it if a definition of one of the symbols in terms of the others can be inferred from the unproved propositions; that is, as Padoa puts it, if "a relation of the form $x = a$, where x is one of the undefined symbols and a is a sequence of other such symbols and logical symbols"¹⁰¹ is a theorem of the theory. This kind of reduction is impossible if, and only if, there are two interpretations of the undefined symbols, both of which satisfy the unproved propositions, differing only in the meaning assigned to the symbol x . In this case, if a is as above, a will have the same meaning in both interpretations. Since the meaning of x is not the same in both, $x = a$ must be false in at least one of the interpretations. Consequently, $x = a$ cannot follow from the unproved propositions of the theory. Padoa formulates this important result as follows:

For demonstrating that the system of undefined symbols is irreducible relatively to the system of unproved propositions it is necessary and sufficient to find, for each undefined symbol, an interpretation of the system of undefined symbols which verifies the system of unproved propositions and which continues to verify it if you suitably change only the meaning of the symbol in question.¹⁰²

3.2.8 Hilbert's Grundlagen

David Hilbert (1862–1943) chose a quotation from Kant as the epigraph for his *Grundlagen der Geometrie*:

All human knowledge begins with intuitions, proceeds to concepts, and ends up with ideas.¹⁰³

With this quotation, Hilbert did not mean to commit himself to Kant's philosophy of geometry. Quite on the contrary. He begins the

book by saying that geometry can be consistently built upon a few simple principles, the axioms of geometry. By listing these axioms and investigating their mutual connection, we perform "the logical analysis of our spatial intuition".¹⁰⁴ Kant's authority is thus invoked to justify a most un-Kantian deed, through which, as Hilbert sees it, we proceed from spatial intuition to its logical, that is, conceptual analysis, a task which Kant believed to be unfeasible (p.31). Hilbert bids us to conceive three different sets of things which we may call, respectively, *points*, *lines* and *planes*. These things must be conceived as standing in certain mutual relations, whose exact description is given in the axioms of geometry. These relations are of five kinds: a binary relation between points and lines, a binary relation between points and planes (both expressed by the verb "to lie on"); a ternary relation between points ("betweenness"); two binary relations between different kinds of point-sets (congruence of segments, congruence of angles). The axioms fall also into five groups, each of which "expresses certain basic related facts of our intuition".¹⁰⁵ The first three groups characterize, respectively, the relations of incidence ("lying on"), betweenness and congruence. The remaining axioms do not introduce new relations, but state additional facts about points, lines and planes, involving the relations we have mentioned. The only axiom in group IV is equivalent to Euclid's Postulate 5. Axiom V 1 is the postulate of Archimedes. Axiom V 2, the "axiom of completeness", is somewhat peculiar and will be discussed later. For our present purposes, it is enough to note that, taken jointly with the axioms which it mentions, Axiom V 2 implies that the set of all points lying on a given line is homeomorphic to \mathbf{R} (assuming that, for every pair of points A, B on that line, the set $\{X \mid X \text{ lies between A and B}\}$ is open). If we grant that the points, lines and planes of classical geometry are somehow intuitively given, the axioms of groups I, II and III can be reasonably said to express the fundamental intuitive facts of incidence, betweenness and congruence. But do the other three axioms state "facts of intuition"? Not Axiom IV, if Proclus was right. And certainly not Axiom V 2. As for the Archimedean Axiom V 1, I wonder whether it is intuitively evident that the segment spanned by the front feet of a gnat, standing on my nose, will, if suitably multiplied by some positive integer, measure out the segment between the gnat itself and Sirius.¹⁰⁶ The full set of Hilbert's axioms offers therefore more than a mere analysis of spatial intuition. Now,

the theory determined by the first three groups of axioms is not categorical, not even in the classical sense (p.198). Consequently, if spatial intuition is reflected by groups I-III only, it is not wholly determinate and it cannot be the manifestation of a definite, unique individual, as some passages of Kant suggest. On the other hand, as noted on p.198 Hilbert's full set can be reformulated to yield a *c*-categorical theory. This theory will unambiguously determine the same abstract structure in every model of it furnished by an interpretation in which 'set' and 'set membership' are understood in their ordinary sense – in every *standard* model of it, as I shall say for short – provided that such models exist and that the ordinary or naïve sense of the set-theoretical predicates is sufficiently precise to determine anything at all. If one can meaningfully speak of *the* object of Euclidean geometry, I know of no better candidate for this name than that structure, *viz.* the unique global relational net that would be discernible in every standard model of a *c*-categorical version of Hilbert's theory if the two foregoing provisos are fulfilled. Every proposition of Euclidean geometry could then be reasonably understood in such way that it is true of that structure and every statement which is true of it as such will be recognized as a proposition of Euclidean geometry. Every Euclidean theorem is a logical consequence of Hilbert's axioms. The latter can therefore be said to provide an exhaustive conceptual analysis of the object of Euclidean geometry. But such an object is not in any sense given in intuition. It might be true, indeed, that we have come to think of it induced by its local, partial, insecure embodiment in our familiar surroundings. Euclidean geometry does indeed regulate our ordering and understanding of what we normally call the spatial features of experience, and it fashions our environment through the commanding influence it exerts upon carpenters and masons, architects and town planners. The relations between certain basic patterns of human behaviour, the articulation of perceptions in the adult mind and the abstract structure deployed in Euclid's *Elements* constitute an important field of philosophical and psychological research. This field, fruitfully explored in our century by Husserl and Becker, Nicod and Piaget, could not even be clearly conceived before the Euclidean structure had itself been neatly isolated and characterized by Hilbert and his predecessors.

Hilbert's chief aim is not, like Pieri's, to exhibit the abstract nature

of geometrical knowledge, or to show that it can be fully expressed in terms of a minimum of undefined notions; but, as he says, “to bring out clearly the significance of the different axiom groups and the scope of the inferences which can be drawn from the several axioms.”¹⁰⁷ This should provide “general information concerning the axioms, presuppositions or resources required to prove a particular elementary geometrical truth”.¹⁰⁸ Before considering some of Hilbert’s findings on this matter, it will be useful to reproduce his axiom system. As I noted above, Hilbert posits three different ‘systems’ of objects (*System* being his German word for set): Points, denoted by capital italics; lines, denoted by lower case italics, and planes, denoted by lower case Greek letters. Planes and lines are not defined as point sets. The relation of a point with the lines or planes on which it is said to lie must therefore be taken as a primitive concept of geometry, which cannot be simply equated with set-membership. This approach is more faithful to Euclid – some will add: more faithful to intuition – than Pieri’s, but it really makes no difference. The fact is that in classical geometry, lines and planes matter only in so far as points are found to lie on them, and every statement about lines or planes can be replaced by an equivalent statement concerning the sets of their respective points. To have seen this clearly was an undoubted merit of Peano and his school. Besides the two relations of incidence or “lying on”, Hilbert assumes, as I said, three more undefined relations: *betweenness*, which is a ternary point relation, and two sorts of *congruence*, which are binary relations between *segments* and between *angles*, respectively. These are two kinds of sets which I now define. In Hilbert’s terminology, a *finite* set of points is called a *figure*. A two-point figure is a *segment* (*Strecke*). If AB is any segment (i.e. if A, B are two distinct points), the set $\{X \mid A \text{ lies between } B \text{ and } X\}$ is a *ray* (*Halbstrahl*) from A . A ray is an infinite set (by Axiom II 1), all of whose points lie on the same line (I 1, II 1). The set formed by two rays from the same point O is called an *angle*; O is the *vertex* of the angle. I give below a literal translation of the axioms, as they appear in the 7th edition, the last which Hilbert himself revised. The comments in parenthesis after some of the axioms are mine. If point A lies on line m , Hilbert says sometimes that A is a point of m and that m goes through A or belongs together with (*zusammengenhört mit*) A . If A lies on two lines m, m' , these are said to meet at A or to have A in common. Similar expressions are used to indicate that A lies on a

plane α . Whenever Hilbert speaks of two, three or more objects, we must understand that these objects are all distinct.

I. *Axioms of Connection (Verknüpfung)*

(I 1) If A, B are two points, there is always a line a which belongs together with each of the points A, B .

(I 2) If A, B are two points, there is not more than one line which belongs together with each of the points A, B .

(I 3) On a line there are always at least two points. There are at least three points which do not lie on one line.

(I 4) If A, B, C are any three points which do not lie on the same line, there is always a plane α which belongs together with each of the three points A, B, C . On each plane there is always a point.

(I 5) If A, B, C are any three points which do not lie on the same line, there is not more than one plane which belongs together with each of the three points A, B, C .

(I 6) If two points A, B of a line a lie on a plane α , every point of a lies on the plane α . (In this case, we say that line a lies on plane α , etc.)

(I 7) If two planes α, β have a point A in common, they have at least another point B in common.

(I 8) There are at least four points which do not lie on one plane.

II. *Axioms of Order (Anordnung)*

(II 1) If a point B lies between a point A and a point C , A, B and C are three different points of a line, and B lies also between C and A .

(II 2) If A and C are two points, there is always at least one point B on the line AC , such that C lies between A and B .

(II 3) Among any three points of a line there is not more than one which lies between the other two. (Hilbert inserts at this point the definition of *segment*. He adds that points A, B are called the endpoints of segment AB . Every point between A and B is called a point of AB and is said to lie within AB .)

(II 4) Let A, B, C be three points not on one line, and let a be a line on the plane ABC which does not go through any of the points A, B, C . If the line a goes through a point of the segment AB , it certainly goes also through a point of the segment AC or through a point of the segment BC . (The axioms of order enable us to discern, on each line

m through a point A , two groups of points (besides A): the points which lie on one side of A and the points which lie on the other side of A . A lies between each point on one side and each point on the other. The sides can be immediately identified by picking one point on m , distinct from A . Likewise, we can distinguish on each plane α which contains a line m , two groups of points, one on each side of m . A point of m lies within every segment formed by a point of one side and a point of the other side. The sides can be identified by picking a point of α not on m).

III. Axioms of Congruence

(III 1) If A, B are two points on a line a and A' is a point on a line a' (possibly identical with a), one can always find on a' , on a prescribed side of A' , a point B' such that segment AB is congruent with segment $A'B'$. In symbols: $AB \equiv A'B'$.

(III 2) If a segment $A'B'$ and a segment $A''B''$ are congruent with the same segment AB , then segment $A'B'$ is also congruent with segment $A''B''$. Briefly: if two segments are congruent with a third one, they are congruent with each other.

(III 3) Let AB and BC be two segments without common points on a line a , and let $A'B'$ and $B'C'$ be two segments without common points on a line a' (possibly identical with a). If $AB \equiv A'B'$ and $BC \equiv B'C'$, then, always, $AC \equiv A'C'$. (The definition of *angle* is given at this point. The angle formed by rays h, k is denoted by $\sphericalangle(h, k)$. Let h comprise points on a line h' , k points on a line k' . We say that h is a ray of h' , etc. Rays h, k , plus their vertex divide the remaining points on plane $h'k'$ into two groups: those which lie on the same side of k' as the points of h and on the same side of h' as the points of k are the inner points of $\sphericalangle(h, k)$ and are said to lie inside this angle; the others are its outer points and are said to lie outside it.)

(III 4) Let $\sphericalangle(h, k)$ be an angle in a plane α and let there be given a line a' on a plane α' and a definite side of a' on α' . Let h' denote a ray of line a' from a point O' . There is then in plane α' one and only one ray k' such that angle $\sphericalangle(h, k)$ is congruent with angle $\sphericalangle(h', k')$ and all the inner points of angle $\sphericalangle(h', k')$ lie on the given side of a' . In symbols: $\sphericalangle(h, k) \equiv \sphericalangle(h', k')$. Every angle is congruent with itself. (Let $\sphericalangle(h, k)$ be an angle with vertex B . If A is a point of h and C is any point of k , $\sphericalangle(h, k)$ will be denoted by $\sphericalangle ABC$. Three points not on one line form a figure called a *triangle*.)¹⁰⁹

(III 5) If, for two triangles ABC and $A'B'C'$, we have that $AB \equiv A'B'$, $AC \equiv A'C'$, $\sphericalangle BAC \equiv \sphericalangle B'A'C'$, then, always, $\sphericalangle ABC \equiv \sphericalangle A'B'C'$.

IV. Axiom of Parallels

(Euclidean Axiom.) Let a be a line and A a point not on a . On the plane determined by a and A there is at most one line which goes through A and does not meet a . (Hilbert defines: two lines are parallel if they lie on one plane and do not meet.)

V. Axioms of Continuity

(V 1) (Axiom of Measurement or Archimedean Axiom.) If AB and CD are any segments, there is a positive integer (*Anzahl*) n , such that, by successively copying CD n times from A on the ray through B , you pass beyond B . (The meaning of this axiom will be clear to everybody, though Hilbert employs in its formulation some expressions which he has not defined and are not sufficiently characterized by the axiom itself. To copy CD successively k times ($k \geq 1$) from A on the ray through B is to find the unique point A_k of that ray, such that $A_{k-1}A_k = CD$, where $A_{k-1} = A$ if $k = 1$, and is determined by the aforesaid condition if $k > 1$ (on the uniqueness of A_k : Hilbert, GG, p.15). You pass beyond B by successively copying CD n times from A on the ray through B , if B lies between A and A_n .)

(V 2) (Axiom of Linear Completeness.) The system of points of a line, with their relations of order and congruence, cannot be extended in such a manner that the relations between the former elements, and the fundamental properties of linear order and congruence which follow from Axioms I-III and Axiom V 1, are all preserved.¹¹⁰

My translation of V 2 badly needs a paraphrase. M. Kline (MT, p.1013) gives the following: "The points of a line form a collection of points which, satisfying Axioms I 1, I 2, II, III and V 1, cannot be extended to a larger collection which continues to satisfy these axioms". This sounds much better, but is not essentially clearer. What does it mean to *extend* the collection of points on a line? In any given interpretation of Hilbert's axioms, each object called *line* is associated with a set of objects called *points*, which are said to lie on it. To extend this set, one must change the interpretation. V 2 is thus seen to differ substantially from the other axioms. Instead of stating some new fact about incidence, betweenness or congruence or introducing a new property of points, lines or planes, V 2 takes, so to

speak, a stand outside the axiom system and says something about its relation to the sets of objects which might conceivably satisfy it. V 2 is what nowadays one would call a metatheoretical statement; though one of a rather peculiar sort, since the theory with which it is concerned includes Axiom V 2 itself. To show this, let me paraphrase V 2 once more. Let H denote Hilbert's axiom system without V 2. Every model of H includes many things called lines (I 1, I 8). On each such line and the set of points lying on it, H induces a structure which we may call line geometry. This structure is determined by Axioms I 3 (first sentence), II 1–3, III 1–3 and V 1, plus the following three propositions which are theorems of H : (i) If A and B are points on the line, there is a point C which lies between A and B ; (ii) any four points on the line can be labelled A_1, A_2, A_3 and A_4 in such way that A_2 and A_3 lie between A_1 and A_4 , A_2 lies between A_1 and A_3 and A_3 lies between A_2 and A_4 ; (iii) the point B' whose existence is postulated in Axiom III 1 is unique. Let L designate this axiom system, while L^* designates the system obtained by adding V 2 to L . It is not hard to find two interpretations I and I' such that (i) in I the set of 'points on the line' is a given set m , while in I' it is the set $m \cup \{z\}$, where z is some object not belonging to m ; (ii) if u, v and w belong to m , v 'lies between' u and w in I if, and only if, v 'lies between' u and w in I' ; (iii) if t, u, v and w belong to m , $\{t, u\}$ and $\{v, w\}$ are 'congruent' in I if, and only if, they are 'congruent' in I' ; (iv) I and I' are modellings of L . Thus, for instance, one may take m to be the field of rational numbers and z to be π and stipulate that in both I and I' ' v lies between u and w ' means that $u < v < w$ and ' $\{t, u\}$ is congruent with $\{v, w\}$ ' means that $|t - u| = |v - w|$. Now, Axiom V 2 says in effect that two interpretations I and I' fulfilling conditions (i)–(iii) cannot both be modellings of L^* , even if they happen to be modellings of L . The curious thing is that V 2 does not introduce any new determination of congruence or betweenness that might preclude two modellings of L satisfying (i)–(iii) from simultaneously satisfying L^* . V 2 merely declares that the addition of itself to axiom system L restricts modellings in the stated manner. There is something highly unsatisfactory about the inclusion of a statement of this kind in an axiom system. Richard Baldus (1930) showed, however, that the full import of Axiom V 2 is given by the following Cantorean axiom:

There exists a segment A_0B_0 with the following property: If A_n, B_n is a sequence of point-pairs such that (i) for every positive integer n , A_n and B_n lie between A_{n-1} and

B_{n-1} , and (ii) for every positive integer n there is a positive integer m such that A_n and B_n do not lie between A_m and B_m , there exists a point X which lies between A_n and B_n for every positive integer n .¹¹¹

Though Hilbert says in the *Grundlagen* that the axioms of geometry state “fundamental facts of our intuition”, he took a very different stance in his private correspondence. Shortly after the publication of the *Grundlagen*, Gottlob Frege (1848–1925) had written to him:

I give the name of axioms to propositions which are true, but which are not demonstrated, because their knowledge proceeds from a source which is not logical, which we may call space intuition (*Raumanschauung*). The truth of the axioms implies of course that they do not contradict each other. That needs no further proof.¹¹²

Hilbert replied:

Since I began to think, to write and to lecture about these matters, I have always said exactly the contrary. If the arbitrarily posited axioms do not contradict one another or any of their consequences, they are true and the things defined by them exist. That is for me the criterion of truth and existence.¹¹³

We say that a set of sentences K is inconsistent if its logical consequences include a sentence S and its negation $\neg S$ (that is, if, for some sentence S , both $K \models S$ and $K \models \neg S$). Otherwise K is said to be consistent. We say that a set of sentences K is satisfiable if there exists an interpretation which satisfies it, that is an interpretation in which every sentence in K is true. Now, it is easy to see that a set of sentences K is inconsistent if and only if it is not satisfiable, so that consistency amounts indeed to existence and truth, as Hilbert maintained.¹¹⁴ However, if this is the true purport of Hilbert’s contention, it is not really opposed to Frege’s. The consistency of a set of sentences can generally be proved only by producing an interpretation which satisfies it, that is, by showing that, on that interpretation, every sentence of the set is true. Hence consistency, though equivalent to truth and existence, cannot be properly said to be their *criterion*, because we must normally infer consistency from truth, not the other way around.

Consistency can be proved directly (i.e. without having to produce a modelling) in certain cases which we now discuss. Let K be the set of axioms of a theory soundly formalized within a calculus C , in which negation can be expressed.¹¹⁵ We say that K , as formalized in C , is syntactically inconsistent if every sentence in C is provable from K in C . Otherwise K (as formalized in C) is syntactically consistent.

Now, if the formalization of K in C is, as we shall say, *semantically complete*, that is, if every logical consequence of K can be proved from K in C , the syntactical consistency of K (as formalized in C) is a necessary and sufficient condition of the consistency of K . It is necessary, because if every sentence S in C can be proved from K in C , both S and $\neg S$ can be proved from K in C . Hence, since our theory is soundly formalized, $K \models S$ and $K \models \neg S$. It is sufficient, because if K is inconsistent, every modelling of K (that is, none at all) satisfies any sentence S of C ; i.e. $K \models S$. Consequently, since our formalization is semantically complete, S can be proved from K in C . The consistency of a set of sentences K can therefore be established without having to produce a modelling of K , by demonstrating the syntactical consistency of K in a sound and semantically complete formalization of the theory determined by K .¹¹⁶ We know, however, that, if Peano's axiomatic arithmetic is consistent, neither it nor any theory which contains it can be given a formalization which is both sound and semantically complete (Gödel, 1931). The consistency of such theories can therefore be demonstrated only by producing a modelling of them, that is, by showing that there exists, in fact, a set of objects which, on a given interpretation, fulfils the theory.¹¹⁷

As a matter of fact, Hilbert proves the consistency of his axiom system by proposing an interpretation which satisfies it, that is, a modelling. He first constructs a modelling of what we have called H , i.e. the system without the axiom of completeness. This is then easily modified to yield a modelling of the full system. Hilbert's models are numerical. The entities assigned to the object variables which occur in the axioms are not such as you might meet in the street or point at with your finger. They are objects we know of only in so far as they are characterized by other mathematical theories. If these theories are inconsistent, Hilbert's models of geometry are void. Hilbert proves therefore only the relative consistency of his axiom system: it is consistent *if* some other axiom system is consistent. Specifically, H is consistent if the arithmetic of natural numbers is consistent; the full Hilbert system is consistent if classical real analysis is consistent. In later life, Hilbert devoted much effort to prove the consistency of arithmetic directly, by constructing a sound, complete, syntactically consistent formalization of it. Hilbert's project foundered on Gödel's discovery of 1931.

Let us sketch Hilbert's modelling of H. Ω will denote a set of real numbers determined as follows: (i) $1 \in \Omega$; (ii) if $a, b \in \Omega$, $b \neq 0$, then $a + b$, $a - b$, ab , $a : b \in \Omega$; (iii) if $a \in \Omega$, then $\sqrt{1 + a^2} \in \Omega$. Plainly, all elements of Ω belong to the countable set of algebraic numbers. We interpret 'x is a point' to mean that $x \in \Omega^3$ (i.e. $x = (x_1, x_2, x_3)$, where $x_i \in \Omega$). A *plane* is understood to be any set of relations $(u_1 : u_2 : u_3 : u_4)$, where $u_i \in \Omega$ and u_1, u_2 and u_3 are not all zero. (x_1, x_2, x_3) lies on $(u_1 : u_2 : u_3 : u_4)$ if $u_1x_1 + u_2x_2 + u_3x_3 + u_4 = 0$. A *line* can be understood to mean a pair of planes which have two points in common, or a set of points which lie on two different planes. The interpretation of *lying on a line*, *betweenness* and *congruence* is a fairly easy matter. If we substitute R for Ω in the foregoing description, we obtain a modelling of Hilbert's full set of axioms. Hilbert observes that this is a modelling of the ordinary Cartesian geometry.

In the introduction to the first edition of his book, Hilbert said he intended to give an independent system of axioms for geometry. This declaration of intent was withdrawn in the second edition, after E.H. Moore (1902) had shown how to derive one of the axioms from the others. Hilbert demonstrates however the independence of the strongest and most characteristic principles of classical geometry: not only the axiom of parallels, but also the Archimedean axiom and Axiom III 5 on the congruence of triangles. The independence of the axiom of completeness evidently follows from the existence of the above modelling of H, which does not satisfy that axiom.

As Peano had shown, in order to prove that a sentence S is independent (i.e. is not a logical consequence) of a set of sentences K, one must give a modelling of $K \cup \{\neg S\}$. Thus, if K comprises Axioms I, II, III and V and S is the parallel axiom IV, the familiar Beltrami-Klein sphere (p.133; substitute 'sphere' for 'circle') is a model of $K \cup \{\neg S\}$. The following interpretation satisfies Axioms I, II, IV, V and all axioms of congruence except III 5: *Points* are elements of \mathbb{R}^3 ; *lines*, *planes*, *lying on*, *betweenness* and *congruence of angles* are interpreted as is usual in analytic geometry. Two segments $AB, A'B'$ are *congruent*, as always, if they have the same length, but the length of AB , where $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, is defined as $((a_1 - b_1 + a_2 - b_2)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2)^{1/2}$.

Hilbert's model of non-Archimedean geometry (Axioms I, II, III, IV and the negation of V 1) is more far-fetched. Let $\Omega(t)$ denote the

set of all functions $f: \mathbf{R} \rightarrow \mathbf{R}$ which fulfil one of the following conditions, for every $t \in \mathbf{R}$: (i) $f(t) = t$; (ii) for some $g, h \in \Omega(t)$, $f(t) = g(t) + h(t)$ or $f(t) = g(t) - h(t)$, or $f(t) = g(t)h(t)$, or, provided that $h(t) \neq 0$ for all $t \in \mathbf{R}$, $f(t) = g(t)/h(t)$; (iii) for some $g \in \Omega(t)$, $f(t) = \sqrt{1 + (g(t))^2}$. If $f \in \Omega(t)$, f is an algebraic function on \mathbf{R} . Consequently, either f is identically zero, or $f(t) = 0$ for, at most, a finite set of values of the argument t . In other words, unless $f \equiv 0$, there is a real number t_f , such that $t = t_f$ is the largest solution of $f(t) = 0$. For every real number $t > t_f$, $f(t)$ is positive, in which case we shall say that f is positive, or negative, in which case we shall say that f is negative. Let f_1, f_2 belong to $\Omega(t)$. We stipulate that $f_1 > f_2$ if $f_1 - f_2$ is positive, and that $f_1 < f_2$ if $f_1 - f_2$ is negative. Let t denote the function $t \mapsto t$; n , the constant function $t \mapsto n$ (n a non-negative integer). By the above stipulation, $n < t$, since, for sufficiently large values of t , $n - t < 0$ always. We obtain a modelling of non-Archimedean geometry by substituting $\Omega(t)$ for Ω in our earlier description of Hilbert's modelling of H . We define as usual the length $|AB|$ of a segment AB by the Pythagorean theorem. If O, X, Y are respectively the points $(0, 0, 0)$, $(1, 0, 0)$ and $(t, 0, 0)$, there is no positive integer n such that $n|OX| \geq |OY|$.

The Archimedean axiom enters into Euclid's *Elements* as a presupposition of the theory of proportions developed in Book V. We saw on page 11 how Euclid, following Eudoxus, defined a linear ordering on the set of ratios between magnitudes. If a and b are two lengths, $a < b$ implies that $a/a > a/b$ (Euclid, V, 8). Consequently, for any two lengths such that $a < b$, there must exist positive integers m, n , such that $ma > na$ but $ma \leq nb$. Obviously m, n fulfil this condition only if it is also fulfilled by $n + 1$ and n . But then $(n + 1)a \leq nb$. Hence $a \leq n(b - a)$. This presupposes, however, that for any pair of lengths (areas, volumes) a and $d = b - a$, there exists a positive integer n such that $nd \geq a$.

In Chapter III of the *Grundlagen*, Hilbert builds a new theory of proportions, which can be used to compare lengths and areas (but not volumes) in the space determined by Axioms I-IV, without assuming the Archimedean axiom V 1. This theory rests on an algebra of segments, which is essentially the same as developed by Descartes in his *Géométrie* (see Section 1.0.4). But, while Descartes bases the construction of the product of two segments on the theory of proportions, *via* Euclid VI, 4, Hilbert shows that the uniqueness of the product

is ensured by a special case of Pascal's theorem.¹¹⁸ He can therefore use the product of two segments for defining proportions between lengths ($a/b = a'/b'$ if and only if $ab' = ba'$) and for proving Euclid VI, 4.

The *Grundlagen* contain also some very interesting investigations concerning the significance of the theorems of Desargues and Pascal, and the geometrical constructions which can be justified by Axioms I–IV without involving the axioms of continuity. But a detailed discussion of these matters would be out of place here.

3.2.9 Geometrical Axiomatics after Hilbert

The two great questions raised and studied in Hilbert's *Grundlagen* can be concisely formulated thus: to find out the simplest properties and relations which suffice to determine the rich structures of geometry, and to investigate which aspects of a given geometrical structure depend on each of its determining properties and relations. The publication of Hilbert's book stimulated many researchers to probe deeper into these two questions. Among those who dealt with the latter, I shall only mention Max Dehn (1878–1952), who studied the effect of a joint denial of the axiom of parallels and the Archimedean axiom.¹¹⁹ He provides a modelling of Axioms I, II, III, which does not satisfy Axiom IV, nor Axiom V 1. In this space, there are infinite lines parallel¹²⁰ to a line m through each point P outside it, yet the three angles of every triangle add up to more than two right angles. Dehn calls this system non-Legendrean geometry, because Legendre's first theorem – the three angles of a triangle are equal to or less than π – does not hold in it. This theorem, which does not depend on the axiom of parallels, cannot be proved without the Archimedean axiom. Even more interesting perhaps is Dehn's semi-Euclidean geometry. This is a modelling of Axioms I, II, III, where there are infinite parallels to each line through every point outside it, but the three angles of a triangle are equal to two right angles. Hence, the latter proposition entails Euclid's fifth postulate only if it is asserted jointly with the Archimedean axiom.

Oswald Veblen (1880–1960) published in 1904 "A system of axioms for geometry" which has several important methodological features. Undefined notions are *point* and a ternary relation of *order* between points.¹²¹ Lines and planes are conceived as sets of points. Veblen's twelve axioms include a (Euclidean) axiom of parallels and a topological axiom which implies the continuity of lines. Veblen proves the

independence of the system and carefully notes, beside each theorem, the axioms on which it depends. He proves also that the system is categorical, in the classical sense which I tried to make precise on pp.198f. With the lines and planes of the space determined by his axioms, Veblen constructs a projective space. Projective points are line bundles, i.e. sets of sets of points of the original space; projective lines are pencils of planes; etc. A parabolic metric is introduced in the projective space after the manner of Cayley and Klein (Section 2.3.6). This is used to define the congruence of angles and segments in the original space. Since the parabolic metric can be specified in many different ways, there appears to be something inherently arbitrary about the Euclidean concept of congruence. Axioms I–XII, which only speak of points and their order, completely determine the structure of three-dimensional affine space, where through every point outside a given line there goes one and only one parallel to that line. These same axioms, however, will only determine the full Euclidean structure when supplemented by the conventional choice of a polarity on the ‘plane at infinity’ of the attached projective space. Segments which are mutually congruent relatively to a polarity Σ , are not congruent relatively to a different polarity Σ' .

Earlier axiom systems could not be proved independent because some of the axioms involved others in their very formulation. That is why Pieri (1899a) could only prove the “ordinal independence” of his system: no axiom belonging to it is a consequence of those that precede it. Veblen overcomes this difficulty with a very simple and elegant move: he formulates most of his axioms as conditional statements, whose antecedents are entailed by other axioms. To see how this works, consider a system of two axioms A, B, such that B is not a consequence of A but A is involved in the formulation of B. Substitute $A \rightarrow B$ for B. The new system is just as strong, since B is a theorem of it. It is also independent: since B, by hypothesis, is not a consequence of A, there exists a modelling of $\{A, \neg B\}$ in which, evidently, $A \rightarrow B$ is false; on the other hand, every modelling of $\neg A$ trivially satisfies $A \rightarrow B$. (Where \rightarrow signifies material implication.)

Veblen credits John Dewey with the expression “categorical axiom system”, though the idea can be traced back to E.V. Huntington.¹²² Veblen explains it as follows:

Inasmuch as the terms *point* and *order* are undefined one has a right [...] to apply the terms in connection with any class of objects of which the axioms are valid pro-

positions. It is part of our purpose however to show that there is essentially only one class of which the twelve axioms are valid. In more exact language, any two classes K and K' of objects that satisfy the twelve axioms are capable of a one-to-one correspondence such that if any three elements A, B, C of K are in the order ABC , the corresponding elements of K' are also in the order ABC . Consequently, any proposition which can be made in terms of points and order either is in contradiction with our axioms or is equally true of all classes that verify our axioms. The validity of any possible statement in these terms is therefore completely determined by the axioms; and so any further axiom would have to be considered redundant (even were it not deducible from the axioms by a finite number of syllogisms). Thus, if our axioms are valid geometrical propositions, they are sufficient for the complete determination of Euclidean geometry. A system of axioms such as we have described is called *categorical*.¹²³

Of course, all the modellings admitted by Veblen interpret set-theoretical predicates in the same naïve commonsense way. 'Is a set' and 'is a member of' are not regarded as interpretable words. The proof that the axiom system is categorical in this limited sense is very easy. All models of the system can be charted globally into R^3 by a Cartesian mapping. Let K, K' be two such models, or, as Veblen puts it, "two classes that verify Axioms I–XII". Let f be a Cartesian mapping of K , f' a Cartesian mapping of K' . Then, $g = f^{-1}f'$ maps K' bijectively onto K . If $A, B, C \in K'$ are in order ABC , $g(A), g(B)$ and $g(C)$ are in order $g(A)g(B)g(C)$.

The following description of Veblen's method of defining congruence complements our discussion of projective metrics (Section 2.3.6). The set of all lines on a plane α which are coplanar with a line m not on α is called a *pencil*. Two coplanar lines a, b determine a pencil (ab) . They also determine the set of lines $\{x \mid x \text{ is the intersection of a plane through } a \text{ and a plane through } b\}$. The union of this set and the pencil (ab) is called a *bundle*. If X and Y are two bundles, through every point O of space there passes one line of each bundle. If these lines are distinct, they determine a plane. The set of all planes thus determined by two bundles is called a *pencil of planes*. A bundle every one of whose lines lies on a plane of a given pencil of planes is said to be incident with this pencil. A bundle will be called a projective point or *p-point*, a pencil of planes a *p-line*. *p-points* incident with the same *p-line* are said to be collinear. A *p-point* A and a *p-line* b determine the set of *p-points* $\{X \mid X \text{ is collinear with } A \text{ and with a } p\text{-point incident with } b\}$. Such a set is called a *p-plane*. Any point of a *p-plane* is said to be incident with it.

p -points incident with the same p -plane are said to be coplanar. A p -point is said to be *proper* if the lines belonging to it meet at a point. A p -line or p -plane is *proper* if there is incident with it a proper p -point. Veblen's Axioms I–XI (i.e. the full set, minus the axiom of parallels) induce on the set of p -points the structure of three-dimensional real projective space. The figure consisting of four coplanar p -points, no three of which are collinear, and the six p -lines incident with them by pairs, is called a *complete quadrangle*. The four p -points are called the *vertices*, the six p -lines, the *sides*; two sides not incident with the same vertex are said to be *opposite*. A p -point incident with two opposite sides of a complete quadrangle is a *diagonal point* of the quadrangle. If A and C are diagonal points of a quadrangle, and B and D are the intersections of the remaining pairs of opposite sides with the p -line AC, D is called the fourth harmonic or harmonic conjugate of B with respect to A and C. (Cf. the construction of the fourth harmonic to three given lines on p.143). The p -points and p -lines of a p -plane constitute a *polar system* if they are set in such a reciprocal one-to-one correspondence that to the p -point (p -line) incident with any two p -lines (p -points) corresponds the p -line (p -point) incident with the corresponding pair of p -points (p -lines). Given a polar system on a plane α , we say that two p -lines (p -points) of α are conjugate if one of them is incident with the p -point (p -line) that corresponds to the other. A polar system is *elliptic* if no element is self-conjugate. A collineation is a bijective mapping of the set of p -points onto itself, which maps p -lines onto p -lines. (Cf. Veblen's Definition 39 and Theorem 71). Let α be a p -plane, A a p -point incident with α . The *reflection* ($A\alpha$) is the collineation that maps each p -point X on its fourth harmonic with respect to A and the intersection of AX with α . A collineation which maps every pair of conjugate elements of a polar system onto a pair of conjugate elements of the system is said to leave the polar system *invariant*. If we now assume the parallel axiom XII, we can easily prove that there is one and only one improper p -plane, to which all improper p -points belong. Let Σ denote an *arbitrarily chosen* polar system of the improper p -plane. A proper p -plane α and a proper p -line m are mutually *perpendicular* if their intersections with the improper p -plane are a pair of corresponding elements of Σ . Two intersecting proper p -lines are perpendicular if they meet the improper p -plane at conjugate p -points of Σ . A *perpendicular reflection*

by a proper p -plane α is the reflection $(A\alpha)$, where A is the p -point which corresponds in Σ to the p -line a along which α meets the improper p -plane. The set of all collineations which leave Σ invariant is plainly a group, each of whose elements maps proper p -points on proper p -points, proper p -lines onto proper p -lines. Call this group $G(\Sigma)$. Every proper p -point (i.e. every bundle of concurrent lines) A , determines a unique ordinary point A^* , where all lines of A meet. On the other hand, every ordinary point determines a unique bundle or proper p -point. Let i denote the bijection $A \mapsto A^*$. Then $\{igi^{-1} \mid g \in G(\Sigma)\}$ is a group of bijective mappings of ordinary space onto itself, which maps ordinary lines onto ordinary lines. Call it $G^*(\Sigma)$. The set $H^*(\Sigma) = \{ihi^{-1} \mid h \in G(\Sigma) \text{ and } h \text{ is the product of a finite number of reflections by proper } p\text{-planes}\}$ is evidently a subgroup of $G^*(\Sigma)$. We define: Two angles are *congruent* if there is a mapping in $G^*(\Sigma)$ which maps the sides of one onto the sides of the other. Two segments are *congruent* if there is a mapping in $H^*(\Sigma)$ which maps one onto the other. Hilbert's third group of axioms can be derived from Veblen's Axioms I–XII and this definition of congruence.

Veblen remarks: "That the choice of Σ is arbitrary is one of the important properties of space; one tends to overlook this if congruence is introduced by axioms". (Veblen (1904), p.382n.). On the other hand, one should not overlook that *congruence* is defined by Veblen in terms of the two undefined concepts of *point* and *betweenness*, plus the arbitrarily designated polar system Σ . As Tarski observed in 1935, Euclidean congruence cannot be defined in terms of *point* and *betweenness* alone (Tarski, LSM, p.306). The proof of this result follows almost immediately, by Padoa's method, from Veblen's own definition of congruence. If you change the polar system of the improper p -plane denoted by Σ while you allow the primitives *point* and *between* (or "are in order ABC") to retain their meaning, you obtain two modellings of Veblen's system (with *congruence*) which satisfy the conditions of Padoa's theorem (p.227).

"A set of postulates for abstract geometry" (1913) by Edward V. Huntington (1874–1952) has, in part, a philosophical motivation. The author, like Pasch and other empirically-minded mathematicians before him, had some qualms about the construction of extension from unextended points. He proposes an axiom system with two undefined predicates, which, in the intended interpretation, mean "x is a sphere", "x contains y". The latter is characterized by the axioms

as an antisymmetric irreflexive binary relation.¹²⁴ Huntington's system is not exactly a "geometry without points", since a sphere which contains no sphere is called a point and behaves like one. But, as Huntington remarks, there is nothing in this terminology "which requires our 'points' to be small; for example, a perfectly good geometry is presented by the class of all ordinary spheres whose diameters are not less than one inch; the 'points' of this system are simply the inch spheres".¹²⁵ If A and B are points, the set $\{X \mid X \text{ is a point and every sphere which contains } A \text{ and } B \text{ also contains } X\}$ is called the *segment* $[AB]$. A and B are its *endpoints*. The *line* AB is the union of $[AB]$ and the sets $\{X \mid A \text{ belongs to } [XB]\}$ and $\{X \mid B \text{ belongs to } [AX]\}$. If A, B, C are points, the set $\{X \mid X \text{ is a point and every sphere which contains } A, B \text{ and } C \text{ also contains } X\}$ is called the *triangle* $[ABC]$. A triangle $[ABC]$ determines three *vertical extensions* like $\{X \mid X \text{ is a point and } A \text{ belongs to the triangle } [XBC]\}$, and three *lateral extensions* like $\{X \mid X \text{ is a point and } [AB] \cap [CX] \text{ is not empty}\}$. (The remaining extensions are defined by permuting A, B, C in these two descriptions.) The union of $[ABC]$ and its six extensions is the *plane* ABC . A tetrahedron and a 3-space are defined analogously. This method of definition can be extended to any number of dimensions.

These definitions are very elegant. Things grow unpleasantly complicated, however, when we come to the concept of *congruence*. Its definition in Huntington's system depends essentially on the properties of parallels. Moreover, not all the properties of congruence follow from its definition: some depend on axioms, which look very simple when stated in terms of congruence, but must sound horribly complex in terms of spheres and inclusion. Two lines are parallel if they are part of the same plane but have no point in common. Four points A, B, C, D form a parallelogram with diagonals $[AC]$ and $[BD]$ if AB is parallel to CD and BC is parallel to DA . A point M is the midpoint of a segment $[AB]$ (noted: $M = \text{mid } AB$) if $[AB]$ is a diagonal of a parallelogram and M belongs to $[AB]$ and to the other diagonal. A segment $[AB]$ is a chord of a sphere S if S contains every point of $[AB]$, but no other point of the line AB . If the sphere S contains a point O such that every pair of chords of S which simultaneously include O are the diagonals of a parallelogram, O is a centre of S . Huntington postulates that if one sphere has a centre, then every sphere which is not a point also has a centre. He does not bother to

mention that a sphere does not have more than one centre, but this follows easily from his Theorems 15 ("Through a given point there is not more than one line parallel to a given line") and 17 ("A segment cannot have more than one middle point"). These theorems depend on rather strong axioms, which imply that every plane is both Arguesian and Euclidean. With these elements, Huntington can proceed to define the congruence of segments. Two segments $[AB]$ and $[CD]$ are congruent if, and only if, one of the following conditions is satisfied: (1a) If line $AB =$ line CD , either $[AB] = [CD]$ or $\text{mid } AC = \text{mid } BD$ or $\text{mid } AD = \text{mid } BC$. (1b) If AB is parallel to CD , $ABCD$ is a parallelogram with diagonals $[AC]$ and $[BD]$. (2a) If $[AB]$ and $[CD]$ have a common midpoint, that midpoint is the centre of a sphere of which $[AB]$ and $[CD]$ are chords. (2b) If $[AB]$ and $[CD]$ have a common endpoint, that endpoint is the centre of a sphere of which $[AB]$ and $[CD]$ are radii (a segment is a radius of a sphere S if one of its endpoints is the centre of S and the other is the endpoint of a chord of S). (3) There exist two segments $[OX]$, $[OY]$ which are mutually congruent by (2) and are congruent by (1) with $[AB]$ and $[CD]$, respectively.

Huntington classifies his axioms into *existence postulates*, which demand the existence of some entity satisfying certain conditions, and *general laws*, which say that *if* such and such entities exist, *then* such and such relations will hold between them. Except for postulate E 1, which posits the existence of at least two distinct points, the remaining existence postulates are conditional statements of the type 'if this exists, then that exists as well'. But then, Huntington's general laws suffice to prove many existential theorems of this kind. This explains perhaps why his classification has not been adopted by other authors. By means of novel and ingenious models (which he calls *pseudogeometries*), Huntington proves that the general laws are independent of each other, while the existence postulates are independent of each other and of the general laws. The system's full independence could be easily achieved by slight changes in wording, but, the author observes, "such changes would tend to introduce needless artificialities".¹²⁶ The consistency of the system is proved by constructing a numerical model, in which the spheres are the closed balls in \mathbb{R}^3 , with radius $r \geq |k|$ ($k \in \mathbb{R}$). Huntington stresses that, since k need not be zero, "we may speak of a perfectly rigorous geometry in which the 'points', like the school-master's chalk-marks on the

blackboard, are of definite, finite size, and the 'lines' and 'planes' of definite, finite thickness".¹²⁷ It should be noted, however, that Huntington's finite points behave in everything like their classical counterparts: there are indenumerably many of them in any segment, any two of them determine a unique line, etc. Indeed, any line can be endowed with the structure of a complete ordered field by arbitrarily choosing a zero point and a unit point on it. There is, thus, some unwitting mockery in Huntington's reference to school blackboards. Let us finally mention that, like Veblen, Huntington makes a point of showing that his axiom system is categorical (in the classical sense).

The reader has probably observed that Huntington's space of spheres is partially ordered by the relation of inclusion or containment. If we agree to add to it a universal sphere U , which contains every other sphere, and a void sphere V which is contained in every other sphere, we can easily conceive Huntington's space as a lattice.¹²⁸ But, though the idea of a lattice had been developed (under a different name) by Dedekind in 1900, it went unnoticed until it was independently rediscovered by Karl Menger and Garrett Birkhoff some thirty years later. These authors immediately based on it a revolutionary approach to the foundations of geometry.¹²⁹ n -dimensional projective and affine geometries can be entirely built in terms of the lattice operations of joining and intersecting. Special postulates differentiate projective from affine lattices. A similar foundation can be provided for BL geometry, but not for Euclidean geometry. This means that, in a definite sense, the latter is less simple than the former.

We cannot study these matters here, but a few indications might stimulate the reader's curiosity.¹³⁰ We consider a domain of entities called *flats*. We define two associative, commutative operations which assign to every pair of flats A, B a flat $A \sqcap B$ called their *meet* and a flat $A \sqcup B$ called their *join*. There is a unique flat U such that, for every flat A , $A \sqcap U = A$ and a unique flat V such that, for every flat A , $A \sqcup V = A$. If $A \sqcap B = A$ and $A \sqcup B = B$ we say that A is a *part* of B and write $A \subseteq B$. If $A \neq B$ and $A \subseteq B$, A is a *proper* part of B (noted $A \subset B$). A flat whose only proper part is V is called a *point*.¹³¹ Our operations satisfy the "law of absorption",

$$A \sqcup (A \sqcap B) = A \sqcap (A \sqcup B), \quad (1)$$

and the "law of intercalation": if P is a point and A and B are two

flats such that $A \subseteq B \subseteq A \sqcup P$, then $B = A$ or $B = A \sqcup P$. Since the operations are associative, it makes good sense to speak of the join and the meet of more than two flats. A finite set of points P_1, \dots, P_n is said to be *independent* if none of them is a part of the join of the others. A flat A has dimension n ($\dim A = n$) if it is the join of $n + 1$ independent points. In particular: $\dim V = -1$; if A is a point, $\dim A = 0$; if $\dim A = 1$, A is called a *line*; if $\dim A = 2$, A is a *plane*; if $\dim U = n$, then any flat with dimension $n - 1$ is called a *hyperplane*. If we now postulate that $\dim U$ is, in fact, equal to a given positive integer n , we have almost everything we need to build the systems of n -dimensional affine and projective geometry. The latter is fully determined by the foregoing assumptions and the following *projective postulate*:

PP. If H is a hyperplane and $V = A \cap H \subseteq B \subseteq A$, then $B = V$ or $B = A$.

It follows that if $\dim A \geq 1$ and H is any hyperplane, there exists always a point $P \subseteq A \cap H$. The projective postulate PP does not hold in affine geometry, which is determined instead by this strong version of the parallel postulate:

AP. If P, Q, R are independent points, there exists one and only one flat L such that $R \subseteq L \subseteq (P \sqcup Q \sqcup R)$ and $L \cap (P \sqcup Q) = V$.

To show that BL geometry can also be founded upon lattice theory, I shall define the basic concepts of betweenness, congruence of segments and parallelism (in the sense of Lobachevsky) in terms of *point, line, meet* and *join*. The reader should try to apply the definitions to the Beltrami-Klein (BK) model of the BL plane (p.133), in order to verify their propriety. (The BL plane, you will recall, is represented in that model by the interior of a Euclidean circle which we shall denote by Z . BL points are the points in the interior of Z ; BL lines are the chords of Z (minus their endpoints, which are not BL points); two BL lines are parallel if their Euclidean extensions meet on Z ; two BL segments, PQ and $P'Q'$ are congruent if the cross-ratio of P, Q and the two endpoints of the chord PQ is equal to the cross-ratio of P', Q' and the endpoints of the chord $P'Q'$.) We assume the lattice-theoretical groundwork common to affine and projective geometry, as explained above. Let points and lines be denoted respectively by capital Romans and by small italics. P lies on m and m goes through P if $m \cap P = P$ and $m \sqcup P = m$. If $m \cap n = V$ we say that m and n do not meet. Three distinct lines a_1, a_2, a_3 form an *asymptotic triangle* if none of them meets any of the others and

through every point P on a_i there goes a unique line p such that $p \neq a_i$; $p \cap a_j = V = p \cap a_k$ ($1 \leq i, j, k \leq 3$; $i \neq j \neq k \neq i$). (In the BK model, asymptotic triangles are triangles inscribed in the circle Z .) We call p the *asymptotic transversal* through P . a is *parallel* to b if, and only if, there exists a line c such that a, b, c form an asymptotic triangle. Let P, Q, R be points on a line m . Q lies *between* P and R if, and only if, given any asymptotic triangle abc such that $a \cap m = P$ and $b \cap m = R$, every line through Q meets at least one of the sides of abc . A segment PQ on a line a is *congruent* to a segment $P'Q'$ on a line a' if, and only if, one of the following two conditions is fulfilled: (i) a is parallel to a' and both lines are parallel to the join of the meets of the asymptotic transversals through P and P' and through Q and Q' ; (ii) a is not parallel to a' but they are both parallel to a line a'' on which there is a segment $P''Q''$, which is congruent with PQ and with $P'Q'$. (To see that Condition (i) is justified, consider the BK model; let A and B be the endpoints of the chord a , A and C the endpoints of the chord a' ; denote by P^* the meet of the asymptotic transversals through P and P' ; by Q^* the meet of the asymptotic transversals through Q and Q' . Since the chord determined by P^* and Q^* is 'parallel' to both a and a' , it must go through A ; let it meet BC at D . A, P, Q, B and A, P^*, Q^*, D are perspective from C ; A, P^*, Q^*, D and A, P', Q', C are perspective from B . Consequently, the cross-ratios $(P, Q; A, B)$ and $(P', Q'; A, C)$ are equal.) With the aid of these definitions, we could formulate a set of axioms for BL geometry, in a more or less cumbersome manner, as conditions imposed on a lattice. Euclidean geometry, on the other hand, cannot be built in this way, because Euclidean congruence cannot be defined in terms of joins and meets alone. (See Tarski (1935), and our brief reference on p.243.)

We have taken a glance at a very small sample of the rich and varied literature of geometrical axiomatics.¹³² Let me mention in passing one further development which shows, like the one we have just examined, that the classical structures of geometry can be made to rest on a rather slender algebraic basis. I refer to the foundation of geometry on the concept of reflection, which can be traced back to J. Hjelmslev (1907), was completed for plane geometry by F. Bachmann (1936, 1951), and was extended to space geometry by J. Ahrens (1959).¹³³ A different development, which has attracted the attention of philosophers, though it is a good deal more tedious and less

beautiful than the aforesaid, goes under the name of *elementary geometry*. It originated with A. Tarski (1951) and is concerned with that part of Euclidean (or non-Euclidean) geometry “which can be formulated and established without the help of any set-theoretical devices”.¹³⁴ Elementary geometry can be formalized in the first-order predicate calculus, no predicator being specifically intended to signify set-membership. W. Schwabhäuser (1956, 1959) has shown that formalized elementary Euclidean geometry is semantically complete, while elementary BL geometry is both semantically complete and decidable.¹³⁵ This tends to confirm our earlier remark that BL geometry is structurally simpler than Euclidean geometry.

3.2.10 *Axioms and Definitions. Frege’s Criticism of Hilbert*

We mentioned earlier that Dugald Stewart held that mathematical theories are deductively built on definitions. A similar statement was made later by Grassmann,¹³⁶ and, in the 1890’s, specifically with regard to geometry, by Georges Lechalas, who, with Auguste Calinon, developed a “general geometry”, embracing the three classical geometries of constant curvature.¹³⁷ Pasch’s axiomatics is quite foreign to these views: a clear distinction is made between defined and undefined notions, and all geometrical propositions can be deduced from principles which state relations between the latter. Hilbert faithfully follows Pasch’s example in the formal set-up of the *Grundlagen*, but he makes a few remarks which seem to line him up with Stewart and Grassmann in this matter. The Axioms of groups II and III are said to *define* (*definieren*), respectively, the concepts *between* and *congruent*.¹³⁸ As Frege was quick to notice, the idea that axioms might *define* anything clashes with Hilbert’s previous statement that they express “fundamental facts of our intuition”. If the axioms express facts, they assert something. In order to do so, says Frege, every expression which occurs in them must have a definite meaning, fixed beforehand, instead of waiting to be defined by the axioms themselves.¹³⁹ In his letter to Frege of December 1899 (quoted on p.235), Hilbert insists in his view of axioms as definitions. He is even willing to change their name, and not call them *axioms* any longer, though this would “conflict with the usage of mathematicians and physicists”.¹⁴⁰ The axioms, say, of group II could be brought into a better agreement with the traditional style of definitions if we reformulated them thus: “*Between* is a relation connecting points of a

line, which has the following features: $\Pi 1, \dots, \Pi 5$ ".¹⁴¹ Frege countered that, even in this new version, Hilbert's axioms fail to render the main service which we expect from a definition, since they do not enable us to tell whether, or not, a given object falls under the concepts allegedly defined by them.¹⁴² In order to know whether something is a point, in Hilbert's sense, we must know already what is meant by a *line*, what is meant by *lying on*, etc. However, if we allow $P, L, \Pi, b, i_1, i_2, k_1, k_2$ to stand, respectively, for *point, line, plane, between*, Hilbert's two kinds of incidence and his two kinds of congruence, we may say that Hilbert's axioms do indeed define the octuple $\langle P, L, \Pi, b, i_1, i_2, k_1, k_2 \rangle$. We can restate them to read: 'A Euclidean 3-space is an 8-tuple $\langle P, L, \Pi, b, i_1, i_2, k_1, k_2 \rangle$, where P, L and Π are sets, b is a relation on P^3 , i_1 is a relation on $P \times L$, i_2 is a relation on $P \times \Pi$, and k_1 and k_2 are relations on such and such subsets of $(\mathcal{P}(P))^2$, which fulfil the conditions stated in Axioms I, ..., V'.¹⁴³ Axioms I-V certainly enable us to tell whether a given octuple is, or is not, a Euclidean 3-space, in this sense. Thus, Hilbert's numerical model of Cartesian geometry (p.237) is such a space, but the octuple $\langle \textit{persons, cities, countries, is a child of, lives in, is a citizen of, got married on the same day as, has the same life-expectancy as} \rangle$ is not. There is, at any rate, one big difference between a 'definition' such as the foregoing and the sentences which Hilbert actually calls by that name, like "two lines are said to be *parallel* if they lie on the same plane and do not meet each other".¹⁴⁴ The axiom system will not enable us to substitute expressions built from known terms for the unknown terms P, L, Π , etc., which the system is supposed to define. On the other hand, given Hilbert's definition of *parallel*, we can eliminate this word from every sentence in which it occurs, by substituting for it some phrase like *coplanar non-intersecting*, which, in its turn, can be easily replaced by a more cumbersome phrase built exclusively from P, L, Π, i_1, i_2 , *there is a ... such that, not, and*. Mario Pieri neatly expressed this difference by distinguishing *nominal* and *real* definitions (*definizione del nome, definizione di cosa*).¹⁴⁵ A nominal definition merely "imposes a name on something already familiar", while a real definition lists a collection of properties which suffice to characterize a concept for the purpose at hand. Hilbert's definition of *parallel* belongs to the former kind; the 'definition' of Euclidean space by the axiom system, to the latter. Following Peano, Pieri prefers to reserve the word *definition* to signify nominal

definitions, and to say that a term is undefined when its meaning is determined by an axiom system. Such is nowadays the ordinary usage of logicians, which we have followed throughout this chapter. But, although axiom systems are not really definitions in this strict sense, this should not blind us to the fact that they do indeed determine (and, hence, in the etymological sense of the word, they do de-fine or de-limit) the undefined concepts which occur in them. To demand like Frege that the meaning of these concepts be intuitively elucidated (*erläutert*)¹⁴⁶ shows a lack of understanding of the nature of logical consequence that is indeed astonishing in the founder of modern logic. Such elucidations are not only unnecessary, but altogether pointless. The logical consequences of a set of axioms will not change an iota because you replace a given elucidation of their undefined terms by another, radically different from the first. As Hilbert put it, in his reply to Frege:

Every theory is naturally only a scaffolding or schema of concepts, together with their necessary mutual relations, and the basic elements (*Grundelemente*) can be conceived in any way you wish. If I conceive my points as any system of things, e.g. the system *love, law, chimney-sweep, . . .* and I just assume all my axioms as relations between these things, my theorems, e.g. the theorem of Pythagoras, will also hold of these things. In other words, every theory can always be applied to infinitely many systems of basic elements. It suffices to apply an invertible univocal transformation [i.e., a bijection] and to stipulate that the axioms hold correspondingly for the transformed things. [. . .] This property is never a shortcoming of a theory and is, in any case, inevitable.¹⁴⁷

Frege's failure to understand abstract axiomatics comes out very clearly in his criticism of Hilbert's independence proofs. He observes that if the axiom system determines the meaning of the undefined terms which occur in it, the elimination of one of the axioms will not fail to alter that meaning. After suppressing, for instance, the Axiom IV of parallels, we no longer stand before the same axiom groups I, II, III and V, which, together with it, constituted Hilbert's system. What remains is a set of statements which merely sound like the axioms of those four groups, but do not say the same as they did. Frege's obtuseness is truly baffling. If Hilbert's system does indeed determine the meaning of $\langle P, L, II, b, i_1, i_2, k_1, k_2 \rangle$ – which Frege is willing to grant for the sake of the argument – two things can happen when we cross out an axiom such as IV, that does not contain any basic term not occurring in the others: *either* the axiom we have

eliminated is a logical consequence of the others, in which case the meaning of $\langle P, L, \dots, k_2 \rangle$, i.e. the range of 8-tuples it may be taken to stand for, is not altered; *or* the said axiom is independent of the others, in which case the meaning of $\langle P, L, \dots, k_2 \rangle$ becomes less specific, that is, the range of 8-tuples it may be allowed to represent becomes wider. In neither case does the meaning of the remaining axioms undergo a radical change. Indeed, we may say that their *contribution* to the determination of $\langle P, L, \dots, k_2 \rangle$ will not be modified by the addition and subsequent suppression of an independent axiom. Frege's resistance to admit this may have been motivated by the seemingly enormous difference between, say, Euclidean straight lines and BL 'straights', in the shape of, for example, the semicircles centred on the edge of the Poincaré half-plane (p.136). But the fact that these semi-circles, in a suitable interpretation of $\langle P, L, \dots, k_2 \rangle$, behave exactly like Euclidean lines with regard to every logical consequence of Hilbert's Axioms I 1–3, II, III and V, bespeaks a deep analogy between them, which can come as a shock only to the mathematically uneducated. To maintain that *line* means something entirely different in BL geometry and in Euclidean geometry, is not more reasonable than to say that *heart* has a completely different meaning in the anatomy and physiology of elephants and in that of frogs.

For a long time, it was fashionable to describe axioms as *implicit definitions* of the undefined or primitive terms which occur in them. Mathematicians were seduced by the analogy with a system of simultaneous equations which implicitly determines its roots. However, this very analogy makes it advisable to avoid that description. For a system of n equations can be said to determine the n unknowns x_1, \dots, x_n which occur in them only if it can be solved for them, i.e. if you can derive from it, through algebraic manipulation, a set of n equations of the form $x_i = f$, where f is an expression in which no unknown occurs. Our analogy would demand therefore that axiom systems be 'solvable' for the primitive terms they contain; in other words, that they yield ordinary nominal 'explicit' definitions of them. But this is, of course, impossible.¹⁴⁸

Mario Pieri was one of the first to describe axioms as "implicit definitions" and primitive terms as "the roots of a system of simultaneous logical equations" (see p.224f.). The phrase, however, and the algebraic analogy, had been introduced eighty years before – with

a different purpose – by Gergonne. In his “Essai sur la théorie des définitions” (1818), he propounds the rule, later defended by both Peano and Frege, that all definitions should be nominal. “A definition does merely establish an identity of meaning between two expressions of the same aggregate of ideas, of which the simpler is new and arbitrary, while the other, more complex one, is formulated in words whose meaning is already fixed, either by usage or by a prior convention.”¹⁴⁹ It is obviously impossible to define all words. How, then, can one learn the meaning of those which must remain undefined? Some words can be explained ostensively. Others, such as those which express “a simple intellectual idea, such as *desire, fear, memory*”, or an “idea of relation, such as *above, below, inside, outside*”, can only be understood after “a long attentive observation of the several circumstances in which the word is used by those who know well its meaning”.¹⁵⁰ Gergonne observes that a single sentence which contains an unknown word may suffice to teach us its meaning. Thus, if you know the words *triangle* and *quadrilateral* you will learn the meaning of *diagonal* if you are told that “a quadrilateral has two diagonals each of which divides it into two triangles”.

Such phrases, which provide an understanding of one of the words which occurs in them *by means of the known meaning of the others*, might be called *implicit* definitions, in contrast with the ordinary definitions, which we would call *explicit*. There is evidently between the latter and the former the same difference as between solved and unsolved equations. One sees also that, just as two equations with two unknowns simultaneously determine both, two sentences which contain two new words, *combined with other known words*, can often determine their sense. The same can be said of a greater number of words *combined with known words* in a like number of sentences; but, in this case, *one must perform a sort of elimination* which becomes more difficult as the number of words in question increases.¹⁵¹

Gergonne has grasped well a familiar linguistic phenomenon and has given it an appropriate name. But his systems of simultaneous implicit definitions are something evidently very different from abstract axiom systems. In these, all designators and predicators behave, if you wish, as unknowns, and no process of elimination can lead to fix their meanings, one by one. We ought not to burden Gergonne with the paternity of the rather unfortunate description of axioms as implicit definitions.

EMPIRICISM, APRIORISM, CONVENTIONALISM

In the context of 19th-century physics, geometry was quite naturally interpreted as the science of space, space itself being conceived as a self-subsisting entity, no less real than the spatial things moving across it. Paradoxically, however, the propositions of this science did not seem to be liable to empirical corroboration or refutation. Since the times of the Greeks, no geometer had ever thought of subjecting his conclusions to the verdict of experiment. And philosophers, from Plato to Kant, viewed geometry as the one unquestionable instance of non-trivial a priori knowledge, i.e. knowledge relevant to things that exist, yet not dependent on our experience of them. Even such an extreme empiricist as Hume regarded geometry as a non-empirical science, concerned not with matters of fact, but with relations of ideas. The discovery of non-Euclidean geometries shattered the unanimity of philosophers on this point. The existence of a variety of equally consistent systems of geometry was immediately thought to lend support to a different view of this science. The established Euclidean system could now be regarded as a physical theory, highly corroborated by experience, but liable to be eventually proved inexact. We have seen that Gauss and Lobachevsky, Riemann and Helmholtz took this empiricist view of geometry.

In Part 4.1, we shall study two authors, John Stuart Mill and Friedrich Ueberweg, who developed an empiricist philosophy of geometry before 1850, while still unacquainted with the new geometrical discoveries. We deal next with the empiricist philosophies of Benno Erdmann and Auguste Calinon, who were directly influenced by non-Euclidean geometry. Finally, we take a look at the novel viewpoints contributed by Ernst Mach to the empiricist philosophy of geometry. We shall see that all these philosophies are beset by one great difficulty, namely, that geometrical objects – points lines, etc. – are nowhere to be found in experience exactly as geometry conceives them. Our authors did not overlook this difficulty. Their persistent yet, in my opinion, unsuccessful struggle to overcome it deserves our attention, because a similar difficulty is bound to

arise at some point within every empiricist epistemology, as long as science includes and even clusters around mathematical physics.

Apriorists did not yield without resistance to the onslaught of the new geometrical empiricism. Most of them dismissed non-Euclidean geometry as a logically viable but physically meaningless intellectual exercise, and sought the unshakeable foundation of established geometry in a geometrical intuition which they believed was common to all mankind. Apriorists were not alone in their rejection of the physical and, consequently, the philosophical significance of the new geometries, but were joined by some empiricists, who staunchly defended the exclusive validity of Euclidean geometry. Together they formed the chorus of Boeotians whose uproar Gauss anticipated and tried not to arouse. Their philosophies are often handicapped by an insufficient knowledge of geometry, new and old. In Part 4.2, we study a small sample of these authors. This includes the well-known philosophers Hermann Lotze, Wilhelm Wundt and Charles Renouvier, and the less well-known Joseph Delboeuf, whose opinions about non-Euclidean geometry do not add much to those of the former three, but whose views on the relationship between geometry and reality are much bolder than theirs.

In Part 4.3, we examine the aprioristic philosophy of geometry propounded by Bertrand Russell in 1897. Strongly influenced by Kant, but making due allowance to the new developments, especially to the findings of Helmholtz, Russell maintained that geometry must ascribe a constant curvature to space, but that the actual value of this curvature can only be determined by experience. According to him it is a priori certain that physical space is maximally symmetric, but it is only empirically likely that it is flat as well.

Part 4.4 is devoted to the philosophy of geometry of Henri Poincaré. This great philosopher-scientist, hailed by historians of mathematics as the last man to have a universal knowledge of this discipline and its applications, refused to walk the trodden paths of apriorism and empiricism and defended, in a series of articles published between 1889 and 1912, an entirely new view of geometry. According to him, the principles of geometry cannot be true or false, because they are conventions adopted for reasons of expediency. Poincaré's geometrical conventionalism is directly linked to his own mathematical researches, which led him to stress the mutual relations and the interchangeability of the several geometrical systems. We may regard

it, therefore, as the only philosophical conception which, in a sense, actually arose from the new developments in geometry. Though scientists and philosophers have generally rejected it, it has exerted an unmistakable, sometimes openly acknowledged, more often barely concealed, influence upon 20th-century epistemology.

4.1 EMPIRICISM IN GEOMETRY

4.1.1 *John Stuart Mill*

Gauss' discovery of BL geometry led him to think that geometry is an empirical science. "The necessity of our geometry cannot be proved – he wrote to Olbers in 1817 – at least neither *by* nor *for* our HUMAN understanding [. . .]. We should class geometry not with arithmetic, which stands purely a priori, but, say, with mechanics."¹ The British philosopher John Stuart Mill (1806–1873), in his *System of Logic* of 1843, went even further, maintaining that all deductive sciences rest upon inductive foundations, and that this applies not only to geometry, but also to arithmetic.² His almost solitary stance on arithmetic has overshadowed his less exclusive philosophy of geometry. The latter, however, is of some interest for us because, though it was apparently developed in complete ignorance of non-Euclidean geometry, it anticipates some of the tenets of latter-day geometric empiricism.

Geometry is built by deduction or "ratiocination". This is identified by Mill with syllogistic inference. The major premises of the syllogisms of geometry are the axioms (these apparently include Euclid's postulates) and some of the so-called definitions. "In those definitions and axioms are laid down the whole of the marks, by an artful combination of which men have been able to discover and prove all that is proved in geometry."³ The main effort in geometrical proof consists in finding the minors by means of which new, unforeseen cases are subordinated to the definitions and axioms.

Mill is aware that from a definition as such, no proposition, unless it be a proposition concerning the meaning of a word, can ever follow. But the definitions which supply some of the major premises in geometry involve existential assumptions, to wit, "that there exists a real thing, conformable to the definition".⁴ Thus, Mill defines parallels as equidistant straight lines.⁵ This definition presupposes that such

lines exist, i.e. that given a straight line you can find another line, which is straight like the first, and all of whose points lie at a fixed distance from it. Mill believes, however, that the existential assumptions of the definitions of geometry are actually false "There exist no real things exactly conformable to the definitions. There exist no points without magnitude; no lines without breadth, or perfectly straight; no circles with all their radii exactly equal, nor squares with all their angles perfectly right."⁶ Mill denies that these geometric objects are even possible: their existence would seem to be inconsistent with the physical constitution of the universe. He also rejects the interpretation which regards them as purely mental entities. Our ideas are copies of the things which we have met in our experience. "Our idea of a point, I apprehend to be simply our idea of the *minimum visible*, the smallest portion of surface which we can see. A line, as defined by geometers, is wholly inconceivable."⁷ Mill, on the other hand, will not admit that a science like geometry might deal with non-entities. How does he reconcile this Platonic thesis with his former remarks about the objects described by the definitions of geometry? Not indeed after the fashion of Plato, by claiming that these objects, because they are the concern of a genuine science, possess some sort of being of their own. In Mill's opinion, the objects characterized in the definitions are simply "such lines, angles, and figures as really exist"⁸; only that in geometry we disregard all their properties except the geometrical ones, and we even ignore the "natural irregularities" in these. Mill says that his position is that of Dugald Stewart, who maintained that geometry is built on hypotheses.⁹ But he remarks that the term *hypothesis* has here a somewhat peculiar sense, meaning, not "a supposition not proved to be true, but surmised to be so, because if true it would account for certain facts", but a proposition "known not to be literally true, while as much of [it] as is true is not hypothetical but certain". Indeed "the hypothetical element in the definitions of geometry is the assumption that what is very nearly true is exactly so. This unreal exactitude might be called a fiction, as properly as an hypothesis"¹⁰. On the character of these scientific fictions, which other writers have called *idealizations*, Mill observes the following:

Since an hypothesis framed for the purpose of scientific inquiry must relate to something which has real existence (for there can be no science respecting non-entities) it follows that any hypothesis we make respecting an object, to facilitate our study of

it, must not involve anything which is distinctly false, or repugnant to its real nature; we must not ascribe to the thing any property which it has not; our liberty extends only to *slightly exaggerating some of those which it has (by assuming it to be completely what it is very nearly) and suppressing others*, under the indispensable obligation of restoring them whenever, and in as far as, their presence or absence would make any material difference in the truth of our conclusions. Of this nature, accordingly, are the first principles involved in the definitions of geometry.¹¹

While definitions are simplifications or exaggerations of experience and their existential presuppositions must be regarded as only approximately true, the axioms, Mill claims, “are true without any mixture of hypothesis”.¹² They are experimental truths, inductions from the evidence of our senses. Some of them are common to geometry and other sciences, e.g. that things which are equal to the same thing are equal to one another. Others are peculiar to geometry. Mill mentions the following two instances of the latter. Two straight lines cannot enclose a space; two straight lines which intersect each other cannot both be parallel to a third straight line.¹³ It might seem strange that propositions which speak about the very entities described in the definitions should be regarded as “exactly and literally true”,¹⁴ while the latter are true, so to speak, *cum grano salis*. Those who raise this objection, says Mill,

show themselves unfamiliar with a common and perfectly valid mode of inductive proof; proof by approximation. Though experience furnishes us with no lines so unimpeachably straight that two of them are incapable of inclosing the smallest space, it presents us with gradations of lines possessing less and less either of breadth or of flexure, of which series the straight line of the definition is the ideal limit. And observation shows that just as much, and as nearly, as the straight lines of experience approximate to having no breadth or flexure, so much and so nearly does the space-inclosing power of any two of them approach to zero. The inference that *if they had no breadth or flexure at all, they would inclose no space at all*, is a correct inductive inference from these facts.¹⁵

A different objection refers specifically to the axiom discussed in the foregoing text:

That two straight lines *cannot* inclose a space, that after having once intersected, if they are prolonged to infinity they do not meet, but continue to diverge from one another. How can this, in any single case be proved from actual observation? We may follow the lines to any distance we please, but we cannot follow them to infinity: for aught our senses can testify, they may, immediately beyond the farthest point to which we have traced them, begin to approach, and at last meet.¹⁶

Mill's answer to this objection deserves to be considered carefully. It rests on a premise that we ought to rule out as psychologically naïve, namely, that geometrical forms possess the "capacity of being painted in the imagination with a distinctness equal to reality".¹⁷ But this premise is not really essential. It serves him merely to avoid the necessity of examining a real pair of intersecting straight lines in order to conclude that they will not meet again. It is enough to consider a pair of imaginary lines. Now, if the intersecting lines were ever to meet a second time, they ought to begin to approach at some point, after diverging from one another. Let us transport our minds to this point and frame a mental image of the appearance which one or both lines must present there. "Whether we fix our contemplation upon this imaginary picture, or call to mind the generalizations we have had occasion to make from former ocular observation, we learn by the evidence of experience, that a line which, after diverging from another straight line, begins to approach it, produces the impression on our senses which we describe by the expression *a bent line*, not by the expression *a straight line*."¹⁸ The modern reader will see at once that Mill's argument is not really based on the supposed exactitude of geometrical images or on the "evidence of experience". In fact, he argues from the accepted meaning of the expression *a straight line*, which, we may grant, implies what he says. But if the axiom is true by virtue of the meaning of the word *straight*, it is what Mill calls a verbal proposition,¹⁹ not an induction from experience. And the assumption that straight lines, in approximately that sense, actually do exist is indeed an adventurous hypothesis.²⁰

It is hard to understand why Mill insists in claiming that the axioms have no admixture of hypothesis. After all, if the definitions or their existential assumptions do not lack this admixture, and they are indispensable in geometrical proof, the science derived from them will be hypothetical throughout. That its propositions are nevertheless usually regarded as necessary truths, says Mill, is only due to the fact that they *follow necessarily* from the assumptions from which they are derived. These assumptions are not themselves necessary, indeed they are not even true, so that the necessity of geometrical theorems is conditional or hypothetical: *if* the assumptions were true, the theorems could not be false without contradiction. "I conceive – adds Mill – that this is the only correct use of the word *necessity* in science; that nothing ought to be called necessary, the denial of

which would not be a contradiction in terms.”²¹ And at another place he remarks: “This inquiry into the inferences which can be drawn from assumptions, is what properly constitutes Demonstrative Science”.²²

Geometry, thus conceived, is on a par with the physical sciences, and ought to be counted as one of them. According to Mill, this has not been acknowledged because of two facts. In the first place, the truths of geometry can be gathered from our mental pictures as effectually as from the objects themselves; this has induced men to believe that geometry is concerned not with physical entities, but with the objects of an internal intuition. In the second place, geometry can be entirely deduced from a few obvious principles. But, says Mill, the advance of knowledge has “made it manifest that physical science, in its better understood branches, is quite as demonstrative as geometry [...] the notion of the superior certainty of geometry being an illusion arising from the ancient prejudice which, in that science, mistakes the ideal data from which we reason, for a peculiar class of realities, while the corresponding data of any deductive physical science are recognised for what they really are, mere hypotheses”.²³

Mill is clearly a forerunner of the modern empiricists, who identify geometrical necessity with the logical necessity of geometrical proofs, while reducing the undemonstrated premises upon which those proofs are built to the status of empirically verifiable and, if need be, falsifiable hypotheses. But Mill does not seem to have thought that those premises might be downright false. He still shares the old unshaken faith in the irrevocable truth of Euclid.

Every theorem in geometry – he writes (and geometry means here, of course, Euclid’s geometry) – is a law of external nature, and might have been ascertained by generalizing from observation and experiment, which, in this case, resolve themselves into comparison and measurement. But it was found practicable, and being practicable, was desirable, to deduce these truths by ratiocination from a small number of general laws of nature, the certainty and universality of which was obvious to the most careless observer, and which compose the first principles and ultimate premises of the science.²⁴

4.1.2 *Friedrich Ueberweg*

The Principles of Geometry, Scientifically Expounded, by Friedrich Ueberweg (1826–1871), written in 1848, published in 1851, takes a stance similar to Mill’s on the nature and the foundations of geometry. Ueberweg has understood, however, that Euclid’s axioms

and postulates are not empirically evident, so that the empiricist position on this matter must be made persuasive by substituting other, really obvious, principles from which the former can be inferred. The empirical facts which Ueberweg proposes as the foundation of geometry clearly anticipate the axioms given by Helmholtz in 1866 (Section 3.1.2).

Ueberweg has explained the purpose of his work in two introductions.²⁵ The first, written when the author was very young, is more conciliatory towards the aprioristic philosophy of geometry which prevailed in Germany at that time. He remarks, however, that before setting up an aprioristic deduction of the principles of geometry from the essence of space, one must derive those principles from a concrete, empirical intuition (*Anschauung*); because, even if space is a priori, at no time in our lives are we aware of the pure intuition of space, unless we manage to isolate it from the whole of empirical perception. Even less than space itself do we have the fundamental concepts, axioms and postulates of geometry in our consciousness before distilling them, by abstraction and idealization, from experience. The second introduction is more polemical, the main target of its attacks being Kant's apriorism. Ueberweg accepts Kant's contention that geometry is an apodictic science. But he does not see why this should imply that space is known a priori. In the first place, Kant has failed to show how the a priori nature of space might ensure the validity of the principles of geometry: Kant claims that this is so, but he does not derive these principles from that nature. In the second place, Kant has never proved that there cannot be an apodictic science concerning an empirical object. He knows only the dilemma *empirical or a priori*; but there is a third alternative, namely, "rational elaboration of the empirically given, in accordance with logical norms, without an a priori contents of knowledge".²⁶ In fact, apodictical certainty belongs to the *system* of geometry, not to its several principles, regarded in isolation. The latter possess merely assertoric, i.e. factual, certainty. Kant failed to see that the theorems derived from the principles, though supported by them, can also serve to strengthen them. This, however, is a common character of all sciences built by deduction from hypothetical premises: "The agreement of all consequences among themselves and with experience confirms the presuppositions and bestows on them an increasing certainty, which becomes absolute as soon as one can

prove that the factually given can be explained *only* from these premises."²⁷ This modern-sounding epistemological conception was already held by Ueberweg when he wrote his first introduction, though he stated it less neatly there. Even if a philosopher might succeed in deriving geometry from the essence of space – he says – he would not thereby discover the foundation of the general belief in its validity. "This is, in the case of geometrical axioms, the same in fact as in the case of physical hypotheses, namely, the uninterrupted approximate confirmation of their consequences by experience. Innumerable propositions derived from the geometrical axioms allow comparison with experience through factual construction. Absolute agreement is, of course, impossible, because we cannot construct with absolute exactness; but we find that, *within the reach of our experience*, the more exact our construction, the more exact is the agreement."²⁸

But Ueberweg's main purpose is not to determine wherein lies the certainty of the indemonstrable principles of geometry, but, as a preliminary contribution thereto, to exhibit a connection between the familiar axioms, postulates and fundamental concepts in Euclid, by deriving them from a common source. This is found by (1) *analysing* our global sensory awareness, in order to obtain general concepts and propositions; (2) *idealizing* the latter by ascribing them absolute, infinite precision. If we can show that the whole of geometry can be built upon this basis just as well or even better than upon Euclid's principles, we shall have paved the way "for the right opinion concerning the logical character of the Euclidean axioms and for the *recognition of geometry as a natural science*".²⁹

Ueberweg observes that "space is separated from the whole of sensory intuition only through the perception of movements".³⁰ The main facts revealed thereby can be stated in many ways. Ueberweg formulates them as follows:

According to the evidence of sense, a solid material body can:

- (I) If unfixd, be carried anywhere, if no other solid body is previously located there.
- (II) If fixed at one place (*Stelle*) only, it can no longer move everywhere, without limitations, but it will not be deprived of all movement.
- (III) If fixed at a second place, no part of it can be moved any longer in all the ways that were possible in Case II, but it can still be moved.
- (IV) But if we fix the body at a third place, which could still be moved in Case III, the movement of the body becomes altogether impossible.³¹

These obvious, familiar empirical facts are now *idealized*. That is, we assume that the stated properties are true with absolute precision. Ueberweg justifies this assumption, as we might expect, by the empirical truth of its consequences, especially by the fact that the propositions inferred from the idealized statements I–IV are mutually consistent and agree with experience with increasing precision as we carry out our constructions more exactly.³² Lie has shown that axioms essentially equivalent to Ueberweg's are sufficient for characterizing three-dimensional maximally symmetric spaces.³³ But Ueberweg thought he could characterize Euclidean space with them. His derivations are therefore inevitably defective.

We shall only discuss his proof of the parallel postulate. Ueberweg defines a point as "the absolutely simple space element"; he characterizes it also as that element of a body which is such that any two—but not any three—of them can be fixed without altogether impeding the movement of the body. A movement with a fixed point P is called a rotation about P . The set of all points occupied by a figure F during a movement m is the *path* (*Weg*) of F during m . A line is the path of a moving point. A straight line is a line whose path during rotation about two of its points coincides with itself. Given two points P, Q , there is one and only one straight line through P and Q .³⁴ A straight line can also be defined as a line of constant direction. In order to explain this, Ueberweg goes into a detailed discussion of the concept of direction (*Richtung*). A moving figure changes its *place* (*Ort*). Two places differ only in their *position* (*Lage*). If a point P is carried to a point Q over a path absolutely determined by P and Q "that determination of the transit of P to Q which depends on the position of Q relatively to the other points which Q can take while rotating about P [in other words, on the position of Q within the sphere through Q centred at P] is called *linear direction* (*Linienrichtung*)".³⁵ In the light of this definition, Ueberweg concludes that a straight line is a line of *one* direction. The difference between the directions of two straight lines meeting at a point P is called *angle*. Ueberweg defines circles and arcs and shows how to use the latter to measure angles. Two straight lines m, n which make equal corresponding angles with a third line t are said to have the same direction. Ueberweg defines parallels as straight lines that have the same direction. Under this definition, two lines m, n which are parallel relatively to a transversal t (with which they make equal

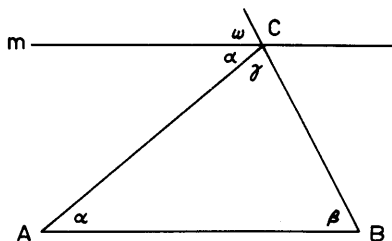


Fig. 19.

corresponding angles), need not be parallel relative to a second transversal t' . The statement that parallelism, as defined by Ueberweg, does not depend on the choice of the transversal is equivalent to Euclid's fifth postulate. This statement is neither proved nor postulated by Ueberweg. He proves instead that given two points P, Q and a direction m at P , there is a unique direction n at Q which is equal to m (in the sense that the straight lines in directions m and n make equal corresponding angles with the straight line PQ).³⁶ In Ueberweg's terminology, this may be stated thus: Given a straight line m and a point Q outside it, there is one and only one straight line n through Q which is parallel to m (relatively to a fixed transversal). Ueberweg omits the proviso I have added in parenthesis, and concludes that *any* straight line meeting a pair of parallel lines (in his sense) makes equal corresponding angles with them.³⁷ Euclid I, 32 (the three interior angles of a triangle are equal to two right angles) follows easily from the last proposition, but, contrary to Ueberweg's belief, this theorem is not a logical consequence of his premises. Let ABC be a triangle with interior angles α, β, γ and let m be a line through C which makes an angle α with AC , as shown in Fig. 19. Let ω denote the angle corresponding to β which m makes with BC at C . It is plain that $\alpha + \omega + \gamma = \pi$. According to Ueberweg's definition, m is the unique parallel to AB through C , relatively to AC . But this does not imply that m is also the unique parallel to AB through C relatively to BC . We do not know, therefore, whether $\omega = \beta$. Consequently, we cannot conclude that $\alpha + \beta + \gamma = \pi$.

4.1.3 Benno Erdmann

The next significant contributions to geometrical empiricism in the 19th century were made by Riemann and Helmholtz. We have already

dealt with them in Parts 2.2 and 3.1. They were extensively discussed and made known to the philosophical public in *The Axioms of Geometry*, a book published in 1877 by Benno Erdmann (1851–1921). The chief purpose of this work is to show that the new geometric theory of space, which Erdmann ascribes to Riemann and Helmholtz, confirms the empiricist theory of spatial intuition and refutes Kant's philosophy of space and geometry. In order to show this, Erdmann first presents the said theory as a successful attempt to provide a *definition* of space. The definition arrived at, after a carefully motivated exposition, is simply a restatement of Helmholtz's axioms for Euclidean geometry.³⁸ Erdmann's exposition is somewhat naïve and is marred by several mathematical misconceptions. Since the book was very popular among philosophical readers, these features have probably exerted a damaging influence on philosophical discussions about space and geometry.

According to Erdmann, Riemann's lecture showed how to define the concept of space by specification of the general concept of an n -fold extended manifold. The procedure, he says, is analogous to that used in analytical geometry for providing the concepts of intuitively given spatial figures. It is merely a matter of finding analytic determinations which correspond to every essential trait of our intuitive representation of space. Thus, the intuitive feature usually described by saying that space is three-dimensional "is characterized by the fact that the position of every point is univocally determined by its relations to three mutually independent spatial quantities, e.g. to a system of three orthogonal coordinate axes". This is expressed analytically by the dependence of every point upon three independent real variables (coordinates).³⁹ The continuity of space is manifested intuitively by its infinite divisibility.⁴⁰ Analytically, this is expressed as follows: as an object moves in space from a point A to a point B, the coordinates of its position must take all real values between their value at A and their value at B; if two coordinates change together, while the third remains fixed, their quotient approaches a limit as their variation tends to zero.⁴¹

Erdmann proposes the following general concept of space: Space is a continuous quantity whose elements are univocally determined by three mutually independent (real) variables.⁴² Erdmann classifies such 3-fold determined continuous quantities into two kinds: those which have and those which do not have interchangeable coordinates.

According to him, the former kind exactly corresponds to Riemann's concept of a 3-fold extended manifold. Space, of course, belongs to that kind.⁴³ Riemann's notion of curvature is all that Erdmann uses to specify this concept.⁴⁴ 3-fold extended manifolds can have a constant or a variable curvature. A constant curvature can be positive, negative or equal to zero. The choice between these alternatives must depend, Erdmann says, "on the properties that we, in fact, observe in our spatial intuition". Erdmann concludes without further ado that these properties are the same as "the conditions which provide the basis for the congruence relations of our geometry".⁴⁵ He accepts Helmholtz's analysis: the free mobility of rigid bodies implies that space has a constant curvature. Which is the value of this curvature? "The answer seems obvious, since the theorems of space geometry show that all metric determinations of the plane can be transferred without any material (*inhaltliche*) modification to our three-dimensional space. [...] From this agreement, however, we cannot conclude immediately [...] that we may ascribe a constant zero curvature to space, because geometrical measurements would give the same results if that curvature possessed an infinitely small positive or negative value. [...] All we can do, therefore, is to determine by means of very carefully performed measurements the sum of the angles of empirically given triangles of the largest possible size." As far as we can tell, the constant curvature of space is indeed zero.⁴⁶ Space may be defined, therefore, as a threefold extended manifold with constant zero curvature. Erdmann believes that this strictly conceptual definition can be retranslated into the language of intuition which was our starting point: to every analytic character thus singled out there must correspond a unique intuitive meaning. He is apparently unaware of the fact that one and the same abstract mathematical structure can have many very different intuitive embodiments. This fact however should have been obvious in the light of contemporary projective geometry and was amply discussed in Felix Klein's Erlangen Programme, a work with which Erdmann apparently was acquainted.

Mathematical misconceptions are not uncommon among *soi disant* scientific philosophers. In this, as in other things, Erdmann is their forerunner. Let us briefly mention a few of his confusions. (i) Metric relations on a continuous manifold (in Riemann's sense) concern *the way how each particular point is determined by the coordinates*

(Erdmann, AG, p.49). Erdmann apparently believes that Riemann's charts, like the classical Cartesian mapping, are isometric mappings of space onto \mathbb{R}^3 , or that they induce a metric on space. (ii) Geodetic lines on a cylindrical surface "exhibit exactly the same curvature (*genau dieselben Krümmungsverhältnisse*) as the straight line on the plane" (Erdmann, AG, p.52). Since some of these geodetic lines are circles, while others are spirals, it follows that the new geometry, as expounded by its philosophical spokesman, conflicts with common sense. (iii) "The straightest line in a spherical space is that which possesses the same constant curvature at every point." (Erdmann, AG, p.155). If this were correct, every circular arc would be a straightest line in a spherical space. This would apply to a very good approximation to semicircles drawn on the earth's surface, any one of which would then not be longer than its diameter! (iv) Actual measurements show that the curvature of space certainly falls within a very small interval about zero. Consequently the probability that it is exactly zero is very high, so high indeed that we may conclude that it is zero (Erdmann, AG, p.70). Using this method of statistical inference we should be able to assign a fixed real value to any physical parameter which can be measured with a passably narrow margin of error.

Erdmann's discussion of the philosophy of geometry is set in the context of a rather primitive ontological framework. Erdmann assumes it quite uncritically but he, at least, has the courage to make it explicit.

All our intuitions of external things and relations are the product of an interaction, whose conditions depend partly on the [...] constitution of things, partly on the essence of psychical events. We are in total ignorance of the manner in which this interaction takes place, but we can derive the following conclusions from the fact of its existence. In the first place, the constitution of every element of our intuition must depend in part on the nature of the stimulating processes, in part on the way how these stimuli are received and elaborated by the psychical activities. Consequently [...] the entire material of our sensations is merely a system of signs for things, since the properties which we ascribe to the latter are nothing but the results of an interaction, one of whose terms, namely, the constitution of our mental activities, we simply take for granted. [...] Also the forms in which that material of sensations is ordered – the spatial forms not more and not otherwise than the intellectual – can only be a system of signs for the relations and situations of things.⁴⁷

The main conflicting theses of the theory of knowledge can now be easily described: *Empiricism* maintains that our representations

wholly depend on things, while *rationalism* holds that they are wholly independent from them. Both persuasions have three varieties. *Sensualist* empiricism believes that our representations agree with things absolutely. *Formalist* empiricism claims only a partial agreement, covering “the quantitative relations of space, time and lawfulness (*Gesetzlichkeit*)”, while representations differ from things in all their qualitative aspects. There is finally a brand of empiricism which Erdmann proposes to call *apriorism* (*Apriorismus*); this seeks to show that all our representations are completely different from the constitution and the relations of things, but correspond to them in every part. The rationalist can also maintain that representations, though uncaused by things, entirely agree with them (*preestablished harmony*); or that they agree only partially, so that, say, the forms of thought are identical with the forms of being (*formal rationalism*); or, finally, that every element in our representations is not only entirely independent of things, but specifically different from them (*nativism*).

Geometric knowledge concerns the properties, specifically the metric properties of our representation of space. The philosophy of geometry may take any of the above forms, as applied to this particular representation. According to Erdmann, “the mathematical results force us to conclude that our representation of space must be unambiguously conditioned by the actually experienced effects of things upon our consciousness”.⁴⁸ In other words, a rationalist philosophy of geometry is incompatible with the results of geometric research. Three arguments back this conclusion:

(i) The logical possibility of n -dimensional manifolds ($n > 3$) shows that “the influence of experience, i.e. of the things affecting us from outside” does not merely awaken but actually determines our particular representation of space as a three-dimensional manifold.⁴⁹ (This argument is valueless: in Kant’s philosophy, the a priori nature of our intuition of space is certainly compatible with the logical viability of other spaces with a different structure.)

(ii) The foundations of geometry involve the empirical concepts of rigid body and movement.⁵⁰ (The existence of freely movable rigid bodies is the *fact* which, according to Helmholtz, lies at the foundation of geometry. But perfectly rigid bodies do not actually exist. To say that the concept of such a body is obtained from experience is therefore somewhat far-fetched.)

(iii) If the representation of space were generated independently of

experience "by the spontaneous force of the soul", if it were only "the universal intuitive form of receptivity towards external things, in Kant's sense", it would not be possible for us to form intuitive representations of other three-dimensional manifolds with different metrical properties. But Helmholtz has shown that this is possible.⁵¹ (In the end, this is the mainstay of Erdmann's empiricism. The reader will judge whether it is imposed by mathematical results. Since we are apparently unable to imagine a space of more than three dimensions, one might feel inclined to conclude, by inversion of the foregoing argument, that three-dimensionality is not an empirical feature of space. But Erdmann knows for certain—presumably by a special revelation—that "the particular constitution of things outside us *compels* us to develop exactly three dimensions, neither more nor less".⁵²)

Geometry excludes rationalism but it will not assist our choice between the three varieties of empiricism. Sensualism, however, is utterly discredited by modern research on the psychophysiology of perception. Most scientists favour empiricist formalism: the qualitative contents of our sensory perceptions may be quite foreign to the actual nature of external things, but their relational structure, especially insofar as it can be quantitatively conceived, reflects the structure of things themselves. Both Riemann and Helmholtz hold this position in their philosophies of space and geometry.⁵³ Erdmann, on the other hand, inclines to the third alternative, which he calls by the unorthodox name of empiricist apriorism. He apparently believes that the very rejection of sensualism inevitably leads to it, at least in the philosophy of geometry (so that empiricist formalism in this field would be intrinsically untenable). Erdmann reasons thus:

Physiological research concludes no less surely than psychological analysis that no process in the central organ is conceivable which might bridge the gap between the observable extensive stimuli and the intensive formation of representations. This makes it understandable that the explanation of the psychological origin of the representation of space is entirely independent of the assumption of a spatially extended world of things; even though it is, of course, undeniable that some inducements (*Anlässe*) which prompt our psychical activities to develop the representation of space must be present in the relations of things themselves. That these inducements cannot themselves consist in spatial relations follows from the fact that *we group sensations spatially*. The very thought that one and the same form of space should mediate (*vermitteln*) the relations of things while, on the other hand, it also effects (*bewerkstelligen*) an order among sensations which are altogether different from those things,

seems contradictory. The contradiction looks even sharper when we consider that this form, as the form of intuition of our sensations, must be, like these, the product of an interaction, so that it cannot exist apart from this interaction, as the form of a part of the interacting elements. The whole question depends therefore on the subjectivity of sensations. If this is granted – and there can no longer be any doubt about this point – the subjectivity of the binding forms, especially of our representation of space, will follow.⁵⁴

Erdmann's reasoning is of course inconclusive. A relational structure such as that defined by a geometry is just the sort of thing that can subsist equally well in wholly disparate embodiments. The same abstract ordering introduced among sensations by some cause can be introduced by a different cause among other very different entities. In the light of modern axiomatics all this is obvious. But it should have been clear also in 1877 to anyone familiar with the Erlangen Programme or with projective geometry.⁵⁵

Erdmann, like Mill and Ueberweg before him, must face the fact that the better known geometrical objects, such as circles and right angles, are not exactly like anything actually perceived. He makes things even more difficult for himself by assuming that we cannot have an intuitive representation of surfaces – not to mention lines or points – but only of very thin bodies.⁵⁶ This is of course quite wrong: if we can have a definite representation of a body we must have a representation of its limits, and these are surfaces. But the fact remains that actually perceived surfaces and bodies show irregularities which are ignored by elementary geometry. Erdmann makes one very important remark which imperils the whole system of geometrical empiricism: *we cannot recognize those irregularities as such unless we have a concept of the rule from which they diverge.*⁵⁷

We can construct in thought (*in Gedanken*), with perfect assurance, lines which are exactly straight and circles whose peripheries have an entirely uniform curvature, though we would never be able to have an intuitive representation of them as extended in only one dimension. The metrical properties of the geometric concepts of construction are therefore neither factual properties of bodies nor concepts directly abstracted from them, but empirical ideas [*Ideen*, i.e. regulative ideas in Kant's sense]; they modify the observable properties of the elementary shapes of bodies in such way that they become ideal models (*ideale Musterbilder*) which can be indefinitely approached but are never attained by reality.⁵⁸

Erdmann declares that “the ideality of the concepts of construction does not exclude their empirical origin”.⁵⁹ His proof of this statement

is surprisingly weak. It runs as follows: "empirical ideas" analogous to these – i.e. such that no empirical entity will ever exactly satisfy their requirements – are also found in mechanics and, generally speaking, in all mathematical physics; now, one can hardly doubt that the latter is an empirical science. Indeed one can hardly doubt it, until one's attention is drawn to this remarkable fact. The ideality of the fundamental concepts of modern physics was indeed one of the chief motivations of 17th-century rationalism. The abandonment of this philosophical position will not suffice to justify the invocation of that very fact as an argument for empiricism. If geometry shares this property with every branch of applied mathematics, this implies only that we face an epistemological problem concerning the latter discipline as a whole: how can it be so helpful in the investigation of nature if it deals with entities which are never actually realized in nature? Erdmann does not entirely bypass this problem, but his attempted solution is very unsatisfactory. It applies specifically to geometry and it is based on the alleged homogeneity of the elements of our spatial intuition, i.e. of "the smallest particular parts of space and the line and surface elements derived from them".⁶⁰ This "factual homogeneity of the geometrical elements [...] makes it possible to conceive the construction concepts of geometry as ideals, since all factual divergences from them need not be thought of as essential differences, but as divergences from the pure concept, which in each particular case can be strictly taken into account both in our intuition and in our calculations. The ideality of the geometric concepts of construction is therefore quite compatible with their empirical origin, since it does not depend on the peculiarity of their source but on the homogeneity of the spatial elements".⁶¹ I do not find that this approach makes the empiricist position any stronger. After all, even if we do possess such an "intuition" of space as a set of homogeneous elements or as a union of homogeneous parts, we can hardly claim that it reproduces the contents of any actual sense perceptions, such as we would expect to lie at the root of any empirically generated representation.

Erdmann's final remarks further undermine geometrical empiricism. Geometry is not an empirical discipline in the same sense as the sciences that deal with quality. Even in its remotest and most complicated parts, it needs no other materials than the definitions and the axioms, on the one hand, and "the pure, i.e. indeterminately filled

representation of space”, on the other. It can attain all its results by purely deductive methods, thus bestowing on every one of its theorems the same generality, necessity and immutability which it claims for its principles. Geometry does not consist however in a mere analysis of the intension of its basic concepts. It is a synthetic science. “The synthetic character of geometrical propositions lies in the fact that in each of them the axioms are applied to new complications of the concepts of construction.”⁶² The development of geometry is independent of every particular experience. This has been usually understood as an argument for rationalism. But, says Erdmann,

it only follows from it that the intuition, upon which the synthetic progress of geometry depends, is not conditioned by the variegated, heterogeneous material of the qualitatively determined sense perceptions, but by the manifold of our representation of space, which lies equally at the basis of every particular experience. Geometry is independent of experience in all its developments because it presupposes that the representation of space, whose relations of construction are studied by it, is equally valid for every experience.⁶³

This passage is Kantian in style and contents, in its assertions and in its choice of words. But Erdmann ends on a Riemannian note. The independence of geometry from experience is not absolute, but only relative.

An exact investigation of limit cases of our metrical relations might reveal a divergence from the constancy or from the null-value of the curvature. As soon as this divergence is established, this corrected representation of space will become the subject-matter of geometrical research, until we are eventually driven by further progress, in case this new result turns out to be unsatisfactory, to make a revision of the properties of congruence and flatness.⁶⁴

4.1.4 *Auguste Calinon*

Non-Euclidean geometries and their epistemological implications were briskly debated in France in the 1890's. In this section, we shall refer to one of the participants in that debate, Auguste Calinon (born 1850). It is only with some reservations that I class him as an empiricist. He resolutely ascribes the source of geometrical concepts to our idealizing faculty, which follows the suggestions of sense perception, but is not enslaved to it. He believes that we learn through experience whatever we can know about the actual geometry of the universe, but he radically questions the possibility of

ascertaining it, except locally and approximately. Calinon anticipates, at any rate, many views typical of 20th-century geometric empiricism and on the matter of physical geometry he shows an open-mindedness comparable to Riemann's.

Calinon published his philosophical views in two short papers on "The geometric spaces" (1889, 1891) and in an article about "The geometric indeterminacy of the universe" (1893). Like most 20th-century empiricists, he makes a neat distinction between mathematics and physics, "two sciences [...] absolutely different in their object and their method, and also in the type and degree of certainty appropriate to each".⁶⁵ Geometry is a branch of mathematics, and it would preserve "its full logical value if the physical world did not exist or if it were other than it is".⁶⁶ Geometry is the deductive theory of forms (lines, surfaces). A geometric theory begins with the exact definition of one or more such forms. The theory is valid (*légitime*) if no contradiction follows from these definitions. Geometry thus conceived does not rest upon an experimental basis.⁶⁷ Geometry, on the other hand, should not be confused with mathematical analysis. The properties of every geometrical figure may be expressed analytically by means of equations. But not every equation can be made to correspond to a conceivable figure or form. We can only conceive clearly such forms as are very similar to those we see about us. All such forms have three dimensions at most. Thus, geometry is the branch of mathematics which takes as its starting-point the notion of the forms conceivable to us, that is, the forms of one, two or three dimensions, that are very similar to the forms surrounding us.⁶⁸ Does this mean that the fundamental concepts of geometry have their source in experience? Not at all. "Our knowledge of real forms is experimental; hence it is incomplete and only approximate. But the ideal forms of geometry are given by exact definitions, which enable us to know those forms absolutely and completely."⁶⁹

The first form defined by Euclidean geometry is the straight line. According to Calinon, its definition includes two properties: (a) for any two points P, Q, there is one and only one straight line through P and Q; (b) given a straight line *m* and a point P outside *m*, there is one and only one straight line through P on the same plane as *m*, which does not meet *m*. Thanks to Lobachevsky and others we know that property (b) does not follow from (a). Let us call the form defined by (a) and (b) the Euclidean straight. It is clearly a special case of a more

general concept, defined by property (a) alone. Let us call this the general straight. The theory of this form has been called non-Euclidean geometry. Calinon prefers to call it *general geometry*, because "far from being the negation of Euclidean geometry, it includes the latter as a special case".⁷⁰ The analytic development of general geometry supplies formulae which contain an undetermined parameter. The familiar formulae of Euclidean geometry are obtained by assigning a definite value to this parameter. Calinon points to a seeming paradox. The general straight through two given points P, Q depends on the undetermined parameter, so that we shall have infinitely many straights through P and Q, one for each value of the parameter. This contradicts however the definition of the general straight.⁷¹ Calinon overcomes this difficulty by treating each value of the parameter as the characteristic of a different three-dimensional space. On each space there is just one straight between two given points P and Q. This leads Calinon to an even broader conception of general geometry: "General geometry is the study of all spaces compatible with geometrical reasoning", i.e. of all consistent systems of one-, two- and three-dimensional forms. The spaces studied by Euclid and the classical non-Euclidean geometries are only a proper subset of such systems, which may be called *identical spaces* (*espaces identiques*), for "every figure constructed at a given point of such a space can be reproduced identically at any other point of the same space".⁷² Euclidean space is not only *identical* but also *homogeneous*, in the following sense: in it alone, *shape* is independent of *size*; two figures can be *similar* even if they are not *equal*.

General geometry is independent of experience. We ask now: which is the particular geometry that is realized in the material world? The several geometric spaces studied by general geometry cannot exist simultaneously, since they cannot contain the same forms. "In order to know which of these spaces contains the bodies we see about us, we must necessarily resort to experience."⁷³ Ordinary facts suggest that space is Euclidean. We constantly see bodies which preserve the same shape as they move from one place to another. Moreover, men have always imitated on a larger or a smaller scale the shape of the things surrounding them. These facts are indeed so familiar that we tend to believe that they are inevitable, being somehow rooted in the essence of the world or in the nature of the human mind. But we must not forget that observed data are never quite

exact. Even if experience shows that the space we live in is identical and homogeneous this means only that it is very nearly Euclidean. All that we may conclude from this is that "the differences that might exist between Euclidean geometry and the actual geometry of the world (*celle qui réalise l'Univers*) are smaller than the errors of observation".⁷⁴ This conclusion is compatible with any of the following three hypotheses:

- (i) Our space is, and will remain, strictly Euclidean.
- (ii) Our space differs slightly from the Euclidean space, but is always the same.
- (iii) Our space realizes in the course of time several different geometric spaces; in other words, the spatial parameter varies with time, either by deviating more or less from the Euclidean parameter, or by oscillating about a given parameter not too different from the Euclidean one.

Calinon apparently believed at first that it made sense to ascribe a definite geometry to the universe, even if such geometry changes with time. Nevertheless, he held that we are in no position to know it even approximately. Thus, if the limited region of space in which all our measurements are performed happens to be very small in comparison with the whole of space, the whole may possess any geometric structure whatsoever, even though our experiments show that limited region to be approximately Euclidean. Calinon adds: "This hypothesis that our measurable universe is contained in an infinitely small part of an arbitrary (*but otherwise well determined*) space, is the most general hypothesis we can make within the limits prescribed by observed facts".⁷⁵ In his paper of 1893 he takes a more radical stance. "The space in which we locate the geometric facts of the universe is indeterminate; this is a fundamental fact."⁷⁶ This geometric indeterminacy of the universe results from the fact that our measurements are only approximate and have a local scope. Calinon draws several consequences from this fact. Thus, though he is apparently unaware of Klein's work on Clifford surfaces, he says that many, very different spaces can be locally isometric to Euclidean space, just as the surface of a cylinder or a cone is locally isometric to a Euclidean plane. Even if our observations could show that the space surrounding us is exactly Euclidean, that would not teach us much about the global geometry of the world. Calinon's conception of geometric indeterminacy does not imply – like Grünbaum's – that space, by its

own nature, cannot possess a definite geometric structure. The indeterminacy is purely epistemological, not ontological. But Calinon, in 1893, does not appear to have had much use for the real, though unknowable, structure of space.

When Calinon wrote his paper, he had read Poincaré's famous article on non-Euclidean geometries,⁷⁷ in which the choice of a physical geometry was compared to the choice of a coordinate system (both were said to be a matter of convenience, not of truth). Calinon apes Poincaré's language. But he emphasizes that the different geometries are not simply equivalent. What is more important, he expressly rejects Poincaré's contention that we are bound to prefer Euclidean geometry because it is by far the simplest and most convenient. The great advantage of geometric indeterminacy is that it permits us to approach each problem with the geometric representation which is most likely to provide the simplest solution.⁷⁸ Thus, Newton's law of gravitation is verified only within certain limits. At the distance which separates two neighbouring molecules of the same body, the law seems to be different. A difference as yet unknown might also become apparent at astronomical distances. "We may therefore very well conceive that at such large distances the law of attraction [. . .] could find its simplest expression in another geometric representation of the universe, different from the Euclidean representation."⁷⁹

Calinon's conception of general geometry is taken up by Georges Lechalas (1851–1919) in his *Etude sur l'espace et le temps* (1896), where several points of it receive further clarification. The traditional postulates of geometry, says Lechalas, are hidden definitions (*définitions méconnues*). If we continue to ignore their real nature, we shall insist in demonstrating them. But they cannot be proved from the definitions which are usually taken as a starting-point of geometry. These are too general, so that many different surfaces or lines fulfil the familiar definitions of the plane or the straight line. "Now, if this is so, it is plain that one can only overcome this indeterminacy by adding, under the guise of postulates, the characteristic supplementary properties required to specify the line or the surface one has in mind." (Lechalas, ET, p.12). Though Lechalas maintains with Calinon that geometry must somehow attach *images* to its *concepts*, he is on the verge of dismissing this restriction as untenable. Euclidean, he says, will object that non-Euclidean

geometries are not genuine geometries, because we are unable to form adequate images of figures incompatible with Euclidean space. But in truth we are just as incapable of forming adequate images of Euclidean figures. "All our images are imperfect [...] and geometrical reasoning concerns the ideas (*idées*) with which the images are associated, and not the images themselves. Thus it matters little if the disagreement is large or small. If it is so large that we can no longer follow on the image the conclusions drawn by analysis, we can still conceive that other beings, with a different sensibility, could have images agreeing with those conclusions as nearly as our own images agree with Euclidean geometry." (Lechallas, ET, p.43; see also Lechallas, IGG, pp.16f.) This fantasy of other, differently organized, sensitive beings is not really needed to back general geometry, as it is conceived by Lechallas. This is clear, I think, in the light of his explanation of Calinon's concept of a plurality of spaces, each of which admits some forms, while excluding others. "For us, a space is nothing but the verbal substantialization (*la substantialisation verbale*) of mutually compatible spatial relations. To say that a figure cannot enter into a space is tantamount to saying that it constitutes a system of relations which is incompatible with a more general system, embellished with the name of space (*décoré du nom d'espace*)." (Lechallas, ET, p.52 n.2). From this point of view, the mainstay of geometry is no longer intuition or imagination, but the set of conceptual relations determined by the definitions.

Like Calinon, Lechallas rejects the empirical origin of geometrical notions. "Since we do not regard geometry as an innate science in the proper and strict sense of this word, it is clear that we must look for a starting-point or rather a mental stimulus (*un excitant pour l'esprit*) among perceptions or experiences. The mind, working with sense-data which lack all precision, applies to them general notions which were perhaps aroused in it on the occasion of those data. Thus it is able to build an a priori science, which might even be in fact inapplicable to actual phenomena without thereby losing any of its value." (Lechallas, ET, p.23). This intellectual exercise generates an infinity of geometries which are equally rational (*également rationnelles*). Reason cannot prefer one of them to the others. But experience can reveal which of them is fulfilled in our universe. (Lechallas, ET, p.64). Since this particular system of geometry is in no way necessary we can only get to know it by observation. "Only

through measurements shall we be able to determine the parameter of our universe, assuming that our space is identical [in Calinon's sense, i.e. that it is a maximally symmetric space], an assumption which is not prescribed a priori. Consequently that determination can be carried out, like every experimental determination, only up to a certain degree of approximation." (Lech alas, ET, p.88). Though Lech alas does not expect us to learn by experimental research which is the exact geometrical structure of the world, he believes that it possesses one and that we can determine it with ever increasing precision. He criticizes Calinon's thesis on the geometrical indeterminacy of the universe. He argues somewhat unconvincingly that we can determine to a very good degree of approximation that light-rays travel along Euclidean straight lines, at least within the solar system and even as far as the nearest stars (the stars that have an observable parallax). He admits next that no experiment can show whether we live in a Euclidean space or in another three-dimensional space which is isometric to Euclidean space (Lech alas, ET, p. 92). But, he adds, two isometric 3-spaces differ only in the way they lie in a four-dimensional space, just as two isometric 2-spaces or surfaces, such as the plane and the cylinder, differ only in the way they lie in 3-space. From the human point of view, it does not make much sense to speak of geometric indeterminacy merely because we cannot distinguish between several three-dimensional structures which are intrinsically indiscernible. (Lech alas, ET, p.98f.). I am afraid that Lech alas is wrong on this point. Apparently, he had not yet heard about the Clifford-Klein space problem (Section 2.3.10). And he pays no attention to the global topological properties of the several spaces, which, generally speaking, are no less intrinsic than their local isometry. By taking them into account, even a Flatlander should be able to distinguish between a cylinder and a plane, though he may have some trouble in actually telling one from the other if the cylinder is very large or if he stubbornly contests the identity of indiscernibles.

4.1.5 *Ernst Mach*

Ernst Mach (1838–1916) explains his thoughts on geometry in the last chapters of his book *Knowledge and Error* (1905).⁸⁰ His analyses, which attain a level of concreteness never found in Erdmann or Mill, contribute new viewpoints and insights to geometrical empiricism. Mach believes, like John Locke, that the empiricist philosopher

shows his mettle by exhibiting the actual development of our ideas from experience. The psychological origin and evolution of geometry had been the subject of some valuable studies by Henri Poincaré.⁸¹ Mach undertakes a more systematic treatment of this matter. Geometry has three sources: intuition, experience and reasoning. "Our notions of space are rooted in our *physiological* constitution. Geometric concepts are the product of the idealization of *physical* experiences of space. Systems of geometry, finally, originate in the *logical* classification of the conceptual materials so gathered."⁸² Spatial intuition is composed of "space sensations" (*Raumempfindungen*), which is Mach's name for the spatial ingredient present in ordinary sensations. A spotlight seen in complete darkness moves upward or downward, forward or backward, to the left or to the right. These characters of the movement are immediately given and do not depend upon a previous intellectual organization of the perceptual field⁸³ – as a matter of fact, the intellectual or scientific idea of space, the space of geometry, is isotropic (equal in every direction) and does not possess those characters. If someone applies a pin at a point on my naked back and then pricks other points on my back with another pin, I feel a second prick nearer or farther from the first, above or below it, to its left or to its right. There is also a neat *spatial* difference between the feeling of being pricked with a pin and that of having, say, the back of a spoon rubbed against one's back. In every impression received by our senses we can distinguish, according to Mach, a "sense-impression" (*Sinnesempfindung*), which depends on the quality of the stimulus, and an "organ-impression" (*Organempfindung*), which varies with the place of the skin, the eye, the tongue, etc., upon which the stimulus acts. Organ-impressions are regarded by Mach as identical with space sensations. Intuitive or "physiological" space is "a system of *graduated organ-impressions* (*abgestufte Organempfindungen*), which certainly would not exist without sense-impressions, but which, when it is aroused by the changing sense-impressions, forms a *permanent* register, wherein those variable sense-impressions are ordered".⁸⁴ Physiological space is quite different from the infinite, isotropic, metric space of classical geometry and physics.⁸⁵ First and foremost, it is not a metric space. We can describe some regions of it as contained in others, and we can set up neighbourhood relations between its points, but any assignment

of real-valued distances to point-pairs in physiological space is arbitrary, unstable and, in the end, pointless. Physiological space can, at most, be structured as a topological space. When viewed in this way, it naturally falls into several components: visual or optic space, tactile or haptic space, auditive space, etc. Mach makes some remarks about the first two. Optic space is anisotropic, finite, limited.⁸⁶ It is certainly not metric. "The places, distances, etc., of visual space are qualitatively, not quantitatively different. What we call visual size (*Augenmass*) is developed only on the basis of primitive physico-metrical experiences."⁸⁷ Mach regards the optic space as three-dimensional.⁸⁸ However, since the direct perception of visual depth depends, to a considerable extent, on the coordinated movements of the two eyes, we might wish to maintain that three-dimensional vision is not originally *given* in intuition, but *developed* through behaviour. From this point of view, the third dimension of optic space should be classed, like visual size, as a product of experience. On the other hand, the different ocular movements required to bring a near or a distant object into focus are hardly ever perceived *as movements*, let alone as deliberate acts. The 'experiences' that lie at the source of our perception of visual depth must therefore be distinguished from those which give rise to the idea of physico-geometrical space, such as the experience that it takes me twenty steps to get to the front-door and two hundred to go to the nearest bakery. Haptic space or "the space of our skin corresponds to a two-dimensional, finite, unlimited (closed) Riemannian space".⁸⁹ This is nonsense, for *R*-spaces are metric while haptic space is not. I take it that Mach means to say that the latter can be naturally regarded as a two-dimensional compact connected topological space. Mach does not emphasize the disconnectedness of haptic from optic space, nor the role of physico-geometric space in the integration of data supplied by the different senses. It is obvious, however, that the region in optic space where I see the tip of my fingers and the region in haptic space where I feel the pressure of my pen are mutually related only through their association (or identification) with one and the same region of my room, located at so many feet from the walls and the floor.

Mach claims that if man were a strictly sedentary animal, like an oyster, he would never attain the representation of Euclidean space. But the possibility of freely moving and reorienting the body as a whole makes us understand "that we can perform the same

movements everywhere and in every direction, that space has everywhere and in every direction the *same* constitution (*gleich beschaffen ist*) and that it can be represented as *unlimited* and *infinite*".⁹⁰ Experiences with bodies and their movements lie at the root of geometry and inspire its fundamental postulate: the perfect homogeneity of space.

Let a body K move away from an observer A by being suddenly transported from the environment FGH to the environment MNO. To the optical observer A the body K decreases in size and assumes generally a different form. But to an optical observer B who moves along with K and retains the same position with respect to K, K remains unaltered. An analogous sensation is experienced by the *tactical observer*, although the perspective diminution is here wanting for the reason that the sense of touch is not a telepathic sense. The experiences of A and B must now be harmonised and their contradictions eliminated, – a requirement which becomes especially imperative when the same observer plays alternately the part of A and of B. And the only method by which they can be harmonised is to attribute to K certain *constant* spatial properties independently of its position with respect to *other* bodies. The space-sensations determined by K in the observer A are recognised as *dependent* on other space-sensations (the position of K with respect to the body of the observer A). But these same space-sensations determined by K in A are *independent* of other space-sensations, characterising the position of K with respect to B, or with respect to FGH...MNO. In *this* independence lies the *constancy* with which we are here concerned. The fundamental assumption of geometry thus reposes on an *experience*, although of the idealised kind.⁹¹

The role played by bodies and by the handling of bodies in the constitution of geometry is repeatedly emphasized by Mach, following the tradition initiated by Ueberweg and Helmholtz. "Geometrical concepts are obtained through the mutual comparison of physical bodies."⁹² "The visual image must be enriched by physical experience concerning corporeal objects to be geometrically available."⁹³ This viewpoint leads to a curious distortion of historical fact: solid geometry is said to have preceded plane geometry. We know however that this is not so, that the beginnings of stereometry were slow and difficult and came *after* plane geometry was a well-developed science. Mach's bias is so strong that he even claims that "every geometrical measurement is at bottom reducible to measurements of *volumes*, to the *enumeration of bodies*. Measurements of lengths, like measurements of areas, repose on the comparison of the volumes of very thin strings, sticks and leaves of constant thickness."⁹⁴ The last remark is preposterous. We can compare the flat polished surfaces of two stone slabs without paying any attention to the thickness of the

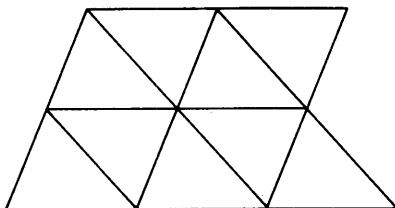


Fig. 20.

slabs; we can mark two points on each surface and compare their distances by superposition, without having to employ a “very thin string” as an instrument of comparison. Mach observes that propositions equivalent to Euclid’s fifth postulate can be proved approximately by means of easy experiments. Thus, with a set of congruent triangular floor-tiles we can construct the figure shown in Fig. 20 which is possible only if equidistant lines are straight. With a triangular piece of paper, folded as in Fig. 21, we can prove that the three internal angles of a triangle are equal to two right angles (when the paper is folded along EF , FG , GH , the vertices A , B , C meet at X and the angles at A , B , C appear as the parts of a single straight angle). The first of these two experiences was probably familiar to men of the earliest civilizations. On the intuitive origin of the idea of straightness Mach repeats a trite remark: “A stretched thread furnishes the distinguishing *visualization* of the *straight line*. The straight line is characterized by its physiological simplicity. All its parts induce the *same* sensation of direction; every point evokes the mean of the space-sensations of the neighbouring points; every part, however small, is similar to every other part, however great”.⁹⁵ This characterization however is but of little use to geometers. It is a

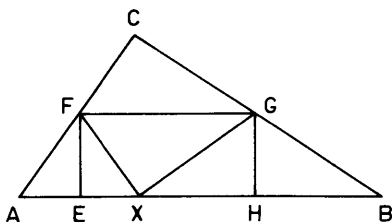


Fig. 21.

mistake to believe that the straight line is known to be the shortest line through mere intuition. "The mere *passive contemplation of space* would never lead to such a result. Measurement is experience involving a physical reaction, a superposition-experiment."⁹⁶ That a straight line is determined by two of its points is also an experimental notion, which Mach motivates as follows:

If a wire of any arbitrary shape be laid on a board in contact with two upright nails, and slid along so as to be always in contact with the nails, the form and position of the parts of the wire between the nails will be constantly changing. The straighter the wire is, the slighter the alternation will be. A straight wire submitted to the same operation slides *in itself*. Rotated round two of its own fixed points, a crooked wire will keep constantly changing its position, but a straight wire will maintain its position, it will rotate within itself. When we define, now, a straight line as the line which is completely determined by two of its points, there is nothing in this *concept* except the *idealization* of the empirical notion derived from the physical experience mentioned,—a notion by no means directly furnished by the physiological act of visualization.⁹⁷

Optical experiences with light-rays have probably aided the rapid development of geometry, but we should not regard them as the essential foundation of this science. "Rays of light in dust or smoke-laden air furnish admirable *visualizations* of straight lines. But we can derive the *metrical properties* of straight lines from rays of light just as little as we can derive them from *imaged* straight lines."⁹⁸ Mach is the first author I know of, who took notice of how planes are actually built in practice and pointed it out to his readers. "Physically a plane is constructed by rubbing three bodies together until three surfaces, A, B, C, are obtained, each of which exactly fits the others—a result which can be accomplished [...] with neither convex nor concave surfaces, but with plane surfaces only."⁹⁹ This practical procedure which unambiguously defines one of the basic figures of geometry will play an important role in Dingler's pragmatic foundation of this science. In fact, if you construct two adjacent planes by this method, their common edge will provide a better approximation to the straight line than any taut string or light-ray.

The rational ingredient of geometry consists, according to Mach, in the deductive organization of the concepts and insights supplied by experience. He is well aware that geometrical concepts are not just *abstracted* from experience but are formed by *idealization*. But his treatment of this subject is not more satisfactory than what we have met in other empiricists. Mach stresses the need for idealization in the

construction of geometry, but he does not see that it may require a peculiar, independent, non-sensory principle of knowledge. We must grant that on this point that old poet Plato showed a keener sense of facts when he proclaimed that geometry was non-empirical, not only because of its certainty, nor mainly because of its deductive structure, but because of the ostensibly non-empirical nature of its subject-matter, its points and lines and planes. Mach attempts to explain geometrical idealization psychologically:

The same economic impulse that prompts our children to retain only the *typical* features in their concepts and drawings, leads us also to the schematization and conceptual idealization of the images derived from our experience. Although we never come across in nature a perfect straight line or an exact circle, in our thinking we nevertheless designedly abstract from the deviations which thus occur.¹⁰⁰

Economy may indeed justify the use of idealization but it cannot explain its possibility. We would be hard put indeed to say how we can recognize and eliminate observed "deviations" from the ideal figures, unless we know beforehand what they deviate from. A few passages show, however, that Mach was ready to go beyond the classical empiricist posture and to acknowledge our intellectual aptitude for autonomously generating concepts: The choice of our geometric concepts is *suggested* by empirical facts, but it finally rests upon the free elaboration of those facts by thought. This intellectual freedom in the formation of concepts is indeed required for their eventual ordering in a deductive system: "For our logical mastery extends only to those concepts of which we have ourselves determined the contents."¹⁰¹ In this, however, geometry does not differ from mathematical physics. Like the latter, it becomes an exact deductive science only through the representation of empirical objects by means of schematic, idealizing concepts. "Just as mechanics can assert the constancy of masses or reduce the interactions between bodies to *simple accelerations only within the limits of errors of observation*, so likewise the existence of straight lines, planes, the amount of the angle sum, etc., can be maintained only on a similar restriction."¹⁰² The imperfect correspondence between geometrical concepts and empirical facts has one important implication:

Different ideas can express the facts with the same exactness in the domain accessible to observation. The *facts* must hence be carefully distinguished from the *intellectual* constructs the formation of which they suggested. The latter – the concepts – must be

consistent with observation, and must in addition be *logically* in accord with one another. Now, these two requirements can be fulfilled *in more than one manner*, and hence the different systems of geometry.¹⁰³

This ambiguity is shared by geometry with physics, but, in Mach's opinion, the former has one signal advantage over the latter. While the ideal concepts of physics, such as the concept of a perfect gas, can be experimentally realized only up to a certain point, beyond which they require adjustment, "we can conceive a sphere, a plane, etc., constructed *with unlimited* exactness, without running counter to any fact".¹⁰⁴

From this, Mach concludes that if the progress of physical experience should eventually require us to modify our scientific concepts, we will rather sacrifice the less perfect concepts of physics, than the simpler, more perfect, firmer concepts of geometry. Scientists can therefore rest assured that they will never need to replace Euclidean geometry in their descriptions of phenomena. This surprisingly conservative conclusion is followed immediately by a no less startlingly revolutionary statement. Physicists, says Mach, can benefit in another sense from the study of unorthodox geometries.

Our geometry refers always to objects of sensuous experience. But the moment we begin to operate with mere things of thought like atoms and molecules, which from their very nature *can never be made the objects of sensuous contemplation*, we are under no obligation whatever to think of them as standing in spatial relationships which are peculiar to the Euclidean three-dimensional space of our sensuous experience.¹⁰⁵

In other words, if we postulate invisible, intangible objects for explaining perceived phenomena, we need not feel compelled to locate those objects in Euclidean 3-space. On the contrary, we are free to set them in any geometrical framework we think fit. Since Mach was read by most young German physicists in the first quarter of the 20th century, it is not unlikely that passages like this one have positively contributed to liberate dynamics from its dependence on classical geometry and kinematics.

4.2 THE UPROAR OF BOEOTIANS

4.2.1 Hermann Lotze

The uproar Gauss had feared came after his death. The strongest protests were made by philosophers who would not admit any

tampering with Euclidean geometry, the established paradigm of scientific knowledge. Many of the objections raised against the novel geometric conceptions merely showed ignorance (and a remarkable readiness to believe mathematicians guilty of the wildest nonsense). Thus, Albrecht Krause, in his book *Kant und Helmholtz* (1876), observes that “lines, surfaces and the axes of bodies in space have a direction and consequently a curvature, but space as such has no direction because everything directed lies in space, and therefore it has *no* curvature; this is not the same as to say that it has a curvature equal to zero”.¹ In his *Grenzen der Philosophie* (1875), W. Tobias rebukes Riemann for believing in the possibility “that the observable world with its actually existing three dimensions ends at an incalculable distance from the earth, where another cosmic space (*Weltenraum*) begins, with a different curvature and perhaps with more than three dimensions”.² Not all critics were so obscure as Krause and Tobias. The Boeotian chorus was joined by some highly regarded thinkers, such as Lotze and Wundt in Germany and Renouvier in France.

Hermann Lotze (1817–1881) is perhaps the most noteworthy among late 19th-century German philosophical system-builders. To his mind, the new geometric speculations were “just one big connected mistake”.³ His criticism of them is set in the context of his metaphysical theory of space. This theory, like Erdmann’s, is conceived in terms of the duality of Mind and Things. According to Lotze, space can exist only as space intuition, that is, only insofar as the Mind is aware of it.⁴ But space is not a mere appearance to which nothing corresponds in Reality (*im Reellen*). “Every particular trait of our spatial intuitions corresponds to something that is its ground in the world of things.” But such ground does not in any way resemble spatial relations. “Not relations, spatial or intelligible, *between* things, but only immediate interactions, which things inflict one another as internal states, are the actual fact whose perception is woven by us into a spatial phenomenon.”⁵ This conception of space raises three questions:

(i) Why must the soul intuit the variegated impressions it receives from things – which originally can be only non-spatial states of mind – under the form of a spatial expanse (*eines räumlichen Nebeneinanders*)? This question admits of no answer. The spatial character of our perception of things must be taken as an inexplicable fact of life,

like the perception of air-waves as sounds or of light-waves as colours.

(ii) What inner states are organized by the soul in this peculiar form? What conditions govern the assignment of a definite spatial position to each particular sense-impression? These questions are the subject of Lotze's theory of local signs, which need not concern us here.

(iii) Which is the geometric structure of the full expanse developed by drawing every consequence that is necessitated or permitted by the given nature of the original intuition of space? Mathematicians have hitherto answered this question by reasoning deductively from unhesitatingly accepted premises. These they take from what they call intuition. This procedure has given rise to Euclidean geometry, which, Lotze says, was never questioned until modern times. Recent speculations on the matter compel him, however, to deal with the question of geometric structure in his metaphysical treatise.

Lotze believes that doubts concerning the validity of Euclidean geometry are motivated by the philosophical thesis that space is purely subjective. Space, as we know it, may be conceived as a special case of the more general concept of an "order system of empty places".⁶ Nothing prevents us from conceiving several different species of this generic concept, structured by other rules than those that govern space. Other beings might exist, who perceive the same world of things as we do, but under one of these alternative order systems. It is possible that they perceive in a different fashion the same aspects of things which we perceive in space, or that the peculiar structure of their intuition enables them to perceive other aspects of things, which are inaccessible to us. Lotze will not dispute these possibilities. There is, in fact, no way of knowing whether they are fulfilled or not. But Lotze emphatically rejects the contention that other beings, unknown to us, could have a *spatial* intuition different from ours.

It might seem at first sight that this is merely a matter of words. We may just as well reserve the name *space* for the order system of our own perception of things. Riemann himself had done so. But according to Riemann we cannot be sure that this order system is adequately represented by Euclidean geometry. Most probably it is not, since many other such order systems are possible, and our perceptions are far too imprecise for us to determine exactly which is true of space.

Lotze has apparently missed the point. If r denotes a straight line and w an angle, *as intuited by us*, then, Lotze maintains, r and w as elements of space determine its global configuration and internal structure completely and unambiguously in full agreement with traditional geometry. "It would be unfair to demand another proof of it than the one provided by the actual development of science until now."⁷ That the elements r and w admit of other combinations that are not made intuitive in our space and that such combinations are not just verbally describable abstract possibilities, but lead to spatial intuitions different from our own, can only be proved by actually producing the intuitions.

Lotze believes that the Euclidean system is a perfectly satisfactory expression of our space intuition and he will not enter into any discussion on this matter. He knows that the theory of parallels is regarded by many as a weak spot in the Euclidean system. That the theory rests upon an undemonstrable postulate is, of course, no objection to it. Geometry, as a description of intuition, is necessarily built upon undemonstrable premises. Like other philosophers before him, Lotze thinks that he can improve the intuitive obviousness of Euclid's theory by redefining parallels. He provides two new definitions, which he apparently believes to be equivalent:

[i] We call two straight lines a and b parallel if they have the same direction in space, and we verify that their direction is the same if a and b make the same angle w with a third line c on the same plane e and towards the same side s .

[ii] a and b are parallel if the endpoints Q and R of any pair of equal segments OQ and PR , measured on a and b from their [respective] origins O and P , lie at the same distance from each other.⁸

We know that these definitions are not equivalent and that none of them is equivalent to Euclid's. The first definition implies, of course, that b is the only parallel to a on plane e through the meet of b and c ; but it does not imply that b is the only straight line which fulfils the italicized condition but does not meet a . The second definition does not guarantee that parallel lines are straight. If we include this requirement we must prove (or postulate) that such lines exist. Lotze probably regarded it as intuitively obvious.

Lotze was, as far as I know, the first one to make the following important remark, which Poincaré later used in support of conventionalism. In Euclidean geometry, the three internal angles of a

triangle are equal to two right angles. This fact, Lotze claims, is not subject to experimental verification or refutation. If astronomical measurements of very large distances showed that the three angles of a triangle add up to less than two right angles, we would conclude that a hitherto unknown kind of refraction has deviated the light-rays that form the sides of the observed triangle. In other words, we would conclude that physical reality in space behaves in a peculiar way, but not that space itself shows properties which contradict all our intuitions and are not backed by an exceptional intuition of its own.⁹

Lotze completes his discussion of modern philosophico-geometrical speculations with a criticism of some of the concepts which turn up in them, such as *intrinsic geometry*, *fourth dimension*, *space curvature*. Lotze's remarks suggest that he had not actually studied the works of Gauss and Riemann, but tried to reconstruct their meaning *proprio Marte* from a few vague hints. Thus, in his opinion, it is impossible to define or measure a curve without presupposing the intuition of the straight line from which that curve deviates. (Lotze, M, p.246). Lotze is probably thinking of the classical definition of the length of a curve as the limit of a sequence of lengths of polygonal lines inscribed in it (see p.69). But this definition had been discarded by Riemann when he demanded that every line should measure every other line (Riemann, H, p.12; see p.90f.). Lotze discusses Helmholtz's Flatland at length. He maintains that a two-dimensional rational being living upon a surface would develop the notion of a third dimension in order to understand the fact that straightest lines in his world return upon themselves. That notion would arise in him, not as a product of immediate perception, "sondern auf Grund des unerträglichen Widerspruchs, der in dieser sich selbst wiedererreichenden Geraden läge" (Lotze, M, p.252). Now, I do not see why, if we are willing to grant Helmholtz's fiction, we cannot also admit that straightest lines in spherical Flatland are ordered cyclically like projective lines. It is difficult to make sense of Lotze's refutation of the fourth dimension (Lotze, M, pp.254-260). It rests upon the notion that the number of dimensions of a space is equal to the number of mutually orthogonal straight lines that meet at an arbitrary point of it. Lotze maintains that this number cannot possibly exceed three. He apologizes for his inability to substantiate this claim with stronger arguments than the one proposed by him (Lotze, M, p.257; Lotze's argument is explained and criticized in Russell, FG, p.106f.). His discussion of space curvature

is quite remarkable. He apparently believes that a three-dimensional space with a positive curvature is constructed onionwise out of many curved surfaces. It is then easy to show that no such space exists; thus, for instance, the space built from the spheres centred at point P, whose radii take all real values, is none other than ordinary Euclidean flat 3-space. "Für jede der in Gedanken an diesem Raume unterscheidbaren, in ihm selbst aber völlig ausgelöschten Oberflächen hat der Begriff eines Krümmungsmasses seinen guten und bekannten Sinn; aber es ist unmöglich, sich eine Eigenschaft des Raumes zu denken, auf die er Anwendung finden könnte." (Lotze, M, p.263). Lotze finally criticizes Riemann's concept of a space with variable curvature, wherein rigid bodies do not enjoy free mobility. An inhomogeneous space, some of whose parts are structurally different from the others "would contradict its own concept and would not be what it ought to be, namely, the neutral background for the variegated relations of that which is ordered in it". (Lotze, M, p.266). Spaces which by their very structure do not admit at one place a figure which can be constructed at another place "can only be conceived as real shells or walls, whose resistance denies admission to an approaching real form, but which can be eventually broken by the increasing impact of the latter (*durch den heftigeren Anfall dieser müssten zersprengt werden können*)." (Lotze, M, p.266). Lotze's criticism of the new geometric concepts is not untypical of a certain kind of philosophical literature. It may help understand why many scientists are impatient of philosophy.

Lotze's metaphysics of space is very similar to Erdmann's but their philosophies of geometry are conspicuously different. Both authors hold space to be the peculiar form in which the human mind perceives the things acting upon it. Both believe – though not for the same reasons – that this form is wholly foreign to things themselves, but they both maintain that the spatial properties and relations of sense appearances are necessarily grounded on the non-spatial properties and relations of things. Though Erdmann quotes Klein's Erlangen Programme and Lotze regards space as an *order* system of *empty* places, they have not grasped the potentialities of the structural viewpoint. They fail to see that the *same* order system whose empty places are filled by sense-appearances could also be embodied in the world of things. Space, as conceived by Erdmann and Lotze, possesses a definite geometrical structure. Since this structure is contingent

upon the factual nature of the Mind and no 'transcendental deduction' of it is attempted,¹⁰ we must conclude that it can only be known empirically. At this point, the ways of Erdmann and Lotze diverge. The former holds that we can know the actual structure of space only approximately, by studying spatial phenomena, while the latter maintains that Euclidean geometry provides an irrefutable exact description of that structure. Lotze's claim is consonant with the philosophical prejudice that self-knowledge is absolutely evident and indisputable. Lotze apparently believes that the geometrical images which we can form with closed eyes are perfectly definite and possess an inner necessity of their own, independently of our experiences with physical objects. Geometrical concepts merely reflect what we 'see' in those images, instead of determining or regulating them.

4.2.2 Wilhelm Wundt

As early as 1877, Wilhelm Wundt (1832–1920) criticized the use of non-Euclidean geometries for grinding philosophical axes.¹¹ His final position on the matter is stated in the 4th edition of his *Logik* (1919, 1920).¹² "Space" is, first of all, the name for "the immediately given order of our sense perceptions".¹³ This is also called our intuition of space (*Raumanschauung*). It is due to "the actualization of original conditions of our physical and mental organization".¹⁴ As such, it may be regarded as "a necessary form of intuition." But its necessity is not the expression of an inborn idea, "but the result of the *constancy* with which all sensations referred to external objects are bound to their spatial order".¹⁵ Since space is originally given as an *order* of sensations, but is not itself a sensation, we can grasp this order abstractly and thus develop the *concept of objective space*. This is not immediately given to us, but we arrive at it by eliminating in thought "the subjective components involved in every particular spatial intuition".¹⁶ By thus freeing space from all ingredients whose subjective origin has been established, we obtain "the conceptual order of an objectively given manifold corresponding to that intuitive form".¹⁷ The determination of the concept of objective space is the task of geometry. Geometry is therefore unquestionably an empirical science. However, this does not detract from its apodictic validity. "The proposition that empirical statements are *never* apodictic is groundless." "If there are any experiences which have no exceptions, we must regard them as necessary. Spatial representations are among the

experiences which are free from exceptions. They must be viewed as the inalterable ingredients of every external experience [...]. The unexceptionable empirical validity of geometrical propositions is thus a sufficient ground for their necessity."¹⁸

Modern mathematics rightly places the concept of objective space among many other related concepts, all of which are comprised under the general notion of a manifold. This has given rise to the mistaken belief that these other concepts can also be associated, like the former, to an intuition. Wundt emphatically rejects this idea.

We can only represent to ourselves as a simultaneously given manifold, the space of our intuition with some concrete contents, which we regard as homogeneous and indifferent, and which we can use, by subjecting it to a different ordering, for constructing a different representable manifold. Every space which differs from our space is the object of a conceptual abstraction or of an analogy based on a conceptual abstraction; in either case, the concepts thus developed do not agree with our actual representations.¹⁹

Our thought can ignore some definite properties of reality or it can transfer notes from some definite concepts to other concepts. But these operations do not have the slightest power to change anything in real facts. For this reason, we cannot allow the supposition that astronomical or physical experiences might teach us some day that our geometrical system is not valid in some regions of the universe.²⁰ The order of the objects of the real world according to the laws of our three-dimensional flat geometry [...] is the factual expression of the real order of phenomena, which cannot, as such, be replaced by any other order.²¹

No arguments are given by Wundt to support these claims. Surprisingly enough, he defines the concept of objective space in terms that are in fact compatible with BL geometry: "*Space is a continuous self-congruent infinite quantity wherein indivisible particulars are determined by three directions.*"²²

Wundt has realised that the subject-matter of Euclidean geometry, "the concept of objective space", though based in our space intuition, is not identical with it. But he continues to think that the intuitively grasped "order of sense perceptions" exactly corresponds to that concept. This is highly questionable. Wundt's "order of sense perceptions" is not the same as the so-called perceptual fields in which the data of the several senses are thought to be separately ordered. He obviously conceives it as a unified order system, common to all sense-data. But if it is meaningful to speak of such a common order of *sense-data* (not, mark you, of *things* known through them), it certainly will not resemble the infinite Euclidean

space. No sense-data can be located beyond those tiny twinkling spots which children call stars, which are all affixed on a dark hemisphere that meets the ground at the horizon. Wundt reasserts the thesis, first stated by Kant, that unorthodox geometries depend parasitically on Euclidean geometry, because they must avail themselves of our ordinary intuition of space.²³ Now, even if the last statement were true, it would not suffice to validate that thesis. A geometrical theory may concern structural properties of intuitive space which are not peculiar to Euclidean geometry. Thus, for example, spherical trigonometry, insofar as it reflects our spatial intuitions, cannot be said to depend on their Euclidean nature, for it agrees equally well with spherical or with BL geometry.

Like other empiricists before him, Wundt does not ignore the fact that no perceived entities exactly correspond to the basic geometrical concepts. These cannot be formed by ordinary abstraction, by which we merely disregard some of the properties of perceived objects, "because points, straight lines and planes, as they are presupposed by geometry, do not exist objectively, neither in isolation nor in connection with other objective properties of bodies".²⁴ Instead of speaking, like some of his predecessors, of idealization, Wundt proposes a completely different approach to mathematical concept formation. In mathematical abstraction, he says, we deliberately ignore all the objective properties of things and we pay attention only to "the logical function of grasping them (*die logische Funktion ihrer Auffassung*)".²⁵ Unfortunately, Wundt does not explain what this means nor how it applies to the vast realm of mathematical concepts.

While dismissing Poincaré's doctrine of the conventionality of metrics, Wundt defyingly proclaims that dimension number is conventional. This is indisputably true if we define dimension number, as he does, by "the number of elements required to determine the position of a point in space".²⁶ Since \mathbb{R}^3 can be bijectively mapped onto \mathbb{R}^n (for every positive integral value of n), any number of real coordinates can be used to specify a particular point in space. But this is not the reason given by Wundt to support his claim. He mentions the fact that if we regard space as the set of its straight lines, instead of viewing it as the set of its points, we shall need four coordinates, instead of three, for determining each element of space. Now, if S denotes space regarded as the set of its points, the set of straight lines of S is not S itself but a subset of the power set $\mathcal{P}(S)$, so that Wundt's

argument fails to prove that *the same set* can be viewed indifferently as having three or four dimensions (in Wundt's sense). Dimension number is defined after Brouwer (1913) as a topological property, ascribable to a wide variety of topological spaces (Section 4.4.6). If a set S with topology T has dimension n , it is generally possible to define on S a topology T' , which makes S into an m -dimensional topological space ($m \neq n$). From this point of view, the dimension number of physical space can be regarded as conventional if, but only if, we are free to endow it with several incompatible topologies, which bestow on it different dimension numbers.

4.2.3 Charles Renouvier

Charles Renouvier (1815–1903), head of the French Neokantian school, developed his views on the old and the new geometries in an essay entitled “La philosophie de la règle et du compas”.²⁷ It aims at “demonstrating the illogical character of non-Euclidean geometry”.²⁸ This aim is directly pursued in the second half of the essay, devoted to “the sophisms of general geometry”. Renouvier carries his animosity towards the new geometries to the point of saying that “anyone who believes that he may question the objective foundation of the old geometry [...] cannot consistently think that the objective foundation of morality is better safeguarded against doubt.”²⁹ The “illogical character” of non-Euclidean geometry is conceived rather broadly. Renouvier grants that BL geometry is exempt from contradiction. Indeed if BL geometry were contradictory, Euclid's fifth postulate would be a demonstrable truth, instead of an indemonstrable principle of geometry. The contradictory, however, is only a species of the absurd, a much vaster genus of inadmissible notions and untrue propositions, including, in particular, “the ideas and propositions which contradict the regulative principles of the understanding”. BL geometry belongs precisely to the latter variety of the absurd, “because it rests on the supposition that one of the principal laws of our representation of space and figures does not express a real relation”.³⁰ The actual development of an absurd but noncontradictory geometry by Lobachevsky provides a welcome confirmation of Kant's thesis that some of the principles of geometry are synthetic judgments, in other words, that geometry must be based on undemonstrable postulates.

For Renouvier, the ultimate foundation of geometry is intuition. He

thus calls the "ideas of space and spatial relations insofar as they consist of intellectual phenomena which can be analyzed, no doubt, and from which consequences can be drawn, but which cannot be demonstrated nor reduced to other phenomena without begging the question."³¹ In contrast with Kant, who restricted analysis to the elucidation of concepts, and regarded spatial intuition as the source of a priori synthetic judgments, Renouvier maintains that the contents of intuition is expressed in analytic judgments, and treats *intuitive* and *analytic* as synonymous. The first "fact of intuition" which lies at the foundation of geometry is three-dimensional space itself (*l'étendue elle-même, à trois dimensions*), "in which every figure is imagined, defined in its internal relations and placed by the mind as in a shapeless medium (*comme en un milieu lui-même sans figure*)". This primary fact of intuition immediately implies, according to Renouvier, that "a figure can be transported everywhere in space, without altering its elements or the relations of its parts".³² This "law of the conservation of figures" – which is tantamount to Helmholtz's free mobility of rigid bodies – "contains in principle every other fact of geometrical intuition". Does it suffice to determine the Euclidean character of true geometry? On this point, Renouvier's position is ambiguous. On the one hand, he maintains that Euclidean geometry cannot be established by analysis alone, but demands an intellectual synthesis, which apparently operates upon intuitive data but is somehow superimposed on them. Thus, while the statement that two straight lines never enclose a space merely analyzes, in Renouvier's opinion, the intuitive notion of a straight line (defined as a line of constant direction), the statement that the straight line is the shortest between two points involves a synthesis which can only be due to the understanding. On the other hand, when he discusses Riemann and Helmholtz, he concludes that the "law of conservation of figures" is fulfilled only in Euclidean space.

This conclusion is somewhat deviously stated at the end of p.52. Renouvier quotes from Helmholtz (1866). He does not seem to be aware of the big mistake in that work (corrected in Helmholtz, 1869; see p.162 of this book). Helmholtz had ignored BL geometry, maintaining that free mobility is compatible only with Euclidean and spherical geometry. Since the latter is automatically excluded by Renouvier's contention that it is analytically true that two straight lines cannot enclose a space, Renouvier's conclusion, though false, is

certainly not unreasonable. But his own statement of it reveals a misunderstanding which it is harder to excuse. Renouvier apparently believes that Riemann's line element (the square root of a quadratic differential expression) is true only of Euclidean space (see Renouvier, *loc. cit.*, pp.49, 52, and also p.297 of this book). If this were so, of course, Helmholtz's proof that the requirement of free mobility of rigid bodies leads to Riemann's definition of the line element would imply that free mobility is incompatible with a non-Euclidean space.

If the law of conservation of figures is obtained by an analysis of intuition and if it implies that space is Euclidean, ordinary geometry is analytically true of the intuitively given space, and synthetic judgments, in Renouvier's sense, play no role in its foundation. Nevertheless, throughout most of his essay, Renouvier remains faithful to his initially stated position and insists in the synthetic, indemonstrable nature of certain basic geometric truths, such as the postulate that the straight line defined by two points is the shortest line that joins them or the postulate that all right angles are equal. Among the indemonstrable, synthetic principles of geometry, he counts the following postulate, from which – if we assume the Archimedean postulate – Euclid's fifth postulate can be derived:

Let A_1, \dots, A_n denote the vertices of a convex polygon of n sides $A_1A_2, A_2A_3, \dots, A_nA_1$. Let m be a straight line which initially covers A_nA_1 and is successively rotated about the points A_1, A_2, \dots, A_n , so that after the j th rotation it covers A_jA_{j+1} ($1 \leq j < n$) and after the n th rotation it returns to its initial position. The angles described by m at A_1, A_2, \dots, A_n add up to four right angles.³³

Renouvier takes this for an equivalent of the parallel postulate, which he apparently prefers to the traditional formulations because it brings out more clearly its quantitative import. The postulate is thus placed on a par with the two manifestly quantitative postulates we mentioned earlier. But Renouvier's attack against BL geometry is not directly based on his version of the parallel postulate, but on one of its equivalents, namely, the existence of similar figures of unequal size. No such figures can exist if we deny the postulate. But this, Renouvier believes, would bring about "the total ruin of geometrical thought".³⁴ Size would be absolute, not relative to the choice of a unit. This he regards as absurd. "Since every measurement is the determination of a relation and the numbers which give the quantitative values vary in proportion to the quantity of the arbitrarily

chosen unit of each kind, every quantity of a given kind can be multiplied by the same factor without changing anything in their comparative sizes, which is all that can be grasped by our senses, imagination and reason."³⁵ Consequently, the multiplication of linear dimensions by an arbitrary constant should not alter the geometrical properties of figures. It is not easy to see why this argument does not apply to the size of angles; why, for instance, if we try to duplicate every angle of a triangle, we do not merely distort the triangle but we downright destroy it. (See p.317.)

After a long diatribe against other aspects of general geometry – in particular against Riemann's concept of a manifold – Renouvier ends upon a conciliatory note. There is no objection to the new geometry if its cultivators acknowledge that their only aim "is to exercise themselves in mathematical analyses of diverse hypotheses, without paying attention to any truth except that regarding the relation between conclusions and their premises".³⁶ In view of the preceding discussion, one should certainly expect Renouvier to prove that Euclidean geometry differs essentially from the non-Euclidean systems in this respect; to show, in other words, that Euclidean geometry cares for something more than just logical consequence. Such proof is nowhere to be found in Renouvier's essay (unless we regard the above argument concerning similar figures as providing it).

Renouvier tends to be quite unreliable when it comes to technical matters. We mentioned on p.296 his incredible confusion regarding Riemann's line element. Renouvier's position on this point can be formally stated as follows: given an R -manifold M with metric μ , there exists a chart x defined on all M , such that $\mu(\partial/\partial x^i, \partial/\partial x^j) = \delta_j^i$ (i.e. 1 if $i = j$, 0 if $i \neq j$). This means, of course, that every R -manifold is a Euclidean space! On p.54, Renouvier refers to BL geometry (without naming it) as a theory which contests "the impossibility of following, upon a plane, from a given point, several straight lines having *the same direction* as a given line". Now, according to Renouvier, two straight lines have the same direction if they make equal corresponding angles with a transversal. But, unless the parallel postulate is true, lines making equal corresponding angles with a given transversal might make different corresponding angles with a different transversal. The denial of the stated impossibility is not therefore so preposterous as Renouvier thinks. *Unless Postulate 5 is true*, given a line m and a point P outside it, several lines n, n', \dots

through P may be said to have the same direction as m , *relatively to different transversals* t , t' . (See our discussion of Ueberweg on p.263f.)

4.2.4 Joseph Delboeuf

The Belgian professor J. Delboeuf (1831–1896) is not so well-known as Lotze, Wundt or Renouvier; but his ideas about geometry and science are, in some respects, more interesting than theirs. We have already mentioned his *Prolégomènes philosophiques à la géométrie* (1860).³⁷ Thirty-five years later, he returned to the subject in four articles on “The old and the new geometries”, published in the *Revue Philosophique*. Delboeuf’s thought cannot be easily classed with a philosophical school. I have chosen to deal with him here because he maintains that general geometry, as expounded by Calinon and Lechalas (Section 4.1.4), does not encompass Euclidean geometry as a special case, but is subordinate to it. In my opinion, he fails to substantiate this claim, which is apparently based on a misunderstanding; but other theses explained in those articles and in the earlier book deserve a close attention.

Unfortunately, Delboeuf’s basic epistemological views are not very clearly set forth by him. He conceives reality as a vast, endlessly diversified happening. Whatever is *here* and *now* differs from what is *there* and *then*. Human intelligence attempts to grasp reality by ignoring the particular and minding the general. Delboeuf apparently thinks that this is merely a matter of abstraction, of disregarding some aspects of phenomena and concentrating upon other aspects. At times he speaks, however, as if notions thus abstracted from experience could never attain sufficient generality, so that the mind must posit ideal facts of its own making in order to build a truly general science. This is developed by logical deduction from these ideal facts or *hypotheses*. The science thus constructed is *true* if the consequences derived from the hypotheses agree with real facts.

According to Delboeuf, the first step towards a scientific grasp of reality consists in regarding the spatio-temporal locus of the universal happening as homogeneous, in the sense that identical bodies can be found at different places and identical events can occur at different times. In this way we obtain the universe of inert things, studied by physics and chemistry. A second step consists in ignoring the differences between the bodies and seeing in them only one and the

same nature. The universe now appears as "an aggregate of bodies subject to reciprocal actions and reactions; their differences consist only in the sum of the actions they exert". This is the subject-matter of mechanics. From this point of view, space and time are still inhomogeneous, in a way, "in so far as the position of bodies and their mutual relations change from one moment to the next, from one place to another".³⁸ The abstract cause of movement and change, says Delboeuf, is *force*. If we ignore the differences, changes and movements which arise from inequalities in force, the universe is reduced to an aggregate of figures. This is the subject-matter of geometry. The space to which geometrical figures belong is absolutely homogeneous, in a double sense. In the first place, if we are given a figure lying about a point in space we can always find another *equal* (i.e. congruent) figure lying in any way whatsoever about another point. This property Delboeuf calls *isogeneity*. In the second place, any figure can be increased or reduced in *size* while preserving its *shape*. This property Delboeuf calls *homogeneity*. Euclidean space is homogeneous in this strict sense. In his articles of the 1890's, Delboeuf proudly points out that this character suffices to distinguish it among all spaces conceived by general geometry, whose subsequent development he had not anticipated when, in 1860, he described "the mutual independence of shape and size" as the first principle of geometry.³⁹

Delboeuf repeatedly claims that his notion of homogeneous space is obtained, in the manner described, by abstraction. To my mind, this is not altogether clear. Indeed, I fail to see why a spatial figure, conceived by ignoring every peculiarity of a body, must possess a shape independent of its size. But homogeneous space can, of course, be freely posited, and its properties can be deduced from its definition and compared with those of real space. On this point, Delboeuf is clear enough. Geometric space, whether we regard it as posited or as abstracted from reality, is a far cry from real space. It should not surprise us to find that, in nature, no line is absolutely straight, no circle is perfect, no ellipse is exact. "How could we draw a circle in heterogeneous space and time, if the arms of the compass expand or contract from one moment to another, from one place to another, due to the ceaseless variations of temperature; if the points are worn, the paper is not flat, etc.?"⁴⁰ Delboeuf's first article on the old and the new geometries aims at showing that real space is utterly different

from Euclidean space. Most of his arguments are highly questionable, but the main one is worthy of consideration. "Real space is nowhere identical with itself; it does not admit equal figures; the smallest grain of sand, the smallest speck of dust in space are altered by the slightest displacement. [...] Real space is necessarily variable and none of its parts will ever return to a state through which it has gone once."⁴¹ In the light of these statements it is hard to understand why Delboeuf insists on the privileged status of Euclidean geometry. Why not allow that other concepts of space, departing from strict homogeneity, can be legitimately posited, and that real space may even agree better with them than with homogeneous space? Delboeuf apparently had not read Riemann. He seems to be acquainted only with maximally symmetric spaces (through the mathematical works of Calinon and Lech alas). Speaking about spaces of constant positive or negative curvature, he rightly observes that

they are artificial spaces, just like Euclidean space; from this point of view, they are no less geometric than the latter. But they possess no special quality that would enable them to represent real space better than it does. Real space [...] certainly has a curvature, but this curvature is different at each one of its points and it changes at every moment. Real figures, that is, bodies, change in it with time and place. The *constant* curvatures of meta-Euclidean spaces are therefore just as far from reality as the homogeneity of Euclidean space.⁴²

Was Delboeuf aware of the full import of his words? Taken literally, they are an invitation to physicists to discard Euclidean geometry and to try out a space of variable curvature to represent physical space. But Delboeuf does not pursue this idea any further. His declared aim is to establish the absolute preeminence of the "old" over the "new" geometry (which, as I said, he apparently knows only in the guise of spherical and BL geometry). His lengthy argument for proving this amounts in the end to the following: (i) Euclidean geometry is the sole guarantee of the consistency of non-Euclidean geometries. (ii) The geodesic arc, which, in non-Euclidean geometries, plays the same fundamental role as the straight segment plays in ordinary geometry, can only be defined in terms of the Euclidean straight line. Both statements are false. The second one rests on the (mistaken) characterization of a geodesic arc as the shortest line joining its extremes and on the classical definition of the length of an arc as the limit of a sequence of lengths of straight segments. Riemann, as we saw, had been able to discard this definition and to substitute for it another one

which in no way presupposes the existence of Euclidean straights in the manifold to which the arc belongs (Section 2.2.8). The first statement arises out of a misunderstanding of the true significance of Beltrami's pseudospherical model of the BL plane. This model guarantees the consistency of BL plane geometry *quoad nos*, because we are willing to believe that Euclidean space geometry is consistent. But it is not the *only* guarantee of that consistency. This can also be proved, to our satisfaction, by means of numerical models, if we take the consistency of arithmetic for granted. Numerical models can also be used for proving the consistency of BL and spherical space geometry.

4.3 RUSSELL'S APRIORISM OF 1897

4.3.1 *The Transcendental Approach*

An Essay on the Foundations of Geometry (1897) was the first in the long series of books published by Bertrand Russell (1873–1970). It is based on the dissertation he submitted at the Fellowship Examination of Trinity College, Cambridge in 1895, when he was twenty-two years old. As it so often happens in philosophy, Russell's ideas look very attractive in their broad lines, but turn out to be quite disappointing when worked out in detail. Russell very soon abandoned the philosophical position maintained in the book, which was not reissued until 1956, when the author, at 83, was a living classic, and everything published under his name was rightly regarded as deserving attention.¹ The book reflects a much more accurate knowledge of the new geometries than any of the writings we have discussed in Part 4.2. Its historical Chapters I and II are still useful, and contain valuable criticisms of the authors we have been studying.² But our main concern here is with Chapter III, on the axioms of projective and metrical geometry, which, as we shall see, promises much more than it is able to fulfil.

Like most of his contemporaries, Russell believes that the main task of a philosophy of geometry consists in determining how much in geometry is necessary, apodictic or a priori knowledge, i.e. knowledge which under no circumstances can be other than it is, so that no conceivable experience can ever clash with it. Russell characterizes a priori knowledge in the best Kantian vein, as knowledge of the conditions required by all experience or by a definite genus of

experience. The psychological concept of the a priori as 'the subjective', i.e. as knowledge arising from the nature of our minds (a concept which can be traced back to Kant's less felicitous texts), Russell dismisses as philosophically irrelevant. Indeed, such knowledge could hardly be said to be necessary, unless we could prove that this or that aspect of our mental functions cannot be exercised in a different way; but such proof would establish that the knowledge in question is a priori in the former objective, 'logical' or 'transcendental' sense. Russell's declared aim is to show that projective geometry (PG) and the general metric geometry of n -dimensional maximally symmetric spaces (GMG) are entirely a priori. On the other hand, the fact that physical space has exactly three dimensions and that its (necessarily constant) curvature is approximately equal to zero is, according to Russell, a contingent empirical fact.

The a priori nature of a branch of geometry will be established if we can (i) find the axioms from which every proposition of that branch of geometry can be derived by ordinary logical deduction; (ii) show that these axioms state general conditions of the possibility of experience, or of a definite genus of experience—in other words, if we can give a transcendental deduction of the axioms themselves. Such is, indeed, Russell's programme. He submits two lists of three axioms each for PG and GMG. He assumes that every kind of experience involves awareness of diversity in unity. This requires at least one "principle of differentiation", something, that is, by which whatever is experienced is distinguished as diverse. "*This element, taken in isolation, and abstracted from the contents which it differentiates, we may call a form of externality.*"³ Russell claims that the axioms of PG state properties common to every conceivable form of externality. GMG, on the other hand, has a more restricted scope. Its axioms express the conditions required for the quantitative determination of a form of externality. Russell apparently believes that every form of externality admits a quantitative determination, but he makes no attempt to prove this. At any rate, if Russell's arguments are sound, the axioms of GMG, though not necessarily true of all experience, will certainly govern that kind of quantitative experience, of experience based on measurement, which is the foundation of modern natural science.

Russell's transcendental deduction of the axioms of geometry is a much more ambitious enterprise than Kant's. The latter claimed in his

“transcendental exposition of the notion of space” that our ordinary intuitive representation of space is independent of experience because it is the source of Euclidean geometry, which he assumed to be necessarily true.⁴ But he never attempted to prove that every particular Euclidean axiom was a necessary condition of every conceivable experience or of every conceivable quantitative experience. He acknowledged that we are unable to explain why the space of our experience has precisely the structure set forth by Euclid.⁵ Now, after the new developments in geometry, Kant’s transcendental argument for the a priori nature of space is no longer available. Consistent systems of geometry very different from Euclidean geometry and also from Russell’s PG and GMG can be found in the text-books. If we wish to establish the necessity of a specific, non-trivial geometrical system, we must give some sort of transcendental proof of its axioms. The failure of Russell’s attempt to demonstrate the necessity of PG and GMG has doubtless contributed to discredit apriorism in the philosophy of geometry. It seems to me, however, that if we reason more carefully and less high-handedly than Russell, we can prove that this or that geometrical theory necessarily belongs to the conceptual framework presupposed by a specific, historically known variety of experience (e.g. physical experience as it was organized in 19th-century laboratories, astronomical experience as it is gathered in present-day observatories, etc.). But it is very unlikely that a geometrical system less general than abstract set theory can ever be shown to be a universal presupposition of all experience.⁶

4.3.2 *The ‘Axioms of Projective Geometry’*

Let us examine Russell’s transcendental deduction of projective geometry. The reader will wish to know whether it concerns real or complex projective geometry. In Chapter I, where he deals with the history of modern geometry, Russell is well aware of the existence of these two kinds of projective geometry,⁷ but no such awareness is noticeable in the systematic discussion of Chapter III. Here, projective geometry is described as dealing “only with the properties common to all spaces”,⁸ a most remarkable statement, since complex projective space $\mathcal{P}_{\mathbb{C}}^n$ and real projective space \mathcal{P}^n do not have the same properties, and in both of them every straight line meets every other straight line, a property not shared by n -dimensional Euclidean

or BL spaces. But perhaps we are being too pedantic. Russell only attempts to prove the a priori truth of three principles, which he calls "the axioms of projective geometry", but which are plainly insufficient to characterize either \mathcal{P}_C^n or \mathcal{P}^n . These axioms read as follows:

(I) We can distinguish different parts of space, but all parts are qualitatively similar, and are distinguished only by the immediate fact that they lie outside one another.

(II) Space is continuous and infinitely divisible; the result of infinite division, the zero of extension, is called a *point*.

(III) Any two points determine a unique figure, called a straight line, any three in general determine a unique figure, the plane. Any four determine a corresponding figure of three dimensions, and for aught that appears to the contrary, the same may be true of any number of points. But this process comes to an end, sooner or later, with some number of points which determine the whole of space.⁹

In the light of Axiom I, each *division* (in the sense of Axiom II) of space or of a part of space must consist in its partition into two or more disjoint but otherwise indiscernible proper parts. Every part into which division may divide a space is again a space liable to division in exactly the same terms as any other space. In this axiom system there are therefore no grounds for the idea that an infinite sequence of divisions might converge to a definite result. The second clause of Axiom II is nonsense. That clause, however, is meant to provide the definition of the term *point* used in Axiom III. If we wish to make some sense of Russell's axioms we must take *point* as a primitive term and postulate some relationship between it and the other primitive of the system, namely, *space*. I suggest that we simply regard *space* as the set of all *points*.¹⁰ The term *continuous* in Axiom II cannot be viewed as primitive. Otherwise, we might just as well substitute for it any other word or sound, since this is its only occurrence in the system (it would thus make no difference to write, for example, (II) *Space is slithy and infinitely divisible*). This term must connect the system with other established mathematical theories. What does it exactly mean? The following translation of the first sentence of Axiom II into current mathematical language is no doubt anachronistic but it is probably not too far from Russell's intended meaning: *Space is continuous = Space is a topological space every point of which has a neighbourhood homeomorphic to \mathbb{R}^{n-1}* . Here n (> 1) is the number of points which, according to Axiom III, suffice to "determine the whole of space". Under this interpretation, not every proper subset of space is a *part* of it in the sense of

Axiom I, since not all subsets of a topological space are qualitatively similar to each other. A reasonable proposal would be to equate the *parts* mentioned in Axiom I to the open connected proper non-empty subsets of space (all of which are homeomorphic and hence indiscernible from a topological point of view). But this conflicts with Axiom II, since a connected topological space cannot be partitioned into two or more non-empty open subsets. As a makeshift solution of this difficulty, I suggest that we regard a *part* of space in the sense of Axiom I as being any open connected proper non-empty subset of space, or its closure, and that we consider two such parts as qualitatively similar if their interiors are homeomorphic. Some version of Axiom III is usually included in the standard axiom systems for Euclidean and related geometries. But then it is followed by other axioms which further determine the properties of lines, planes, etc. Taken all by itself, the statement that there exist in space, say, subsets of type A, B, C . . . , determined, respectively, by two, three, four . . . points, is not of much use.¹¹

Russell's transcendental deduction of Axioms I-III attempts to show that they are a prerequisite of all experience because every conceivable "form of externality" shares the properties which these three axioms ascribe to *space*. A successful achievement of this undertaking would not establish the a priori truth of projective geometry, in the usual meaning—or meanings—of this expression, since the latter contains much more than what goes into those axioms. But it would be a very important epistemological achievement. Unfortunately, Russell's execution of the programme leaves much to be desired.

Let us recall that the expression "form of externality" designates the element in perception by which perceived things are distinguished as various, when the said element is taken in isolation and abstracted from the contents which it differentiates. Russell describes it as the bare possibility of diversity (of perceived contents) and as the "principle of bare diversity". The notion is indeed quite general, and it would seem that no specific structure can be regarded as necessarily belonging to a "form of externality" in this sense. On the other hand, if we conceive such a form less broadly, we shall be able to 'deduce' that it must have this or that structural property, but we can hardly claim any necessity for the "form of externality" itself. In order to avoid this dilemma, Russell resorts to a standard method of

transcendental deduction, which had been used *ad nauseam* by German post-Kantian idealists. This consists in calling the concepts which play an essential role in the argument by names whose ordinary meaning is much richer than the defined meaning of those concepts, and allowing the aura of meaning suggested thereby to strengthen the premises in which such concepts occur. Since “form of externality” is introduced by Russell as a defined concept, we ought to be able, in principle, to give it any name. However, if we substitute, say, the word ‘juggerwogg’ for ‘form of externality’ in all the occurrences of this expression in Russell’s book, Russell’s argument is stopped dead, for it depends on the familiar connotations of ‘form’ and ‘external’ in ordinary English.¹²

Axiom III is understood to mean that *space* has a finite number of dimensions, in the sense defined below. This is justified as follows:

Positions, we have seen, are defined solely by their relations to other positions. But in order that such definition may be possible, a finite number of relations must suffice, since infinite numbers are philosophically inadmissible. A position must be definable, therefore, if knowledge of our form is to be possible at all, by some finite integral number of relations to other positions. Every relation thus necessary for definition, we call a dimension. Hence we obtain a proposition: *Any form of externality must have a finite integral number of dimensions.*¹³

Russell argues further that every form of externality worthy of this name must have more than one dimension. We shall not stop to examine his argument, but shall only remark that, with Russell’s definition of dimension, the Euclidean plane \mathbb{R}^2 – and generally every Euclidean space \mathbb{R}^n – can be regarded as one-dimensional. Let k denote a Peano curve which covers \mathbb{R}^2 (this implies that k is the image of a continuous mapping of \mathbb{R} onto \mathbb{R}^2).¹⁴ Let P denote the origin of k (i.e. the image of 0 under the said mapping). Then every point Q in \mathbb{R}^2 is unambiguously determined by the arc (or arcs) of k joining Q to P , hence by a relation of Q to a single position in \mathbb{R}^2 .

The insufficiency of Russell’s axioms for supporting the full weight of projective geometry was pointed out by Henri Poincaré in a critical article about Russell’s book (Poincaré, 1899). Russell replied in his essay “Sur les axiomes de la géométrie” (1899). He acknowledged that Poincaré was right on this point and proposed a new set of six axioms. These are stated with great precision. Letters are used instead of familiar words for designating the undefined concepts of the system. Russell’s six axioms are axioms of incidence. Since no

axioms of order are given, the new system is again insufficient to derive the theorems of projective geometry. But it is designed in accordance with the modern idea of a deductive theory, as developed by Pasch and the Italian school. (This shows, by the way, that Russell did not have to wait, as some suggest, until the Parisian philosophical congress of 1900 in order to learn about this conception—or to develop it on his own.) Russell makes no attempt to prove that his new axioms state necessary properties of every form of externality. But then, as he declares at the beginning of his reply to Poincaré, he had changed his mind on several matters after the publication of his book (Russell (1899), p.684).

4.3.3 *Metrics and Quantity*

Russell defines metrical geometry as “the science which deals with the comparison and relations of spatial magnitudes”.¹⁵ Russell does not define *magnitude* but apparently he regards this concept as one of those basic familiar notions which everybody understands without further explanation. He usually treats it as synonymous with *quantity*. It is far from obvious that magnitudes must be found in every form of externality or that any such form must possess quantitative properties or relations. Russell however makes no attempt to prove this. He merely asserts that metrical geometry, as conceived by him, is true of space “if quantity is to be applied to space at all”.¹⁶ Russell tries to show that the axioms of metrical geometry state the necessary conditions under which, alone, quantity is applicable to a form of externality. If Russell’s arguments are conclusive, these axioms will have been shown to be necessary, though not in an absolute sense, but only relatively to a space where magnitudes exist and the concept of quantity is applicable. They would then express the a priori requirements of a definite kind of experience, namely, experience based on spatial measurements. Russell’s claims are more ambitious. According to him, each of the axioms of metrical geometry states a necessary property of any form of externality. But the arguments put forth to substantiate this claim in the case of two of the axioms are plausible only if we restrict their scope in the manner described above. The arguments purport to prove that those two axioms—namely, the axiom of free mobility and the axiom of distance—are implied by the possibility of spatial measurement, not that such measurement must always be possible.

We have seen that, according to Russell, projective geometry deals with "the properties common to all spaces".¹⁷ Its axioms are "a priori deductions from the fact that we can experience externality, i.e. a coexistent multiplicity of different but interrelated things".¹⁸ We might therefore expect that Russell will introduce metrical geometry after the manner of Cayley and Klein, through the definition of a distance function on point-pairs in projective space. Cayley-Klein projective metrics do indeed bestow some plausibility on Russell's thesis that a priori metrical geometry is the general theory of n -dimensional spaces of constant curvature k (a theory we have designated above by GMG), but that the actual values of n and k in physical space must be ascertained empirically. It is not unlikely that Russell himself considered the possibility of justifying metrical geometry in this manner, but he rejects it in his book. Cayley-Klein metrics are based on an assignment of numerical coordinates to the points of projective space which, Russell claims, is quite foreign to the proper use of coordinates in a quantitative science of space. The coordinates assigned by the von Staudt-Klein procedure (Section 2.3.9), says Russell, "are not coordinates in the ordinary metrical sense, i.e. the numerical measures of certain spatial magnitudes. On the contrary, they are a set of numbers, arbitrarily but systematically assigned to different points, like the numbers of houses in a street, and serving only [. . .] as convenient designations for points which the investigation wishes to distinguish."¹⁹ In fact, the von Staudt-Klein coordinatization involves more than a mere labelling of points, since it presupposes (or induces) a topological structure in projective n -space which agrees locally with that of R^n . Nevertheless, Russell is quite right in maintaining that the coordinates assigned to any given point P do not have a quantitative meaning, insofar as they do not in any way depend on the actual distance between P and another point. This is indeed a truism, since no distance function has been defined on projective space. But I fail to see why this fact should prevent us from introducing one or more such functions, as Klein did, *via* the von Staudt-Klein coordinate functions. This leads, of course, as Russell rightly observes, to a conventionalist conception of metric geometry: the distance between two points in space is made to depend on the arbitrary choice of a distance function. Russell rejects it because he assumes that distance is a metaphysical relationship between points, which the distance function merely expresses. Klein

defined the distance between two arbitrary points P, Q in a projective space in terms of the cross-ratio between P, Q and two fixed points on the straight line through P and Q . But, Russell objects, "before we can distinguish the two fixed points [...] from which the projective definition [of distance] starts, we must already suppose some relation between any two points on our line, in which they are independent of other points; and this relation is distance in the ordinary sense".²⁰ In another passage, Russell describes distance as "a spatial quantity [...] completely determined by two points".²¹ The relation between two points mentioned in the former text consists in the fact that they determine this particular spatial quantity. The distance function assigns to the two points a number which, so to speak, measures that quantity. The notion of a spatial quantity determined by a point-pair independently of every other point is obscure indeed; but we may reasonably expect that, if the point-pair belongs to a projective space, the real-valued function which expresses that quantity will be a two-point projective invariant. We know, however, that there are no two-point projective invariants.²² Consequently, Russell's claim that every point-pair in space has a relation in which they are independent of other points and which consists in determining a quantity measured by a real-valued function, openly clashes with his assertion that every form of externality is a projective space.²³ Russell will argue that the quantitative study of a form of externality presupposes the existence of distance. If this is right, it means simply that the purely projective structure of a form of externality F must be enriched with a metric structure before such a quantitative study can begin. This is done, as we know, by defining a suitable real-valued function on $F \times F$. In this way, we obtain a metric space F' , which is no longer the same as F . This consequence is unavoidable, if F is originally given as a projective space.

4.3.4 *The Axiom of Distance*

The general system of metric geometry proposed by Russell (GMG) depends, he says, on three axioms: the axiom of free mobility, the axiom of dimensions and the axiom of distance. The axiom of dimensions is essentially the same as Projective Axiom III:

If Geometry is to be possible, it must happen that, after enough relations have been given to determine a point uniquely, its relations to any fresh known point are deducible from the relations already given. Hence we obtain as an *a priori* condition of

Geometry, logically indispensable to its existence, the axiom that *Space must have a finite integral number of Dimensions*. For every relation required in the definition of a point constitutes a dimension, and a fraction of a relation is meaningless. The number of relations required must be finite, since an infinite number of dimensions would be practically impossible to determine.²⁴

But the number of dimensions of real space is a contingent matter which must be empirically determined. It is not liable however "to the inaccuracy and uncertainty which usually belong to empirical knowledge. For the alternatives which logic leaves to sense are discrete [...] so that *small errors* are out of the question".²⁵

The axiom of distance is stated thus: "Two points must determine a unique spatial quantity, distance".²⁶ No further conditions are imposed on this quantity, but Russell would probably have agreed that it is adequately represented by a non-negative real number which is equal to zero if, and only if, the two points are identical, that it does not depend upon the order in which the two points are taken and that it satisfies the triangle inequality (the distance determined by points P and Q is equal to or less than the distance determined by P and R plus the distance determined by R and Q). Russell holds that the axiom is a priori in a double sense: (i) it is involved in the possibility of measurement and (ii) it is necessarily true of any possible form of externality. This he regards as a consequence of four propositions which he intends to prove: (1) spatial magnitude is not measurable unless distance exists; (2) two points determine a distance only if they determine a unique curve in space; (3) "the existence of such a curve can be deduced from the conception of a form of externality"; (4) "the application of quantity to such a curve necessarily leads to a certain magnitude, namely distance, uniquely determined by any two points which determine the curve".²⁷ It is clear that (i) follows immediately from (1). But (ii) does not follow from our four propositions alone; we must add: (5) every form of externality invites – or demands – the application of quantity. As we observed earlier, this last premise is neither mentioned nor proved by Russell.

*Russell's proofs of Propositions (1)–(4) are long and inconclusive. They are interesting chiefly as illustrations of some philosophical prejudices. Since the original text is easily available (Russell, FG, pp. 164–175), we shall sketch them cursorily. The proof of (1) requires an additional premise: *Spatial figures can be freely moved without distortion*. This is the axiom of free mobility, which, Russell claims, is

presupposed in all spatial measurement (we shall deal with it in Section 4.3.5). It seems to me, however, that this axiom makes no sense unless we take distance for granted, since *distortion* means precisely a change in the distances between the points of a figure. It is surprising, therefore, that Russell should use this axiom to *prove* that distances exist. He argues more or less as follows: Two points must have some relation to each other, for such relations alone constitute position. It follows from the axiom of free mobility that two points, forming a figure congruent with the given pair, can be constructed in any part of space. Consequently, the relation between the two point-pairs is "*quantitatively the same* [...] *since congruence is the test of spatial equality*. Hence the two points have a quantitative relation" which is not altered by motion. This implies that the relation depends on the two points alone, because if it also depended on a third point, there would be some motion of the first two points which would alter it. "Hence the relation between the two points [...] must be an intrinsic relation, a relation involving no other point or figure in space; and this relation we call distance." (Russell, FG, p.165). The italicized passage marks the point where Russell openly begs the question, immediately after invoking the axiom of free mobility: it is assumed that the relation between the two point-pairs can be judged from a quantitative point of view. Russell asks: why should not there be more than one such intrinsic quantitative relation between two points? His reply is fantastic: "A point is defined by its relations to other points, and when once the relations necessary for definition have been given, no fresh relations to the points used in definition are possible, since the point defined has no qualities from which such relations could flow." (Russell, FG, p.166). If relations between points must flow from their qualities, one must ask for the qualities of the as yet undefined points whence the relations defining them are supposed to flow.

*The proof of (2) runs thus: "Some curve joining the two points is involved in the above notion of a combined motion of the two points, or of two other points forming a figure congruent with the first two. For without some such curve, the two point-pairs cannot be known as congruent, nor can we have any test by which to discover when a point-pair is moving as a single figure. Distance must be measured, therefore, by some line which joins the two points." (Russell, FG, p.166f.). This line must be determined by the two points alone,

because if it depended on still another point, distance would not be a quantity completely determined by two points. I confess that Russell's reasoning bewilders me. Why should the curve used for testing that a pair of points moves as a single figure actually *measure* the *distance* between them? How does such a test work? Russell apparently believes that we can claim that two point-pairs (P, Q) , (P', Q') are congruent only if a particular arc k uniquely determined by P and Q is congruent with the arc k' uniquely determined by P' and Q' . It seems clear, however, that in order to establish the congruence between k and k' we must first bring P and Q into coincidence with P' and Q' . Thus the congruence between point-pairs is presupposed by the test of the congruence between arcs.

*The proof of (3) presupposes that the axiom of free mobility is true of every conceivable form of externality (see Note 36). This implies that (1) is true of every such form as well. (3) is then inferred as follows: "Since our form [of externality] is merely a complex of relations, a relation of externality must appear in the form, with the same evidence as anything else in the form; thus if the form be intuitive or sensational, the relation must be immediately presented, and not a mere inference. Hence, the intrinsic relation between two points must be a unique figure in our form, i.e. in spatial terms, the straight line joining the two points". (Russell, FG, p.172). The last step clearly implies that, in Russell's opinion, a point-pair, as such, is not a figure in space (in order to make a figure we must draw a line joining the points). Now, if a point-pair is not a figure, the axiom of free mobility does not apply to it, and Russell's proof of (1) breaks down. Hence, we would not be entitled to assert the "intrinsic relation between two points" which is presupposed by the present argument.

*(4) asserts that the application of quantity to a curve uniquely determined by two points leads to a magnitude, namely distance, uniquely determined by those two points. Through (3) and (4), we can tie the axiom of distance to the possibility of a quantitatively determined form of externality. Since the same can also be done through (1), we have that (3) and (4) are superfluous unless they offer a genuine alternative to (1). But both (3) and (4) are inferred from the existence of the "intrinsic relation" between point-pairs of which (1) is an immediate consequence. (4) is proved as follows: two arbitrary points P, Q have a unique intrinsic relation (by the proof of (1)); P

and Q determine a unique line that joins them ((2)); all points in this line are qualitatively equal; but "if one point be kept fixed, while the other moves, there is obviously some change of relation"; such change must be a change of quantity. "If two points, therefore, determine a unique figure, there must exist, for the distinction between the various other points of this figure, a unique quantitative relation between the two determining points. [...] This relation is *distance*." (Russell, FG, p.172). We find again the childish notion that a figure determined by two points cannot consist of those two points alone, but must be a line through them. It is clear that the moving point must move along this line, otherwise its motion would introduce a qualitative difference between it and the other points on the line (namely, that it no longer belongs to the line). But if all points on the line are qualitatively equal the motion of one of them along the line cannot be defined, unless we presuppose some non-qualitative difference between them. In Russell's terminology, whatever is non-qualitative is quantitative. The argument therefore begs the question: unless we assume that the two points which determine the line sustain a unique quantitative relation, we cannot make any sense of the motion of one of these points against a fixed background of other points which are qualitatively equal to it.

*I wish to discuss finally Russell's assertion that all the points of the unique curve determined by P and Q are qualitatively equal. Until now, we have understood that this curve is an arc from P to Q. On this arc, P and Q, being the extremes, differ qualitatively from the points which lie between them. But Russell assumes here a different interpretation: the curve determined by P and Q is the straight line through these points. This interpretation agrees with Projective Axiom III, which says that two points determine a straight line. Russell consistently identifies qualitative properties and relations in space with projective properties and relations. The relation between P and Q is projectively equivalent to the relation between P and any other point R on the straight line PQ. Hence, according to Russell, Q and R are qualitatively equal (at least as far as their relation to P is concerned). On Russell's assumptions, the argument is sound. But if the unique curve determined by P and Q is the (full) straight line PQ, we cannot claim that the length of this curve measures the distance between P and Q (as Russell concluded in the proof of (2)).

4.3.5 *The Axiom of Free Mobility*

The mainstay of Russell's theory of metric geometry is the axiom of free mobility. He states it thus:

*Spatial magnitudes can be moved from place to place without distortion; or, as it may be put, Shapes do not in any way depend upon absolute position in space.*²⁸

A similar principle had been placed at the foundation of geometry by Ueberweg (Section 4.1.2) and by Helmholtz (Section 3.1.1). These authors assumed a space in which the distance between points was defined, and tried to ascertain the conditions which such a space must fulfil in order to satisfy the principle of free mobility. Helmholtz concluded that, if the space is a differentiable manifold, it must be an *R*-manifold of constant curvature. Russell, on the other hand, uses the axiom of free mobility for proving that a point-pair must determine a distance. This might make sense if we deal with a physical space, populated by material bodies, and there happens to exist a non-geometrical test of the deforming forces which act on bodies, i.e. a method for ascertaining the presence or absence of such forces without measuring geometrical magnitudes (volumes, distances). Then, if a body *B*, which fills a region *R*, is moved in the absence of deforming forces to a region *R'* we may conclude that *R'* is congruent with *R*. In particular, if two marks *M*, *N* on *B*, which originally lie upon the points *P*, *Q* on *R*, are carried over to points *P'*, *Q'* on *R'*, we shall say that (*P*, *Q*) and (*P'*, *Q'*) are equidistant and we shall require that any distance function which we might wish to define will agree with this fact. We thereby treat geometry as inseparable from physics, and as founded upon physical facts. Such was the main tenet of Helmholtz's empiricist philosophy of geometry (Section 3.1.3). Russell criticizes it vigorously. "But for the independent possibility of measuring spatial magnitudes, none of the magnitudes of Dynamics could be measured. Time, force, and mass are alike measured by spatial correlates: these correlates are given, for time, by the first law [of Newtonian mechanics]; for force and mass, by the second and third [. . .]. Geometry, therefore, must already exist before Dynamics becomes possible: to make Geometry dependent for its possibility on the laws of motion or any of its consequences is a gross *hysteron proteron*."²⁹ Nevertheless, Russell does not conceive geometrical motion, after the fashion of pure mathematics, merely as a space

transformation subject to certain conditions. Such a conception presupposes indeed a metric function, which motions are required to preserve. Russell regards motion as actual transport of matter. For geometry, however, matter is "merely kinematical matter", matter deprived in thought of all its dynamical properties. Such matter, Russell maintains, is a priori rigid, because, being "devoid, *ex hypothesi*, of causal properties, there remains nothing, in mere empty space, which is capable of changing the configuration of any geometrical system".³⁰ This "geometrical rigidity", which is fully sufficient for the theory of geometry, "means only that a shape, which is possible in one part of space, is possible in any other".³¹ Let us consider more carefully what sameness of shape can mean under the conditions (or rather, the absence of conditions) assumed by Russell. Let M denote a lump of "kinematical matter", which fills a region R in space S . A movement f takes M to a different region R' . I imagine that Russell would have expected f to represent some sort of continuous process. That makes sense only if S is a topological space. If R is a connected subspace of S , R' is also a connected subspace. More generally, we may require R and R' to have homeomorphic interiors. I do not think that on Russell's assumptions we can impose any further restrictions on R' . Indeed, since both matter and space are entirely devoid of causal properties, any continuous process which carries M from one region of S to another takes place in the absence of deforming forces and may therefore claim the status of a rigid motion. In other words, on the stated assumptions, sameness of shape is tantamount to topological equivalence. We could hardly have expected a different outcome, since S is not defined *ab initio* as a metric space and M is not subject to non-geometrically testable shape-preserving forces (which could have been used for introducing a metric *a posteriori*). In fact, if M is held together during the movement f it is due only to the postulated continuity of f , for kinematical matter does not by itself possess any dynamical properties to prevent M from flying apart. Russell's "kinematical bodies" are thus seen to be mere abstract sets, endowed with such structure as they can pick from the *previously defined* space in which they are placed.³²

We need not dwell long on Russell's proof that the axiom of free mobility states a prerequisite of spatial measurement, since we have already seen this point argued by Helmholtz (Section 3.1.1). Russell

gives a “philosophical” and a “geometrical” argument. According to the former, a figure will change its shape as a result of motion only if space itself exercises a definite action upon it. But this is absurd, since space is passive. “Space must, since it is a form of externality, allow only of relative, not of absolute position, and must be completely homogeneous throughout.”³³ The “geometrical” argument is given as a refutation of the possibility, claimed by Benno Erdmann, of constructing a geometry in which sizes vary with motion according to definite law.³⁴ Russell understands that in such geometry “the fundamental proposition that two magnitudes which can be superposed in one position can be superposed in any other, still holds”.³⁵ In other words, he fails to see that if the magnitudes change size with motion and are transported along different routes they might no longer coincide when they meet for a second time. The refutation of Erdmann proceeds as follows:

A judgment of magnitude is essentially a judgment of comparison [. . .]. To speak of differences of magnitude, therefore, in a case where comparison cannot reveal them, is logically absurd. Now in the case contemplated above, two magnitudes, which appear equal in one position, appear equal also when compared in another position. There is no sense, therefore, in supposing the two magnitudes unequal when separated, nor in supposing, consequently, that they have changed their magnitudes in motion [. . .]. Since, then, there is no means of comparing two spatial figures, as regards magnitude, except superposition, the only logically possible axiom, if spatial magnitude is to be self-consistent, is the axiom of Free Mobility.³⁶

The argument is powerless, since, as we remarked above, it rests on a false assumption. Its interest is mainly historical: it involves an early version of the notorious verifiability criterion of meaning.

Russell’s insistence in shape preservation and the homogeneity of space suggested an interesting objection to Louis Couturat. Russell’s space is merely isogeneous, not fully homogeneous in Delboeuf’s sense (p.299). But, says Couturat, most of Russell’s arguments for the isogeneity of space could also be made for its homogeneity. According to Russell, space is relative, passive, indifferent to figures and bodies placed in it. But

these three characters seem to imply homogeneity and not only isogeneity. Can you say that space is a pure, empty form, indifferent to its content, unless you can construct in it two similar figures of different size? [. . .] Can you maintain that it is the amorphous, passive receptacle of every possible figure if you can neither construct the same figure on diverse scales, nor enlarge it without deforming it, as if space reacted upon it in the manner of a rigid form?³⁷

Couturat concludes that Russell's arguments do not merely establish the *a priori* truth of GMG, but of *n*-dimensional Euclidean geometry as well. Russell replies with an argument which we have already met (p.297):

Those who assert that it is *a priori* evident that the sides of a triangle can be increased in a given proportion without changing the angles, should also claim [...] that it is equally possible to change all the angles in a fixed proportion without changing the sides. But this is, as we know, impossible in all geometries. If we admit the logically relative nature of every magnitude, I cannot see why the argument should apply only to linear dimensions and not to angles which are magnitudes as well.³⁸

A stronger objection was made by Lech alas (1898). He believes that the axiom of free mobility, as understood by Russell (and by Helmholtz) is unnecessarily strong. To set up a metrical geometry, it should suffice to assume (with Riemann) that *the length of an arc* is preserved during displacement. Indeed, if the free mobility of *n*-dimensional figures were a necessary condition of *n*-dimensional metric geometry, we could not study the intrinsic geometry of an arbitrary surface, as taught by Gauss. On such a surface, say, on the surface of an egg, it is impossible to transport a 2-dimensional figure undeformed. On the other hand, if we are content to postulate the preservation of arc-length in motion, admissible geometries need not fit into the framework of Russell's GMG, but, as in Riemann's lecture, they may even extend beyond the much broader framework of *R*-manifolds of arbitrary curvature. Russell had discussed in his book the example of egg-geometry, but had refused to draw from it any conclusions regarding higher-dimensional spaces. He reasons thus:

What, I may be asked, is there about a thoroughly non-congruent Geometry, more impossible than this Geometry on the egg? The answer is obvious. The geometry of non-congruent surfaces is *only* possible by the use of infinitesimals, and in the infinitesimal all surfaces become plane. The fundamental formula, that for the length of an infinitesimal arc, is only obtained on the assumption that such an arc may be treated as a straight line, and that Euclidean Plane Geometry may be applied in the immediate neighbourhood of any point. If we had not our Euclidean measure, which could be moved without distortion, we should have no method of comparing small arcs in different places, and the Geometry of non-congruent surfaces would break down. Thus the axiom of Free Mobility, as regards three-dimensional space, is necessarily implied and presupposed in the Geometry of non-congruent surfaces; the possibility of the latter, therefore, is a dependent and derivative possibility, and can form no argument against the *a priori* necessity of congruence as the test of equality.³⁹

This passage contains a gross misunderstanding of the fundamentals of Gauss' and Riemann's differential geometry. In Riemann's theory (Part 2.2), the geometry of an arbitrary n -dimensional R -manifold M is locally approached at each point P by the geometry of the n -dimensional Euclidean space $T_P(M)$, but this has nothing whatsoever to do with a possible embedding of M in an $(n + 1)$ -dimensional (or, if you wish, in an $(n + k)$ -dimensional) Euclidean space. $T_P(M)$ is the tangent space of M at P , which is certainly not conceived after the intuitive analogy of a plane which touches a surface at a point and extends into the surrounding space (Section 2.2.7). Indeed, if Russell's argument were sound, we ought to conclude that spherical and pseudospherical geometries can be constructed in two dimensions because "our Euclidean measure" is available in the circumambient Euclidean space, but that, contrary to Russell's beliefs, a three-dimensional space of constant positive or negative curvature is impossible, unless there actually exists a higher-dimensional Euclidean space in which it is imbedded. That would imply the subordination of GMG to n -dimensional Euclidean geometry which Russell rejected in his discussion with Couturat.

4.3.6 *A Geometrical Experiment*

We said earlier that according to Russell the determination of the constant curvature of physical space must be left to experience. Couturat defied him to mention one experiment that could serve for this purpose. Russell replied that no experiment can give the exact value of space curvature, but that the following, very simple procedure, can fix an upper and a lower bound to that value: Take a circular disc, e.g. a coin; make a mark on its edge; let it run along a geodetic arc in space until the mark makes a full revolution; we can thus determine the ratio of the circumference to the diameter of a circle and compute from it the value of the space curvature.⁴⁰ This experiment presupposes that we can recognize a circular disc and a geodetic arc and that we can determine the length of the latter. Russell is apparently sure that the experiment will show that we live in an approximately Euclidean space. But he emphasizes a fact we have repeatedly suggested in this book: "The image we actually have of space is not sufficiently accurate to exclude, in the actual space we know, all possibility of a slight departure from the Euclidean type".⁴¹ Indeed, if this were not so, Euclid's fifth postulate would have

appeared obvious from the outset and probably nobody would have chanced upon the idea of developing a non-Euclidean geometry.

4.3.7 *Multidimensional Series*

Soon after the publication of the *Foundations of Geometry*, Russell took a very different approach to the problems of space and geometry, based on the analysis of the “logical” ideas of *series* and *order*. The new approach is briefly sketched toward the end of Russell’s reply to Poincaré,⁴² it provides the main support for the criticism of the relationist theories of time and space which he read at the Paris Congress of Philosophy in 1900,⁴³ and it determines the treatment of geometry in his great book, *The Principles of Mathematics* (1903). In Russell’s usage, the idea of order covers both linear and cyclical order.⁴⁴ According to him, order can only arise in a set with more than two elements. Order is generated in a set S if a transitive antisymmetric binary relation is defined in S so that, for any three distinct elements, $x_1, x_2, x_3 \in S$, there is always a permutation σ of $\{1, 2, 3\}$ such that $x_{\sigma(1)}$ stands in the said relation to $x_{\sigma(2)}$ and $x_{\sigma(2)}$ stands in it to $x_{\sigma(3)}$. A *self-sufficient simple series* is an ordered set. Russell speaks also of a simple series *by correlation*, which is a set indexed by an ordered set, or, as we would rather say, the graph of a mapping of an ordered set onto an arbitrary set. A self-sufficient simple series is also described as “a series of one dimension”. The ordered elements of such a series are called *terms*. A series of two dimensions is a series of one dimension whose terms are series of one dimension. Generally, a series of $(n + 1)$ dimensions ($n \geq 1$) is a series of one dimension whose terms are series of n dimensions.

Geometry – says Russell – may be considered as a pure *a priori* science, or as the study of actual space. In the latter sense, I hold it to be an experimental science, to be conducted by means of careful measurements. [...] As a branch of pure mathematics, Geometry is strictly deductive, indifferent to the choice of its premisses and to the question whether there exist (in the strict sense) such entities as its premisses define. Many different and even inconsistent sets of premisses lead to propositions which would be called geometrical, but all such sets have a common element. This element is wholly summed up by the statement that Geometry deals with series of more than one dimension.⁴⁵

Russell’s definition of geometry as “*the study of series of two or more dimensions*” is inordinately restrictive and has never been heeded by philosophers or mathematicians.

Russell's mature views on geometry and space, presented in *Our Knowledge of the External World* (1914) and *The Analysis of Matter* (1926), owe a great deal to the influence of A.N. Whitehead and fall outside the scope of this study.

4.4 HENRI POINCARÉ

4.4.1 *Poincaré's Conventionalism*

Henri Poincaré (1854–1912) had an agile, keen intelligence and a masterful command of French prose. The very ease with which novel ideas and similes came to his mind and flowed from his pen caused him, at times, to state his philosophical views with less care than he deemed necessary, say, when formulating mathematical equations. This has given rise to some misunderstandings and unfair criticisms of his position. The core of his epistemology seems to be the following: Science is concerned with hard facts and their relations.¹ Hard facts are known through our senses and are completely independent of the scientist's will. In order to report such facts, to reason about them and to state their common features and mutual connections, scientists must agree on certain conventions, regarding the manner and method of description. Some of the conventions are older than science, and the scientist cannot help agreeing with them as they stand. Such are the grammatical rules of the languages used in scientific literature, French, English, German, etc. Even in this field, however, scientists can show some initiative, e.g. by ascribing an unambiguous technical meaning to an ordinary word, or by adhering faithfully to a few standard constructions (this is often observed in 20th-century mathematical prose). Other descriptive conventions, pertaining exclusively to science, lie entirely in the scientists' hands. Thus, the choice of a definite set of generalized coordinates when stating a problem in mechanics is not imposed by the facts of the matter, though the nature of the problem will normally make one choice more advisable than others. Or, to quote another, more controversial example: in Poincaré's opinion, two distant events can be said to be simultaneous only by virtue of a freely stipulated rule.²

The main idea of Poincaré's conventionalism is thus seen to be a piece of sound common sense, and it is hard to imagine that anyone could disagree with it. Difficulties arise however as soon as we wish to draw a line between the conventional and the factual ingredients in

scientific statements. When shall we say that two sets of sentences differ only in their manner of putting the very same facts? When, that they convey different, possibly incompatible items of information? Consider first what is seemingly the simplest example: sentences in different languages. The sentences "Das Freiburger Münster hat einen schönen gothischen Turm" and "La catedral de Friburgo tiene una hermosa torre gótica" plainly convey the same fact, which can also be expressed in English as follows: "Freiburg Cathedral has a beautiful gothic tower". But when it comes to a more complex text, such as Thomas Mann's *Zauberberg* or Gracián's *Criticón*, we would be hard put to find a set of English sentences capable of rendering their entire content, with every nuance. This generally acknowledged impossibility of faithfully translating literary works is not regarded as epistemologically significant because it is tacitly agreed that those aspects of reality which cannot be grasped and reported equally well in every civilized language are not a proper subject matter for science. That is why the study of scientific discourse is normally pursued in the light of sentences and expressions drawn from a single language, such as English, which are regarded as standing for their equivalents in any viable language of science. Yet the impossibility of literary translation should make us expect analogous situations also within the limited field of scientific discourse. Thus, it may happen that a particular method of description is alone suited to give a satisfactory idea of a certain kind of facts, either because no better method has ever occurred to anyone or—why not?—because it really is the best conceivable. In such a case, the scientist's preference for that manner of speaking about those facts would be no less compulsory than, say, Shakespeare's 'choice' of the English language for writing *King Lear*.

Two more examples will bring out another aspect of the subject which is often overlooked in philosophical discussions. It is generally admitted that the measurement units employed in registering and reporting quantitative data belong to the conventional ingredient of science. Indeed, such units are fixed by explicit agreement in international scientific congresses and national parliaments. This should mean, apparently, that the same data, say, the distance between two points at a given moment, can be registered and reported in metres or in yards. It is evidently so if both the yard and the metre are defined as different multiples of the same wavelength. Metre and yard stand then to each other in a relation analogous to that between inch and

foot: they are different derived units of the same metrical system. A length stated in terms of one of them can be exactly expressed in terms of the other just by multiplying it by a rational factor. Such is not the case however if the yard is defined as the length of a metal rod kept at some governmental institute, while the metre is given its present official definition as so many wavelengths.³ The conversion factor cannot then be expressed exactly. Moreover, it cannot be determined to any desired degree of approximation, because there are practical limits to the accuracy with which a wavelength can be measured with a metal rod or vice versa. Since most lengths are measured less accurately, you can indifferently use one or the other unit to report them; to express, say, the height of a child, or the distance flown by a plane from Heathrow to Kennedy. Measurements based on an optical standard can attain, however, a greater precision than those based on a rigid standard. As a consequence of this, quantitative data which can be registered with instruments calibrated by an optical standard cannot be registered with the same degree of exactness with instruments calibrated by a rigid bar. Increase in accuracy was indeed one of the reasons why the scientific community discarded the original geodesic metre in 1889, adopting instead the platinum-iridium standard kept at Breteuil, and in 1960 replaced the latter by today's optical metre. The newer unit was, in each case, defined so as to make it equal to its immediate predecessor within the latter's range of accuracy. But its introduction opened up the possibility of registering and reporting quantitative data which were, so to speak, beyond the pale of the system of measurement based on the earlier unit.

We turn now to our second example. Think of the theories of gravitation propounded by Newton in 1689 and by Einstein in 1915. All scientists and most philosophers will grant that the choice between them is not merely a matter of convention. Though these two conceptually very different theories agree within the bounds of experimental error in nearly all their predictions, there are some cases in which their discrepancy can be experimentally controlled. Thus, for example, while gravitation, according to Newton's theory, does not affect the frequency of electromagnetic waves, Einstein's theory predicts that an electromagnetic signal sent from a point P where the gravitational potential is lower to a point Q where it is higher will be seen to have, upon reception at Q, a lower frequency than a signal

emitted under otherwise identical conditions at Q itself. This effect, known as "gravitational redshift", was experimentally verified by R.V. Pound and G.A. Rebka (1960) and by R.V. Pound and R.L. Snider (1965), and is usually regarded as sufficient ground for preferring Einstein's theory to Newton's theory. Nevertheless, in most applications, both theories yield practically equivalent predictions, so that any of them can be used to calculate the evolution of the more familiar gravitational phenomena. Newton's theory is ordinarily adopted, because its mathematics are more manageable. (As a matter of fact, the intractability of Einstein's field equations will, in some cases, make it not just advisable, but even imperative to employ the Newtonian framework in actual calculations.)

Though our first example concerned the choice between two freely instituted standards of measurement, while the second refers to the choice between two physical theories which purportedly describe the factual texture of phenomena, they show a striking analogy. Within a specifiable range of experimental accuracy, the choice is in either case epistemically indifferent and can be based on expediency. Outside that range, one of the proposed alternatives must be preferred for purely epistemic reasons.

The presence and significance of conventional elements in human knowledge was emphasized in the 17th century by Thomas Hobbes, but most philosophers took little or no notice of it. Attention was again devoted to this issue in the last thirty years of the 19th century, in connection with the problem of the definition and identification of inertial systems in mechanics. This problem was raised by Carl Neumann (1870) and was brilliantly dealt with by Ludwig Lange (1885). Newton conceived true motion as a change of position in absolute space. An object can appear to move and yet be truly at rest, if, say, it constantly changes its position in the relative space determined by the walls and the ceiling of our room, but stays fixed in absolute space. However, Newton's laws of motion imply that "the motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forwards in a right line without circular motion". (Corollary V to the Laws of Motion). This conclusion puts an end to any hope one might have entertained of determining which bodies really move and which are at rest, through the observation of bodily motions. For absolute space itself is not directly observable. Moreover, since it is

supposedly infinite and homogeneous, it is not easy to attach a definite meaning to the idea of keeping or changing places in it. The truly important thing, for the interpretation and application of Newtonian mechanics is to identify the class of relative spaces moving “uniformly forwards in a right line without circular motion” with respect to one another, which are mentioned in Corollary V. A relative space is determined by a system of bodies mutually at rest. The systems which determine the relative spaces of Corollary V are known as *inertial systems*. We can pick any inertial system and postulate that it is at rest, without prejudice to Newton’s laws or to the predictions derived from them. On the other hand, if we have identified an inertial system, we can easily deduce the others: they are all those systems which are at rest or move uniformly in a straight line and without rotation relative to it. 19th-century astronomers knew how to construct systems of stars which can fill the role of an inertial system to an excellent approximation. But on the strength of Newton’s gravitational theory, no particular collection of bodies can actually behave exactly as an inertial system. Carl Neumann proposed therefore to postulate a fictitious “alpha body”, relative to which any free particle (that is, any body of insignificant size upon which no external forces are acting) is either at rest or moves in a straight line, traversing equal distances in equal times. The time scale involved in the characterization of the alpha body was introduced by Neumann through an ostensibly conventional definition: *two times are equal if a free particle traverses in them equal distances.*⁴ Such equal distances must of course be measured with respect to an inertial system, so that Neumann’s construction appears to be circular. But Neumann’s definition of the inertial time scale inspired Ludwig Lange with his own purely conventional characterization of an inertial system:

*An inertial system is any coordinate system in which three free particles projected non-collinearly from a given point will have straight-line motion.*⁵

Neumann’s definition of *equal times* comes in quite naturally after this. The physical contents of Newton’s law of inertia is expressed in Lange’s two “theorems”:

(I) Relative to an inertial system [determined by three freely moving particles] any additional free particle also moves in a straight line.

(II) Relative to an inertial time scale [determined by one freely moving particle] every other free particle traverses in any inertial system equal distances in equal times.⁶

According to Lange, this rendering of Newton's first law has a twofold advantage: it makes us aware of the "partial convention" involved in it, and it shows at once that there are infinitely many different inertial systems (not mutually at rest).

It is clear that Lange and other like-minded critics of Newton would not have thought it necessary to include such conventions among the principles of mechanics if positions in absolute space could somehow be perceived. On the other hand, Lange's solution obviously assumes that, at least in principle, one can always tell a straight spatial trajectory from a curved one. Poincaré was acutely aware of the impossibility of observing absolute spatial positions and motions and of its importance for the methodology of science. His repeated reminders of this impossibility have probably done more for the eventual development of relativistic mechanics than his direct contribution to the study of the Lorentz group and its application to physics. The total irrelevance of absolute space to scientific observation and experiment led him early to a most radical conclusion: experience cannot teach us anything about the true structure of space; consequently, the choice of a geometry for the description of physical phenomena is a purely conventional matter. This implies, of course, that a given spatial trajectory will be regarded as straight or not depending on our free selection of a geometry. Indeed, if all of geometry, and not just its metrical aspect, is conventional, even our judgment that a given collection of points can be construed as a possible trajectory depends on our previous conventions; a trajectory must be the range of a continuous mapping of a real interval into space, and the continuity of such a mapping depends on the topology of space.

4.4.2 *Max Black's Interpretation of Poincaré's Philosophy of Geometry*

Poincaré's conventionalist philosophy of geometry has not been understood by everybody in the same way.⁷ Before explaining my own view of it, it will be useful to take a look at an interpretation proposed by Max Black in 1942. He claimed that there are two sides to Poincaré's doctrine, that concern pure and applied geometry,

respectively. Pure geometry consists of a collection of formal axiomatic theories.⁸ Applied geometry arises when one of these theories is given a purportedly physical interpretation. The conventionality of applied geometry follows from that of pure geometry. Pure geometry is conventional because every axiomatic theory is translatable into its contrary.⁹ To see what this means, let us consider an axiomatic theory T , which is expressed in m -English.¹⁰ Let T be determined by a set A of independent axioms. Let A' denote the set obtained by replacing one of the sentences of A by its negation. The theory T' determined by A' is said to be *contrary* to T . T is *translatable* into T' if all the undefined interpretable words of T can be defined in T' , so that upon replacing the interpretable words of any provable sentence of T by the expression which defines them in T' one obtains a provable sentence of T' .¹¹

Even if all the theories of pure geometry were actually translatable into any of their contraries, it would hardly make sense to say that pure geometry is conventional.¹² We say that an intellectual discipline is conventional when statements are adopted or rejected in it for reasons other than their (presumed) truth or falsity. But in pure geometry no such decisions are made. Each axiomatic theory coexists with its contraries and does not stand in their way. They all enjoy equal epistemic rights, but there is no need to choose between them, except insofar as we might wish at a given moment to study or to teach one of them and not the others – a choice which evidently does not involve a dismissal of the latter, but only their temporary neglect by one or more men. On the other hand, if each axiomatic theory is translatable into any of its contraries, applied geometry and, generally speaking, applied mathematics are obviously conventional. If one such theory T provides a satisfactory framework for the description of some kind of natural phenomena P , the same phenomena can be described just as faithfully (though perhaps more clumsily) within the framework of any other theory T' into which T can be translated. It is merely a matter of interpreting T' so that the expressions used to render the interpretable words of T come to mean the same as these. One may prefer T to its contraries as an appropriate means of describing P because it is more beautiful or because it is easier to work with it, but not because the description provided by it is truer. However, if a given theory T , which suitably describes phenomena P , can only be translated into some of its contraries, but not into all of

them, we are faced with a quite different situation and we can no longer maintain that applied mathematics and geometry are conventional. Let $a, b, c \dots$ denote the independent axioms of a theory T which can only be translated into one of its contraries, say, that which results from replacing a by its negation $\neg a$. The choice between T and T' will then be a matter of convention, as before, but the choice between $T - \{a\}$ and its several contraries can still be a matter of truth and error. (I denote by $T - \{a\}$ the theory determined by the remaining axioms of T). If T is well-corroborated by experience, we ought to conclude that the contraries of $T - \{a\}$ are downright false. If $T - \{a\}$ is a geometrical theory, we cannot say that applied geometry is conventional; not, at any rate, for the reason given by Black.

There is a particularly apposite example of a geometrical theory which is not translatable into all its contraries. Plane BL geometry is contrary, in Black's sense, to plane Euclidean geometry. Now, plane BL geometry can be obtained, in the manner sketched in p.247f., by adding a few axioms and definitions, but no new primitive terms, to lattice theory. On the other hand, plane Euclidean geometry cannot be obtained in this way, without introducing a new primitive term.¹³ Consequently, plane Euclidean geometry is not translatable into plane BL geometry. The same is true, for similar reasons, of BL and Euclidean space geometries.

Our counterexample suffices to refute Black's version of geometrical conventionalism, but it does not dispose of it as a reading of Poincaré. The following considerations, however, should make it implausible. In the first place, Poincaré makes no use of the distinction between pure and applied geometry when explaining his doctrine, though it was current in contemporary French literature. More important still: Black's approach implies that not only applied geometry but all applied mathematics is conventional, so that any theory in mathematical physics can be replaced by its negation, *salva veritate*, provided we suitably reinterpret some of its terms. But there is no trace in Poincaré of such an extreme posture. He only contends that the geometrical ingredient of physical theories, that is, all that pertains specifically to the description of the spatial features of phenomena, is not prescribed by experience, but can be chosen freely by scientists. And his contention rests mainly on the peculiar way how we get to know these features, and not on the semantic adaptability of the theories used to describe them.

4.4.3 Poincaré's Criticism of Apriorism and Empiricism

I have just spoken somewhat loosely of the spatial features of phenomena, trusting that the reader is sufficiently familiar with ordinary English to know what I mean. But, though a mastery of everyday language is the necessary presupposition and the starting-point of philosophy, the philosopher cannot rest content with it. In our particular case, at least, we must try to state more precisely what we mean by spatial features in order to grasp and evaluate Poincaré's thesis. The following rough partial inventory will do for our purpose. Size and distance are probably the first things to come to one's mind when thinking about spatial features: Buckingham Palace is larger than the White House; Selfridge's is nearer to the Wallace Collection than Macy's is to the Frick Collection. No less conspicuous is shape: lines are straight or curved; surfaces are flat or concave or saddlelike; all circles, all spheres, all squares, all cubes have the same shape. There are still other, less readily mentioned, yet possibly more fundamental spatial features of phenomena; such are betweenness, orientation (think of a right shoe and a left shoe), continuity, dimension number (three for a body, two for a surface, one for a line), and last but not least, the relation of spatial containment (the proverbial skeleton is *in*, that is, *inside* the cupboard). Does Poincaré's thesis refer to all these kinds of spatial features of things and events, or only to some of them? When he states it in its full generality he never seems to place any restriction on its scope. Taken literally, this would mean that these spatial features can be described just as faithfully by any system of geometry which is sufficiently rich to encompass them all, even though two such systems will probably differ in what they term large or small, straight and crooked, contiguous or separate, interior or exterior, etc. This will sound less startling if we bear in mind that, according to Poincaré, the spatial features ascribed to physical objects by the mathematical theories of physics—which depend on the location of those objects in what Poincaré calls "geometrical space"—are wholly foreign to the spatial features exhibited by phenomena as they appear to our senses—which Poincaré collects under the name of "sensible space" or "*espace représentatif*". And it is, of course, only to the former that the conventionalist thesis is explicitly applied by him. Now, though our construction of geometrical space is suggested and even guided by

our actual experience of sensible space, Poincaré believes that after that construction is perfected it can be suited to describe any experience, however different it might be from that which originally inspired it. To make his meaning clear, he tells a story:

Beings with minds like ours, and having the same senses as we, but without previous education, would receive from a suitably chosen external world impressions such that they would be led to construct a geometry other than that of Euclid and to localize the phenomena of that external world in a non-Euclidean space, or even in a space of four dimensions. As for us, whose education has been accomplished by our actual world, if we were suddenly transported into this new world, we should have no difficulty in referring its phenomena to our Euclidean space. Conversely, if these beings were transported into our environment, they would be led to relate our phenomena to non-Euclidean space.¹⁴

Though Poincaré only asserts here the interchangeability of two geometries which differ in their metrics but might agree in their topologies, he never denied the possibility of employing topologically unusual geometries in mathematical physics. And he explicitly declared that one topological property, namely, dimension number, is conventionally stipulated, though, of course, it is suggested by experience.¹⁵

Before discussing Poincaré's positive reasons for upholding the conventionalist thesis, let us examine the grounds of one powerful negative reason he adduced in support of it. In his opinion, geometrical conventionalism is the only alternative which is still open, given that apriorism and empiricism are false. His case against apriorism is stated very briefly. If any system of geometry were true a priori, one could not conceive a contrary, yet equally rational system (i.e. a system which consistently denies one of the independent principles of the former). Since this is always possible, no system of geometry can be true a priori. This argument shows quite plainly that Poincaré is not at all concerned with what we call pure geometry. A priori knowledge of one system of pure geometry (that is, a priori knowledge of the relations of logical consequence between its axioms and its theorems) does not preclude the possibility of knowing a priori other such systems. Poincaré's argument refutes the thesis that the actual geometrical structure of the physical world, as it is described, say, in Euclid's system, is logically necessary. I wonder whether this thesis has ever been literally held by anybody. Leibniz and Hume

apparently believed in something of the sort, but they never made their meaning altogether clear. Had they done so, they would probably have realised that their position was untenable. The discovery of BL geometry, of course, made it obvious. Poincaré's argument is powerless, however, against Kant's brand of apriorism, which presupposes the very fact invoked by Poincaré. In Kant's philosophy, the necessity of geometry is not an absolute, logical necessity, but is contingent on the changeless but unfathomable constitution of the human mind.¹⁶ Poincaré apparently misunderstood Kant when he first argued his case against geometrical apriorism.¹⁷ But he developed later a more adequate strategy. He denied that we have a non-empirical yet immediate awareness of space as a universal framework in which every object of sense perception must be located (that is to say, in Kantian terms, he denied that we have an a priori 'intuition' of space as a 'form of the outer sense'), and he sought to show how space and geometry arise from the purely intellectual enterprise of comparing sense perceptions and reflecting upon them. Towards the end of his life, he did assert however that there exists such a thing as a "geometrical intuition", which is the source, e.g. of Hilbert's axioms of order and to which he, Poincaré, had continually resorted in the course of his topological researches. But such intuition is nothing but the awareness of our faculty of constructing an n -dimensional continuum. The decision to put $n = 3$ and the definition of a metric must be based on experience.¹⁸

The case against geometrical empiricism is argued at greater length, in a manner which suffices, in my opinion, to turn the tables on it, as it was advocated in the 19th century.¹⁹ Poincaré's approach, on the other hand, has certainly contributed to prepare the new, subtler forms of empiricism which have prevailed after him. Let us mention, first of all, two arguments of an heuristic nature, which Poincaré always states together. Geometry cannot be an empirical science because it is not subject to revision in the light of increasing experience. Moreover, geometry is an exact science, whereas empirical sciences are always approximative. The first statement may foster the idea that Poincaré is really talking about pure geometry or, perhaps, that he is utterly confused. If, as the context shows, he speaks, in fact, about the geometrical groundwork of mechanics, the second statement might be taken to imply that mechanics itself and, more generally, every theory of mathematical physics, are not a whit

more empirical than geometry. For are they not, considered in themselves, just as exact? The last remark, however, suggests an interpretation of Poincaré's meaning which I think will remove our doubts regarding both statements. As we saw in Section 4.4.1, Poincaré believed, like every practising scientist, that physical theories, notwithstanding their mathematical exactness, can be compared with and be corroborated or refuted by the inevitably imprecise data supplied by observation and experiment. Why did he maintain that geometry – and that means, as I take it, applied or physical geometry – was exempt from such condition? Because geometry must mediate, so to speak, between theories and data. The rough facts of observation can be compared with the neat predictions of theory only if they are described in terms akin to the latter. The geometrical description of phenomena (strictly speaking, their kinematic, that is, *geochronometrical*, description) provides the terms of comparison required for the evaluation of physical theories. The translation of the “Book of Nature” into “mathematical language” can be performed in many different ways; as many as the different systems of geometry which are rich enough for the purpose. The formulation of a scientific theory must, of course, be adapted to suit the chosen system of description, but its predictive contents will remain unaltered throughout its “translations”. I have not been able to find a passage in Poincaré that directly bears witness to my interpretation, but I think that his account of the manner how “geometrical space” – which is none other than the space of mechanics and the rest of physics – is constructed ratifies it indirectly.²⁰ If my interpretation is accepted, we at once see why physical geometry must be exact and cannot be revised in the light of experience. Insofar as geometry itself supplies the scheme according to which the data of experience must be displayed if they are to make any scientific sense, it is impossible that it should ever clash with them. Some geometries are, of course, more manageable than others, because of their own structure and because of the peculiar features of the empirical material which we try to bring under their sway. Poincaré believed that Euclidean geometry was unexcelled on both counts.²¹

The main argument for Poincaré's rejection of empiricism was mentioned earlier (at the end of Section 4.4.1): empirical information has no bearing whatsoever on the structure of geometrical space. Or, as he puts it:

Experiments only teach us the relations of bodies to one another; none of them bears or can bear on the relations of bodies with space, or on the mutual relations of the different parts of space.²²

In *La Science et l'Hypothèse* this remark is placed immediately after a very interesting discussion of "the law of relativity", which Poincaré obviously regards as having a close relation to it. Poincaré proposes that we consider an isolated material system. The laws of the phenomena taking place in this system may depend on the state of the component bodies and their mutual relations, but "because of the relativity and the passivity of space" they cannot depend on the absolute position and orientation of the system. In other words, "the state of the bodies and their mutual distances at any given moment will depend only on the state of these same bodies and their mutual distances at the initial moment", but not on their relations with (absolute) space. Poincaré calls this *the law of relativity*. This law is ordinarily verified by experiences described according to Euclidean geometry. The same experiences can certainly be described according to a non-Euclidean geometry. But the non-Euclidean distances between the different bodies will not generally be the same as their Euclidean distances. Might not our experiences, when described according to a non-Euclidean geometry, clash with the law of relativity? Our preference for Euclidean geometry could then perhaps be empirically grounded, after all. Poincaré remarks that a strict application of the law of relativity demands that one consider the universe as a whole. But if our material system is the entire universe, experience cannot say anything about its absolute position and orientation in space. All that our instruments can reveal to us is the state of the different parts of the universe and their mutual distances. The law of relativity should therefore be stated thus:

The readings we shall be able to make on our instruments at any instant will depend only on the readings we could have made on these same instruments at the initial instant.²³

Since this statement is independent of the geometrical interpretation of the readings, the "law of relativity" cannot by itself enable us to decide between Euclidean and non-Euclidean geometry.

That experience cannot teach us anything about the "mutual relations of the several parts of space" is certainly true of absolute space

as it was conceived in classical mechanics. But if phenomena exhibit nothing but the mutual relations between material bodies, it is difficult to understand why their geometrical description should ever put them in connection with an elusive immaterial transcendent space. Such connection, says Poincaré, is never revealed by experience. Why then, must we make it at all? Poincaré seems to aim at a different conception of space, which he never quite succeeded in clarifying. Suppose we regard physical geometry as a mathematical structure whose underlying set is formed by material bodies ('particles') or perhaps by phenomena ('events') themselves. On this view, experience obviously reveals "the mutual relations between the several parts of space", and Poincaré's statement is trivially false. It would seem, however, that this conception of space agrees much better than the classical Newtonian one with his overall approach. Of course, it is not just a matter of wishing to see things in this way; the whole of mechanics must be consistently reformulated in accordance with the new view before one can finally adopt it. It is not likely that any attempt in that direction – any attempt, that is, to treat geometry as a structure of matter and to rid physics of the spook of absolute space – could have succeeded while physicists persisted in conceiving space and time separately. On the other hand, disembodied absolute space vanished as soon as it was seen that the genuine "geometrical space", the "mathematical continuum" which underlay the exact representation of phenomena since the beginning of mathematical physics, was not the three-dimensional Euclidean space, but four-dimensional space-time, the former having always been treated as a subspace (or rather, as a class of homeomorphic subspaces) of the latter. This insight is usually credited to Minkowski.²⁴ Though Poincaré was well acquainted with Minkowski's work – indeed he even anticipated some of its technical aspects – he apparently failed to appreciate its great significance for the philosophy of geometry.²⁵ From the new vantage point, it is quite natural and perhaps inevitable to allow some outstanding physical processes to determine the characteristic features of physical geometry. Thus, in Minkowski's version of special relativity, the geodesics of the semi-Riemannian spacetime manifold are given (in part) by the spacetime trajectories or "world-lines" of material particles and light-rays travelling unperturbed by external forces. In Cartan's version of Newtonian gravitational theory, spacetime is an affine 4-manifold in which the geodesic

joining a pair of non-simultaneous points is determined by the world-line of a small test particle falling freely between those points. If geodesics are fixed in this way, the details of the affine structure of spacetime can only be settled by experience. Since geodesics are the straightest curves of a geometry, we may say that in these theories certain remarkable physical processes provide a standard of straightness. Such standards are freely chosen, in a sense, and one can therefore claim that the spacetime geometry determined by them is conventional. But it can happen that, as a matter of empirical fact, the only processes which are sufficiently regular and ubiquitous to serve as standards are precisely those actually chosen. Thus, one could hardly define spacetime geodesics as the world-lines of amoebae, although, as Poincaré would probably have been quick to point out, this choice cannot be objected to on principle. In this case, as in others we have met before, the actual circumstances of life can restrict the scientist's decisions so much that it makes little sense to call them conventional.

Our last remarks have a bearing on an anti-empiricist argument which Poincaré took over from Lotze.²⁶ We mentioned earlier that Gauss and Lobachevsky thought that one could test Euclidean geometry by astronomical triangulations (p.63). Poincaré writes:

If Lobachevsky's geometry is true, the parallax of a very distant star will be finite; if Riemann's is true, it will be negative. These are results which seem within the reach of experiment, and there have been hopes that astronomical observations might enable us to decide between the three geometries. *But in astronomy 'straight line' means simply 'path of a ray of light'.* If therefore negative parallaxes were found, or if it were demonstrated that all parallaxes are superior to a certain limit, two courses would be open to us; we might either renounce Euclidean geometry, or else modify the laws of optics and suppose that light does not travel rigorously in a straight line.²⁷

The sentence in italics shows that in 1891 Poincaré knew very well that physical geometry actually identifies its characteristic elements with some reproducible physical prototypes. Had he taken Minkowski's standpoint he would probably have concluded that there is no viable substitute for light-rays as a prototype of (spacetime) straightness.²⁸

A final anti-empiricist argument, based on the alleged possibility of constructing bodies which "move according to the Lobachevskian

group”,²⁹ will be understood better in the context of Section 4.4.4 (p.339f.)

4.4.4 *The Conventionality of Metrics*

The earliest statement of Poincaré’s thesis on geometry referred only to the choice between the geometries of Euclid and Lobachevsky (one might add, perhaps, the “geometry of Riemann”, i.e. the geometry of a maximally symmetric space of constant positive curvature). It occurs at the end of the paper on the fundamental hypotheses of geometry (1887), which we considered in Section 3.1.6. Although this is concerned with pure geometry, the conventionalist thesis refers explicitly to physical geometry. The conclusion of the article is, as we know, that the two-dimensional geometries of constant negative, positive and null curvature can be characterized respectively by a different group of motions. Poincaré rightly takes for granted that a similar conclusion applies to the corresponding space geometries. Since “geometry is nothing but the study of a group”, “one might say that the truth of the geometry of Euclid is not incompatible with the truth of the geometry of Lobachevsky, for the existence of a group is not incompatible with that of another group”.³⁰

Among all possible groups, we have chosen one in particular, in order to refer to it all physical phenomena, just as we choose three coordinate axes in order to refer to them a geometrical figure.³¹

The choice of this particular group is motivated in the first place by its simplicity: in contrast with the groups characteristic of BL and spherical geometries, the group of Euclidean motions contains a proper normal subgroup; “translated into analytical language, this means that there are fewer terms in the equations”.³² But it is chosen also because

there exist in nature some remarkable bodies which are called *solids*, and experience tells us that the different possible movements of these bodies are related to one another much in the same way [*à fort peu près*] as the different operations of the chosen group.³³

On the other hand, “the chosen group is merely more comfortable than the others”. To say that Euclidean geometry is true and BL geometry is false makes so much sense as to say that Cartesian coordinates are true and polar coordinates are false.

It is perhaps no accident that in his remarks of 1887 on the status of physical geometry, Poincaré should have mentioned only the latter two geometrical systems, though he considered several others in the purely mathematical part of his article. The Euclidean group and the BL group are topological groups which can be assumed to act transitively and effectively on one and the same topological space.³⁴ Each is indeed isomorphic with a different subgroup of the group of continuous transformations of \mathbb{R}^3 .³⁵ Obviously, every item of information concerning figures in \mathbb{R}^3 can be conveyed in terms of the invariants of either group (p.176). Consequently, if, as in classical mechanics, physical space is assumed to be homeomorphic with \mathbb{R}^3 , both Euclidean and BL geometry can be used as a framework for the geometrical description of physical phenomena. The foregoing argument does not apply to every system of geometry. Thus, the group of motions of spherical 3-space cannot act transitively on \mathbb{R}^3 .³⁶ Nevertheless, in subsequent discussions of the conventional status of geometry, Poincaré always treats spherical geometry (“the geometry of Riemann”) on a par with Euclidean and BL geometry. I can think of two possible explanations of his attitude.

The first is fairly simple. The group of motions of spherical space geometry can act transitively on the three-dimensional sphere S^3 . \mathbb{R}^3 is homeomorphic with the punctured sphere (S^3 minus a point). Any physical system which is accessible to observation in its entirety will remain, during the whole period in consideration, within some open proper subset of physical space. If, as we have assumed, this is homeomorphic with an open set of \mathbb{R}^3 , it is also homeomorphic with an open set of S^3 . We may therefore describe its contents in terms of the invariants of the spherical group, just as we could do it in terms of those of the other two groups. Of course, this will no longer do if one attempts to speak about the whole physical world. One must then face the alternative of a compact or a non-compact space, and it does not seem likely that this can be settled by convention.³⁷

The second explanation which I wish to propose for Poincaré’s equal treatment of the three classical geometries is more involved. It is suggested by his discussion of the origin of Euclidean geometry.³⁸ Poincaré was one of the first mathematicians to distinguish neatly between the abstract structure of a group – its “form”, as he called it – and its embodiments in diverse “materials”. Thus, e.g. the group of permutations of $\{1, 2, 3, 4\}$ is isomorphic with the group of

isometries of a regular tetrahedron (i.e. the distance preserving mappings of the tetrahedron onto itself) and with the groups of motions of a cube and of a regular octahedron (i.e. the distance and orientation preserving transformations of the cube and of the octahedron). We may therefore regard these groups as four embodiments of the same "form" in different "materials". For greater precision, allow me to introduce a few new terms. Let G be a group, S a set, T_S the group of permutations of S (i.e. the set of all bijective mappings of S onto S , with composition of mappings as group product). A *realization* of G in S is an homomorphism of G into T_S , that is, a mapping $\varphi: G \rightarrow T_S$, such that, for any $g, g' \in G$, $\varphi(g) \cdot \varphi(g') = \varphi(gg')$. S is said to be the *basis* of the realization φ . A realization φ of G in S is said to be *transitive* if for every $x, y \in S$ there is a $g \in G$ such that $\varphi(g)$ maps x on y . It is said to be *faithful* if φ is an isomorphism of G onto $\varphi(G)$. Obviously, what Poincaré calls the "material" or "matter" of a group is the set providing the basis for a given realization of the group. We note, in particular, that if S is a topological space and G is a topological group which acts on S , the action $\Phi: G \times S \rightarrow S$ determines a realization of G in S , namely, the mapping which assigns to each $g \in G$ the permutation $x \mapsto \Phi(g, x)$ ($x \in S$). The realization is transitive if the action Φ is transitive; it is faithful if the action is effective.³⁹ Each of the classical geometries is concerned with a faithful and transitive realization of one of the three classical groups of motions. The basis for the realization is provided by the same topological space in the case of Euclidean and BL geometry; by a different space in the case of spherical geometry. This fact might make an epistemologically significant difference between the latter geometry and the other two if groups had to obtain their "materials" so to speak from the outside. Our preference for the latter geometry or for one of the former would then depend on the nature of the available "material". Such was, as Poincaré observes, the position of his predecessors Helmholtz and Lie, who believed that "the matter of the group existed previously to the form" and that "in geometry the matter is a *Zahlenmannigfaltigkeit* of three dimensions". For Poincaré, "on the contrary, the form exists before the matter".⁴⁰ Moreover, as Poincaré certainly knew, any group has a realization in itself or in a set which is given together with it (i.e. in one of the sets that exist if the group exists). Let us call a realization for which the group itself directly or indirectly provides a basis, an immanent

realization of the group. On pp.350f. we shall construct an immanent realization of the Euclidean group which is a model of Euclidean geometry. Models of the other two classical geometries can be built analogously. Poincaré apparently thought that any of them could be used to describe the "brute facts" of sense experience, if the latter are suitably idealized.

*As an exercise which will be of some use to us later, I give here a general method of constructing an immanent realization of a group. We make first some agreements on notation and terminology. Let φ be a realization of a group G in a set S . We agree to write φ_g instead of $\varphi(g)$ for the value of φ at $g \in G$. If $x \in S$, the set $\{g \mid g \in G \text{ and } \varphi_g(x) = x\}$ is a subgroup of G , called the *stability group* of x . If φ' is a realization of G in a set S' , and there is a bijective mapping $f: S \rightarrow S'$ such that, for every $g \in G$, $\varphi'_g = f \cdot \varphi_g \cdot f^{-1}$, we say that φ' is *similar* to φ .

*Let G be any group, H a subgroup of G . If $g \in G$, we denote by gH the set $\{x \mid x = gh \text{ for some } h \in H\}$; we call this set the *left coset* of H by g . Let G/H be the set $\{gH \mid g \in G\}$ of all the left cosets of H . The reader should satisfy himself that each $g \in G$ belongs to one and only one element of G/H . We now define a mapping $g \mapsto f_g$ of G into the set of permutations of G/H . It is enough to determine the value of f_g , for any $g \in G$, at an arbitrary point kH of G/H . We fix it as follows: $f_g(kH) = gkH$.

*It is not hard to see that f is a transitive realization of G in G/H .⁴¹ Since G/H is given together with G , this is indeed an immanent realization. It can also be shown easily that H (regarded as a subgroup of G) is the stability group of H (regarded as a point in G/H).⁴² Moreover, f is similar to every realization of G in which H is a stability group. If H does not contain a proper normal subgroup of G , f is a faithful realization of G .

The choice between Euclidean and BL geometry can be viewed as a choice between two definitions of distance in \mathbb{R}^3 , namely, the two-point invariant of the Euclidean group

$$d(x, y) = \left(\sum_{i=1}^3 (x_i - y_i)^2 \right)^{1/2}$$

and the corresponding two-point invariant of the BL group. Poincaré enlarged upon his ideas on the conventionality of distance in his article of 1891 on non-Euclidean geometries and in two polemical

articles of 1899 and 1900, motivated by Russell's *Foundations of Geometry*. In 1891, he introduced a dictionary to translate BL geometry into Euclidean geometry, an idea that in diverse variations and generalizations has had great success in 20th-century epistemology. Since we have seen already on pp.81f. how a dictionary of this kind works, I shall not dwell upon this subject any longer.⁴³ In the article of 1900, Poincaré agrees with Russell that one cannot define everything, but he will not admit that distance is one of the notions which you cannot or need not define. There is no such thing as a direct intuition of distance. Moreover, Poincaré will not grant that the distance from Paris to London is greater than one metre. He readily admits of course that any definition which would make that distance equal to or less than one metre runs counter to common sense. For I can encompass the standard platinum-iridium metre within my arms, while it is impossible for me to place at the same time one hand in London and the other in Paris. It is also so much more difficult to go from Paris to London than to traverse the length of the standard metre.

That is why a method of measurement which would show the distance from Paris to London to be equal to one metre would be inadequate for all practical uses and anyone who proposed to adopt it by convention would lack common sense. But to say that the distance from Paris to London is greater than one metre *absolutely*, independently of every method of measurement, is neither true nor false; I find that it does not mean anything.⁴⁴

A golf-ball in a golf hole is certainly smaller than the earth. But I cannot infer from this that it will remain smaller than the earth while it flies across the air to the next hole. To grant that the ball preserves its volume as it moves is tantamount to making it into a measurement instrument, thereby conventionally adopting a system of measurement.

The article of 1900 develops also the anti-empiricist argument which I mentioned at the end of Section 4.4.3 and whose discussion I postponed. The argument purports to show that the fact that ordinary solid bodies move approximately in accordance with the Euclidean group (that is to say, that the Euclidean distances between their parts, measured by the usual methods, do not change appreciably as the bodies move), cannot tell us anything about the geometrical structure of physical space. We shall consider two material bodies K_1 , K_2 .

K_i ($i = 1, 2$) consists of eight thin steel rods $OA_1^i, \dots, OA_6^i, OB_1^i, OB_2^i$, permanently joined at O ; it also includes some device regulating the relative positions of the vertices A_j^i, B_k^i as K_i moves. Let P, Q_1, Q_2 be three points marked, say, on a piece of wood, so that A_j^i, B_k^i and O can be simultaneously placed upon P, Q_2 and Q_1 , respectively ($j = 1, \dots, 6; k = 1, 2$). We assume also that the point-pairs $A_j^i A_{j+1}^i$ ($1 \leq j \leq 5$) and $A_6^i A_1^i$ can be brought into coincidence with PQ_i ($i = 1, 2$). Poincaré asserts that K_1 “moves according to the Euclidean group or at least that it does not move according to the Lobachevskian group”, while K_2 “does not move according to the Euclidean group”, but might move according to the BL group. Now K_1 can be built quite easily. K_2 on the other hand, requires some ingenuity, but, Poincaré says, any mechanic can contrive it.⁴⁵ Since K_1 and K_2 can exist simultaneously, their properties cannot teach us anything about the true geometry of the world.

There is a fallacy in the foregoing argument that it will be instructive to expose. We must distinguish between the body K_i , consisting of eight steel rods plus a regulating device, and the particle system formed by the eight vertices A_j^i, B_k^i ($i = 1, 2; 1 \leq j \leq 6; k = 1, 2$). Let us denote the latter by K_i' . Now it is only K_2' , but not K_2 , that can be held to move according to the BL group. The eight particles of K_2' will preserve their BL distances because they are artfully joined by eight or more bodies, each of which moves according to the Euclidean group. We can imagine of course a BL polyhedron (a double hexagonal pyramid) whose vertices are, at each moment, A_j^2, B_k^2 , and we can also make plastic models of it in its several positions. But there is no known material which, when moulded into one of these shapes, will take of itself the whole series of them, as it is pushed around.

4.4.5 *The Genesis of Geometry*

Poincaré's deepest speculations on our subject are contained in his long essay “On the foundations of geometry”, published in English in *The Monist* of October 1898. It is a rather ambitious attempt to show how our idea of geometrical space—as it occurs, say, in classical mechanics—arises *with* experience, though certainly not *from* it. Poincaré's entire construction rests upon an untenable theory of perception, according to which all our knowledge of physical facts can be ultimately traced to a variegated and changing aggregate of

elementary sensations, each of which is caused by the momentary stimulation of an afferent nerve. Because they rest on such foundations, many statements in the essay are unclear or simply unlikely, which is probably the reason why it has often been neglected in recent discussions of Poincaré's conventionalism.⁴⁶ It is indeed a pity that the great mathematician and mathematical physicist should have been thus seduced by his philosophical colleagues into believing their psychological fantasies; but we must put up with this fact and its often irritating manifestations, if we wish to follow Poincaré's thought. The latter is instructive in spite of its shortcomings, because it provided some essential ingredients and a rudimentary prototype for other, more sophisticated, psychologically sounder inquiries into the origin of geometry that have been pursued in the 20th century. Due to its greater simplicity and naïveté, Poincaré's theory can throw light on the work of his successors and aid us to understand the very problem which they all set out to solve. In the rest of this section, I shall outline Poincaré's genetic construction step by step, without questioning his sensationist assumptions any further. I shall, however, bring out several points which merit criticism in Poincaré's own terms. Throughout our exposition, we must try to avoid the chief danger involved in such investigations, namely, the premature and surreptitious application of the very ideas of geometry and space whose genesis is being re-enacted. Because our language is pervaded with spatial idioms, this danger is very difficult to avoid. I am not quite sure that Poincaré himself always stayed clear of it.

According to Poincaré, "the crude data of experience, which are our sensations,"⁴⁷ "have no spatial character" and "cannot give us the notion of space."⁴⁸ This is confirmed, in particular, in the case of visual sensations. Imagine a man who only has such sensations, say, a paralytic with an anaesthetized skin, who stares at the world through a single fixed eye.⁴⁹ A red sensation caused by the stimulation of the upper edge of his retina and a red sensation caused by the stimulation of its lower edge will appear to him as qualitatively different, essentially incomparable sensations. They do not appear so to us, but as qualitatively similar, though diversely located sensations, because we can transform one into the other and vice versa by merely moving our eyes up and down.⁵⁰ (Attention: "moving my eyes up and down" is here shorthand for "contriving to feel such and such a succession of muscular sensations"; remember that we do not have yet a space in

which to move.) Sensations arising from different nerves will therefore exemplify diverse incomparable qualities. But "sensations furnished by the same nerve-fibre" can be ordered according to their intensity. The aggregate of our sensations can be thus referred, through "the active intervention of the mind", to "a sort of rubric or category" which Poincaré calls *sensible space*.⁵¹ He claims that this space has as many dimensions as we possess nerve-fibres, but in fact, if n is the number of such fibres, and we regard every distinct sensation as a point of sensible space, it would be more accurate to say that the latter is a *one-dimensional* topological manifold with n components. Even this would not be perfectly accurate, however, since, as we shall explain later, the different intensity scales which constitute sensible space are not *continuous* in the exact mathematical, but only in the rough physical, sense of the word, so that none of them can actually be mapped injectively onto a real interval. On the other hand, we might consider each point of sensible space to be an *aggregate* of simultaneous sensations, that is, what we shall call hereafter a *state of sense awareness*. According to sensationism any such state will vary continuously in a unique manner as the stimulus acting on one particular nerve-tip is gradually modified while those acting on all the other tips remain unchanged. The space of such states is roughly an n -dimensional topological manifold, because the set of all the states into which any state of sense awareness can thus develop can be roughly charted into \mathbb{R}^n . We shall refer to sensible space, in the second acceptance, as the *space of sense awareness*.

Poincaré assumes that the so-called muscular sensations can be clearly distinguished from the rest. Some of the muscular sensations are of a static kind, and can last a short time, just like a sweet taste or a red after-image; e.g. the sensation of holding a 30 lb. bag, ten inches above the ground (note however that the sensation will tend to change in quality as one grows tired). But most of them can only be had in a fleeting succession, woven as it were into a particular 'melodic' pattern of such sensations. A voluntary change in the state of sense awareness which is accompanied by a succession of muscular sensations is called an *internal change*. Poincaré regards internal changes as identical if they are accompanied by the same melodic pattern of muscular sensations. Thus, turning your head to the right at the pole or on the Equator, rising from your seat after a concert or after a faculty meeting, will cause the same internal changes. It is essential to

bear this in mind in the sequel. An involuntary change in our state of sense awareness which is not accompanied by a succession of muscular sensations is called an *external change*. External changes are viewed as being identical only if they are qualitatively indistinguishable. Consider an external change A which transforms a state of sense awareness α into a state β . There might be an internal change A' which transforms a state indistinguishable from β into a state indistinguishable from α . If such an A' exists, we say that it *cancels* A , and we call both A and A' *locomotions*.⁵² We also give this name to any combination of locomotions succeeding one another. Such a combination is said to be cancelled by an internal change which transforms its final state into its initial state. If locomotion A transforms state α into state β , and locomotion B transforms a state qualitatively indistinguishable from β into state γ , there is a conceivable locomotion which is qualitatively indistinguishable from A followed by B , which transforms a state qualitatively indistinguishable from α into a state qualitatively indistinguishable from γ . We denote this locomotion by $A + B$. A locomotion whose initial and final states are the same will be said to be *neutral*. Since the same internal change can transform a wide variety of initial states into a wide variety of final states, it can obviously cancel many different locomotions. All locomotions cancelled by the same internal change are regarded as equivalent. We also regard two internal changes as equivalent if they cancel the same locomotion. It might seem that such equivalences are relative to a particular locomotion, which cancels or is cancelled by every member of an equivalence class. Poincaré asserts, however, that it is *an empirical fact* that if locomotions A and B are cancelled by locomotion A' and B is also cancelled by locomotion B' , then B' cancels A .⁵³ If he is right, it follows that the equivalence classes of locomotions mutually determine one another, and equivalence is not relative to a particular locomotion.⁵⁴ We may also regard all neutral locomotions as equivalent. Moreover, if A is equivalent to P and B is equivalent to Q , and if $A + B$ and $P + Q$ can be defined as above, then $A + B$ is equivalent to $P + Q$. Hereafter an equivalence class of locomotions will be called a *displacement*.⁵⁵ The class of all neutral locomotions will be denoted by 0 . The set of all displacements will be denoted by \mathcal{D} . Poincaré tacitly assumes that if $a, b \in \mathcal{D}$, there is always some $A \in a$ and some $B \in b$ such that $A + B$ is a conceivable locomotion. We define the sum $a + b$ of a and b as

the equivalence class of $A + B$. Evidently $a + b$ is defined for every $a, b \in \mathcal{D}$ only if every displacement a includes a locomotion which ends up in a state where some locomotion belonging to any given displacement b might begin. Such assumption is not inconsiderable and does not follow from our definitions. It is implied, however, by an even stronger assumption which Poincaré makes as a matter of course, namely, that *if a is any displacement and α any state of sense awareness, there is a locomotion A in a , whose initial state is α* . Since both assumptions can be refuted or corroborated by experience, but neither is liable to final verification, we must regard them as *empirical hypotheses*. Given that the operation $+$ is clearly associative and that every displacement a has an inverse a^{-1} such that $a + a^{-1} = 0$, the weaker hypothesis is sufficient to prove that $(\mathcal{D}, +)$ is a group. The stronger hypothesis implies moreover that this group has a realization in the space of sense awareness. We call $(\mathcal{D}, +)$ the group of displacements. Hereafter, we denote it by the name of its underlying set \mathcal{D} . Poincaré's theory of geometry rests on the existence of group \mathcal{D} . An example will show how he understands it. Let N_1 and N_2 be two street corners in midtown New York, ninety yards apart; let V_1 and V_2 be two street corners in Venice, on a straight lane, with sidewalk and canal, also ninety yards apart. Let N, V denote the external changes which I experience as I am carried from N_1 to N_2 and from V_1 to V_2 , respectively. Both N and V are cancelled by the internal change which I feel as I walk ninety yards in a straight line. N and V are therefore equivalent. They both belong to the same displacement, which we may call a 'translation' of 90 yards. Our example suggests how the group of displacements can provide a foundation for geometry, but it also raises a problem which Poincaré has apparently overlooked. V can also be cancelled by the internal change caused by getting into a boat and *rowing* ninety yards. We shall not mind the fact that this change will not cancel N — one could after all dig a canal in Madison Avenue. The real difficulty lies elsewhere. The feeling of rowing 90 yards might not differ from the feeling of rowing 150 yards with a second oarsman to help you, or even from the feeling of rowing 0 yards if the boat is tied to a pier; moreover, there does not appear to be any difference in our muscular sensations whether we row the 90 yards in a straight line or in a circle with another person at the rudder. It might seem that we can overcome the difficulty by eliminating from the class of locomotions all internal changes which

have neutral instances (displacement 0 could still be built from the neutral external changes). But this will not do: the feeling of walking any distance can be neutral if the ground happens to be moving under your feet in the opposite direction (think of the 'mechanical carpets' or passenger conveyor-belts which have been set up in some airports). It does not seem possible to distinguish such cases from the more familiar instances of walking, in terms of muscular sensations alone.⁵⁶ Poincaré could have objected that such anomalous situations have no part in the formation of our geometrical ideas, and that when they arise those ideas are already available and provide a suitable framework for their interpretation. Be that as it may, I shall proceed with my outline as if the difficulty did not exist.

Poincaré's next step is to show that the group of displacements is, as he says, *continuous*. In his paper "Le continu mathématique" (1893)⁵⁷ he had distinguished between the *physical* and the *mathematical* continuum. His prototype of the latter is the ordered field \mathbf{R} of real numbers, with, I presume, the standard topology.⁵⁸ However, he also mentions "the mathematical continuum of n dimensions", by which I imagine he means \mathbf{R}^n , also with the standard topology. And I dare say he would have regarded every topological manifold – that is, every topological space which is *locally* homeomorphic with \mathbf{R}^n – as a mathematical continuum.⁵⁹ His idea of the physical continuum is presented through an example. Let A, B, C denote weights of 10, 11 and 12 grammes, respectively, and assume that we can only perceive differences in weight that are equal to or greater than two grammes. "The crude results of experience can then be expressed by the following relations:

$$A = B, \quad B = C, \quad A < C$$

which can be regarded as the formula of the physical continuum."⁶⁰ It should be noted that the relation between A and B and between B and C is not one of identity, but of perceptual indiscernibility, and that it is misleading to represent it by the symbol "=", since it is not even an equivalence (it is symmetric and reflexive, but not transitive). Bearing this in mind, I propose the following tentative characterization of Poincaré's physical continuum (which, by the way, might perhaps be more properly called a mental or psychological continuum):

- (A) A *simple physical continuum* is a triple $\langle S, R, < \rangle$ such that
- (i) S is a non-empty set;

(ii) R is a binary reflexive symmetric non-transitive relation in S (read " aRb " as " a is indiscernible from b ");

(iii) $<$ is a binary antisymmetric transitive relation in S (read " $a < b$ " as " a precedes b ");

(iv) if $a, b \in S$ and aRb , then neither $a < b$ nor $b < a$;

(v) if $a, b \in S$ and not aRb , then either $a < b$ or $b < a$;

(vi) if $a, b, c \in S$ and $a < c < b$, there is no $x \in S$ such that aRx and xRb ;

(vii) S contains a non-empty subset L such that, if $a, b \in L$ and $a \neq b$, either $a < b$ or $b < a$ (in other words, L is linearly ordered by $<$);

(viii) if $a, b \in L$ and $a < b$ and there is no $c \in L$ such that $a < c < b$, there is an $x \in S$ such that aRx and xRb .

(ix) if $a \in S$, there is a $b \in L$ such that aRb .

(B) If a, b are two objects, we say that a simple physical continuum $\langle S, R, < \rangle$ joins a and b if $a, b \in S$ and for every $x \in S$, either aRx or xRb or $a < x < b$ or $b < x < a$.

(C) A *connected physical continuum* is a triple $\langle S, R, < \rangle$ such that

(i) conditions A(i)–(iv) are fulfilled;

(ii) if $a, b \in S$, a and b are joined by a simple physical continuum $\langle J, R, < \rangle$, where $J \subset S$ and R and $<$ are the restrictions to J of the homonymous relations in S .

(D) A *physical continuum* is the union of a family of connected physical continua.

In the *Monist* essay of 1898, Poincaré argues that the group of displacements is a physical continuum, but that, since such an entity is "repugnant to reason", we must regard it as a mathematical continuum. He reasons thus: A displacement can be added to itself any number of times. In this way, we obtain different displacements which may be regarded as multiples of the same displacement (if d is a displacement and k a positive integer, we write kd for $d + d + \dots + d$, k times).

Now we soon discover that any displacement whatever can always be divided into two, three, or any number of parts whatever; I mean that we can always find another displacement which, repeated two, three times will reproduce the given displacement. This divisibility to infinity conducts us naturally to the notion of mathematical continuity; yet things are not so simple as they appear at first sight.

We cannot prove this divisibility to infinity, directly. When a displacement is very small, it is inappreciable for us. When two displacements differ very little, we cannot

distinguish them. If a displacement D is extremely small, its consecutive multiples will be indistinguishable. It may happen then that we cannot distinguish $9D$ from $10D$, nor $10D$ from $11D$, but that we can nevertheless distinguish $9D$ from $11D$. If we wanted to translate these crude facts of experience into a formula, we should write

$$9D = 10D, \quad 10D = 11D, \quad 9D < 11D.$$

Such would be the formula of physical continuity. But such a formula is repugnant to reason. It corresponds to none of the models which we carry about in us. We escape the dilemma by an artifice; and for this physical continuity – or, if you prefer, for this sensible continuity, which is presented in a form unacceptable to our minds – we substitute mathematical continuity. Severing our sensations from that something which we call their cause, we assume that the something in question conforms to the model which we carry about in us, and that our sensations deviate from it only in consequence of their crudeness.⁶¹

The formula quoted by Poincaré is “repugnant to reason” only if reason is foolish enough to read ‘=’ as a symbol of identity, which in this context it certainly is not. Hence, Poincaré’s ground for treating the supposedly physical continuum of displacements as an ideal mathematical continuum lacks cogency. I believe, however, that the step into ideality had been taken earlier, when we defined displacements as equivalence classes of locomotions. That definition was predicated on the assumption that *if locomotions A , B are cancelled by the same locomotion, then A is cancelled by any locomotion which cancels B* . But this assumption will not be true if the group of displacements is a physical continuum. For let d be a displacement such that kd is discernible from $(k+2)d$ but not from $(k+1)d$ (where k is a positive integer). An instance of $(k+1)d$ is cancelled then by an instance of kd^{-1} and by an instance of $(k+2)d^{-1}$, but only the former and not the latter will cancel an instance of kd .⁶² As a matter of fact, the italicized assumption is false (see Note 54). But unless we make it, the set of locomotions cannot be partitioned into displacements. Such partition is therefore an idealization, which draws neat boundaries where experience is fuzzy. The group \mathcal{D} obtained by the partition is certainly not a physical continuum and there is no immediately apparent reason why it should be a mathematical one. Poincaré’s ground for conceiving it as such might still be defended however in a modified form: since the set of locomotions which provides the empirical basis for our idealization is indeed a physical continuum, the ideal set of displacements must be thought of as a mathematical continuum. This argument will not do, however, because displacements

are classes of which locomotions are elements.⁶³ When we partition the latter into the former, introducing an unreal definiteness by *fiat*, we must arbitrarily distribute doubtful cases between bordering classes, but the resulting classification need not be a mathematical continuum (think of the partition of the rainbow into seven colours). However, in order to proceed with our outline of Poincaré's doctrine we shall assume hereafter that \mathcal{D} is indeed a mathematical continuum, i.e. a topological manifold. Poincaré evidently assumes as well, without saying so, that the group product $+$ and the mapping $d \mapsto d^{-1}$ ($d \in \mathcal{D}$) are continuous in the agreed topology, so that the group of displacements is a topological group.

The observable features of the physical continuum of locomotions suggest the specific structure of the topological group \mathcal{D} . Let A be a locomotion which transforms state α into state β . Suppose that there is a sensation σ , common to α and β , which remains unchanged throughout A . In practice, of course, a sensation will only remain *roughly* unchanged throughout a locomotion, but we idealize this into perfect constancy. We say then that A fixes σ . Let a be the equivalence class of A . If every element of a fixes some sensation, we call a a rotative displacement or a *rotation*. A non-rotative displacement will be called a *translation*. In order to simplify some statements we agree to regard 0 as being both a rotation and a translation (although not every locomotion in 0 fixes a sensation). A locomotion belonging to a rotation is called a rotative locomotion. We now list some properties of \mathcal{D} which, according to Poincaré, are suggested by experience. I take him to mean that any reasonable partition of locomotions into displacements will roughly agree with the following statements:

(i) Let H be the set of all rotative locomotions which start from a particular state of sense awareness α and fix a particular sensation σ ; let \mathcal{K} be the set of the displacements to which the locomotions in H belong; then \mathcal{K} is a subgroup of \mathcal{D} , which we term a rotative subgroup; \mathcal{D} contains rotative subgroups.

(ii) Any two rotative subgroups of \mathcal{D} have more than just the neutral displacement 0 in common.

(iii) If \mathcal{K} is the intersection of two rotative subgroups of \mathcal{D} , \mathcal{K} is an Abelian subgroup of \mathcal{D} called a *rotative sheaf* (i.e. if $a, b \in \mathcal{K}$, $a + b = b + a$).

(iv) The rotative sheaf determined by two rotative subgroups of \mathcal{D} is contained in an infinity of rotative subgroups of \mathcal{D} .

(v) A rotative sheaf is contained in a maximal Abelian group, the *helicoidal subgroup* of \mathcal{D} determined by the sheaf.

(vi) Any rotation belonging to a rotative subgroup \mathcal{K} is the sum of three rotations $x, y, z \in \mathcal{K}$, belonging to three different rotative sheafs.

(vii) If a displacement a commutes with every element of a rotative subgroup \mathcal{K} , $a \in \mathcal{K}$.

(viii) Any displacement is the sum of two rotations belonging to two given rotative subgroups.

(vi) and (viii) imply that, given three rotative sheaves $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ belonging to a rotative subgroup \mathcal{K} , and three rotative sheaves $\mathcal{K}_4, \mathcal{K}_5, \mathcal{K}_6$ belonging to a second rotative subgroup \mathcal{K}' , any displacement d can be represented as a sum of displacements $h_1 + h_2 + h_3 + h_4 + h_5 + h_6$, with $h_i \in \mathcal{K}_i$.

Poincaré asserts that if \mathcal{D} has these properties it is isomorphic with one of the three classical groups of motions, characteristic of the geometries "of Euclid, . . . Lobatchevsky and Riemann".⁶⁴ \mathcal{D} is isomorphic with the Euclidean group if, and only if, it contains a proper normal subgroup. Experience agrees well with the assumption that the set of translations is such a normal subgroup (in other words, that for every displacement d and every translation t there is a translation t' such that $d + t = t' + d$).

Poincaré's reconstruction of the group of displacements as a Euclidean group of motions is not sufficient to explain the origin of geometry. In the terminology of p.337, we can say that we need to find a transitive faithful realization of the Euclidean group in a topological space homeomorphic with \mathbb{R}^3 . But Poincaré conceives \mathcal{D} as a group acting on the space of sense awareness described on p.342, which, as I said there, is roughly homeomorphic with \mathbb{R}^n ($n \geq 3$).⁶⁵ The action of \mathcal{D} on such space is a mapping assigning to every state of sense awareness α and to every displacement d a state of sense awareness $d\alpha$, which is the transform of α by some locomotion in d , and which we may call the *effect* of d on α . The existence of such a mapping obviously presupposes (i) that for any given state of sense awareness α , each displacement d includes some locomotion which begins at α (this is the 'strong assumption' which I attributed to Poincaré on p.344); (ii) that any two locomotions belonging to the same displacement

d and beginning by the same state α , transform α into one and the same state $d\alpha$. The properties of group action (p.172) preclude \mathcal{D} from acting on the space of sense awareness as it is really given. The latter is a *physical* continuum. In order that \mathcal{D} may act on it it must be idealized into a proper mathematical continuum, which is not just roughly, but exactly, homeomorphic with \mathbf{R}^n . The idealized manifold provides a basis for the realization of \mathcal{D} , which however obviously differs from the realization studied in ordinary geometry, for n is much larger than 3. "How shall we escape the difficulty? Evidently by replacing the group which is given us, together with its form and its material, by another isomorphic group, the material of which is simpler."⁶⁶ We shall not dwell on the allegedly psychological descriptions through which Poincaré tries to show how this simplification is suggested by human experience. They are lengthy and not altogether clear.⁶⁷ I propose instead a construction which I regard as essentially equivalent to his. I proceed in two steps: I outline first a mathematical argument which I deem necessary to support the mathematical conclusions of his psychological inquiry; this will enable me to explain then, simply and concisely, what I take to be the gist of his conclusions. We assume, as before, that \mathcal{D} is isomorphic with the Euclidean group of motions. \mathcal{D} is therefore a topological group homeomorphic with \mathbf{R}^6 (p.174). We denote the subgroup of translations by \mathcal{T} . \mathcal{T} is a normal subgroup homeomorphic with \mathbf{R}^3 . Let Σ be the set of all the rotative sub-groups of \mathcal{D} . Let \mathcal{H}_0 be a given element of Σ . Then each $\mathcal{H} \in \Sigma$ is equal to $t\mathcal{H}_0t^{-1}$, for some $t \in \mathcal{T}$. Moreover, the mapping $f: t \mapsto t\mathcal{H}_0t^{-1}$ is a bijection of \mathcal{T} onto Σ . Let $Z \subset \Sigma$ be open in Σ whenever $f^{-1}(Z)$ is open in \mathcal{T} . With this topology, Σ is homeomorphic with \mathcal{T} , and hence with \mathbf{R}^3 . Let $d \in \mathcal{D}$. Let f_d denote the mapping $\mathcal{H} \mapsto d\mathcal{H}$ ($\mathcal{H} \in \Sigma$), where $d\mathcal{H}$, as on p.338, is the left coset of \mathcal{H} by d . f_d is a permutation or transformation of Σ (p.337). It is not difficult to see that $f: d \mapsto f_d$ ($d \in \mathcal{D}$) is a realization of \mathcal{D} in Σ . We see at once that in such realization every $\mathcal{H} \in \Sigma$ is its own stability group.⁶⁸ Consequently, f is similar to the standard realization of \mathcal{D} in \mathcal{D}/\mathcal{H} described on p.338. f is therefore transitive. Moreover, f is faithful, because no rotative subgroup of \mathcal{D} contains a proper normal subgroup. We have found thus a faithful, transitive and, moreover, immanent realization of \mathcal{D} in a space homeomorphic with \mathbf{R}^3 . I fear, however, that many a reader will look down upon it as just another piece of algebraic abracadabra, incapable of giving an insight into the origin of geometry.

Let us therefore inject some psychological blood into it. Choose an arbitrary state of sense-awareness α . If \mathcal{H} is a rotative subgroup of \mathcal{D} , we let \mathcal{H}_α stand for the set $\{h\alpha \mid h \in \mathcal{H}\}$, where $h\alpha$ denotes, as before, the effect of the rotation h on the state α . According to our definition of a rotative subgroup (p.348, (i)) there will be a unique (idealized) sensation which is common to all the elements of \mathcal{H}_α . We call it the sensation fixed by \mathcal{H} at α and denote it by $\sigma(\mathcal{H}, \alpha)$. Let Σ_α be the set of all sensations fixed at α by some rotative subgroup $\mathcal{H} \in \Sigma$. The definition of a rotative subgroup also implies that $\sigma(\mathcal{H}, \alpha) = \sigma(\mathcal{H}', \alpha)$ if, and only if, $\mathcal{H} = \mathcal{H}'$. Consequently $g: \mathcal{H} \mapsto \sigma(\mathcal{H}, \alpha)$ is a bijective mapping of Σ onto Σ_α . By stipulating that $Z \subset \Sigma_\alpha$ is open whenever $g^{-1}(Z)$ is open, we make Σ_α into a topological space homeomorphic with \mathbb{R}^3 . Let f denote once more the realization of \mathcal{D} in Σ which was described above. The mapping $d \mapsto gfg^{-1}(d \in \mathcal{D})$ is a realization of \mathcal{D} in Σ_α , which is clearly similar to f . Since α is arbitrary, the realization does not depend on its particular nature. Σ_α may stand, therefore, for the pure three-dimensional space of Euclidean geometry.

Such is the genesis of geometry according to Poincaré. Experience plays in it an essential but not a decisive role. Empirical data have been repeatedly idealized to fit our schemata: first, in the constitution of the displacement group and the refinement of the space of sense-awareness into a topological space on which this group can act; then, in order to ensure that the said group is one of the classical groups of motions and indeed is none other than the Euclidean group. The process of idealization has so divorced geometry from sense-experience that, though we can very well *conceive* the infinite, isotropic, homogeneous, mathematically continuous space of geometry, we are unable to *visualize* it by any stretch of the imagination. "We cannot represent to ourselves objects in geometrical space, but can merely reason upon them as if they existed in that space."⁶⁹ Our idealizations follow, so to speak, the line of least resistance. They are, Poincaré insists, *suggested* by experience. But they are nonetheless free. A different set of idealizations would yield a clumsier, more contorted, but equally admissible framework for the registration and communication of scientific facts. "Transported to another world, we might undoubtedly have a different geometry, not because our geometry would have ceased to be true, but because it would have become less convenient than another."⁷⁰ Might not a broadened experience eventually lead us to substitute a different geometry for

our Euclidean system? In 1891, Poincaré starkly denied it: “Euclidean geometry is and will remain the most comfortable one.”⁷¹ But in 1898, he readily granted that “if our experiences should be considerably different, the geometry of Euclid would no longer suffice to represent them conveniently, and we should choose a different geometry.”⁷²

4.4.6 *The Definition of Dimension Number*

Poincaré is one of the Founding Fathers of topology.⁷³ In his philosophical writings, he repeatedly stressed the importance of the “qualitative geometry”, *analysis situs*, underlying the more familiar “quantitative geometry”.⁷⁴ In this discipline, he observes, two figures are equivalent whenever any of them can be made to take the shape of the other by a process of continuous deformation.⁷⁵ When viewed topologically, space appears cohesive like rubber, but not rigid like glass. It is a *continuum amorphe*, as Poincaré, anticipating Grünbaum, is pleased to call it.⁷⁶ One might be tempted to think that even if the choice of a metric geometry is a conventional matter, the underlying “formless continuum” presupposed by such geometry is somehow imposed on us by experience or by our mental make-up. But Poincaré will have none of it. In particular, “the fundamental proposition of *analysis situs*”, namely that *space has three dimensions*,⁷⁷ is, according to him, no less conventional than the definition of distance.

Poincaré tried twice to prove that the number of dimensions of physical space is fixed by convention, in 1903 and in 1912.⁷⁸ The substance of his argument was given in Section 4.4.5 (p.350). The only space that may be said to be imposed on us by the nature of sense data or by our psychophysiological constitution is the space of sense awareness – viewed of course as a physical, not as a mathematical continuum. This space has much more than three dimensions; according to Poincaré it has as many dimensions as we have independently excitable nerve-tips. The reduction of spatial dimensions to three is the outcome of a process of idealization and simplification involving several free decisions.

In our discussion of this matter in Section 4.4.5, we did not attempt to probe into the nature of spatial dimensions. We regarded a topological space as n -dimensional whenever it was globally or at least locally homeomorphic with \mathbf{R}^n . This definition of dimension number would be ambiguous and hence useless if \mathbf{R}^n could be mapped homeomorphically into \mathbf{R}^{n+p} for some $p \neq 0$. Brouwer proved in 1911

that this is impossible. Brouwer's proof justifies the use of the preceding concept of dimension number in certain contexts, but it throws no light on the structural property shared by \mathbb{R}^n and its homeomorphic images, by virtue of which they are said to be n -dimensional. As early as 1893, Poincaré had proposed a characterization of this property which, though unsatisfactory, provided one of the starting points of modern topological dimension theory.⁷⁹

Poincaré does not define dimension, but dimension number, that is, a correspondence assigning a characteristic positive integer to each continuum. The correspondence is defined recursively: Poincaré determines which continua have dimension 1, and stipulates that an arbitrary continuum has dimension n when certain subcontinua of it have dimension $n - 1$. Poincaré's definition of dimension number is motivated by a familiar fact, which any child might state thus: in order to divide a thread into two separate parts it is enough to cut it at one or more points; in order to divide a leaf of paper you must cut it along one or more lines; in order to divide a solid body, you must cut it across one or more surfaces. The same fact was expressed in more general terms by Euclid when he said that the boundaries of bodies are surfaces, the boundaries of surfaces are lines and the boundaries of lines are dimensionless points.⁸⁰ In his paper of 1893 on the mathematical continuum and again in his paper of 1903 on space and its three dimensions, Poincaré defines an n -dimensional *physical* continuum in a manner that is clearly inspired by the said fact.

The reader will recall our axiomatic characterization of physical continua on p.345f. Poincaré seeks to define the dimension number of a connected physical continuum by means of the idea of a cut which divides it or disconnects it. He describes a cut C in a connected physical continuum K as an arbitrary subset of K , which can therefore consist of one or more distinct and separate elements of K or of one or more subcontinua of K . We might feel inclined to say that a cut C in a connected physical continuum K divides or disconnects K if removal of C deprives K of its structure as a connected physical continuum, (specifically, if $K - C$ does not satisfy the condition (C)(ii) prescribed on p.346 for any connected physical continuum S). We could then define a one-dimensional connected physical continuum K as one which is disconnected by a cut consisting of separate points, i.e. by a subset of K whose elements do not coalesce with each other to form a continuum. A connected physical continuum K would be

said to be n -dimensional if n is the least positive integer such that K is disconnected by a cut consisting of a collection of one or more separate $(n - 1)$ -dimensional connected physical continua. Though seemingly adequate, this characterization cannot satisfy us. Simple physical continua, which we certainly wish to regard as one-dimensional, may remain connected after the removal of a finite and even of a countable collection of separate elements. For any such element E belonging to a simple physical continuum K is indiscernible from elements E_1, E_2, \dots which are not removed from K by the excision of E . If, as is usual, some of these elements E_1, E_2, \dots are also indiscernible from each other, removal of E will not break the connectedness of K . Similar considerations apply to non-simple physical continua. Take, for instance, the continuum of sounds, ordered by pitch and intensity. One would naturally expect it to be two-dimensional. One would also regard the continuum of all sounds of a given pitch as a one-dimensional subcontinuum of it. Yet the continuum of sounds is not disconnected, in the sense defined above, by the continuum of all sounds of a definite pitch. For every sound S of the chosen pitch there exist sounds S' and S'' such that S, S' and S'' are indiscernible from one another but S' is also indiscernible from a sound S_L of perceptibly lower pitch than S , and S'' is indiscernible from a sound S_H of perceptibly higher pitch than S . S' and S'' cannot therefore be said to have the same pitch as S and are not removed from the continuum of sounds together with all the sounds of that pitch. Yet, being indiscernible from one another, they can represent the excised pitch, and thus preserve the connectedness of the continuum of sounds. This example shows that a definition of dimension number of physical continua, based on the idea of a cut, cannot ignore the fact that any element of a physical continuum K has a 'fringe' formed by all the elements of K which are indiscernible from it. We define the fringe of a subset $S \subset K$ as the set $S^* = \{x \mid x \in K \text{ and } x \text{ is indiscernible from some element of } S\}$. We define, as before, a cut in a connected physical continuum K as an arbitrary subset of K . If $S \subset K$ and the continuum structure of K is determined, as on p.346, by two binary relations R and $<$, we designate by $\langle S, R, < \rangle$ the structure determined on S by the restriction of R and $<$ to S . Let C be a cut in a connected physical continuum K , structured by relations R and $<$. Let C^* be the fringe of C . We say that K is disconnected by C if and only if $\langle K - C^*, R, < \rangle$ is not a connected physical continuum.

C will be said to be 0-dimensional if it consists of a non-empty collection of elements of K , which does not contain a subcontinuum of K . C will be said to be n -dimensional if it can be represented as a collection of n -dimensional connected physical continua but it cannot be represented as a collection of $(n + p)$ -dimensional connected physical continua for any positive integer p . We can now define: A connected physical continuum K is n -dimensional if, and only if, n is the least positive integer such that K is disconnected by an $(n - 1)$ -dimensional cut in K . A physical continuum K is n -dimensional if n is the largest positive integer such that K can be represented as a family of n -dimensional connected physical continua. These definitions, as the reader can verify, agree substantially with those given by Poincaré.⁸¹

In the two papers of 1893 and 1903 that we mentioned above, Poincaré attempts to show how the concept of a mathematical continuum is built by idealization from that of a physical continuum. (See p.346.) However, in neither of them does he extend to mathematical continua his new definition of dimension number, but is content to repeat the commonplace that a mathematical continuum is n -dimensional if n real-valued coordinate functions are required for labelling its points.⁸² Yet the application of the new definition to mathematical continua is immediate, as Poincaré showed in 1912, in the paper entitled "Why space has three dimensions?" There, he first defines the dimension number of mathematical continua using the idea of a cut, and then adds that the proposed definition can also be extended to physical continua. Since each element of a mathematical continuum is distinct and discernible from the others, the definition of dimension number for such continua need not use the concept of a fringe. Poincaré does not state exactly what he means by a mathematical continuum. His notion of it is akin to but probably narrower than that of a topological space. (See p.360.) Assuming that any mathematical continuum in Poincaré's sense is also a topological space, we can define a path in a mathematical continuum K as a continuous mapping of an interval of the real line R into K . In particular, a mapping c of a closed interval $[a, b]$ into K will be said to *join* points x and y in K if $x = c(a)$ and $y = c(b)$. We say that K is a path-connected mathematical continuum or a pmc if each pair of elements of K is joined by a path in K . For simplicity's sake, we agree to describe any singleton contained in a pmc K , i.e. any subset of K

which has only a single member, as a '0-dimensional subcontinuum of K '. If n is any positive integer, an ' n -dimensional subcontinuum of K ' is simply a subset of K which is a pmc in its own right (with the structure induced in it by the continuum structure of K) and is n -dimensional according to the definition we shall now give. These agreements suffice to make sense of the following recursive definition:

A pmc K is n -dimensional (n a positive integer) if there is a sequence $X = (X_1, X_2, \dots)$ of $(n - 1)$ -dimensional subcontinua of K , such that for some pair a, b of elements of K , not belonging to the union of X , the range of every path in K joining a and b intersects the union of X .

The dimension number of an arbitrary mathematical continuum can of course be determined in an obvious manner by the dimension number of its path-connected components.

According to the definition we have given, \mathbb{R}^n and every topological space which is globally homeomorphic to \mathbb{R}^n are n -dimensional. The definition certainly throws light on the structural properties shared by such spaces by virtue of which they are said to be n -dimensional. On the other hand, the definition leads to unreasonable results if applied to topological spaces which are only locally homeomorphic to \mathbb{R}^n (i.e. to spaces in which each point has a neighbourhood homeomorphic to \mathbb{R}^n). Consider, for example, the surface S generated in Euclidean 3-space by the revolution of two intersecting straight lines about the bisector of one of the two pairs of vertically opposite angles formed by them. S can be endowed in a fairly obvious way with a topology by virtue of which it is locally homeomorphic to \mathbb{R}^2 .⁸³ Yet S , with this topology, is not two-dimensional according to our definition, but one-dimensional, since the intersection P of the generators of S is a point of S such that the repetitious sequence $\langle \{P\}, \{P\}, \{P\}, \dots \rangle$ of 0-dimensional subcontinua of S satisfies the description of sequence S in our definition of dimension number.

The preceding example is mentioned by L.E.J. Brouwer in his paper on the natural concept of dimension (1913) as a telling objection against Poincaré's definition of dimension number. Brouwer proposed in that paper a new definition which was later perfected by Urysohn (1922, 1925, 1926) and Menger (1923, 1924). The concept thus obtained is known in contemporary dimension theory as inductive dimension. Let S be any topological space. If S' is a subset of S we

agree to regard S' as a topological space whose open sets are the intersections of S' with the open sets of S . With this topology, S' is a subspace of S . In particular, the empty set \emptyset is a subspace of every space. We agree that \emptyset (and only \emptyset) has inductive dimension -1 . A topological space S is said to have inductive dimension n if n is the least non-negative integer such that for each point $x \in S$ and each open neighbourhood V of x there exists an open neighbourhood U of x such that $U \subset V$ and the boundary of U has inductive dimension $n - 1$.⁸⁴ Having been defined exclusively by means of general topological concepts, inductive dimension is evidently a topological invariant; in other words, any two topologically equivalent (homeomorphic) spaces have the same inductive dimension. Inductive dimension possesses two more features which one would naturally expect any satisfactory concept of dimension number to share: (i) the inductive dimension of \mathbb{R}^n and of every space locally homeomorphic to \mathbb{R}^n is n ; (ii) if S is an arbitrary topological space and S' is a subspace of S , the inductive dimension of S' is less than, or equal to, the inductive dimension of S . Unfortunately, as far as we can tell, inductive dimension does not lend itself to the development of a rich theory. That is probably the main reason why mathematicians have proposed other concepts of dimension number for topological spaces. The two most important of them – namely the so-called “large” inductive dimension defined in Note 84 and the covering dimension characterized below – agree with inductive dimension on an important class of spaces⁸⁵ but not on all. These two concepts are, of course, topological invariants and they both share the above mentioned feature (i). They can also be shown to possess feature (ii) if S is assumed to be a totally normal space.⁸⁶

In 1911, commenting on Brouwer’s up to that time unpublished proof that \mathbb{R}^n cannot be mapped homeomorphically onto \mathbb{R}^{n+p} unless

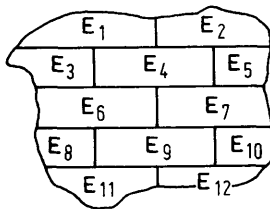


Fig. 22.

$p = 0$, Henri Lebesgue suggested a completely different approach to the concept of dimension number.⁸⁷ He observed that if D is a finite open connected subset of \mathbb{R}^n its closure \bar{D} can be covered by a finite collection of closed sets E_1, \dots, E_k such that some points of \bar{D} belong to $n + 1$ of these sets but no point of \bar{D} belongs to more than $n + 1$ of them. (Lebesgue (1911), p.166.) Lebesgue's remark is illustrated in Fig. 22 for the case $n = 2$. In 1933, Čech introduced a concept of dimension number applicable to general topological spaces which is based on Lebesgue's remark. This is now known as *covering dimension*. In order to characterize it, we need a few new terms. A cover of a set S is a family of subsets of S such that each element of S belongs to at least one of the members of the family. The cover is said to be 'of order equal to or less than n ' if each point in S belongs to at most $n + 1$ members of the cover. If K and K' are two covers of S and every member of K is contained in some member of K' , K is said to be a refinement of K' . A cover of a topological space or a refinement of such a cover are said to be open if they consist of open sets. We now define the *covering dimension* of a topological space S as the least integer n such that every finite open cover of S has a finite open refinement of order equal to or less than n .

APPENDIX

1. MAPPINGS

Let A and B be sets, such that at least B is non-empty. A *mapping* $f: A \rightarrow B$ assigns to each element x of A a unique element $f(x)$ of B . We denote f by $x \mapsto f(x)$. A is the *domain* of f ($\text{dom } f$), B its codomain. $f(x)$ is the *value* of f at *argument* x . The collection of all values of f is its *range* or *image* ($\text{im } f$). If the codomain of f equals its range, f is said to be *surjective* or a *surjection*. The collection of all arguments at which f takes a given value is the *fibre* of f over that value. If each fibre of f is a singleton (i.e. if it contains only one element of A), f is said to be *injective* or an *injection*. If A is a subset of B , the mapping $x \mapsto x$ is called the *canonical injection* of A into B . A mapping both injective and surjective is said to be *bijective* or a *bijection*. If f is bijective, its *inverse* $f^{-1}: B \rightarrow A$ is the mapping $f(x) \mapsto x$. If U is any subset of A and V is any subset of B we denote by $f(U)$ the set of all values of f at arguments belonging to U and by $f^{-1}(V)$ the set of all arguments at which f takes values belonging to V (note that the latter set may well be empty). The restriction of f to U , denoted by $f|_U$, is the mapping $g: U \rightarrow B$ which is such that $g(x) = f(x)$ for every x in U . If $f: A \rightarrow B$ and $g: B \rightarrow C$ are mappings, the *composite mapping* $g \cdot f: A \rightarrow C$ assigns the value $g(f(x))$ to each x in A .

2. ALGEBRAIC STRUCTURES. GROUPS

Let S be a non-empty set. Let n denote the set of the first n natural numbers. An ordered n -tuple or n -list of elements of S is a mapping $n \rightarrow S$. We denote it by $\langle a_0, a_1, a_2, \dots, a_{n-1} \rangle$, where a_j stands for the value of the list at j . The collection of all such lists is designated by S^n . An n -ary *operation* on S is a mapping $S^n \rightarrow S$. An *algebraic structure* is an $(r+1)$ -list $\langle S, f_1, \dots, f_r \rangle$, where f_i is an n_i -ary operation on the set S ($1 \leq i \leq r$). S is the structure's *underlying set*. One often designates an algebraic structure by the name of its underlying set, or vice versa.

A *group* is an algebraic structure $\langle G, f \rangle$, where f is a binary operation or *product* on set G that fulfils the following requirements:

(i) f is *associative*, that is, $f(x, f(y, z)) = f(f(x, y), z)$ for any x, y, z in G .

(ii) for every x and y in G there are elements u and v in G such that $f(x, u) = f(v, x) = y$.

These requirements imply that there exists a *neutral element* in the group, that is, an element $e \in G$ such that $f(e, x) = f(x, e) = x$ for every $x \in G$; and that for every $x \in G$ there exists a unique *inverse* $x^{-1} \in G$, such that $f(x, x^{-1}) = f(x^{-1}, x) = e$.

Let $H \subset G$. Let $i: H \rightarrow G$ be the canonical injection. If $\langle H, g \rangle$ is a group and the restriction of f to H^2 equals the composite mapping $i \cdot g$, $\langle H, g \rangle$ is a subgroup of $\langle G, f \rangle$. If this condition is satisfied and for every $a \in H, b \in G, f(b, f(a, b^{-1}))$ belongs to H , $\langle H, g \rangle$ is said to be a *normal* subgroup of $\langle G, f \rangle$. Note that both $\langle G, f \rangle$ itself and the group whose underlying set consists of the neutral element $e \in G$ alone are subgroups of $\langle G, f \rangle$ according to our definition. They are referred to as improper subgroups while all other subgroups of $\langle G, f \rangle$ are said to be *proper*.

Let $\langle G, g \rangle$ and $\langle H, h \rangle$ be groups. A mapping $f: G \rightarrow H$ is a group *homomorphism* if for any x and y in $G, h(f(x), f(y)) = f(g(x, y))$. If f is bijective, it is a group *isomorphism*. Two groups are said to be *isomorphic* if there is a group isomorphism that maps the underlying set of one onto that of the other.

If x, y belong to a group $\langle G, f \rangle$, one usually writes xy for $f(x, y)$.

Further information on groups and other algebraic structures can be obtained from any good textbook of algebra, such as S. MacLane and G. Birkhoff, *Algebra* (New York: Macmillan, 1967). The reader will do well to take a look at the definitions of rings, fields, modules, vector spaces and lattices if he is not already familiar with them.

3. TOPOLOGIES

Let S be any set and T a collection of subsets of S . T is a *topology* on S and the pair $\langle S, T \rangle$ is a *topological space* if the following four conditions are satisfied: (i) $S \in T$; (ii) the empty set $\emptyset \in T$; (iii) the union of every collection of members of T belongs to T ; (iv) the intersection of any two members of T belongs to T . A collection B of subsets of S is a *base* of the topology T if every member of B belongs to T and every non-empty member of T is a union of members of B .

A member of T is called an *open set*, its complement (relative to S) is a *closed set*. The elements of S are its *points*. If $x \in S' \subset S$ and there is an open set S'' such that $x \in S'' \subset S'$, S' is said to be a *neighbourhood* of x . If S' itself is open it is called an open neighbourhood. Let $W \subset S$. The set $\{x \mid x \text{ has a neighbourhood entirely contained in } W\}$ is called the *interior* of W or $\text{Int } W$. W is open if, and only if, $\text{Int } W = W$. The set $\{x \mid \text{each neighbourhood of } x \text{ contains an element of } W\}$ is called the *closure* of W and is denoted by \bar{W} . W is closed if, and only if, $\bar{W} = W$. Let Y be the complement of W (relative to S). The intersection of \bar{W} and \bar{Y} is the *boundary* of W (and of Y).

If T and T' are two topologies on the same set S and $T \subset T'$, T is *weaker* or *coarser* than T' (and T' is *stronger* or *finer* than T).

A topological space $\langle S, T \rangle$ is *connected* unless it is the union of two non-empty disjoint open sets.

Let S be any set. A collection of sets whose union contains S is called a *cover* of S . If K and K' are two covers of S and $K' \subset K$, K' is a *subcover* of K . If T is a topology on S , T is obviously a cover of S . A subcover of T is called an open cover of S . A topological space $\langle S, T \rangle$ is *compact* if every open cover of S has a finite subcover.

Let $\langle S, T \rangle$ be a topological space. Let S' be any subset of S and let T' designate the set $\{S' \cap W \mid W \in T\}$. It can be easily verified that T' is a topology on S' , known as the topology induced on S' by T . $\langle S', T' \rangle$ is a topological *subspace* of $\langle S, T \rangle$.

Let $\langle S, T \rangle$ and $\langle S', T' \rangle$ be topological spaces. A mapping $f: S \rightarrow S'$ is said to be *open* if f maps every open set of S onto an open set of S' ; it is said to be *continuous* if, for every open set Y of S' the set $\{x \mid f(x) \in Y\}$ is an open set of S . An open and continuous bijective mapping $f: S \rightarrow S'$ is called a *homeomorphism*. Two topological spaces are homeomorphic if there is a homeomorphism that maps the underlying set of one into that of the other.

Further information on topological spaces can be gathered from J.R. Munkres, *Topology*.

4. DIFFERENTIABLE MANIFOLDS

I assume that the reader is familiar with the rudiments of linear algebra and analysis. (Behnke *et al.*, *Fundamentals of Mathematics*, Vol. I, Chapter 3 and Vol. III, Chapters 1, 2, 3, 4 and 5 contain all the information needed.) The following notes summarize some basic

definitions and results concerning differentiable manifolds. To make some matters simpler we shall aim at less than maximum generality. For the sake of brevity some concepts (e.g. the Lie bracket) will not be introduced in the most natural manner. The reader may turn for further details to the books by Spivak (CIDG), Matsushima (DM) and Malliavin (GDI) mentioned in the Reference list.

Let us recall that a mapping f of an open subset of \mathbb{R}^n into \mathbb{R}^m is said to be of class C^k at a point P in its domain if all its partial derivatives of order k exist and are continuous at P . If this is true of every integer k , f is said to be of class C^∞ at P . If f can be expanded into a power series on a neighbourhood of P , f is said to be analytic at P . The qualification 'at P ' is dropped if the aforesaid conditions hold good for every point in $\text{dom } f$.

Let S be a set. An n -dimensional real-valued *chart* of S is a bijective mapping of a subset of S onto an open subset of \mathbb{R}^n . Let f and g be two n -dimensional real-valued charts of S . f and g are C^k -compatible if the sets $f(\text{dom } g \cap \text{dom } f)$ and $g(\text{dom } f \cap \text{dom } g)$ – that is, the ranges of the restrictions of f and g to the intersection of $\text{dom } f$ and $\text{dom } g$ – are open in \mathbb{R}^n , and the restrictions of the composite mapping $g \cdot f^{-1}$ to the former set and of $f \cdot g^{-1}$ to the latter are mappings of class C^k . An n -dimensional real-valued C^k -atlas of S is a collection A of n -dimensional real-valued charts of S such that (i) any two charts in A are C^k -compatible, and (ii) the collection of the domains of the charts in A is a cover of S . An n -dimensional real C^k -differentiable manifold is a pair $\langle S, A \rangle$, where S is a set and A is an n -dimensional real-valued C^k -atlas of S . The set of all conceivable n -dimensional real-valued charts of S which are C^k -compatible with each chart in A is the maximal C^k -atlas determined by A and will be denoted by A_{\max} . The weakest topology on S which makes every chart in A_{\max} into a continuous mapping is the natural topology of the differentiable manifold $\langle S, A \rangle$. If $P \in S$ and x is a chart defined on a neighbourhood of P , x is said to be a chart at P .

An n -dimensional complex differentiable manifold is defined analogously. (Substitute *complex* for *real* and \mathbb{C} for \mathbb{R} throughout the foregoing paragraph.)

Henceforth, we shall consider only n -dimensional real C^∞ -differentiable manifolds with a natural Hausdorff topology. We call them simply n -manifolds. (Recall that a topological space is Hausdorff if for every pair of points P and Q there is a neighbourhood N_P of P and

a neighbourhood N_Q of Q such that $N_P \cap N_Q = \emptyset$.) We regard \mathbb{R}^n (for every positive integer n) as an n -manifold with the differentiable structure determined by the atlas whose only chart is the identity mapping $x \mapsto x$, defined on all \mathbb{R}^n .

Let M be an m -manifold with an atlas A and let N be an n -manifold with an atlas B . We define the *product manifold* $M \times N$ by the following stipulations: (i) the underlying set of $M \times N$ is the Cartesian product of the underlying sets of M and N ; (ii) for each chart x in A_{\max} and each chart y in B_{\max} the maximal atlas of $M \times N$ contains an $(m+n)$ -dimensional chart z which assigns to each pair (P, Q) in $\text{dom } x \times \text{dom } y$ the $(m+n)$ -tuple $(x(P), y(Q))$.

If M is an m -manifold and N is an n -manifold, a mapping $f: M \rightarrow N$ is said to be C^k -differentiable at a point $P \in M$ if for any chart x at P and any chart y at $f(P)$ the composite mapping $y \circ f \circ x^{-1}$ is of class C^k at $x(P)$. (The reader should verify that this property does not depend on the choice of x and y .) A C^k -differentiable bijection is called a *C^k -diffeomorphism* if its inverse is C^k -differentiable. Two manifolds are said to be *C^k -diffeomorphic* if there exists a C^k -diffeomorphism of one onto the other.

Let M be an n -manifold and let $f: M \rightarrow \mathbb{R}$ be C^1 -differentiable at a point $P \in M$. If x is a chart at P , the mapping $f \circ x^{-1}$ is defined and has continuous first-order partial derivatives on a neighbourhood U of $x(P)$. Let this mapping be denoted by F . F maps an open subset of \mathbb{R}^n into \mathbb{R} . We introduce the following notation:

$$F_{,i}(a) = \lim_{h \rightarrow 0} \frac{F(a_1, \dots, a_i + h, \dots, a_n) - F(a)}{h},$$

$$\frac{\partial f}{\partial x^i} = F_{,i} \circ x, \quad (1)$$

(where $a = (a_1, \dots, a_n)$ is an arbitrary point of U).

Let M be an n -manifold and let (a, b) be an open interval of \mathbb{R} . A C^k -differentiable mapping $c: (a, b) \rightarrow M$ is called a *C^k -path* in M . We can also consider C^k -paths defined on a closed interval $[a, b]$ provided that we regard them as the restrictions to $[a, b]$ of C^k -paths defined on some open interval that contains $[a, b]$. If $c: (a, b) \rightarrow M$ is a C^k -path and $h: \mathbb{R} \rightarrow \mathbb{R}$ is a homeomorphism, the mapping c' given by $c' = c \circ h^{-1}$ is a *reparametrization* of c by h .

Let M be an n -manifold. Consider a C^1 -path c in M , defined on an

open interval (a, b) such that $a < 0 < b$. Let $c(0) = P$. A path fulfilling these conditions will be called 'a suitable path through P'. We now define a ring of real valued functions which we shall denote by $\mathcal{F}^k(P)$: (i) for every neighbourhood U of P (in M), the underlying set of $\mathcal{F}^k(P)$ includes every C^k -differentiable mapping $f: U \rightarrow \mathbb{R}$; (ii) the ring operations are given by the following rules:

$$\begin{aligned}(f + g)(Q) &= f(Q) + g(Q), \\ fg(Q) &= f(Q)g(Q),\end{aligned}\tag{2}$$

for every $f, g \in \mathcal{F}^k(P)$ and every $Q \in \text{dom } f \cap \text{dom } g$. If α is any real number, we let α denote the constant function $M \rightarrow \mathbb{R}; Q \mapsto \alpha$ which obviously belongs to $\mathcal{F}^k(P)$ for all $P \in M$ and every positive integer k . t shall designate the identity mapping of \mathbb{R} onto itself. The *vector $\dot{c}(0)$ tangent to path c at $P = c(0)$* is the function $\mathcal{F}^1(P) \rightarrow \mathbb{R}$ which maps each $f \in \mathcal{F}^1(P)$ on the real number $(\partial(f \cdot c)/\partial t)(0)$. It is clear that for every $f, g \in \mathcal{F}^1(P)$ and every $\alpha, \beta \in \mathbb{R}$,

$$\dot{c}(0)(\alpha f + \beta g) = \alpha \dot{c}(0)(f) + \beta \dot{c}(0)(g).\tag{3}$$

The set of all vectors tangent at P to suitable paths through P is given the structure of a real vector space by the following rules: For every v, w belonging to that set, every $f \in \mathcal{F}^1(P)$ and every $\alpha \in \mathbb{R}$,

$$\begin{aligned}(v + w)(f) &= v(f) + w(f), \\ (\alpha v)(f) &= \alpha v(f).\end{aligned}\tag{4}$$

Endowed with this structure, the said set is called *the tangent space of M at P* and is denoted by $T_P(M)$.

Let x be a chart at $P \in M$, where M is, as before, as n -manifold. We denote by x^i the i th 'coordinate function' $p_i \cdot x$, where p_i is the i th projection function of \mathbb{R}^n (p_i assigns to each element (a_1, a_2, \dots, a_n) of \mathbb{R}^n its i th term a_i). Let

$$\begin{aligned}c_i(u) &= x^{-1}(u_1, \dots, u_n), \\ u_j &= u\delta_j^i + x^j(P).\end{aligned}\tag{5}$$

(Here, δ_j^i is the 'Kronecker delta' which equals 1 if $i = j$ and equals 0 otherwise.) Equations (5) evidently define n suitable paths through P , c_1, c_2, \dots, c_n . Now, for any $f \in \mathcal{F}^1(P)$

$$\dot{c}_i(0)(f) = \frac{\partial(f \cdot c_i)}{\partial t}(0) = \lim_{h \rightarrow 0} \frac{1}{h} (f \cdot c_i(h) - f \cdot c_i(0))$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} (f \cdot x^{-1}(x^1(P), \dots, x^i(P) + h, \dots, x^n(P)) \\
 &\quad - f \cdot x^{-1}(x^1(P), \dots, x^n(P))) \\
 &= \frac{\partial f}{\partial x^i}(P).
 \end{aligned} \tag{6}$$

It is therefore reasonable to write $\partial/\partial x^i|_P$ for $\dot{c}_i(0)$. If $\alpha_1, \dots, \alpha_n$ are n real numbers such that at least one of them, say α_j , is not zero, the linear combination $\sum_i \alpha_i \partial/\partial x^i|_P$ is not identically zero, since it is equal to α_j at x^j . The family $(\partial/\partial x^i|_P)_{1 \leq i \leq n}$ is therefore a free family of vectors in $T_P(M)$. It can be shown that it is a basis of $T_P(M)$, the *canonical basis* relative to chart x . $T_P(M)$ is therefore an n -dimensional vector space. Associated with it are its dual, the cotangent space $T_P^*(M)$ of real valued linear functions on $T_P(M)$ and the whole array of spaces of covariant, contravariant and mixed tensors of all orders which the reader will find defined in any textbook of linear algebra. Hereafter, if v belongs to a tangent space of some manifold M and f belongs to its dual space, we write $(f \cdot v)$ instead of $f(v)$.

If M and M' are manifolds and $\varphi: M \rightarrow M'$ is C^k -differentiable at $P \in M$, ($k > 0$), φ determines a mapping of $\mathcal{F}^1(\varphi(P))$ into $\mathcal{F}^1(P)$ which we shall denote by $\varphi^*\mathcal{F}$, and a mapping of $T_P(M)$ into $T_{\varphi(P)}(M')$ which we shall denote by φ_{*P} . These mappings are defined for every $f \in \mathcal{F}^1(\varphi(P))$ and every $v \in T_P(M)$ by:

$$\begin{aligned}
 \varphi^*\mathcal{F}(f) &= f \cdot \varphi, \\
 \varphi_{*P}(v) &= v \cdot \varphi^*\mathcal{F}.
 \end{aligned} \tag{7}$$

Consequently, $\varphi_{*P}(v)$, a vector tangent to M' at $\varphi(P)$, maps a function f in $\mathcal{F}^1(\varphi(P))$ on the same real number assigned by v , a vector tangent to M at P , to the function $f \cdot \varphi$, which belongs to $\mathcal{F}^1(P)$. If x is a chart at P and y is a chart at $\varphi(P)$,

$$\varphi_{*P} \left(\frac{\partial}{\partial x^i} \Big|_P \right) = \sum_j \frac{\partial(y^j \cdot \varphi)}{\partial x^i} \Big|_P \frac{\partial}{\partial y^j} \Big|_{\varphi(P)}. \tag{8}$$

Let c be a C^k -path in an n -manifold M , defined on a neighbourhood of 0. Equation (8) implies that if x is a chart at $c(0)$, then

$$c_{*0} \left(\frac{\partial}{\partial t} \Big|_0 \right) = \sum_j \frac{\partial(x^j \cdot c)}{\partial t} \Big|_0 \frac{\partial}{\partial x^j} \Big|_{c(0)} = \dot{c}(0). \tag{9}$$

This result motivates the following terminological and notational convention: if c is any path in M , defined on an arbitrary open interval (a, b) , and $a < u < b$, we call $c_{*u}(\partial/\partial t|_u)$ the vector tangent to c at $c(u)$ and we denote it by $\dot{c}(u)$. It will be readily seen that if M and M' are manifolds and $\varphi: M \rightarrow M'$ is C^k -differentiable and c is a C^k -path in M , $\varphi \cdot c$ is a C^k -path in M' . Let $P = c(u)$ for some real number u in the domain of c . Then $\varphi_{*P}(\dot{c}(u))$ is the vector tangent to $\varphi \cdot c$ at $\varphi(P)$.

Let TM denote the collection of all vectors tangent to the n -manifold M . The *projection* mapping $\pi: TM \rightarrow M$ assigns to each $v \in TM$ the point πv at which v is tangent to M . The fibre of π over each $P \in M$ is the tangent space $T_P(M)$. TM is endowed with a canonical $2n$ -dimensional real valued C^∞ -atlas – and thus made into a $2n$ -manifold – by a simple device. Let A be an atlas of M . For each chart x in A_{\max} we define a $2n$ -dimensional chart \bar{x} of TM as follows: if v is an element of TM such that $\pi v \in \text{dom } x$,

$$\bar{x}^i(v) = x^i(\pi v), \quad \bar{x}^{n+i}(v) = v(x^i), \quad (1 \leq i \leq n). \quad (10)$$

Evidently $\text{dom } \bar{x} = \pi^{-1}(\text{dom } x)$. It can be readily shown that the set of charts $\{\bar{x} \mid x \in A_{\max}\}$ is an atlas of TM that makes π into a C^∞ -differentiable mapping.

A triple $\langle A, f, B \rangle$, where A and B are manifolds and $f: A \rightarrow B$ is a surjective C^∞ -differentiable mapping is called a *fibre bundle* over B . In particular, $\langle A, f, B \rangle$ is a *vector bundle* over B if the fibre of f over each $b \in B$ is a vector space which is mapped onto \mathbb{R}^n (for some fixed positive integer n) by a linear isomorphism L_b , and each $b \in B$ has an open neighbourhood U such that $f^{-1}(U)$ is mapped diffeomorphically by $x \mapsto (f(x), L_{f(x)}(x))$ onto $U \times \mathbb{R}^n$. It is clear that, under the stipulations of the preceding paragraph, $\langle TM, \pi, M \rangle$ is a vector bundle over M . We call it the *tangent bundle* of M . The cotangent bundle of M and the infinite array of its tensor bundles of all types and orders are built analogously from the diverse linear spaces attached to each point of M (p.365).

A C^k -*section* over B in the vector bundle $\langle A, f, B \rangle$ is a C^k -differentiable mapping $g: B \rightarrow A$ such that $f \cdot g$ is the identity mapping $b \mapsto b$ of B onto itself. A C^k -section over the n -manifold M in the tangent bundle $\langle TM, \pi, M \rangle$ is called a C^k -*vector field* on M . A C^k -section over M in its cotangent bundle is a C^k -*covector field* on M . More generally, a C^k -section over M in one of its tensor bundles is a C^k -*tensor field*

on M of the same type and order as its values. (Thus, a (j, k) -tensor field on M assigns to each $P \in M$ a (j, k) -tensor on $T_P(M)$, i.e. a mixed tensor of $(j + k)$ th order, contravariant on the first j indices and covariant on the last k indices, which maps suitable lists of cotangent and tangent vectors at P into \mathbb{R} .) A C^k -differentiable mapping of M into \mathbb{R} will be called a C^k -scalar field. If X is any C^k -field on M and $P \in M$, we usually write X_P instead of $X(P)$ for the value of X at P . If V and F are, respectively, a C^k -vector and a C^k -covector field on M , the C^k -scalar field $\langle F \cdot V \rangle$ is given by the equation:

$$\langle F \cdot V \rangle_P = \langle F_P \cdot V_P \rangle, \quad (P \in M). \quad (11)$$

Let M be an n -manifold. We denote by $\mathcal{F}^k(M)$ the ring of C^k -scalar fields on M (ring operations as in eqns (2)). We denote by $\mathcal{V}^k(M)$ the collection of all C^k -vector fields on M . $\mathcal{V}^k(M)$ is made into a module over $\mathcal{F}^k(M)$ by the following rules: For every V, W in $\mathcal{V}^k(M)$, every f in $\mathcal{F}^k(M)$ and every P in M ,

$$\begin{aligned} (V + W)_P &= V_P + W_P, \\ (fV)_P &= f_P V_P. \end{aligned} \quad (12)$$

Each $V \in \mathcal{V}^k(M)$ determines a mapping \bar{V} of $\mathcal{F}^k(M)$ into $\mathcal{F}^{k-1}(M)$. \bar{V} assigns to each C^k -scalar field f a C^{k-1} -scalar field $\bar{V}f$, whose value at $P \in M$ is given by

$$(\bar{V}f)_P = V_P f. \quad (13)$$

We usually write V for \bar{V} . With this notation, if f and V are, respectively, a scalar and a vector field on M , fV is a vector field on M and Vf is a scalar field on M . A similar notation is used for covector and tensor fields.

Let U be an open subset of the n -manifold M . We naturally regard U as an n -manifold on its own right, whose maximal atlas includes the restriction to $U \cap \text{dom } x$ of each chart x in the maximal atlas of M . With this structure, U is termed an *open submanifold* on M . Let $i: U \rightarrow M$ be the canonical injection. It can be shown that for every $P \in U$, i_{*P} is a linear isomorphism of $T_P(U)$ onto $T_P(M)$. We may therefore identify $T_P(U)$ with $T_P(M)$ for each $P \in U$ and also TU , the tangent bundle of the manifold U , with $\pi^{-1}(U)$, the inverse image of $U \subset M$ by the projection $\pi: TM \rightarrow M$. We shall henceforth consider vector and tensor fields defined on open submanifolds of a given manifold. Thus, if x is a chart of M , $\partial/\partial x^i: P \mapsto \partial/\partial x^i|_P$ is a C^∞ -vector

field on $\text{dom } x$ ($1 \leq i \leq n$). If $V \in \mathcal{V}^k(M)$, then on $\text{dom } x$

$$V = \sum_i Vx^i \frac{\partial}{\partial x^i}. \quad (14)$$

The scalar fields Vx^i are called the *components* of V relative to chart x .

Let U_1 and U_2 be open submanifolds of n -manifold M . If V_1 is a C^∞ -vector field on U_1 and V_2 is a C^∞ -vector field on U_2 , the *Lie bracket* $[V_1, V_2]$ of V_1 and V_2 is the C^∞ -vector field on $U_1 \cap U_2$ defined by

$$[V_1, V_2]f = V_1(V_2f) - V_2(V_1f), \quad (15)$$

where f is any C^∞ -scalar field on $U_1 \cap U_2$.

A *real n -parameter Lie group* is an n -manifold endowed with a group structure such that the group product is a C^∞ -differentiable mapping. It is often assumed that a Lie group is an analytic manifold with analytic group product. Many important properties depend on the further assumption that the natural topology of the underlying manifold has a countable base; this is always true if the manifold is connected. A *complex Lie group* is defined analogously. \mathbf{R}^n is a real Lie group with the differentiable structure given on p.363 and the group structure determined by vector addition.

Let G be a Lie group and M an n -manifold. G is said to *act* on M if there is a C^∞ -differentiable surjection $F: G \times M \rightarrow M$ such that for every $m \in M$ and every $g, h \in G$, $F(g, F(h, m)) = F(gh, m)$. F is called the *action* of G on M . G acts *transitively* on M if for every $m_1, m_2 \in M$ there is a $g \in G$ such that $m_2 = F(g, m_1)$. G acts *effectively* on M if $F(g, m) = m$ for every $m \in M$ only if g is the neutral element of G . If F is the action of G on M , there is associated with each $g \in G$ a diffeomorphism $f_g: M \rightarrow M$ whose value at each $m \in M$ is given by:

$$f_g(m) = F(g, m). \quad (16)$$

Let G' be the set $\{f_g \mid g \in G\}$. G' is obviously a group with group product given by the composition of mappings. If G acts effectively on M , the mapping $J: f_g \mapsto g$ is plainly a group isomorphism of G' onto G . If x is a chart of G , $x \cdot J$ is a chart of G' . With the differentiable structure given by the collection of such charts, G' is a *Lie group of transformations* of M .

An example will illustrate the foregoing notions. Let M be an

n -manifold and let X be a C^∞ -vector field on M . Let c be a C^∞ -path in M , defined on an open interval about zero, such that $c(0) = P$. If $\dot{c} = X \cdot c$ on $\text{dom } c$, c is said to be an *integral path* of X with *origin* P . There always is an open neighbourhood U of P and an open interval about zero J such that each $Q \in U$ is the origin of an integral path of X defined on J . Such path agrees on a neighbourhood of zero with every other integral path of X with origin Q . Moreover, if c_1 and c_2 are integral paths of X originating at the same point $P \in M$, c_1 and c_2 agree on the intersection of their domains. The union of the domains of the integral paths of X with origin P is therefore an open interval J_P which is the domain of the maximal integral path of X with origin P . If $J_P = \mathbb{R}$ for every $P \in M$, X is said to be a *complete vector field* on M . It can be shown that X is complete if, and only if, there exists a neighbourhood of zero that is part of the domain of each integral path of X . If M is compact, X is necessarily complete. Let $L_s: \mathbb{R} \rightarrow \mathbb{R}$ be the mapping $u \mapsto u + s$. If $c_P: J_P \rightarrow M$ is the maximal integral path of X with origin P and if $s \in J_P$, the path $c_P \cdot L_s$ is the maximal integral path of X with origin $c_P(s)$ and its domain is $L_{-s}(J_P)$. Let D be the set $\{(s, P) \mid P \in M, s \in J_P\}$. The mapping $F: D \rightarrow M$ given by

$$F(s, P) = c_P(s) \quad (17)$$

is called the *flow* of vector field X . If $(s + t, P) \in D$, $F(s + t, P) = F(s, F(t, P))$. Consequently, if X is complete, the flow of X is an action on M of the Lie group \mathbb{R} and each real number t determines a diffeomorphism f_t of M onto M , defined by

$$f_t(P) = F(t, P) = c_P(t). \quad (18)$$

$\{f_t \mid t \in \mathbb{R}\}$ is the underlying set of a Lie group of transformations of M , known as the *one-parameter group generated by vector field X* .

Let M be an n -manifold. By a *field of M* we shall hereafter mean a C^∞ -scalar, vector or tensor field on an open submanifold of M . A C^∞ -linear connection ∇ on M assigns to every pair (X, Y) of vector fields of M a vector field $\nabla_X Y$ on $\text{dom } X \cap \text{dom } Y$ meeting the following conditions: if f is any scalar field of M and X, Y and Z are any vector fields of M , eqns. (19i)–(19iv) hold wherever the expressions on their left-hand sides are defined:

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z, \quad (19i)$$

$$\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z, \quad (19ii)$$

$$\nabla_{fX}Y = f\nabla_XY, \quad (19\text{iii})$$

$$\nabla_XfY = (Xf)Y + f\nabla_XY. \quad (19\text{iv})$$

∇_XY is called the *covariant derivative of Y in the direction of X*. The linear connection ∇ determines for every vector field Y of M a (1,1)-tensor field ∇Y on dom Y, which satisfies the following equation for every vector field X and every covector field F on dom Y:

$$\nabla Y(F, X) = \langle F, \nabla_XY \rangle. \quad (20)$$

∇Y is called the *covariant derivative of Y*.

Let x be a chart of M. For every pair of vector fields X and Y of M, ∇_XY can be expressed, on the intersection of its domain with dom x , as a linear combination of the vector fields $\partial/\partial x^i$ ($1 \leq i \leq n$). In particular,

$$\nabla_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}. \quad (21)$$

The Γ_{ij}^k are n^3 scalar fields on dom x , called the *components of the linear connection ∇ relative to chart x* . They obviously suffice to determine ∇ on dom x . We now introduce n covector fields dx^i on dom x by stipulating that

$$\left\langle dx^i, \frac{\partial}{\partial x^j} \right\rangle = \delta_j^i, \quad (1 \leq i, j \leq n), \quad (22)$$

where δ_j^i is the constant scalar field equal to 1 if $i = j$ and equal to 0 otherwise. The components of the covariant derivative ∇Y relative to chart x are the values of ∇Y at $(dx^i, \partial/\partial x^j)$. We denote them by $Y^i_{;j}$. The reader will verify that, writing Y^i for Yx^i ,

$$Y^i_{;j} = \frac{\partial Y^i}{\partial x^j} + \sum_k \Gamma_{jk}^i Y^k. \quad (23)$$

Let c be a C^∞ -path in M which maps a real interval (a, b) onto a curve k . By a vector field on c we shall understand a vector field of M that is defined on every point $P \in k$. Two vector fields on c will be said to be equivalent if they agree on k . If V is a vector field on c and $t \in (a, b)$, $V \cdot c(t) = V_{c(t)}$. We shall therefore write V_c instead of $V \cdot c$, whenever V is a vector field on c . If $V_c = \dot{c}$ on (a, b) , V is said to be a *tangent field* of c . Let X be a tangent field of c . A vector field Y is said to be *parallel along c* if Y is a vector field on c and $(\nabla_X Y)_c = 0$ on

(a, b). By suitably restricting the domain of c , we can always find a chart z defined on its entire range. On $\text{dom } z$, $Y = \sum_i Y^i \partial/\partial z^i$ and $X = \sum_i X^i \partial/\partial z^i$. We shall write Y^i for Yz^i , etc. Noting that $X^i \cdot c = (Xz^i) \cdot c = X_c z^i$ and that $X_c = \dot{c}$, we obtain:

$$\begin{aligned} 0 &= (\nabla_X Y)_c = X_c \sum_i Y^i \frac{\partial}{\partial z^i} \Big|_c + \sum_i \sum_j (Y^i \cdot c)(X_c z^j) \left(\nabla_{\partial/\partial z^j} \frac{\partial}{\partial z^i} \right)_c \\ &= \sum_i \sum_j \frac{\partial z^j \cdot c}{\partial t} \frac{\partial Y^i}{\partial z^j} \Big|_c \frac{\partial}{\partial z^i} \Big|_c \\ &\quad + \sum_i \sum_j \sum_k (Y^i \cdot c) \frac{\partial(z^j \cdot c)}{\partial t} (\Gamma_{ji}^k \cdot c) \frac{\partial}{\partial z^k} \Big|_c \\ &= \sum_i \sum_j \sum_k \frac{\partial}{\partial z^k} \Big|_c \left(\frac{\partial(Y^k \cdot c)}{\partial t} + (Y^i \cdot c) \frac{\partial(z^j \cdot c)}{\partial t} (\Gamma_{ji}^k \cdot c) \right), \end{aligned} \quad (24)$$

where the Γ_{ji}^k are the components of the linear connection ∇ relative to z . Consequently, a vector field V is parallel along c if, and only if, it is a vector field on c and $V_c = \sum_i v^i \partial/\partial z^i|_c$, where the functions $v^i: \mathbf{R} \rightarrow \mathbf{R}$ are solutions of the system

$$\frac{dv^k}{dt} + \sum_i \sum_j v^i \frac{d(z^j \cdot c)}{dt} (\Gamma_{ji}^k \cdot c) = 0 \quad (25)$$

The v^i are uniquely determined for each choice of initial values $v^i(t_0)$ ($t_0 \in \text{dom } c$). This implies that for each vector v tangent to M at $c(t_0)$ (for each $v \in T_{c(t_0)}(M)$) there is a unique equivalence class $[V]$ of vector fields on c , such that for any $V \in [V]$, $V_{c(t_0)} = v$ and V is parallel along c . The mapping $V_{c(t_0)} \mapsto V_{c(t)}$ ($t \in \text{dom } c$) is a linear isomorphism of $T_{c(t_0)}(M)$ onto $T_{c(t)}(M)$ called *parallel transport along c from $c(t_0)$ to $c(t)$* . If c has a tangent field which is parallel along c , c is said to be a *geodesic* of ∇ . Equation (25) implies that, if z is a chart at each point of $\text{im } c$, c is a geodesic of M if, and only if,

$$\frac{d^2(z^k \cdot c)}{dt^2} + \sum_i \sum_j \frac{dz^i \cdot c}{dt} \frac{dz^j \cdot c}{dt} (\Gamma_{ji}^k \cdot c) = 0, \quad (26)$$

where Γ_{ji}^k are the components of the linear connection of M relative to z .

We shall now define the *curvature* or *Riemann tensor* of an n -manifold M endowed with a linear connection ∇ . To each pair (X, Y) of vector fields of M we assign an operator $\bar{R}(X, Y)$, which

maps vector fields on vector fields. If Z is a vector field of M , then on $\text{dom } X \cap \text{dom } Y \cap \text{dom } Z$

$$\bar{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z. \quad (27)$$

$\bar{R}(X, Y)Z$ depends linearly on X , Y and Z . Its value at each point of its domain depends only on the values of X , Y and Z at that point. The *curvature* R of M is a (1,3)-tensor field such that, for any covector field F and any vector fields X , Y and Z , on the intersection of the domains of F , X , Y and Z ,

$$R(F, Z, X, Y) = \langle F, \bar{R}(X, Y)Z \rangle. \quad (28)$$

If x is a chart of M , the n^4 scalar fields $R^h_{ijk} = R(dx^h, \partial/\partial x^i, \partial/\partial x^j, \partial/\partial x^k)$, defined on $\text{dom } x$, are the components of the curvature R relative to x . Using eqns. (21), (27) and (28), one calculates that

$$R^h_{ijk} = \frac{\partial \Gamma^h_{ki}}{\partial x^j} - \frac{\partial \Gamma^h_{ji}}{\partial x^k} + \sum_s \Gamma^s_{ki} \Gamma^h_{js} - \sum_s \Gamma^s_{ji} \Gamma^h_{ks}. \quad (29)$$

If R is everywhere equal to 0, the connection ∇ and the manifold endowed with it are said to be *flat*.

The *torsion* of the manifold M endowed with the linear connection ∇ is a (1, 2)-tensor field on M which can be characterized thus. For each pair (X, Y) of vector fields of M , $T(X, Y)$ is the vector field on $\text{dom } X \cap \text{dom } Y$ given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \quad (30)$$

Let X, Y be two vector fields of M and let F be a covector field of M . The torsion T assigns to the triple (F, X, Y) the scalar field $\langle F, T(X, Y) \rangle$ wherever the latter is defined. The components of T relative to a chart x of M are given by

$$T^i_{jk} = \left\langle dx^i, T \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) \right\rangle = \Gamma^i_{jk} - \Gamma^i_{kj}, \quad (31)$$

where the Γ^i_{jk} are the components of the linear connection ∇ relative to x . If T vanishes everywhere, ∇ is said to be *torsion-free*. It follows from eqns. (26) and (31) that a torsion-free linear connection is fully determined if its geodesics are given.

Let M be an n -manifold. A *semi-Riemannian C^∞ -metric* g on M is a (0,2)-tensor field on M which is (i) *symmetric* and (ii) *non-degenerate*. g assigns to every $P \in M$ a scalar product g_P on $T_P(M)$. Let

V and W be vector fields of M . $g(V, W)$ denotes the scalar field $P \mapsto g_P(V_P, W_P)$, defined on $\text{dom } V \cap \text{dom } W$. (i) means that $g(V, W) = g(W, V)$ irrespective of the choice of V and W ; (ii) means that for each $P \in M$, $g(V, W)_P = 0$ for every vector field W defined on a neighbourhood of P if, and only if, V_P is the vector 0 (in $T_P(M)$).

The components of the metric g relative to a chart x of M are given by

$$g_{ij} = g \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right). \quad (32)$$

Since g is non-degenerate, the matrix of the g_{ij} is non-singular on $\text{dom } x$. There exists therefore a unique $(2, 0)$ -tensor field on M , whose components g^{ij} relative to the chart x are given by

$$\sum_j g^{ij} g_{jk} = \delta^i_k. \quad (33)$$

The signature of the metric g at a point $P \in \text{dom } x$ is the number of positive, minus the number of negative, eigenvalues of the matrix $[g_{ij}(P)]$. Since g is continuous and non-degenerate, its signature is constant on M . We denote it by $sg(g)$. If $sg(g) = n$, g is said to be a *Riemannian metric* and (M, g) is called a *Riemannian manifold*. The most familiar example of a Riemannian metric is the Euclidean metric δ on \mathbb{R}^n . Its components relative to the identity chart $x: u \mapsto u$ are given by

$$\delta \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (34)$$

A metric g with signature $sg(g) = n - 2$ is called a *Lorentz metric*. Einstein's theory of relativity conceives physical spacetime as a 4-dimensional real manifold endowed with a Lorentz metric.

If g is a semi-Riemannian metric on the n -manifold M there is a unique torsion-free linear connection ∇ on M that satisfies the condition

$$\nabla g = 0. \quad (35)$$

This connection is said to be the only linear connection *compatible* with g and is called the *Levi-Civita connection* of g . Let x be a chart of M and let g_{ij} and Γ^i_{jk} stand, respectively, for the components of g and ∇ relative to x . It can be shown that

$$\Gamma^h_{ij} = \frac{1}{2} \sum_k g^{hk} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right). \quad (36)$$

The components Γ_{ij}^h of the Levi-Civita connection of a semi-Riemannian metric \mathbf{g} on a manifold M , relative to a chart of M , are thus equal to the Christoffel symbols of the second kind $\left\{ \begin{smallmatrix} h \\ ij \end{smallmatrix} \right\}$ defined on p.94 (eqn. (5) of Section 2.2.8). Comparing eqn. (9) of Section 2.2.8 on p.95 with eqn. (26) above, we verify that the geodesics defined in terms of a Riemannian metric by the former equation are precisely the geodesics of the Levi-Civita connection of that metric.

NOTES

CHAPTER 1. BACKGROUND

- ¹ Aristotle, *Metaph.*, 981^b20–25.
- ² Aristotle, *Metaph.*, 983^b17–20; Proclus, *Comm.*, ed. Fr., pp.157, 250, 299, 352.
- ³ Plato, *Meno*, 80d–86c.
- ⁴ Proclus, *Comm.*, ed. Fr. p.207. Cf. Berkeley, *Principles of Human Knowledge*, §16.
- ⁵ Arpad Szabó maintains that the origin of this new style of proof can be traced to the Eleatic philosophers Parmenides and Zeno. If Szabó is right we ought to hail Parmenides of Elea, rather than Thales or Pythagoras, as the true father of scientific mathematics. But, as it often happens in philology, Szabó's arguments are not altogether convincing. See his *Anfänge der griechischen Mathematik* (1969), pp.243–293, etc., Szabó (1964) presents some of Szabó's ideas in English.
- ⁶ See Becker (1934), Reidemeister (1940), Van der Waerden (1947/49). Strictly speaking, the word *number* (*arithmos*) means in Greek an integer greater than one. Yet the propositions proved of numbers generally hold good also if the *unit* (*monas*) is counted as a number. Hereafter, when dealing with Greek mathematics, we shall mean by *number* a positive integer.
- ⁷ Heath, EE, Vol.II, p.414.
- ⁸ Plato, *Respub.*, 511^a1; cf. 526^a6.
- ⁹ Van der Waerden, SA, p.144.
- ¹⁰ According to the numbering of the Basel edition of 1533. Modern editors have excised this proposition as apocryphal.
- ¹¹ Plato, *Respub.*, 533^b7–^c5.
- ¹² Plato, *Respub.*, 533^a8.
- ¹³ Aristotle, *Eth. Nicom.*, 1039^b1; cf. *Anal. Post.*, 100^b12.
- ¹⁴ Aristotle, *Anal. Post.* Book II, Chapter 19; *Eth. Nicom.*, Book VI, Chapter 6; Oxford translation.
- ¹⁵ Aristotle, *Anal. Post.*, 72^a14–18, 76^a37–41; *Metaph.*, 1005^a19–25.
- ¹⁶ Aristotle, *Metaph.*, 1005^b18–20.
- ¹⁷ Aristotle, *Anal. Post.*, 72^a18–24. A. Gómez-Lobo (1977) argues persuasively that this passage describes *hypotheses* as singular statements of the form "This here is a such and such" and not – as it has been usually understood – as existential generalizations of the form "There is a such and such". Since Aristotle's logic is not a 'free' logic, such singular statements are existential all the same.
- ¹⁸ Aristotle, *Topica*, Book VI, Chapter 4; cf. *Anal. Post.* 93^b22.
- ¹⁹ Kurt von Fritz (1955); Arpad Szabó, AGM, 3. Teil. Additional reasons for doubting Euclid's Aristotelianism flow from Imre Tóth's investigations mentioned in Note 2 of Part 2.1.
- ²⁰ Heath, EE, Vol.I, pp.222–224, 232.
- ²¹ Aristotle, *Metaph.*, 1005^a20.

²² E.g. Aristotle, *Metaph.*, 1061^b20; *Anal. Post.*, 77^a31.

²³ Heath, EE, Vol.I, pp.195, 196, 199, 200, 202. I have modified Heath's translation slightly to make it more literal. The numbers are not found in the manuscript and are included for reference. Postulate 5 is analysed in Section 2.1.1.

²⁴ It says that the intersection of two lines exists when certain conditions are fulfilled.

²⁵ Aristotle, *Phys.*, Book III, Chapter 6.

²⁶ Zeuthen (1896), who interpreted all five postulates as existential statements, maintained that the fourth really means that there is a unique straight line of which a given segment is a part, or, in other words, that the construction postulated by the second *aitema* can be unambiguously carried out. O. Becker (1959) takes the sounder view that all the constructions postulated by the first three *aitemata* are required by these to be unambiguous (this is an essential part of their meaning). Becker remarks that Postulate 4 can then be proved by means of Postulates 1 and 2. He concludes that Postulate 4 was not really meant as a postulate or unproved assumption, but as a reminder that had to be inserted before Postulate 5 because this speaks of "two right angles" as if it were a perfectly definite quantity.

²⁷ Ἡεῖρησθῶ was rendered above as "let it be postulated". It is the third person singular aorist imperative of the verb αἰτέομαι, that in ordinary Greek means 'to ask for one's own', 'to claim', 'to beg'. Αἴτημα is the noun corresponding to this verb, and in ordinary Greek it meant 'a request'; cf. e.g. Plato, *Republica*, 566^b5.

²⁸ Aristotle, *Anal. Post.*, 76^b27–30; 32–34. G.R.G. Mure thought it appropriate to render *aitema* in the present context as *illegitimate postulate*.

²⁹ Aristotle, *Metaph.* 1005^a29–31.

³⁰ Aristotle, *Metaph.*, 985^b32.

³¹ Aristotle, *Metaph.*, 986^a1.

³² Euclid, V, Definitions 4, 5, 7.

³³ *Astron* is the Greek word for 'heavenly body' (including sun, moon and planets). Plato's demand that astronomy "let heavenly things alone" was not so far-fetched as it might seem to us. After all, geometry had by then divorced itself from the art of surveying, from which it took its name.

³⁴ Plato, *Respub.*, 529^d7–530^b4. I reproduce, with slight changes, Sir Thomas Heath's translation in his *Greek Astronomy*.

³⁵ Plato, *Respub.* 530^d1.

³⁶ Plato, *Respub.*, 531^b2–4.

³⁷ Plato, *Respub.*, 429^b5–^c1.

³⁸ I must stress, however, that Eudoxian models do not involve any assumptions regarding the absolute or relative sizes of the spheres. Since all that matters are the spherical *motions*, every sphere pertaining to a given planet might just as well have the same radius.

³⁹ Simplicius, *In Aristotelis de Caelo*, ed. Heiberg, p.488.16–18.

⁴⁰ Plato, *Leges*, 821^e1–822^b8; 897^c4–6.

⁴¹ Plato, *Leges*, 967^b2–4. The reader will not fail to notice that Plato reasons like the 20th-century engineer who ascribes intelligence to his computer.

⁴² *Epinomis*, 983^b7–^c4. Cf. *Ibid.*, 982^c4–^d2.

⁴³ Aristotle, *Metaph.*, 1073^b38–1074^a15. If, like Aristotle, we assign to each planet its full Callippean model, we must end up with 61 spheres, because every planet will have

a sphere of its own that rotates about the North-South axis with the same angular velocity as the fixed stars, and this rotation must be compensated by the rotation of an additional sphere before it is again introduced with the next planet. Aristotle overlooked this fact and counted only 55 spheres (loc. cit., 1075^a11). Seven of the 55 rotate like the first (i.e., like the fixed stars). Since these seven rotations have none to counter them, the Aristotelian moon must rise in the East seven times a day. Apparently nobody noticed this before Norwood Russell Hanson (*Constellations and Conjectures*, pp.66-80). Earlier scholars had pointed out that six spheres were dispensable, but none said that, unless their motions were neutralized (by suppressing them or by adding new spheres), the lower planets would be going too fast.

⁴⁴ Aristotle, *Eth. Nicom.*, 1094^b23-25.

⁴⁵ Aristotle, *Metaph.*, 995^a14-17.

⁴⁶ Aristotle, *Phys.*, 193^b24f.

⁴⁷ Aristotle, *Phys.*, 193^b35.

⁴⁸ Aristotle, *Metaphys.*, 1025^b28-1026^a6; cf. *Physica*, 194^a1-6.

⁴⁹ This result can be very easily proved in the special case of a (fictitious) planet whose trajectory lies on a plane through the centre of the earth. Let this plane be represented by the complex plane, with the centre of the earth at the origin. An arbitrary planetary trajectory on that plane will be represented by a continuous periodic function $z = f(t)$. Uniform circular motion about the point designated by the complex number k is represented by the function $z' = k + a \exp(ibt)$ (where a and b are real numbers depending on the radius of the circle and the angular velocity of the motion). Consequently, the position at time t of a body moving with n th degree uniform geocentric epicyclic motion is given by $z'' = a_1 \exp(ib_1 t) + \dots + a_n \exp(ib_n t)$. By a suitable choice of n and of the $2n$ constants a_i, b_i , the difference $|z - z''|$ can be made to remain less than any assigned positive real number.

⁵⁰ Derek J. de S. Price (1959), p.210. Pre-Keplerian astronomy never attained the optimal accuracy indicated on p.20; Price's calculations show, however, that its failings were due to a wrong choice of parameters rather than to the inadequacy of the epicyclic models.

⁵¹ Averroes, *Metaphysica*, lib.12, summae secundae cap.4, comm.45, quoted by Duhem, SP, p.31.

⁵² Maimonides, *The Guide of the Perplexed* (transl. by Shlomo Pines), pp.325f.

⁵³ Maimonides, GP, p.327.

⁵⁴ John of Jandun, *Acutissimae quaestiones in XII libros Metaphysicae*, 12.20, quoted by Duhem, SP, p.43.

⁵⁵ Kepler, *Epitome astronomiae copernicanae* (1618); GW, Vol.VII, p.23.

⁵⁶ Kepler, *Ibid.*; GW, Vol.VII, p.25.

⁵⁷ Kepler, Letter to Johann Georg Brenger, October 4, 1607; GW, Vol.XVI, p.54.

⁵⁸ Kepler, Letter to Herwart von Hohenburg, April 10, 1605; GW, Vol.XV, p.146. *Machina* means here 'structure' or 'edifice'; if it is rendered as 'machine', Kepler's programme sounds trivial - indeed, he ought then to prove first that the heavens *are* a machine.

⁵⁹ Kepler, *Mysterium cosmographicum* (1596); GW, Vol.I, p.26.

⁶⁰ Kepler, *Astronomia nova* (1609); GW, Vol.III, p.241.

⁶¹ Kepler, *Harmonices Mundi* (1619); GW, Vol.VI, p.223.

- ⁶² Kepler, Letter to Michael Mästlin, April 9, 1597; GW, Vol.XIII, p.113.
- ⁶³ Galilei, Letter to Fortunio Liceti, January 1641; EN, Vol.XVIII, p.295.
- ⁶⁴ Descartes, *Principia Philosophiae*, II,21; AT, Vol.VIII, p.52; cf. AT, Vol.V, p.345.
- ⁶⁵ Descartes, *Principia Philosophiae*, II.21. Cf. I,26 in AT, Vol.VIII, pp.14f. See Henry More's rebuke in Descartes, AT, Vol.V, pp.242f.
- ⁶⁶ Poincaré, SH, p.78. An exact definition of these properties would be out of place here. The reader probably has an intuitive idea of the first three. Homogeneity means that there are no privileged points in space; isotropy, that there are no privileged directions through any point.
- ⁶⁷ Plato, *Timaeus*, 52a–b.
- ⁶⁸ Aristotle, *Phys.*, 212^a6 (as amended by Ross following the ancient commentators).
- ⁶⁹ Strato, fr.55, in Wehrli, *Die Schule des Aristoteles*. Cf. Aristotle, *Phys.*, 211^b8.
- ⁷⁰ Philoponus, *In Aristotelis Phys. IV–VIII* (ed. Vitelli), p.567.29–33.
- ⁷¹ Simplicius, *In Aristotelis Phys. I–IV* (ed. Diels), p.618.21ff.
- ⁷² See for example Lucretius, *De rerum natura*, I.1002–1007.
- ⁷³ Bradwardine, *De causa Dei*, p.179A.
- ⁷⁴ H.A. Wolfson, *Cresca's Critique of Aristotle*, pp.147, 187–189, 417 (n.31).
- ⁷⁵ Bruno, *De Immenso et Innumerabilibus*, I.8; *Op. Lat.* Vol.I.1, p.231.
- ⁷⁶ Leibniz, Fifth Paper to Clarke, No.47, in H.G. Alexander, LCC, p.69.
- ⁷⁷ Henry More, OO, Vol.II.2, p.162.
- ⁷⁸ Kant, KrV, A39/B56.
- ⁷⁹ Kant, *Gedanken von der wahren Schätzung der lebendigen Kräfte*, §9; *Ak.*, Vol.I, p.23.
- ⁸⁰ Kant, *Ibid.*, §10; *Ak.*, Vol.I, p.24.
- ⁸¹ Thus, some species of snails have a left-handed spiral shell; others a right-handed one. The orientation of the spiral is preserved from one generation to the next. The ontological significance of this fact must have loomed large before Darwin.
- ⁸² Kant, *Ak.*, Vol.II, p.403.25.
- ⁸³ Kant, *Ak.*, Vol.II, p.398.
- ⁸⁴ Kant, *Ak.*, Vol.II, p.403.
- ⁸⁵ "Illud inane rationis commentum." Kant, *Ak.*, Vol.II, p.404.2.
- ⁸⁶ Kant, *Ak.* Vol.II, p.402f. Kant adds: "Ceterum Geometria propositiones suas universales non demonstrat: obiectum cogitando per conceptum universalem, quod fit in rationalibus, sed illud oculis subiiciendo per intuitum singularem, quod fit in sensitivis" (*Ak.*, Vol.II, p.403). It is hard to imagine what the eyes – even if we take them to be the "eyes of the mind" – can see in pure, i.e. sensation-free, intuition, unless the latter is determined by concepts. Nevertheless, Kant's view of geometrical proof was incredibly popular among philosophers throughout the 19th century (see, e.g., Schopenhauer, WW, Vol.VII, pp.62–67; Vol.II, pp.82–99). J. Hintikka has recently tried to make sense of it by drawing a parallel between Kant's appeal to "singular intuition" and the use of existential instantiation, which we now know to be indispensable in most mathematical demonstrations. But existential instantiation, far from being the opposite of "thinking an object by means of a universal concept", presupposes a concept with which the proposed instance must exactly agree. See Hintikka (1967) and the other references listed in Hintikka, LLI, p.23 n.38.
- ⁸⁷ Kant, *Ak.*, Vol.II, pp.404f. It is not altogether unlikely that Kant had been apprised

by his friend J.H. Lambert of the possibility of modelling a two-dimensional non-Euclidean geometry on a surface imbedded in Euclidean 3-space (see p.50).

⁸⁸ Kant, *Ak.*, Vol.II, p.404.

⁸⁹ Kant, *KrV*, B274ff.

⁹⁰ Kant, *KrV*, A210/B255. See "On the subjectivity of objective space", Torretti (1970).

⁹¹ Kant, *KrV*, B129f.

⁹² The latter text was substituted for the former in the 1787 edition. Compare *KrV*, A20, B34.

⁹³ Kant, *KrV*, B160n.

⁹⁴ Kant, *Prolegomena*, §38; *Ak.*, Vol.IV, pp.321f.

⁹⁵ If (D) is false, m can be partitioned into two classes of points, a_1 and a_2 , such that every point in a_i lies between two points in a_i and no point in a_i lies between two points in a_j ($i, j = 1, 2; i \neq j$). Let m lie on plane Π . There are infinitely many pairs (ζ_1, ζ_2) of disjoint open half-planes of Π , such that $a_i \subset \zeta_i$. Let m' be the common boundary of one such pair; m' is then a straight line joining points of Π which lie on either side of m and yet m' does not meet m . (If the intersection of m and m' were not empty it would contain a point belonging to a_1 or to a_2 but not lying between any two points of its own class; this contradicts the above characterization of a_1 and a_2 .) Descartes would doubtless have judged this result incompatible with the continuity of line m .

⁹⁶ A linear order on a set S is determined by a binary relation R such that, if x, y and z are any three elements of S , the following conditions are fulfilled: (i) either Rxy or Ryx (R is universal); (ii) Rxy precludes Ryx (R is antisymmetric); (iii) if Rxy and Ryz , then Rxz (R is transitive). If R determines a linear order one usually writes ' $x < y$ ' instead of Rxy (read: ' x precedes y ' or ' y follows x '). The reader should verify that conditions (i), (ii) and (iii) on page 35 determine a linear ordering of the points of m , and that this order is preserved whenever a positive segment $O'E'$ is substituted for OE and is merely reversed whenever OE is replaced by a negative segment.

⁹⁷ One should bear in mind that the field structure of Σ depends on the choice of a segment OE and of a direction on each Euclidean line.

CHAPTER 2. NON-EUCLIDEAN GEOMETRIES

2.1 *Parallels*

¹ Heath, *EE*, Vol.I, p.155.

² Imre Tóth (1966/7) has classed and analysed numerous passages in the *Corpus Aristotelicum* which, according to him, allude to a pre-Euclidean discussion of parallels and related matters. Tóth claims to have shown that: (i) the existence of a parallel to a given line through a point outside it can be proved by construction; (ii) alleged direct proofs of the uniqueness of the said parallel are question-begging; (iii) if the parallel to a given line through a point outside it is not unique, the three interior angles of a triangle are not equal to π ; (iv) it can be proved by *reductio ad absurdum* that they are not greater than π (Aristotle repeatedly mentions the proof of this statement as a familiar example of apagogic proof); (v) attempts to prove that they are not less than π are inconclusive (something never mentioned by Aristotle but which should be obvious

from the fact that Euclid included Postulate 5 among the unproved assumptions of geometry). If Tóth is right, the debate on Postulate 5 from the 1st century B.C. to the 19th century originated in ignorance or lack of understanding of the geometrical tradition leading up to Euclid. Although my pedestrian imagination is not always able to follow Tóth's in its sometimes frenzied flight, I believe that his work deserves careful study.

³ Proclus, *Comm.*, ed. Fr., p.191.

⁴ Proclus, *Comm.*, ed. Fr., p.192.

⁵ Proclus, *Comm.*, ed. Fr., pp.363–373.

⁶ Wallis, *Operum mathematicarum volumen alterum* (1693), p.678; Stäckel and Engel, TP, p.30.

⁷ Wallis, *ibid.*, p.676; Stäckel and Engel, TP, p.26.

⁸ Wallis, *ibid.*, p.676; Stäckel and Engel, TP, pp.26f.

⁹ *Euclides ab omni naevo vindicatus (Euclid freed from all flaw)*, Milan 1733.

¹⁰ Klügel (1763), p.16; quoted in Stäckel and Engel, TP, p.140.

¹¹ Stäckel and Engel, TP, p.162.

¹² Lambert's quadrilateral is identical with one of the two congruent parts into which a Saccheri quadrilateral is divided by the perpendicular bisector of its base. Lambert's three hypotheses as numbered by him are respectively equivalent to Saccheri's as I have numbered them.

¹³ Stäckel and Engel, TP, p.202.

¹⁴ Lambert writes: "Ich möchte daraus fast den Schluss machen, die dritte Hypothese komme bei einer imaginären Kugelfläche vor." (Stäckel and Engel, TP, p.203).

¹⁵ Stäckel and Engel, TP, p.200. A similar remark by Gauss is quoted below (Section 2.1.5, p.55).

¹⁶ Dehn (1900). See Section 3.2.9.

¹⁷ Legendre's investigations on the theory of parallels are brought together in his "Refléxions sur différentes manières de démontrer la théorie des parallèles ou le théorème sur la somme des trois angles du triangle." (1833). Two of his attempted proofs of Postulate 5 are reported in Bonola, NEG, pp.55–60. More on Legendre below, Section 3.2.4. Bolzano's attempt at deriving Postulate 5 from the essential properties of the straight line is also discussed in Section 3.2.4.

¹⁸ Stäckel and Engel, TP, p.262.

¹⁹ Reprinted in Gauss, WW, Vol.8, pp.180f. English translation in Bonola, NEG, p.76.

²⁰ Gauss to Gerling, March 16, 1819. (Gauss, WW, Vol. 8, p.181.)

²¹ Taurinus, *Theorie der Parallellinien*, p.86; Stäckel and Engel, TP, p.258.

²² Norman Daniels (1972) claims that the Scottish philosopher Thomas Reid (1710–1796) discovered a non-Euclidean geometry in the 1760's, having published his results in his *Inquiry into the Human Mind* (1764). All I find in the passages mentioned by Daniels (see Reid, PW, Vol.I, pp.142–153) is a discussion of the geometric structure of the visual field, which, according to Reid, is spherical. Reid's approach is certainly original, but he has not made a contribution to geometry. The geometry of the sphere which he sees embodied in his "geometry of visibles" (Reid, PW, Vol.I, pp.147ff.) was familiar to Euclid and was used profusely in ancient astronomy. There is in Reid's book no suggestion that the characteristic properties of the sphere (e.g. that straightest lines meet twice) might also be realized in a three-dimensional space. Reid cannot even be

said to have anticipated Gauss' 'intrinsic' study of surfaces imbedded in Euclidean space (see below, Section 2.2.4); at any rate, I do not find in his book anything remotely reminiscent of Gauss' methods.

²³ Gauss to Bessel, January 27, 1829. (Gauss, WW, Vol.8, p.200.)

²⁴ Gauss to Schumacher, May 17, 1831 (Gauss, WW, Vol.8, p.216). Stäckel conjectures that the text written by Gauss in those days is that given in WW, Vol.8, pp.202–209.

²⁵ Gauss to Farkas Bolyai, March 6, 1832. (Gauss, WW, Vol.8, p.221).

²⁶ Gauss, WW, Vol.8, p.220.

²⁷ Gauss to Schumacher, November 28, 1846 (Gauss, WW, Vol.8, pp.238f.).

²⁸ Gauss, WW, Vol.8, p.159. The relevant passage is quoted in English translation in Kline, MT, p.872. On Gauss and the Bolyais, see Stäckel and Engel (1897).

²⁹ Gauss, WW, Vol.8, p.169. Recall Lambert's remark to this effect, quoted on p.51, Section 2.1.4.

³⁰ Gauss, WW, Vol.8, p.177.

³¹ Gauss, WW, Vol.8, p.182.

³² Gauss, WW, Vol.8, p.187.

³³ Gauss, WW, Vol.8, pp.202, 208; Bolyai, SAS, p.5, Lobachevsky, ZGA, p.11.

³⁴ Lobachevsky, ZGA, pp.174ff. (*Novye nachala geometrii*, §102).

³⁵ See Gauss, WW, Vol.8, p.187; Lobachevsky, ZGA, p.24 (*O nachalakh geometrii*, §15).

³⁶ C is equal to Schweikart's constant mentioned on p.52, Section 2.1.4.

³⁷ Bolyai, SAS, p.18 (§21).

³⁸ Lobachevsky, ZGA, pp.193, 195.

³⁹ Lobachevsky, GRTP, pp.11f.

⁴⁰ Gauss, WW, Vol.8, p.182. This text of 1819 should establish that there is no truth in the story that Gauss undertook geodetic measurements at mounts Brocken, Hohehagen and Inselsberg in order to decide experimentally whether Euclid's geometry was true of physical space. The ultimate source of this rather naïve tale appears to be a passage in W. Sartorius von Waltershausen's memoir *Gauss zum Gedächtnis* (1856) p.80 (quoted in Gauss, WW, Vol.8, p.267). But Sartorius says only that Gauss maintained that "we know from experience, e.g. from the angles of the triangle Brocken, Hohehagen, Inselberg, that [Postulate 5] is approximately correct". Sartorius does not say that Gauss, who had indeed measured that triangle in the course of his geodetical work in the early 1820's, did it with *the aim of testing* Euclidean geometry. From the above quotation we learn that Gauss knew already in 1819 that it would be hard to test it even on an astronomical scale. Let me add, by the way, that in his *Disquisitiones generales circa superficies curvas* (1827) Gauss himself mentions the Hohehagen–Brocken–Inselberg triangle, though not as a *plane* triangle differing imperceptibly from a *Euclidean* triangle, but as a *terrestrial* triangle which differs so little from a spherical triangle, that we cannot determine from it, let alone from a smaller such triangle, the difference between a perfect sphere and the actual shape of the earth. (Gauss, GI, p.43). See Arthur I. Miller (1972), (1974).

⁴¹ Lobachevsky, ZGA, p.22.

⁴² Lobachevsky, ZGA, p.2.

⁴³ Lobachevsky, ZGA, p.80.

⁴⁴ Lobachevsky, ZGA, p.82.

⁴⁵ Lobachevsky, ZGA, p.76.

⁴⁶ Lobachevsky, ZGA, p.65.

2.2 Manifolds

¹ "In pulcherrimo geometriae corpore duo sunt naevi", viz. the theory of parallels and the theory of proportions. (Savile, *Praelectiones*, p.140.)

² Bessel to Gauss, February 10, 1829 (Gauss, WW, Vol.8, p.201).

³ The above definition of *path* presupposes, in fact, that space is Euclidean. Unlimited differentiability is required in order to avoid qualifications that would distract the reader from the conceptual issues which are our concern. A more general characterization of paths is given in the Appendix, p.363. A mapping is said to be injective if it assigns distinct values to distinct arguments. (See p.359.)

⁴ The *i*th projection function on \mathbb{R}^n maps each ordered *n*-tuple of real numbers on its *i*th member.

⁵ Such a convention may be set as follows: $c_1(t) = \Phi(t, b)$ and $c_2(t) = \Phi(a, t)$ describe curves on $\Phi(\zeta)$ meeting at $\Phi(a, b) = P$; $c'_1(a) = H_1$ and $c'_2(b) = H_2$ are the tangential images of these curves at *P*. We choose Q_i as the normal image of $\Phi(\zeta)$ at *P* if, and only if, whenever the right thumb and right forefinger point, respectively, from *O* toward H_1 and from *O* toward H_2 , the right middle finger can be made to point toward Q_i . This convention clearly determines our choice of Q_i at each point *P*.

⁶ That is, by the area of $n(\zeta)$ if the "part of a curved surface" is denoted by $\Phi(\zeta)$.

⁷ Gauss, GI, pp.9f. (§6).

⁸ A rational reconstruction of Gauss' definition of the (local) (curvature of a surface is furnished by the theory of differential manifolds. (See Appendix, pp.361ff.) Our surface $\Phi(\zeta)$ and the unit sphere can both be conceived as 2-manifolds. *n* is then a differentiable mapping of the former into the latter. Consequently, *n* determines a mapping n_* of the tangent bundle over $\Phi(\zeta)$ into the tangent bundle over the unit sphere such that, if *c* is any path in $\Phi(\zeta)$ and c_1 denotes the path $n \cdot c$, $n_*c = \dot{c}_1$. *n* also determines a mapping n^* of each bundle of $(0, k)$ tensor fields on the unit sphere into the bundle of $(0, k)$ -tensor fields on $\Phi(\zeta)$, such that, for any $(0, k)$ -tensor field *T* on the former and any *k*-tuple (v_1, \dots, v_k) of vector fields on the latter, $n^*T(v_1, \dots, v_k) = T(n_*v_1, \dots, n_*v_k)$. The area of a region *U* of a 2-manifold *M* is measured by integrating over *U* a 2-differential form (i.e. an alternating $(0, 2)$ -tensor field) on *M*, called the surface element of *M*. Let φ and ω be the surface elements of $\Phi(\zeta)$ and of the unit sphere, respectively. Then, at any point *P* in $\Phi(\zeta)$, φ_P and $(n^*\omega)_P$ both belong to the one-dimensional vector space of alternating $(0, 2)$ -tensors on the tangent space of $\Phi(\zeta)$ at *P*, so that their quotient exists and is a real number, just as in Gauss' definition. The mapping $P \mapsto \varphi_P / (n^*\omega)_P$ is then a scalar field on $\Phi(\zeta)$, the Gaussian curvature of the surface. A more straightforward vindication of Gauss' definition can be achieved within the theory of formal differential geometry currently being developed by Kock, Wraith and Reyes. (On this theory, see, for example, A. Kock and G.E. Reyes (1977).)

⁹ An arc *s* in $\Phi(\zeta)$ joining two points *A* and *B* is said to be an arc of *shortest length* or a *shortest arc* if its length is less than or equal to that of any other arc joining *A* and *B* which is contained in a (suitably narrow) strip of $\Phi(\zeta)$ covering *s*.

¹⁰ Surfaces of constant negative curvature had been studied by Ferdinand Minding (1839, 1840) in two papers published in the same journal (Crelle's *J. für die reine u.*

angewandte Mathematik) that had printed Lobachevsky's "Géométrie imaginaire" in 1837.

¹¹ Gauss, GI, p.46.

¹² I.e. such that, for any $x \in \mathbb{R}^2$, $|x| = |f(x)|$.

¹³ "Neue allgemeine Untersuchungen über die krummen Flächen", Gauss, WW, Vol.8, pp.408–442; English translation in Gauss, GI, pp.81–110.

¹⁴ Intuitively speaking, the isometrically invariant structure of a surface includes all those properties of the latter which do not change if it is moved or bent in any way whatsoever, without stretching or shrinking or tearing it. The term 'intrinsic geometry' seems quite appropriate to denote this set of properties. This concept of an intrinsic geometry ought not to be confused with Adolf Grünbaum's notion of an "intrinsic metric". (see Grünbaum, PPST, pp.501f.).

¹⁵ On arcs of shortest length see above, Note 9. Solutions of the differential equations set up by Gauss are paths in $\Phi(\zeta)$ which are known as geodesics; their ranges are called geodetic arcs. It can be shown that if two points A, B on $\Phi(\zeta)$ can be joined by a shortest arc, that arc is a geodetic arc; but it is not necessarily the only geodetic arc joining A and B. It can also happen that *no* shortest arc joins A and B, *even though* they are joined by a geodetic arc; e.g. if $\Phi(\zeta)$ is a punctured sphere and A and B lie on a punctured meridian on the same hemisphere as but on opposite sides of the excised point.

¹⁶ Of course, in the induced geometry, the 'shortest' arc joining two points A and B of Q is not necessarily straight: it is the Ψ -image of the shortest arc joining $\Psi^{-1}(A)$ and $\Psi^{-1}(B)$ on S. Also a 'right' angle is not necessarily equal to its adjacent angle, unless Ψ is conformal, i.e. angle-preserving.

¹⁷ See e.g. Poincaré, VS, p.58. Compare Hausdorff (1904), pp.14–17.

¹⁸ The main advances in this development were made by Riemann himself (in his prize-essay of 1861; see Riemann, WW, pp.401–404), Beltrami (1868/69), Christoffel (1869), Schur (1886), Ricci and Levi-Civita (1901), Levi-Civita (1917), Weyl (1918), Cartan (see his LGER; references to some of his original papers are given in Cartan, ERS), Ehresmann (1950) and Koszul (reported in Nomizu (1954), p.35, n.2).

¹⁹ See the illuminating commentary on Riemann's lecture in Spivak, CIDG, Vol.II, from which I draw abundantly in what follows.

²⁰ Riemann, H, p.8.

²¹ Riemann, H, p.8. Experience could not help us to discover the metrical properties that single out space among threefold extended quantities if space were indeterminate in this regard or metrically "amorphous". For this reason, I cannot agree with Adolf Grünbaum's reading of Riemann (PPST, pp.8ff.) nor can I regard the latter as a forerunner of the former's conventionalist philosophy of space and geometry. It is one thing to maintain that the generic concept of which our physical space is a species does not involve a definite metric and quite another to hold that such a metric is not a structural property of physical space. In a much quoted passage, Riemann (H, p.23) asserted that if physical space is a continuous manifold, its metric relations cannot be derived from the mere conception of this manifold as such, and their foundation must be sought in the nature of the physical forces that keep the manifold together. But this not imply that the metric of physical space is conventional but, on the contrary, that it is natural and hence cannot be known a priori.

²² Riemann, H, p.8.

²³ Points of a continuous "manifold" are therefore conceived by Riemann as the specifications of a genus (*Bestimmungsweisen eines allgemeinen Begriffs*); each point of a "manifold" differs from every other point as a species from another—each point possesses, so to speak, the individuality of an angel. This is indeed a far cry from the lack of differentiation traditionally ascribed to the points of space.

²⁴ He gives two examples, however: the "manifold" of functions defined on a given domain, and the possible figures of a closed region in space.

²⁵ I mean by $\text{dom } f$ the domain of f . See p.359.

²⁶ If $m \neq n$, the neighbourhood relations induced on M by its B -structure will be different from those induced on M by its A -structure. More surprising is the fact that incompatible neighbourhood relations can be induced on M by different atlases even if $m = n$, if only $n \geq 7$. (Milnor (1956)).

²⁷ The reader will notice how naturally our previous definition of a path in space fits in with this one (p.68). See also Appendix, p.363.

²⁸ A differentiable mapping f of a differentiable manifold into another is said to be an *imbedding* if (i) f maps its domain homeomorphically onto its range and (ii) at each point P in $\text{dom } f$, the mapping f_{*P} maps its domain (i.e. the tangent space at P) isomorphically onto its range. On the tangent bundle of a manifold, see Appendix, p.366.

²⁹ Riemann, H, p.9.

³⁰ Such is the case with time intervals, unless the intervals compared are one a part of the other, the trivial case that Riemann explicitly excludes from his considerations.

³¹ An arc is the range of a path defined on a closed interval. See Appendix, p.363.

³² The straight segment joining $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in \mathbb{R}^n is the set of all points $z = (z_1, \dots, z_n)$ that satisfy the equations $(x_i - z_i) = k(x_i - y_i)$ ($1 \leq i \leq n$), for some positive real number $k < 1$.

³³ The *norm* of a vector space V assigns to each $v \in V$ a non-negative real number $\|v\|$, such that (i) $\|v\| = 0$ if and only if v is the zero vector of V ; (ii) $\|v + w\| \leq \|v\| + \|w\|$ for every $v, w \in V$; (iii) $\|\alpha v\| = |\alpha| \|v\|$ for every $\alpha \in \mathbb{R}$ and every $v \in V$.

³⁴ It is essential the arc be smooth, i.e. that it be the range of a *differentiable* mapping (at least of class C^1) of an interval of \mathbb{R} into our manifold. As noted by Killing (EGG, Vol.II, p.9), not every line can be measured by any other line, by Riemann's methods, if by *line* we understand the range of a *continuous* mapping of a real interval into the manifold.

³⁵ That is, the tangent space of the one-dimensional submanifold $c((a, b))$ at point $c(s)$. This submanifold does not include the points $c(a)$ and $c(b)$. These are included however in the range of an extension of c . See p.363.

³⁶ A *metric space* is a pair (S, d) , where S is a set and d is a mapping of $S \times S$ into \mathbb{R} such that for any $x, y, z \in S$ (i) $d(x, x) = 0$; (ii) $d(x, y) > 0$ whenever $x \neq y$; (iii) $d(x, y) = d(y, x)$, and (iv) $d(x, y) + d(y, z) \geq d(x, z)$. d is called the distance function; $d(x, y)$, the distance between x and y .

³⁷ Appendix, p.365.

³⁸ The covariant tensor field μ and the vector fields $(\partial/\partial x^i)$.

³⁹ The expressions on the left-hand side of equations (5) are known as the Christoffel symbols of the first and the second kind, respectively. Christoffel introduced these functions in his paper of 1869, denoting them by $\begin{bmatrix} i & j \\ k \end{bmatrix}$ and $\left\{ \begin{matrix} i & j \\ k \end{matrix} \right\}$.

⁴⁰ The right-hand side of eqn. (4) on p.80 will read like the positive square root of the right-hand side of eqn. (8) on p.95 if we make a few notational adjustments. Take $n = 2$. Substitute Φ^{-1} for x . (Remember that if $\Phi(\zeta)$ is a surface, Φ^{-1} is a chart on it.) c in eqn. (4) stands for $x \cdot c$ in eqn. (8). Put $E = g_{11} \cdot c$, $F = g_{12} \cdot c = g_{21} \cdot c$, $G = g_{22} \cdot c$. In eqn. (4) the argument t was omitted.

⁴¹ A note on notation: If f and g are covector fields on a manifold M , $f \otimes g$ is the $(0, 2)$ -tensor field which assigns to every pair (v, w) of vector fields on M the scalar field $(f \otimes g)(v, w) = f(v)g(w)$. The 'form' $f \wedge g$ introduced on p.99 is the alternating $(0, 2)$ -tensor field defined by $f \wedge g = (1/2)(f \otimes g - g \otimes f)$.

⁴² On geodesics, see Note 15 and Appendix, pp. 371-374.

⁴³ In other words, the restriction of Exp_P to the said neighbourhood of 0 in $T_P(M)$ is a differentiable injection whose inverse is likewise differentiable.

⁴⁴ Riemann, H, p.14.

⁴⁵ Riemann, H, p.16.

⁴⁶ That is, as differentiable mappings assigning to each point Q in M a linear function on $T_Q(M)$. See Appendix, p.366.

⁴⁷ Hence, if $X \in T_P(M)$, $dx'(X)$ denotes the value at X of the linear function $dx'(P)$.

⁴⁸ Spivak, CIDG, Vol.II, proposition 4 B-4.

⁴⁹ We introduce a factor of -3 instead of Riemann's $-3/4$ because we divide $F(X, Y)$ by the area of the parallelogram formed by X and Y , not by the area of a triangle.

⁵⁰ There are counterexamples to this claim, even for $n = 2$. But R.S. Kulkarni (1970) has shown that if $n > 3$, Riemann's claim is true, except in a special family of cases. To be more precise, let us say that two R -manifolds M and M' are isocurved if there exists a diffeomorphism $f: M \rightarrow M'$ such that, for every $P \in M$ and every two-dimensional subspace α of $T_P(M)$, $k(\alpha) = k(f_{*P}(\alpha))$. Kulkarni's theorem states that if M and M' are isocurved manifolds of dimension $n > 3$ they are globally isometric, unless it happens that they are diffeomorphic not globally isometric manifolds of the same constant curvature.

⁵¹ Riemann, H, p.19. Riemann adds: "Consequently in the manifolds with constant curvature, figures may be placed in any way we choose."

⁵² Riemann's short discussion of surfaces of positive constant curvature in Section 2.2.5 plus an important remark in Section 3.1.2 regarding the finiteness of three-dimensional manifolds of constant positive curvature are, as far as I can see, the only justification for giving the name "Riemann geometry" to the intrinsic geometry of a sphere. Riemann geometry, in this sense, should not be confused with *Riemannian geometry*, or the general theory of R -manifolds. Since the latter, due to its importance and generality, is a better candidate for preserving and honouring the name of its creator, I recommend against the use of "Riemann geometry" for conveying the former sense. We do better just to speak of *spherical geometry*, even if such geometry is true also of manifolds that are not spheres.

⁵³ Submitted to the Paris Academy in 1861, to compete for a prize on a question concerning heat conduction. In this work, Riemann arrives at a result which, according to M. Spivak, "amounts to another invariant definition of the curvature tensor" (Spivak, CIDG, Vol.II, p.4D-24). See Riemann, WW, pp.402f.

⁵⁴ The use of the connection symbol ∇ is explained in the Appendix, pp.369ff. $[X, Y]$ is the Lie bracket of X and Y (Appendix, p.368). If X and Y are vector fields on M (Appendix, p.366) $\mu(X, Y)$ denotes the function $P \mapsto \mu_P(X_P, Y_P)$.

⁵⁵ Clifford, "On the space-theory of matter", abstract of a paper read to the Cambridge Philosophical Society on February 21, 1870.

⁵⁶ According to the definition we have given, an n -dimensional differentiable manifold M cannot contain a boundary. For let P be any point of M and x a chart defined on a neighbourhood U of P . Then $x(U)$ is an open subset of \mathbb{R}^n , so that $x(P)$ has an open neighbourhood V which is entirely contained in $x(U)$. Since P lies in $x^{-1}(V)$ and $x^{-1}(V)$ is open in M , P cannot be a boundary point of M . For a definition of a *manifold with boundary* see Munkres, EDT, pp.3, 43–57.

⁵⁷ Riemann, H, p.22. See Einstein (1917) in Lorentz *et al.*, R, pp.130ff.

⁵⁸ Riemann, H, p.22.

⁵⁹ Riemann, H, p.23.

⁶⁰ Riemann, WW, p.508.

⁶¹ Russell, FG, pp.62f.

⁶² Kant, *KrV*, A 76/B 102, A 100, B 134.

⁶³ Herbart's psychological theory of space is presented in his *Psychologie als Wissenschaft neu gegründet auf Erfahrung, Metaphysik und Mathematik* (1824/25), Herbart, WW, Vol.VI, pp.114–150. See also Herbart, WW, Vol.V, pp.480–514. Space is described by Herbart as a sort of sediment left behind by the flux of our ideas of sense and providing a neutral background, somewhat like a river bed, wherein they flow. See especially Herbart, WW, Vol.VI, p.134.

⁶⁴ Herbart, WW, Vol.IV, p.159.

⁶⁵ Herbart, WW, Vol.IV, p.171.

⁶⁶ Herbart, WW, Vol.IV, p.193.

⁶⁷ Compare Riemann, H, p.10, with Grassmann, WW I.1, pp.47–49, 299.

⁶⁸ Grassmann, WW, I.1, p.297.

⁶⁹ Grassmann, WW, I.1, p.297.

⁷⁰ Grassmann, WW, I.1, p.325.

⁷¹ In an appendix to the second edition of the *Ausdehnungslehre* of 1844 (1878), Grassmann explains that non-Euclidean geometries easily fit in his general theory of extension because three-dimensional non-Euclidean spaces may be regarded as hyper-surfaces of a "region" of higher level, i.e. of a vector space of more than three dimensions (Grassmann, WW, I.1, pp.293f.). This is certainly true. Moreover every connected n -dimensional differentiable manifold can be imbedded into \mathbb{R}^{2n+1} (Whitney (1936)). However this does not imply that the theory of differentiable manifolds can be absorbed by the theory of vector spaces – as Grassmann seems to believe – for a manifold can always be regarded as a *submanifold* but is not usually a *subspace* of a vector space of higher dimension.

2.3 Projective Geometry and Projective Metrics

¹ It is true that projective geometry deals with cross-ratio, which may be defined as a function of the distance between four points on a line, or of the angles between four coplanar lines through a point. But in Sections 2.3.5 and 2.3.10 we shall show how this form of dependence of projective geometry on (Euclidean) metrics can be avoided.

² For greater precision, let α be a plane angle at P and let α' be the angle opposite to α . We call $\alpha \cup \alpha'$ a double angle at P . Let q_α denote the set of all meets of q with lines

through P that lie within the double angle $\alpha \cup \alpha'$. Then the set $\{q_\alpha \mid \alpha \cup \alpha'\}$ is a double angle at P} is a base of a topology on q (regarded as the set of its meets). This is the topology induced on q by a flat pencil through a point outside it. It does not depend on the choice of that point (P in the preceding discussion). (On topologies, see Appendix, pp.360f.)

³ Relative to the topology defined in Note 2.

⁴ Appendix, p.361.

⁵ Appendix, p.361.

⁶ Take the weakest topology that makes every collineation into a homeomorphism. (A collineation is a bijective mapping of projective space onto itself that maps collinear points onto collinear points.)

⁷ Let a_1, \dots, a_n be elements of a vector space over field F (or, more generally, of a module over ring F). a_1, \dots, a_n are said to be linearly dependent if there are elements k_1, \dots, k_n in F, not all equal to zero, such that $k_1 a_1 + \dots + k_n a_n = 0$. (In the text above, regard \mathbb{R}^3 as a vector space over \mathbb{R} .)

⁸ A more satisfactory approach to complex projective geometry, which does not depend, like ours, on numerical representations, was proposed by von Staudt in 1856. See Staudt, BGL, pp.76ff.; Lüroth (1875). The significance of von Staudt's proposal in the history of mathematics is clearly brought out by Freudenthal (1974).

^{8a} Solutions exist unless a_{11} , its cofactor A_{11} and the determinant $|a_{ij}|$ all have the same sign. If that condition is fulfilled the polarity is elliptic.

⁹ As an immediate corollary of the stated characteristic of conics let us mention that a collineation always maps a conic onto a conic. Let f be a collineation and ζ a conic which is the locus of self-conjugate points under a polarity g ; then fgf^{-1} is also a polarity and its locus of self-conjugate points is $f(\zeta)$.

¹⁰ That is, if none of them can be represented by a set of complex homogeneous coordinates $(a_j + b_j i)$ such that $b_j = 0$ for $j = 1, 2, 3$.

¹¹ An interesting exception occurs if the left-hand side of the linear equation is a factor of the left-hand side of the quadratic equation. In that case, the conic represented by the latter is a degenerate one composed of two lines, one of which is the one given by the linear equation. Trivial exceptions occur if any of the left-hand sides is identically zero.

¹² But of course these "circles" in \mathcal{P}^2 are not the loci of points equidistant from a given point. We have, as yet, no concept of distance in our projective plane.

¹³ Lie, VCG, pp.130f., 216f.

¹⁴ Lie, VCG, pp.131ff. By 'reducible' I mean that its value can be calculated from the value of the cross-ratio (thus, the square or cube of the cross-ratio are reducible to the cross-ratio, etc.).

¹⁵ To ensure that f_ζ is defined on every point-pair (P, Q) we index the points $(PQ/\zeta)_1$ so that $(PQ/\zeta)_1 \neq P$ and $(PQ/\zeta)_2 \neq Q$.

¹⁶ Since P_1, P_2, P_3 are collinear their joins meet ζ at the same two points, say $(p_i), (q_i)$. We may therefore calculate $f_\zeta(P_j, P_h)$ ($j, h = 1, 2, 3$) using eqn. (5) of Section 2.3.5 for the cross-ratio. Let $P_j = (k_j p_j + m_j q_j)$. Then $f_\zeta(P_j, P_h) = m_j k_h / k_j m_h$. Consequently

$$d_\zeta(P_1, P_2) + d_\zeta(P_2, P_3) = c \left(\log \frac{m_1 k_2}{k_1 m_2} + \log \frac{m_2 k_3}{k_2 m_3} \right) = c \log \frac{m_1 k_3}{k_1 m_3} = d_\zeta(P_1, P_3).$$

- ¹⁷ See Note 36 to Part 2.2 (p.384).
- ¹⁸ Laguerre (1853). See Coolidge, HGM, p.80.
- ¹⁹ Its dual, i.e. the corresponding degenerate point conic, is none other than the ideal line $x_3 = 0$, which joins the two circular points.
- ²⁰ Arthur Cayley, "Sixth memoir upon quantics" (1859).
- ²¹ Strictly speaking, the duals of segments are the double angles defined in Note 2.
- ²² Cayley did not in fact employ our function d_t . Instead of the natural logarithm he used the cyclometric function *arc cos*. Cayley welcomed Klein's substitution of a logarithmic function for his cyclometric function as "a great improvement, for we at once see that the fundamental relation $\text{dist}(PQ) + \text{dist}(QR) = \text{dist}(PR)$, is satisfied." (Cayley, CP, Vol.II, p.604). A clear sketch of Cayley's theory is given in Kline, MT, pp.907-909.
- ²³ Cayley, CP, Vol.II, p.592.
- ²⁴ By a real conic, I mean a conic some of whose points are real, i.e. such that they can be represented by a set of homogeneous coordinates with their imaginary part equal to zero (see Note 10).
- ²⁵ See Klein (1871) and Klein, VNG, Chapters VI and VII.
- ²⁶ They are: two real lines meeting at a real point, two conjugate imaginary lines meeting at a real point; one real line with two distinguished real points on it, one real line with two distinguished conjugate imaginary points on it (the only case considered by Klein in 1871), and one real line with one distinguished point on it. See Klein, VNG, pp. 74, 85, 181-184.
- ²⁷ See Klein, *Elementary mathematics from an advanced standpoint: Geometry*, pp.181ff.
- ²⁸ Klein, VNG, p.189.
- ²⁹ E.g., Klein, VNG, p.221.
- ³⁰ See Cayley's remarks in Cayley, CP, Vol.II, pp.604f.
- ³¹ Klein (1871), pp.582f.
- ³² Borsuk and Szmielew, FG, pp.245ff.
- ³³ This is equal to the absolute value of $d_t(Q, R)$ if we choose $c = 1/2$. Angle measure is defined analogously.
- ³⁴ Beltrami, OM, Vol.I, p.375.
- ³⁵ Beltrami confirms this view in his paper of 1869 where he develops the general theory of n -dimensional spaces of constant curvature. "Every concept of non-Euclidean [BL] geometry finds a perfect equivalent in the geometry of the space of constant negative curvature. It should be observed, however, that, while the concepts belonging to simple planimetry receive in this manner a true and proper interpretation, since they turn out to be *constructible* upon a *real* surface, those which embrace three dimensions will only admit an analytical representation, because the space in which such a representation could materialize (*verrebbe a concretarsi*) is different from that to which we generally give that name." (Beltrami, OM, Vol.I, p.427.)
- ³⁶ The equation of a tractrix referred to the axis of rotation $x = 0$ and to a suitably chosen orthogonal axis $y = 0$ is

$$y = k \log \frac{k + \sqrt{k^2 - x^2}}{x} - \sqrt{k^2 - x^2}.$$

The curvature of the pseudosphere is $-1/k^2$.

³⁷ Reprinted in Hilbert, GG, Anhang V, pp.231–240.

³⁸ See Poincaré's own brief exposition in a note appended to the 7th edition of Rouché and Comberousse's *Traité de Géométrie*, Vol.II, pp.581–593.

³⁹ A stereographic mapping of a sphere S from a point $P \in S$ onto a plane α tangent to S is a mapping assigning to each point $Q \in S$ the point Q' where the straight line PQ meets plane α . See Fig. 15 on p.137.

⁴⁰ There is a four-page introduction in which Klein describes the contents and background of the two parts of his article and makes some remarks on the general aim of investigations concerning non-Euclidean geometry. Their aim is not "decision on the validity of the axiom of parallels"; their only concern is with the question "*whether the axiom of parallels is a mathematical consequence of the other axioms listed by Euclid*" (Klein, 1873, p.113). Other benefits reaped from such investigations are (i) "enlarging of the circle of our mathematical concepts" and (ii) "our being provided with material for judging the necessity of our familiar geometrical representations and modifying them in the appropriate way in case this should turn out to be desirable"; therein lies "the significance of these investigations for physics". (Klein (1873), p.114.)

⁴¹ Klein (1873), p.116.

⁴² See Appendix, p.362.

⁴³ Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Erlangen: A. Düchert, 1872. This is Klein's celebrated Erlangen Programme. I shall quote from the revised edition of 1893, hereafter designated by EP. A note in Klein (1873), p.121, suggests that this paper was written before the Programme, though the latter appeared first.

⁴⁴ Readers unfamiliar with the concept of a group ought to read carefully the definitions given in the Appendix, p.360.

⁴⁵ Strictly speaking, d_ζ is not defined on any pair $(x, y) \in (\mathcal{P}_\mathbb{C}^n \times \mathcal{P}_\mathbb{C}^n)$ such that x or y lies on ζ . We pointed this out above for $n = 2$. If ζ is degenerate it may happen that d_ζ is invariant not under the group of collineations that map ζ onto itself, but only under a proper subgroup of that group.

⁴⁶ The reader should observe that when we transfer to M' the G -geometry of M via an arbitrary bijection f , we ignore all structure that M' might possess on its own. Disregard of this point has often caused confusion. Thus, it is well known that \mathbb{R}^2 can be mapped bijectively onto \mathbb{R} (Cantor, GA, pp.119–133). E. Stenius mentions this fact to prove that we can define ordinary *plane* geometry on a *line*. But such a statement is misleading. We can certainly do as he says, but only on the condition that we forget that the line is a one-dimensional topological space and that we treat it as an abstract set (with the power of the continuum). When we define plane geometry on this set we endow it with the structure of a two-dimensional topological space. See Stenius, *Critical Essays*, p.54.

⁴⁷ Klein (1873), p.123. See p.100, Section 2.2.8; and Section 3.1.1.

⁴⁸ Klein (1873), p.124. See above, p.134.

⁴⁹ Klein (1873), p.124.

⁵⁰ Cf. p.93; Appendix, pp.372ff.

⁵¹ Schouten (1926), p.143.

⁵² Klein (1873), pp.132ff. He returns to the subject in Klein (1890), pp.565–570.

⁵³ Klein (1871), pp.623f. Von Staudt's construction of the fourth harmonic to three

given points or lines is presented in his *Geometrie der Lage* (1847), pp.43ff. The assignment of homogeneous coordinates to space by means of this construction was introduced in von Staudt's *Beiträge zur Geometrie der Lage* (1856), pp.261ff.

⁵⁴ Zeuthen's proof, contained in a letter addressed to Klein after the publication of Klein (1873), is reproduced in Klein (1874), §2. A similar proof was sent to Klein by Lüroth. Let us recall that, if p, q, r and s are four different lines of a flat pencil, p and q are said to separate r and s if p is contained in one of the two pairs of vertically opposite angles formed by r and s , and q is contained in the other. Cf. p.411, n.44.

⁵⁵ The lines of net (uvw) fall into two classes: those which have been assigned a rational number whose absolute value is less than π , and those which have been assigned a rational number whose absolute value is greater than π . No pair of lines in the first class is separated by a pair of lines in the second class. There is a unique line in pencil X which, together with the line labelled ∞ , separates the remaining lines of one class from the remaining lines of the other class. This line cannot be associated with a rational number and consequently does not belong to (uvw) .

⁵⁶ Take, for example the last case we mentioned. Point O is mapped on $[0, 0, 1]$; point Y on $[s, t, 1]$. Every triple of the class $[s, t, 0]$ is a solution of the equation

$$\begin{vmatrix} x_1 & s & 0 \\ x_2 & t & 0 \\ x_3 & 1 & 1 \end{vmatrix} = 0.$$

Consequently, this class lies on the join of $[0, 0, 1]$ and $[s, t, 1]$. (See p.124.)

⁵⁷ See, in particular, his Evanston lecture of 1893, "On the mathematical character of space intuition and the relation of pure mathematics to the applied sciences" (in English in Klein, GA, Vol.II, pp.225–231). See also the following: his 1873 lecture "Ueber den allgemeinen Funktionsbegriff", in Klein, GA, Vol.II, pp.214–224; Klein (1890), pp.570–572; (1897), pp.584–587; (1902) pp.146–148; EP, p.94, etc.

⁵⁸ Klein (1897), p.593. Projective intuition is mentioned in Klein (1890), p.570f.; it is contrasted with "ordinary intuition" in Klein, EP, p.75.

⁵⁹ Klein, GA, Vol.II, p.250.

⁶⁰ Klein, GA, Vol.II, p.226.

⁶¹ Apparently Klein believed that intuition did more than just suggest this. He writes: "The straight line of ordinary intuition (*der gewöhnlichen Anschauung*) contains only one infinitely distant point. We can approach this point from either side without ever reaching it." (Klein (1871), p.593).

⁶² Clifford explained his discovery in a lecture that Klein attended in 1873. The text of this lecture was never published but Klein reconstructed its mathematical contents in Part I of his paper of 1890. A very clear account of Clifford's discovery and its implications is given in Klein, VNG, pp.233–238, 241–249, 254–270. There is a short description in Klein (1897), pp.591–593. The matter has been further dealt with by Killing (1891) and Hopf (1926).

⁶³ The polar of a is the line assigned to a by the involutory correlation which defines ζ . (Correlations in 3-space map points on planes, lines on lines.)

⁶⁴ Let JKLM be such a quadrilateral, with JK opposite and equal to LM, JM opposite and equal to LK. The geodesic LJ (on Σ) divides JKLM into two congruent geodesic

triangles, whose six angles add up to 2π . Consequently these triangles have neither an excess nor a defect. (See p.74.)

⁶⁵ Wolf, *Spaces of Constant Curvature*, p.123. The classification of all compact 3-dimensional Euclidean space-forms was given by W. Hantzsche and H. Wendt (1935).

⁶⁶ Klein (1897), p.595.

⁶⁷ Klein, VNG, p.270.

CHAPTER 3. FOUNDATIONS

¹ Hegel, *Logik; Werke*, ed. Glockner, Vol.V, p.306.

² Delboeuf, PPG, pp.75ff.

³ Since the axiom of completeness is lacking – it was added in the French translation of 1900 – the system can be modelled in the denumerable set of algebraic number triples. See p.237.

3.1 Helmholtz' Problem of Space

¹ Helmholtz (1866), p.197.

² Riemann, H, p.8. See above, p.84.

³ Riemann, H, p.21. See above, p.104.

⁴ R4 is of course implied by R5. I state it separately in order to bring out the increasingly restrictive character of these hypotheses. The infinity of space follows from R5, if space is unlimited; in Part 2.2, Note 56, we argued that space must be unlimited if it is a manifold.

⁵ Helmholtz (1870), G, p.19.

⁶ Helmholtz (1868), G, p.60.

^{6a} Obviously Helmholtz has no qualms about the truth of R1, which he probably regarded as an analysis of the meaning of "space".

⁷ See above, p.100. A proof was given by Lipschitz (1870).

⁸ Helmholtz (1866), pp.199, 201; (1868), G, pp.36, 60; (1870), G, p.29. Helmholtz is so convinced of the necessity of this requirement that he does not even perceive that Riemann was of another mind. He simply overlooks the fact that Riemann, who shared with him the assumption that physical measurement involves superposition of physical magnitudes, inferred from this assumption, not that *every body* must be superposable with every other body, but that *every line* must be measurable by – and to this end totally or partially superposable with – every other line (p.91). We might thus say that Riemann's physical geometry requires ideally thin, perfectly flexible and inextensible strings, but does not require, like that of Helmholtz, absolutely rigid bodies.

⁹ Helmholtz (1870), G, p.29.

¹⁰ See p.177.

¹¹ Helmholtz, G, p.41.

¹² Helmholtz's reluctance to extend his axioms concerning rigid point systems to systems of any arbitrary size is motivated perhaps by a fact he learned through his investigations on the psychophysiology of vision: "The visual field exhibits a more restricted mobility of the images on the retina". (Helmholtz, G, p.40).

¹³ "All rotations of the system about the point $r = s = t = 0$ " (Helmholtz, G, p.57) really

means all infinitesimal rotations about any arbitrarily fixed *line element* through that point.

¹⁴ Helmholtz (1868), G, p.57. For a reconstruction of Helmholtz's proof, which introduces clarity and precision into it, while remaining faithful to its spirit, see Weyl, MAR, pp.29-43.

¹⁵ Helmholtz (1866), p.201.

¹⁶ Kant, *Ak.*, Vol.IV, p.312.

¹⁷ Helmholtz, G, p.71f.

¹⁸ Helmholtz, G, p.8; PSL, p.227.

¹⁹ Helmholtz, G, p.25; PSL, p.241. Cf. Poincaré, FS, pp.416f.

²⁰ Helmholtz, G, p.28; PSL, p.244.

²¹ Helmholtz, G, p.81.

²² Helmholtz (1878), p.213; G, p.62.

²³ Schlick in Helmholtz, SE, p.162.

²⁴ "As all our means of sense-perception extend only to space of three dimensions, and a fourth is not merely a modification of what we have, but something perfectly new, we find ourselves by reason of our bodily organization quite unable to represent a fourth dimension." Helmholtz, G, p.28; PSL, p.244.

²⁵ I cannot accept, however, Schlick's alternative suggestion that Helmholtz's "general form of spatial intuition" exhibits space as a three-dimensional continuous manifold wherein quantitative comparisons are possible (Schlick in Helmholtz, SE, p.161). The possibility of quantitative comparisons is not a consequence of the general form of extendedness, since it presupposes the existence of rigid bodies. A liquid mathematician living in a liquid world cannot, according to Helmholtz, develop a geometry, but he could very well share our general form of spatial intuition.

²⁶ Helmholtz, G, p.29; PSL, p.244.

²⁷ Helmholtz, G, p.17; PSL, p.234. Formerly, he had tried to make this intuitively clear in the light of the two-dimensional case. In that context, he introduces a two-dimensional country inhabited by two-dimensional rational beings who "have not the power of perceiving anything outside" the surface they live on, but have, upon it, "perceptions similar to ours"! (Helmholtz, G, p.8; PSL, p.227). These beings will supposedly develop the intrinsic geometry of their country. But if they happen to live, say, upon an egg, they will be unable to build transportable rigid figures and, consequently, according to Helmholtz, they will be incapable of defining a geometry. Though Helmholtz introduces the example of an egg, he does not draw the latter consequence; had he drawn it, it would probably have shocked him out of his operationist bias, for he certainly knew that one can define with Gaussian methods the intrinsic geometry of an egg-like surface. Let us remark that the country of Flatland is used by Helmholtz merely as a didactic prop, and ought not to be taken too seriously. Greater significance must be attributed to the Helmholtzian country we mentioned on p.165 *viz.* Through-the-Convex-Looking-Glass.

²⁸ Helmholtz (1866), p.197; cf. Helmholtz (1868), G, pp.32f.

²⁹ Helmholtz (G, pp.29f.; PSL, p.245) says "transcendental in Kant's sense". But the sense intended is certainly not Kant's, for Helmholtz says in the same sentence that experience "need not exactly correspond therewith".

³⁰ Helmholtz, G, p.30; the English version in PSL, p.245 is very free.

³¹ The expression is not really Kantian, but a shibboleth of the disreputable philosophies of Fichte and Schelling.

³² Helmholtz, G, p.29; PSL, pp.244f.

³³ Poincaré (1898), p.40, acknowledges that he owes much to Helmholtz.

³⁴ Helmholtz, G, p.29; PSL, p.245 (quoted in Section 3.1.2, p.160).

³⁵ Helmholtz, G, p.30; PSL, p.245.

³⁶ Lie (1890), p.359 n.2.

³⁷ That Lie's continuous groups are connected follows from the definition in Lie, TT, Vol.I, p.3.

³⁸ It is clear that L_g is surjective, since f is surjective. We prove that L_g is injective: $L_g(m) = L_g(n) \rightarrow f(g, m) = f(g, n) \rightarrow f(g^{-1}, f(g, m)) = f(g^{-1}, f(g, n)) \rightarrow f(e, m) = f(e, n) \rightarrow m = n$. (Read ' \rightarrow ' as 'implies that'.)

³⁹ This is now the usual definition of transitive action. But Lie's own definition, in modern terms, would read as follows: G acts transitively on M if there is an open subset $M' \subset M$ such that, for every pair x, y in M' , there is a $g \in G$ with $gx = y$ (see Lie, TT, Vol.I, p.212).

⁴⁰ This is said of R_3 in Lie (1890), p.20. See also Lie, TT, Vol.I, p.572.

⁴¹ Lie, TT, Vol.III, pp.385, 387, etc. Observe that by fixing once and for all the nature of the space R_n on which his groups are allowed to act, Lie was able to ignore the complications that arise from considering the action of groups on manifolds of diverse global topologies.

⁴² This statement of the problem is paraphrased from Lie, TT, Vol.III, p.397. Lie says, however, that the sought for properties should distinguish the three groups mentioned above from "all other possible groups of motions of a number-manifold (*Zahlenmannigfaltigkeit*)". In the light of Lie's actual performance I have judged it appropriate to broaden "groups of motions" to "groups of analytic transformations", while narrowing down "eine Zahlenmannigfaltigkeit" to \mathcal{P}_C^n . It is indeed not easy to say what Lie (or Engel) understands by "eine Zahlenmannigfaltigkeit". In TT, Vol.III, p.394, this expression is introduced by saying that, in his lecture of 1854, "Riemann let all his research be guided by the proposition that space is a *Zahlenmannigfaltigkeit*, so that the points of space can be determined by coordinates." It might seem, therefore, that a *Zahlenmannigfaltigkeit* is for Lie any n -dimensional, real or complex, presumably analytic, differentiable manifold. This is somehow confirmed by the fact that Lie calls any regular submanifold of R_n "eine Mannigfaltigkeit des R_n " (Lie, TT, Vol.I, p.133). On the other hand, there is no doubt that "the space R_n " discussed in the subsequent chapters on Helmholtz's problem and the foundations of geometry is \mathcal{P}_C^n , \mathcal{P}^n , or an open subset of either. This is very clearly brought out in Lie's own statement of some of his results in Lie (1890), pp.303f., 308, 312. Unfortunately, the text of TT, edited by Engel, is not so careful on such conceptual matters.

⁴³ The first of these, in Lie, TT, Vol.III, pp.452f., is Lie's "translation" of Helmholtz's axioms into group-theoretical language. Lie criticizes Helmholtz's own deductions from his axioms because Helmholtz "nimmt [...] stillschweigend und ohne ein Wort der Begründung an, dass alle seine Axiome, die er über die nach Festhaltung eines Punktes noch möglichen Bewegungen aufgestellt hat, auch auf die Punkte anwendbar bleiben, die dem festen Punkte unendlich benachbart sind". (Lie, TT, Vol.III, p.454). This would imply that, if G is a group which fulfils the axioms in TT, Vol.III, pp.454f., and if

G_P is a subgroup of G which leaves invariant a point P , then the group $G_{*P} = \{g_{*P} \mid g \in G\}$, acting on the tangent space at P , also fulfils those axioms (See Note 44). Lie quotes several counterexamples which show that this assumption is false (one of them is very clearly explained by Paul Herz in Helmholtz, SE, p.63). In TT, Vol.III, pp.465–471, Lie shows however that the axioms in TT, Vol.III, pp.452f. do characterize the Euclidean and the non-Euclidean motions of R_3 , if these axioms apply to *every* point (not just to the points of general position – cf. Lie (1890), p.386) in an open connected subset of R_3 . This proof does not depend on the group-theoretical equivalent of the axiom of monodromy H4. On the other hand, if the axioms apply only to the points of general position, they do not suffice to characterize those three groups of motions, *even if we add H4*. A second set of axioms is given in TT, Vol.III, p.461; the proof that they constitute a solution of the HL problem was suggested to Lie by Helmholtz's calculations. Lie's main solutions of the HL problem are given in TT, Vol.III, pp.471–523. The first one, which is described in the text, was also given in Lie's first paper of 1890. The second one is proved for R_3 only. It is based on a set of axioms (in TT, Vol. III, pp.506f.) which make no assumptions concerning infinitesimal displacements. It might seem, therefore, that this solution is more faithful than the first one to Helmholtz's empiricist spirit.

⁴⁴ Let us recall that if G is a Lie group acting on a manifold M , to every g in G there is associated a unique analytic transformation of M which we have agreed to call g (also L_g – see p.173). An analytic transformation $g: M \rightarrow M$ induces at each $P \in M$ a mapping $g_{*P}: T_P(M) \rightarrow T_{gP}(M)$, which is defined as follows: given any analytic function f defined on a neighbourhood of gP , the value of $g_{*P}(v)$ at f , for any $v \in T_P(M)$, is equal to the value of v at $f \circ g$ (the latter composite function is obviously defined on a neighbourhood of P).

⁴⁵ Lie, TT, Vol.III, p.481. Two groups are similar, in Lie's idiom, if one can be obtained from the other through a coordinate transformation of the underlying manifold (*ibid.*, p.364). I assume that a "real point transformation" of the complex manifold R_n is a coordinate transformation that maps real coordinates onto real coordinates, non-real coordinates onto non-real ones (thus preserving the division between real and imaginary points).

⁴⁶ Lie, TT, Vol.III, p.538. The terms 'proper subgroup' and 'normal subgroup' are defined in the Appendix, p.360.

⁴⁷ Our definition of free mobility in the infinitesimal is inapplicable if $n < 3$. For $n = 2$, we may restate it thus: Let G be a group acting on R_2 ; G has free mobility in the infinitesimal at a real point $P \in R_2$ if G contains a one-dimensional subgroup which fixes P , but no subgroup of G , except $\{e\}$, fixes a non-zero tangent vector at P . The Euclidean and non-Euclidean groups of motions of R_2 possess free mobility in the infinitesimal in this sense at a point of general position, but there is another group of R_2 which also possesses it. If x is a chart of R_2 , we can express a basis of the Lie algebra of this group in terms of this chart as follows:

$$\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} + c \left(x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} \right) \right\}.$$

Here, c is any real number $\neq 0$ (Lie (1890), p.290). This group is excluded by the two-dimensional analogon of the axiom of monodromy, which is probably the reason

why Helmholtz thought this axiom necessary also in the general n -dimensional case. (See Helmholtz, G, p.21).

⁴⁸ Poincaré (1887).

⁴⁹ Axiom D does the job Poincaré expects it to do only if we assume that the *distance* between two points P, Q is the same as the *length* of the segment PQ . This assumption is implicit in Poincaré's own statement of feature (a) of the geometry defined by the one-sheet hyperboloid.

⁵⁰ Killing (1892), p.129.

⁵¹ Killing (1892), p.153. I understand he means transitive n -dimensional connected Lie groups.

⁵² Killing (1892), p.131.

⁵³ Killing (1892), p.137. Compare Killing's treatment of dimension number in EGG, Vol.I (1893), Absch.3, especially pp.238–265.

⁵⁴ Killing (1892), p.167.

⁵⁵ One-parameter groups and their generators are defined in the Appendix, p.369.

⁵⁶ In other words, if there is a g in G which maps the A_i on points arbitrarily near (A_i) ($i = 1, 2, 3$), there is an h in G which maps (A_i) on (A_i) .

⁵⁷ Essential steps in the argument show that: (i) If K is a true circle, $x(K)$ is a closed Jordan curve. (ii) The rotations about a point P constitute a group isomorphic to the group of rotations about a point in \mathbb{R}^2 . (iii) If P, Q are two points of π , there is a unique point $M \in \pi$, the midpoint of PQ , such that there exists a rotation g about M with $gP = Q$ and $gQ = P$. (g is called a half-rotation.) (iv) We can choose two points O, E in π with the following property: let m denote the set generated from O and E by the operations of taking midpoints and performing half-rotations; let \bar{m} denote the set $\{P \mid P \text{ is the limit of a convergent sequence of points in } x(m)\}$; then \bar{m} is the range of an injective continuous mapping $\mathbb{R} \rightarrow \mathbb{R}^2$; if $g \in G$, $gx^{-1}(\bar{m})$ is called a *true line*.

⁵⁸ Hilbert, GG, p.181n.

3.2 Axiomatics

¹ See also Descartes, AT, Vol.VII, pp.160–170; Vol.IX, pp.124–132.

² "Les géomètres supposent toutes ces propositions sans le dire." I have not seen a copy of Lamy's book; I take this quotation from P. Rossier, "Les axiomes de la géométrie et leur histoire" (1967), pp.379f., who obtained it from the 6th edition (1734).

³ See G. Goe (1962).

⁴ See the passage quoted on p.49.

⁵ Plücker, *Neue Geometrie des Raumes* (1868–69); see Nagel (1939), pp.188–192.

⁶ Strictly speaking, I demand that the set of axioms be computable. For a definition of *computable set* see, e.g., M. Davis, *Computability and Unsolvability*. Informally, we may say that a set S is computable if an ordinary computer with unlimited memory can be programmed to determine whether any given object belongs to S or not.

⁷ Let me recall that Leon Henkin, in his justly celebrated paper on "Completeness in the theory of types" (1950), uses a different concept of interpretation (and hence, a different notion of logical consequence). In Henkin's semantics, object variables range over an arbitrary set of entities, as above, but first-order n -ary predicate variables range

over an arbitrary class of n -ary relations between the entities of that set. The latter are freely stipulated in each interpretation. Henkin's semantics underlies Abraham Robinson's "non-standard analysis", but is otherwise foreign to mathematical English and its formalizations, as they are normally understood.

⁸ See Oswald Veblen (1911), pp.5f.

⁹ The informal characterization of the structural equivalence of models of first-order theories given above can be made more precise with the aid of a few auxiliary concepts. We consider first a theory T in which all variables are of one kind. Let T^* , I_1 , I_2 , D_1 , D_2 and f be as above. Let $T_{\#}^*$ be the set $\{s \mid S \text{ is a basic sentence and } S \text{ belongs to } T^*\}$. Let $FT_{\#}^*$ be the collection of all sentence schemes ("formulae") obtainable by replacing in each sentence of $T_{\#}^*$ none or one or more than one of the object constants that occur in it by free variables, in every conceivable combination. A *variant* of I_1 is an interpretation I_1' which differs from I_1 at most in the denotations assigned to one or more object variables. For each variable x , if the variant I_1' of I_1 assigns to x the denotation $m \in D_1$, the *corresponding variant* of I_2 (relative to bijection f) assigns to x the denotation $f(m)$. The two models of T , I_1 and I_2 , are *structurally equivalent* or *isomorphic* if, and only if, for every formula φ in $FT_{\#}^*$, φ is satisfied in a variant I_1' of I_1 if, and only if, it is satisfied in the corresponding variant of I_2 . This characterization can be easily extended to theories with object variables of several kinds, V_1, \dots, V_n . I_1 will then allow the variables of type V_j to range over a domain D_j^i . f must be a bijection of $\cup_j D_j^i$ onto $\cup_j D_j^i$ that maps D_j^i onto D_j^i ($1 \leq j \leq n$).

¹⁰ A more precise characterization will be found in Grzegorzcyk, OML, pp.347ff.

¹¹ See below, Notes 74 and 124.

¹² See John Mayberry (1977).

¹³ The better known axiom systems for set theory derive from those propounded by Zermelo in 1908 and by von Neumann in 1925. (In English in Heijenoort, FFG, pp.199ff., 393ff.) All such systems contain axioms to the effect that if some definite set or sets exist, then another set also exists. Particularly far reaching is the so-called Axiom of the Power Set: If a set S exists, the set of all subsets of S – called the power set of S and denoted in this book by $\mathcal{P}(S)$ – also exists.

¹⁴ Stewart, WW, Vol.III, p.117. See also the quotation *ibid.*, p.123, n.1.

¹⁵ Stewart, WW, Vol.III, p.114.

¹⁶ See, however, the passage from Buffon's *Histoire Naturelle* (Vol.I, Paris 1749, pp.53f.) quoted in Tonelli (1959), p.47 n.52.

¹⁷ Stewart, WW, Vol.III, pp.26, 31, 32 n.1.

¹⁸ Grassmann, WW, I.1, p.10.

¹⁹ Grassmann, WW, I.1, p.65.

²⁰ See the references in Note 67 of Part 2.2.

²¹ If (e_1, e_2, e_3) is a basis of such a vector space, the scalar product of two vectors $v = \sum_i a_i e_i$, $w = \sum_i b_i e_i$ is $\sum_i a_i b_i$ ($a_i, b_i \in \mathbb{R}$). The norm assigns to each vector v the real number $\|v\|$ equal to the positive square root of the scalar product of v with itself. A clear sketch of the structure of a Grassmann third level system is given in Kline, MT, pp.782ff.

²² Veronese, GG, p.674.

²³ Let V be a three-dimensional vector space with scalar product f . A basis (e_1, e_2, e_3) of V is orthonormal if $f(e_i, e_j) = \delta_{ij}$ ($i, j = 1, 2, 3$).

- ²⁴ Plücker, *Gesammelte wissenschaftliche Abhandlungen*, Vol.I, p.173; quoted by Nagel (1939), pp.191f.
- ²⁵ Plücker, *ibid.*, p.149; quoted in Nagel (1939), p.191.
- ²⁶ This was first given in the 3rd edition and maintained until the 8th edition. In the 9th edition, Legendre withdrew the proof, because he found it defective; but in the 12th he gave a new one.
- ²⁷ Bolzano (1804), p.46. (My italics.)
- ²⁸ An additional assumption introduced in §21 can be easily seen to follow from that in §24. It can be paraphrased thus: Let a , b and c be three points; if $D(a, b)$ is either the same as or opposite to $D(a, c)$, then $D(b, a)$ is either the same as or opposite to $D(b, c)$ and $D(c, a)$ is either the same as or opposite to $D(c, b)$.
- ²⁹ On Staudt, see Freudenthal (1974).
- ³⁰ Delboeuf, PPG, p.127.
- ³¹ Delboeuf, PPG, p.128.
- ³² Delboeuf's philosophical justification of this idea is examined in Section 4.2.4.
- ³³ Delboeuf, PPG, p.129.
- ³⁴ Delboeuf, PPG, pp.223, 229, 222.
- ³⁵ Hoüel, PFGE, p.2. I quote after the second edition (1883), which is equal to the first, except for a few changes in the introduction and the inclusion of the essay I mention on p.209.
- ³⁶ This had appeared already in *Archiv der Mathematik und Physik*, Vol.40 (1863), under the name "Essai d'une exposition rationnelle des principes fondamentaux de la géométrie élémentaire".
- ³⁷ Hoüel, PFGE, pp. 63, 64.
- ³⁸ Hoüel, PFGE, pp.64f.
- ³⁹ Hoüel, PFGE, p.66. Compare Hilbert's letter to Frege, of December 1899, quoted on p.235.
- ⁴⁰ Rossier (1967), p.896.
- ⁴¹ Méray (1869). Méray's work is summarized in Mannheim, GPST, pp.80–82.
- ⁴² Pasch, *Vorlesungen über neuere Geometrie*, first edition, Leipzig 1882 (VNG); second edition, Berlin 1926 (VNG²). The second edition was revised by Pasch himself, not by Max Dehn, as we read in Kline, MT, p.1008. Dehn wrote the valuable historical study appended to this edition.
- ⁴³ Pasch, VNG, p.3.
- ⁴⁴ Pasch, VNG, p.III. Pasch adds that these original concepts of geometry have later "been covered, little by little, with a net of artificial concepts, designed to favour the theoretical development".
- ⁴⁵ Pasch, VNG², p.3. In the first edition they were called *fundamental concepts* (*Grundbegriffe*).
- ⁴⁶ Pasch, VNG, p.16.
- ⁴⁷ Pasch, VNG, p.4.
- ⁴⁸ Pasch, VNG², p.16. In the first edition they were called *principles* (*Grundsätze*). (VNG, p.17).
- ⁴⁹ Pasch, VNG, p.43. Please note that this remark applies to *every* axiom, not just to some of them, as we read in Kline, MT, p.1008.
- ⁵⁰ Pasch, VNG², p.18.

- ⁵¹ Pasch, VNG, p.45.
- ⁵² Pasch, VNG, p.6.
- ⁵³ Pasch, VNG, p.5.
- ⁵⁴ Pasch, VNG, p.17.
- ⁵⁵ Pasch, VNG, p.43.
- ⁵⁶ Pasch, VNG, p.98.
- ⁵⁷ Pasch (1917), p.185.
- ⁵⁸ Pasch, VNG, p.3.
- ⁵⁹ Pasch, VNG, p.IV; VNG², p.VI.
- ⁶⁰ Pasch, VNG, pp.5–7; VNG², pp.5–7. Axioms S I–S VIII are *exactly the same* in the first and in the second edition. S IX is added in the latter. I have been unable to trace the source of a different list of eight axioms which Paul Rossier (1967), pp.400f., says he found in VNG². I have prefixed the letter S (for *Strecke, segment*), to the axioms of Pasch's first group, in order to distinguish them from the other two groups, which I call E (for *Ebene* = plane) and K (*Kongruenz*).
- ⁶¹ Pasch, VNG, pp.20–21; VNG², pp.19–20. (Same text in both editions.)
- ⁶² Pasch, VNG, p.34.
- ⁶³ Pasch, VNG, p.40.
- ⁶⁴ Pasch, VNG, p.51.
- ⁶⁵ Pasch, VNG, p.58.
- ⁶⁶ Pasch, VNG, p.58.
- ⁶⁷ Pasch, VNG, pp.103–110; VNG², pp.94–101. The text of these axioms is the same in both editions. My version of them is more a paraphrase than a translation, except for K VI, which I have translated literally.
- ⁶⁸ Pasch, VNG, §11, pp.83ff. See Section 2.3.9, pp.143ff.
- ⁶⁹ Schur (1899) showed how to introduce homogeneous coordinates using axioms of congruence, but without having to assume the Archimedean axiom or any axiom of continuity.
- ⁷⁰ Pasch, VNG, p.120.
- ⁷¹ Pasch, VNG, pp.125f.
- ⁷² Pasch, VNG, p.127. Theorem 1.8 says that if two proper points A, B lie on a line m , you can always choose a proper point C on m which lies between A and B (Pasch, VNG, p.10). Pasch remarks however that this theorem cannot be applied to a given line an indefinite number of times (*Ibid.*, p.18). Theorem 1.8 follows immediately from Axiom S II. It can also be derived from Axioms S I, S VI, S IX, E IV (Hilbert, GG, p.5, Satz 3). Consequently, one or more of these axioms have a restricted application. Such are the miseries of mathematical empiricism.
- ⁷³ Pasch, VNG², p.174 (added in the 2nd edition).
- ⁷⁴ This corresponds to our distinction between interpretable and non-interpretable words. Peano regarded set theory as a part of logic.
- ⁷⁵ Peano (1894), p.52. These ideas must have a name in every language, at least in all languages known to be suitable for geometrical discourse. Consequently, says Peano, *space* cannot be a basic concept of geometry, because there is no word for it in the language of Euclid and Archimedes.
- ⁷⁶ Peano (1889), p.24.
- ⁷⁷ I prefix P (for *position*).

- ⁷⁸ Peano (1894), p.62.
- ⁷⁹ Fano (1892), p.108; quoted by Freudenthal (1957), p.112n.
- ⁸⁰ Veronese, GG, p.656; my italics.
- ⁸¹ Veronese, GG, p.XVI. Veronese prided himself upon his genuinely geometrical approach to n -dimensional space, which he did not view merely as a "number-manifold" (*Zahlenmannigfaltigkeit*), in the manner of Klein and Lie. He was the first to study non-Archimedean geometries, in which the Archimedean postulate (p.45) does not hold. We shall deal later with these geometries, in connection with Hilbert's *Grundlagen* (p.237). Evidently, the space posited by a non-Archimedean geometry cannot be identified with an open subset of \mathbb{R}^n or \mathbb{C}^n , and is therefore quite foreign to the *Zahlenmannigfaltigkeit* of late 19th-century mathematics.
- ⁸² Peano (1894), p.75.
- ⁸³ Pieri (1900), p.373.
- ⁸⁴ Pieri (1900), pp.375f.
- ⁸⁵ Pieri (1900), p.387. "Axiom XII" is Euclid's Postulate 5.
- ⁸⁶ Pieri (1900), p.387. See Section 3.2.10, pp.252f.
- ⁸⁷ Pieri (1900), pp.387f. Cf. Freudenthal (1957), p.116.
- ⁸⁸ Pieri (1900), pp.388f.
- ⁸⁹ Pieri (1900), pp.373, 374. See also Pieri (1899a), p.2.
- ⁹⁰ Mario Pieri (1899a), "I principii della geometria di posizione composti in sistema logico deduttivo"; (1899b) "Della geometria elementare come sistema ipotetico deduttivo. Monografia del punto e del moto." Pieri mentions three earlier papers of his as preliminary stages of (1899a): "Sui principii che reggono la geometria di posizione" (*Atti della R. Accademia di Scienze di Torino*, Vols. 30, 31), "Un sistema di postulati per la geometria proiettiva astratta degl'iperspazii" (*Revista di Matematica*, Vol.6), and "Sugli enti primitivi della geometria proiettiva astratta" (*Atti della R. Acc. di Sc. di Torino*, Vol.32). These papers were published in 1895-97.
- ⁹¹ Pieri (1899a), pp. 1, 2.
- ⁹² Pieri (1899a), p.2.
- ⁹³ Finite geometries were first introduced by G. Fano, Pieri quotes Fano's paper "Sui postulati fondamentali della geometria proiettiva", *Giornale de Matematica*, Vol.30 (1891).
- ⁹⁴ Pieri (1899b), p.177. Cf. Pieri (1899a), pp.4f., Pieri (1900), pp.381f.
- ⁹⁵ Peano (1900), p.279.
- ⁹⁶ Burali-Forti (1900), p.290. "Definitions by abstraction" characterize a mapping f by indicating its domain and the partition of this into subsets on which f is constant. (*Ibid.*, p.295).
- ⁹⁷ Frege, *Grundlagen der Arithmetik* (1884).
- ⁹⁸ Padoa (1900), p.318. Padoa's phrasing is awkward. Axioms do not impose conditions on the *symbols* which occur in them, but on the *objects* (properties, etc.) which these symbols may stand for.
- ⁹⁹ Padoa (1900), p.319f.
- ¹⁰⁰ Padoa (1900), p.323.
- ¹⁰¹ Padoa (1900), p.321.
- ¹⁰² Padoa (1900), p.322. See Padoa (1900b).
- ¹⁰³ Kant, *KrV*, A702/B730. I quote Hilbert's *Grundlagen* after the 10th edition of 1968

(abbreviated GG). Save for one small but significant change on p.120, this reproduces the text of the 7th edition of 1930 (substantial changes have been introduced however in the appendices and supplements). For the 7th edition, Hilbert revised the text quite thoroughly. Important changes had been made earlier in the second edition (1903). All changes are carefully noted in Hilbert–Rossier, FG.

¹⁰⁴ Hilbert, GG, p.1.

¹⁰⁵ Hilbert, GG, p.2.

¹⁰⁶ On these matters, however, opinions are not unanimous. Thus, the great Danish geometer J. Hjelmslev, who considered Axiom I1 (“Two points determine a straight line”) intuitively false, because intuitive points and lines are not widthless, counted the Archimedean axiom as intuitively evident, apparently because he expected his “natural geometry” or “geometry of reality” to be displayable on a drawing-board. See Hjelmslev (1923), p.7.

¹⁰⁷ Hilbert, GG, p.1.

¹⁰⁸ Hilbert, GG, pp.125.

¹⁰⁹ Hilbert’s definition of *triangle* (GG, p.9) covers the degenerate case in which the three vertices are collinear, but he tacitly assumes that every triangle he mentions is not degenerate. This assumption was made explicit by Bernays in a note added to the 8th edition (GG, p.14).

¹¹⁰ Axiom V2 did not appear in the first edition. Its necessity was pointed out by Hilbert in a lecture “Ueber den Zahlbegriff” delivered in 1899. (Appendix VI of GG, 7th edition; suppressed in later editions.) The French translation of the *Grundlagen* by Laugel, published in 1900, includes a note, signed by Hilbert, which reads, in part, as follows: “Remarquons qu’aux cinq précédents groupes d’axiomes l’on peut encore ajouter l’axiome suivant qui n’est pas d’une nature purement géométrique [...]: *Au système des points, droites et plans, il est impossible d’adjoindre d’autres êtres de manière que le système ainsi généralisé forme une nouvelle géométrie où les axiomes des cinq groupes (I) à (V) soient tous vérifiés; en d’autres termes; les éléments de la géométrie forment un système d’êtres qui, si l’on conserve tous les axiomes, n’est susceptible d’aucune extension.* [...] La valeur de cet axiome au point de vue des principes, tient [...] à ce que l’existence de tous les points limites en est une conséquence et que, par suite, cet axiome rend possible la correspondance univoque et réversible des points d’une droite et de tous les nombres réels.” (Hilbert–Rossier, FG, pp.43f.). Editions 2–6 carry the following version of V2: “The elements of geometry, points, lines and planes, constitute a system of objects which, if you assume the foregoing axioms, does not admit any extension.” In the 7th edition, a statement similar to this is proved as Theorem 32. A discussion in depth of the axiom of completeness is given in Bernays (1955). See also Baldus (1928).

¹¹¹ I paraphrase this axiom from Baldus (1937), p.225. It implies that every segment P_0Q_0 will share the property ascribed to A_0B_0 . Copy A_0B_0 on a plane on which P_0 and Q_0 lie. Let lines A_0P_0 and B_0Q_0 meet at Z . Lines ZA_i and ZB_i meet line P_0Q_0 at P_i and Q_i , respectively. The sequence $\langle P_i, Q_i \rangle$ fulfils conditions (i) and (ii). Line ZX meets P_0Q_0 at Y . Y lies between P_n and Q_n , for every positive integer n .

¹¹² Frege, KS, p.409.

¹¹³ Frege, KS, p.411.

¹¹⁴ If K is inconsistent, i.e. if $K \models S$ and $K \models \neg S$, K is unsatisfiable, for every

interpretation which satisfies K must satisfy $\{S, \neg S\}$, so that there is no such interpretation at all. On the other hand, if K is unsatisfiable, every interpretation which satisfies K (i.e. none at all) will satisfy any arbitrary sentence. Consequently, for any sentence S , $K \models S$ and $K \models \neg S$. Hence, K is inconsistent.

¹¹⁵ We say that an axiomatic theory T , with axioms K , is soundly formalized within a calculus C if a sentence S can be proved from K in C only if $K \models S$. See p.192, Section 3.2.2.

¹¹⁶ Let K be formalized in a calculus C . The syntactical consistency of K in this formalization can be established by showing that a proof from K in C cannot possibly terminate in a particular sentence of C . This can usually be done by studying the possible outcome of a few well-defined operations on finite strings of symbols.

¹¹⁷ The following remark might prevent some misunderstandings. The syntactical consistency of the arithmetic of natural numbers in its standard formalization was demonstrated by Gentzen in 1936. He had to resort to methods of proof that one would normally regard as less unquestionable than those of arithmetic itself. But if Gentzen's result is correct, then, on the strength of Gödel's theorem (Gödel (1931), Theorem XI), that formalization cannot be complete. Consequently, the syntactical consistency of arithmetic in that formalization does not ensure its consistency in our (semantical) sense of the word.

¹¹⁸ Pascal's theorem is a theorem in projective geometry which says that, if six distinct points $P_1, P_2, P_3, P_4, P_5, P_6$ lie on a conic, the points of intersection $P_1P_2 \cap P_4P_5$, $P_2P_3 \cap P_5P_6$, and $P_3P_4 \cap P_6P_1$ lie on a line. The theorem used by Hilbert says that if P_1, P_3, P_5 lie on a line and P_2, P_4, P_6 lie on another line, then, if P_1P_2 is parallel to P_4P_5 , and P_2P_3 is parallel to P_5P_6 , P_3P_4 is parallel to P_6P_1 . This is clearly the restriction of Pascal's theorem to the affine plane, with a degenerate conic equal to two straight lines and the points of intersection on the line 'at infinity'.

¹¹⁹ Dehn (1900).

¹²⁰ In Euclid's and Hilbert's sense, i.e. coplanar and not concurrent.

¹²¹ A modified version of Veblen's axioms of order was reproduced on p.197. This modified version is taken from the axiom system of Veblen (1911), which, following the example of R.L. Moore (1908), includes *congruence of segments* among the undefined concepts.

¹²² Huntington (1902), p.277.

¹²³ Veblen (1904), p.346.

¹²⁴ Like Peano and Veblen, Huntington regards the signs that stand for the set-theoretical predicates "x is a set", "x is an element of set y", as "symbols which are necessary for all logical reasoning" (Huntington (1913), p.526), and treats them as non-interpretable words.

¹²⁵ Huntington (1913), p.530.

¹²⁶ Huntington (1913), p.524.

¹²⁷ Huntington (1913), p.540.

¹²⁸ A *lattice* is a triple $\langle S, \sqcup, \sqcap \rangle$, where S is a set and \sqcup and \sqcap are two associative and commutative binary operations on S , such that for any u and v in S , $u \sqcup (u \sqcap v) = u \sqcap (u \sqcup v) = u$, $u \sqcap v$ is usually called the *meet* and $u \sqcup v$ the *join* of u and v . In the case mentioned in the text above, for any two spheres A and B , let their join be the smallest sphere which contains A and B while their meet is the largest sphere which is

contained in A and in B. A point should now be defined as a sphere which contains no sphere except V.

¹²⁹ Menger (1928); Birkhoff (1935).

¹³⁰ I follow Menger (1940), Lecture II. See also Blumenthal and Menger, *Studies in Geometry* (1970), Parts I and II; Birkhoff, *Lattice theory* (1967), Chapter IV.

¹³¹ Compare Martianus Capella's Latin translation of Euclid's definition of a point: "Punctum est cuius pars nihil est." (Quoted by Heath, EE, Vol.I, p.155.)

¹³² For a concise description and classification of its several currents, see P. Bernays (1959).

¹³³ See Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff*, 2nd enlarged edition (1973).

¹³⁴ Tarski (1959), p.16.

¹³⁵ A formalized axiomatic theory is said to be semantically complete if every logical consequence of the axioms is provable from them; it is said to be decidable if the set of all theorems is computable (see Note 6).

¹³⁶ Grassmann, WW, Vol.I.1, p.65.

¹³⁷ See Section 4.1.4.

¹³⁸ Hilbert, GG, pp.4, 11.

¹³⁹ Frege, KS, p.265. Cf. Frege, KS, p.282.

¹⁴⁰ Frege, KS, p.411.

¹⁴¹ Frege, KS, p.411. In the first edition of Hilbert's *Grundlagen*, group II comprised five axioms.

¹⁴² Frege, KS, p.416. Frege made this point again in the second of his two articles on Hilbert's *Grundlagen* (Frege, KS, p.268), and in the first of his three articles in reply to Korselt (Frege, KS, p.291). These five articles (as well as Korselt's) bear the same title: "Ueber die Grundlagen der Geometrie".

¹⁴³ Cf. Freudenthal (1957), p.116.

¹⁴⁴ Hilbert, GG, p.28. In the first edition, Hilbert used indifferently the words *Definition* and *Erklärung*. After Frege suggested that these two words might express two different ideas, Hilbert changed all occurrences of *Definition* into *Erklärung*. (See Frege, KS, p.407).

¹⁴⁵ Pieri (1899a), p.3 n.6. Cf. Pieri (1899b), p.173n.; (1900), p.378n.1.

¹⁴⁶ Frege, KS, p.288.

¹⁴⁷ Frege, KS, p.412. The kind of "invertible univocal transformation" which Hilbert has in mind has been illustrated above, on pp.81f.

¹⁴⁸ As a matter of fact, the analogy between Hilbert's axiom system and a system of simultaneous equations was used by Frege as an argument against the viability of the former: "If we survey the whole of Mr. Hilbert's definitions and axioms, they will be seen to be comparable to a system of equations with many unknowns; for in each axiom you normally find several of the unknown expressions "point", "line", "plane", "lie on", "between", etc., so that only the whole, not particular axioms or groups of axioms, can suffice to determine the unknowns. But does the whole suffice? Who can say that this system is solvable for the unknowns, and that these are unambiguously determined?" (Frege, KS; p.268; cf. p.416.) In his *Philosophy of Space and Time* (1928), Hans Reichenbach claims that there are equations that determine an unknown implicitly and are not solvable for it. Axioms, according to him, are comparable to such

equations. He mentions by way of example the equation $x = \sin y$, which, he says, is not solvable for y . Though this equation can also be written “in the apparently solved form $y = \arcsin x$. . . the meaning of the function \arcsin is defined only by the previously given implicit equation.” (Reichenbach, PST, p.93.) The champion of Scientific Philosophy evidently overlooked that (i) $x = \sin y$ does not determine y ; (ii) the system $x = \sin y$, $-\pi/2 < y < \pi/2$, which does determine y , is solved explicitly by

$$y = \int_0^x \frac{dt}{\sqrt{1-t^2}} \quad (-1 < x < 1).$$

¹⁴⁹ Gergonne (1818), p.13.

¹⁵⁰ Gergonne (1818), p.22.

¹⁵¹ Gergonne (1818), p.23; my italics, except for the words *implicit* and *explicit*, italicized by the author.

CHAPTER 4. EMPIRICISM, APRIORISM, CONVENTIONALISM

4.1 *Empiricism in Geometry*

¹ Gauss, WW, Vol.8, p.177. Quoted above, p.55.

² J.S. Mill, SL, pp.209–261, 607–621; cf. Early Draft of the Logic, *ibid.* pp.1083–1097. That the first principles of arithmetic are generalizations from experience is stated on p.257. Arithmetic, however, differs from geometry in so far as the former “is deduced wholly from propositions exactly true”, while the latter partly depends, as we shall see, on “hypotheses or assumptions which are only approximations to the truth”. (J.S. Mill, SL, p.1092; cf. pp.258f.)

³ J.S. Mill, SL, p.218.

⁴ J.S. Mill, SL, p.1088.

⁵ J.S. Mill, SL, p.618n. I imagine that he expects them to be coplanar as well.

⁶ J.S. Mill, SL, p.225.

⁷ J.S. Mill, SL, p.225. “Inconceivable” for Mill is apparently everything that we would repute unimaginable.

⁸ J.S. Mill, SL, p.225.

⁹ See Section 3.2.3, pp.199f.

¹⁰ J.S. Mill, SL, p.227n. (The note from which our quotation is taken was added by Mill in 1872.)

¹¹ J.S. Mill, SL, pp.228f. In the first three editions the text enclosed by the asterisks read as follows: “suppressing some of those which it has”.

¹² J.S. Mill, SL, p.230.

¹³ J.S. Mill, SL, pp.229f. The reader will not fail to observe that the last of the two axioms is incompatible with BL geometry while the first excludes spherical geometry. That Mill, who apparently knew nothing of the new developments in mathematics, should have chosen precisely such examples shows to what extent the subject was in the air at that time.

¹⁴ J.S. Mill, SL, p.1089.

¹⁵ J.S. Mill, SL, p.232n. (note added in 1865; my italics).

- ¹⁶ J.S. Mill, SL, p.233.
- ¹⁷ J.S. Mill, SL, p.234.
- ¹⁸ J.S. Mill, SL, p.235.
- ¹⁹ J.S. Mill, SL, pp.115f.
- ²⁰ Perceived lines are very rarely straighter than a pair of perpendiculars carefully drawn on a frozen lake. But these, if prolonged, will meet at the antipodes.
- ²¹ J.S. Mill, SL, p.1089.
- ²² J.S. Mill, SL, p.259.
- ²³ J.S. Mill, SL, p.616 (text of the first four editions).
- ²⁴ J.S. Mill, SL, pp.616f.
- ²⁵ The first one, which was published with the essay in 1851, appears in Ueberweg, PG, pp.263–269. The second one, written for Delboeuf's French translation of 1860 (printed as an appendix to Delboeuf, PPG), is reproduced, in German, in Ueberweg, PG, pp.308–316.
- ²⁶ Ueberweg, PG, p.314.
- ²⁷ Ueberweg, PG, pp.312f.
- ²⁸ Ueberweg, PG, p.268.
- ²⁹ Ueberweg, PG, p.315.
- ³⁰ Ueberweg, PG, p.269.
- ³¹ Ueberweg, PG, pp.269f.
- ³² Ueberweg, PG, p.313.
- ³³ This is Lie's second main solution of Helmholtz's problem, mentioned above in Part 3.1, Note 43, See Lie, TT, Vol.III, pp.506f.
- ³⁴ Theorem 25 (Ueberweg, PG, p.287). The 'proof' of the theorem involves many tacit assumptions, whereby, among other things, elliptic and spherical geometry are implicitly excluded.
- ³⁵ Ueberweg, PG, p.291.
- ³⁶ Theorem 47 (Ueberweg, PG, p.304).
- ³⁷ Theorem 49b (Ibid., p.305).
- ³⁸ See Section 3.1.2. Erdmann's version of the axioms is considerably less precise than Helmholtz's (Erdmann, AG, p.83).
- ³⁹ Erdmann, AG, p.38. In the same year, 1877, in which Erdmann's book was published, Georg Cantor discovered that the above expression of dimension number is inadequate, because \mathbb{R}^n can be mapped bijectively onto \mathbb{R} . See Cantor, GA, pp.119–133. Cantor's proof is sketched in Kline, MT, pp.997f.
- ⁴⁰ "Die Teilbarkeit ins unendliche, die jeder Raumschauung anhaftet" (Erdmann, AG, p.38). This is, of course, highly questionable as a description of spatial *perception*. We must assume that Erdmann, like most empiricist philosophers, allows the bounty of imagination to make up for the stinginess of the senses.
- ⁴¹ Erdmann, AG, p.39. The alleged intuitive fact of infinite divisibility does not warrant the employment of such strong analytical means for expressing it. It would suffice to let the coordinates take all *rational* values between their value at A and their value at B, or merely all values expressible by means of a terminating decimal fraction.
- ⁴² Erdmann, AG, p.40.
- ⁴³ Coordinates x_1, x_2, x_3 are 'interchangeable' in Erdmann's sense if distances measured along each coordinate axis (i.e. along each line $x_i = x_j = 0; i \neq j$) are comparable with

distances measured along the other two. (Erdmann, AG, p.42). The distinction in the text presupposes, therefore, the concept of distance. Erdmann apparently takes it for granted. As an example of the second kind of 3-fold determined continuous quantity he mentions the manifold of sounds, each of which is defined by its height, intensity and timbre. Cf. the critical remarks in Ernst Mach, EI, pp.392ff.

⁴⁴ Erdmann does not seem to be aware of the difference between the general Riemannian concept of a (continuous) manifold and the concept of an *R*-manifold, i.e. one endowed with a Riemannian metric. Erdmann alludes to the latter in AG, p.59.

⁴⁵ Erdmann, AG, pp.58f.

⁴⁶ Erdmann, AG, p.69. An additional argument for the flatness of space is given on p.99. It is based on "the perception (*die Wahrnehmung*) that the direction of a moving body can be regarded, within the limits of our experience, as completely independent of its position (*Ort*)". Erdmann surely refers to a body moving in *empty* space, for the direction of a body moving in the presence of other bodies appears to be strongly influenced by its position relative to them, especially if they are very large. One may wonder indeed how Erdmann managed to *perceive* a body moving all by itself in absolutely empty space.

⁴⁷ Erdmann, AG, pp.93f.

⁴⁸ Erdmann, AG, p.116.

⁴⁹ Erdmann, AG, p.146.

⁵⁰ Cf. Erdmann, AG, pp.59ff., 91, 141ff.

⁵¹ Erdmann, AG, p.91; cf. pp.115, 135, 145, 152f. On Helmholtz, see Section 3.1.3.

⁵² Erdmann, AG, pp.144f. Erdmann refers to the dimension number of space on pp.38, 49, 83, 95, 143; nowhere do I find an argument for the above quoted statement.

⁵³ Erdmann, AG, pp.118f.

⁵⁴ Erdmann, AG, pp.94f.; cf. p.128.

⁵⁵ Erdmann extensively quotes the Erlangen Programme in AG, pp.124f., n.2. Had he read it? That he remained untouched by Klein's views can be gathered from the peculiar distinction he makes between the *form* and the *contents* of space: The form comprises those properties which are common to all *n*-dimensional manifolds, while the contents of our space consists in "the triplicity of dimensions and the flatness of metrical relations" (Erdmann, AG, p.143). Projective geometry is not mentioned in the whole book.

⁵⁶ Erdmann, AG, p.157; cf. p.44n.2, p.50.

⁵⁷ Erdmann, AG, p.158; (my italics). Cf. Helmholtz (1866), p.197; quoted above, p.167, Section 3.1.3.

⁵⁸ Erdmann, AG, p.159. I take it that the *geometric concepts of construction* (*die geometrische Constructionsbegriffe*) are concepts such as *straight line* and *circle*, which enter into the description of geometrical constructions.

⁵⁹ Erdmann, AG, p.159.

⁶⁰ Erdmann, AG, p.160.

⁶¹ Erdmann, AG, p.161.

⁶² Erdmann, AG, p.169.

⁶³ Erdmann, AG, p.170.

⁶⁴ Erdmann, AG, p.170. Erdmann admits here that experimental results may show that space curvature is not constant. According to the Helmholtzian stance taken by

Erdmann in the rest of the book, such experimental discovery would imply that the ideal of a perfectly rigid motion cannot be indefinitely approached, and that, as a consequence of this, there is an upper bound to the accuracy of actual geometrical measurements.

⁶⁵ Calinon (1891), p.375.

⁶⁶ Calinon (1891), p.375.

⁶⁷ Calinon (1889), p.589. Cf. Calinon (1891), p.368; "La Géométrie générale est l'étude de tous les groupes de formes dont les définitions premières sont astreintes à une condition unique, qui est de ne donner lieu à aucune contradiction, lorsqu'on les soumet au raisonnement géométrique indéfiniment prolongé".

⁶⁸ Calinon (1891), p.375. Cf. Klein (1890), p.571, quoted above, p.149, Section 2.3.10.

⁶⁹ Calinon (1891), p.375.

⁷⁰ Calinon (1889), p.590. Calinon's "general geometry" embraces Euclidean geometry, BL geometry and the so-called geometry of Riemann, that is, spherical or elliptic geometry. It is, in other words, a theory of maximally symmetric spaces (p.184). The name "general geometry" indicates that Calinon was less open-minded than he thought.

⁷¹ The same paradox had motivated Taurinus' rejection of BL geometry. See p.53.

⁷² Calinon (1891), pp.368f. Identical spaces, in this sense, were called *isogenous* (*isogènes*) by Delboeuf, a term often used in the French literature of this period. See pp.206f.

⁷³ Calinon (1889), p.594.

⁷⁴ Calinon (1889), p.595.

⁷⁵ Calinon (1891), p.374 (my italics).

⁷⁶ Calinon (1893), p.605. The paper of 1893 is written in what the author calls an "idealistic" language; that is, space is identified there with our perceptually based representation of space. Calinon says that his views on the matters discussed by him can also be set forth in a realistic language, a task which he proposes as an exercise to the reader. In a realistic version of Calinon's paper we ought perhaps to ascribe a definite though unknown geometric structure to physical space. But it seems clear that in Calinon's opinion it makes no scientific sense to state hypotheses about such a structure. The choice of a geometry in physics must be made in the light of the requirements of the problem at hand, not with a view to determining the essence of things in themselves.

⁷⁷ Published in the *Revue générale des sciences* on December 15, 1891. See Part 4.4.

⁷⁸ Calinon (1893), p.607. Such are also the criteria usually applied when choosing a coordinate system. It is amazing that the great mathematician Poincaré should have overlooked this side of the analogy drawn by him.

⁷⁹ Calinon (1893), p.607.

⁸⁰ Mach, EI, pp.337-448. The two longest chapters (pp.353-422) had previously been published in English in *The Monist*. See References.

⁸¹ H. Poincaré, "Le continu mathématique" (1893); "L'espace et la géométrie" (1895); "On the foundations of geometry" (1898). See Part 4.4. Federico Enriques published, shortly before Mach, a study explaining the psychological development of geometry from an empiricist point of view. (Enriques, "Sulla spiegazione psicologica dei postulati della geometria" (1901).) I do not think it would be rewarding to discuss Enriques' views here. They are readily accessible in English in Enriques, PS, pp.199-231.

- ⁸² Mach, EI, p.389; cf. pp.371, 381.
- ⁸³ Mach, EI, p.337. Compare Mach's illuminating description of time-sensations in EI, p.423.
- ⁸⁴ Mach, EI, p.345.
- ⁸⁵ Mach distinguished the *geometrical* and the *physiological* properties of a figure in space in his *Beiträge zur Analyse der Empfindungen* (1886), pp.44, 54, long before Poincaré introduced the concepts of *espace géométrique* and *espace représentatif* in "L'espace et la géométrie" (1895).
- ⁸⁶ Mach, EI, p.337.
- ⁸⁷ Mach, EI, p.339.
- ⁸⁸ I gather this from the statement in Mach, EI, p.343, that physiological space as such is three-dimensional. Since haptic space is said to be two-dimensional (Ibid., p.340), the third dimension can only be given by the optic space (unless we view it as belonging only to the acoustic or, say, the olfactory component of physiological space).
- ⁸⁹ Mach, EI, p.340.
- ⁹⁰ Mach, EI, p.347. I fail to see how the experiences described by Mach might justify the conclusion that space is infinite, and not just unlimited. This is one of those wild speculative jumps that empiricists must make every now and then to reconcile their philosophy with experience.
- ⁹¹ Mach, EI, p.355. The reader will probably notice that in order to *transport* a body from a neighbourhood FGH to a neighbourhood MNO, as required by the experiment described above, we must regard it as located in the unitary objective physical space whose very idea supposedly arises out of such experiments.
- ⁹² Mach, EI, p.434.
- ⁹³ Mach, EI, p.367.
- ⁹⁴ Mach, EI, p.380.
- ⁹⁵ Mach, EI, p.367.
- ⁹⁶ Mach, EI, p.368.
- ⁹⁷ Mach, EI, pp.368f.
- ⁹⁸ Mach, EI, p.380. The last remark, in fact, agrees with the opinion of authors such as Klein and Enriques, who emphasized the optical origin of our idea of straightness, for they saw in visual intuition the root of projective, not of metric geometry (see above, Section 2.3.10, p.147; Enriques, PS, pp.205ff.).
- ⁹⁹ Mach, EI, p.370.
- ¹⁰⁰ Mach, EI, p.371.
- ¹⁰¹ Mach, EI, p.385.
- ¹⁰² Mach, EI, p.409.
- ¹⁰³ Mach, EI, p.414.
- ¹⁰⁴ Mach, EI, p.418.
- ¹⁰⁵ Mach, EI, p.418.

4.2 *The Uproar of Boeotians*

¹ Krause, loc. cit., p.84; quoted by Schlick in Helmholtz, SE, p.29, n.25.

² Tobias, loc. cit., p.81; quoted by Erdmann (AG, p.118). Helmholtz's critic J.P.N. Land contrasts geometrical intuition and algebraic analysis; it remains to be seen how

much of the latter will bear translation into the former (Land (1877), p.40). Indeed, while analytical geometry is ready to grapple with any number of dimensions, intuition is confined, for a mind like ours, to Euclidean 3-space. "All other 'space' contrived by human ingenuity may be an aggregate with fictitious properties and a consistent algebraical analysis of its own, but space it is called only by courtesy." (Land (1877), p.42.)

³ "Nur ein einziger grosser und zusammenhängender Irrtum". Lotze, M, p.234.

⁴ Lotze rejects Kant's arguments for the "transcendental ideality" of space, but supports it with another argument of his own: space as an aggregate of equal points can only be held together by consciousness. (Lotze, M, pp.211f.)

⁵ Lotze, M, p.223.

⁶ "Eines Ordnungssystems leerer Plätze". (Lotze, M, p.235).

⁷ Lotze, M, pp.241f.

⁸ Lotze, M, p.247. I have changed Lotze's lettering to make the meaning clearer.

⁹ "Käme es aber einmal dazu, dass astronomische Messungen grosser Entfernungen nach Ausschluss aller Beobachtungsfehler eine kleinere Winkelsumme des Dreiecks nachwiesen, was dann? Dann würden wir nur glauben, eine neue sehr sonderbare Art der Refraction entdeckt zu haben, welche die zur Bestimmung der Richtungen dienenden Lichtstrahlen abgelenkt habe; d.h. wir würden auf ein besonderes Verhalten des physischen Realen im Raume, aber gewiss nicht auf ein Verhalten des Raumes selbst schliessen, das allen unseren Anschauungen widerspräche und durch keine eigene exceptionelle Anschauung verbürgt würde." (Lotze, M, pp.248f.). See the comments in Russell, FG, p.100. Poincaré make a similar remark, without quoting Lotze, in "Les géométries non-Euclidiennes" (1891), p.774 (now in Poincaré, SH, pp.95f.).

¹⁰ A transcendental deduction proves the necessity of a statement by showing that it expresses a condition of the possibility of an established branch of knowledge. See Part. 4.3.

¹¹ In his paper "Ueber das kosmologische Problem" (1877). In pp.105–113, Wundt discusses Zoellner's use of "transcendent geometry" for constructing a temporally eternal but spatially finite model universe. He mentions in passing what he believes are the absurd consequences of a non-zero space curvature. In his opinion, the motion of rigid bodies would be impossible in a curved space, even if its curvature is constant. "In einem sphärischen Raum von drei Dimensionen, z.B würde, da diejenigen Linien, die den Parallelen im Euklidischen Raum entsprechen, grösste Kreise sind, ein Körper bei der Translocation von einem bestimmten Punkte an zuerst sich ausdehnen und dann wieder zusammenziehen." (Wundt (1877), p.109.)

¹² Wundt, L, Vol.I, pp.478–506; Vol.II, pp.176–208. I quote Vol.I after the fifth (posthumous) edition of 1924.

¹³ Wundt, L, Vol.I, p.489: "Die unmittelbare gegebene Ordnung unserer Wahrnehmungsinhalte." Ibid., p.504: "Eine Ordnung von Empfindungen".

¹⁴ Wundt, L, Vol.I, p.497.

¹⁵ Wundt, L, Vol.I, p.501.

¹⁶ Wundt, L, Vol.I, p.503.

¹⁷ Wundt, L, Vol.I, p.504.

¹⁸ Wundt, L, Vol.I, p.494.

¹⁹ Wundt, L, Vol.I, p.483.

- ²⁰ Wundt, L, Vol.I, p.485.
- ²¹ Wundt, L, Vol.I, p.482.
- ²² Wundt, L, Vol.I, p.492. "Self-congruence" means that a figure congruent with a given figure can be constructed anywhere in space. I have quoted Wundt's shortened definition in the second paragraph of p.492. In the first paragraph we find a more explicit definition. In it, instead of *unendlich* (infinite), Wundt writes *unbegrenzt* (unlimited). This longer definition is therefore compatible with spherical and elliptic geometry; it also covers all the hypersurfaces of elliptic and hyperbolic 4-space which are isometric to Euclidean 3-space, etc.
- ²³ Kant, *Ak.*, II, pp.404f. (quoted on p.31).
- ²⁴ Wundt, L, I, p.488.
- ²⁵ Wundt, L, I, p.488.
- ²⁶ Wundt, L, I, p.484.
- ²⁷ First published in Renouvier's *Critique philosophique* in November 1889, as a reply to Lechalas' paper on general geometry, published in the same journal in September 1889 (see *ibid.*, p.337n.). I quote from Renouvier (1891), a considerably revised and enlarged version published in Pillon's *Année philosophique*.
- ²⁸ Renouvier (1891), p.3.
- ²⁹ Renouvier (1891), p.64.
- ³⁰ Renouvier (1891), p.43.
- ³¹ Renouvier (1891), p.4.
- ³² Renouvier (1891), p.4.
- ³³ Renouvier (1891), p.20. (My paraphrase.)
- ³⁴ "La ruine entière de l'idéation géométrique". (Renouvier (1891), p.45.)
- ³⁵ Renouvier (1891), p.45.
- ³⁶ Renouvier (1891), p.66.
- ³⁷ See pp.153 and 206.
- ³⁸ Delboeuf, PPG, p.67.
- ³⁹ Delboeuf, PPG, p.129.
- ⁴⁰ Delboeuf, PPG, p.50.
- ⁴¹ Delboeuf, ANG, I, p.456. Delboeuf concludes: "L'espace géométrique ou euclidien est donc un espace imaginaire, un espace hypothétique, n'ayant de commun avec la réalité que ceci qu'elle en a fourni l'idée, parce que les solides ont l'air de s'y transporter sans altération sensible." (*Ibid.*, pp.464f.) As we know, this empirical fact which according to Delboeuf "provides the idea of Euclidean space" fits equally well with every space of constant curvature.
- ⁴² Delboeuf, ANG, II, p.372.

4.3 Russell's Apriorism of 1897

- ¹ Russell, FG, New York: Dover, 1956.
- ² Chapter I, "A short history of metageometry" (FG, pp.7-53); Chapter II, "A critical account of some previous philosophical theories of geometry" (pp.54-116). I feel inclined to believe that these chapters were written after the rest of the book, at a more advanced stage of Russell's mathematical education. Russell's *Autobiography* and *My*

Philosophical Development say nothing on this particular point. In the latter book, pp.39f. Russell dismisses FG as “somewhat foolish”, because Einstein’s space-time geometry of the General Theory of Relativity “is such as I had declared to be impossible”.

³ Russell, FG, p.136.

⁴ Kant, *KrV*, B 40.

⁵ Kant, *KrV*, B 246. This is one of the many passages where Kant might seem to favour a psychological conception of the a priori.

⁶ In *The Principles of Mathematics* (1903), Russell will try to prove that abstract set theory is only a branch of logic and is therefore, so to speak, trivially a priori. Later developments have shown that this thesis can only be upheld if we broaden logic well beyond its traditional limits in such a way that we can no longer regard its apriority as trivial. (The triviality of pre-Russellian – or rather pre-Fregean – logic is shown by the fact that we can encode its truths and programme a computer to recognize them; the truths of modern logic can also be encoded, but it is demonstrably impossible to design a general method of computation for deciding whether a formula in the appropriate code does or does not represent a logical truth.)

⁷ Russell, FG, p.28.

⁸ Russell, FG, p.118.

⁹ Russell, FG, p.132.

¹⁰ This suggestion conflicts with the thesis held by Russell in FG that the concept of a point is self-contradictory. Russell refutes this silly notion in his *Principles*, Chapter 51. But even in FG he is able to treat space as a point-set by introducing the concept of a purely geometrical matter, whose elements or “atoms” are unextended and supply the terms for spatial relations. See Russell, FG, pp.77ff., 190ff.; and below, p.315, Section 4.3.5.

¹¹ Replying to a critical review by Poincaré (1899), Russell writes: “Je ne prétends pas que deux points *définissent* une droite. Je soutiens, au contraire, que chaque droite est simple et inanalysable, et par suite indéfinissable. Ce que je veux dire est ceci: ‘L’ensemble complet des lignes droites étant donné, il suffit de mentionner deux points de l’une d’elles pour la distinguer de toutes les autres, en tant qu’elle est spécialement liée à ces deux points’”. (Russell, 1899, p.696). This explanation, however, will not increase the deductive power of Axiom III, or of the other axioms.

¹² The reader is advised to read the argument in Russell, FG, pp.136f.

¹³ Russell, FG, pp.139f. Russell’s rejection of infinite-dimensional spaces is not an instance of mathematical ignorance – such spaces had not yet been discovered – but of a typically unmathematical narrow-mindedness which often crops up in philosophical literature.

¹⁴ See Peano (1890).

¹⁵ Russell, FG, p.149.

¹⁶ Russell, FG, p.46.

¹⁷ Russell, FG, p.118.

¹⁸ Russell, FG, p.147.

¹⁹ Russell, FG, p.119; cf. p.30.

²⁰ Russell, FG, p.35.

²¹ Russell, FG, p.164.

²² See above, p.124, Section 2.3.5.

²³ As a matter of fact, Russell actually asserts that every form of externality satisfies what he calls "the axioms of projective geometry". Since these axioms fail to characterize projective space, the clash between Russell's statements is more apparent than real.

²⁴ Russell, FG, p.161.

²⁵ Russell, FG, p.163. Riemann had made the same remark; see above, Section 2.2.9, p. 104.

²⁶ Russell, FG, p.164.

²⁷ Russell, FG, p.164.

²⁸ Russell, FG, p.150.

²⁹ Russell, FG, pp.78, 79.

³⁰ Russell, FG, p.79.

³¹ Russell, FG, p.81.

³² That Russell's "geometrical matter" is composed of abstract sets is confirmed by the discussion in FG, §199. Cf. especially p.192, where geometrical matter is said to be composed of simple unextended atoms, which contain no real diversity and whose only function is to supply terms for spatial relations. A kinematical body can only be a collection of such atoms.

³³ Russell, FG, p.152.

³⁴ Erdmann, AG, p.60.

³⁵ Russell, FG, p.155.

³⁶ Russell, FG, pp.153f. The deduction of the axiom of free mobility from the relativity of position in Russell, FG, p.160, begs the question. Relativity of position implies that the form of externality is homogeneous, says Russell, and free mobility follows from homogeneity "for our form would not be homogeneous unless it allowed, in every part, shapes or systems of relations, which it allowed in any other part" (*Ibid.*). This argument makes sense only if metrically determined shapes have been defined in our form of externality.

³⁷ Couturat (1898), p.373f.

³⁸ Russell (1898), p.763.

³⁹ Russell, FG, pp.158f.

⁴⁰ Russell (1898), p.760. Russell says that the experiment was suggested to him by A.N. Whitehead.

⁴¹ Russell, FG, p.74.

⁴² Russell (1899), pp.703-707.

⁴³ Russell, "L'idée d'ordre et la position absolue dans l'espace et le temps" (1900). Russell (1901) is the corrected and slightly expanded English version of this paper.

⁴⁴ Linear order was defined in Note 96 to Chapter 1 (p.379). A 4-ary relation R is said to determine a cyclical order on a set S if, for any elements x, y, z, v and w of S the following six conditions hold:

- (i) $R(x, y, v, w)$ if and only if $R(v, w, x, y)$;
- (ii) $R(x, y, v, w)$ if and only if $R(x, y, w, v)$;
- (iii) $R(x, y, v, w)$ only if not $R(x, v, y, w)$;
- (iv) $R(x, y, v, w)$ only if $x \neq y$;
- (v) $R(x, y, v, w)$ and $R(x, v, y, z)$ only if $R(x, v, w, z)$;

(vi) If x, y, z and v are four distinct elements of S , then either $R(x, y, z, v)$ or $R(x, z, y, v)$ or $R(x, v, y, z)$.

If $R(x, y, v, w)$ one usually says that x and y separate v and w . (Cf. p.390, n.54.)

⁴⁵ Russell, *Principles*, p.372.

4.4 Henri Poincaré

¹ "Faits bruts", he calls them, with an expression apparently taken from LeRoy (Poincaré, VS, p.155). This author claimed that "le savant crée le fait", if not "le fait brut, [...] du moins [...] le fait scientifique". Poincaré remarks: "Le fait scientifique n'est que le fait brut traduit dans un langage commode [...]. Cette convention étant donnée, si l'on me demande: tel fait est-il vrai? je saurai toujours que répondre, et ma réponse me sera imposée par le témoignage de mes sens [...]. Les faits sont des faits, et s'il arrive qu'ils sont conformes à une prédiction ce n'est pas par un effet de notre libre activité. [...] Je ne puis admettre que le savant crée librement le fait scientifique puisque c'est le fait brut qui le lui impose [...]. En résumé, tout ce que crée le savant dans un fait, c'est le langage dans lequel il l'énonce". (Poincaré (1902), now in Poincaré, VS, pp.161, 158, 163, 156, 162.)

² Poincaré (1898b), now in Poincaré, VS, Chapter II, especially pp.47–54.

³ On October 14, 1960, the XI Conférence Générale des Poids et Mesures agreed to define the metre as "that length which is equal to 1,650,763.73 times the wavelength *in vacuo* of the radiation corresponding to the transition between the $^2p_{10}$ and the 5d_5 levels of an atom of krypton 86". This is, of course, to an excellent approximation, the length of the old platinum-iridium rod which had been defined to be one metre long in 1889.

⁴ Neumann, PGNT, p.18.

⁵ Lange (1885), p.337. Instead of translating Lange's text, I have quoted the English paraphrase given in Robertson and Noonan, RC, p.13. Lange motivated his definition as follows: "Just as the *one-dimensional* inertial time scale could be defined by means of a *single* free particle (*sich selbst überlassener Punkt*), we can define the *three-dimensional* inertial system by means of *three* free particles. [...] Given three (or less than three), not necessarily free, particles P, P', P'', in relative motion, it is always possible to construct a coordinate system [...] relatively to which those particles move in a straight line. On the other hand, if we are given more than three points, this possibility exists only accidentally, under particular circumstances. It follows that the law of the unchanging direction of motion of free particles is a *mere convention* in the case of three such particles, and expresses a remarkable result of scientific research only in so far as it is true of more than three, indeed of arbitrarily many particles, with respect to one and the same system. The *physical* condition of freedom from outer disturbance (*Unbeeinflusstsein*) has precisely this very remarkable *kinematic* consequence: for *arbitrarily many* particles subject to that condition there exists a coordinate system wherein they all move in a straight line." (Lange (1885) p.337.)

⁶ Lange (1885), p.338 (my paraphrase). These so-called *theorems* are of course postulates. Lange's "proof" of the first is really a consistency proof: given an inertial system relatively to which a particle moves in a straight line, one cannot construct a second inertial system relatively to which that same particle does not move in a straight line.

⁷ The most complete study of Poincaré's philosophy of geometry is still Rougier, *La philosophie géométrique de Henri Poincaré* (1920). The book by Mooij, *La philosophie des mathématiques de Henri Poincaré* (1966) devotes to it only a short rather unilluminating chapter (pp.3–29). A penetrating analysis of its foundations is given in Vuillemin (1972).

⁸ On axiomatic theories, see above, Section 3.2.2, pp.191ff. Black's concept of an axiomatic or, as he says, a *deductive* theory is not quite the same as ours. The theory determined by a set of axioms A is not, as with us, the set of logical consequences of A, but the set of sentences provable from A in accordance with the agreed rules of inference. The latter are not specified by Black, but he presumably expected them to be effective and to ape the relation of logical consequence. Practically any such theory qualifies as a 'geometry' in Black's sense, since, as he says, "the term *geometry* [...] when applied to a deductive theory indicates little more than an indefinite degree of resemblance of formal structure in relation to certain familiar and historically important deductive theories." (Black (1942), p.340.)

⁹ "There can be little doubt that *any* deductive theory is capable of translation into a 'contrary' deductive theory." (Black (1942), p.345). Black does not say "into *any* contrary theory", but, as we shall see, his thesis on the translatability of axiomatic theories does not lend support to geometrical conventionalism unless we understand it in this strong sense.

¹⁰ On *m*-English, see pp.193ff. Black, following the fashion of the thirties, initially describes a deductive theory as "a 'game' played with symbols, arranged in given initial positions, and manipulated in accordance with certain rules of play (or rather the totality of positions obtained in some such game)". But he adds immediately: "Not every game will qualify as a deductive system, however; the initial positions must bear some resemblance to the sentences of ordinary language, and the rules of play are to approximate more or less closely to 'rules of inference'. More precisely, any specimen of a deductive theory will consist of sentences, i.e. strings of symbols, designating either logical connectives, relations and predicates, or individual variables." (Black (1942), p.339.) This is almost equivalent to saying that any deductive theory can be rendered in *m*-English directly and without paraphrase. But even if it were not so, it is enough to show that a theory which can be thus rendered cannot be translated into one of its contraries, to destroy Black's contention (in the strong sense in which we agreed to understand it in Note 9).

¹¹ Black (1942), pp.341f.

¹² I agree on this point with Sklar (1969), p.45, though I do not share the grounds proposed by him for this opinion. He says that, since the propositions of pure geometry are neither true nor false, they cannot be *conventionally* true or false. But Poincaré maintained that geometry was conventional precisely because it was neither true nor false. The expression "true by convention" is never used by Poincaré. I imagine that he would, quite rightly, have judged it meaningless.

¹³ See Tarski, LSM, p.306; also pp.243, 248, Section 3.2.9.

¹⁴ Poincaré (1891), p.774; reprinted in SH, p.77.

¹⁵ "Je considère l'axiome des trois dimensions comme conventionnel au même titre que ceux d'Euclide" (Poincaré (1900), p.73). "L'expérience ne nous prouve pas que l'espace a trois dimensions; elle nous prouve qu'il nous est commode de lui en attribuer trois."

(Poincaré, VS, pp.93f.; cf. p.98.) In a paper written shortly before his death, Poincaré wrote however that “it is the external world, it is experience, which *determines* us” to choose a three-dimensional model of space (Poincaré, DP, p.157; my italics).

¹⁶ Kant, *KrV*, B146, A557/B585, A613f./B641f.

¹⁷ In its original context (in the paper of 1891 on non-Euclidean geometries) the argument against apriorism read as follows: Are geometrical axioms “synthetic *a priori* judgments, as Kant said? They would then impose themselves on us with such force that we should be unable to conceive the contrary proposition or to build a theory upon it. There would not be a non-Euclidean geometry”. (Poincaré, SH, p.74.) In 1895, he proposed a different argument, directed, it would seem, more specifically against Kant’s doctrine: “If geometrical space were a frame imposed on *each* of our representations individually considered it would be impossible to represent an image stripped of this frame and we could not change anything in our geometry”. (Poincaré, SH, p.88.) Even this second argument falls wide of the mark, insofar as Kant did not view space as a frame imposed on *each* of our sense-perceptions considered individually, but on *all* of them considered as a manifold. Both texts were reprinted without alterations in *La science et l’hypothèse* (1903).

¹⁸ Poincaré, DP, p.157: “I shall conclude that we all have in us the intuition of a continuum of an arbitrary number of dimensions, because we have the faculty of constructing a physical and mathematical continuum; that this faculty precedes all experience, because without it experience properly so called would be impossible and would reduce itself to crude sensations, incapable of being organised in any way; that this intuition is nothing but our awareness (*conscience*) of this faculty”. This sounds like a modified Kantianism, quite consonant with its original spirit. See, e.g., Kant, *Ak.*, Vol.XVII, p.639, where the *a priori* intuition of space is described as awareness (*Bewusstsein*) of one’s own aptitude to perceive things according to certain relationships.

¹⁹ Poincaré, SH, Chapter V, “L’expérience et le géométrie”, a chapter composed of passages pieced together from Poincaré (1891), (1899), (1900), plus a short summary of some of the ideas presented in Poincaré (1898).

²⁰ See Section 4.4.5.

²¹ Poincaré, SH, p.76; Poincaré (1898), pp.42f. See below, pp.351f., Section 4.4.5.

²² Poincaré, SH, p.101; cf. pp.75, 95.

²³ Poincaré, SH, p.99. Compare Einstein’s formulation of the general *principle of relativity*: “Natural laws are only statements about spatio-temporal coincidences; therefore their natural expression must consist in generally covariant equations”. (Einstein (1918), p.241.)

²⁴ Minkowski (1908). In fact, Minkowski’s contribution concerned only the underlying manifold of special relativity, which he showed to be a semi-Riemannian 4-manifold (see Appendix, p.372). But he paved the way for the four-dimensional formulation of Newtonian mechanics by Cartan (1923, 1924).

²⁵ In 1912, he described Minkowski’s spacetime geometry as “a new convention” which some physicists had adopted not because they were “constrained” to do so, but because they found it “more comfortable”. Physicists who did not share that opinion could, of course, adhere to their earlier convention and preserve their “old habits”.

Poincaré added: "Je crois, entre nous, que c'est ce qu'ils feront encore longtemps". (Poincaré, DP, p.109.)

²⁶ See p.289 and Note 9 to Part 4.2. It is likely that Poincaré reinvented this argument independently.

²⁷ Poincaré (1891), p.774; SH, pp.95f. (my italics). On "Riemann's geometry" see Part 2.2, Note 52.

²⁸ Poincaré foresaw the rise of "an entirely new mechanics chiefly characterized by the fact that no velocity can surpass that of light". (Poincaré, VS, pp.138f.; the passage belongs to a lecture given in St. Louis in 1904.)

²⁹ Poincaré, SH, p.103.

³⁰ Poincaré, *Oeuvres*, Vol.XI, p.90.

³¹ Poincaré, *Oeuvres*, Vol.XI, p.91.

³² Poincaré (1898), p.43. The normal subgroup in question is the group of translations; see above, p.178. Normal subgroups are defined in the Appendix, p.360. Today's mathematicians call a group 'simple' if it contains no proper normal subgroups. (See, e.g., Kurosh, TG, Vol.I, p.68.) If we adopt this terminology, Poincaré's thesis must be restated thus: the Euclidean group is simpler than the two classical non-Euclidean groups because the latter are simple, while the former is not. This seeming paradox bears witness to the ambiguity of simplicity.

³³ Poincaré, *Oeuvres*, Vol.XI, p.91. The "operations" of the group are not what we would call thus (namely, the group product and the mapping which assigns to every element its own inverse), but simply the elements of the group; the group is viewed as "operating" through them on an underlying space.

³⁴ A topological group is a group G endowed with a topology by virtue of which the mapping $(g, h) \mapsto gh^{-1}$ ($g, h \in G$) is a continuous mapping. The effective and transitive action of a topological group on a topological space is defined like that of a Lie group on a differentiable manifold (make the action continuous in the definition on p.172).

³⁵ I cannot understand why Sklar (1969), p.47, stresses the fact that in the usual Euclidean models of BL geometry the BL group is allowed to act only on a *fragment* of \mathbb{R}^3 (an open ball, a half-space; see Section 2.3.7). Let B denote the interior of a sphere in \mathbb{R}^2 ; B is homeomorphic with \mathbb{R}^2 . Let $f: B \rightarrow \mathbb{R}^3$ be a homeomorphism. Let G be the set of all continuous transformations of B which map chords onto chords and preserve a given orientation of B . G is a group isomorphic with the BL group (p.140). $\{fgf^{-1} \mid g \in G\}$ is then a group of continuous transformations of \mathbb{R}^3 which is plainly isomorphic with G , and hence with the BL group.

³⁶ This group – call it G – is a compact topological group with a countable base of its topology (Appendix, p.360). Suppose that G acts transitively on \mathbb{R}^3 , choose a point $p \in \mathbb{R}^3$ and let H denote the set $\{g \mid g \in G \text{ and } gp = p\}$. H is a closed subgroup of G known as the *stability* (or *isotropy*) *group* of p . The quotient space G/H is compact. Since \mathbb{R}^3 is Hausdorff and locally compact (i.e. every point in \mathbb{R}^3 has a compact neighbourhood), it follows that G/H is homeomorphic with \mathbb{R}^3 . But this is absurd, because \mathbb{R}^3 is not compact. Consequently, our supposition that G acts transitively on \mathbb{R}^3 cannot be true. (See Matsushima, DM, p.185.)

³⁷ See however Glymour (1972).

³⁸ Poincaré (1898), (1903). See below, Section 4.4.5.

³⁹ See pp.172f., where this is explained in connection with Lie groups.

⁴⁰ Poincaré (1898), p.40. Lie's concept of a *Zahlenmannigfaltigkeit* is discussed in Part 3.1, Note 42.

⁴¹ Let $u, v, w \in G$. $f_u f_v(wH) = uvwH = f_{uv}(wH)$. Consequently, f is a realization. $f_{uv}^{-1}(vH) = uH$. Consequently, f is transitive.

⁴² Let $g \in H$. $f_g(H) = gH = H$. On the other hand, if $g \notin H$, $f_g(H) = gH \neq H$. Consequently $\{g \mid g \in G \text{ and } f_g(H) = H\} = H$.

⁴³ It is very likely that Poincaré's dictionary of 1891 inspired Max Black with the interpretation of conventionalism in geometry that was discussed in Section 4.4.2. I must point out, however, that Poincaré's dictionary cannot be used *mutatis mutandis* for "translating" (in Black's sense) a formalized version L of Lobachevskian geometry into a formalized version E of Euclidean geometry. The dictionary presupposes that we have picked out a Euclidean plane π and a side $+$ of π . Only then can we define with Poincaré: *Space* = half-space on side $+$ of π ; *plane* = sphere meeting π at right angles; *straight line* = circle meeting π at right angles; *distance* between points x and y = logarithm of the cross-ratio of x, y and the two points where a circle through x and y meets π at right angles (Poincaré, SH, p.68). But E need not, and indeed should not, include any logical constants, because there are no distinguished points, lines or planes in Euclidean space. Consequently, E will not have a proper name for our plane π , let alone for its preferred side.

⁴⁴ Poincaré (1900), p.81.

⁴⁵ Poincaré (1900), p.85; SH, p.104.

⁴⁶ A valuable recent exception is Vuillemin (1972).

⁴⁷ Poincaré, SH, p.51.

⁴⁸ Poincaré (1898), p.1.

⁴⁹ I do not think that the poor wretch would be able to *see* anything, unless he is aided by the recollection of better times; but then, as I said, I reject sensationism. For someone who accepts it, the man in my example must enjoy 'pure' visual sensations.

⁵⁰ Poincaré, VS, pp.73f.; (1898), p.2.

⁵¹ Poincaré, (1898), p.3.

⁵² Instead of *cancel*s, Poincaré says *correct*s. Instead of *locomotion* he says *displacement*. See Note 55.

⁵³ As a matter of fact, Poincaré reasons as if a given internal change could set in in *any* state of sense awareness. This sounds impossible. Thus, it would seem that in order to experience the feeling of walking one must feel first that one is standing on the ground. An astronaut falling freely in outer space cannot suddenly start walking. We must bear in mind, however, that an obliging demon could, in principle, always suitably modify the gravitational field and place a magic carpet beneath our astronaut's feet at the very moment in which he sets out to walk.

⁵⁴ Poincaré, VS, pp.72f. Poincaré's assertion betrays the mathematician's *penchant* to conceive things with greater neatness than they will bear. It is actually false. Suppose A, B, C are the internal changes felt when one steps forward 2 yards, 2 yards k inches and 2 yards $2k$ inches, respectively. Choose the value of k so that you can distinguish A from C, but you cannot perceive the difference between A and B, nor that between B and C. There is some external change A' which is cancelled by A and not by C. There is also some external change B' which is cancelled by B. Obviously B' is cancelled by A

and by C, since both are indistinguishable from B. This contradicts Poincaré's assertion.

⁵⁵ In 1898, Poincaré used the word *displacement* both in this sense and as a synonym of our term *locomotion*. This creates some confusion. In 1903, he called external locomotions *changements de position*; internal locomotions, *déplacements du corps en bloc* (Poincaré, VS, p.79), and reserved *déplacement* as a name for an equivalence class of locomotions. He lacks a word for locomotions composed of internal and external changes.

⁵⁶ There is one noticeable difference between walking on a conveyor-belt and rowing in a tied boat: the latter activity can be interrupted without giving rise to any external change; on the other hand, whenever the former stops, one will be rapidly carried backwards (unless the conveyor is arranged to stop as soon as you cease stamping your feet on it). But according to our definitions the nature and identity of an internal change depends exclusively on the muscular sensations involved in it.

⁵⁷ Now Chapter II of *La science et l'hypothèse*.

⁵⁸ I.e. the topology consisting of the empty set and all the unions of open intervals.

⁵⁹ An n -dimensional topological manifold is a topological space every one of whose points has a neighbourhood homeomorphic with \mathbb{R}^n . A concept substantially equivalent to this was introduced by Poincaré himself in his pioneering work "Analysis situs" (1895).

⁶⁰ Poincaré, SH, p.51.

⁶¹ Poincaré (1898), pp.14f. Poincaré's argument implies that every displacement is joined to 0 by a simple physical continuum; it follows easily that any two displacements are also joined thus.

⁶² The italicized assumption is made in Poincaré (1903), but not in Poincaré (1898). Instead of it, the earlier work assumes the following: If locomotions A, A' are cancelled by X, and B, B' are cancelled by Y, then $A + B$ and $A' + B'$ are cancelled by $Y + X$. This also conflicts with the physical continuity of the group of displacements. For let A and B be instances of kd , A' and B', instances of $(k + 1)d$, where k and d are as above; then $A + B$ and $A' + B'$ are instances of $2kd$ and $(2k + 2)d$, respectively, and they will not be cancelled by the same locomotion.

⁶³ Poincaré fails to make this distinction (in 1898, he even calls locomotions and displacements by the same name). But it is unavoidable, because locomotions do not form a group (not every locomotion X can be added to a given locomotion A to make $A + X$, but only those whose initial state is indistinguishable from the final state of A).

⁶⁴ Poincaré (1898), p.21. Poincaré's assertion depends on Lie's second main solution of the Lie-Helmholtz space problem (Lie, TT, Vol.III, pp.506f.; see Part 3.1, Note 43, *ad finem*). This presupposes that the group in question is a Lie group. Can this be said of \mathcal{D} ? It has been observed lately that "it is difficult to estimate to which degree [Lie's] proofs [in this second main solution] can withstand modern demands of rigour". Freudenthal (1965), p.152.) Lie's first main solution is generally acknowledged to be rigorous, but it depends on the assumption of free mobility in the infinitesimal (p.178) which Poincaré has not proved for \mathcal{D} .

⁶⁵ Poincaré puts this as follows: "Displacements, we have seen, correspond to changes

in our sensations, and if we distinguish in the present group between form and material, the material can be nothing else than that which the displacements cause to change, viz., our sensations. Even if we suppose that what we have above called sensible space has already been elaborated, the material would then be represented by as many continuous variables as there are nerve-fibres." (Poincaré (1898), pp.22f.)

⁶⁶ Poincaré (1898), p.23. On the *form* and *material* of a group, see above, pp.336f.

⁶⁷ They will be found in Poincaré (1898), pp.23–32 and in a much improved version in Poincaré, VS, pp.80–93.

⁶⁸ See Note 36.

⁶⁹ Poincaré (1898), p.5; cf. Poincaré (1895), now in SH, p.82. By drawing our attention to the fact that Euclidean space is not a whit more *imaginable* than the non-Euclidean spaces Poincaré has finally disposed of a staple Boeotian argument.

⁷⁰ Poincaré (1898), p.42.

⁷¹ Poincaré, SH, p.76.

⁷² Poincaré (1898), p.42.

⁷³ His main contribution is contained in his long paper "Analysis situs" (1895) and its five supplements, published between 1899 and 1904. But the "qualitative theory" of differential equations developed in his four papers "Sur les courbes définies par une équation différentielle" (1881–1886) also follows an essentially topological approach. See Kline, MT, pp.732ff., 1170ff.

⁷⁴ Poincaré, VS, pp.59f.; DP, pp.132ff.

⁷⁵ Poincaré, DP, p.134. We say, today, that two figures F, G (belonging to the same or to different topological spaces) are topologically equivalent if there is a continuous bijection $f: F \rightarrow G$ whose inverse f^{-1} is also continuous. This concept is wider than the one proposed by Poincaré in DP. It should be noted, however, that the name *homeomorphism* given today to such bijections as f was introduced by Poincaré. See Georg Feigl (1928), p.274.

⁷⁶ Poincaré, VS, p.59; cf. VS, p.55; FS, p.417; DP, pp.100, 136.

⁷⁷ Poincaré, DP, p.135; VS, p.60.

⁷⁸ "L'espace et ses trois dimensions", now in Poincaré, VS, Chapters III and IV; "Pourquoi l'espace a trois dimensions", now in DP, pp.133–157. Both articles appeared originally in the *Revue de métaphysique et de morale*.

⁷⁹ Poincaré (1893); reprinted in Poincaré, SH, pp.58–60.

⁸⁰ Euclid, Book I, Definitions 3 and 6; Book XI, Definition 2. (Heath, EE, Vol.I, pp.165, 171; vol.III, p.263.) Compare Menger (1926), pp.118f.

⁸¹ Poincaré, SH, pp.58–60; VS, pp.61–64.

⁸² Poincaré, SH, p.60; VS, pp.64f.

⁸³ Define an ϵ -neighbourhood of a point $Q \in S$ as the set of all points that lie together with Q on one of the two sheets of S , at a distance from Q less than ϵ . Let ϵ range over all positive real numbers and Q over all S . Take ϵ -neighbourhoods as a base of the topology of S . Note that since P lies on both sheets of S , P has two distinct ϵ -neighbourhoods for each positive real number ϵ .

⁸⁴ Cf. Hurewicz and Wallman, DT, p.24. Inductive dimension as defined above is called by recent authors "small inductive dimension" or *ind*, and is distinguished from "large inductive dimension" or *Ind*, which is defined as follows: $Ind(\emptyset) = -1$; $Ind(S) = n$ if n is the least non-negative integer such that for each closed set E and each open set G of

S such that $E \subset G$ there is an open set F such that $E \subset F \subset G$ and the large inductive dimension of the boundary of F is $n - 1$. (See Pears, DT, p.155.)

⁸⁵ Namely, separable metrizable spaces. A topological space S is *separable* if it contains a countable subset C whose closure \bar{C} is identical with S . S is *metrizable* if there exists a distance function $d: S \times S \rightarrow \mathbb{R}$ such that for every x in S and every r in \mathbb{R} , the 'open ball' $\{y \mid y \in S \text{ and } d(x, y) < r\}$ is open in S . Distance functions were defined in Part 2.2, Note 36 (p.384).

⁸⁶ A topological space S is *normal* if (i) S is T_1 -separable (i.e. if for every pair of points $x, y \in S$ there is a neighbourhood X of x and a neighbourhood Y of y such that $y \notin X$ and $x \notin Y$); (ii) for each pair of disjoint closed sets A and B of S there exist disjoint open sets U and V of S such that $A \subset U$ and $B \subset V$. A normal space S is *totally normal* if every subspace of S is normally situated in S , that is, if every $A \subset S$ satisfies the following requirements: for every open set U of S which contains A there exists an open set G such that $A \subset G \subset U$; G is the union of a family $(G_i)_{i \in I}$ of open sets of S , each of which is the union of a countable family of closed sets of S , and each $x \in G$ has a neighbourhood which intersects only a finite number of members of the family $(G_i)_{i \in I}$.

⁸⁷ Lebesgue's comments are contained in a letter to O. Blumenthal and were published in *Mathematische Annalen*, Vol.70, right after Brouwer's proof. Their main purpose was to show that Brouwer's result could be demonstrated in a simpler way. Lebesgue's argument, however, is unsatisfactory.

NOTE ADDED IN PROOF. Randall R. Dipert, in a recent article about "Peirce's theory of the geometrical structure of physical space" (*Isis*, 68 (1977) 404–413), quotes a fragment, written c.1893, in which Peirce argues that physical space can hardly be Euclidean. "It is now agreed – Peirce says – that there is no reason whatever to think the sum of the three angles of a triangle precisely equal to 180 degrees. [...] There is an infinite number of different possible values, of which 180 degrees is only one; so that the probability is as 1 to ∞ or 0 to 1, that the value is just 180 degrees. In other words, it seems for the present impossible to suppose the postulates of geometry precisely true." (Peirce, *Collected Papers*, 1.131.) While Peirce's argument from probability is presumably original and compares very favourably with the statistical *non sequitur* we detected in Erdmann (p.267), the claim to which it lends support follows immediately from a passage in Riemann's lecture of 1854, as we noted on pp.104f. As a matter of fact, the empirical falsehood of the principles of geometry had been proclaimed by John Stuart Mill in his *System of Logic* (quoted on p.257), and had been forcefully expressed by Joseph Delboeuf in the *Revue Philosophique* for 1893 (quoted on p.300). I cannot therefore subscribe to R.R. Dipert's statement that "among scientifically oriented philosophers writing in the second half of the nineteenth century, Peirce was alone in maintaining that [physical] space cannot be Euclidean." (Dipert, loc. cit., p.404.)

REFERENCES

Books: Abbreviations referring to books are obtained from the book titles and are given after the respective author's name(s).

Journals: The following abbreviations are applied.

BJPS: *British Journal for the Philosophy of Science*

JDMV: *Jahresbericht der Deutschen Mathematiker Vereinigung*

JP: *Journal of Philosophy*

JraM: *Journal für die reine und angewandte Mathematik*

JSL: *Journal of Symbolic Logic*

KS: *Kantstudien*

MA: *Mathematische Annalen*

MZ: *Mathematische Zeitschrift*

PS: *Philosophy of Science*

RMM: *Revue de Métaphysique et de Morale*

RPh: *Revue Philosophique de la France et de l'Étranger*

In both the case of books and journals, when more than one edition, or issue, is listed under a particular entry, quotations given refer to the one mentioned first.

Ahrens, J.: 1959, 'Begründung der absoluten Geometrie des Raumes aus dem Spiegelungsbegriff', *MZ* 71, 154–185.

Alexander, H.G. (*LCC*), *The Leibniz-Clarke Correspondence*. Manchester: Manchester University Press, 1956.

Angell, R.B.: 1974, 'The geometry of visibles', *Noûs* 8 87–117.

Aristotelis Opera edidit Academia Regia Borussica. Aristoteles graece ex recognitione Immanuelis Bekkeri. Berlin: G. Reimer, 1831. 2 Vols.

Aristotle: *Physics*. A revised text with introduction and commentary by W.D. Ross. Oxford: Clarendon Press, 1936.

Bachmann, Friedrich: 1936, 'Eine Begründung der absoluten Geometrie in der Ebene', *MA* 113 424–451.

Bachmann, Friedrich: 1951, 'Zur Begründung der Geometrie aus dem Spiegelungsbegriff', *MA* 123 341–344.

Bachmann, Friedrich: *Aufbau der Geometrie aus dem Spiegelungsbegriff*. Zweite ergänzte Auflage. Berlin: Springer, 1973.

Baldus, Richard: 1928, 'Zur Axiomatik der Geometrie. I. Ueber Hilberts Vollständigkeitsaxiom'. *MA* 100 321–333.

Baldus, Richard: 1930, 'Zur Axiomatik der Geometrie. III. Ueber das Archimedische und das Cantorsche Axiom'. *Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Math.-Naturwiss. Kl.*, 1930, 5. Abhandlung.

- Baldus, Richard: 1937, 'Zur Axiomatik der Geometrie. V. Vereinfachung des Hilbertschen Axiomensystems der Euklidischen Geometrie.' *Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Mathem.-Naturwiss. Abteilung*, 1937, 189–228.
- Becker, Oskar: 1934, 'Die Lehre vom Geraden und Ungeraden im Neunten Buch der Euklidischen Elemente. (Versuch einer Wiederherstellung in der ursprünglichen Gestalt).' In Becker, *GGM* pp.125–145. (Originally published in *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, Abt. B, Bd. 3, S. 533–553.)
- Becker, Oskar: (*GGM*), *Zur Geschichte der griechischen Mathematik*. Darmstadt: Wissenschaftliche Buchgesellschaft, 1965.
- Behnke, H., et al.: (*FM*), *Fundamentals of Mathematics*. Translated by S.H. Gould, Cambridge, Mass.: The MIT Press, 1974. 3 Vols.
- Beltrami, Eugenio: (*OM*), *Opere matematiche*. Milano: Ulrico Hoepli, 1902, Vol.I.
- Beltrami, Eugenio: 1868, 'Saggio di interpretazione della geometria non-euclidea', *Giornale di matematiche*, 6 284–312. (Now in Beltrami, *OM* Vol.I, pp.374–405.)
- Beltrami, Eugenio: 1868/9. 'Teoria fondamentale degli spazii di curvatura costante', *Annali di Matematica Pura ed Applicata*, serie II, tomo II, 232–255. (Now in Beltrami, *OM* Vol.I, pp.406–429.)
- Bernays, Paul: 1955, 'Betrachtungen über das Vollständigkeitsaxiom und verwandte Axiome.' *MZ* 63 219–229.
- Bernays, Paul: 1959, 'Die Mannigfaltigkeit der Direktiven für die Gestaltung geometrischer Axiomensysteme.' In Henkin et al., *AM* 1–15.
- Birkhoff, Garrett: 1935, 'Combinatorial relations in projective geometry.' *Annals of Mathematics*, 36 743–748.
- Birkhoff, Garrett: (*LT*), *Lattice Theory*. Providence: American Mathematical Society, 1967.
- Black, Max: 1942, 'Conventionalism in geometry and the interpretation of necessary statements', *PS*, 9 335–349.
- Blumenthal, L.M. and Menger, K.: *Studies in Geometry*. San Francisco: W.H. Freeman, 1970.
- Bolyai, Farkas: *Tentamen juventutem studiosam in elementa matheseos purae elementaris ac sublimioris, methodo intuitiva, evidentiaque huic propria, introducendi*. Tomus Primus. Maros Vasarhely: J. et S. Kali, 1832.
- Bolyai, Janos: (*SAS*), *The Science Absolute of Space*. Translated by G.B. Halsted. Printed as a Supplement to Bonola, *NEG*.
- Bolzano, Bernard: (*O*), *Oeuvres* éditées par la Société Royale des Lettres et des Sciences de Bohême. Tome 5. Mémoires Géométriques, publiées et commentées par Dr. Jan Vojtěch. Praha: Královská Česká Společnost Nauk, 1948.
- Bolzano, Bernard: (*BGE*), *Betrachtungen über einige Gegenstände der Elementargeometrie*. Prag: Karl Barth, 1804. (Now in Bolzano, *O*, Vol.5, pp.9–49.)
- Bonola, Roberto: (*NEG*), *Non-Euclidean Geometry. A critical and historical study of its development*. English translation with additional appendices by H.S. Carslaw. New York: Dover, 1955.
- Borsuk, Karol and Szmielew, Wanda: (*FG*), *Foundations of Geometry. Euclidean and Bolyai-Lobachevskian Geometry. Projective Geometry*. Translated from Polish by Erwin Marquit. Amsterdam: North-Holland, 1960.

- Bradwardine, Thomas: *De Causa Dei contra Pelagium et de Virtute causarum ad suos Mertonenses libri tres . . . opera et studio Di. Henrici Savillii . . . ex scriptis codicibus nunc primum editi*. Londini: Ex officina Nortoniana, apud Ioannem Billium, 1618.
- Brouwer, L.E.J.: 1911a, 'Ueber eineindeutige, stetige Transformationen von Flächen in sich', *MA*, 69, 176–180.
- Brouwer, L.E.J.: 1911b, 'Beweis der Invarianz der Dimensionenzahl', *MA* 70 161–165.
- Brouwer, L.E.J.: 1913, 'Ueber den natürlichen Dimensionsbegriff', *JraM* 142 146–152.
- Bruno, Giordano: (*Op. lat.*), *Opera latine conscripta*. Faksimile-Neudruck der Ausgabe von Fiorentino, Tocco und anderen, Neapel und Florenz, 1879–1891. Stuttgart-Bad Canstatt: Frommann-Holzboog, 1962. 3 Vols. in 8.
- Brunschvicg, Léon: *Les étapes de la philosophie mathématique*. Paris: P.U.F., 1947.
- Burali-Forti, Cesare: 1900, 'Sur les différentes méthodes logiques pour la définition du nombre réel', *Bibliothèque du Congrès International de Philosophie. Paris 1900*. Paris: A. Colin, 1901, Vol.3, pp.289–307.
- Busemann, Herbert: (*GG*), *The Geometry of Geodesics*. New York: Academic Press, 1955.
- Calinon, Auguste: 1889, 'Les espaces géométriques', *RPh* 27 588–595. (English translation in Čapek, *CST* pp.297–303.)
- Calinon, Auguste: 1891, 'Les espaces géométriques', *RPh*, 32 368–375.
- Calinon, Auguste: 1893, 'Etude sur l'indétermination géométrique de l'univers', *RPh* 36 595–607.
- Cantor, Georg: (*GA*), *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*. Hildesheim: Olms, 1962.
- Cantor, Georg: 1878, 'Ein Beitrag zur Mannigfaltigkeitslehre'', *JraM* 84 242–258.
- Čapek, Milič: (*CST*), *The Concepts of Space and Time. Their Structure and Development*. Dordrecht: D. Reidel, 1976.
- Cartan, Elie: 1923, 1924, 'Sur les variétés à connexion affine et la théorie de la relativité généralisée', *Annales de l'Ecole Normale Supérieure*, 40 325–412; 41 1–25.
- Cartan, Elie: 1927, 'La théorie des groupes et la géométrie', *L'enseignement mathématique*, 26 200–225.
- Cartan, Elie: (*LGER*), *Leçons sur la géométrie des espaces de Riemann*. Paris: Gauthier-Villars, 1928.
- Cartan, Elie: (*ERS*), 'Les espaces riemanniens symétriques', *Verhandlungen des Internationalen Mathematiker-Kongresses, Zürich 1932*, Vol.I, pp.152–161.
- Carus, Paul: 1902/3, 'The foundations of geometry', *The Monist*, 13 370–397, 493–522.
- Cayley, Arthur: (*CP*), *The Collected Mathematical Papers of Arthur Cayley*. Cambridge: at the University Press, 1889–1897. 13 Vols.
- Cayley, Arthur: 1859, 'A sixth memoir upon quantics', *Philosophical Transactions of the Royal Society of London*, 149 61–90. (Also in Cayley, *CP*, Vol.II, pp.561–592.)
- Christoffel, Elwin Bruno: 1869, 'Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades', *JraM* 70 46–70.
- Christoffel, Elwin Bruno: 1869b, 'Ueber ein die Transformation homogener Differentialausdrücke zweiten Grades betreffendes Theorem', *JraM* 70 241–245.
- Clagett, Marshall (Ed.): (*CPHS*), *Critical Problems in the History of Science*, Madison, University of Wisconsin Press, 1959.

- Clifford, William Kingdon: 'On the space-theory of matter (abstract)', Cambridge Philosophical Society, Proceedings, II (1876), pp.157-158. (Also in Clifford, *MP*, pp.21-22.)
- Clifford, William Kingdon: (*MP*), *Mathematical Papers*, edited by Robert Tucker, London: Macmillan, 1882.
- Coolidge, Julian Lowell: (*HGM*), *A History of Geometrical Methods*. Oxford University Press, 1940.
- Cornford, Francis M.: 1936, 'The Invention of Space', in M. Čapek, *CST*, pp.3-16. (Originally in *Essays in Honour of Gilbert Murray*, London: Allen and Unwin, 1936, pp.215-235.)
- Couturat, Louis: 1893, 'Note sur la géométrie non-euclidienne et la relativité de l'espace', *RMM* 1 302-309.
- Couturat, Louis: 1893a, 'L'Année Philosophique de F. Pillon (2^e Année, 1891)', *RMM* 1 63-85.
- Couturat, Louis: 1896, 'Etudes sur l'espace et le temps de MM. Lechalas, Poincaré, Delboeuf, Bergson, L. Weber et Evellin', *RMM* 4 646-669.
- Couturat, Louis: 1898, 'Essai sur les fondements de la géométrie par Bertrand Russell', *RMM* 6 354-380.
- Daniels, Norman: 1972, 'Thomas Reid's discovery of a non-Euclidean geometry', *PS* 39 219-234.
- Davis, Martin: (*CU*), *Computability and Unsolvability*. New York: McGraw-Hill, 1958.
- Dedekind, Richard: 1900, 'Ueber die von drei Moduln erzeugte Dualgruppe', *MA* 53 371-403.
- Dehn, Max: 1900, 'Die Legendreschen Sätze über die Winkelsumme im Dreieck', *MA* 53 404-439.
- Dehn, Max: 'Die Grundlegung der Geometrie in historischer Entwicklung', in Pasch, *VNG*² pp.185-271.
- Delboeuf, Joseph: (*PPG*), *Prolégomènes Philosophiques de la Géométrie et Solution des Postulats*. Liege: J. Desoër, 1860.
- Delboeuf, Joseph: (*ANG*), 'L'ancienne et les nouvelles géométries', *RPh* 36 (1893) 449-484; 37 (1894) 353-384; 38 (1894) 113-147; 39 (1895) 345-371.
- Delboeuf, Joseph: *Mégamicros ou les effets sensibles d'une réduction proportionnelle des dimensions de l'univers*. Paris: Alcan, 1893.
- Delboeuf, Joseph: *La Géométrie Euclidienne sans le Postulatum d'Euclide*. Paris: Hermann, 1897.
- Descartes, René: (*AT*), *Oeuvres*, Publiés par Charles Adam et Paul Tannéry. Paris: L. Cerf, 1897-1910. 12 Vols.
- Duhem, Pierre: (*SP*), *To save the phenomena. An Essay on the Idea of Physical Theory from Plato to Galileo*, translated by Edmund Doland and Chaninah Mahler. Chicago: The University of Chicago Press, 1969.
- Ehresmann, Ch.: 1950, 'Les connexions infinitésimales dans un espace fibré différentiable', *Colloque de Topologie (Espaces Fibrés)*, Bruxelles 1950. Paris: Masson, 1957, pp.29-50.
- Einstein, Albert: 1916, 'Die Grundlage der allgemeinen Relativitätstheorie', in Lorentz *et al.*, *R*, pp.81-124. (Originally in *Annalen der Physik*, 49 769.)

- Einstein, Albert: 1917, 'Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie', in Lorentz *et al.*, *R*, pp.130–139. (Originally in *Sitzungsberichte der Königl. Preussischen Akademie der Wissenschaften*, Jahrgang 1917, pp.142ff.)
- Einstein, Albert: 1918, 'Prinzipielles zur allgemeinen Relativitätstheorie', *Annalen der Physik*, **55** 241–244.
- Enriques, Federico: *Lezioni di geometria proiettiva*. Bologna: Zanichelli, 1898.
- Enriques, Federico: 1901, 'Sulla spiegazione psicologica dei postulati della geometria', *Rivista Filosofica* (Pavia), **4** 171–195.
- Enriques, Federico: (PS), *Problems of Science*. Translated by Katharine Royce. LaSalle, Ill.: Open Court, 1943.
- Erdmann, Benno: (AG), *Die Axiome der Geometrie. Eine philosophische Untersuchung der Riemann-Helmholtzschen Raumtheorie*. Leipzig: Leopold Voss, 1877.
- Euclid: *Elementa*, edidit I.L. Heiberg. Leipzig: B.G. Teubner, 1883–1888. 5 Vols. (See also under Heath).
- Fano, G.: 1892, 'Sui postulati fondamentali della geometria in uno spazio lineare a un numero qualunque di dimensioni', *Giornale Matem. Battaglini*, **30** 106–132.
- Feigl, Georg: 1928, 'Geschichtliche Entwicklung der Topologie', *JDMV* **37** 273–286.
- Frege, Gottlob: (KS), *Kleine Schriften*, herausgegeben von I. Angelelli. Darmstadt: Wissenschaftliche Buchgesellschaft, 1967.
- Frege, Gottlob: 1903, 'Ueber die Grundlagen der Geometrie', *JDMV*, **12** 319–324, 368–375. (Also in Frege, KS, pp.262–272.)
- Frege, Gottlob: 1906, 'Ueber die Grundlagen der Geometrie', *JDMV*, **15** 293–309, 377–403, 423–430. (Also in Frege, KS, pp.281–323.)
- Freudenthal, Hans: 1956, 'Neuere Fassungen des Riemann-Helmholtz-Lieschen Raumproblems', *MZ*, **63** 374–405.
- Freudenthal, Hans: 1957, 'Zur Geschichte der Grundlagen der Geometrie. Zugleich eine Besprechung der 8. Auflage von Hilberts Grundlagen der Geometrie.' *Nieuw Archief v. Wiskunde*, 3^e série, **5** 105–142.
- Freudenthal, Hans: 1960, 'Die Grundlagen der Geometrie um die Wende des 19. Jahrhunderts', *Math.-phys. Semesterbericht*, **7** 2–25.
- Freudenthal, Hans: 1965, 'Lie Groups in the Foundations of Geometry', *Advances in Mathematics*, **1** 145–190.
- Freudenthal, Hans: 1974, 'The Impact of von Staudt's Foundations of Geometry', in R.S. Cohen *et al.*, *For Dirk Struik*, Dordrecht: D. Reidel, 1974, pp.189–200.
- Fritz, Kurt von: 1955, 'Die APXAI in der griechischen Mathematik', *Archiv für Begriffsgeschichte*, **1** 13–103.
- Galilei, Galileo: (EN), *Le Opere*. Nuova ristampa della Edizione Nazionale. Firenze: G. Barbera, 1964–1966. 20 Vols.
- Gauss, Carl Friedrich: (WW), *Werke*. Achter Band. Herausgegeben von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Leipzig: B.G. Teubner, 1900. (Pp.157–268 are devoted to the foundations of geometry.)
- Gauss, Carl Friedrich: *Disquisitiones generales circa superficies curvas*. Göttingen: Dieterich, 1828. (Also in *Comm. Soc. Reg. Sc. Göttingensis, cl. math.*, tomus VI, pp.99–146.)
- Gauss, Carl Friedrich: (GI), *General Investigations of Curved Surfaces*. Translated by A. Hildebrandt and J. Morehead. Hewlett, N.Y.: Raven Press, 1965.

- Gauss, Carl Friedrich: *Briefwechsel zwischen C.F. Gauss und H.C. Schumacher*, herausgegeben von C.A.F. Peters. Altona: G. Esch, 1860–1865. 6 Vols.
- Gentzen, Gerhard: 1936, 'Die Widerspruchsfreiheit der reinen Zahlentheorie', *MA* 112 493–565.
- Gergonne, Joseph-Diez: 1818, 'Essai sur la théorie des définitions', *Annales des Mathématiques*, 9 1–35.
- Glymour, Clark: 1972, 'Topology, Cosmology and Convention', *Synthèse*, 24 195–218.
- Gödel, Kurt: 1931, 'Ueber formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme. I.' *Monatshefte für Mathematik und Physik*, 38 173–198.
- Goe, George: 1962, 'Kästner, forerunner of Gauss, Pasch and Hilbert', *Proceedings of the Tenth International Congress of the History of Science, Ithaca 1962*. Paris: Hermann, 1964, pp.659–661.
- Gómez-Lobo, Alfonso: 1977, 'Aristotle's Hypotheses and the Euclidean Postulates', *Review of Metaphysics*, 30 430–439.
- Gonseth, Ferdinand: (GPE), *La Géométrie et le Problème de l'Espace*, Neuchâtel: Griffon, 1945–1955. 6 fasc.
- Grassmann, Hermann: (WW), *Gesammelte mathematische und physikalische Werke*. Ersten Bandes erster Theil. Die Ausdehnungslehre von 1844 und die Geometrische Analyse. Unter der Mitwirkung von E. Study hrsg. von F. Engel. New York: Chelsea, 1969. (Reprint of Leipzig edition of 1894.)
- Grassmann, Hermann: *Die Wissenschaft der extensiven Grösse oder die Ausdehnungslehre, eine neue mathematische Disciplin*. Erster Theil, die lineale Ausdehnungslehre enthaltend. Leipzig: Otto Wigand, 1844.
- Grünbaum, Adolf: (PPST), *Philosophical Problems of Space and Time*. Second, enlarged edition. Dordrecht: D. Reidel, 1973.
- Grzegorzcyk, A.: (OML), *An Outline of Mathematical Logic. Fundamental Results and Notions Explained with All Details*. Dordrecht: D. Reidel, 1974.
- Hanson, Norwood Russell: (CC), *Constellations and Conjectures*, edited by W.C. Humphreys. Dordrecht: D. Reidel, 1973.
- Hantzsche, W. and Wendt, H.: 1935, 'Dreidimensionale euklidische Raumformen', *MA* 110 593–611.
- Hausdorff, Felix: 1904, 'Das Raumproblem', *Annalen der Naturphilosophie*, 3 1–23.
- Heath, Thomas L.: (EE), *The Thirteen Books of Euclid's Elements*, translated from the text of Heiberg with introduction and commentary. Second edition, revised with additions. New York: Dover, 1956. 3 Vols.
- Heath, Thomas L.: (GA), *Greek Astronomy*. London: J.M. Dent, 1932.
- Hegel, Georg Wilhelm Friedrich: (Werke), *Sämtliche Werke*. Jubiläumsausgabe in 20 Bänden, herausgegeben von H. Glockner. Stuttgart: Frommann-Holzboog, 1958.
- Heijenoort, Jean van: (FFG), *From Frege to Gödel. A Source-Book in Mathematical Logic, 1879–1931*. Cambridge, Mass.: Harvard University Press, 1967.
- Heilbronner, Johann Christoph: *Versuch einer mathematischen Historie*. Erster Theil. Frankfurt u. Leipzig: S. Wohler, 1739.
- Helmholtz, Hermann von: (WA), *Wissenschaftliche Abhandlungen*. Leipzig: J.A. Barth, 1882–1895. 3 Vols.
- Helmholtz, Hermann von: (SE), *Schriften zur Erkenntnistheorie*. Herausgegeben und erläutert von Paul Hertz und Moritz Schlick. Berlin: Springer, 1921.

- Helmholtz, Hermann von: (G), *Ueber Geometrie*. Darmstadt: Wissenschaftliche Buchgesellschaft, 1968.
- Helmholtz, Hermann von: 1866, 'Ueber die tatsächlichen Grundlagen der Geometrie', *Verhandlungen des naturhistorisch-medicinischen Vereins zu Heidelberg*, 4 197-202 (22 Mai, 1866). 'Zusatz', *Ibid.*, 5 31-32 (30 April, 1869). (Also in Helmholtz, WA, Vol.II, pp.610-617.)
- Helmholtz, Hermann von: 1868, 'Ueber die Tatsachen, die der Geometrie zum Grunde liegen.' In Helmholtz, G, pp.32-60. (Originally published in *Nachrichten von der Kgl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität aus dem Jahre 1868*, Nr.9, pp.193-221.)
- Helmholtz, Hermann von: 1870, 'Ueber den Ursprung und die Bedeutung der geometrischen Axiome'. In Helmholtz, G, pp.1-31. (Lecture delivered in the *Docentenverein* in Heidelberg in 1870; published in Helmholtz, *Vorträge und Reden*, Vol.II, Braunschweig, 1884, pp.1-31.)
- Helmholtz, Hermann von: 1878, 'The origin and meaning of geometrical axioms (II)', *Mind*, 3 212-225.
- Helmholtz, Hermann von: 'Ueber den Ursprung und Sinn der geometrischen Sätze: Antwort gegen Herrn Professor Land.' in Helmholtz, G. pp.61-81. (German text of Helmholtz (1878); includes some additions to the English version published in *Mind*; originally published in Helmholtz, WA, Vol.II, pp.640-660.)
- Helmholtz, Hermann von: (PSL), *Popular Scientific Lectures*, selected and edited by Morris Kline. New York: Dover, 1962.
- Henkin, Leon: 1950, 'Completeness in the theory of types', *JSL* 15 81-91.
- Henkin, Leon, Suppes, Patrick and Tarski, Alfred: (AM), *The Axiomatic Method with special reference to Geometry and Physics. Proceedings of an International Symposium held at the University of California, Berkeley, December 26, 1957-January 4, 1958*. Amsterdam: North-Holland, 1959.
- Herbart, Johann Friedrich: (WW), *Sämmtliche Werke*, herausgegeben von G. Hartenstein. Leipzig: Leopold Voss. Vol.4 (1851); Vols.5 and 6 (1850).
- Hessenberg, G.: 1905, 'Beweis des Desargueschen Satzes aus dem Pascalschen', *MA* 61 161-172.
- Hessenberg, G.: 1905b, 'Begründung der elliptischen Geometrie', *MA* 61 173-184.
- Hilbert, David: (GG), *Grundlagen der Geometrie*. 10^e Auflage. Mit Supplementen von Dr. Paul Bernays. Stuttgart: B.G. Teubner, 1968. (The first edition was published in *Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmal*, Leipzig: B.G. Teubner, 1899, pp.3-92.)
- Hilbert, David: 1902, 'Ueber die Grundlagen der Geometrie', *MA* 56 381-422. (Reprinted in Hilbert, GG, pp.178-230.)
- Hilbert, David and Cohn-Vossen, S.: (GI), *Geometry and the Imagination*, translated by P. Nemenyi. New York: Chelsea, 1952.
- Hilbert, David-Rossier, Paul: (FG), *Les Fondements de la Géométrie*. Edition critique avec introduction et compléments préparée par Paul Rossier. Paris: Dunod, 1971.
- Hintikka, Jaakko: 1967, 'Kant on the mathematical method', *The Monist*, 51 352-375.
- Hintikka, Jaakko: (LLI), *Logic, Language-Games and Information. Kantian Themes in the Philosophy of Logic*. Oxford: Clarendon Press, 1973.

- Hjelmslev, J.: 1907, 'Neue Begründung der ebenen Geometrie', *MA* 64 449–474.
- Hjelmslev, J.: 1923, 'Die natürliche Geometrie', *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität*, 2 1–36.
- Hopf, Heinz: 1926, 'Zum Clifford-Kleinschen Raumproblem', *MA* 95 313–339.
- Hoüel, J.: (PFGE), *Essai critique sur les principes fondamentaux de la géométrie élémentaire, ou commentaire sur les XXXII premières propositions des Eléments d'Euclide*. 2^e édition. Paris: Gauthier-Villars, 1883. (The first edition was published in Paris by Gauthier-Villars in 1867.)
- Hoüel, J.: *Etudes géométriques sur la théorie des parallèles par N.I. Lobatchewsky, suivi d'un Extrait de la Correspondance de Gauss et de Schumacher*. Paris: Gauthier-Villars, 1866.
- Huntington, Edward V.: 1902, 'A complete set of postulates for the theory of absolute continuous magnitude', *American Mathematical Society Transactions*, 3 264–279.
- Huntington, Edward V.: 1913, 'A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion', *MA* 73 522–559.
- Hurewicz, Witold and Wallman, Henry: (DT), *Dimension Theory*. Princeton: Princeton University Press, 1948.
- Jammer, Max: (CS), *Concepts of Space. The History of the Theories of Space in Physics*. Cambridge, Mass.: Harvard University Press, 1957.
- Jasińska, E.J. and Kuchrzewski, M.: 1974, 'Grundlegende Begriffe der Kleinschen Geometrie', *Demonstratio Mathematica*, 7 381–402.
- Kant, Immanuel: (Ak.), *Kants gesammelte Schriften*, herausgegeben von der Königlich Preussischen Akademie der Wissenschaften. Berlin, 1902ff.
- Kant, Immanuel: (KrV, A/B), *Kritik der reinen Vernunft*. Riga: J.F. Hartknoch, 1781 (A), 1787 (B).
- Kepler, Johannes: (GW), *Gesammelte Werke*, herausgegeben unter der Leitung von Walther von Duck und Max Caspar. München: C.H. Beck, 1937ff.
- Killing, Wilhelm: 1891, 'Ueber die Clifford-Klein'schen Raumformen', *MA* 39 257–278.
- Killing, Wilhelm: 1892, 'Ueber die Grundlagen der Geometrie', *JrAm* 109 121–186.
- Killing, Wilhelm: (EGG), *Einführung in die Grundlagen der Geometrie*. Paderborn: Ferdinand Schöningh, 1893–1898. 2 vols.
- Klein, Felix: (GA), *Gesammelte mathematische Abhandlungen*. Berlin: Springer, 1921–1923. 3 Vols.
- Klein, Felix: 1871, 'Ueber die sogenannte Nicht-Euklidische Geometrie', *MA* 4 573–625. (Also in Klein, *GA*, Vol.I, pp.254–305.)
- Klein, Felix: 1872, 'Ueber einen Satz aus der Analysis situs', in Klein, *GA*, Vol.I, pp.306–310. (Originally published in *Nachrichten von der Kgl. Gesellschaft der Wissenschaften zu Göttingen*, Nr.14, June 5, 1872.)
- Klein, Felix: 1873, 'Ueber die sogenannte Nicht-Euklidische Geometrie (Zweiter Aufsatz)', *MA* 6 112–145. (Also in Klein, *GA*, Vol.I, pp.311–343.)
- Klein, Felix: 1873b, 'Ueber den allgemeinen Functionsbegriff und dessen Darstellung durch eine willkürliche Curve', *MA* 22 (1883), 249–259. (Lecture delivered on December 8, 1873. Reprinted in Klein, *GA*, Vol.II, pp.214–224.)
- Klein, Felix: 1874, 'Nachtrag zu dem "zweiten Aufsatz" über Nicht-Euklidische Geometrie', *MA* 7 531–537. (Also in Klein, *GA*, Vol.I, pp.344–350.)

- Klein, Felix: 1890, 'Zur Nicht-Euklidischen Geometrie', *MA* 37 544–572. (Also in Klein, *GA*, Vol.I, pp.353–383.)
- Klein, Felix: (EP), 'Vergleichende Betrachtungen über neuere geometrische Forschungen', *MA* 43 (1893), 63–100. (Revised text of the so-called Erlangen Programme, first published in Erlangen by A. Düchert in 1872.)
- Klein, Felix: 1893, 'On the mathematical character of space-intuition and the relation of pure mathematics to the applied sciences', in Klein, *GA*, Vol.II, pp.225–231.
- Klein, Felix: 1894/5, 'Riemann und seine Bedeutung für die Entwicklung der modernen Mathematik', *JDMV* 4 72–82. (Also in Klein, *GA*, Vol.III, pp.482–497.)
- Klein, Felix: 1897, 'Gutachten, betreffend den dritten Band der Theorie der Transformationsgruppen von S. Lie anlässlich der ersten Verteilung des Lobatschewsky-Preises', *MA* 50 583–600. (Also in Klein, *GA*, Vol.I, pp.384–401.)
- Klein, Felix: 1902, 'Auszug aus dem Gutachten der Göttinger philosophischen Fakultät betreffend die Beneke-Preisaufgabe für 1901', *MA* 55 (1902), 143–148. (Also in Klein, *GA*, Vol.II, pp.241–246.)
- Klein, Felix: (VNG), *Vorlesungen über Nicht-Euklidische Geometrie*. Für den Druck neu bearbeitet von W. Rosemann. New York: Chelsea, n.d.
- Klein, Felix: (VEM), *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*. New York: Chelsea, 1967. 2 Vols. in one.
- Klein, Felix: (EMHS), *Elementarmathematik vom höheren Standpunkte aus*. 3^e Auflage. Dritter Band. Präzisions- und Approximationsmathematik. Berlin: Springer, 1928.
- Klein, Felix: *Elementary Mathematics from an Advanced Standpoint. Geometry*. Translated from the Third German Edition by E.R. Hendrick and C.A. Noble. New York: Dover, n.d.
- Kline, Morris: (MT), *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press, 1972.
- Klügel, Georg Simon: *Conatum praecipuorum theoriam parallelarum demonstrandi recensio quam publico examini submittent Abrah[am] Gotthelf Kaestner... et auctor respondens Georgius Simon Klügel...* Gottingae: Ex officina Schultziana, curante F.A. Rosenbusch, 1763.
- Kock, A. and Reyes, G.E.: 1977, 'Connections in Formal Differential Geometry', Aarhus Universitet, Matematisk Institut (mimeographed).
- Koyré, Alexandre: 1949, 'Le vide et l'espace infini au XIV^e siècle', *Archives d'Histoire Doctrinale et Littéraire du Moyen Age*, 24 45–91.
- Koyré, Alexandre: (CWIU), *From the Closed World to the Infinite Universe*. New York: Harper, 1958.
- Koyré, Alexandre: (RA), *La révolution astronomique. Copernic Kepler Borelli*. Paris: Hermann, 1961.
- Krause, Albrecht: *Kant und Helmholtz über den Ursprung und die Bedeutung der Raumanschauung und der geometrischen Axiome*. Lehr: M. Schauenburg, 1876.
- Kulkarni, R.: (1970, 'Curvature and metric', *Annals of Mathematics*, 91 311–331.
- Kurosh, A.G.: (TG), *The Theory of Groups*. Translated from the Russian and edited by K.A. Kirsch. New York: Chelsea, 1955–1956. 2 Vols.
- Laguerre, Edmond: 1853, 'Note sur la théorie des foyers', *Nouvelles Annales des Mathématiques*, 12 57–66.

- Lambert, Johann Heinrich: *Theorie der Parallellinien*, in Stäckel and Engel, *TP*, pp.152–207. (Originally published in *Magazin für die reine und angewandte Mathematik*, Nr.2, pp.137–164 and Nr.3, pp.325–358 (1786).)
- Lamy, Bernard: *Les Eléments de Géométrie; ou, De la mesure du corps. Qui comprennent tout ce qu'Euclide en a enseigné; les plus belles propositions d'Archimede et l'analyse*. Paris: A. Pralard, 1685.
- Land, J.P.N.: 1877, 'Kant's space and modern mathematics', *Mind*, 2 38–46.
- Lange, Ludwig: 1885, 'Ueber das Beharrungsgesetz', *Berichte über die Verhandlungen der K. Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Math.-Phys. Klasse*, 37 333–351.
- Lebesgue, H.: 1911, 'Sur la non-applicabilité de deux domaines appartenant à des espaces de n et $n + p$ dimensions. (Extrait d'une lettre a M.O. Blumenthal.)' *MA* 70 166–168.
- Lechalas, Georges: 1889, 'La géométrie générale', *Critique Philosophique*, n.s., 5 II, 217–231.
- Lechalas, Georges: 1890a, 'La géométrie générale et les jugements synthétiques a priori', *RPh* 30 157–169.
- Lechalas, Georges: 1890b, 'Les bases expérimentales de la géométrie', *RPh* 30 639–641.
- Lechalas, Georges: 1890c, 'La géométrie générale et l'intuition', *Annales de Philosophie Chrétienne*, n.s., 23 57–74.
- Lechalas, Georges: 1893, 'La géométrie non-euclidienne et le principe de similitude', *RMM* 1 199–201.
- Lechalas, Georges: 1894, 'M. Delboeuf et le problème des mondes semblables', *RPh* 37 73–78. (Followed by 'Réponse de M. Delboeuf', *Ibid.*, pp.78–84.)
- Lechalas, Georges: 1896, 'Note sur la courbure et la distance en géométrie générale', *RMM* 4 194–202.
- Lechalas, Georges: (ET), *Etude sur l'Espace et le Temps*. Paris: Alcan, 1896.
- Lechalas, Georges: 1898, 'L'axiome de libre mobilité d'après M. Russell', *RMM* 6 746–758.
- Lechalas, Georges: 1898b, 'Les fondements de la géométrie d'après M. Russell', *Annales de Philosophie Chrétienne*, n.s., 38 646–660; 39 75–93, 179–197, 317–334.
- Lechalas, Georges: 1901a, 'Un paradoxe géométrique', *RMM* 9 361–367.
- Lechalas, Georges: 1901b, 'Les fondements de la géométrie. A propos d'un livre récent', *Revue Néoscholastique*, 8 (1901), 338–354; 9 (1902), 19–34.
- Lechalas, Georges: (IGG), *Introduction a la géométrie générale*. Paris: Gauthier-Villars, 1904.
- Legendre, Adrien-Marie: *Eléments de géométrie*, avec des notes. Paris: F. Didot, 1794.
- Legendre, Adrien-Marie: 1833, 'Réflexions sur les différentes manières de démontrer la théorie des parallèles', *Mémoires de l'Académie des Sciences, Paris*, 12 367–410.
- Levi-Civita, Tullio: 1917, 'Nozione di parallelismo in una varietà qualunque', *Rendiconti del Circolo Matematico di Palermo*, 42 173–205.
- Lie, Sophus: 1886, 'Bemerkungen zu v. Helmholtz's Arbeit: Ueber die Tatsachen, die der Geometrie zu Grunde liegen', in Lie, *Gesammelte Abhandlungen*, II. Band, 1. Teil, Leipzig: B.G. Teubner, 1935, pp.374–379. (Originally published in *Berichte über die Abhandlungen der Kgl. Sächsischen Gesellschaft der Wissenschaften in Leipzig, Math.-Phys. Classe*, Supplement, abgeliefert am 21.2.1887, pp.337–342.)

- Lie, Sophus: 1890, *Ueber die Grundlagen der Geometrie*, Darmstadt: Wissenschaftliche Buchgesellschaft, 1967. (Originally published in *Berichte über die Abhandlungen der Kg. Sächsischen Gesellschaft der Wissenschaften in Leipzig, Math.-Phys. Classe*, 42 (1890), pp.284–321, 335–418.)
- Lie, Sophus: (TT), *Theorie der Transformationsgruppen*. Unter Mitwirkung von Dr. Friedrich Engel. New York: Chelsea, 1970. 3 Vols. (Reprint of Leipzig edition of 1888–1893).
- Lie, Sophus: (VCG), *Vorlesungen über continuierliche Gruppen, mit geometrischen und anderen Anwendungen*, bearbeitet und herausgegeben von Dr. Georg Scheffers. New York: Chelsea, 1971. (Reprint of Leipzig edition of 1893.)
- Lipschitz, R.: 1870, 'Fortgesetzte Untersuchungen in Betreff der ganzen homogenen Functionen von n Differentialen', *JraM* 72 1–56.
- Lobachevsky, Nikolai Ivanovich: 'O nachalakh geometrii', *Kasanski Vestnik*, Feb.-March 1829, pp.178–187; April 1829, pp.228–241; Nov.-Dec. 1829, pp.227–243; March-April 1830, pp.251–283; July-Aug. 1830, pp.571–636.
- Lobachevsky, Nikolai Ivanovich: 'Novye nachala geometrii', *Uchonia sapiski Kasanskava Universiteta*, 1835, III, pp.3–48; 1836, II, pp.3–98; III, pp.3–50; 1837, I, pp.3–97, 1838, I, pp.3–124; III, pp.3–65.
- Lobachevsky, Nikolai Ivanovich: (ZGA), *Zwei geometrische Abhandlungen*, aus dem Russischen übersetzt, mit Anmerkungen und mit einer Biographie des Verfassers von Friedrich Engel. Leipzig: B.G. Teubner, 1898–1899. 2 Vols. (German annotated translation of the two preceding items.)
- Lobachevsky, Nikolai Ivanovich: 1837, 'Géométrie Imaginaire', *JraM* 17 295–320.
- Lobachevsky, Nikolai Ivanovich: *Geometrische Untersuchungen zur Theorie der Parallellinien*, Berlin: F. Fincke, 1840.
- Lobachevsky, Nikolai Ivanovich: (GRTP), *Geometrical Researches on the Theory of Parallels*, translated by G.B. Halsted. Printed as a Supplement to Bonola, NEG. (English translation of the preceding German monograph.)
- Lobachevsky, Nikolai Ivanovich: *Pangéométrie ou précis de géométrie fondée sur une théorie générale et rigoureuse des parallèles*. Kazan: Universitet, 1856.
- Lorentz, H.A., Einstein, A., Minkowski, H.: (R), *Das Relativitätsprinzip. Eine Sammlung von Abhandlungen*. Darmstadt: Wissenschaftliche Buchgesellschaft, 1958. (Reprint of 1923 edition.)
- Lotze, Hermann: (M), *Metaphysik. Drei Bücher der Ontologie, Kosmologie und Psychologie*. Leipzig: S. Hirzel, 1879.
- Lucretius: *De rerum natura*, with an English translation by W.H.D. Rouse. London: William Heinemann, 1924. (The Loeb Classical Library.)
- Lüroth, J.: 1875, 'Das Imaginäre in der Geometrie und das Rechnen mit Würfeln. Darstellung und Erweiterung der v. Staudt'schen Theorie', *MA* 8 145–214.
- Lüroth, J.: 1877, 'Das Imaginäre in der Geometrie (Zweite Abhandlung)', *MA* 11 84–110.
- Mach, Ernst: (BAE), *Beiträge zur Analyse der Empfindungen*. Jena: Gustav Fischer, 1886.
- Mach, Ernst: 1901/2, 'On the psychology and natural development of geometry', *The Monist*, 12 481–515.
- Mach, Ernst: 1903/4, 'Space and geometry from the point of view of physical inquiry', *The Monist*, 14 1–32.

- Mach, Ernst: (*EI*), *Erkenntnis und Irrtum. Skizzen zur Psychologie der Forschung*. Darmstadt: Wissenschaftliche Buchgesellschaft, 1968. (Reprint of 5th edition, published in Leipzig in 1925; the first edition appeared in 1905.)
- MacLane, Saunders and Birkhoff, Garrett: *Algebra*. New York: Macmillan, 1967.
- Maimonides, Moses: (*GP*), *The Guide of the Perplexed*. Translated with an introduction and notes by Shlomo Pines. Chicago: The University of Chicago Press, 1963.
- Malliavin, Paul: (*GDI*), *Géométrie différentielle intrinsèque*. Paris: Hermann, 1972.
- Mannheim, Jerome H.: (*GPST*), *The Genesis of Point-Set Topology*. London: Pergamon, 1964.
- Matsushima, Yozo: (*DM*), *Differentiable Manifolds*. Translated by E.T. Kobayashi. New York: Marcel Dekker, 1972.
- Mayberry, John: 1977, 'On the Consistency Problem for Set Theory: An Essay on the Cantorian Foundations of Classical Mathematics', *BJPS* 28 1-34, 137-170.
- Menger, Karl: 1923, 1924, 'Ueber die Dimensionalität von Punktmengen', *Monatshefte für Mathematik und Physik*, 33 148-160; 34 137-161.
- Menger, Karl: 1926, 'Bericht über die Dimensionstheorie', *JDMV* 35 113-150.
- Menger, Karl: 1928, 'Axiomatik der endlichen Mengen und der elementargeometrischen Verknüpfungsbeziehungen', *JDMV* 37 309-325.
- Menger, Karl: 1938, 'Non-Euclidean Geometry of Joining and Intersecting', *American Mathematical Society Bulletin*, 44 821-824.
- Menger, Karl: 1940, *Three Lectures on Mathematical Subjects - Delivered at the Rice Institute, December 11, 12 and 13, 1939*. The Rice Institute Pamphlet, XXVII, No. 1.
- Menger, Karl, Alt, F. and Schreiber, O.: 1936, 'New foundations of projective and affine geometry', *Annals of Mathematics*, 27 456-482.
- Méray, Charles: 1869, 'Remarques sur la nature des quantités définies par la condition de servir de limites à des variables données', *Revue des Sociétés Savantes*, 4 (1869), 280-289.
- Méray, Charles: (*NEG*), *Nouveaux éléments de géométrie*, Paris: F. Savy, 1874.
- Mill, John Stuart: (*SL*), *A System of Logic Ratiocinative and Inductive, being a Connected View of the Principles of Evidence and the Methods of Scientific Investigation*. Edited by J.M. Robson. Toronto: University of Toronto Press, 1973-1974. 2 Vols.
- Miller, Arthur I.: 1972, 'The myth of Gauss' experiment on the Euclidean nature of physical space', *Isis*, 63 345-348.
- Miller, Arthur I. et al.: (1974, 'Comments on Miller's "The myth of Gauss' experiment on the Euclidean nature of physical space"', *Isis*, 65 83-87. (Comment I by George Goe; Comment II by B. L. van der Waerden; Reply by Arthur I. Miller.)
- Milnor, J.: 1956, 'On manifolds homeomorphic to the 7-sphere', *Annals of Mathematics*, 64 399-405.
- Minding, Ferdinand: 1837, 'Ueber die Biegung gewisser Flächen', *JraM* 17 295-320.
- Minding, Ferdinand: 1839, 'Wie sich entscheiden lässt, ob zwei gegebene krumme Flächen aufeinander abwickelbar sind oder nicht, nebst Bemerkungen über die Flächen von unveränderlichen Krümmungsmasse', *JraM* 19 370-387.
- Minding, Ferdinand: 1840, 'Beiträge zur Theorie der kürzesten Linien auf krummen Flächen', *JraM* 20 323-327.
- Minkowski, Hermann: 1908, 'Raum und Zeit', in Lorentz et al., *R*, pp.54-66. (Lecture

- read before the 80. Versammlung Deutscher Naturforscher und Aerzte in Cologne on September 21, 1908.)
- Montucla, Jean Etienne: *Historie des mathématiques, dans laquelle on rend compte de leurs progrès depuis leur origine jusqu'à nos jours*. Paris: C.A. Jombert, 1758. 2 Vols.
- Mooij, J.J.A.: *La philosophie des mathématiques de Henri Poincaré*, Paris: Gauthier-Villars, 1966.
- Moore, Eliakim Hastings: 1902, 'On the projective axioms of geometry', *American Mathematical Society Transactions*, 3 142-158.
- Moore, G.E.: 1899, Review of Russell, *An Essay on the Foundations of Geometry*. *Mind*, n.s., 8 397-405.
- Moore, Robert L.: 1908, 'Sets of metrical hypotheses for geometry', *American Mathematical Society Transactions*, 9 487-512.
- More, Henry: (OO), *Opera Omnia*. Hildesheim, Olms, 1967. 2 Vols. in 3. (Reprint of the London edition of 1679.)
- Mouret, G.: 1892, 'Sur les géométries non-euclidiennes', *Revue générale des Sciences pures et appliquées*, 3 39-40.
- Munkres, James R.: (EDT), *Elementary Differential Topology*. Princeton: Princeton University Press, 1963.
- Munkres, James R.: *Topology. A First Course*. Englewood Cliffs: Prentice-Hall, 1975.
- Nagel, Ernest: 1939, 'The formation of modern conceptions of formal logic in the development of geometry', *Osiris*, 7 142-224.
- Neumann, Carl Gottfried: (PGNT), *Ueber die Principien der Galilei-Newton'schen Theorie*, Leipzig: B.G. Teubner, 1870.
- Nomizu, K.: 1954, 'Invariant affine connections on homogeneous spaces', *American Journal of Mathematics*, 76 33-65.
- Padoa, Alessandro: 1900, 'Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque', in *Bibliothèque du Congrès International de Philosophie, Paris 1900*, Paris: Armand Colin, 1901, Vol. 3, pp.309-265.
- Padoa, Alessandro: 1900b, 'Un nouveau système irréductible de postulats pour l'algèbre', *Compte rendu du Deuxième Congrès International des Mathématiciens tenu à Paris du 6 au 12 août 1900*. Paris: Gauthier-Villars, 1902, pp.249-256.
- Pasch, Moritz: (VNG), *Vorlesungen über neuere Geometrie*. Leipzig: B.G. Teubner, 1882.
- Pasch, Moritz: (VNG²), *Vorlesungen über neuere Geometrie*. Zweite Auflage. Berlin: Springer, 1926.
- Pasch, Moritz: 1917, 'Grundfragen der Geometrie', *JraM* 147 184-190.
- Peano, Giuseppe: 1889, *I Principii di Geometria logicamente esposti*. Torino: Fratelli Bocca, 1889.
- Peano, Giuseppe: 1890, 'Sur une courbe, qui remplit toute une aire plane', *MA* 36 157-160.
- Peano, Giuseppe: 1894, 'Sui fondamenti della geometria', *Rivista di Matematica*, 4 51-90.
- Peano, Giuseppe: 1900, 'Les définitions mathématiques', *Bibliothèque du Congrès International de Philosophie, Paris 1900*. Paris: A. Colin, 1901, Vol.3, pp.279-288.
- Pears, A.R.: (DT), *Dimension Theory of General Spaces*. Cambridge: Cambridge University Press, 1975.

- Philoponus, Ioannes: *In Aristotelis Physicorum libros quinque posteriores commentaria*, edidit H. Vitelli. Berlin: G. Reimer, 1888.
- Pieri, Mario: 1899a, 'I principii della geometria di posizione composti in sistema logico deduttivo', *Memorie della Reale Accademia delle Scienze di Torino, Classe di Sc. Fisiche, Matematiche e Naturali*, Serie 2, 48 1-62.
- Pieri, Mario: 1899b, 'Della geometria elementare come sistema ipotetico-deduttivo; monografia del punto e del moto', *Memorie della Reale Accademia delle Scienze di Torino, Classe di Sc. Fisiche, Matematiche e Naturali*, Serie 2, 49 173-222.
- Pieri, Mario: 1900, 'Sur la géométrie envisagée comme un système purement logique', *Bibliothèque du Congrès International de Philosophie*, Paris 1900, Paris: A. Colin, 1901, Vol.3, pp.367-404.
- Platonis Opera recognovit brevique adnotatione critica instruxit Ioannes Burnet. Oxonii: e Typographeo Clarendoniano, 1900-1907. 5 Vols.
- Plücker, Julius: *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement*. Leipzig: B.G. Teubner, 1868-1869. 2 Vols.
- Plücker, Julius: *Gesammelte wissenschaftliche Abhandlungen*, im Auftrag der Kgl. Gesellschaft der Wissenschaften zu Göttingen herausgegeben von A. Schoenflies und F. Pockels. Leipzig: B.G. Teubner, 1895-1896. 2 Vols.
- Poincaré, Henri: (*O*), *Oeuvres*, publiées sous les auspices de l'Académie des Sciences. Paris: Gauthier-Villars, 1916-1956. 11 Vols.
- Poincaré, Henri: (*SH*), *La Science et l'Hypothèse*. Paris: Flammarion, 1968. (First published in 1902.)
- Poincaré, Henri: (*VS*), *La Valeur de la Science*. Paris: Flammarion, 1970. (First published in 1905.)
- Poincaré, Henri: (*SM*), *Science et Méthode*. Paris: Flammarion, 1908.
- Poincaré, Henri: (*DP*), *Dernières Pensées*. Paris: Flammarion, 1963. (First edition: 1913; second enlarged edition: 1926.)
- Poincaré, Henri: (*FS*), *Foundations of Science. Science and Hypothesis, The Value of Science, Science and Method*. Translated by G. Halsted. New York: The Science Press, 1929.
- Poincaré, Henri: 'Sur les courbes définies par une équation différentielle', *Journal des Mathématiques pures et appliquées*, (3) 7 (1881), 375-422; 8 (1882), 251-296; (4) 1 (1885), 167-244; 2 (1886), 151-217. (Also in Poincaré, *O*, Vol.I, pp.3-84, 90-161, 167-221.)
- Poincaré, Henri: 1887, 'Sur les hypothèses fondamentales de la géométrie', *Bulletin de la Société Mathématique de France*, 15 203-216. (Also in Poincaré, *O*, Vol.XI, pp.79-91.)
- Poincaré, Henri: 1891, 'Les géométries non-euclidiennes', *Revue générale des Sciences pures et appliquées*, 2 769-774.
- Poincaré, Henri: 1892, 'Sur les géométries non-euclidiennes', *Revue générale des Sciences pures appliquées*, 3 74-75. (Reply to G. Mouret.)
- Poincaré, Henri: 1893, 'Le continu mathématique', *RMM* 1 26-34.
- Poincaré, Henri: 1895, 'L'espace et la géométrie', *RMM* 3 631-646.
- Poincaré, Henri: 'Analysis situs', *Journal de l'Ecole Polytechnique* (2) 1 (1895), 1-121. 'Premier complément à l'analysis situs', *Rendiconti del Circolo Matematico di Palermo*, 13 (1899), 285-343. 'Deuxième complément...', *London Mathematical Society Proceedings*, 32 (1900), 277-308. 'Troisième complément...' *Bulletin de la*

- Société Mathématique de France*, 30 (1902), 49–70. 'Quatrième complément...' *Journal des Mathématiques pures et appliquées*, (5) 8 (1902), 169–214. 'Cinquième complément...', *Rendiconti del Circolo Matematico di Palermo*, 18 (1904), 45–110. (Also in Poincaré, *O*, Vol.VI, pp.193–288, 290–498.)
- Poincaré, Henri: 1897, 'Réponse a quelques critiques', *RMM* 5 58–70.
- Poincaré, Henri: 1898, 'On the foundations of geometry', *The Monist*, 9 1–43. (Poincaré, *Des fondements de la géométrie*, Paris: Chiron, n.d. [1921], is a French translation of this article).
- Poincaré, Henri: 1898b, 'La mesure du temps', *RMM* 6 1–13.
- Poincaré, Henri: 1899, 'Des fondements de la géométrie. A propos d'un livre de M. Russell', *RMM* 7 (1899), 251–279.
- Poincaré, Henri: 1900, 'Sur les principes de la géométrie. Réponse à M. Russell', *RMM* 8 73–86.
- Poincaré, Henri: 'Sur la géométrie non-euclidienne', Note II, Sections III–VII of E. Rouché and Ch. de Comberousse, *Traité de Géométrie*, 7^e édition, Paris: Gauthier-Villars, 1900, Vol.II, pp.581–593.
- Poincaré, Henri: 1902, 'Sur la valeur objective de la science', *RMM* 10 263–293.
- Poincaré, Henri: 1902b, 'Compte rendu de Hilbert, *Les fondements de la géométrie*', *Bulletin des Sciences Mathématiques*, 2^e série, 26 249–272. 'Rectification', *Ibid.*, 27 (1903), 115. (Also in Poincaré, *O* Vol.XI, pp.92–113.)
- Poincaré, Henri: 1902c, 'Les fondements de la géométrie', *Journal des Savants*, mai 1902, pp.252–271. (This is a review essay on Hilbert's *Grundlagen*, largely coincident with the preceding item; it omits, however, the more technical passages of the latter. It is reproduced in Poincaré, *DP*, pp.161–185.)
- Poincaré, Henri: 1903, 'L'espace et ses trois dimensions', *RMM* 11 281–301, 407–429.
- Poincaré, Henri: 1907, 'La relativité de l'espace', *L'Année Psychologique*, 13 1–17.
- Poincaré, Henri: 1912, 'Pourquoi l'espace a trois dimensions?', *RMM* 20 483–504.
- Poincaré, Henri: 1912b, 'L'espace et le temps', *Scientia*, 12 159–170.
- Pound, R.V. and Rebka, G.A.: 1959, 'Gravitational Red Shift in Nuclear Resonance', *Physical Review Letters*, 3 439–441.
- Pound, R.V. and Rebka, G.A.: 1960, 'Apparent weight of photons', *Physical Review Letters*, 4 337–341.
- Pound, R.V. and Snider, J.L.: 1965, 'Effect of gravity on gamma radiation', *Physical Review*, 140 B 788–803.
- Price, Derek J.: *The Equatorie of the Planets*. Edited from Peterhouse Ms. 75.I. Cambridge: Cambridge University Press, 1955.
- Price, Derek J.: 1959, 'Contra-Copernicus. A critical reestimation of the mathematical planetary theory of Ptolemy, Copernicus and Kepler', in Clagett, *CPHS*, pp.197–218.
- Proclus Diadochus: (*Comm.*, ed. Fr.), *In primum Euclidis Elementorum librum Commentarii*, ex recognitione G. Friedlein. Hildesheim: Olms, 1967. (Reprint of the Leipzig edition of 1873.)
- Proclus: *A Commentary on the First Book of Euclid's Elements*. Translated with Introduction and Notes by Glenn R. Morrow. Princeton: Princeton University Press, 1970.
- Rédei, L.: (*FENG*), *Foundation of Euclidean and Non-Euclidean Geometries according to F. Klein*. Budapest: Akadémiai Kiadó, 1968.

- Reichenbach, Hans: (PST), *The Philosophy of Space and Time*. Translated by M. Reichenbach and J. Freund. New York: Dover, 1958.
- Reid, Constance: *Hilbert*. Berlin: Springer, 1970.
- Reid, Thomas: (PW), *Philosophical Works*, with notes and supplementary dissertations by Sir William Hamilton. Hildesheim: Olms, 1967. 2 Vols. (Reprint of the 8th edition, Edinburgh 1895.)
- Reid, Thomas: *An Inquiry into the Human Mind on the Principles of Common Sense*. Dublin: A. Ewing, 1764.
- Reidemeister, Kurt: 1940, 'Die Arithmetik der Griechen', in Reidemeister, *Das exakte Denken der Griechen*, Darmstadt: Wissenschaftliche Buchgesellschaft, 1972, pp.15-43. (First published as a pamphlet in Leipzig in 1940.)
- Renouvier, Charles: 1889, 'La philosophie de la règle et du compas ou des jugements synthétiques a priori dans la géométrie élémentaire', *La Critique Philosophique*, n.s., 5 II, 337-348.
- Renouvier, Charles: 1891, 'La philosophie de la règle et du compas. Théorie logique du jugement dans ses applications aux idées géométriques et à la méthode des géomètres', *L'Année Philosophique*, 2 1-66.
- Resnik, Michael David: 1974, 'The Frege-Hilbert controversy', *Philosophy and Phenomenological Research*, 24 386-403.
- Ricci, G. and Levi-Civita, T.: 1901, 'Méthodes de calcul différentiel absolu et leurs applications', *MA* 54 125-201.
- Riehl, A.: 1904, 'Helmholtz in seinem Verhältnis zu Kant', *KS* 9 261-285.
- Riemann, Bernhard: (WW), *Gesammelte mathematische Werke und wissenschaftlicher Nachlass*. Herausgegeben unter Mitwirkung von Richard Dedekind von Heinrich Weber. Nachträge herausgegeben von M. Noether und W. Wirtinger. Leipzig: B.G. Teubner, 1902.
- Riemann, Bernhard: (H), *Ueber die Hypothesen, welche der Geometrie zugrunde liegen*. Darmstadt: Wissenschaftliche Buchgesellschaft, 1959. (Reprint of first edition, published in 1867 in *Abhandlungen der Kgl. Gesellschaft der Wissenschaften zu Göttingen*, Vol.13.)
- Riemann, Bernhard: *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*, neu herausgegeben und erläutert von H. Weyl. Dritte Auflage. New York: Chelsea, 1973. (reprint of 1923 edition.)
- Robertson, H.P. and Noonan, Thomas W.: (RC), *Relativity and Cosmology*. Philadelphia: W.B. Saunders, 1968.
- Rosenthal, Artur: 1910, 'Über das dritte Hilbertsche Axiom der Verknüpfung', *MA* 69 223-226.
- Rosenthal, Artur: 1912, 'Vereinfachungen des Hilbertschen Systems der Kongruenzaxiome', *MA* 71 257-274.
- Rossier, P.: 1967, 'Les axiomes de la géométrie et leur histoire', *Archives des Sciences de Genève*, 20 369-414.
- Rougier, Louis: 1916, 'La démonstration géométrique et le raisonnement déductif', *RMM* 23 809-858.
- Rougier, Louis: (PGHP), *La philosophie géométrique de Henri Poincaré*. Paris: Alcan, 1920.
- Russell, Bertrand: 1896, 'The a priori in geometry', *Proceedings of the Aristotelian Society*, 3 97-112.

- Russell, Bertrand: 1896b, 'The logic of geometry', *Mind*, n.s., 5 1-23. (This article is almost identical with the preceding one; their text is reproduced with minor changes in Russell, *FG*, Chapter III. B, Sections I and II.)
- Russell, Bertrand: (*FG*), *An Essay on the Foundations of Geometry*. New York: Dover, 1956. (Reprint of the first edition, published by Cambridge University Press in 1897.)
- Russell, Bertrand: 1898, 'Les axiomes propres à Euclide sont-ils empiriques?', *RMM* 6 759-776. (Reply to Couturat, 1898.)
- Russell, Bertrand: 1899, 'Sur les axiomes de la géométrie', *RMM* 7 684-707. (Reply to Poincaré, 1899.)
- Russell, Bertrand: 1900, L'idée d'ordre et la position absolue dans l'espace et le temps'. *Bibliothèque du Congrès International de Philosophie*, Paris 1900, Paris: A. Colin, 1901, Vol.3, pp.241-277.
- Russell, Bertrand: 1901, 'Is position in time and space absolute or relative?' *Mind*, n.s., 10 293-317.
- Russell, Bertrand: 1901b, 'On the notion of order', *Mind*, n.s., 10 30-51.
- Russell, Bertrand: (*Principles*), *The Principles of Mathematics*. Cambridge: Cambridge University Press, 1903.
- Russell, Bertrand: (*KEW*), *Our Knowledge of the External World as a Field for Scientific Method in Philosophy*. Chicago: Open Court, 1915.
- Russell, Bertrand: (*AM*), *The Analysis of Matter*. London: Allen and Unwin, 1927.
- Russell, Bertrand: *My Philosophical Development*. London: Allen and Unwin, 1959.
- Russell, Bertrand: *The Autobiography of Bertrand Russell, 1872-1914*. Boston: Little, Brown, 1967.
- Russo, François: 1969, *Groupes et géométrie: La genèse du Programme d'Erlangen de Félix Klein*. Paris: Palais de la Découverte, 1969.
- Saccheri, Girolamo: *Euclides ab omni naevo vindicatus sive conatus geometricus quo stabiliuntur prima ipsa universae geometriae principia*. Mediolani: Ex Typographia Pauli Antonii Montani, 1733.
- Saccheri, Girolamo: *Euclides vindicatus*, edited and translated by G.B. Halsted. Chicago: Open Court, 1920.
- Savile, Henry: *Praelectiones tresdecim in principium Elementorum Euclidis*. Oxonii: Iohannes Lichfield & Iacobus Short, 1621.
- Scholz, Heinrich: 1930, 'Die Axiomatik der Alten', *Blätter für deutsche Philosophie*, 4 259-278.
- Schopenhauer, Arthur: (*WW*), *Sämtliche Werke*, nach der ersten, von Julius Frauenstädt besorgten Gesamtausgabe neu bearbeitet und herausgegeben von Arthur Hübscher. Wiesbaden: E. Brockhaus, 1948-1950. 7 Vols.
- Schouten, J.A.: 1926, 'Erlanger Programm und Uebertragungslehre. Neue Gesichtspunkte zur Grundlegung der Geometrie', *Rendiconti del Circolo Matematico di Palermo*, 50 142-169.
- Schur, Friedrich: 1886a, 'Ueber die Deformation der Räume constanten Riemannschen Krümmungsmaasses', *MA* 27 163-176.
- Schur, Friedrich: 1886b, 'Ueber den Zusammenhang der Räume constanten Riemannschen Krümmungsmaasses mit den projectiven Räumen', *MA* 27 537-567.
- Schur, Friedrich: 1891, 'Ueber die Einführung der sogenannten idealen Elemente in die projective Geometrie', *MA* 39 113-124.

- Schur, Friedrich: 1899, 'Ueber den Fundamentalsatz der projectiven Geometrie', *MA* 51 401–409.
- Schwabhäuser, Wolfram: 1956, 'Ueber die Vollständigkeit der elementaren euklidischen Geometrie', *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 2 137–165.
- Schwabhäuser, Wolfram: 1959, 'Entscheidbarkeit und Vollständigkeit der elementaren hyperbolischen Geometrie', *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 5 132–205.
- Simplicius: *In Aristotelis Physicorum Libros Quattuor Priores Commentaria*, edidit H. Diels. Berlin: G. Reimer, 1882.
- Simplicius: *In Aristotelis de Caelo Commentaria*, edidit I.L. Heiberg. Berlin: G. Reimer, 1894.
- Sklar, Lawrence: 1969, 'The conventionality of geometry', in N. Rescher, *Studies in the Philosophy of Science*, Oxford: Basil Blackwell, 1969, pp.42–60.
- Sklar, Lawrence: (ST), *Space, Time and Spacetime*. Berkeley: University of California Press, 1974.
- Spivak, Michael: (CIDG), *A Comprehensive Introduction to Differential Geometry*, Boston: Publish or Perish, 1970ff. (Mimeographed).
- Stachowiak, Herbert: (RU), *Rationalismus im Ursprung. Die Genesis des axiomatischen Denkens*. Wien: Springer, 1971.
- Stäckel, Paul and Engel, Friedrich: (TP), *Die Theorie der Parallellinien von Euklid bis Gauss. Eine Urkundensammlung zur Vorgeschichte der Nichteuklidischen Geometrie*. Leipzig: B.G. Teubner, 1895.
- Stäckel, Paul and Engel, Friedrich: 1897, 'Gauss, die beiden Bolyai und die nicht-Euklidische Geometrie', *MA* 49 149–206.
- Staudt, Georg Karl Christian von: (GL), *Geometrie der Lage*. Nürnberg: Friedrich Korn, 1847.
- Staudt, Georg Karl Christian von: (BGL), *Beiträge zur Geometrie der Lage*. Nürnberg: Verlag von Bauer und Raspe, 1856–1860. 3 fascicles.
- Stenius, Erik: *Critical Essays*. Amsterdam: North-Holland, 1972.
- Stewart, Dugald: (WW), *The Collected Works*, edited by Sir William Hamilton. Volume III. Edinburgh: Thomas Constable, 1854. (Contains the second part of his *Elements of the Philosophy of the Human Mind*.)
- Szabó, Arpád: 1964, 'The transformation of mathematics into deductive science and the beginnings of its foundation on definitions and axioms', *Scripta Mathematica*, 27 27–49, 113–139.
- Szabó, Arpád: (AGM), *Anfänge der griechischen Mathematik*. München: Oldenbourg, 1969.
- Tannéry, Paul: 1876, 'La géométrie imaginaire et la notion d'espace', *RPh*, 2 433–451; 3 (1877), 553–575.
- Tarski, Alfred: (LSM), *Logic, Semantics, Metamathematics. Papers from 1923 to 1938*. Translated by J.H. Woodger. Oxford: Clarendon Press, 1956.
- Tarski, Alfred: 1935, 'Einige methodologische Untersuchungen über die Definierbarkeit der Begriffe', *Erkenntnis*, 5 80–100. (English translation in Tarski, *LSM*.)
- Tarski, Alfred: 1951, *A Decision Method for Elementary Algebra and Geometry*. Berkeley: University of California Press, 1951.
- Tarski, Alfred: 1959, 'What is elementary geometry?', in Henkin *et al.*, *AM*, pp.16–29.

- Taurinus, Franz Adolf: *Theorie der Parallellinien*. Köln am Rhein: J.P. Bachem, 1825.
- Taurinus, Franz Adolf: *Geometriae Prima Elementa*. Colonia Agrippinae: Typis J.P. Bachemii, 1826.
- Tits, J.: 1952, 'Etude de certains espaces métriques', *Bulletin de la Société Mathématique de Belgique*, 4 pp.44–52. ('Addenda', *Ibid.*, 5 (1953) 124–125.)
- Tobias, Wilhelm: *Grenzen der Philosophie, constatirt gegen Riemann und Helmholtz, vertheidigt gegen von Hartmann und Lasker*. Berlin, 1875.
- Tonelli, Giorgio: 1959, 'Der Streit über die mathematische Methode in der Philosophie der ersten Hälfte des 18. Jahrhunderts und die Entstehung von Kants Schrift über die "Deutlichkeit"', *Archiv für Philosophie*, 9 37–66.
- Torretti, Roberto: 1970, 'On the Subjectivity of Objective Space', *Proceedings of the Third International Kant Congress, Rochester 1970*, Dordrecht: D. Reidel, 1972, pp.569–573.
- Tóth, Imre: 1966/7, 'Das Parallelenproblem im Corpus Aristotelicum', *Archive for the History of Exact Sciences*, 3 249–422.
- Ueberweg, Friedrich: (PG), *Die Prinzipien der Geometrie wissenschaftlich dargestellt*. In *Die Welt- und Lebensanschauung Friedrich Ueberwegs*, hrsg. v. Moritz Brasch. Leipzig: Gustav Engel, 1889, pp.263–316. (Originally published in *Archiv für Philologie und Pädagogik*, 17 (1851).)
- Urysohn, P.S.: 1922, 'Les multiplicités cantorienes', *Comptes Rendus de l'Académie des Sciences (Paris)*, Sér. A–B, 157 440–442.
- Urysohn, P.S.: 1925, 1926, 'Mémoire sur les multiplicités cantorienes', *Fundamenta Mathematicae*, 7 30–137; 8 225–359.
- Van der Waerden, B.L.: 1947/9, 'Die Arithmetik der Pythagoreer', *MA* 120 127–153, 676–700. (Also in Becker, *GGM*, pp.203–254.)
- Van der Waerden, B.L.: (SA), *Science Awakening*. English translation by Arnold Dresden with additions of the author. New York: Oxford University Press, 1961.
- Veblen, Oswald: 1904, 'A system of axioms for geometry', *American Mathematical Society Transactions*, 5 343–384.
- Veblen, Oswald: 1911, 'The foundations of geometry', in J.W.A. Young, *Monographs on Topics in Modern Mathematics relevant to the Elementary Field*, New York: Dover, 1955, pp.3–51. (This is a reprint of the 1911 edition of Young's book, in which Veblen's article was originally published.)
- Veronese, Giuseppe: 1891, *Fondamenti di Geometria a piu dimensioni e a piu specie di unità rettilinee, esposti in forma elementare*. Padova: Tipografia del Seminario, 1891.
- Veronese, Giuseppe: (GG), *Grundzüge der Geometrie von mehreren Dimensionen und mehreren Arten gradliniger Einheiten in elementarer Form entwickelt*. Mit Genehmigung des Verfassers nach einer neuen Bearbeitung des Originals übersetzt von Adolf Schepp. Leipzig: B.G. Teubner, 1894. (Pp.631–684, Anhang: Historisch-kritische Untersuchungen über die Principien der Geometrie.)
- Vuillemin, Jules: 1972, 'Poincaré's philosophy of space', *Synthese*, 24 161–179.
- Wallis, John: *Opera mathematica*. Oxonii: e theatro Sheldoniano, 1695–1699. 3 Vols.
- Waltershausen, W. Sartorius von: *Gauss zum Gedächtnis*, Leipzig, 1856.
- Wehrli, Fritz: *Die Schule des Aristoteles. Texte und Kommentar. Straton von Lampsakos*. Basel: Schwabe and Co., 1950.
- Weyl, Hermann: 1918, 'Reine Infinitesimalgeometrie', *MZ* 2 384–411.

- Weyl, Hermann: (MAR) *Mathematische Analyse des Raumproblems*. Berlin: Springer, 1923.
- Whitney, Hassler: 1936, 'Differentiable Manifolds', *Annals of Mathematics*, 37 645-680.
- Wolf, Joseph A.: (SCC), *Spaces of Constant Curvature*. New York: McGraw-Hill, 1967.
- Wolfson, Harry A.: *Cresca's Critique of Aristotle. Problems of Aristotle's Physics in Jewish and Arabic Philosophy*. Cambridge, Mass.: Harvard University Press, 1929.
- Wundt, Wilhelm: 1877, 'Ueber das kosmologische Problem', *Vierteljahrsschrift für wissenschaftliche Philosophie*, 1 80-136.
- Wundt, Wilhelm: (L), *Logik. Eine Untersuchung der Prinzipien der Erkenntnis und der Methoden wissenschaftlicher Forschung*. I.Bd. 5. Aufl. Stuttgart; Enke, 1924. II. Bd. 4. Aufl. Stuttgart: Enke, 1920.
- Zeuthen, H.G.: 1896, 'Die geometrische Construction als "Existenzbeweis" in der antiken Geometrie', *MA* 47 222-228.

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