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EDITED BY
OLIPHANT SMEATON

Euclid

His Life and System

By Thomas Smith, D.D., LL.D.

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EDITORIAL PREFACE

THE volume on Euclid is one of those which have been added to the series of the WORLD'S EPOCH-MAKERS since the prospectus was issued. Although I had long cherished the desire to include the founder of what may be called the "Science of Geometry" in the series, I did not see my way at the outset to do so, first, owing to the difficulty of getting any one to undertake the subject under the limitations of space necessarily imposed; and, second, owing to the doubt I entertained whether the subject could be treated in a manner sufficiently popular to warrant its inclusion in a Series such as this.

After several conversations with my friend, Emeritus Professor Thomas Smith, D.D., LL.D., and after hearing him read some parts of the work, I felt assured that the plan of treatment proposed by Dr. Smith was not only exhaustive from a scientific point of view, but was sufficiently popular in style to win the interest and attention of the non-scientific reader.

Dr. Smith has had to encounter and overcome difficulties neither few nor small in the accomplishment of his task, and it is not the least interesting feature of this volume that it has been produced after its venerated author had reached his eighty-fifth year. In his case, however, the intellectual bow has abode in strength long after the time when it might reasonably have been expected to become relaxed.

O. S.

EUCLID

I

In order to justify the inclusion of the name of Euclid in the list of epoch-making men, we must *first* of all intimate who Euclid was; *then*, what influence he exerted in his own day and in subsequent times; *thirdly*, we must inquire in what sense, or to what extent, the introduction and the general study of geometry formed an epoch in the history of Europe, and consequently of the world; *lastly*, it may not be out of place to hazard a forecast as to the future cultivation of the science, and to consider how far its development is to be effected by adherence to Euclid's methods, how far by a modification of them, or whether by a virtual abandonment of them. Such, then, briefly, is a summary of our present undertaking, a condensed table of contents of the present volume.

In all these branches of our task we shall have serious difficulties to encounter. In the earlier ones we shall have to regret the paucity of authentic information, and the inconsistency of such as might have been expected to be authentic. In two of our branches we shall have not easy argumentative work, in the course of which we shall have to deal with living opponents far more than "worthy of our steel," and with some of the mighty men of the past whose authority it seems almost profanity to question. But our main difficulty throughout will be to determine the character which our work is to assume with reference to the class of readers who may be expected to take interest in its subject. Certainly the book is not designed for mathematicians; and if any such deign to peruse it, they will find in it much that will be to them unedifying, and will even seem trivial. Yet we see not, on the other hand, how it will be possible to treat some parts of our subject without introducing technicalities which will be repulsive to such as are altogether ignorant of even elementary mathematics. In these circumstances we must endeavour to steer a middle course. Our aim shall be to write for the reader who has just the amount of knowledge of, and interest in, mathematical subjects which may be reasonably expected to pertain to intelligent, though not necessarily intellectual, men and women of the twentieth century; while it is evident that, for the last section, as indicated above, we shall have to bespeak especially the attention of such as are interested, professionally or otherwise, in education. Be it frankly said that we have no expectation of producing a great or classical work. But we are not without hope that our little volume may be suggestive of thoughts which may be conducive to the augmentation and diffusion of intelligence, and even, in some cases, to the quickening of mathematical tastes, and so ultimately to the advancement of mathematical science. At the least, we shall have to present for the contemplation of our readers a probably unique picture of a man, leading a humble and obscure life, yet in subsequent ages exercising a mighty influence over the thoughts and lives and destinies of mankind,—an influence the more remarkable because it acts not on the passions and affections, nor directly on the interests, of men, but on their minds as distinguished from their hearts; and operates on these minds not with a stimulant, but rather with a sedative power. Even so, for "God hath chosen . . . things which are not, to bring to nought things that are."

II

Who was Euclid? We do not know. No one knows. We do know, however, and ought to state, that he was *not* Euclid of Megaera, with whom he has been disastrously confounded. Euclid of Megaera was a disciple of Socrates, after whose death, in 399 B.C., he left Athens and founded a school at Megaera. He was therefore, probably, a century prior to our Euclid.

If the same question were put respecting any other man, substantially the same answer might be given, and would be true in a certain sense. For not only of the men who lived long ago, and of those who lived recently, or are living now, in lands far off and little known, but even of those whose doings are chronicled in our newspapers and magazines; yea, of those with whom we are maintaining constant and intimate intercourse, it is true that we know them only in part, and that, in many cases,—probably in all,—the part which we know is insignificant in comparison with that which is beyond our ken. But it is not merely in this sense that we are compelled to give the answer which is here given when the question is put concerning Euclid. For we know neither the place nor, except within rather broad limits, the time of his birth. We know nothing of his parentage or ancestry, or of the influences which, in his early years, acted on the development of his mind and the formation of his character. Very little is known of the events of his life or of his habits, and simply nothing of his physical and moral characteristics; while even his mental and intellectual powers can only be inferred from his writings. Then, as to the writings imputed to him, it is not known concerning some of them whether they are his or not; while, even respecting those of undoubted authenticity, it is uncertain whether he was their author, in a strict sense, or only their compiler. All that we can do, therefore, towards answering the question which introduces this section, is to summarise the scanty records which have come down to us, none of which are contemporaneous, while some contain statements which are certainly inaccurate. Homer and Euclid are both dead. In this respect there is entire accord between the fates which have befallen the father of poetry and the father of geometry respectively. But here the accordance ends. Seven cities have contended for the honour of having given birth to the poet; whereas it does not appear that any city, town, village, hamlet, villa, or cottage has advanced any pretence to stand in that relation to the geometer. A thorough-going mythist might even doubt, and consequently deny,—for, with the proper mythist, a very small measure of doubt warrants a very decided denial,—that Euclid was ever born, or ever existed as a man at all. The argument might stand in some such fashion as this. Manifestly the name *Eukleides* is compounded of the two words *eu* = "well," and *kleis, kleidos* = "a key." Now what name could have been given more appropriately to the science of geometry than *the happy key?* Or to what could the name have been more appropriately given? Did not Plato virtually give the name to the science when he proclaimed—

> "Πρὸ τῶν προθύρων αὑτοῦ γράψας ὑπῆρχε Πλάτων,
> Μηδεὶς ἀγεωμετρῆτος εἰσίτω μου τὴν στέγην,"[1]

that the want of it made entrance into the abode of knowledge impossible? Could any language have expressed more emphatically the conviction that geometry was the key, the only key, the auspicious key, before whose potent touch the portals of the temple of knowledge would roll back, and disclose a patent access to the sacred shrine? And then, was it not in accordance with all ancient usage that this science-key should be personified; that the science should become a man; and that in the course of time innumerable legends should cluster around his name? If this reasoning (!) be not sound and conclusive, then a great amount of precisely parallel or virtually identical reasoning, which has been used by some and accepted as valid by many, with reference to other persons and things, is inconclusive and unsound. What is the use of living in the twentieth century if we are to be bound by the traditionary beliefs of the centuries which were notoriously uncritical?[2]

[1] It is true that for this inscription on the gates of the "olive-groves of Academe" we have no earlier authority than a monk of the twelfth century. The question of its genuineness is therefore simply a weighing of probabilities: whether it is likelier that it had come down through channels of tradition which have now disappeared, or that this monk invented it.

[2] Since this was written, we have found that what we wrote as an extravagant burlesque is, almost to the letter, descriptive of what was actually done long ago. The following is an extract from Mr. Ball's *Short History of Mathematics*:—"Some of the mediæval writers went so far as to deny his existence, and, with the ingenuity of philologists,

There is no reasonable doubt that Euclid was a Greek by birth, though it has been said by some that he was a native of Tyre; and there is considerable probability that he was an Athenian, or, at all events, an Attican. He was born about 330, and died about 275 B.C. If we assume these approximate dates to be the actual ones, it will appear that he was fully a century later than Plato (429–348 B.C.). He was contemporary with Aristotle (384–322 B.C.), having been born fifty-four years later, and having died forty-seven years later than the Stagyrite. Thus Euclid was about eight years old when Aristotle died. It may help the reader to "locate" Euclid in the history of Greece, if it be stated that the year which is given as that of his birth was seven years before the death of Alexander the Great, who survived his famous tutor by a year. Archimedes also was contemporary with Euclid, having been born thirty-three years later, and survived him by sixty-three years (297–212 B.C.). All

they explained that the term was only a corruption of *ukli*="a key," and *dis*="geometry." The former word was presumably derived from *kleis*. I can only explain the meaning assigned to *dis* by the conjecture that, as the Pythagoreans said that the number two symbolised a line, possibly a Schoolman may have thought that it could be taken as indicative of geometry." With all humility, we submit that our theory is better than that of the mediæval writers referred to, and our explanation better than Mr. Ball's. For, *first*, the name is not Ukleidis, but Eukleides. *Secondly*, there is no such word as *ukli;* but there are two such words as *eu* and *kleis*. *Thirdly*, *d* is part of the root of the word of which *kleis* is the nominative; and *es* is merely a termination necessary in order to give the word a Greek form. Some may reckon it curious to note that Euclid received at the hand of these mediæval writers precisely the same treatment that Homer afterwards received from Wolf, and that various persons of antiquity are now receiving at the hands of the "higher critics"; the latest victim being the prophet Malachi.

these dates are probably correct, or nearly so. Their statement is sufficient to show the untruthfulness of a legend narrated by a Greek author, to the effect that the Athenians were visited by a plague in 430 B.C.; that they consulted the oracle at Delos as to its removal, and were told that it could be stopped only by their making a cubical golden altar, which should be just twice as large as the existing altar of Apollo at Delos; that, on failing to solve the problem, they sought counsel of Plato, who referred them to the geometers, *especially Euclid.* Now it appears that Plato was not born till the year after 430, nor Euclid till a century later. This "Delian problem," as it was called, of the duplication of the cube stands out prominently in the history of Greek mathematics. We shall have occasion to refer to it afterwards, probably more than once. At present we notice it only to point out the inaccuracy of this particular version of the legend as to its origin.

When Euclid became a young man, the *Academy* was flourishing, though not with all the lustre which had been imparted to it by its founder. It would appear that Euclid sought admission within its gates, and that, being already "ouk ageometretos," his claim was allowed. Of his academic career, and of the nature of his studies and his measure of success in the prosecution of them, we have no record. These we can only infer from his subsequent achievements; for it is certain that the foundation of these achievements must have been laid while he was yet what would now be called first a schoolboy and then an undergraduate.

III

What was that foundation? It was, of course, his knowledge of the work of his predecessors. Materials do not exist for ascertaining either the nature or the extent of that knowledge. But the assumption may be safely made, that he profited to the full extent by the opportunities which he possessed; and we are able to estimate, with an approach to certainty, the character and the extent of these opportunities. We have now, therefore, to institute an inquiry as to the amount of mathematical science that was within the reach of the student of the time. This being ascertained,—as it can be with good measure of accuracy,—the assumption just stated will bring us as near as it is possible for us to get to a knowledge of the attainments which he actually made. The conduct of this inquiry is virtually a review of the advancement of geometry before and up to Euclid's days; the execution and exhibition of a sketch of the history of pre-Euclidean geometry.

This part of our task is much facilitated by the recent publication of two books, which contain in brief compass, and in admirably lucid form, a statement of all that is known—all that in all probability will ever be known—of this most important and most interesting subject. They are Dr. Allman's *Greek Geometry from*

Thales to Euclid,[1] and Mr. Ball's *Short History of Mathematics*.[2] Of these admirable books we shall make free use, while we shall occasionally have recourse to original authorities as Proclus and Pappus, and to Montucla's ponderous *Histoire des Mathematiques*.

As there lived brave men before Agamemnon, whose names and deeds are unremembered because of the lack of Homeric recorders, so it is quite possible that there may have been geometers before the Egyptians. Yea, it is not altogether impossible that the names and achievements of pre-Egyptian geometers may be redeemed from oblivion by the stupendous researches of our time, and the probably more stupendous of times yet to come. But hitherto no one has a right to question the truth of the commonly accepted opinion, that geometry had its origin in ancient Egypt. Of course, it will be understood that we speak of geometry as more or less systematised; of geometry not, indeed, as a science, but at least as an embryo, which was to be developed into a science. In any other sense the statement were not true. For we can hardly conceive of any man destitute of a knowledge of *some* geometric truth, as, for example, of that truth which now takes rank as a proposition, that any one side of a triangle is less than the sum of the two others. We might go further, and say that

[1] "Dublin University Press" Series, *Greek Geometry from Thales to Euclid*. By George Johnston Allman, LL.D., D.Sc., F.R.S., Professor of Mathematics in Queen's College, Galway; Member of the Senate of the Royal University of Ireland. Dublin, 1889.

[2] *A Short Account of the History of Mathematics*. By W. W. Rouse Ball, Fellow and Tutor of Trinity College, Cambridge. Third edition. London, 1901

no animal above the rank of the immovable molluscs is without a knowledge, consciously or unconsciously, of this important truth, on which the eagle in his flight, the tiger in his spring, and the slug in his silent progression, all proceed. Nor beast nor bird, nor fish nor reptile voluntarily proceeds from point to point otherwise than by the straight line which joins those points. We speak of geometry, then, not as an instinct, but as an embryo science, when we say that there is no reason to disbelieve that its origin was in Egypt.

It is stated by Proclus that the Egyptians were led to the study of geometry by the necessity of restoring the boundaries of the fields after their yearly destruction by the inundation of the Nile; and that the science took its name from this, its first application. But it is pointed out by Montucla that the inundation does not obliterate the boundaries, and that any damage which it may produce can be far more simply remedied, or altogether prevented, by driving stakes into the ground. To this we may add, but with much deference, that it does not seem at all likely that the Greeks would have given the name of geometry, or measure of the *earth*,—for, of course, it was the Greeks and not the Egyptians who gave it this name,—to a mere method of *land* measuring. To us it appears that the Egyptians aimed at, and to a certain extent attained, a far more ambitious object to which the name of *earth* measure might be more appropriately given. For we find Eratosthenes, a Greek living in Egypt about 200 B.C.,—in fact, he committed suicide there in 194 B.C.,—engaged in the measure of a circle of longitude, with the view of ascertaining the earth's circumference in order to the calculation of its volume. It is interesting to note that, with infinitely inferior means, he attained results not differing very widely from those come to two millennia later by the French, British, and Indian trigonometrical surveys.

But is it not possible that the Egyptian knowledge of geometry and of many other sciences, and of the fine and the industrial arts, was but the traditional remnant of a primeval knowledge which was once the common property of the human family? Is it quite proved that absolute ignorance of science and of religion was the primeval condition of mankind? What if there were a previous *descent* to that position from which the slow and laborious *ascent* confessedly began? Be that as it may, it cannot be doubted that the initial knowledge of the Egyptians was extremely limited. Whether there were any mathematicians before them or no, it is certain that for our present purpose they are to be regarded as the first, inasmuch as it was from them that the Greeks derived their first knowledge of the science. From a very early time Greeks went into Egypt for the purpose of studying the philosophy and science of that country; or, having gone for other purposes, they took advantage of the opportunities afforded them for studying these.

The earliest Greek of whose sojourn in the land of the Nile we have any record, was Thales (640–546 B.C.). "He," says Dr. Allman, "was engaged in trade, is said to have resided in Egypt, and, on his return to Miletus in his old age, to have brought with him from that country the knowledge of geometry and astronomy." It were to be devoutly wished that we could learn with any approach to certainty what was the amount of his knowledge of these subjects. Several writers—unhappily, however, long after his time—have referred to this subject. Proclus in his *Commentary* on Euclid says, under I. 5, "Thanks, therefore, are to be given to the ancient Thales for the invention of this theorem, as well as a multitude of others. For he is said to have first perceived and affirmed that the angles at the base of every isosceles triangle are equal." Again, under I. 15, Proclus says, "This, therefore, is what the present theorem evinces, that when two right lines mutually cut each other, the vertical angles are equal. And it was first invented (according to Eudemus) by Thales, but was thought worthy of a demonstration producing science by the Institutor of the *Elements*."[1] To these two Mr. Ball adds four others to make up "the chief propositions that can now with reasonable probability be ascribed to Thales," namely, III.: "A triangle is determined if its base and base angles be given." (This is part of Euc. I. 26.) IV. "The sides of equiangular triangles are proportional." (Euc. VI. 4.) V. "A circle is bisected by every diameter." VI. "The angle subtended by a diameter at any point in the circumference is a right angle." (Might be a corollary from Euc. III. 22.)

The subject is so interesting, and its bearing on our present inquiry is so important, that we shall be justified in subjecting this catalogue of the acquirements or inventions of Thales to some examination.

[1] Thus does Proclus in his *Commentary* constantly designate Euclid. Indeed, he devotes a chapter to the vindication of his claim to the epithet.

It must be first of all remarked that Proclus lived in the fifth century after Christ (412–485 A.D.), and Thales in the sixth century before Christ (640–540 B.C.). They are therefore separated from each other by more than a thousand years. The independent testimony of Proclus as to the attainments of Thales is therefore manifestly worth nothing. But he refers to the work of Eudemus, who was contemporary with Euclid, and therefore some seven hundred years nearer to Thales.[1] Our next remark is that, if the statement be correct that Thales knew that a triangle is determined if its base and base angles are given, he could scarcely fail to conjecture that the sum of the three angles is constant, though he may easily have fallen short of the perception that that sum is two right angles. The belief that he knew that the sides of equiangular triangles are proportional, is founded on a narrative as to his having measured

[1] Mr. Ball elsewhere says, "We possess no copies of the Histories of Mathematics written about 325 B.C. by Eudemus (who was a pupil of Aristotle) and Theophrastus respectively. Luckily Proclus, who about 450 A.D. wrote a *Commentary* on Euclid's *Elements*, was familiar with the History of Eudemus, and gives a summary of that part of it which dealt with geometry." We think it is rather an over-statement to say that Proclus gives a summary of any part of the work of Eudemus, though he often refers to him as an authority. But the book is a strange farrago of metaphysics, heathen theology, and geometry; and we do not profess to have read it fully, or to have fully understood what we have read. In another respect the statement is not strictly accurate; for the *Commentary* of Proclus is only on the First Book of the *Elements*. At the close he says, "But we, indeed, shall give thanks to the gods, should we be able to comment on the other books in a similar manner. In the meantime, if other cares should prevent the execution of our design, it is my opinion that such as are studious of these contemplations ought to expound the other books after the same mode." No continuation of the work is extant; nor is there reason to believe that any was ever produced.

or calculated the height of a pyramid by means of its shadow. Of the narrative there are two versions, only one of which will justify the belief. According to Diogenes Laertius, he chose the time when his own shadow was equal in length to his own height. (That, of course, would happen once a day, when the sun's altitude was 45°.) Then, assuming that the shadow of the pyramid must also be equal to *its* height, he measured the former, and so deduced the latter. The amount of geometrical knowledge involved in this deduction is merely that, if two right-angled triangles have one of the acute angles in each equal, and the one triangle be isosceles, the other will also be isosceles. But Plutarch gives the narrative otherwise. According to him, Thales, without reference to the sun's altitude at the time, measured a staff and its shadow, and also the shadow of the pyramid, and then applied the proposition that the measured shadow of the staff was to the measured length of the staff as the measured shadow of the pyramid to its unmeasured height. This would involve that knowledge of proportion with which Mr. Ball credits him. Plutarch goes on to state that Amasis the king, who was present, was greatly astonished at the ingenuity displayed. From this Montucla infers that Thales had outstripped his teachers, and had done what the Egyptian geometers could not do. It does not seem that the inference is warranted. It confounds the assertion that the king did not know that the geometers could do it, with the very different assertion that the king knew that the geometers could not do it. We wish Mr. Ball had stated on what authority he ascribes to Thales the knowledge that the angle in a semicircle is a right angle. We know that it is stated by Pamphila that Thales inscribed a right-angled triangle in a circle. But Pamphila was a lady who lived in the time of Nero, and who, according to her own account, composed her book by diligently noting whatever she heard from her husband and many learned visitors, as well as whatsoever she read in books. It is not likely that Mr. Ball would accept her statement as authoritative. Most probably, therefore, he had some better authority. Such, then, is all that we know as to the amount of geometrical science which Thales brought from Egypt and taught to the Greeks. A meagre amount truly. It ought, however, to be remembered that the Commentaries of Proclus are not, as he designed them to be, on Euclid's *Elements*, but only on the First Book of these *Elements*, and only refer, and that incidentally, to the achievements of earlier geometers.

But we certainly know that the Egyptian geometry before the time of Thales was more extensive than this, and we may reasonably suppose that so competent a scholar as he was made more extensive acquisitions. For example, we know that they were aware of the fact that a triangle whose sides are in the proportion of 3, 4, and 5 is right angled; and we can scarcely imagine that they failed to see that the square on the hypotenuse of *such* a right-angled triangle is equal to the sum of the squares of its sides. Then, with tesselated pavements constantly under their feet, they must surely have seen that the same relation holds between the squares on the sides and the square on the hypotenuse of an isosceles right-angled triangle; in other words, that the square on the diagonal of any square is double of the square itself. Almost as clearly would a pavement formed of equilateral triangles show to them that the same thing is true respecting a right-angled triangle which has one of its acute angles double of the other, or which has its hypotenuse double of its shorter side. It seems, then, very probable that the Egyptians, before the sojourn of Thales among them, were aware of the truth of these three cases of what afterwards became Euc. I. 47, though it is very likely that they had not proved them by geometric methods.

It would appear that Thales, having returned to Greece and founded the "Ionian School," occupied himself mainly with applied, rather than with pure mathematics. There is no doubt that he taught the rotundity of the earth; and calculated, though not with much approach to accuracy, the obliquity of the ecliptic. It seems also beyond doubt that he predicted a solar eclipse which actually occurred. It is not likely, however, that his prediction was founded on calculation. On the one hand, the oliquity of the ecliptic is an important element in the calculation of a solar eclipse, and it could only have been by a very improbable "fluke" that, with his erroneous estimate of that element, he could have attained a correct result. Then, on the other hand, we know that the Egyptians, as well as the Hindus and the Chinese, kept careful records of past eclipses; and from these they must have seen that there is a cycle of eclipses; that, if the sun or the moon is eclipsed on any day at a particular hour, it will be again eclipsed after the lapse of eighteen years, eleven days, and some hours. It can scarcely be doubted that it was only from the Egyptian catalogue that Thales made his happy prediction, by adding this period, or some multiple of it, to the recorded time of some earlier eclipse.

2

It is not quite pertinent to our subject, but it may be mentioned, that Thales was aware of the fundamental fact in what is now the science of electricity, namely, that friction imparts to amber the power of attracting light substances. In fact, he knew of electricity all that any man knew who lived between him (say 600 B.C.) and our countryman Gilbert (1600 A.D.).[1] Who shall determine whether Thales in those early days, or Gilbert two thousand two hundred years after, were the founder of that science which we, three hundred years later still, regard as perfected by Lord Kelvin, but to which those who shall live three hundred years hence may haply consider Kelvin to have stood in nearly the same relation which we assign to Gilbert?

The Ionian School continued to flourish till about 400 B.C., but its members seem to have done little or nothing for the advancement of geometrical knowledge. As in the later days of its founder pure mathematics gave way to applied mathematics, so these under his successors seem to have been superseded by the study of philosophy, whatever may have been the precise meaning of that term in those days. Indeed, it does not appear that the Ionian School contributed anything to the extension of geometrical knowledge. Rather it may be questioned whether its later members maintained even the position—little advanced as that was—attained by its founder. It could scarcely be otherwise, for Thales had no *scientific* knowledge of geometry. In other words, the geometry of which he knew a little

[1] Many years ago, when we first read the *Novum Organum*, we were at once surprised and amused by the scorn—not to say the animosity—with which Bacon repeatedly refers to Gilbert. Can this be accounted for?

was not a science, in fact had in it none of the essential elements of science. The block which, under the sculptor's chisel, begins to show some faint resemblance to the human figure, gives promise of the statue which it is destined to become; but a bushel of pebbles or of marble chips gives no such promise. Mere knowledge has not in itself any capacity of growth. It is system or method—systematised method or methodised system—that converts knowledge into science, and imparts to it the possibility, if not the necessity, of extension.

Pythagoras gave some measure of cohesion to the geometric chips of Thales, and may be regarded as the father of geometric science. Born about 570 B.C.,[1] the first twenty years of his life were coincident with the last twenty of the life of Thales. There is no one of the characters of old times about whom so many irreconcilable statements are made by ancient writers. It seems that he, like Thales, travelled and studied in Egypt. This is stated by some, it is not denied by any; but it is not mentioned by some who might have been expected to mention it. "The details of his life," says Mr. Ball, "are somewhat doubtful, but the following account is, I think, substantially correct. 'He studied first under Pherecydes of Scyros, and then under Anaximander. By the latter he was recommended to go to Thebes; and there, or at Memphis, he spent some years." Montucla says that he was first under the teaching of Thales, who conceived great hopes of the ingenuity (*penetration*) of his young pupil. This is scarcely possible, unless the date of his

[1] This, in the opinion of Mr. C. P. Mason (Smith's *Dict. of Gr. and Rom. Biog.*), is the most probable date. Some old writers, however, put the date of his birth about forty years earlier—608 B.C.

birth was considerably earlier than that given above. Montucla gives 590 B.C. as the probable date of his birth. That would make him forty years old at the death of Thales. It seems more probable that the estimate of Mr. Ball is accurate, or at all events less inaccurate, though he does not give his authorities. Now Pherecydes of Scyros was a rival of Thales, and Anaximander was one of the successors, if not the immediate successor, of the founder of the Ionian School. To us, then, it seems the most probable supposition that Pythagoras began his academic course under Pherecydes, and continued it in the Ionian School under Anaximander; and on its completion went, like the modern "Travelling Fellow," to Egypt. How much knowledge of mathemathics he took with him, and how much he brought back, can only be matter of conjecture. But if we are to credit the old story of his excessive joy, expressing itself in a munificent sacrifice, at his discovery of the proposition that is called by his name, and is otherwise called Euc. I. 47, or more commonly "*the* 47th," his knowledge could not have been very extensive.

At the same time, we should be in error if we concluded that he knew only this geometric truth and the other truths involved in it, or indispensable for its proof. We have seen that even Thales is reputed to have known that the angle in a semicircle is a right angle; and it was impossible, one would think, even to draw a figure for the proof of this, without perceiving that the angle in a segment less than a semicircle is obtuse, and that in a segment greater than a semicircle is acute. Now we are not entitled to assume that Pythagoras knew all that Thales knew; but it is highly probable that, what with the teaching of Anaximander and what with that of the Egyptian priests, his knowledge included the substance of what Thales knew, with important additions derived from his own study.

But his position among geometers is not to be assigned him simply on the ground of the extent of his knowledge of mathematical truth, for then he would take a place behind even the tyros of later times. He did better than merely learn what his predecessors had to teach him, and add to it a small contribution of original discovery. It does not appear that the Egyptian priests, or Thales, or his successors in the Ionian School, ever set themselves to elucidate the connection of the geometric truths which they knew; in other words, that they ever contemplated geometry as a science, or that they ever entered on such a work as demonstration, in the proper sense of that term. But it seems that Pythagoras demonstrated the propositions of elementary geometry. The steps by which he demonstrated the equality of the square on the hypotenuse to the sum of the squares on the sides of a right-angled triangle, are not known. We know that in Euclid's *Elements* almost every one of the preceding forty-six propositions is indispensable for the proof of the 47th; and although Pythagoras may have proved it, and probably did prove it, otherwise, he also must have erected extensive scaffolding ere he could rear the beautiful structure. It is interesting to note that this high praise is heartily accorded him by Proclus, worshipper as he was of Euclid, "the Institutor," as he fondly calls him, "of the *Elements*." "After them (the geometers of the Ionian School) Pythagoras changed that philosophy which is conversant about geometry itself into the form of a liberal doctrine, considering its principles in a more exalted manner, and investigating its theorems immaterially and intellectually."

There is no biography more involved in contradiction than that of Pythagoras. We have seen that the date of his birth cannot be ascertained, the authorities varying to the extent of forty years. There is equal uncertainty as to the events of his life. "It was the current belief in antiquity that Pythagoras had undertaken extensive travels, and had visited not only Egypt, but Arabia, Phœnicia, Judea, Babylon, and even India, for the purpose of collecting all the scientific knowledge that was obtainable; and especially of deriving from the fountainheads instruction respecting the less public or mystic cultus of the gods." There is no improbability in this; but neither is there any possibility of verifying it. On his return from his travels he began teaching in his native Samos. Shortly afterwards he went to Sicily, thence to Tarentum, and ultimately to Crotona, in the south of Italy. Thus his teaching was not carried on actually in Greece; but Tarentum and Crotona were in Magna Grecia, a Greek colony in Italy more Grecian than Greece itself. His disciples seem to have been exclusively Greeks. At all events, all of them that are mentioned have Greek names. In fact, in the Grecian colony at Crotona, as in other such colonies, there was probably little intercourse between the colonists and their Italian neighbours. Thus the Sicilian, Tarentine, and Crotonian Schools may be regarded as Grecian Schools. These Pythagorean Schools seem to have been to a great extent of a monastic character. Among the disciples there was a community of goods, even to the extent that the discoveries, mathematical or other, made by any one of the members, were not regarded as his, but as belonging to the order, or even to the founder of the order, though they may have been made long after his death. The mode of life in these institutions was highly ascetic. Their founder seems to have anticipated even the La Trappist system, if it be true that he required a silence of seven, five, or three years as a preliminary to initiation into the order. One important point of monastic discipline he did not adopt, celibacy to wit. He married Theano, one of his pupils, a young lady of great beauty and many accomplishments. Clement of Alexandria quotes Didymus as relating, in his work *On the Pythagorean Philosophy*, that this "Theano of Crotona was the first woman who cultivated philosophy and composed poems." Other writers speak of her as author of many books, among others a Life of her husband. Unhappily, none of these are extant. Elsewhere, in the *Stromata*, Clement quotes several of her sayings. The schools founded by Pythagoras were essentially and intentionally religious institutions, in which mathematical studies were mixed with theological and metaphysical questions. It is certain, however, that the Pythagoreans gave much attention to the nature and relations of numbers, and to music, which they regarded as a branch of mathematics. Their arithmetical researches were, for the time, of great importance. We can only refer to one of them as bearing on geometry. We have already seen that the Egyptians, long before the time of Thales, were aware that a triangle whose sides were in the ratio of 3, 4, and 5 is right-angled. Now we have the authority of Proclus for affirming that Pythagoras was aware that this is but a particular case of a more general proposition. In stating that proposition, we may be excused

Namely that the legs are propor-tional to 2MN and M^2-N^2 where M and N are prime to one another, and one of them is even,

for making use of modern notation. It is this—If n be any odd number, then the lines in the proportions of n, $\frac{1}{2}(n^2-1)$ and $\frac{1}{2}(n^2+1)$ will form a right-angled triangle. This, in its turn, might be easily shown to be itself only a particular case of a still more general proposition. Nevertheless, its discovery was of no little moment.

The Pythagoreans had much in common with the Freemasons. Like them, they bound themselves by an oath not to reveal to the uninitiated the esoteric teaching of their school. Like them, too, they had secret signs by which they could recognise one another. One of these was called the pentagram. It was a regular pentagon with its sides produced both ways, so that their points of meeting would be the angular points of a larger regular pentagon. It is probable that to this figure they ascribed other properties than its geometrical ones. This probability is enhanced by the consideration that they put at the points of the figure the letters of the word *ὑγιθα*, which, indeed, is not a word, but which is supposed to be a modification of the word *ὑγίεια*, *health*; *θ* being substituted for *ει* in order to reduce the number of letters in the word to the five which the figure required.

We borrow from Mr. Ball an anecdote which he takes from Iamblichus, which seems to illustrate happily the way in which a Pythagorean could make himself known to a brother Pythagorean as being such, and, as such, having a claim to brotherly help. "Iamblichus, to whom we owe the disclosure of this symbol, tells us how a certain Pythagorean, when travelling, fell ill at a roadside inn, where he had put up for the night. He was poor and sick, but the landlord, who was a kind-hearted fellow, spared no trouble or expense to relieve his pains. However, in spite of all efforts, the student got worse. Feeling that he was dying, and unable to make the landlord any pecuniary recompense, he asked for a board, on which he inscribed the pentagram-star. This he gave to his host, begging him to hang it up outside, so that all passers-by might see it, and assuring him that the result would recompense him for his charity. The scholar died and was honourably buried, and the board was duly exposed. After a considerable time had elapsed, a traveller one day saw the sacred symbol. Dismounting, he entered the inn, and, after hearing the story, handsomely remunerated the landlord."

In an insurrection at Crotona, 501 B.C., many of the Pythagoreans were killed. Pythagoras himself escaped to Tarentum. But he was killed the next year, 500 B.C., in another insurrection. The school continued to flourish at Tarentum for a long time—150 years, it is said. The man who gave it lustre in its later years was Archytas of Tarentum. From the first the Pythagorean doctrine had been largely political, and, both in his sufferings at Crotona and in his death at Tarentum, its founder was a martyr in the cause of politics,—sound politics, indeed,—and not in the cause of science or philosophy. But in the century which elapsed between his death and the time of Archytas the school would seem to have degenerated into a mere secret political association, and to have had but little influence even in that character, and less in any other. Its organisation must have greatly changed during that century, or else it must have been revolutionised at its close by Archytas himself. In his time we find no trace of its original monastic character. Archytas was acknowledged as head of the school, and, at the same time, he was a prominent "man of affairs." He was for eight years chief magistrate of his city, which was in reality a small but not unimportant State; a brave and successful commander of its army, and a wise, large-minded statesman.

The glory has been assigned to Socrates of bringing down philosophy from heaven to earth. So did his contemporary[1] Archytas by science, especially mathematical science. He was the first to apply mathematics to theoretical and practical mechanics. He is said to have been the inventor of the pulley; but it is not likely that there were no pulleys used before his time. Probably it was he who first investigated the principle of this mechanical power, and the efficacy of its various forms. He is also said to have constructed a wondrous flying dove and other automata. His great achievement in pure mathematics was the solution of the Delian problem of the duplication of the cube. This he effected by means of the intersection of a cylinder and a cone. His solution, therefore, lies outside of the territory to which we confine ourselves, plane geometry.[2]

[1] Socrates, born in 468 B.C., died 398 B.C. Archytas flourished 400 B.C.

[2] It may be well here to indicate the accepted meaning of this term. *Plane* geometry is, of course, opposed to *solid* geometry; and about this there is no ambiguity. But the conic sections as they are now generally treated, without reference to their being sections of the cone, and many other curves, such as the cycloid and multitudes of others, are strictly plane curves, as much as the circle itself. But, conventionally, these are excluded from the domain of plane geometry; and it is restricted to the consideration of the properties of rectilinear figures and of the circle. Plane geometry is therefore "the geometry of the ruler and the compasses."

It is pointed out by Dr. Allman that in the solution he makes use of the truths now demonstrated in Euc. III. 18, III. 35, and XI. 19.

We may notice that the first and third of these are so simple that Archytas may possibly have regarded them as axiomatic. The second, one of the most important propositions in plane geometry, he must have proved in some way. Whether his proof amounted to a rigid demonstration, we have no means of ascertaining. The problem of the duplication of the cube belongs, of course, to solid geometry, and the solution of it by Archytas was perfectly legitimate. But it resolves itself into a problem of plane geometry, namely, to insert two means between a line and its double. If that problem could be solved, the duplication of the cube would follow at once. To the solution of this problem by plane geometrical means the Greek geometers set themselves with great zeal, and failed. It was solved afterwards by many of them in various ways by means of conic sections. Only in modern times it has been shown to be incapable of solution by means of the ruler and compasses. "After a life," says Mr. Philip Smith (in Dr. W. Smith's *Biographical Dictionary*), "which secured to him a place among the very greatest men of antiquity, he was drowned while upon a voyage on the Adriatic. He was greatly admired for his domestic virtues. He paid particular attention to the comfort and education of his slaves. The interest which he took in the education of children is proved by the mention of a child's rattle (πλαταγή) among his mechanical inventions." Plato is said to have been one of his Tarentine students. His tragic death is referred to by Horace in the following lines:—

> "Te, maris et terræ numeroque carentis arenæ
>   Mensorem cohibent, Archyta,
> Pulveris exigui prope litus parva Matinum
>   Munera, nec quidquam tibi prodest
> Aërias tentasse domos animoque rotundum
>   Percurrisse polum, morituro!"

IV

Very similar to the relation between the schools of Miletus and Crotona (or Tarentum) was that between the latter and the school of Athens. Eudoxus (409–355 B.C.) was a pupil of Archytas at Tarentum, but set up on his own account at Cyzicus, and ere long removed to Athens. Plato also, if not a pupil of Archytas, was an intimate friend. It is recorded that when Plato was in Sicily, and got into a "difficulty" with Dionysius, Archytas saved his life by intervention with the tyrant. It seems hopeless to attempt a reconciliation between various statements as to the relations between Eudoxus and Plato, their friendship and their enmity. The following is, upon the whole, perhaps the likeliest account of the matter; but it must be admitted that it is in direct opposition to statements which may possibly be true. Our supposition is that the two were friends in their youth, and were fellow-travellers in Egypt; that, on their return, Eudoxus went to Cyzicus and Plato to Athens; that their friendship lasted as long as Eudoxus remained in Cyzicus; but that Plato resented his removal to Athens, and his opening of a rival school to his own Academy. *Tantæne animis cœlestibus iræ?*

Eudoxus was a mathematician of a high order, though it ought in fairness to be stated that Professor De Morgan, than whom there are few higher authorities, speaks most disparagingly of him as an astronomer; and it is difficult to believe that so utterly incompetent an astronomer as De Morgan represents him to have been, could have been an accomplished geometer. Yet such he certainly was. He founded some beautiful theorems on the division of a line in Medial Section (Euc. II. 11, VI. 30). These are now known as Euc. XII. 4, 5, 6, and are given in many modern editions of Euclid as corollaries from II. 12, or as additional propositions in Book II. He gave special attention to the subject of proportion, and is supposed by some to have been virtually the author of the Fifth Book of Euclid. This supposition rests on a statement in an anonymous fragment of a commentary on Euclid which has been ascribed to Proclus, but which was probably later than his time. The conjecture seems to have grown, after the manner of the classical Three Black Crows, and to have been consolidated into positive assertion. Thus we have now before us a folio edition of Euclid, by a Jesuit of the seventeenth century, which in its title-page[1] professes to be a commentary on the thirteen books of Euclid's *Elements*, and on certain treatises by Isidorus, Hypsicles, and Proclus,

[1] Euclidis Elementorum Geometricorum Libros tredecim; Isidorum et Hypsiclen et recentiores de corporibus regularibus; et Procli propositiones Geometricas immissionemque duarum rectarum linearum continue proportionalium inter duas rectas, tam secundum antiquos, quam secundum recentiores, Geometras (qu. Geometros?) novis ubique fere demonstrationibus illustravit, et multis definitionibus, axiomatibus, propositionibus, et animadversionibus ad Geometriam recte intelligendam necessariis, locupletavit Claudius Ricardus, e Societate Jesu sacerdos, patria Ornacensis, in libero Comitatu Burgundiæ, et Regius Mathematicarum Professor.—Antverpiæ, ex officina Hieronymi Verdussii, 1645.

but which gives the titles and headlines of Books V. and VI. respectively as *Commentarius in Librum Quintum (Sextum) Elementorum Geometricorum Eudoxi et Euclidis.* The idea of a joint-authorship of Book V., here assigned to the two geometers, is at once set aside by the simple fact that Eudoxus was born 408 B.C. and Euclid about 330 B.C.—an interval of some eighty years. Eudoxus died in 355 B.C.—a quarter of a century before Euclid was born. But, apart from joint-authorship, which is impossible, there is no reason to question that Euclid made the same use of the labours of Eudoxus in the construction of his Fifth Book that, in the construction of his other books, he made of those of Pythagoras and Archytas and other predecessors.

It is interesting to note that the geometers of this period exercised themselves greatly in attempts to solve the problem of the duplication of the cube, the rectification and quadrature of the circle. These attempts were unsuccessful, yet it is not to be supposed that they were made in vain. The old fable of the treasure pretended to be buried in the vineyard by the old man, in order that his sons might trench deeply in order to find it, has its application here as elsewhere. The fabulist tells us that the diggers found no treasure, for the very good reason that there was none to find, but that they were rewarded for their toil by a series of abundant vintages. But he omits to mention the additional gain that they derived from the labour, in their own brawny arms and more firmly knit loins, and habits of patient perseverance. It has been thus with the diggers in several departments of the scientific vineyard. Astronomy and chemistry might have been elevated to their actual high position by other and more satisfactory means; but as a matter of actual fact it was in astrology and alchemy respectively that they had their beginning. Honest labour is never wholly useless.

But the attempts of these early geometers to solve these problems were not altogether unproductive even of direct results. Hippocrates[1] of Chios, in attempting to square the circle, succeeded in squaring certain figures bounded by circular arcs. This is of so much interest that we must endeavour to make it "understanded" by the non-mathematical reader. The simplest case is this. If a semicircle be described on the chord of a quadrant in any circle as diameter, there will be formed a figure called a *lunula* or *lune*, bounded by a quadrantal arc of the one circle and a semicircular arc of the other. Now, with the assumption that the areas of circles are proportionate to the squares of their diameters, it is very easily proved that this lune is equal to a certain rectilineal figure. It may be presumed that Hippocrates so proved it. He probably believed that he had made a near approach to the squaring of the circle; and some who ought to have known better asserted that he had. He had made no approach at all, but had left that problem just where he found it. But it is his proud fame to have been the first to square a figure with circular boundaries, though he necessarily failed to square the circle itself, that is, the figure with one circular

[1] He was a contemporary with his namesake of Cos, the celebrated physician, with whom he has been unhappily confounded, even as we have seen that *our* Euclid has been confounded with his namesake of Magaera, with whom he was not even a contemporary.

boundary. What concerns us more immediately, however, is that he wrote a text-book on elementary geometry, and therefore was the legitimate predecessor of Euclid in the department with which we are specially concerned. Presumably Euclid was acquainted with this work of Hippocrates; but what use he made of it, or whether any, we have no means of ascertaining. We have not noticed that Proclus makes any mention of it or its author; but it is possible that there may be some references which have escaped us, as in the absence of an index may easily have happened. It may be stated in passing that in Dr. Smith's *Dictionary*, Hippocrates the mathematician is dismissed with thirteen lines of a column, whereas thirteen columns are devoted to the physician of the same name.[1] Not too many these, but surely too few those.

Plato, as we have seen, was a contemporary and associate of Archytas and Eudoxus. He was certainly a mathematician, and set a high value on mathematical study; yet individually he does not seem to have done much, if aught, for the extension or consolidation of geometrical science. But the Academic School, of which he was the founder, gave much heed to mathematics, and what is of great importance, to the principles of mathematical reasoning. There is no doubt that before the time of Plato there was considerable laxity in the modes of demonstration. In particular, it seems to have sometimes been assumed that if a proposition is true, its converse is necessarily true, and needs no demonstration. We do not suppose that this was ever stated

[1] There are seventy-one lines in a column, $\therefore$ Geom. : Med. $= 13 : 13 \times 71 = 1 : 71$!

in so many words. It had been better that it had been so stated, for then the fallacy would have been manifest. It may be well to illustrate the perniciousness of the fallacy by an extreme case—too extreme to have ever occurred, but differing in degree only and not in kind from many that have occurred. It is demonstrated that if all the sides of a triangle are produced, the sum of the exterior angles is four right angles. Now the converse of this proposition is that if the sum of the exterior angles made by producing the sides of a rectilineal figure be four right angles, then the rectilineal figure is a triangle. Now every tyro knows that this property belongs to every rectilineal figure, and not to the triangle only. It were needless to point out that the logical error consists in the simple conversion of a proposition which is not capable of such conversion.[1] If it be so, then—as it is—that one great function of the study of geometry is to cultivate the faculty of logical reasoning, and if Plato contributed largely to securing that the reasonings of geometry be strictly logical, then we must acknowledge that geometry owes to him far more than it would have owed to him had he discovered some important theorems,

[1] A term has come into use since the days when we studied logic, and which is helpful for the avoidance or for the detection of this fallacy, the QUANTIFICATION OF THE PREDICATE. Thus the proposition, "all horses are animals," asserts only that ALL horses are SOME animals; therefore its proper converse is SOME animals are horses, not ALL animals are horses. So the proposition stated in the text is properly that ALL triangles are SOME of the figures having this property, and it can only be converted into, SOME figures having this property are ALL triangles. The proposition that all rectilinear figures have this property is true, but it is not a legitimate inference from the proposition that all triangles possess the property, any more than would be the false proposition which in the text we have imagined to be inferred from it.

or solved some important problems. Even Euclid himself, although he is generally regarded as carrying preciseness to the extent of finicalness, has been charged with this fallacy in a few instances. Fortunately, in these instances the converse propositions were true, and could have been easily proved; so that his illegitimate conversions, if he really made them, did not vitiate the subsequent propositions in whose establishment he employed them. We apprehend, then, that in the Platonic School geometry was cultivated as a means rather than as an end, a philosophy rather than a science, a discipline conducive to the right study of all philosophy; yet "historical criticism" tells us that on the academic portals there was no inscription debarring the entrance of the non-geometer, thereby evincing that itself is without geometry in the Platonic sense, since the most that a geometer in that sense would say is that it is *not proved* that such an inscription was there,—a very different thing from its being proved that it *was not* there. This same historical criticism, forsooth, tells us that old Knute uttered no rebuke to Ocean's radical ripple for lack of respect for his royal feet; that Alfred never was unfaithful to his charge of oat-cakes, nor ever exhibited alacrity in dealing with the same cakes when carbonised by such neglect; that Galileo muttered no protest on behalf of the earth's unfixedness; that Newton saw no apple fall; that Wellington did not at Waterloo issue the curt command, "Up, Guards, and at 'em!" We are ready to weep in very sorrow for our dear posterity of a few centuries hence, whom that historical criticism will tell that the most illustrious statesman of our times never wore a collar and never felled a tree. That his successor on the Treasury bench could not distinguish between a driver and a putter, that to him a brassie and a niblic, a cleek and a mashee, were all as one, or differed only by a negligible quantity; while to the Colonial Secretary of the end of Victoria's reign and the beginning of Edward's, eyeglass and orchid were things unknown!

And now, having launched a whole quiverful of shafts against this iconoclastic criticism, let us return in all seriousness to the Academic School and its relation to geometry. We cannot, of course, hold that a statement which could only have been founded on tradition which had come down without record for fifteen hundred years, has any historical value; yet we cannot but think it improbable that a monk of the twelfth century could have invented a motto which so happily describes the mode of treatment of philosophical and ethical and political themes in the Platonic School.[1] In the numerous writings of Plato which have come down to us, there are not many references to geometry; but they are all pervaded with a geometric air: a constant effort is perceptible to give continuity and ever-increasing force to the argument; while there is the constant inculcation of the truly geometric principle, that the object of the teacher is "not to communicate instruction, but to lead to the spontaneous discovery of it." Let us fortify this assertion by a short quotation from one[2] who had no such object in view as we have, but

[1] Such is the state of the case regarding the inscription on the porch of the Academy. It cannot be traced farther back than to the couplet which we have (p. 6) quoted at second-hand from Tzetzes, who wrote in 1176. The question of its authenticity is therefore only one of probability, as stated in the text.

[2] Dr. Christian A. Brandis, University of Bonn, art. "PLATO" in Smith's *Dictionary*.

who has breathed what we have called the geometric air, and has felt it to be peculiar, but has probably not recognised it as distinctively geometric. "With all the admiration which from the first has been felt for the distinctness and liveliness of the representation, and the richness and depth of the thoughts, it is impossible not to feel the difficulty of rendering to oneself a distinct account of what is designed and accomplished in any particular dialogue, and of its connection with others. And yet, again, it can hardly be denied that each of the dialogues forms an artistically self-contained whole, and at the same time a link in a chain." If in this extract we substitute *proposition* for *dialogue*, it would be exactly descriptive of the impression made on us by the perusal of a good system of geometry. In each proposition we feel distinctness and liveliness of representation, and that each one has its place as a link in a chain. We must confess that our acquaintance with the writings of Plato is not intimate; but from our general knowledge of them we should say that his ambition was not to "turn out" a few great mathematicians, or a few great philosophers or moralists or dialecticians or politicians, but rather so to imbue the minds of all with the principle of sound research, that their being should become identified with it; and this principle he found embodied in geometry. Thus, if we had the prospect of being introduced to a perfect specimen of the result of Plato's teaching—a Senior Wrangler of the Academy of Athens—we should not expect him to talk mathematics, but we should be disappointed if he did not talk mathematically, that is, accurately, with a distinct though scarcely conscious apprehension of what is true, what probable, what improbable, and what false. It is thus that we expect the Frenchman and the German, the peer and the peasant, the cleric and the laic, the man of letters and the man of art, the soldier and the civilian, the sailor and the landsman, the lawyer and the farmer, the blacksmith and the tailor, to view from different points, and to express in variously modified terms, their views on subjects having no immediate connection with their own nationalities or professions. They all speak prose, but the prose of each is different from that of every other. No one is aware that he speaks a prose of his own. Rather, no one knows that he speaks prose at all. These distinctions, of course, admit of innumerable combinations, and, indeed, none of them can exist apart from some combination. A man cannot be a Frenchman and nothing else—an abstract Frenchman. He may be French cleric or French laic. So a German may be emperor, and soldier, and sailor, and orator, and patriot, and everything else; and all these elements may combine, and each, if any one had skill enough to discern it, in its exact proportion to make up the man and to differentiate his views of all matters of thought. In point of fact, the school of Plato did not send out any great mathematician. The only one of any note is Theaetetus, who gave special attention to the subject of incommensurable magnitudes, and to whom, we learn from Mr. Ball, we are indebted for the proposition which is now Euc. X. 9. He must have been a special favourite with his master, who gave his name to one of his dialogues, and who introduces him as an interlocutor both in that dialogue and another.

Aristotle was also a disciple of Plato, and a most favourite one. But as he became himself the founder of a great school, and as his doctrines are in some respects opposed to those of Plato, he is generally thought of and spoken of not as a pupil of the Academy, but as the founder of the Lyceum or the Aristotelian or the Peripatetic School. In his exceedingly voluminous works there is one section which commentators distinguish as mathematical. But there is little or nothing in it of what we reckon as *pure* mathematics. It is rather an application of mathematics to physical science, mainly what we now call mechanics. From the immense influence that the writings of Aristotle exercised over the thought of the Western Church from the days of Augustine downward, and thereby over the thought of Europe throughout the Middle Ages, he might well be regarded as an epoch-making man, the maker of an epoch which has had its beginning, its middle, and happily[1] its end; although it is not altogether certain that he does not still exert a real, if tacit, influence over our thought.

[1] It was not Aristotle, however, but the *abuse* of his teaching, that exerted so deleterious an influence on mediæval thought. Had he never existed, the same result would probably have been produced by some other means.

V

We have to confess that we have never been able to form an apprehension altogether satisfactory to ourselves of the way whereby knowledge got itself diffused among men when there were no printing-presses to multiply copies of books, and no *Publishers' Circulars* to make known their production. But we are bound to suppose that there were ways whereby the thoughts and inventions and discoveries of men became known to those who had interest in their subjects. We must therefore assume that in some way Euclid had it in his power to become acquainted with the work of his precursors in the race of geometric discovery; and while we frankly confess our ignorance as to what that way might be, an assumption less questionable—indeed, wholly unquestionable—is that he made full use of the opportunities afforded him. Putting these two assumptions together, we infer, then, that Euclid, when he entered on the study of geometry, acquired a knowledge of it so far as it had then advanced; and when he composed his *Elements* he possessed that knowledge. It would, of course, be absurd to suppose that such an inference would be warranted with respect to other men and in other times. The student in our time, for example, has not only as good facilities of becoming acquainted with mathematical science as it is in our day, as Euclid could have enjoyed for ascertaining to what extent it had attained in his day; not only as good, but far better. And it is not to be doubted that in our time we have as diligent students as Euclid could be. Yet no man would say regarding *every*, yea, regarding *any*, student or teacher of to-day that he is acquainted with the whole science of mathematics as it now is. But the cases are not parallel. To learn all the mathematics of our day were a task far beyond the power of any man. To learn all the pre-Euclidean geometry were a task which the most ordinary student could accomplish in a few days, provided that it were presented to him as, according to our assumption, it was presented to Euclid. Therefore we submit that our inference is warranted in the case with which we have to do, while it would be glaringly absurd in many other cases. Believing, then, that Euclid, before he began the composition of his *Elements*, had a fair, if not an absolutely complete, knowledge of the geometrical achievements of his predecessors, it will be well to summarise what has been ascertained as to the extent of these achievements. It should be premised that almost certainly the ascertainment of these achievements is coextensive with the achievements themselves; for, although many works which might have contributed to the ascertainment have not come down to us, yet these have been referred to by subsequent writers so as to leave no uncertainty as to their contents, so far as these bear on our present design.

We have seen, then, that Euclid had the means of knowing from others—and therefore, if our assumption be legitimate, he did know—(1) That the angles at the base of an isosceles triangle are equal; probably also the converse of this, that a triangle which has two equal angles has two equal sides. (2) That the vertical opposite angles made by the intersection of two straight lines are equal. (3) That the three angles of every triangle are, in their sum, equal to two right angles. (4) That the square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares described on its two sides. These truths, which he demonstrated in his I. 5, 15, 32, 47, were known to his predecessors, and were taught by them, and it were unreasonable to doubt that he learned these truths from them. But it was impossible to give Euclid's demonstration of Prop. 32,[1] or to give his construction of Prop. 47, without reference to the doctrine of parallels. We are therefore shut up to the alternative conclusion, either that the pre-Euclidean geometers knew that doctrine substantially as Euclid taught it, or else that their demonstrations of the propositions referred to were not rigorous, and even their construction of the latter of them was not accurate. We suspect that the latter conclusion is the likelier. Of course they had a notion of parallel lines; but we have no reason to believe that they had such a systematised knowledge of their properties as would have formed the basis of a rigorous demonstration or a geometrically accurate construction.

Resuming our summary, we come to—(5) The means of dividing a straight line so that the rectangle con-

[1] Here and elsewhere we use this mode of expression for the sake of brevity. Every reader will perceive that we mean, not the propositions, which had not their present order till Euclid gave it them, but the truths demonstrated in the propositions.

tained by the whole line and one of the parts shall be equal to the square of the other part. As the very conception of a rectangle and a square involves a reference to parallel lines, our remark in the preceding paragraph is equally applicable to this. (6) That a triangle having the diameter of a circle as its base, and having its vertex in the circumference, is a right-angled triangle; in other words, that the angle in a semicircle is a right angle. (7) That an equilateral triangle, a square, a regular hexagon (probably also a regular pentagon), can be inscribed in a circle. (8) A doctrine of proportion, which referred to the distinction between commensurable and incommensurable quantities. Another important proposition, one case of which—but that the simplest—is used by Hippocrates. The proposition is that the circle described on the hypotenuse of a right-angled triangle as diameter is equal to the circles described on its sides. To us it seems that no one could know this without knowing also what is now Euc. VI. 31, and that Hippocrates inferred it from the analogy of the circle to the rectilineal figure.

For reasons which will afterwards appear, we restrict our summary to the matter contained in the first six books of the *Elements*. With solid geometry, as taught in Books XI. and XII., we do not concern ourselves here, though one part of it—that respecting the five regular solids—received much attention at the hands of geometers before Euclid.[1]

[1] What are called Books XIV. and XV. of the *Elements* treat largely of these solids. But there is no reason to believe that Euclid was their author. They are now generally ascribed to Hypsicles, who was generally believed to have lived in the second century of our era, but whom Professor De Morgan, with apparently good reason, assigns to the sixth.

VI

"AT Babylon, 21st April, Alexander III., king of Macedonia, in the thirty-third year of his age and the thirteenth year of his reign." Such a notice might have appeared in the death column of the *Macedon Times* of 22nd April 323 B.C. An extra-Macedonian reader—say a Londoner—on reading this notice, would not have regarded the event so chronicled as one of very great importance, since Macedonia was, at the best, but a third-rate power, as it had had twenty-one kings, not one of whom, with the exception of Philip II., father of this Alexander, was ever heard of beyond the bounds of the small kingdom. The supposed reader, when he saw that the paper was broadly black-bordered, and contained a "leader," two columns long, setting forth that this Alexander had subdued not only neighbouring European provinces, but had waged brilliant and successful war against Persian and Syrian and Indian and Egyptian rulers,—had, in fact, created a "larger Macedonia" in Europe and Asia and Africa,—would doubtless have perceived that this hero's *life* was an important event in human history. But he would scarcely have realised that the *death* of this man was to mark one of the most signal epochs in the political and intellectual history of the world. The partition of his empire into four kingdoms, the mutual wars and treaties between these kingdoms, their perpetual combinations and divisions, recombinations and redivisions, and their subjugation in succession by all-devouring Rome, constitute the staple of the world's history for many centuries. With these wars and tumults, amalgamations and separations, we have nothing to do; nor with any of the sections of the Alexandrian empire save one—the African; nor with that one, save in respect of an event in its early history, which at the time of its occurrence was probably not regarded as of much importance.

The kingdom of Egypt, with no very definite boundaries, and apparently including, along with Egypt proper, Libya to the west and a portion of Arabia to the east, fell to the lot of Ptolemy, who is generally supposed to have been an illegitimate son of Philip, and therefore step-brother of Alexander. Be this as it may, he was certainly a special favourite of the father, and one of the bravest and most distinguished generals in the army of the son. It had been a "hobby" of Alexander to erect in the lands which he conquered monuments to himself, in the form of cities called by his name. Thus in Asia there was a multitude of Alexandrias, no one of which had any lengthened existence, or left any history behind it. It was otherwise with the Egyptian Alexandria. Built on one of the finest maritime sites in the world, it soon became a most noted seaport, so that in old times it was regarded as the great granary of Rome. It has been the scene of most eventful contests in ancient and modern times; and although in our day the construction of the Suez Canal has materially diminished its importance, it continues, and in all probability will long continue, to be a great commercial city, and the capital of what was once, and may be again, a great kingdom, whether independent or affiliated to a great empire.

The early years of the reign of the first Ptolemy were spent in wars, in which, it must be admitted, he seems generally to have been the aggressor, or at least, when there were faults on both sides, to be entitled to a full moiety of the blame. He found in his capital a variety of nationalities. We know that even in the time of Alexander a great number of Jews had settled as merchants, traders, and mechanics. Their religion, as with their fathers who dwelt in the same Egyptian land some two thousand years earlier, debarred them from amalgamation with the people of the land, or from other than commercial relations with them. They therefore became, and long continued to be, a separate community, and were the main components of that important body whom their countrymen in Palestine designated as Hellenists. With Ptolemy there came, of course, a large body of virtually Greek soldiers and officials. And as we have seen that from a much earlier time there were many Grecian visitors to Egypt, we may be sure that the number was not diminished when the country came under European rule. The Greeks were not, like the Jews, prevented by their religion from entering into any relations with the native Egyptians to which interest or inclination might prompt them. They were the dominant race, and the manners and habits of thought and life of such of the natives as came in contact with them were naturally assimilated to theirs; and so, though Egypt did not in any sense become a Grecian land, Alexandria soon became virtually a Grecian city.

When Ptolemy was fairly settled on his throne, he set himself vigorously to the device of schemes for elevating his subjects and advancing the glory of his capital. Having in his early days shared with his putative brother in the instructions of Aristotle, he could not fail to have vividly apprehended to what extent the cultivation of science and philosophy would contribute to these ends. He therefore determined to found a great school,—university, we should call it now,—with a library befitting the foreseen glory of his kingdom. The two together were called the MUSEUM. It is with the school that we have now to do; of the library we shall have to speak afterwards. The design was formed about 306 B.C., and by 300 B.C. the institution, with magnificent buildings and rich endowments, and a staff of distinguished teachers, was ready to enter on its illustrious career. We have seen that for centuries, from Thales to Plato, Greek after Greek went to Egypt to learn; now for the first time, so far as appears, they went to teach. Among those who accepted the invitation of Ptolemy was Euclid, who apparently in the year named—the first of the third century B.C.[1]—was installed as the first professor of mathematics in the University of Alexandria. At this time he was about thirty years old, and he occupied the position to which he was now called for twenty-five years, until his death in 275 B.C.

There seems to be no possibility of learning anything as to the date of his mathematical works, or ascertaining whether they were of Athenian or Alexandrian birth, or whether some were of the one and some of

[1] So we maintain, just as we maintain that this year in which we write (1901) is the *second*, not the first, of the twentieth century of our era.

The two cases are not analogous.

the other. We must therefore treat them as a unit. This is a catalogue of them which we take from De Morgan, only omitting the Greek titles:—1. The *Elements* of Geometry, in thirteen books. 2. The *Data.* 3. A *Treatise on Music.* 4. The *Division of the (Musical) Scale.* (These two of doubtful authenticity.) 5. The *Appearances* (of the heavens). 6. On *Optics.* 7. On *Catoptrics.* In addition to these, De Morgan mentions six other works, which are either lost or do not remain in the original Greek. There seems to be good reason to believe that Euclid wrote on *Conic Sections,* but that his work is lost, the four books which Apollonius published as his not being really so. It is certain that he wrote a book on *Fallacies,* of which De Morgan says—and we thoroughly agree with him—"The loss of this book is much to be regretted, particularly on account of the explanations of the course adopted in the *Elements,* which it cannot but have contained."

As we shall have occasion afterwards to refer at some length to the extent of Euclid's acquaintance with the science of numbers, we may borrow here the following paragraph from Mr. Ball:—

"To these works I may add the following little problem, which occurs in the Palatine anthology, and is attributed by tradition to Euclid:—A mule and a donkey were going to market laden with wheat. The mule said, 'If you give me one measure, I should carry twice as much as you; but if I gave you one, we should bear equal burdens.' Tell me, learned geometrician, what were their burdens. It is impossible to say whether the question is due to Euclid, but there is nothing improbable in the suggestion."

If we believe in the attribution of the problem to Euclid, we must be amused at the thought that he should have set for solution by the "learned geometrician" a question which a schoolboy, with some knowledge of arithmetic, would answer, with the help of his slate, in a couple of minutes, and which the algebraical tyro would solve without such aid in as many seconds.

In the absence of all means of assigning the production of these works to the Grecian or the Egyptian portion of Euclid's life, we can only venture upon a rather vague conjecture. The name of Euclid must have become well known in Egypt before Ptolemy could think of him as a desirable member of his professoriate. This must have been by his publication of mathematical works. But his works on *Music*, on *Celestial Phenomena*, and on *Optics* would gain for him a contemporary fame which he could never have attained by means of his *Elements*. It was probably to them, and not to the *Elements*, that he owed the high honour of Ptolemy's choice. Yet it is by the *Elements* that he has made an epoch, while all his other works are utterly forgotten. Thus Sir Walter Scott was known to the people of Selkirkshire only as "the Shirra," and to his young sportsman son as ever first to descry the hare in his lair! *We* know him as the author of *Waverley*. Our conjecture accordingly is that some of Euclid's works, namely, those just mentioned and perhaps his *Conic Sections*, were published in Athens, and formed the ground of his summons to Alexandria, while his *Elements* grew out of his teaching there. It may be noticed that this conjecture fits in with a suspicion of De Morgan founded on quite different grounds. "We could suspect," he says, "that Euclid, having arranged his materials in his own mind, and having completely elaborated the Tenth Book, wrote the preceding books after it, and did not live to revise them thoroughly." With the assumption, then, that the *Elements* are the result of a *growth* rather than of a deliberate constructive purpose, we are able to trace the stages of that growth. His idea of geometric method necessitated his beginning with definitions, axioms, and postulates.[1] With his first six propositions he had perfectly plain sailing. Having proved the equality of two triangles which have two sides and the contained angle in the one equal to two sides and the contained, angle in the other, he had next to prove the equality of two triangles having all the sides of the one equal to all those of the other. This would naturally have followed Prop. 6. But, in nautical phrase, he is "brought up all standing." For, may there not be triangles having their sides equal, but not their angles or their areas? He knew that there cannot; but it must be proved. He therefore interjects Prop. 7 as a mean towards the proof of Prop. 8. Having established the equality of triangles having two sides and the contained angle equal, or having two sides and the base equal, he had next to demonstrate the equality of triangles which have two angles and a side equal. Of this there are two cases, according as the side given as equal is that between the two angles given as equal or that subtending one of these angles. We have seen that the former of these

[1] As to the actual definitions and axioms, we shall have a good deal to say in the sequel; at present we have to do only with the fact of their existence.

cases had been treated by his predecessors, and he could easily have dealt with it just as he did with Prop. 4. But it is otherwise with the second case. A roundabout course must be taken in order to reach the proof of it. Doubtless he saw that this equality of the third angle to the third, on condition of the equality of two angles and a side to two angles and the corresponding side, would make essentially for the proof of the equality of the sum of the three angles to a constant quantity, if not definitely of its equality to two right angles. Now this equality he knew, and the proof of it must be undertaken. In order to this proof, all the propositions from 16 to 31 are interjected. They are not all of them useless in themselves, but most of them—notably Props. 16 and 17—can be regarded only as scaffolding for the construction of Prop. 32. Having demonstrated Prop. 26, Euclid makes a fresh departure by the introduction of parallel lines. We shall have much to say as to his treatment of these lines. At present we have only to point out that by their help he easily proved Prop. 32 and its important corollaries. And now he has but one more stage to traverse in order to attain what he evidently regarded as the terminus of Book I., viz., props. 47 and 48. That stage is the treatment of parallelograms.

Book II. props. 1–8 is little more than the skilful construction of figures. It is very beautiful, and is much spoiled by the modern system of substituting for it merely arithmetical operations. It is, no doubt, quite true that if two numbers represent the times that two lines respectively contain a common measure, say an inch, a foot, or a mile, the product of these numbers will accurately represent the number of times that the rectangle of which these lines are sides contains the square described on that measure.[1] True it is also that with this explanation, $(a+b)^2=a^2+2ab+b^2$ is substantially identical with Euc. II. 4, and the arithmetical or algebraical proposition is helpful for the *remembrance* of the geometrical; but it is a great evil to dispense with the *learning* of the geometrical[2] proposition and substituting for it the learning of the algebraical. One is disposed to regret that Euclid did not demonstrate Props. 9 and 10 in the same way in which he demonstrated Props. 1–8. Of the problem Prop. 11 we have spoken already. Props. 12 and 13 together constitute a beautiful extension of I. 47.

It has always appeared to us that in Book III. Euclid is at his best. The arrangement of the propositions is natural; and we find no appearance of such scaffolding as we found in Book I. It is true, as has often been pointed out, that the enunciation of one of his propositions is inconsistent with one of his definitions. The proposition III. 20 is that "an angle at the centre is double of the angle at the circumference standing on the same arc." Now it is afterwards

[1] Thus, if a represent 4 inches and b represent 5 inches, then ab will represent 20 square inches.

[2] We wish editors of the *Elements* and teachers would set their faces against this method. No doubt it is easier, but the other is not so difficult as to be more than a health-giving mental exercise. A quiet walk along street or road is easier than a tussle at football, and has important uses; yet every intelligent schoolmaster, every healthy schoolboy, aye, and every right-minded mother, would scorn the idea of substituting the easier exercise for the more difficult. Why should it be otherwise with mental exercise?

proved (III. 31) that the angle in a semicircle is a right angle. According to the proposition, then, the angle at the centre must in that case be two right angles. But Euclid's definition of an angle as "the inclination to each other of two straight lines which meet in a point and *are not in the same straight line,*" would imply that there is no such angle as two right angles, since what we call such an angle is made by two straight lines which meet in a point and *are in the same straight line.* It must be admitted that this is a flaw. But it is a very slight one, since it only affects the enunciation of the proposition, and leaves the demonstration all compact. It is not a rift—even a small one—within the lute, but rather a scratch on its surface, marring slightly its beauty, but with no tendency to "make the music mute."

The Fourth Book, unexceptionable as it is and altogether indispensable, is to us the least attractive of all; not, however, on account of any lack of elegance in the constructions or of accuracy in the demonstrations of its propositions. With the exceptions of Props. 10 and 11, there is nothing in it but what one feels he could have done himself; whereas in Books I., III., and VI., one is constantly coming in contact with what he can only recognise as a product of genius of which he possesses none.

Of the Fifth Book—be it Euclidean or Eudoxian—we have spoken already, and shall have to speak at some length again. We shall now, therefore, say nothing more than that it puts a new and potent weapon into the hand of our geometer.

This weapon he wields with giant force in Book VI. Hitherto he could treat only of the equality or inequality of figures.[1] Now he can discuss their proportion, and what results he achieves from the discussion! It may be maintained—and we have no wish to dispute it—that the wrath of Achilles and the wanderings of Ulysses, the loss of Eden and its recovery by One greater Man, the bliss of Paradise and the torments of the Inferno, are higher themes, and appeal to a higher range of faculty and affection than the properties of rectilineal and circular figures can ever do. But none the less do we see in Euclid genius akin to that of Homer and Milton and Dante. And suppose he were not so much an author as a compiler, is not the poet a compiler too?

Of Books VII.–X. we say nothing, because we know very little. However it may be in regard to some other matters, it seems clear that, at all events on such a subject as this, silence is the better course in the absence of knowledge. But while we know little *of* these books, we know *about* them, that the first three are an elaborate treatise on arithmetic. They are now wholly superseded, and we suspect that there are few living men whose knowledge of them is much greater than our own. With respect to the first three of these books, we are quite willing that our knowledge should remain as it is. But we wish that we had not let the time pass when we could have made acquaintance with Book X. This De Morgan regards with admiration. He says that it is "the development of all the powers

[1] It has just occurred to us that in I. 41, where he proves that a parallelogram is double the triangle having the same base and altitude, he in reality introduces proportion in a modified form. The proposition is substantially, paral. : tri. = 2 : 1.

of the preceding books," that it is "one of the most curious of the Greek speculations," that it "has a completeness which none of the others can boast of." But it, too, has been superseded, and few of the men of our time know much of it.

Books XI. and XII. are not altogether superseded. They are given—though generally with much modification—in many of the editions of to-day. They are a good elementary treatise on the geometry of planes and solids. But there are many better. Indeed, this branch of geometric science is not generally taught now in the method of the old geometry, but is treated, when it is treated at all, by means of the modern analysis. Whether this be a matter for gratulation or regret may be questioned; that it is a fact is unquestionable.

The Thirteenth Book is the beginning of a treatise on the five regular solids, the "Platonic" or the "Mundane bodies." This treatise is carried on and completed in Books XIV., XV. But it is admitted on all hands that these are not Euclid's. The tradition is that their author was Hypsicles, who taught in Alexandria about a century later than Euclid's time. As we are writing for non-mathematicians, it may be well to explain that a regular solid is one whose "faces" are equilateral and equiangular rectilineal figures. Of these there are, and, as can be easily proved, can be only five: the *tetrahedron*, whose faces are four equilateral triangles; the *cube*, whose faces are six squares; the *octohedron*, with eight equilateral triangles; the *dodecahedron*, with twelve regular pentagons, the *eikosohedron*, with twenty equilateral triangles for their faces.

The matter stands thus: Euclid's works on the *Conic Sections*, on *Celestial Phenomena*, and on *Music*, are unknown, and do not appear to have ever been much known. His arithmetical work lost all its value before the work itself was known in Europe. His work on *Solid Geometry* has no special value, and probably never had—certainly now has not—much influence on mathematical thought. Therefore, when we venture to speak of Euclid as having made an epoch, we are to be understood as speaking of him as author of the *Elements;* and when we speak of the *Elements,* as meaning Books I.–VI. thereof. In thus restricting ourselves we but fall in with the universal usage of mankind—at least of juvenile mankind. For *Euclid* means not a man but geometry; and geometry means "the first six books."[1]

We shall probably in these critical days be charged with anachronism, if, in limning the first mathematical professor in the University of Alexandria, and, having neither portrait nor faded photograph to guide us, we draw from a model whose perfect likeness to our proper subject cannot be disproved, and ought not to be doubted. Yet this is just what we are going to do. Our model is a mathematical professor in another

[1] Our early recollections furnish us with an exactly parallel case, though on a more limited scale. There must, we suppose, have lived somewhere and somewhen,—presumably in Upper Clydesdale or Tweeddale,—and about the middle or towards the end of the eighteenth century, a Mr. Gray. He had written a book on arithmetic, and locally the name Gray, or *the* Gray, had been transferred from the man to the book, and from the book to the whole subject of which it treated. To "begin the Gray" was to enter on the learning of arithmetic. To be "through the Gray" was to attain a position of which the senior wranglership of Cambridge gives but a faint conception. From our recollection we should say that *the Gray* was as bad a book as could have been written. We hope its use and its influence were confined to the locality we have indicated.

university, under whose guidance we long years ago got our first glimpse into the mathematical vista. Our choice of a model is justified by the fact that the mathematical world pronounced him to be as Euclidean as Euclid; while some, parodying Shakespeare, hinted that he even out-euclided Euclid. Knowing him, then, we knew Euclid well, and can accurately picture him. Euclid was a man somewhat over middle age, with noble forehead and grave aspect; profound of knowledge, and ever seeking more. Accurate himself, and painfully exacting accuracy from others. Occasionally absent-minded as he listens to demonstrations which he has heard hundreds of times before, yet quick as lightning to be "down on" every slip. Patient—within limits—of ignorance and stupidity; and, when these limits are overpast, checking the expression of contempt and scorn which is ready to break forth, and substituting for it some snappish utterance in which humour and sarcasm are happily blended, the humour as genuine as if sarcasm were unknown, the sarcasm—bitter it may be, but never sour—as sharp as if humour had no place in the composition. Such was Euclid, as he stood from day to day in decent academic gown, ever liberally powdered with chalk, and rejoiced in the thought that in the course of his long occupancy of the chair he had turned out one or two who might possibly become mathematicians, unconscious that every session he turned out a whole class of friends.[1]

[1] After much hesitation, we have resolved to state that the "subject" of this sketch is Professor Wallace of Edinburgh, the successor in the occupancy of the Mathematical chair of David and James Gregory, of Maclaurin, of Matthew Stewart, of Playfair, and of Leslie, and the predecessor of Kelland and Crystal. Can any position, sacred or civil, boast a nobler succession of occupants?

This is the completest sketch that has ever been given of the Alexandrian professor. Its only predecessor is that from the pen of Pappus, which we transfer from De Morgan :—

"Pappus states that Euclid was distinguished by the fairness and kindness of his disposition, particularly towards those who could do anything to advance the mathematical sciences; but as he is here evidently making a contrast to Apollonius, of whom he more than insinuates a directly contrary character, and as he lived more than four centuries after both, it is difficult to give credence to his means of knowing so much about either. At the same time, we are to remember that he had access to many records which are now lost."

Two anecdotes have come down to us, whose authenticity, though it has been questioned, is avouched by their accordance with our estimate of his sarcastic humour. It appears that his great patron, Ptolemy, had the ambition to add mathematics to his other acquisitions, but shrank from the labour of mastering the definitions and axioms and postulates which lay betwixt him and the *Pons Asinorum*. He therefore asked the professor whether he might not proceed by a shorter and easier way. "Nay, Sire," was the quick response, "there is no royal road to geometry." The other story is that one of his students petulantly exclaimed, "What shall I ever gain by all this rubbish?" The good man, horrified by such utilitarianism, straightway called his servant, and ordered him to present the young gentleman with three oboli.[1]

[1] Mr. Ball says "some coppers." Instinct ought to have told him that Euclid *could* not give so indefinite an order.

The first-fruits of the Alexandrian School were Archimedes and Apollonius, the former of whom seems to have studied under Euclid himself, and the latter at a somewhat later period. These nobly maintained the reputation of their *Alma Mater*. Archimedes was, without doubt, the greatest geometer of ancient times. It is notable that he seems to have regarded his splendid mechanical inventions as mere toys, and his attention to them as something to be apologised for as a relaxation from his geometrical studies. Apollonius, whatever may have been his *morale*, of which Pappus formed a low estimate, was unquestionably a geometer of the highest class.

Alexandria grew and flourished under the successive Ptolemies—a round dozen of them,—and they all seem to have taken pride in the library and the school. Cleopatra, the last of the dynasty, the daughter of one and the sister of another Ptolemy, with all her vanity and debauchery, was not indifferent to this glory of her capital, and even in her last days she enlarged the library of Alexandria by the annexation of that of Pergamus. It happened to be her vices that brought ruin on her country. But that ruin must have come ere long had she been ever so virtuous. Rome could not afford to have so rich a land as Egypt side by side with herself. Egypt, with the territories all round the Mediterranean which successive Ptolemies had subdued, must be herself subdued, trodden under the iron-shod heel of all-conquering Rome. Yet Rome took pride in the well-being of her provinces. These might groan under the tyrannies and tyrannous exactions of prætors and proprætors. But the traditional policy, both of the republic and the earlier empire, was *parcere* *victis, debellare superbos,* and Egypt was probably more prosperous under the Cæsars than it had ever been under the Pharaohs or the Ptolemies.

The annexation of Egypt to the Roman Empire took place about forty years before the Christian era. The revolution was, of course, attended with much turmoil. But the Alexandrian School had buoyancy enough to keep it afloat, and when "the storm was turned into a calm," was ready to weigh anchor and set sail on a fresh voyage. The second Alexandrian School—or rather the school in its second stage—did not attain to the glory of the first; but it had a glory of its own, brighter than that of any contemporary school. Its brightest luminaries were three P.s, the Pappus and the Proclus whom we have mentioned so often, and Ptolemy, whom we do not suppose to have had any relationship to the many kings of that name, but who has the higher honour of being the founder of Scientific Astronomy. The school seems to have become gradually less mathematical and more philosophical and theological. In this latter department its most distinguished teachers were Origen and Clement, the former of whom made a valuable contribution to the apologetics, and many of more doubtful value to the critical department of theology. The evening stars in the firmament of the Alexandrian School were Theon and his daughter Hypatia, in the beginning of the fifth century of the Christian era. The father taught mathematics as subsidiary to the study of the Platonic philosophy, and carefully trained his daughter at once in mathematics and philosophy. "She soon," says the late Bishop Cotton of Calcutta, "made such immense progress in these branches of knowledge, that she is said to have presided over the Neo-Platonic School of Plotinus at Alexandria, where she expounded the principles of his system to a numerous auditory. She appears to have been most graceful, modest, and beautiful, but nevertheless to have been a victim to slander and falsehood. She was accused of too much familiarity with Orestes, prefect of Alexandria; and the charge spread among the clergy, who took up the notion that she interrupted the friendship of Orestes with their archbishop, Cyril. In consequence of this, a number of them, at whose head was a reader named Peter, seized her in the street, and dragged her from her chariot into one of the churches, and tore her in pieces. Theodoret accuses Cyril of sanctioning this proceeding; but Cave (*Script. Eccl. Hist. Lit.*) holds this to be incredible, though on no grounds except his own opinion of Cyril's general character." We hold no brief for the defence of Cyril. In opposition to Cave's estimate of his general character, we, judging chiefly from his treatment of Nestorius long afterwards, can think of no man with a prominent name in the history of the Church who was less unlikely to have instigated, or at least condoned, this horrible crime. Yet it ought to be borne in mind that Theodoret, by whom the charge is made, had himself suffered grievously at the hands of Cyril, and could not possibly regard him but as a bitter foe. We scarcely think that Cyril instigated the perpetration of the crime, and we do not think there is proof, though we cannot get rid of a suspicion, that he was *particeps post factum*. It has been foolishly attempted to represent this lady as having suffered martyrdom at the hands of the Christians because she was a mathematician, or because she zealously controverted Christianity. This is absurd. Hypatia was neither a martyr of science nor a martyr of antichristian Platonism. She was the victim of a disgraceful outrage, perpetrated, indeed, partly by nominal Christians, on the ground that, as they believed—with more or less reason—she used her influence with the civil authorities to prejudice the interests of the Church. No one possessed of the smallest molecule of the mathematical spirit can fail to peruse with glowing interest the hitherto too short list of lady mathematicians, which begins with Hypatia, and waits for addition to succeed the far greatest name of all—that of our own Mrs. Somerville.[1]

Theon and Hypatia were respectable mathematicians, or more than respectable; but before their time mathematics had, for a time, lost its glory. It had come, chiefly through the strong antichristianism of its teachers in the Alexandrian School, to be regarded by Christians as a handmaid to heathenism, and its cultivators had accepted the estimate.

[1] We should like to call attention to one of the names in this roll. Bhaskar-Acharjya was a mathematician, an astronomer, and astrologer in India in the twelfth century. In the last of these capacities he discovered that some dire calamity was to befall him on the day of the marriage of his daughter Lilavati. To prevent this, he determined that she should never be married, that she should never be out of his sight. The flinty breast of the old Brahman seems to have been softened in time, and as some compensation for enforced celibacy and seclusion—for enforced seclusion was not then the rule in India as it is now—he resolved to devote his life to training her as a mathematician. His instructions have come down to us in a book which he called by her name, *Lilavati*. Many years ago we went through the greater portion of the *Lilavati* with the help of a mathematical pandit. It is amusing, and contains interesting references to the habits of mediæval India. But as an arithmetical text-book it is not superior—it could hardly be inferior—to our old friend *The* Gray.

The title of a work by an earlier Theon (of Smyrna, early in the second Christian century) well illustrates this. It may be rendered—"So much of mathematics as is necessary for the reading of Plato." From this it appears that mathematics had, even in the days of the earlier Theon, come to be regarded as a mere handmaid of Neo-Platonism, and Neo-Platonism in the days of the later Theon was the greatest antichristian force. The policy of Julian contributed greatly to the fostering of this misconception. He strove to exhibit science and Christianity as natural and irreconcilable foes. Unhappily, the Christians were not in a position to controvert this allegation. They accepted it, and deemed themselves bound, as they would defend Christianity, to condemn science. If they had but taken to the study of science for themselves, they would soon have found that it, at all events the mathematical branch of it, had nothing akin to that strange medley of metaphysics and pantheism which the Neo-Platonists had grafted on it! Doubtless there were Christians who "tasted of the good word of God and the powers of the world to come," and who lived quiet lives undisturbed by the contentions alike of Church and school. But the prominent members of the Christian community were more concerned to defend Christianity than to live it. If Julian had much to do with the making of these men, they too had not a little to do with the making of Julian, and so evil became worse and ever worse.

The Alexandrian School continued to exist for two full centuries after this. But its existence was but death in life. From the teaching of Proclus, to whom we have so often referred, who was a disciple of Theon and Hypatia, and who was certainly a competent geometer, we can learn what was the character of the teaching in those times. Literary critics declaim, and not without reason,—though they do not always give due weight to what may be said on the other side,—against "novels with a purpose." All that they say, and more, might be said against science-teaching with a purpose, even if the purpose were not, as in the case of Proclus and his compeers it was, the inculcation of falsehood and absurdity. If Kingsley's specimens of Hypatia's teaching be not purely imaginative, her *purpose* was the same, while she sought to attain it by means as graceful as those of her disciple were clumsy.

As Euclid, a Platonist, went from Athens to Alexandria to take part in the foundation of the great school, so, some seven hundred years after, Proclus, a Neo-Platonist, trained in that school, went back to Athens in the hope of restoring life and verdure to the "olive groves of Academe." Though "the Attic bird" did not through his throat "thrill her sweet warbled note," yet there were those who professed to detect sweet music in his strain, even as many, in a land we wot of, find music in the bagpipe! *De Gustibus.*—Alexandria sent forth shoots also to Rome and Constantinople, and it may be said that whatever mathematical knowledge existed in the early Christian centuries was of Alexandrian origin. But that was not much. With the great mundane events of that long period—the substitution of Imperial for Republican Rome—the degeneration of the weakened empire of Rome into the weaker empire of Constantinople—the subsequent subjugation of a race become contemptible through weakness and luxury, and well-nigh detestable through vice, by a race robuster at least, and adding the virtues of barbarism to vices not so detestable as those of the race which it subdued,—with all these events we have nothing to do. Even of that supramundane yet most blessedly mundane event which took place in our period, the incarnation, the life, the death, the resurrection, the ascension of the Son of God, we have nought to say, save that it was the foundation of that new world in which alone true science can live, and in which the highest science is destined yet to live. Let every lover of science, every lover of humanity, ever think of that event with profoundest thankfulness, ever speak of it with bated breath and reverent awe. The event with which we *have* to do is that which brought our period to its close, the Arab conquest of what was once the kingdom of the Ptolemies, and then was one of the most important of the provinces of Rome.

Questions relating to men and times and their reciprocal action—as to the extent to which men make times and times make men—as to what an epoch would have been but for a man, and what a man would have been had he lived in another age and had been subjected to other influences—are far too complicated to admit of definite solution. The "problem of three bodies," with which mathematical astronomers of a late generation were largely exercised, does not to the same extent exercise their successors, simply because it has been solved in a few of its simplest cases, and because it has become evident that a general solution transcends human powers and human resources. If this be so with respect to only three bodies, and these capable

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of putting forth but one influence, and *that* one of the greatest simplicity, what may be expected of the problem of ten thousand subtle influences, conspiring with and counteracting one another in infinitely diversified combinations, but that, while it is solved by omniscience every day and every hour in the case of every man, it can in no case be grappled with by any created intelligence? Yet it is only in so far as approximation can be effected to the solution of this problem, which is manifestly insoluble, that an answer can be given to the question: How comes it that at practically one and the same time Europe was invaded by the Huns and Goths and Vandals, and Western Asia and Northern Africa were overrun and subdued by the Mohammedans? As we can make no approach to the solution of the problem, we make no attempt to answer the question. Account for it as we may, or leave it, as we intend to do, without attempting to account for it, the fact is that about the middle of the seventh century the former of these events reached its consummation, and the latter was accomplished with astounding rapidity. It is with the latter that we have now to do.

Among the strangest of all human biographies is that of Mohammed. No man ever rose from so lowly to so lofty a position. None ever experienced greater vicissitudes of fortune. None ever attained, during a comparatively short lifetime, such power over contemporary peoples. Of no one has the influence for good or evil been so extensive or so enduring. With that biography we have no concern, else we could amply verify from Sir William Muir's elaborate Lives of the Prophet and his earliest successors, the statements we have ventured to make,—statements which those alone who are ignorant of the facts will deem extravagant. Only one apparently unimportant incident in the history of one of his successors has any bearing on our present theme. The success of Mohammed began with the Hejira, or flight from Mecca to Medina in July 622 A.D. During the ten remaining years of his life he pursued an unexampled career of conquest in his own country, and in Persia and Syria. Ten years after the death of Mohammed, Alexandria was taken by his successor Amru or Omar. The incident which concerns us was the destruction, by his order, of the Museum and its magnificent library. Of the school, which was the other portion of the Museum, we have had much to say. The library had been collected at fabulous expense of labour and money from all countries of the world. Its destruction was a wanton act; but its perpetrator showed, like the "loving spouse" of another noted personage, that "though on pleasure he was bent, he had a frugal mind." He did not consume the books on their shelves, or in whatever repositories contained them, although doubtless they would have made a beautiful blaze. He utilised them as fuel for heating the baths of the city; and we are told that they sufficed to heat the water for 4000 such baths for six months. With an average share of persuasibility, when it is not against our will to be convinced, we stagger at the statement that 730,000 furnaces could have been supplied with fuel from the contents of even that magnificent palace, and therefore venture to suggest that the papyri and palm-leaf manuscripts were used rather as fire-lighters than as fuel. Even this is a rather large order. But undoubtedly the collection was enormous. The reason tradition ascribes to Omar for this act has never, so far as we know, been disputed till quite recently, when "historical criticism" has taken it in hand. The contents of these books are either in accordance with the teaching of the Koran, or they are opposed to it. If in accord, then they are useless, since the Koran itself is sufficient; and if in opposition, they are pernicious, and must be destroyed. Therefore in either case, etc. *Q.E.D.*

But the piecemeal destruction of many hundreds of thousands of manuscripts was no trifling task, even for a despotic Khalif. A few escaped their doom. How, we do not know. Perhaps some officer annexed for himself some manuscript that struck him as specially beautiful; or perhaps some stoker at some bath rejected one as slow of ignition. At all events a few—probably very few—were preserved, and among them must have been copies of the writings of Euclid and Ptolemy, the *Elements* of the one, the *Almagest* of the other. But these were in the Greek language, and those into whose hands they eventually fell knew no Greek. But the Jews, then as now, were ubiquitous. They knew Greek well, and Arabic was of easy acquisition to them, as it was of near kindred to their own sacred tongue. They were thus competent translators, and so were employed by the spoilers to render their booty available for their instruction or amusement.

There seems to be no means of ascertaining whether the Arabs had any knowledge of mathematics before their conquest of Alexandria. The probability is that geometry was *not altogether unknown* to them, else they would not have cared to save the geometrical works from the general burning. Now, it seems to have been mainly, if not exclusively, these works that were actually saved. Equal probability there is that they had *not made much progress* in geometrical science; for when we come to a knowledge of their geometry as it was some centuries later, we find that it was wholly a reproduction of the Greek geometry. It would appear, then, that during these centuries the Arab geometers occupied themselves almost entirely in the study of the works of the Alexandrian mathematicians, as translated by Jewish scholars for their use. In another branch of mathematical science—that relating to numbers—they had nothing to learn from the Greeks, and the Greeks had nothing to teach them. To us nowadays it is incomprehensible how the Greeks and the Romans could conduct even their ordinary business with their miserable arithmetic. It is true that Euc. VII.–X. treat of numbers, but not arithmetically. We have already stated that personally we have very little knowledge of this section of the *Elements*. But those who have carefully studied it marvel at its excellence; very much as people are astonished at the success with which persons without hands have written with pens grasped by their toes, or drawn portraits with pencils held by their lips. That Europe is indebted to Arabia for her initiation in the science of numbers, she acknowledges by designating in all her languages her numerical symbols as *Arabic numerals*, and calling one great branch of the science by an Arabic name, *Algebra*. Many suppose that, as Arabs derived their geometrical knowledge from the Greeks, so they derived their knowledge of arithmetic from the Indians. This is probably the generally prevalent opinion. But we cannot think it proved, or even probable. The earliest connection of the Mohammedans with India and its people which could reasonably be supposed to have enabled the former to learn from the latter, was the invasion and partial conquest of India by Mahmud of Ghizni, 1000 A.D. But we know that the Arabians had used their numerals, and had even introduced them into Europe, a full half century before that date. We think it therefore more likely that the Hindus learned from the Arabs than that the Arabs learned from the Hindus. The Hindus make extravagant claims to antiquity and precedence; and these claims have been too facilely conceded by many European students of their history. We claim brotherhood with the Hindu arithmetician, as equally with him the son of the Arab, but dispute his right to regard himself as our grandfather, the father of our Arab father!

The first generation of Mohammedans cultivated the art of war rather than the arts of peace. But by the middle of the eighth century we find art and literature and science flourishing under the patronage of the Abbaside Khalifs. Their capital, Bagdad, was a worthy successor of the Athens and Alexandria of earlier days. Montucla gives the names of a considerable number of the mathematicians of the Bagdad School. They seem to have trodden in the footsteps of Euclid, excepting in the department of trigonometry, of which they may be regarded as the authors. Ptolemy, indeed, in the *Almagest* meritoriously solves many cases of triangles;[1] but neither he nor any other Greek could have any

[1] Only right-angled triangles, however, if we mistake not.

conception of trigonometry as we now use the term. The arithmetic of Bagdad was essential to the being, and the logarithms of Edinburgh to the completion of *that* trigonometry.

Considering the translation of Euclid into Arabic as most important in itself, and especially in its bearing on our subject, we translate D'Herbelot's short account of the translator :—

"Honain Ben Ishak Ben Honain, a Christian physician celebrated in his profession, but still more illustrious as the translator of Greek books into Syrian and Arabic. He was the son of one Isaac, and the father of another Isaac, who was distinguished as Ben Honain, and he was himself the grandson also of a Honain. He was an Ebadi, or Ebadian, *i.e.* one of those Christians known as *servants of God*, who had gathered themselves together out of many districts of Syria and Arabia, and had settled in the Babylonian or Chaldean Irak, in the neighbourhood of Hirah and Coufah. He was physician to the Khalif Motavakkel, and died under the Khalifate of Motamed, 260 or 261 A.H., having been excommunicated by the Patriarch for an act of great irreverence committed against the images. He had been a disciple of John, son of Massoviah, whom we call Mesué, who appears to have been jealous of his teaching. In his translating work he made great use of his son Isaac and his nephew Hobaiz. We have from him in Arabic, says Ben Schonah, Euclid and the *Almagest* of Ptolemy, which Thabet Ben Corrah the Sabean afterwards revised and corrected."

We confess to something akin to gratification at the thought that Euclid's first translator was a Christian, and one ready to suffer for his contempt of idolatry.

Of the reviser we have not been able to ascertain any particulars. He was probably a Mohammedan. D'Herbelot mentions a family of Corrahs who were related to the Prophet. Probably Thabet belonged to that family, but D'Herbelot mentions none of so late a period.

Long before the days of Honain, indeed so early as A.H. 91 (A.D. 706), a tribe of Arabs had invaded Spain, and laid the foundations of a kingdom which, with constantly varying boundaries, existed for two hundred and sixty years, and contributed much to the material civilisation of Europe. The invaders came immediately from Northern Africa, and therefore the kingdom is commonly spoken of as the Moorish kingdom. But they were Arabs; not Africans, but Asiatics. They were not, however, Abassides but Ommiades, which had their head quarters in Damascus, as the Abassides had theirs in Bagdad. But the lustre of Damascus was gradually outshone by that of Bagdad, and by the time with which we have to do, the Ommiade Khalifate of Bagdad had become the paramount Arabic house. Naturally this led to the transportation of *Euclid* into Europe. But the "Moors" in Spain, while highly accomplished in art, and particularly in architecture,—as witness the Alhambra and other buildings which have come down to us in various degrees of preservation,—did not devote themselves to the study of abstract science. Moreover, they had but little intercourse with the other nations of Europe. Therefore the Euclidean seed planted in European soil had to lie dormant awhile ere it germinated.

We have seen that the school of Alexandria sent off branches to Athens and Rome and Constantinople.

These never had much vitality; but what they had they retained after the destruction of the mother-school. Thus we find that Boethius,[1] who lived in the early part of the sixth century of our era, published the enunciations of the propositions of Euclid's First Book, apparently in order to their being committed to memory by students. This, of course, could not have been done at a time when geometry was unknown. But as little could it have been done at a time when there was any right apprehension of the use of geometry and its study.[2] The geometers who came from Alexandria to Rome had doubtless brought manuscripts with them, probably in no great numbers. These had been preserved, we may suppose, all the better because they were not subjected to the wear and tear of frequent consultation! D'Herbelot tells us that Othman of Damascus saw such a Greek manuscript

[1] There is an interesting notice of Boethius, by the late Dean Stanley, in Smith's *Dictionary of Greek and Roman Biography*. It is very doubtful whether Boethius were a Christian or not. He is generally supposed to have professed Christianity. But the Dean shows that there is no evidence of this, positive or negative. Living in Rome in the sixth century, he could not but know much of the nature and claims of Christianity. Yet in his *Consolatio Philosophiæ* there is "total omission of all mention of Christianity in passages and under circumstances where its mention seemed to be imperatively demanded." Every one knows that the *Consolatio* was translated into Anglo-Saxon by Alfred the Great. It is interesting to know that the Royal translator christianised the book to some extent by "large original insertions relative to . . . Christian history." We suspect that Boethius was a Neo-Platonist, a precursor of those who would give us a "*Christianity without Christ.*" This may be thought to be not out of keeping with his gift of *Euclid without Demonstrations.*

[2] Of all the deleterious perversions of mathematical study, the worst is that of committing the propositions to memory, a habit into which learners not unfrequently fall, and against which, it is to be feared, teachers do not always protest with sufficient vigour.

of Euclid's *Elements* in Rome, and translated it afresh into Arabic, adding many discourses,—a commentary, we presume. And after the study of geometry was revived in Europe, several other Greek MSS. were found; and from these corrections were made from time to time. But it seems to be beyond doubt that it was Honain's version, made from a MS. taken to Arabia from Alexandria on the destruction of the great library, and not Othman's, made from one taken to Rome from Alexandria at an earlier time, that became the text-book of the "Moorish Spaniards." Thus this book, composed by a European man resident in Africa, was translated into an Asiatic language by a Jewish Christian, and brought into Europe by Africanised Asiatics.

But while Euclid was now *in* Europe, it was not yet *of* Europe; for, with the exception of the Arabs resident in Spain, and of many Jews resident there and some in other countries, and a few monks who had been in Eastern monasteries, the Arabic was an unknown tongue to the inhabitants of Europe. A poet might tell of the heart-breaking longings of the caged eagle, conscious of its powers and forecasting its destiny, yet doomed to pine in silent, solitary inaction, till the cage bars shall be smashed and the fetters be dashed from its limbs! There is a measure of satisfaction in the thought that the first blow to the bars and the fetters was "delivered" by the sturdy arm of a countryman of our own, *i.e.* we know that it was given by a British arm, and we think it legitimate to assume that *therefore* it was a strong one. The following is Professor De Morgan's account of the matter:—

"The first European who found Euclid in Arabic and translated the *Elements* into Latin, was Athelard, or Adelard, of Bath, who was certainly alive in 1130. This writer probably obtained his original in Spain; and his translation is the one which became current in Europe, and is the first which was printed, though under the name of Campanus. Till very lately Campanus was supposed to have been the translator. Tiraboschi takes it to have been Adelard as a matter of course; Libri pronounces the same opinion after inquiry; and Scheibel states that in his copy of Campanus the authorship of Adelard was asserted in a handwriting as old as the work itself (A.D. 1482). Some of the manuscripts which bear the name of Adelard have that of Campanus attached to the Commentary. There are several of these manuscripts in existence; and a comparison of any one of them with the printed book which was attributed to Campanus would settle the question."

Mr. Ball's account contains an additional particular of interest:—

"*The Twelfth Century.*—During the course of the twelfth century, copies of the books used in Spain were obtained in Western Christendom. The first step towards procuring a knowledge of Arab and Moorish science was taken by an English monk, Adelard of Bath, who, under the disguise of a Mohammedan student, attended some lectures at Cordova about 1120, and obtained a copy of Euclid's *Elements*. This copy, translated into Latin, was the foundation of all the editions known in Europe till 1533, when the Greek text was discovered." [1]

[1] A somewhat fuller account of Adelard is given in the *Dictionary of National Biography*. The following is that portion of it which relates

Under the next century (the thirteenth) Mr. Ball says:—

"*Campanus.*—The only other mathematician of this century whom I need mention is Giovanni Campano, or, in the Latinised form, Campanus, a canon of Paris. A copy of Adelard's translation of Euclid's *Elements* fell into the hands of Campanus, who issued it as his own; he added a commentary thereon, in which he discussed the properties of a regular re-entrant pentagon. He also, besides some minor works, wrote the *Theory of the Planets*, which was a free translation of the *Almagest*."

We should be quite ready to bare brand in championship of Adelard—first, because he was our country-

to our present subject: "Adelard of Bath (thirteenth century [1]), a writer on philosophy, of English birth, flourished about the beginning of the twelfth century. His English name was Æthelherd. His native place is said to have been Bath. But of the facts of his life little is known beyond the few references of travels contained in his own writings, and an entry in the Pipe Roll, 31 Henry I. (1130), granting a small sum of money from the revenues of Wiltshire. He is said to have studied at Tours and Laon, and to have lectured in the latter school. He then travelled much more widely than was at that time common, and appears to have passed through Spain, the North of Africa, Greece, and Asia Minor. He was one of those Englishmen who lived for a time in the Norman kingdom of Sicily, and he is known to have visited Syracuse and Salerno. Later writers have ascribed to him a profound knowledge of the Greek and Arab science and philosophy; but in regard to this nothing can be laid down with certainty. That Adelard knew Greek is almost certain. But it has not yet been determined whether the translation of Euclid's *Elements* (undoubtedly executed by him, though often ascribed to Campanus of Novara, with whose comments it was published in 1482 at Venice) was made from an Arabic version or from the original. From the character of the translation the former supposition seems the more satisfactory."

[1] Query, twelfth.

man; and, secondly, because his plea is undoubtedly just. But it does seem that Mr. Ball is unfair to Campanus in saying that he issued Adelard's translation as his own. There might have been a question as to what constituted the *issuing* of a book in the days when all books were manuscripts. But Mr. Ball uses the term with reference to the printed book, which contained Adelard's *Euclid*, with the commentary thereon, and some tracts by Campanus. Did Campanus issue the book? Mr. Ball mentions him as flourishing in the latter part of the thirteenth century. It is not likely that he was later than 1280 or 1290. But a few years one way or other are of no consequence for our argument. The book in question was "printed by Ratdolt at Venice in 1482," two hundred years or thereabouts after the death of Campanus. To us the matter seems very clear, Campanus had access to Adelard's version. He transcribed it, or had it transcribed for him. In the same book, and by the same hand—his own or that of a professional transcriber—were written his own commentary and tracts. The book having been written for his own use, he needed not to mention Adelard's name as the translator of Euclid, or his own as the translator of Ptolemy. But he naturally put his own name on the fly-leaf, not as the author, but as the owner of the book. Two hundred years after, this book or a transcript of it came into the hands of the Venetian printer, and formed the "copy" from which the edition of 1482 was produced.

It should be stated that Montucla's account is different. We translate it: "Among the Western Christians, Athelard in England and Campanus of Novarre in Italy laboured at nearly the same time (*à peu-près dans même temps*) at deciphering and translating Euclid from the Arabic versions. It was only then that the Latins began to know this author; for until that time their only masters in geometry were Boethius and St. Augustine, or the author, whoever he was, of the book entitled *De Principiis Geometriæ.* The work of Athelard only exists in manuscript in the Bodleian Library and that of Nuremberg. But the work of Campanus was published in 1482, by the care, I suppose, of Lucas de Burgo, who himself published a new Latin edition of Euclid in 1489. Zamberti gave another edition in 1505, which was reprinted in 1516. He has been charged with not always understanding his original."

If the times of these two men were nearly coincident, Montucla's view may be correct, that they laboured independently of each other; that the edition of 1482 gives the version of Campanus, while that of Adelard has never been printed. But if the dates given by Mr. Ball are correct, and if one hundred and fifty years intervened betwixt the men, then the account of Montucla cannot be accurate, and there is no reason to doubt that given by the two Englishmen. Surely the matter is of interest enough to call for the collation by a competent mathematician, as suggested by Mr. De Morgan, of the edition of 1482 with the MS. of Adelard. Howsoever such a collation might decide the question as to the translation, it would not affect the question with which alone we have to do. Euclid was the author, whoever may have been the translator. But it is somewhat more than a mere matter of curiosity to have any doubt as to the translation removed.

VII

From the nature of the case, it is never possible to determine with precision the date of any invention or discovery. The poet (ποιητής, maker or inventor) is born, not made. But the invention (ποίημα) is not born, but made; and that generally by a slow and scarcely traceable process, many hands and many minds acting independently or in concert. So it was with the invention of printing, the greatest in its results that man has ever made, and, so far as can be anticipated, the only great invention that will never be superseded. The steam-engine has hitherto been next to it in importance and influence; but its supersession by electric power has already begun, and, in the opinion of many of those able to judge, is destined to be complete. But printing, continually improved, and furnished with better and ever better apparatus, shows no symptom of decay, but rather of perpetuity and continuous advance. In order to fix with precision the date of this mighty invention, we should need not only a more definite knowledge than is attainable of the times when the several steps were made in the race whose goal was the invention, but we should need also a definition of terms, which must involve an arbitrary element. Are block-printing, by means of which the *Biblia Pauperum* was produced at a very early period, and lithography and other modern processes, to be regarded as *species* of the *genus* printing? Or is the term to be restricted to typography? Making this restriction, which seems to be in accordance with the most frequent usage, we cannot assign an earlier date to the invention than the middle of the fifteenth century. Naturally, from the prevalent currents of thought of the times, the first-fruits of the press were biblical, liturgical, or ecclesiastical. It was probably due to the fact that Jews had much to do with the earliest printing operations, that in the first of these classes portions of the Hebrew Scripture bore what might now seem a disproportionate part. Thus we find that the Hebrew Psalter, with the Commentary of Rabbi Kimchi, was printed ("probably," says Mr. Hartwell Horne, "at Bologne") in 1477, and the whole Hebrew Old Testament at Soncim in 1488. We have seen that, only five years after the earlier of these dates, and six years before the later, the first edition of the *Elements* (Adelard's) was printed at Venice, and a second edition (De Burgos) only a year later than the issue of the complete Hebrew Bible. We have seen that in the course of thirty-four years (1482–1516) four editions of the *Elements* were printed in Italy, and there may have been others printed elsewhere. We can form no estimate of the number of copies that constituted these editions. They were probably fewer than the numbers printed of Mr. Hall Caine's or Mr. Rudyard Kipling's recent works, but probably also they were not a very small number. Surely it may be conjectured that not fewer than 2000 copies were issued. Verily the epoch-making work had begun.

We have seen that one Greek manuscript of the *Elements* came to light in 1533, and was soon followed by others. We assume that these had been brought from Alexandria before the destruction of the great library, and were substantially the same with that, or those, rescued by Arabs from the conflagration, and from which the Arabic translation was made. Of course there may have been "various readings" between the *codices* taken from Alexandria into Europe and those taken into Asia, both of which may be presumed to have had a long ancestry of copies during the long time—roughly, a thousand years—betwixt Euclid's own day and theirs. And then variations must have been effected by the rendering of the Greek into Arabic, and frequent transcriptions of that rendering, and the subsequent rendering of one of these transcripts into Latin, and the printing of that translation after a long interval.[1] We have no means of estimating the actual differences thus introduced, but they do not seem to have been material.

Almost exactly contemporaneous with the invention of printing was the Italian Renaissance, although there

[1] It may be well to bring the dates together—

Euclid wrote about 300 B.C.

Manuscripts were taken to Europe probably about the fifth century of our era.

The Alexandrian Library was destroyed A.D. 640.

The translation of Ben Honain was made about 250 A.H., say 860 A.D.

Adelard's translation from it was made about 1130 A.D.

And was printed 1482 A.D.

Thus from the first writing to the first printing there were 300 + 1482 = 1782 years, or, as the first of these numbers is uncertain, say eighteen hundred years.

does not seem to have been any very definite causal connection between the two events. We cannot say that either was the cause of the other, or even contributed materially to its production. But each was needed for the production of the full benefit derivable from the other, and the Almighty Ruler decreed the occurrence of both, that by their concert they might bring about the New Era which in His unfathomable goodness He purposed to bring in upon the race of mankind.

The Renaissance did not do much directly for the study or progress of mathematical science. It concerned itself mainly with art and literature (*Literæ Humaniores*). But it awakened the dormant mind of Europe; and that mind "sought and intermeddled with *all* wisdom." The note of the *reveille* was not struck on the mathematical or scientific chord. But all the chords of the grand instrument vibrated in sympathy, and the mind of Europe arose and girded itself for the mighty work set before it.

For a long time there was no occasion for the translation of Euclid into any of the vernacular languages of Europe; nor, indeed, were these languages fit to express the necessities of mathematical thought. We can scarcely think of a Chaucerian Euclid! All who desired to enter on mathematical study naturally and with perfect ease made use of the Latin. It was long ere a translation of the *Elements* into any European vernacular was needed or was executed. So late as the days of the Commonwealth, the mathematical teachings of Wallis, and later still those of Barrow, the immediate predecessor of Newton in the Lucasian chair at Cambridge, were in Latin. Barrow published a complete edition of the *Elements* in 1655. But five years later, in 1660, he published an English version. As personally we owe much to Barrow, both as a theologian and a mathematician, we shall gratify ourselves, while we are sure that we shall not displease any reader, by transcribing a few sentences concerning him from Mr. Ball.

"He is described as low in stature, lean, of a pale complexion, slovenly in his dress, and an inveterate smoker. He was noted for his strength and courage, and once, when travelling in the East, he saved the ship by his own prowess from capture by pirates. A ready and caustic wit made him a favourite of Charles II., and induced the courtiers to respect even if they did not appreciate him. He wrote with a sustained and somewhat stately eloquence, and with his blameless life and scrupulous conscientiousness was an impressive personage of his time."

But while the academical teaching of mathematics, down to the time of Newton inclusive, was in Latin, vernacular teaching had, at a considerably earlier period, found its way into what we would now call the secondary schools. So early as 1570 was published an English translation by Sir Henry Billingsley. Of the translator and the translation we give De Morgan's very interesting notice:—

"In 1570 appeared Henry Billingsley's translation of the *Fifteen Books*, with Candalla's *Sixteenth*, London, folio. This book has a long preface by John Dee, the magician, whose picture is at the beginning, so that it has often been taken for Dee's translation; but he himself, in a list of his own works, ascribes it to Billingsley. The latter was a rich citizen, and was

Mayor (with knighthood) in 1591. We always had doubts whether he was the real translator, imagining that Dee had done the drudgery at least. On looking into Anthony Wood's account of Billingsley (Fast., Oxon., *sub verbo*), we find it stated (and also how the information was obtained) that he studied three years at Oxford before he was apprenticed to a haberdasher, and there made acquaintance with an eminent mathematician named Whytehead, an Augustine friar. When the friar was put to his shifts by the dissolution of the monasteries, Billingsley received and maintained him, and learned mathematics from him. When Whytehead died, he gave his scholar all his mathematical observations that he had made and collected, together with his notes on Euclid's *Elements*. This was the foundation of the translation, on which we have only to say that it was certainly made from the Greek, and not from any of the Arabico-Latin versions, and is, for the time, a very good one. It was reprinted, London, folio, 1661. Billingsley died in 1606 at a great age."

Slight modifications have been made on the *Elements* since those days—the most important by two Scotsmen, Robert Simson, 1756, and John Playfair, 1795; but they do not to any appreciable extent change the character of the book.[1] Attempts have also been made from time to time to pave a more royal road to geometry than that of the grand old Alexandrian. But these happily have been futile.

Before Billingsley's time, Tartaglia published his Italian version, 1543; and somewhat later than his time—the middle of the sixteenth century and begin-

[1] Subsequent English editors have generally followed Simson, and Scotch editors Playfair. The differences between them are unimportant.

ning of the seventeenth—translations were made into the other European languages: German, by Xylander, 1562; French, by Errard, 1598; Dutch, by Dou, 1606; still later were a Spanish translation by Saragoza, 1673, and a Swedish by Stroemer, 1753. These were probably second-hand translations from earlier vernacular versions, the former probably being from the Italian, the latter from the German.

These translations continued, with modifications considerable in number but insignificant in importance, to be the text-books for the students of geometry all over Europe till the time of the French Revolution at the close of the eighteenth century. In this country Euclid still reigns alone. In France he holds a divided, but very unequally divided, sway with Legendre. In Germany and in the New World, we would say—but our knowledge is not definite—the sway is also divided, but with the balance in favour of the old régime. Belgium, with its host of distinguished mathematicians, probably holds with France, and Spain and Italy with England or with Germany. As to Russia, we have failed to get any information. In Denmark, Norway, and Sweden we understand that the text-books are essentially Euclid, or thoroughly Euclidean. The "larger England" of Canada, South Africa, Australia, New Zealand, and India are a "larger Euclid" too. Thus we may say that every man who entered on the study of geometry between the end of the sixth and the end of the eighteenth century, and a large portion of those who have entered on it during the century which has lately closed, have entered on it under the guidance, philosophy, and friendship of the Alexandrian geometer. In order to estimate the significance of this, it must be borne in mind that in the study of geometry, far more than in any other study, the progress and the end take their character from the beginning. As in the study of Euclid itself an inefficient teacher may make good progress impossible to his pupils, so it is with Euclid himself as a teacher as compared with the others who have to a less or greater extent superseded him. Far be it from us to say a word in disparagement of Legendre, the chief of these, one of the most accomplished geometers of his own or of any time. We shall have occasion, at a further stage, to institute some comparison between his method and Euclid's. All that we have to do at present is to state emphatically our conviction that initiation by the one method or the other will modify the mathematical character of the initiated all through his career. A Euclidean and a Legendrian geometer will believe and know precisely the same body of mathematical truth, but they will to a certainty differ in their modes of viewing and stating it. It should be noticed, however, in this connection, that Legendre and other innovators all received their own instruction in mathematics according to the method of Euclid. If it were admitted that their methods are better than his, it would have to be borne in mind that they were indebted to him for the power to make the improvements. The Legendrians—applying that term to all non-Euclidean schools — are a colony of the great Euclidean nation. Though they have claimed and achieved independence, and sometimes display a tendency to "spread-eagleism," they cannot divest themselves of a kindly feeling for their cousins, and it is with pride and not with shame that they acknowledge a common ancestry and common obligations to the venerable father.

There are other deviations from the methods of Euclid which have not been taken seriously, and probably were not intended even by their authors to be so taken. There is, for example, a treatise by an Italian mathematician, Mascheroni, which must have been translated into French, as we have a vivid recollection of having in our student days frequently heard it referred to under a French title, Mascheroni's *Geometrie de Compas.* We understand that it is an exceedingly clever attempt to do by restricted means what others have done, and probably done much better, by unrestricted; as a self-confident pugilist might volunteer to go into the "ring" with his right hand tied behind his back. There is also an English treatise called *Geometry without Axioms,*[1] to which we shall have to refer further on. Meantime we may say that to our thinking it is self-condemned by its title, as geometry or any other science without axioms is an absurdity, a contradiction in terms.

[1] Its author, Col. Peyronnet Thompson, certainly did intend it to be "taken seriously."

VIII

Having proved that Euclid was the "father of all such as handle the rule and compasses," we have to inquire what has been the influence which these have exerted on the generations which have intervened between Adelard and ourselves, between the twelfth and the twentieth centuries. In this, it must be admitted, we are faced by two questions, with only one of which we have any means of dealing. We know, in some sort, what the generations have been under the influence of their mathematicians; but we know nothing, and can know nothing, of what the same generations would have been under the influence of the same men, the Newtons and the Leibnitzes, the Laplaces and Lagranges, had these men not been mathematicians at all. They would doubtless have exerted a potent influence, but it would have been an influence utterly different from that which they did actually exert. The history of the world might have been as interesting as it has been; but it must have been wholly different.

To tell all that mathematics has done for the world during the eight centuries, would be to recount the history of the world and of mathematics—a task impossible of achievement, though we had a hundred lives to spend in its accomplishment, and a hundred volumes in which to record it. To achieve it well would need in addition that our ability were multiplied a hundred-fold. It is but a minute portion of this stupendous task that we can undertake. Any question as to how much of what modern mathematics is, and of what it has done, was known to Euclid, does not concern us. We are quite willing to admit that he knew comparatively very little. We claim not for him that he knew much, but that he handed down to his successors the means of knowing vastly more than he ever knew. He was not the giant to see afar; rather he was the block on which others stood, so that the sphere of their view might be extended far beyond his. Some may deem it a very humble merit that we claim for him. We do not think so; and, what is of far more moment, Lord Bacon would not have thought so. What concerns us is not the amount of the extension of mathematical science, but the manner in which that extension is due to adherence to Euclid's methods.

The progress of mathematical science, as probably of all other sciences, has been partly slow and continuous, and partly rapid and brilliant; now by slow and quiet walking,—the snail's pace, if you like,—and now by a notable bound. That the progress, so far as it has been of the former kind, has been in the line of the Euclidean methods, will not be disputed; to use his own expression, it needs no demonstration. But we must examine the other case at some length.

When we speak in this connection of leaps and bounds, we do not refer at all to the demonstration of some theorem which has not been demonstrated before. A solution of the problem of the duplication of the cube, or the trisection of an angle, or the rectification and quadrature of the circle, or, in another department of mathematics, the reduction of the irreducible case of cubic equations, would not constitute such a leap as we contemplate. We have not in view the advancement of results, however important that may be; but the discovery of new methods, the invention of new instruments which are fitted to increase the power of the geometer, very much as the invention of gunpowder long ago, and the multitudinous inventions of explosives in our own day, have increased the destructive potency of our troops and our warships. The greatest of these inventions are: (*a*) that of logarithms by Napier (or Neper) (1550–1617); (*b*) that of co-ordinate geometry by Descartes (1596–1650); (*c*) that of the higher calculus, simultaneously by Newton (1642–1727) and Leibnitz (1646–1716). We place among these in the meantime (*d*) the innovations of Legendre (1752–1833), although he never regarded them otherwise than as improvements on Euclid. We shall have, in the sequel, to consider whether they were such or the contrary. To these, in their order, with reference to their bearing on the position of Euclid, we have now to direct our attention.

(*a*) Logarithms are not now regarded as belonging at all to geometry. They are essentially an arithmetical instrument, an instrument of arithmetical calculation, and no one now would think of computing them otherwise than algebraically. But it was otherwise in the days of old. Euclid himself regarded arithmetic as a branch of geometry; and, as we have seen, the books of the *Elements*, VII.–X. inclusive, constitute an arithmetical manual, a great portion of whose contents does not go beyond the standard of our ordinary schoolbooks. Things were materially different in the days of Napier; for it cannot be said that algebra was then non-existent, or that arithmetic had no being save as a department of geometrical science. Still algebra had only outlived the embryonic, and had entered on the infantile state of being. We do not quite precisely know what methods Napier adopted in his computation. But it would appear that they were more or less dependent on geometrical relations, and were founded on certain properties of one of the conic sections. Hence his logarithms are still frequently distinguished from other systems by the designation of *hyperbolical*. The relation to the hyperbola is not really distinctive of the Napierian logarithms, but appertains to logarithms generally; and therefore we suppose that the designation was applied to his system while as yet it was the only system. It may be well to explain to the non-mathematical reader what logarithms substantially are. This can be done very simply—

If we take two series of numbers, one consisting of the numbers in their order, and the other of the numbers formed by the continuous multiplication of unity by any number, say 2, we shall form a table of logarithms, thus—

Log.	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14.
Num.	1,	2,	4,	8,	16,	32,	64,	128,	256,	512,	1024,	2048,	4096,	8192,	16,384.

Now, suppose we wish to multiply one by another of the numbers contained in the line marked *Num.* in this perfectly real, though, of course, very brief table, we have to add the logarithms corresponding to these numbers, and below their sum in the line of *Log.* will be their product in the line of *Num.* Thus, to find the product of 16 × 64, we add the corresponding logarithms $4 + 6 = 10$, and the number corresponding to the logarithm 10 is 1024, which is the product required. This table could be very easily continued to any length, but it would not really be of any use, because the line of numbers contains only comparatively few of the actual numbers, and the table gives no logarithm for 3, 5, 6, 7, 15, etc.

Now Napier's work was to compute a table in which a logarithm should be given for every number between 1 and 100,000. Any arithmetician will see that as our little table gives the product of two numbers by addition of their logarithms, so it will give the quotient of two numbers by subtracting the logarithm of the divisor from that of the dividend; that the logarithm of the square root, cube root, etc., of a number will be $\frac{1}{2}$, $\frac{1}{3}$, etc., of the logarithm of the number. Thus, to find the square root of 16384, we halve its logarithm 14, and the number corresponding to the logarithm 7 is 128, the square root required. We have stated that Napier computed his logarithms in quite a different way from this. He did not make the use of any number that we have made of the number 2 in constructing our little table. But it was found that his results would have been got if he *had* made that use of a certain number, greater than 2 and less than 3.[1] After the publication of his work he perceived that a much more convenient table would be produced by the use of 10 as the "base" of the system. The same thought occurred to Henry Briggs, of Oxford, who at once perceived the marvellous potency and the glaring defect of Napier's table. He paid Napier a visit at Merchiston,[2] and at his earnest request agreed to make the transformation, as Napier

[1] The number 2·718281828459. . . .

[2] A suburb of Edinburgh.

himself was too old to undertake it. We suspect that Briggs adopted some method less laborious than Napier's. Still the labour must have been very great, and Briggs has richly merited that the decimal logarithms, the greatest boon ever conferred, or ever likely to be conferred, on mathematical science, should be known in all the world and in all time by his name.

Plane and spherical trigonometry are really branches respectively of plane and solid geometry. They were not unknown to Euclid, and Ptolemy treated them both, especially the latter, in the *Almagest*. But their application to astronomy involved calculations which, till the invention of logarithms, were enormously laborious, and it might almost be said that Napier by that invention made astronomy virtually a new science.[1] It was therefore quite in the line of Euclid's method that Napier made his invaluable contribution to mathematical science, while it may be frankly admitted that with the advance of algebra it must soon have been made on lines unknown to Euclid.

(*b*) The most powerful rival to Euclid is Descartes, a contemporary of Napier, though a younger man. It is not as a rival that we have now to consider him. That we shall have to do further on. At present we have only to regard him as having given a fresh impetus to geometrical progress. His method is sometimes described as the application of algebra to geometry. This is, in our judgment, an erroneous and misleading designation. There had been many applications of algebra to geometry before his day, and there have

[1] It may be stated, though it does not belong to our present subject, that many mathematicians, while regarding Briggs' tables as beyond all price, long for their extension.

been innumerable since, which had nothing in common with the Cartesian geometry. For example, the Second and Fifth Books of Euclid have often been treated by means of algebraic symbols and algebraic processes. Thus, the algebraic proposition $(a+b)^2=a^2+2ab+b^2$ needs only the explanation that a and b represent lines of lengths respectively a and b times any unit of lineal measure, as a hair's-breadth, an inch, a foot, a yard, a mile, the earth's diameter, etc., in order to make it identical with the geometrical proposition Euc. II. 4. But the method of Descartes is something wholly different from this. That method consists essentially in regarding the position of a point as determined by its distances from two intersecting lines, which, in the simplest case, are taken at right angles to one another. There is an indispensable assumption which may be supposed by some to be merely conventional, but which is really dictated by the nature of the case, that the algebraic signs + and − will indicate the direction in which the point lies from the intersecting lines, above or below the horizontal, to the right or left of the vertical. Now, to take the simplest case. We have to do, let us say, with a point whose distances from the intersecting lines are x and y, and it is ascertained that between the co-ordinates, as they are called, the equation obtains $x^2+y^2=r^2$, r being a known or given line, and x and y unknown lines. There is but one equation, and it contains two unknown quantities; it is therefore an indeterminate equation, and there are many points that will fulfil the condition. By putting the equation under the forms $x^2=r^2-y^2$, and $y^2=r^2-x^2$, we see that neither x nor y can be greater than r, for then we should have a square equal to a negative quantity, which is impossible. But each of the co-ordinates may have any value between 0 and $\pm r$. If $y = 0$, then $x = \pm r$; and if $x = 0$, $y = \pm r$. So also, if $y = r$, $x = 0$, and if $x = r$, $y = \pm 0$.

For every intermediate value, as a, of y, we shall have $x = \pm \sqrt{(r^2 - a^2)}$, and every intermediate value, as b, of x, will give $y = \pm \sqrt{(r^2 - b^2)}$. We find, then, that the equation holds for every point in the circumference of a circle whose centre is the intersecting point of the axis of co-ordinates, and whose radius is r, and that it will not hold of any point which is not in the circumference of that circle. We call then $x^2 + y^2 = r^2$, the *equation* of the circle, and we call the circle the *locus* of that equation; and from this equation, combined, when necessary, with the corresponding equation of the straight line, we can very easily deduce all the properties of the circle as demonstrated in Euclid III. For example, since $y = \pm \sqrt{(r^2 - x^2)}$, it follows that every diameter bisects every chord perpendicular to it; and since $y^2 = r^2 - x^2 = (r + x)\ (r - x)$, it follows that the square of every semi-chord is equal to the rectangle contained by the segments into which that chord divides the diameter which bisects it.

Within the province of the "rule and compasses" the propositions are all easy, and there is no temptation to discard the Euclidean method for the Cartesian. But when we come to the conic sections, it must be admitted that their properties are much more easily found by the latter than the former method; and for curves of higher order, represented by equations of a higher degree than the second, the latter is sometimes available where the former fails us altogether. We admit frankly that the advance of geometrical science has been greatly hastened by the method of Descartes, and willingly accord to him the credit of a great and important invention, although we may have to say in the sequel that some deduction must be made from the credit in order to compensate for certain defects; but what we have to do with at present is, that Descartes' invention could only have been made by a well-trained Euclidean. It is founded throughout on Euc. I. 47 and VI. 4, 5. Without these it could have had no existence, and these it could not have proved by its own method. We submit, therefore, that we are entitled to claim for Euclid, over and above the glory which belongs to him independently, some portion, be it more or less, of that which enhaloes the head of Descartes.

(*c*) We are not going into the controversy, which was carried on for many weary years with a degree of bitterness which throws even the proverbial *odium theologicum* into the shade, regarding the priority and mutual dependence or independence of the invention of the fluxional calculus by Newton, and the differential and integral calculi by Leibnitz. The conclusion that is generally accepted now is, that Newton made the discovery first, and communicated it to a few friends; but, with a morbid aversion to publicity, which is known to have been one of his most marked characteristics, refused to publish it. Some years after, Leibnitz made substantially the same discovery, and immediately published it. Thus the priority of discovery belongs to Newton, the priority of publication to Leibnitz. Of course, Newton's supporters maintained that Leibnitz had the discovery made known to him by Newton's friends,—with some of whom he maintained correspondence,—and appropriated it. The supporters of Leibnitz insisted as strenuously that Newton, on receiving the published paper of Leibnitz, vamped up a claim of prior discovery. We venture to suggest that it is possible that Leibnitz may have got, in conversation or correspondence with some friend of Newton, a hint that Newton had made an important discovery in a particular direction, and that this unconsciously may have given a tendency to his thoughts and his investigations. That the one or the other was guilty of base plagiarism, —conveyance, the wise it call,—is simply incredible. Be all this as it may, the two systems are substantially identical. We should say that Newton's proof is decidedly more satisfactory than that of Leibnitz. On the other hand, the notation used by Leibnitz is much more convenient than that of Newton; and on that account only, the form of the discovery with which that notation is associated is universally recognised, and the very name of *fluxions* has gone wholly out of use.[1]

We must attempt to give the reader a general idea of the nature of this discovery. We premise that when two quantities are so related that the magnitude or value of the one depends on the magnitude or value of the other, the one is said to be a *function* of the other. Thus, as the square of a number, or the square described on a line, is larger or smaller according as the number in the one case or the line in the other is larger or smaller, the square is said to be a function

[1] What Newton called a fluxion, and designated as $\dot{x}$, Leibnitz called a differential under the designation dx. The late Professor Babbage wittily expressed the distinction as the *dot*-age of the one and the *d*-ism of the other.

of the number or line. So also the sine of an acute angle is greater or less, and the cosine of an acute angle is less or greater, according as the angle is greater or less. Therefore the sine and cosine are functions of the angle. These may be thus expressed—

$$x^2 = fx, \sin x = \text{F}x, \cos x = \varphi x.$$

Let us now take the very simple function to which we have just referred. If we have two quantities of the same kind, as two lines, two angles, two areas, such as x and $x+h$; we have $x+h-x=h$; $(x+h)^2-x^2 = 2xh+h^2$. Now this difference will be ever the smaller as h is the smaller. If $h=0$, then $2xh+h^2=0$, as it must be, for then $(x+h)^2=x^2$; and as h becomes less and less, *i.e.* approaches 0, the difference will be more and more nearly $=2xh$. This *limit*, then, to which the difference approaches, we call the *differential* of the function. We have spoken of x and $x+h$ as separate quantities, but what we have said will be equally true if we regard x as a *variable* quantity, and as receiving increments (or decrements, *i.e.* negative increments). Then h will be indifferently the difference or differential of x, while $2xh+h^2$ is the difference, and $2xh$ is the differential of x^2. Thus calling h the differential of x, or dx, we have $d(x^2)=2xdx$; or $\frac{d(x^2)}{dx}=2$. This in its general form $\frac{dfx}{dx}$, is called the *differential coefficient* of the function. Now, as every several function of a variable quantity has its own differential, and in general[1] every several differential its own "in-

[1] The exception is that $fx \pm \text{C}$, when C is a constant quantity and has no differential, has its difference, differential, and differential coefficient the same with those of fx. In differentiating, that is, finding the

tegral " from which it has been derived, it is evident that this idea of differentiation introduces us to new relations of quantities, very much as we may suppose that the acquisition of a sixth sense would reveal to us properties of material things which lie beyond alike our cognisance and conception.

In point of fact, the use of the calculus has wonderfully advanced mathematical science generally, and geometric science in its higher departments particularly; and of this advance, as we did in the case of Descartes, we claim a share for Euclid. Newton himself was a Euclidean of the first water, as distinguished from a Cartesian, and his *Principia*, in so far as it is a geometrical book—as it is very largely—is scrupulously Euclidean.

(*d*) Legendre introduced no new principle, but he made important changes in the order of the *Elements* and the methods of proof. While some of these may be admitted to be improvements, we are very strongly of opinion that the aggregate result is not improvement but deterioration. Legendre was a mathematician of the first order. But he seems to have failed to apprehend that methods and conceptions which are evolved in the course of the study of the higher departments are not appropriate for the demonstration of elementary theorems.

The earliest and the most important of his deviations from Euclid s method is in the proof of the

differential from the function, this can give rise to no error, as dx is the differential of $fx+C$ as really as of fx. But in integrating, that is, finding to what function a given differential belongs, we must always conclude that $fx+C$ is the integral of dfx, and we must ascertain by other means whether $C=0$, or whether it has any real value.

theorem that the three angles of a triangle are equal to two right angles (Euc. I. 32). In his text he attempts this by the introduction, as a theorem, of Euclid's disputed axiom regarding non-parallel lines, of which we shall have much to say in the sequel. With respect to his "investigation" of this theorem, he makes an admission which, we suppose, is unique in geometrical treatises. "*Scholium.* The preceding investigation, being founded in a property which is not deduced from reasoning alone, but discovered by measurements made on a figure constructed accurately, has not the same character of rigorousness with the other demonstrations of elementary geometry. It is given here merely as a simple method of arriving at a conviction of the truth of the proposition. For a strictly rigorous demonstration we refer to the second note."[1] The "rigorous demonstration" in Note II. is in substance this. He first states, "By supraposition it can be shown immediately, and without any preliminary propositions, that *two triangles are equal which have two angles and the interjacent sides equal.*" This seems to us quite legitimate, though Mr. Dodgson and others hold that it is not. From this he infers that the third angle C is "entirely determined" when the angles A and B and the side opposite to C are known. Thus the magnitude of C depends only on the magnitudes of these three quantities, A, B, and c. In other words, C is a function of these three quantities. This in analytic notation is expressed thus: $C = f(A, B, c)$. But the value of C cannot be dependent to any degree upon the value of c, for then we

[1] This and other quotations are from Sir David Brewster's translation of Legendre.

should have it that all triangles on the same base must have their vertical angles equal, which were absurd. The function given above is therefore reduced to this other function, $C = f(A, B)$. "This function," he says, "already proves that if two angles of one triangle are equal to two angles of another, the third angle of the former is equal to the third of the latter; and this granted, it is easy to arrive at the theorem we have in view," *i.e.* the theorem Euc. I. 32. Mr. Dodgson objects to the proof of the first case of Euc. I. 26 by supraposition, and Sir John Leslie objects to the reduction of $f(A, B, c)$ to $f(A, B)$. We do not hold either of these objections valid, but go along with Legendre in holding it proved that $C = f(A, B)$. But here we part company with him, and protest against his treatment of that function. That function, and Legendre's treatment of it, we must examine somewhat minutely.

As A, B, and C are angles, they can have no product, therefore AB cannot enter into the function; and as C is an angle, its value cannot contain the quotient $A \div B$, for that quotient is a number, and cannot be the value of an angle. A and B must therefore enter into the function only as sums or differences of multiples of one and of the other. It is possible, also, that the value of C may depend on the sum or difference of the sums or differences of multiples of A and B and a fourth angle, which must be invariable, so that C cannot be a function of it. This fourth angle may be 0, or it may have any value that an angle can have. If we call that fourth angle P, then the possible forms of the function are included in the equation $C = \pm P \pm mA \pm nB$.

This is in reality eight different equations, only one of which is inadmissible, namely, $C = -P - mA - nB$, for C must have a positive value. There remain, however, seven[1] good and true equations, all equally expressed by the formula $C = f(A, B)$, and Legendre had no right to prefer one of these seven to the others. But he assumed that only one of these seven is true, and *that* only with certain assumed limitations, namely, $C = +P - mA - nB$, and that $m = n = 1$. Having by means of these assumptions concluded that $C = P - (A + B)$, he had only further to prove $P = \pi$, and this he thought was easily done. Now, why did he make these assumptions? Because he had learned *from Euclid* that this one alone is true. He *proved* that C is *some* function of A and B, and he assumed that it is *this* function, simply and only because Euclid had proved otherwise, what his formula could not indicate, that it is this function. It is not impossible that that very clever young lady, Miss Portia, gave to Mr. Bassanio a "tip" as to the one of the three caskets which would serve his turn. It is certain that M. Legendre made his choice between seven solely on the information received from Euclid.

But this is not all. Admitting, as we do, that he was right in holding that the first case of Euc. I. 26 might come immediately after I. 4, we must point out that this is but a small part of what he had to prove. Euclid's proposition is that two triangles having two angles equal *each to each*, and the interjacent side equal, have the third angle equal. This was sufficient for Euclid's purpose. But for Legendre's purpose it was necessary to prove the vastly larger proposition, that

[1] There remain, in fact, an "innumerable number" included in these seven, for m and n are any numbers whatever.

if two triangles have *the sum* of two angles in the one equal to *the sum* of two in the other, then they have their third angles equal. This he does not attempt to prove. The matter stands thus, put into language which every reader will understand. He attempts to prove that if one angle of a triangle be, say, 60° and another 70°, then the third angle will be 50°. He does not prove this satisfactorily. But if he had proved it, he would not have been entitled to infer from it that a triangle which has the two angles 52° and 78°, or 23° and 107°, or innumerable other combinations, has equally its third angle 50°.

But even this is not all. Legendre, in order to prove what we have stated as the proposition $P = \pi$, says that it is so in the case of the isosceles right-angled triangle. He may have been able to prove this; but he does not prove it, and it needed proof. He knew it so well as deducible from Euc. I. 32, that he forgot that it must not enter into the proof of that proposition.

All English mathematicians agree that the introduction of the doctrine of functions at this stage of the learner's progress is wholly inadmissible; and with this we thoroughly agree. Legendre admits this, by putting what he acknowledges to be no demonstration into his text, and relegating his demonstration to Note II. With that demonstration we have now dealt. Both the length of that dealing and its method must be distasteful to many—probably to most—of our readers. We are sorry for this, but we cannot help it.

But we agree with all that has been said by critics as to the illegitimacy of the method itself, the ineptitude of employing in the proof of a proposition in elementary geometry a principle derived from another department of mathematics, with which no learner of elementary geometry should be assumed to be conversant. Great mathematicians have adorned the Legendrian School, but we cannot help thinking that as geometers the greatest of them would have been greater still if they had been trained in the stricter school, and habituated to the more elaborate perhaps, but certainly the more demonstrative, methods of Euclid. The English mathematicians are under unspeakable obligations to the French, and it may be admitted that the former owe to the latter no small portion of their power in advancing the science; but we cannot quite divest ourselves of the belief—or is it only the feeling?—that what is distinctively their own is better than what they have consciously or unconsciously appropriated. The most that we can claim for Euclid of the success of the French methods is that the geometry into which Legendre himself was initiated was that of Euclid. But for this, a man of his great powers might have introduced still more revolutionary changes. The arrow that he launched to the old eagle's heart was pinioned with feathers from the eagle's breast.

All this with respect to pure mathematics—geometry by itself geometry. But while it is not a trifle that in all generations that have intervened between that 1570, when Richard Billingsley gave the *Elements* to the English-speaking world, and this present year of grace 1901, some hundreds, say even a few thousands, of our countrymen have enjoyed a pure and high delight in the pursuit and attainment of mathematical science, and in the consciousness of more or less power to advance that science, it must be admitted that this has not very materially affected the aggregate amount of enjoyment or well-being of the many millions of men and women who have composed the generations. And so of the thousands of others than English-speaking geometers. We have seen incidentally that Euclid scorned utilitarianism as an incentive to the study of his science. Rightly so; most rightly. But while utilitarianism were an unworthy incentive *to* the study, utility is a blessed result *of* the study. It is with knowledge as with God's other good gifts: "there is that giveth and yet increaseth, and there is that withholdeth more than is meet, and it tendeth to poverty." And no human knowledge, it may be safely asserted, has been so magnificently "altruistic" as geometrical knowledge.

From the nature of the case, pure geometry has no dependence on, or relation to, material things. The demonstration of Euc. I. 47 might have been precisely as it is had there never been a right-angled triangle or a square of gold, or silver, or brass, or iron, or wood, or paper, or cloth. The property demonstrated holds of all right-angled triangles, of whatever magnitude and of whatever material. The sole requirements are that it be a triangle, and that one of its angles be a right angle. The process, then, by which geometry becomes subservient to our material well-being, depends on the fact that in itself it has no relation to any material thing, and therefore it is capable of application to all material things, and to all alike. From these applications arise a host of sciences and arts, and of scientific arts and art-sciences, which go under the general name of applied or mixed mathematics. The reason of the former designation is obvious; the latter refers to the fact that such sciences must have two foundations—demonstration and observation. Pure geometry has no place for observation or experiment. Whenever that comes in, geometry retires, or at least ceases to be pure. We may ascertain that two straight lines are equal, by measuring them with a foot-rule. This may be very convenient, but it is not geometrical.[1] We venture to lay it down as a principle of such sciences, that their two foundations, demonstration and observation, should ever be regarded as absolutely distinct from each other. In practice this may not be fully

[1] It may be objected that geometry allows the use of the rule and compasses. This is usually said, and for practical purposes it is true. But, strictly speaking, it rather only assumes, as Euclid does in his postulates, that a straight line may be drawn, and a circle may be described. And in this the meaning is even stricter than the expression. The meaning is rather that there *is* a straight line between any two points, than that such a line *can* be drawn by any materially composed rule; and that round every point there *are* an infinite number of circles, each one with a different radius, rather than that any one of these circles *can* be traced by a human hand with material compasses. In addition to this, we might say that man not only cannot draw a *straight* line, but he cannot draw a *line* at all; for whether he write with the finest pencil or scratch with the sharpest needle, he makes a surface with length and breadth, not a line with length only. The use which geometry makes of the rule and the compasses is to mark out within narrow limits the space through which a straight line or the circumference of a circle must pass, and that merely to aid the geometer or the learner in the conception of the actual line. So also the *point* with which the geometer deals is not the figure made with pen or pencil on his paper, approximately a circle, but the centre of that circle or other point within it. Further, it might be noted that the *rule* which geometry contemplates, besides being immaterial, is ungraduated, so that it is incapable of measuring length. It is a rule, but not a foot-rule. The compasses can ascertain the equality or inequality in length of two lines, but takes no cognisance of the actual length of either.

attainable, but in proportion to the measure of its attainment will be the approach of the science to perfection. Mathematics cannot observe, observation cannot prove. From the confounding of their separate functions unspeakable evils have arisen, and may still arise. The fatal defect of the pre-Baconian science was the neglect of observation. It may be feared that much of our modern science has a tendency to fall into the opposite error, of supposing that the observation, which is one of the essential foundations of science, can of itself prove scientific doctrines, or that conjecture can legitimately take the place of demonstration in dealing with observed phenomena. No mathematics can go a step or a hair's-breadth toward the proof of the existence of a planetary system. No observation can approach to the proof of the laws which govern that system whose existence observation has discovered. All honour to the late Mr. Darwin, the greatest of all observers. Let his followers beware of imagining that any observation, however extensive or however accurate, can ever constitute science. Its service is invaluable in laying a foundation. The superstructure will be true science or "science falsely so called," according as mathematics or conjecture be the architect. No detailed account of the applications of geometry to material phenomena were consistent with the nature and object of our present undertaking. We can only mention a few of them, and cannot enter into the details of any.

ASTRONOMY.—The earliest and the most extensive of the mixed sciences is astronomy. It is likely that for a long time observations, more or less systematic, had been made of the heavenly bodies before the observations were subjected to mathematical treatment. To the Babylonians credit has been usually given as having been the earliest observers; and if the reasons for this assignment be not altogether adequate, they have not to encounter any considerable reasons in opposition. It is certain that the Egyptians also occupied themselves from very early times with astronomical observations, the methods of which they probably learned from the Babylonians. The school of Alexandria, so far as it was a school of mathematics, can be regarded as Egyptian only so far as, through what may be called accident, it had its residence in Egypt. Neither its teachers, its scholars, nor its methods were to any appreciable extent Egyptian. The same is true of it as a theological school. But it seems to have been otherwise with its astronomical department. Hipparchus, who studied for some years in Alexandria, seems to have made use of the Egyptian observations and methods, and is acknowledged as the founder of astronomical science. His observations were literally marvellous for their accuracy, when account is taken of the circumstances in which, and the instruments with which, they were made. His writings are all lost. But their chief matter has been transmitted to us by Ptolemy, who generously records his obligations to one who preceded him by fully three hundred years. The *Almagest*,[1] the great work of Ptolemy, was published about the middle of the second century of our era.

[1] The title of this book is *Μέγιστη Σύνταξις τῆς Ἀστρονομίας*. Probably the original title was simply *Σύνταξις*. His disciples may have signified their admiration of it by the epithet *μέγιστη*. This was arabesqued—if we may be pardoned for making such a word—by prefixing to it the Arabic definite article *al*, and dropping the *σύνταξις*. Thus, *The*

Like the *Elements*, it was first translated into European languages from an Arabic translation, and Greek manuscripts were found afterwards. The part of the *Almagest* which is of most interest to modern astronomers, is that which contains the observations of Hipparchus, and of Ptolemy himself, of the places of the fixed stars. To us at present, as most bearing on our subject, the main interest of the book lies in the fact that a large part of it is devoted to geometry and trigonometry, plane and spherical. It is of much interest to note that he uses the sexagesimal division of the circle, dividing the circle into 360°, the degree into 60′, and the minute into 60″. Whether he invented this division or adopted it from Hipparchus is unknown. The geometry is, of course, purely Euclidean. The astronomy of Ptolemy was the sole astronomy from his days to the times of Tycho Brahe, Copernicus, Kepler, and Galileo, in the latter part of the fifteenth and the earlier part of the sixteenth century. Undeserved ridicule has been cast on the Ptolemaic system by the poet, who speaks so scornfully of its "cycles on epicycles, orbs on orbs." It is interesting to know that Ptolemy himself was aware that he could have dispensed with these, if only he could have admitted the rotation [1] of the earth.

The sixteenth century astronomers whom we have named removed the *Almagest* from the place which it had held so long as *the* astronomical text-book of

Almagest, the name by which it is now always designated, means literally *The the Greatest*. This is one of the absurdities which no one of even moderately conservative feeling would desire to have corrected.

[1] So says De Morgan. Does he not mean rather the earth's revolution round the sun?

Europe, and left it to be at once a monument of a great man, and a historical record of the beginning and earliest progress of astronomical science. Their work, however, had been rather destructive than constructive. Mathematics cannot lie. But the application of infallible mathematics to an imperfect observation, or, as in the present case, to an erroneous assumption, will produce imperfect or erroneous results. The chain had snapped in its weakest link, and it was left to Newton to apply the unbroken, the unbreakable, the mathematical link to a new observational link. How ably this was done it were impertinent for us to say. A man must occupy a high position as mathematician or astronomer before he is entitled even to praise Newton. Enough to say that he not only founded mathematical astronomy, but perfected it. Since the publication of the *Principia*, observation has done much and will do more, and in the mathematical department simpler and even better methods have been introduced; but substantially the science remains, and will remain to the end of time, as Newton left it.

It is said—we know not on what authority—that Laplace[1] lamented that Newton had lived before him, and deprived him of the possibility of being a great discoverer. Whether Laplace could have done what

[1] Since writing this, we have found that the complaint was not made by Laplace himself, but by Lagrange on his own behalf and that of Laplace. Lagrange described the *Principia* as the greatest production of the human mind, and said he felt dazed at an illustration of what man's intellect might be capable of. In describing the effect of his own writings and those of Laplace, it was a favourite remark of his that Newton was not only the greatest genius that ever lived, but also the most fortunate; for, as there is but one universe, it can happen but to one man in the world's history to be the interpreter of its Laws.—Ball's *Short History*, p. 362.

Newton did, it is, of course, impossible to say. What he actually did in the *Mecanique Celeste* was to prove in a different order and by different methods the truths which Newton had already proved. It is true that the *Mecanique* contains a great amount of matter that is not in the *Principia*, but that is either outside of the strict limits of the science, or it is the application of Newtonian principles to intervening observations, or the further development of Newtonian propositions. Whatever loss the Frenchman may have sustained at the hands of the Englishman by the latter having forestalled the former in anticipating the discoveries which he would have made, as we all, alas! have to charge the "ancients" with the "theft of our best thoughts," was compensated by the fact that, through living at a later time, Laplace could use a developed differential and integral calculus, whereas Newton had first to invent, and then to use in its nascent stage, the fluxional calculus. It may be stated that, while Newton was well acquainted with the modern analysis, and actually wrote several treatises on several parts of it, yet in the *Principia* he renounced the use of it, and rigidly restricted himself to the synthetic methods of Euclid. Laplace, being under no such restriction, might have been expected to produce a work of greater simplicity than the *Principia*. In this he has not succeeded. Perhaps he did not desire it. The study of the *Mecanique* is not less difficult than that of the *Principia*, while the study of the latter is likely to develop a higher range of faculty. We may be pardoned the egoism of saying that we have tried both tasks, and though we have not *accomplished* either, we have made so much progress as to entitle us to pronounce on their comparative easiness. The study of the subject is greatly facilitated by the use of Mrs. Somerville's *Mechanism of the Heavens*. This is not, as might be supposed from the title, a mere translation or abridgment of Laplace's great work. It is the independent work of a profound student of the *Mecanique Celeste*, of whom it was said by a competent judge, that she was "equally in her element when calculating the aberration of a comet, and when darning her husband's stocking."

That we have reason to say that mathematical astronomy is a perfect science, is evinced by its triumph achieved in our time, when it led Mr. Adams and M. Leverrier to the conclusion that there is a world till then unknown, and that at a particular time it must occupy a certain position in space, and must contain a certain amount of matter; and when they, so directed, simultaneously discovered the planet Neptune. Nothing so great as this has ever occurred in the history of science. Surely not in vain nor extravagantly said the poet, in lines which we quote from memory, and perhaps not with perfect accuracy—

> "Nature and Nature's laws lay hid in night:
> God said, 'Let Newton be'; he was, and there was light."

In so far, then, as mathematical astronomy has contributed to the formation of our modern epoch, we claim for Euclid, acting primarily through Ptolemy, the geometers of the sixteenth century, and Newton, and acting in no small measure, though not so fully, through Descartes, Lagrange, and Laplace, the credit of its initiation and its completion.

Optics.—It is very doubtful whether Euclid was the author of the treatise on *Optics* which has come down to us bearing his name. At any rate it is not on the ground of such authorship that we claim for him any share of the epochal influence of that science. The treatise is a poor affair, containing some important errors, and has had no part in the foundation of optical science as it now is. But that science is geometrical throughout. Like all the mixed sciences, it has an observational or experimental and a mathematical element, and in it, more perhaps than even in astronomy, the mathematical element is essential to its being as a science. In optics the observational department seems in old times greatly to have outrun the mathematical, as was indeed to have been expected. It is evident also that the phenomena of reflected light, the *catoptrics* of modern science, were observed much earlier and much more extensively than those of transmitted light (*dioptrics*). This also is easily accounted for. From a very early period in their history, men had the power both of moulding and of burnishing metals; they could not fail to observe the reflection of light from the polished surfaces of the instruments which they constructed, the plane surfaces of their shields, the convex and concave surfaces of their bowls, and whatever they had corresponding to our modern spoons.[1] At a very early period it is evident that metallic surfaces were polished for the very purposes of the reflection of light. Thus Moses,

[1] In this respect, as in many others, the experience of the microcosmic individual is identical with that of the macrocosmic race. How many of us have derived our earliest idea of a concave and a convex mirror from the contemplation of our features as reflected from the inner and the outer sides of a silver or plated spoon!

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in the furnishing of the tabernacle in the wilderness, made the laver and its stand out of the brazen looking-glasses of the women, who freely consecrated their cherished possessions to the sacred purpose, Ex. xxxviii. 8. The burning-glasses of Archimedes seem to have been plane mirrors arranged on a frame so constructed as to concentrate the rays of light that fell on a large surface; but Euclid (or whoever else was the author of the book ascribed to him) treats of burning-glasses which acted by reflection from the inner surface of a spherical segment. We read that Ptolemy Euergetes (B.C. 224) is recorded to have placed in the tower of the Pharos at Alexandria, a mirror which showed plainly an enemy's fleet at the distance of 600,000 paces. We are safe in saying that the distance —more than 280 miles—is vastly overrated. But there is no good reason to question the fact that a mirror was placed in the tower for this purpose. Now it must have been a convex mirror. But whereas from times long prehistoric men had at their hand metals from which to form reflectors,[1] and had ability with sufficient accuracy to fashion these metals into the desired form, and to polish them when so fashioned, it was not till a much later time—though the actual time cannot be ascertained with any approach to precision—that they had a material which was at once solid and transparent. When glass was invented is unknown, and then it is probable that it was for a

[1] It should be noticed that our reflectors still are all metallic. We speak of a *glass*-mirror and a looking-*glass*; but the glass serves only for a protection for the metallic reflector behind it. The glass contributes nothing to the reflection. It is not a good, but a necessary evil, when the reflecting metal is mercury.

long time of very imperfect transparency, and the power of working it was very limited. Of the three operations, blowing, moulding, and grinding, they seem, till a comparatively late time, to have been restricted to the first. Thus even Seneca (A.D. 64) wrote: "Literæ quamvis minutæ et obscuræ, per vitream pilam aqua plenam, majores clarioresque cernuntur." From this it appears that, even after the middle of the first Christian century, the only "magnifying glass" was a blown bubble filled with water. At a not much later period we learn that the Romans used glass prisms for ornamental purposes. But we suspect that these were not ground, but only moulded. A lens might have been made by blowing, that is, by joining two segments cut from a hollow globe, as many experimenters of straitened means have made good use of two watch-glasses. But a lens fit to serve any purpose of scientific optics must be ground and polished with a degree of accuracy of which the uninitiated can scarcely form a conception.

It is here that geometry first comes in, to dictate the forms of that apparatus which is essential to the study of the science itself, and to the greater part of applications of that science in its advanced state. The old writers on optics were much occupied with discussions on the nature of light and of vision, and of their relation to each other. The prevalent notion among the Greeks was that light is emitted from the eye and falls on the object that it may be seen; that, in fact, the eye takes its walks abroad, and virtually comes into contact with the object. But neither that theory nor the opposite theory, now universally received, that light proceeds from the object to the eye; neither the theory that light is a material emanation from the luminous object, nor the theory that it is an undulation produced by such object in an etherial medium, goes a single step towards the solution of the questions, what light is or what vision is. We say that an unsupported stone falls to the ground, and we say that it does so in consequence of the earth's attraction. But what attraction is, or why the earth attracts and does not repel the stone, Newton knew no more than does the untutored savage. All science must come to the conclusion that however far it may follow the alternate links of causation, it can only come a little nearer to that primary cause which, as Newton himself puts it, "is certainly not mechanical." We see, because a beneficent Creator has given us the power of seeing. It is well that such questions are regarded by modern science—we say not by all that is often regarded as science—as wholly inept. But while neither geometry nor any other science is cognisant of the essence of light or of visual power, every phenomenon of light, its reflection, its refraction, its aberration, its polarisation, is subject to laws which are purely geometrical, and which can be understood only by the application of geometry. To determine the form or colour of the rainbow, we must ascertain or assume the presence of drops of water, and of rays of light capable of being refracted in accordance with certain laws in passing through these drops, and of being reflected according to certain laws from the surface of these drops at their entrance and their emergence. Otherwise the problem is as much geometric as is the construction of an equilateral triangle on a given finite straight line.

We shall probably best suggest to readers thoughts which will enable them to estimate the profit which our age derives from optical science, if we confine ourselves to mentioning, and little more, some of the optical instruments in use, all of them, be it noted, dependent on geometry for their invention, and all of them requiring mathematical precision in their construction.

Spectacles.—It seems to be impossible to attain any knowledge of the date or circumstances of this invention —perhaps the most beneficial that man has ever made. Certainly the artist was guilty of an anachronism, who, in an illustrated Dutch Bible, pictured the child Timothy getting his lesson out of a bound Bible, his instructress being Lois, a venerable old lady in full Dutch attire, and with heavy-rimmed spectacles on her nose! There is doubt as to the reference by St. Paul in 1 Cor. xiii. 12. But, with all deference to others better able to judge, we would suggest that he does not refer to spectacles, but rather to a darkened glass employed to shade the too bright rays of the sun. Though this explanation does not seem to have occurred to any of the commentators, it seems to be more likely to be correct than that which many give, that when the apostle says "we see *through* a glass," he means "we see *in* or *on* a mirror." We are not aware that there is anything in the classical or mediæval writers which would indicate acquaintance on their part with anything of the nature of our spectacles. But *our* ignorance in this department does not go for much. It is evident that in the days of Galileo spectacle-making was a common trade; and we should suppose that the invention was made long before. Of all the good gifts that the beneficent God has bestowed on His creatures, over and above air and light, and food and raiment, surely there is none that contributes more than this to the creature's enjoyment, and to his power of doing good to his fellows. Surely so good a gift is "from above, and cometh down from the Father of lights." The present writer is in a position to *know*, though he is not able to *tell*, how precious is the boon.

The Microscope.—Few men ever forget the emotion excited by their first look through a microscope. Feelings of vague wonder, approaching incredulity, were probably in most succeeded by reverential thoughts of the attributes of the Creator manifested in a virtually new world brought under their ken. As by the telescope we see the immensity and unspeakable magnificence of the Creator's work, so by the microscope we perceive the boundless extent of His care. By the one we are compelled to reflect on the littleness of man, by the other we learn that over creatures millions-fold less the great Creator cares. We reject with scorn the misanthropic satirist's conception of higher intelligences who would "dandle Newton as we might an ape." We know that little as was Newton, far less as are we, far less than we—we will not let modesty prevent our saying—as is the most unintelligent and the most depraved of our fellow-men, yet even he is of more value than many sparrows; while one sparrow may be fairly supposed to be equivalent to a million of microbes. Yet every one of these microbes the Creator of the sun, and the countless stars with their several worlds, has seen meet to make; and for the least of these little ones, He who guides the host of heaven condescends to care. If the lesson of the telescope is, *What is man?* that of the microscope is, *Thou art mindful of him.*

The invention of the microscope had not been in vain if it had only given rise, as in thousands of instances it has done, to such thoughts as these. But, in addition, a new and more visibly useful work has now been assigned to it. That bacteriology, which it has enabled already to rank among the sciences, gives good promise that in the hand of the sanitarian, the pathologist, the agriculturist, and many others, it will ere long take a foremost place amongst the utilitarian sciences.

The Telescope.—To speak in laudatory terms of the telescope in its various forms, as the pocket opera-glass, the tourist's binocular, the field-glass, which the advertisements daily and monthly commend to rifle-markers and deer-stalkers as each more capable than all the others of rendering it possible to ascertain the time from a clock at five miles' distance,—we do not know whether it is claimed for any of them that it is capable of rectifying the error of the clock!—and the somewhat clumsy "day and night" glass, which did yeoman service in its time, but which is now being gradually superseded by the binocular; the astronomical telescope on its brass stand, which shows Saturn's ring and Jupiter's satellites; the gigantic reflector of Lord Rosse, and the gigantic refractor of the Lick Observatory,—to set forth the merits of any of these were verily to "gild refined gold." What we have to do with is the fact that not one of these could have had a being without their inventors' knowledge of those laws of optics which are purely geometric, and that every one of these inventors learned his geometry from Euclid. It may be objected that the original invention of the telescope was empiric, and was suggested by an accident. Yes, but the accident would not have suggested the invention to any but a geometer, as Galileo was, even as there needed not only an apple to suggest the law of universal gravitation, but a Newton to receive the suggestion. And then it must be remembered that the telescope invented by Galileo had probably very little in common with our modern telescopes. Certainly it was destitute of the achromatism which gives them half their value. With our modern reflectors it had simply nothing in common.

The Photographic Camera.—We (the author) had well nigh attained "man's estate" when Daguerre in France and Talbot in England introduced photography. Well do we remember the wonder with which we gazed on the mysterious plates which were called daguerreotypes, and "dodged" to find the angle at which they were to be held in order that the pictures might be seen. The science on which photography is based is chemistry, but for its practice it is largely dependent on optics. Perhaps nothing except cycling and golf has "caught on" as has photography. And no wonder; for nothing has more contributed to the enjoyment of all ranks and conditions of men. Who shall estimate the sorrow-soothing influence exerted in hundreds of stately homes and thousands of cottage homes by the photographs which are the sole memorials of those whom their Lord loved too well to leave on earth? One must have lived in India to apprehend how the pang of separation from the little ones sent "home" is alleviated by the periodical arrival of the annual group which enables loving eyes to trace the development in form and feature of their loved ones. But no sojourn in far-off lands is needed to tell us to what extent this blessed art contributes to the emollience of our modes of thought, and the counteraction of the savagery which an age of commerce and of war tends to produce.[1] The albums on our tables and the groups on our walls are a potent factor in keeping our best affections—love, gratitude, admiration, reverence—alive in our souls. Of late years photography has been much and increasingly used in education, and we have heard much of its use in the teaching of engineering and of many medical departments, as anatomy and pathology. The mention of these subjects recalls to our mind the latest step, the most marvellous step in the march in which every step is a marvel, the discovery of the Röntgen Rays, and their application to surgical purposes. In our military hospitals in South Africa at this day, hundreds of our heroes are saved the horrid pain and dangerous delay of searching for bullets, and so set out on the path towards recovery with brighter prospects of reaching the goal. Of course, geometry had no part directly in making the discovery; but it was made by that optical science of which geometry is an essential part, and it can only be applied by means of instruments which only the advanced geometer could devise; and the advanced geometer owed his advance to Euclid.

NAVIGATION is perhaps the one of all the applied mathematical sciences in which the mathematical element is most largely predominant. The earth, with its seas on which he sails, and its lands where are his desired havens, are prepared for the sailor's use by geometrical and trigonometrical surveys. His place on the sea from day to day is ascertained by operating, by methods purely geometrical, on data furnished by astronomical observation. His main instruments are the compass, the

[1] "Ingenuas didicisse fideliter artes
Emollit mores, nec sinit esse feros."

sextant, the chronometer, and the *Nautical Almanac.* The first of these is independent of geometry; but its rectification is the part of the geometer, without whose investigations of its variation it would not only be useless, but would even be dangerous in long voyages. We thought of introducing the sextant among optical instruments when speaking of them, but we forbore, because it is far more a simply geometrical than an optical instrument. The chronometer is the perfection of workmanship, which depends largely on the application of mathematical principles. Its balance with its compensation, its springs, the very construction of the teeth of its wheels, all lie within the province of the mathematician; and then its testing and its rating, without which the best that could be hoped from it is simply that it should be useless, can be effected only by astronomical, *i.e.* geometrical, means. Let any one read the chapter headed THE TIME DEPARTMENT in Mr. Maunder's account of the Greenwich Observatory,[1] we can promise that he will be astonished, as we were, to learn the magnitude of the great national work of testing and rating. The computation of the *Nautical Almanac* is a magnificent work of pure mathematics. This simple outfit of four small articles is all that the sailor requires so far as navigation is concerned. Seamanship is another question. The whole matter of setting sails is reducible to geometrical formulæ. But of this we do not speak, because, historically, experience and common sense taught it to fishermen long before mathematical reasoning was brought to bear upon it; and in practice it is very much the same with

[1] *The Royal Observatory: a Glance at its History and Work.* By E. Walter Maunder, F.R.A.S. London, 1900.

the ship captains of our day. We read in old Scottish annals that St. Columba was thought to be possessed of miraculous powers, because he could make two boats sail in opposite directions with the same wind. The grand old saint knew only the truth which is embodied in the proposition of "the parallelogram of forces," yet we do not think that he knew the proposition. Rather, we suspect, he knew by experience, which unconsciously grew into instinct, what that proposition might have taught him, very much as we all spoke in prose long before we had ever got a lesson in English composition, or so much as knew that it was prose we spoke.

That "Britannia rules the waves," that her grand old "flag has braved a thousand years the battle and the breeze," is due, under God, to many causes. But one, and not the least important, is the fact that her sons have not been unheedful of the still small voice of the Alexandrian teacher, who, if he said that there is no royal road to geometry, might have said also, had he been gifted with superhuman foresight, that geometry was destined to be a road to royalty.

As we glory in the old flag to which God in His all-wise providence has granted a millennium of glory, and for which our hope is that it may please Him in His goodness to grant her another millennium whose glory shall be still greater than that of the past, shall it be deemed out of place if, writing a day after we have been anguished by a terrible disaster at our very doors,[1] and by the tidings of others scarcely less

[1] H.M. Revenue Cutter *Active* was wrecked at Granton, in the immediate neighbourhood of Edinburgh, on the 12th November 1901. Twenty out of twenty-three men and officers perished. In the same storm nine lives were lost by the capsizing of a boat at Yarmouth.

terrible all round our coasts, we remind ourselves and our readers of the duty at once and the privilege of constantly and fervently commending to the care and protection of Him who "gathers the winds in His fists, and bindeth the waters in a garment," all those His servants who "go down to the sea in ships, and do their business in the great waters," that He may give to His winds and His waves charge concerning them, and bring them and the vessels in which they sail to their severally desired havens. We suppose that it will ever be the prayer of Britons that their sailor-sons be saved not *from*, but *in* the storm.

GEOGRAPHY AND HYDROGRAPHY.—Our sailors find the latitude and longitude of their place at sea by observations of the sun, the moon, and the stars, and by using their altitudes and angular distances as data for working out, by the help of the *Nautical Almanac* and a table of logarithms, certain problems in Spherical Trigonometry. But what good would the ascertainment of his own position do the sailor, if he had no means of knowing the position also of his destined port, and of the rocks and banks which he has to avoid if he is to reach it? Well, this knowledge is provided for him by charts and maps, the result of surveys conducted with unspeakable labour and scarce calculable cost, and all on the lines of the science to which the *Elements* are the introduction.

MINING.—Though we do not accept without important qualification the maxim that the *weal* of a nation and its *wealth* are identical,—albeit the said maxim seems to rest on incontrovertible etymology,—yet it must be admitted that there is much connection between them. Now, next to our navigation and commerce, our wealth depends on our manufactures, and these mainly on our coal supply, and that upon our mining; and mining-engineering is one of the mixed sciences, and in it geometry is an essential agent. The cutlery of Sheffield, the hardware of Birmingham, the textile fabrics of Leeds and Bradford and our Scottish Border towns, the great ships of the Thames and the Clyde and the Belfast Lough, are all drawn from the dark depths of the coal mine by the Euclid-taught engineer.

But incalculable as is the value of the contribution which our mining operations make to our national wealth, through our manufactures and our traffic by sea and land, it is as nothing in comparison with their contribution to our individual weal. If in the monster furnaces of our factories they produce abundant wealth, on our domestic hearths they make life possible, and provide that life with unspeakable blessing. If from the bountiful hand of our Creator we day by day receive the daily bread without which we could not live, it is prepared for us, as by ministering angels, by the fires on these hearths, fed principally with mineral fuel. Come it in the sumptuous banquet, whose materials are gathered from the lands and seas and rivers of many climes, or in the plain meal to which the healthy hunger induced by labour is sufficient seasoning, or in the scanty crust and insipid cup of aged poverty, the food of peer and peasant and pauper may almost without a figure be said to come from the mine, since it is the coal that converts animal and vegetable substances into food. And as to comfort, what would lordly mansion or yeoman's grange, or peasant's cottage or the "room" of the poor widow be without its fire? Certainly not a home. And how, without "the midnight oil" which now is chiefly mineral, or the midnight gas that is wholly so, could books be written; and how, without the same aid, could they in many cases be read?

CHEMISTRY is not to any large extent a mathematical science, but in its department of crystallography it is purely geometrical. To us this fascinating science is unknown. But to an outsider it appears that the atomic theory is destined to reduce all chemistry into crystallography, since to such an one there seems no reason why an atom of one substance should combine with an atom, or with a definite number of atoms, of another substance, save in respect of the form of these atoms. But if speech ought to be only commensurate with knowledge, we have said on the subject not only enough, but too much, by the full amount of what we have said.

Others may not agree with us,—many do not,—but we are convinced that the material results of mathematical science are not more advantageous than is the mental effect, and chiefly of the Euclidean geometry, in so far as it forms a part of our education. We mean not as keeping up the succession of mathematicians, but as training and developing the intellects of all of us, to be used in all departments of infinitely varied life. But this is a separate branch of our theme, and calls for extensive controversial treatment, on which we are now to enter.

IX

Two questions lie before us. It being proved, as we submit we have proved, that mathematics, and especially the mathematics of Euclid, has exercised a mighty impulse in the formation of our actual state, *Is it necessary for the continuance or further improvement of that state that the study of mathematics be continued?* This is our first question, and it is a momentous one. And the second is like unto it, *To what extent is the mathematics of the future to be on the lines of Euclid?*

It may be safely assumed that no man of even moderate intelligence would advocate that our country should recede from the position which she has attained and holds in the world of science. That position she owes mainly to her mathematicians; to mathematicians she must look for its maintenance. The former of these statements we hold to be indisputable. The latter may be questioned on such grounds as these. Mathematics, it may be argued, has done its work in the establishment of science. What we have to do now is to make known the results of mathematical investigations, and to diffuse a knowledge of these results throughout the community, who are and must be incapable of following the processes which have led to the results. Far be it from us to undervalue this diffusion. We would have all men know the facts of science, and we estimate highly the value of such knowledge. But science will languish and will ultimately die, unless we have, besides the many who should know its facts, the few who are thoroughly acquainted with the processes by which these facts have been ascertained, and are able to use these processes, or to improve upon them, for further investigation and discovery.

It is true that we are in the habit of regarding some of the sciences as perfect. If this be true respecting any of them, it is true of astronomy. We have seen that Lagrange considered it so in his day, and lamented that Newton had left nothing for Laplace and him to do. Yet they and the younger Herschell, and Airy and Arago and Bessel, and a host of others, have had noble work to do, and have done it nobly, in the very department of science which was deemed to be complete. He would betray a sadly unscientific mind who should say of any science at any time that it was complete. Lagrange said it only in a very limited sense that there was nothing left for him to do. Had he believed it in its full sense he would have done nothing, would have attempted nothing. We do not regard the discovery of a new asteroid, or a new planet, or a double star, as usually contributing almost anything to the *science* of astronomy. But far otherwise is it with respect to the discovery of Neptune. That was the result of mathematical reasoning, at once the nicest and most profound; and it has gone far to show that what were deemed defects in the Newtonian theory are in reality the strongest confirmation of its absolute accuracy. Who shall say that there are no other important discoveries to be made?

If any one is tempted to despair of the advance of science, let him think of the marvellous connections between the several sciences which have been established in our day. The spectroscope is a purely optical instrument; "spectroscopic analysis," a branch of purely optical science. It was unknown a few years ago. Now it has taken a high place in the science of chemistry, and has given rise to a new department in the science of astronomy which we venture to designate "chemico-astronomy." It enables us to ascertain what are the constituent elements of a star in the Milky Way, just as our expert analyses a half-pint of milk with the view of detecting an infringement of the "Food and Drugs Act"! If this has been the effect of the spectroscope, which takes cognisance only of the simple and long-known phenomena of refraction, who shall assign limits to the expectations which may reasonably be formed of the future achievements of the polariscope, whose province is the recondite and but recently discovered phenomena of polarisation? It has already begun what promises to be a mighty achievement, by giving rise to "stereo-chemistry," which is yet in its infancy, but for which, as we have been, through the kindness of our friend Dr. Crum Brown, enabled to get such a glimpse of it as our untrained eyes enable us to take, we venture to promise a manhood of might. As it is, the connection established between optics, chemistry, and astronomy, departments of science apparently so widely separated, is of intensest interest and of inestimable importance.

Take another example. The sun spots, their movements and changes of figure, their appearances and disappearances, are a mystery. The variations of magnetic attraction are a mystery. Patient observation shows that there is a close connection betwixt these two sets of phenomena, to the extent that the darkening of an unusually large portion of the sun's disc is always coincident with times of magnetic disturbance. Personally, we have no minute knowledge of this most interesting subject. But it may not be presumptuous to suggest that it seems to point to a mighty extension of the Newtonian law. A proof that gravitation, light, heat, magnetism—therefore electricity, which is already identified with magnetism—are but different phases of one mysterious cosmic influence, would be not so much an extension as it would be a revolution of science. But we are reminded of the salutary maxim—*Ne sutor ultra crepidam.*

We are on ground not quite so strange to us when we refer to the fact that undulation may be held as proved to be the mode of the propagation of light, heat, electricity, and magnetism, as well as sound. This gives an opening to—rather a call for—profoundest mathematics to operate on an immense scale. What if it were to turn out that all these, and gravitation too, are but one force, taking one or other of their forms according to the rapidity of its vibrations or the lengths of its waves! We are not so enthusiastic as to believe that this can be done soon or easily. Many subjects of investigation present themselves. As sound is propagated through one medium, and light through another, is it not possible that light, electricity, and magnetism may have each its appropriate ether, from whose vibrations its peculiar phenomena are produced? Or what else, however marvellous, is not possible? Were some such discovery made, one of its effects would be to bring mankind back to the position which was occupied in old times by the sage of Uz: "Lo, these are parts of His ways; but how little a portion is heard of Him! but the thunder of His power, who can understand?" When we speak of returning to a position of old times, we but repeat in concrete form the principle which Bacon enunciated abstractly: "It is true that a little philosophy"—by which he means what is now called science—"inclineth man's mind to atheism; but depth in philosophy bringeth men's minds about to religion; for while the mind of man looketh upon second causes scattered, it may sometimes rest in them and go no further; but when it beholdeth the chain of them confederate and linked together, it must needs fly to Providence and Deity."

All this is no digression. It is legitimate conjecture as to the continuance of the era in whose beginning Euclid had, unconsciously, so large a part. But mathematicians capable of extending scientific discovery, or even of adequately comprehending and appreciating the discoveries already made, cannot be trained in classes. We may have one Newton in some three centuries, and some lesser but not little lights in each generation. We expect no more. We desire no more. And the greater light, and the great though lesser lights, cannot be selected and trained to shine. True, but by enlightening the community generally, we enhance the probability of the great lights being developed. We presume that there are giants among the pigmies as well as among the Anakim. But the pigmy giant is a pigmy still, as the giant pigmy is still a giant. Now, if we could, by gymnastic or whatever other discipline, increase the stature of the pigmy race, their giants would share the development, and would still overtop their fellows; and when the pigmy race became an Anakim race, the pigmy giant would become an Anak giant, that is, a giant positively, not only comparatively and relatively. To us, then, it seems supremely desirable that mathematics, especially geometry, most especially the Euclidean geometry, should be, more than it is, regarded as an essential part of general education in its secondary stage. We would elevate the many, in order that the few, who will ever overtop the many, may attain a better stature.

But is the education of all our youth to be made subservient to the training of the few—the very few—who are to become mathematicians, with the hope that they shall advance science? No, most emphatically No, if thereby the general education will be, in the interests of the many, and without reference at all to the few, deteriorated. But Yes, if it can be shown that the interests of the many will not suffer; emphatically Yes, if we find reason to believe that what we advocate will be beneficial also to the many. We will not rob the multitudinous Peter in order to pay the exceptional Paul. But we should greatly rejoice if a scheme could be devised by which Paul could be paid while Peter was not robbed but enriched.

X

THIS leads us to what we have looked forward to all along as a most important portion of our task, a consideration of the question, What place geometry —of course, it must only be elementary geometry— should occupy in general education. In this country we have never had, never can have, and ought never to try to have, any such hard-and-fast educational curriculum as they have had for a long time in France, and as they seem to be striving to emulate in the United States. In virtue of this inflexible routine, it has been said that the Minister of Instruction in Paris can tell at any minute of any day how every individual of juvenile France is occupied at that minute! We must have a larger scope for spontaneity here. We will not relinquish our innate freedom—madness, if our neighbours insist on calling our love of freedom so. But we will consent that some method be infused into our madness,—a method not imposed by mechanical statutes, but by enlightened public opinion. And therefore we would strive to contribute, in the measure of our ability, to the enlightenment of that opinion.

Be it understood, then, that for the present we have nothing to do with the formation of a school of mathematics, but with the training of a nation of citizens.

All special education lies outside our present view. We have to do only with the general education, which ought in every case to be preliminary to all professional education, and which, in the case of many males, and of a great proportion of females, is to constitute the whole of their education.

The use of general education is not to fit any man for a specific calling or profession. It is to fit every man to be the best that his nature and abilities admit of, for any profession that he may adopt, or any position that he may have to occupy. If it is to do this it must be by cultivating all his powers, physical, mental, moral, and spiritual. Any educational system which neglects any of these is defective to the extent of its neglect; a perfect system would cultivate them all, each to the fullest degree compatible with the due cultivation of the others. The full attainment of this may never have been reached in any case. It may be impossible. Yet not less than this should ever be aimed at.

One of the most important of human powers is reason. We do not confound this with reasoning, or with what is called the reasoning faculty. But "pure reason"—to use Kant's stereotyped nomenclature—cannot be acted on by the educator directly. He must seek to act on it mainly by the cultivation and exercise of the reasoning faculty. Every subject of study, languages, history, geography, poetry, will aid in the culture of this faculty. But each of them seems to want an element indispensable for completeness, the element of continuity. The logician can teach his pupil that it were an illegitimate process from the ascertained truth that all donkeys are animals, to infer that all animals are donkeys, and can demonstrate to him by means of a device which he calls "the quantification of the predicate" the reason of the illegitimacy, inasmuch as the ascertained truth is that *all* donkeys are *some* animals, and that its converse, therefore, only is that *some* animals are *all* donkeys. The lesson is an important one. But precisely the same lesson will be taught—and, to our thinking, much better taught—by the mathematician when he shows that Euclid was not entitled, and knew that he was not entitled, to infer from the proposition that all equilateral triangles are also equiangular, the apparently converse proposition that all equiangular triangles are also equilateral. The illegitimacy of the inference is all the more impressive because in the case of the triangles the inferred proposition is true, whereas in the case of the donkeys it is manifestly false. Yet both inferences are alike erroneous. It is to be remembered that we are not speaking of technical education. We are not training our scholars to determine questions of asininity on the one hand, or of equilaterality on the other. We are training them to a facility, which should become practically intuitive, of discriminating between the conclusions that may and such as may not be drawn from propositions assumed or admitted or proved in any region of thought. We are convinced that there neither is nor can be any department of study which could supply the means of continuous exercise of the reasoning faculty to any extent approaching that to which they are afforded by geometry. It seems impossible that, with any materials other than geometrical, so long a chain could be forged. And then, if the study is rightly pursued, every link of the chain must be tested. The learner ought to apprehend both the grounds on which the geometer draws certain conclusions from certain premisses, and the reasons which debarred him from reaching his goal by other paths. To go back to the instance of which we made use in our last paragraph,—we cannot think of a better testing question to be put in an examination on formal logic than some such one as this: "*Wherein lies the necessity of Euc.* I. 6?" or, "*Why are we not entitled to infer Euc.* I. 6 *from Euc.* I. 5?"

Hitherto on this point we have stated our view dogmatically, as if it were uncontroverted and incontrovertible. It has, however, been the subject of long and keen—we might say bitter—controversy; and we cannot accomplish our task without prolonging that controversy. The late Sir William Hamilton of Edinburgh, whose philosophical genius it would be presumptuous in us even to applaud, seems to have regarded himself as vested with a special mission to refute the error that there is any good in mathematics as a mental exercise or a means of mental training. In his collected works there is an extensive section under the general title *Education.* The first treatise is this section is in the form of a review of a work [1] by the late Professor Whewell. It enters into the subject at such length, and—it goes without saying—treats it with such ability, that it may be safely assumed that all that *can* be said in disparagement of mathematical study *is* said, and said in the most effective way. If the arguments of this *advocatus diaboli* can be answered, the canonisation of Euclid is secure.

We shall first give a general idea of the character of

[1] *Thoughts on the Study of Mathematics as a part of a Liberal Education.* By the Rev. William Whewell, M.A., Fellow and Tutor of Trinity College, Cambridge. 1835.

the treatise, by quoting the headlines of the pages in the edition before us, which happens to be an American one. These are—*Work reviewed* (headline on 1 page); *Question stated—Mr. Whewell's Ground* (1 page); *Mr. Whewell's Ground untenable* (1 page); *Mathematics not Philosophy* (1 page); *Mathematical not an improving Study* (3 pages); *Reasons why Mathematical Study unimproving* (2 pages); *Mathematics do not conduce to Generalisation* (1 page); *Mathematics not a logical Exercise* (8 pages); *Mathematics induce Credulity* (2 pages); *Mathematics induce Scepticism* (3 pages); *Comparative use of Geometric and Algebraic Study* (2 pages); *True use of Mathematical Study* (1 page); *Cambridge System absurd* (1 page).[1] One does not write himself down a coward who confesses to a measure of trepidation when he goes to the brook in quest of smooth stones wherewith to ward off the charge of such a battalion.

The question as Sir William lays it down is precisely that which we have at present in hand. "The question does not regard the value of mathematical *science*, considered in itself or in its objective results, but the utility of mathematical *study*, that is, in its subjective effects, as an exercise of mind." The subject *as thus laid down* is precisely ours. Perhaps it may be due to the pamphlet under his review, which is an uncompromising defence of what was in those days commonly spoken of as the Cambridge system, that he

[1] It should perhaps be noted, that as, in accordance with usual custom, the headlines are given only on the right-hand page, each one on an average refers to a number of pages double of that on which it is printed. Thus the sum of the numbers given in the text is twenty-seven, while the number of pages to which they refer is fifty-six.

does not confine himself by any means to the treatment of the question as stated by himself, but often strays into the ground which he himself declares not to be his. Though in much we are at one with Professor Whewell, yet we are not so in all; so that some of the shafts levelled against him fly wide of us. These we shall not generally notice. The part of the argument which bears most directly on our position, which we hold in common with Whewell, while he holds other positions which we do not share with him, we give in Sir William's own words—

"Mr. Whewell contrasts mathematics and logic, and endeavours to establish the high and general importance of the former by showing their superiority to the latter as a school of *practical* reasoning. Now admitting, what we are far indeed from doing, that the merits of the two sciences are fully produced and fairly weighed against each other, still the comparison itself is invalid. Logic, by a famous distinction, is divided into theoretical or general logic (χωρὶς πράγματων, *docens*), in so far as it analyses the mere laws of thought; and into practical or special logic (ἐν χρησει, *utens*), in so far as it applies these laws to a certain matter or class of objects. The former is one, and stands in the same common relation to all sciences; the latter is manifold, and stands in proximate relation to this or that particular science, with which it is, in fact, identified. Now, as all matter is either *necessary* or *contingent* (a distinction which may be roughly assumed to coincide with *mathematical* and *non-mathematical*), we have thus, besides one theoretical or general logic, also two practical or special logics in their highest universality and contrast.

"I. Theoretical Logic.

"II. 1. Practical logic as specially applied to *necessary matter = mathematical reasoning.*

"II. 2. Practical logic as specially applied to *contingent matter = philosophy and general reasoning.*

"Now the question which Mr. Whewell proposes to handle is—*What is the best instrument for educating men to a full development of the reasoning faculty?* and his answer to that question is—*Mathematics.* But the reasoning faculty in men, being *in all principally, in most altogether,* occupied upon contingent matter, comprising what Mr. Whewell himself calls 'the most important employments of the human mind,' he was bound articulately to prove, what cannot be presumed, that mathematics (the practical logic of necessary material) cultivates the reasoning faculty for its employment on contingent matter, better than philosophy, etc., the practical logic itself of *contingent matter.* But this he does not attempt."

We shrink from saying it, but we cannot help feeling that this passage would have been considerably improved, in intelligibility if not in accuracy, had the logician condescended to take a lesson from the despised mathematician. It may be very pedantic, but it is sometimes useful, to introduce a process of reasoning by defining the terms that are to be employed in the conduct of it. Now there are two terms in this passage—the two which alone are of any moment—which imperatively demand definition if the passage is to become intelligible. The one is *logic,* the other is *philosophy.* If logic be, as it is commonly understood to be, the mental art by which we distinguish between sound reasoning and unsound, by ascertaining wherein the soundness of the one and the unsoundness of the other consist, by bringing them into contact with certain assumed or ascertained canons, then the division into general and practical, we submit, is vicious. Logic is one and indivisible. The logical principle that a universal positive proposition does not admit of simple conversion is identical, whether it be applied to the proposition *Every Y is X*, or *Every ass is a quadruped*, or *Every triangle has the sum of its angles equal to two right angles*, or *Every Briton is a free man*, or to an infinite multitude of others. We are sorely tempted, but strenuously resist the temptation, to add to this list of unconvertible propositions yet this other, *Every mathematician is a logician!* This logic is one and the same, however diverse the applications. The case is analogous to that of pure and applied mathematics. We call it pure mathematics when we prove that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on its sides. We call it applied mathematics when we calculate the height of a tower from the length of a string which reaches from its top to the ground at a measured distance from its foot. But the mathematics is the same. We may speak, if we like, of the morning sun and the evening sun, of the rising sun and the setting sun, of the eastern sun and the western sun; but it is the same sun, "which is as a bridegroom coming out of his chamber, and rejoiceth as a strong man to run his race; his going forth is from the end of heaven, and his circuit unto the ends of it; and there is nothing hid from the heat thereof." And so it is with logic, and so with mathematics. It will be observed that our contention is not for mathematics as opposed to logic; but for mathematics as a particular application of logic, and as the best form for cultivating the habit of sound reasoning, and developing that habit into an unconscious instinct.

If, as Sir William Hamilton argues, the habit acquired by studying logic in one of its applications cannot be transferred to another, then innumerable applications must be separately studied. The politician, the political economist, the social reformer, the moralist, the preacher, must not only have each a logic of his own, but each must have many. The journalist will need a new one every day. Thus viewed, the most thorough Hamiltonian we ever heard of is a youthful arithmetician, who, having realised the extent to which "multiplication is a vexation" by futile attempts to ascertain the number of apples contained in 73 sacks, each containing 587, sought for additional data by requesting to be informed whether the apples in question were "dessert" or "cooking" apples!

Under the heading, "Mathematical not an improving Study," Sir William overloads his treatise by citing a vast number of men who have expressed that opinion more or less strongly. If his object was to exhibit the vast range of his own reading, he has certainly succeeded. But a great proportion of his authorities only suggest the fable of "*sour grapes*." Of the exceptions his treatment is characteristic. Take Descartes as an instance: "Nay, Descartes, the greatest mathematician of his age, and in spite of his mathematics, also its greatest philosopher, was convinced from his own consciousness that these sciences,[1] however valuable as an instrument of external science,

[1] Probably a misprint for *studies*.

are absolutely pernicious as a means of internal culture."

On this passage it may be noted in passing, that it would have been just as easy for an advocate of mathematical study to describe Descartes as "a great mathematician in spite of his philosophy," as it is for a deprecator of that study to designate him as "a great philosopher in spite of his mathematics"; equally easy, equally unprofitable, and equally in violation of good taste and propriety. The mathematician would have this to say in support of his view of the matter, that the Cartesian *geometry* is at this hour a most potent factor in human thought, while very few living men know aught of the Cartesian *philosophy* beyond its endlessly controverted principle, *Cogito, ergo sum.* Of the testimony itself of Descartes, Sir William gave a brief summary in the *Review* as originally published; but in the republication he gives his own words. As the circumstances of his original publication led Sir William to omit this passage, so the circumstances of ours lead us to think that it will be better to translate it.

"In truth there is nothing more *inane* than to be so occupied about bare numbers and imaginary figures as to seem to be willing to rest in the knowledge of such trifles, and to brood over those superficial demonstrations, which are oftener discovered by accident than by skill, which are addressed rather to the eye and the imagination than to the intellect, until we lose in some manner the habit of using our reason itself. At the same time there is nothing more *intricate* than to let loose, by such a mode of proof, new difficulties wrapped up in confused numbers. But when I afterwards considered how it could be that those first authors of philosophy would admit no one to the study of wisdom who was unskilled in mathematics, as if they deemed this discipline the easiest of all, and most necessary for cultivating and preparing the mind to take up other and greater sciences, I plainly suspected that they were acquainted with some sort of mathematics very different from the ordinary mathematics of our age."

We are not aware that any mathematician would hesitate to endorse all that Descartes says in this passage. It is a strong protest against excessive and exclusive mathematical study; but neither is, nor was meant by its author to be used as, an argument against a moderate and well-balanced study. It is not against the use but the abuse. We are quite willing to admit that to some the study of mathematics is exceedingly fascinating. Such will, of course, be tempted to excessive study. If they cannot or do not resist the temptation, they will become mere mathematicians.[1] If they resist it, as did Descartes, they will either abandon the study or will curtail the extent of their application to

[1] These have their important uses, as we have seen, in the extension and even the conservation of the mathematical sciences. The *general* cultivation of elementary mathematics is essential to the formation of a soil in which alone these can grow; but at present we are occupied with consideration of the value of the study as a factor—and that as one of many—in improving the *general* intellect. In this regard we have to do only with elementary mathematics, for that only can be *generally* studied. Our present standpoint is quite different from that of Professor Whewell. Our thought is of the study of mathematics as an item in secondary education. His rôle was to advocate its use as an important item in university education. It is quite possible, therefore, that some of Hamilton's arguments may tell against Whewell, while they leave us unscathed.

it, and betake themselves to philosophy, as did Descartes, or to ascetic piety, as Pascal, or to politics, or literature, or commerce, or whatever else.

We would point out, though we would not make much of it, that the charge which Descartes brings against excessive mathematical study tells much more strongly against his own method than against Euclid's. To us the whole passage seems to be nothing more than one of these groans which statesmen and literary and artistic men utter in their hours of exhaustion, the substance of all of which may be concentrated into the ejaculation: "*Vanitas vanitatum; vanitas omnia.*"

Partly for the reason that Hamilton's arguments are mainly directed against special study of the higher mathematics,—such as might issue in a Cambridge Senior Wranglership,—while we at present are only advocating general study of the elementary, but mainly because limitation of space forbids, we do not enter on a detailed examination of the arguments employed by the erudite author. But there is one of his arguments which, if it has any force as directed against the higher, must have much more if aimed at the lower; and though it had none as against the former, might possibly have some as against the latter. If it wound Whewell at all it must hurt us more, for it cannot reach him but by passing through us; or it may so spend its force on us, that it shall fall unharmful at his feet. We must not pass this argument without special notice. It is substantially this, that mathematical study is unprofitable on account of its too great easiness. We shall state in Sir William's own words, abbreviating it indeed in the interests of space by leaving out some sentences, guaranteeing that this omission does not detract from the force of the argument—

"Nor is the extreme facility of mathematics any paradox. No one almost, says Cicero, 'seems to have intently applied himself to this science, who did not attain any proficiency he pleased.' 'Mathematics are the study of a sluggish intellect,' says the Helvetian Pliny; and Warburton calls 'the routine of demonstration the easiest device of reason, where much less of the *vigour* than of the *attention* of mind is required to succeed.' . . . This leads us to observe that to minds of any talent mathematics are *only difficult because they are too easy.* Pleasure is the concomitant of the spontaneous and unimpeded energy of a faculty or habit; and pain the reflex either of the compulsion of a power to operation beyond its due limits, whether in continuance or degree, or of the compulsory repression of its spontaneous tendency to action. A study, therefore, will be agreeable in proportion as it affords the conditions of an exercise, spontaneous and unimpeded, to a greater number of more energetic faculties; and irksome in proportion as it constrains either to a too intense or too protracted activity, or to no activity at all. It is by reason of this principle that mathematics are found more peculiarly intolerable by minds endowed with the most varied and vigorous capacities; for such minds are precisely those which the study mulcts of the most numerous and vivid pleasures, and punishes with the largest proportion of intensest pains. It cannot, certainly, be said that the cultivation of these sciences fatigues a faculty by urging it to an activity at any moment too intense; in fact, they are felt as irksome, in a great measure because they do not allow even the one power which they partially occupy its highest healthy exercise. In mathematics we attain our end *non vi sed sæpe cadendo.* But the continued and monotonous attention they necessitate to a long concentrated deduction, each step in the lucid series calling forth, on the same eternal relation [1] and to the same moderate amount, the same simple exertion of reason; this, added to the inertion to which they condemn all the nobler and more pleasurable energies of thought, is what renders mathematics, in themselves the easiest of all rational studies, the most arduous for those very minds to which studies, in themselves most arduous, are easiest. In mathematics dulness is thus elevated into talent, and talent degraded into incapacity."

O Æsop, Æsop! how long hast thou misled a too confiding world! Haste thee now; take to thee tablet and stylus; rewrite, reverse thy unveracious history. The grapes thou tellest of owe not their acidity to their occupying a position above the limits of Reynard's most agile bound, but one below the level of his furthest stoop. To the giraffe no crab-apple is so sour as they; to the mole they may be tolerable; only to the dormouse, doing its period of hybernation, are they sweet. You should have taken a hint, good Master Reynard, from the poet—

> "Wisdom is ofttimes nearer when we stoop
> Than when we soar."

Yes, and grapes too. And you too, ye mathematical professors! give your medals and your prizes no longer

[1] What is the meaning of this clause?

to the greatest proficients in your classes, but to those who have failed to pass, or even to reach, a bridge we wot of, but which must bear no longer the name by which it has been known. Are you sure that you are safe from an action of slander and defamation if you give a student a certificate that he has done well the work of your classes? Truly the Hamiltonian doctrine is a most comforting one to many. We have been somewhat surprised occasionally to detect what seemed a tone of self-satisfaction in a friend's announcement that he has no taste for mathematics. It seems that this want of taste warrants not satisfaction merely, but triumphant jubilation. Some of us who have got over the bridge to which we just referred, are mortified at the thought that we have made so little progress as we have made in mathematical study and attainment. Let us turn our mortification into rejoicing. If our passage of the said bridge excludes us from the highest rank of intellect, the smallness of our proficiency at least secures for us a much higher rank than appertains to Newton and Laplace, and Euler and Lagrange, and to certain men whom we have been accustomed to regard as superior in intellect to ourselves. Yes, a most comforting doctrine is this!

But, seriously, mathematical demonstration is not easy, whether to the brilliant intellect or to the crass. It might be so if its process were as Sir William Hamilton describes it. Of course we are to remember that the process, as it now concerns us, is that of *learning* mathematics from a text-book or a teacher, and therefore we are not entitled to speak of the difficulty of mathematical *invention*. But, according to Sir William, the sole paradigm of a geometrical demonstration would be as follows:[1]—

It is assumed, or has been proved, that every A is B, and every B is C, and every C is X.

Also—that every X is Y, and every Y is Z.

∴ Every A is Z—*Q.E.D.*

Now this may be the whole that is logical in a geometric theorem. But to follow these steps is not to learn the theorem. No doubt it is true that every A is B, and that every B is C. But it is equally true that every A is M and every M is N. Why did the teacher or the author of the text-book make use of the one set of equalities rather than the other? In some cases we shall find that the other will either serve the purpose as well as the one chosen, or at all events will serve it, and then we have an alternative demonstration. In such a case, is the one preferable to the other, or are both equally good? If preferable, wherein does its superiority consist? In brevity? or clearness? or the beauty which belongs to a demonstration, no less than to a rose or a landscape or a human face, although in no case does it admit of analysis? Many such questions as these the student must put to himself. Till he has intelligently answered them, he has not learned the proposition. In mathematical invention, in all that distinguishes the mathematician from the mere learner of mathematics, imagination is the faculty chiefly employed, and this faculty must be cultivated in the process of learning if it is to be

[1] The steps may take different forms, as—

Every A is either B or D, but this particular A is not D, ∴ it is B; or—if A be not Z, then it must be Z^1; but it is not Z^1, ∴ it is Z. But these differ not in substance, but only in form, from that in the text.

available for future use. We do not suppose that any teacher of geometry allows his students after the very earliest stage to confine themselves to receptive work. Every one who is competent requires them to enter on inventive work, though from the nature of the case he can only prescribe to them the task of inventing for themselves what has been invented by others, no matter how long ago. Every text-book contains what—for some to us inexplicable reason—are called "riders"; and every competent teacher constantly prescribes exercises. While the performance of these exercises will be impossible without adequate knowledge of the propositions on which they depend, yet the facility with which they will be performed will not necessarily be in proportion to the accuracy of the knowledge. Of two students who have learnt a proposition with perfect accuracy, you may find that one perceives with apparent intuition certain conclusions deducible from it, while the other may be absolutely incapable of making deductions for himself, though he may be as capable as the other of apprehending them when they are set before him. The one gives promise, and the other does not, of becoming a geometer. Both have the receptive faculty; the one has, the other has not, the *poietic* faculty, which differs only by an *iota* from the *poetic*.

Fairness requires us to add that Sir William admits, though somewhat ungraciously, that mathematical invention is not so absolutely the part of the dullard as is mathematical acquisition. Thus he says: "We are far from meaning hereby to disparage the mathematical genius which invents new methods and formulæ, and new and felicitous applications of the old; but this we assert, that the most ordinary intellect may, by means of these methods and formulæ once invented, reproduce and apply, by an effort nearly mechanical, all that the original genius discovered." We doubt what this assertion means; or rather we doubt whether it has any meaning. That "the most ordinary intellect" can *apprehend* methods invented by others, is just what we have been saying, and perhaps that may be what he means by *reproducing* them. Thus apprehended, they take their place among what Sir William rightly calls "the old." But he has just stated that felicitous application of the old is the work of "mathematical genius" and not of "the most ordinary intellect." One would scarcely expect such reasoning from one whose "talent" has not been "degraded into incapacity" by mathematical study. It need scarcely be said that by this we mean no more than that *Quandoque bonus dormitat Homerus.*

Our argument thus far stands thus. For the advancement, and even for the conservation of mathematical science, and of the numerous sciences which are dependent on mathematics, we need mathematicians of the highest order; and these we cannot get but by their outstripping in the mathematical race fleet runners, nor those but by their outrunning others less fleet but more numerous, and so in the series of numerous terms each more extensive than that which is to spring from it. If we may somewhat clumsily change the figure, we cannot have a lofty pyramid on a contracted base. Without a figure, mathematicians of the highest order can be expected to arise only in a community in which a knowledge of elementary mathematics is widely diffused. If we get such mathematicians, we must not be broken-hearted though they should prove to be not of the highest order in the ranks of statesmen, or of orators, or of lawyers, or of bankers, or of journalists, or of poets, or of mechanics, or in any other ranks but their own, for we need some men, though but few, to be mathematicians simply.

It were idle to imagine that a scheme of education on a national scale is to be constructed or maintained, in the hope that out of it may emerge a number of creditable mathematicians in each generation, a few superior in the course of several generations, and mayhap one supreme in the course of a couple of centuries. It is only from the inclusion of mathematics in all our educational grades that this result can be reasonably expected to flow. But we are not entitled to frame, or even to modify, our educational courses with a view to the production of this result. This is a case in which the democratic principle of the greatest good of the greatest number must be accepted by democrat and oligarch alike.

But we undertook to show, and we submit that we have succeeded in showing, that the mental habitudes which are formed by the study of mathematics are conducive to the power of handling aright the infinitely varied questions which must constantly be handled by every intelligent man. We therefore plead that elementary mathematics should have a place in all secondary education. It is not without significance that even of primary education, mathematics is deemed so essential a part, that arithmetic, its only available department, ranks with the other two *R's* as an element of the most primary.

Primary education must contemplate two classes of learners, those to whom it is to be final as well as first, and those to whom it is in a stricter sense *primary* as preliminary to *secondary*. For both these classes it is deemed, and rightly deemed, essential that instruction in elementary arithmetic be provided. This provision is dictated, we suppose, mainly or exclusively by considerations of immediate utility. The boy that is destined to be a day-labourer must be provided with the means of protecting himself from the dishonesty of a fraudulent employer. The young housewife must be enabled to escape the most intense and most graphically delineated misery of Mrs. Copperfield, who so pathetically sobs out the wail that her "sums will not add up"! To the other class of scholars, who are to follow up their primary with secondary education, instruction in arithmetic is equally essential, since without it they would be unfit to enter on training for any profession, while, like their humbler schoolfellows, they will need it in the business of every day of their future lives. These are certainly imperative considerations. We do not suppose that any account is ever taken of the educative value of arithmetic as a branch of mathematics; and in good truth, unless the teaching of it be precisely the reverse of what we sadly remember, the only account to be taken of it should be to chronicle it as the best means yet devised of training in the art of "how not to do it," and to devise measures whereby its evil influence may be minimised.

As primary education is to its recipients either final or preparatory for secondary, so to some of its recipients secondary education is the terminal stage in the educational curriculum, to others it is an intermediate station on the way to a further terminus. This twofold purpose of the institution happily opposes to a certain extent a tendency to convert into a professional or technical training what ought to be the beginning of a general, or what we shall call for convenience, and without inquiring critically into the accuracy of the term, a liberal education. The temptation seems to have been yielded to only so far, that some institutions, public or comparatively private, allow their pupils, or their guardians for them, the option of selecting for the final term of their course a classical, a commercial, or a military "side." We wish this were not so, and also that such institutions did not advertise classes as specially conducted with a view to preparing for certain professional examinations, that is, in plain Saxon, classes for "cram." Just as in the case of the primary education, so in that of the secondary, the interests of two classes of learners must be consulted, those to whom the secondary is to be the final stage of general education, to be followed only by the professional, and those in whose case a higher or university course is to be interjected between the secondary and the professional. Our contention is that from first to last of the general educational course, primary, secondary, and higher, mathematics ought to form an essential part.

We have already touched on the importance of the study of mathematics as a means of cultivating the reasoning faculty, but we must treat this part of the subject rather more fully than we have yet done. The science of logic—for it is a science—has the cultivation of the reasoning faculty for its end. This end it seeks to accomplish by the investigation of all that constitutes sound reasoning, and exhibiting the particulars in which it differs from unsound. With singular and admirable ingenuity it has collected these particulars into formulæ, and has embodied them in doggerel or nonsense verses, which are often sneered at; but only, we should suppose, by such as can appreciate neither their ingenuity nor their utility.[1] These formulæ constitute rules by which sound reasoning must be conducted, and unsound may be detected. It should be specially noted that these rules are in no sense, and to no extent, arbitrary. They are not restrictions imposed upon the reasoning faculty, but statements of what, from the nature of the case, is incompatible with sound reasoning. Reasoning is not sound or unsound because it is logical or illogical; it is logical or illogical according as it is sound or unsound. In the actual conduct of reasoning, recourse is had to logical rules much more frequently for destructive than for constructive purposes. When, in judging of an argumentation submitted to us orally, or in a newspaper or magazine article, or in a book, we suspect a defect or error in the reasoning, we probably have recourse to formal logic. We reduce the proposition to syllogistic form, and seek to ascertain whether there be any violation of the principles of reasoning. Such violation we are able to detect by application of the rules which embody in a convenient form the principles which logic did not make, but only embodied in technical rules. We inquire whether there is any "undistributed middle," or any "illicit process," etc. It is just as, in speaking

[1] We doubt if so much important instruction on any subject has ever been packed into so small compass as is contained in the lines which some deem so ridiculous, Barbara, Celarent, etc.

or writing our own language, we never think of grammatical rules at all, but may sometimes be led to think of them when, in revising our own writings or in reading those of others, we come upon a sentence of doubtful construction. Now, why is this? It is because those principles which have been embodied by the logicians into rules have been first incorporated into our mental constitution, so that ordinarily our mental processes are coincident with the logical processes. The end of this branch of intellectual education is nought else than this incorporation, so that the perception of the principles of reasoning may be practically intuitive, and the application of them may be instinctive.

This incorporation is best effected—perhaps can only be effected—by continuous conversance with the sound reasoning of others. Such reasoning, of course, we meet with in all literature, didactic or imaginative. And we estimate quite as highly as any Hamiltonian can do, the importance of giving heed to the use of subsidiary literary study. But our contention is that we can nowhere find sound reasoning conducted so extensively and so continuously as it is by the geometricians. Hence we conclude that there is no so favourable field for dialectic *training* as is the study of geometry. And as in this study we come into contact not only with geometry, which is infallible, but with geometricians, who are not quite so, we are not without warning to be on our guard against error, nor without occasional opportunities to detect errors into which they have fallen: let us illustrate this by two examples. Our first is taken from Simson's *Euclid:* Def. xxv. is, "An isosceles triangle is that which has only two sides equal"; Prop. i. 5 is, "The angles at the base of an isosceles triangle are equal"; and the corollary from this proposition is, "Hence every equilateral triangle is also equiangular." Now, an equilateral triangle being one "which has three equal sides," and an isosceles triangle being one which has "*only* two equal sides," it is evident that what has been proved of the isosceles does not necessarily follow regarding the equilateral. If we take the small word *only* out of Definition xxv., then the equilateral is a species of the isosceles, and the corollary is all right. We have before us what is probably the latest edition of the *Elements*[1] published in this country. In cursorily looking through this up-to-date edition, which we found to be in many respects a very good one, we came upon a new definition of external and internal contact of circles. When one circle touches another, and lies wholly within it, these circles are usually said to touch each other internally. The edition before us states that in this case the smaller circle touches the larger internally, but the larger touches the smaller externally. Then there can be no such thing as two circles touching *each other* externally (p. 164). Now we have no fault to find with this new definition of internal and external contact. It is, no doubt, more strictly accurate than the usual definition of internal contact, which in substance is that two circles touch one another internally when they have one point

[1] Euclid's *Elements of Geometry*, Books I.–IV., VI., and XI., edited for the Use of Schools by Charles Smith, M.A., Master of Sydney Sussex College, Cambridge, and Sophie Bryant, D.Sc., Head Mistress of the North London Collegiate School for Girls. London : Macmillan, 1901.

in common, and one lies wholly within the other. What we have at present to do with is that the new definition makes it impossible that two circles can touch each other internally. But, on turning to page 182, we read: "Thus, if two circles touch each other internally, every point on the smaller circle except the point of contact, is within the larger." Probably the change in the definition was made by one of the editors, and was not adverted to by the other in dealing with the proposition regarding Contact. "Collaboration" is said to have wrought well in the hands of Messrs. Besant and Rice in the manufacture of novels. We have recently learned that it was found somewhat inconvenient by Lord Kelvin and Professor Tait, "T. & T." as they were wittily designated. It requires to be used with very scrupulous care in the composition of a geometrical treatise. It may be presumed that every student before reaching Book III. has so imbibed the principles of reasoning that he will detect so flagrant an error in the text-book for himself. Certainly every teacher will use it much as temperance lecturers among the Spartans are said to have used the unhappy Helots.[1]

We are not forgetful of the fact that in order to find the construction of an argument two things are necessary, the collection of properly ascertained premisses and the deduction from these of sound conclusions. It may be admitted that, for the ascertainment of truth, the former is quite as essential as the latter, while it is with the latter alone that logic

[1] It occurs to us to say, further, that Leslie gives a definition of a "crooked line" which would justify the assertion that every polygon is bounded by a single line!

has to do. For the ascertainment of premisses, or for the selection from any available premisses of those to be used in the conduct of a particular argument, no rules can be given. Aptitude must be acquired by assiduous attention to the practice of the greatest masters of reason, as it is said that Lord Brougham, by way of preparing himself for his greatest speech, read the *De Corona* right through, and some parts of it many times. Now it cannot be denied that the geometer has a much narrower range for the collection of his premisses than has the literary or the political, the ethical or the theological writer. So far, then, he is, as a pattern to the student, at a disadvantage as compared with these others. But while this is admitted, it must be borne in mind that the geometer, too, in mustering his argumentative force, has to choose between available premisses. He may not have to reduce a host of two and thirty thousand to three hundred, but he must often determine whether of two heroes is the fitter for his present use. For example, the most enthusiastic admirer of Euclid—an epithet to which we humbly aspire—will not deny that it is at least an open question whether even he might not have done better than he has in one or two instances, notably in his treatment of parallel lines.

Viewing geometrical study as a drill for the development of those intellectual powers which are to be used in the actual warfare of ordinary life in whatever of its branches, we have to admit that its range is limited, and that this limitation brings its usefulness below an ideal level. But we maintain that its utility is greater than that of any rival system that has been proposed. The military recruit is trained to marching and to the use of his arms, not because in actual service he will ever have to march in the same way in which he marches on the training-ground, or because on the battlefield he will have to swing dumb-bells or Indian clubs, or will have to repeat in man-to-man encounters the precise cuts and thrusts and parries which he practises with his foil; but that every muscle may be developed to its fullest capacity and knit to metallic firmness, and that muscle and eye and nerve and brain may be trained to instantaneous and almost unconscious obedience to the word of command. If the glory of the victory of Waterloo appertained in any measure to the "professionals" who trained our officers on the cricket and football fields, in no less measure did it belong to that not too popular officer the drill-sergeant, who trained both officers and men by most wearisome lessons in positions, and marking time, and stepping off, and to "the right about face," and all the rest of it. While Euclid, as intellectual drill-sergeant, might be all the more efficient if he could put his recruits through a more extensive range of exercises, we cannot admit that any fault is to be found with him on the ground that his exercises are not those that are to be practised in actual field service. Their utility to all is so great as it is, just because they are not special to any. This is essentially the difference between a liberal and a merely professional education. The proper product of the one is *man*, of the other, *craftsman*. And the craftsman will be a noble product if the professional educator has the liberally educated man as his subject. An enemy deemed it the most crushing reproach he could cast upon us that we are "a nation of shopkeepers"; we regard it as matter of boasting to be a nation of "merchant princes." The difference between the merchant prince and the pedlar, between the statesman and the political hack, between the jurist and the pettifogger, between the physician and the quack, between the preacher and the twaddler, is to a very great degree—we might say, in the main—moral and spiritual. But it is also in no small measure intellectual; and in so far as it is, it depends mainly on the having or not having a liberal education. The lack of a liberal education *anterior* to professional training is, in some cases, compensated by subsequent processes of self-culture. And it were easy to name notable and noble instances. But these cases are exceptional, and must ever be rare. Hence the necessity of providing a liberal education for all who are capable of profiting by it. And it has been our part to show that mathematical training should be a constituent element of such education.

We have spoken of geometrical study as better fitted than any other to exercise, and by exercising to develop, the faculty of accurate reasoning. There is another faculty, as important as any other in the use of the mind, to whatever subject it may have to be applied, and whose cultivation is therefore to be constantly kept in view in a course of mental training. That faculty is attention. The use of this faculty is, of course, necessary in all learning; but none requires it so much as—certainly none more than—geometrical learning. Not only must the connection be apprehended between step and step in each proposition, but the connection also between the successive propositions. In order to this apprehension, every one knows that assiduous attention is imperatively required. The attention must be close and constant; but if the study be judiciously conducted, it will not be painful. Even Sir William Hamilton seems to admit this. We have referred to his opinion that geometrical study is too easy for minds of a high order, and is only fit for those of an inferior class. Well! be it so. It is with just such that, in all but exceedingly few cases, we have to do. Be it that men of consummate genius can afford to dispense with mathematical studies. The number of students will not be materially diminished by their withdrawal. It were, we think, absurd to argue that attention to the steps of proof in a geometrical proposition, and to the steps of the sequence of the propositions in a geometrical system, will not enable us to attend to the steps of an argument on a non-geometrical subject. As well might it be maintained that the power of calculating the price of apples will not avail us in estimating the price of pears. True, a knowledge of the *qualities* of apples will not enable us to judge of the *qualities* of pears. That must be part of the *professional* education of the fruiterer.

The statement that human nature is the same in all ages is not quite so original as it is true. But it is true also—and this statement is as little original as the other—that this human nature is acted upon by infinitely varied influences, and therefore is very variously developed in different ages and under different circumstances, while yet its essential identity is maintained. Equally true and equally lacking originality, is the further statement that each development has advantages and defects peculiar, if not in their kind, at least in their degree, to itself. A natural inference from this is that the education of each age should be so framed as to promote, so far as possible, the advantages, and to counteract, so far as possible, the defects, of the peculiar development belonging to the age. To enumerate the special advantages and special defects of the intellectual development of our age, were a task as far beyond our capacity as it were beyond the bounds of our allotted space. All that we can attempt is to show that mathematical study is fitted to correct some of the most prominent faults of the prevalent mental habits of our day.

Mathematics sets out with, and proceeds throughout on, the postulate that truth, objective truth, exists, that it should be sought for, that in many cases it may be ascertained, while in many more the scarcely less important ascertainment of what is *not* truth may be reached. Next to being able to say, "I know that *this* is true," is the ability to say, "I have not discovered the truth, but I have learned that *this* is false." Now we are prepared to maintain that the very opposite of this postulate pervades much of what is most popular in the literature, the science, the philosophy, the politics, the economics, the ethics, and—what we feel most painfully of all, though this may not be the fittest place for the expression of our feeling—the theology of our day, and the religion based on it.

It will not be denied that the most popular literature of our time is that of the magazine and the novel. Thousands have far more extensive acquaintance with these departments of literature than we can pretend to. But our knowledge of them is sufficient to warrant our forming and expressing a judgment as to their general character. That judgment, deliberately and reluctantly come to, is as we have stated it. The cheaper magazines, consisting mainly of "short stories" and "serials," or rather of an endless repetition of the same story with varied details, do not generally profess to serve any higher purpose than amusement; and this is a perfectly legitimate, and not unimportant, purpose. But there pervades all these stories, or the one story in its multitudinous forms, the assumption that ethical and religious truth—with intellectual they do not deal—is not objective, but subjective and relative. We have no wish to deny their superiority to the literary school which Lord Campbell's Act has suppressed, and to the class represented by the "penny dreadful" of some years back. But does their superiority to the latter consist in much more than the substitution of baseless sentiment for bluster and brutality, the worship of Aphrodite for that of Ares? There are magazines of a higher class than this, which discuss literary and historical and political questions, and, so far as our acquaintance with them goes, they are sufficiently dogmatic; and that may seem to be the opposite of the quality which we are reprehending. But their dogmatism seems to be generally that of assertion rather than that of conviction. At all events, they fall far short of the others in respect of popularity. The larger magazines, with one or two exceptions, are professedly without principle, in so far that their pages are open to articles advocating opposite views on all questions of interest. Now it is unquestionable that on most questions honest inquirers may reach different and even opposite conclusions, and the maxim, *Audi alteram partem* is altogether a salutary one. But is not the fact that this can only be effected by allowing opposing partisans to produce their several opinions side by side, an indication that the leaders of public opinion are pleaders rather than judges?

A similar tone of uncertainty or indifference to truth seems to characterise much of our less ephemeral literature, or what its authors expect to be less ephemeral. Our history is lost in a haze of "historical criticism." That its object is the discovery of truth we doubt not; that it has sometimes been successful in dissipating time-honoured delusions, and to find new theories to be more or less probable, we frankly admit. But it seems to us that it generally sets about its inquiries with a desire—latent probably in the minds of the critics—to produce a twofold harvest of uncertainties, to render uncertain what had not been heretofore doubted, and to introduce new theories, for which the most that can be claimed is that they are plausible, and may probably be true. We freely admit that the too conservative mind may apply too widely the maxim that "what is true is not new, and what is new is not true." But there is another class of minds to whom novelty suggests at least a presumption of truth, and such minds set about the investigation of historical truth with a strong bias in favour of destructive results of criticism. We are not going to discuss any political question. But we do not think we misrepresent the politics of our time, when we say that opportunism, expediency, and partisanship enter largely into their composition.[1] Now we do not mean to deny that the most

[1] On the morning after this was written, we read in the *Scotsman* the following statement by one of the leaders of our political parties. It is in answer to a question put to him at a meeting of his constituents :—

"When there is time for it, and when Parliament is in the humour earnest politician must often be compelled to take the best that he can get, although he would fain have what he holds to be better. Moreover, we believe that in a free State there will always be political parties, and that party-government is well-nigh essential to the right administration of such a State as ours. But politics are in evil case when expediency takes the place of right.

But it is in the department of science that the evil works with most deadly effect. We can hardly believe that any man of a truly scientific spirit can follow the course of modern science without forming the judgment that its methods are in some respects vicious, and many of its conclusions only plausible. To Darwin, who must be regarded as the founder of what is distinctive in modern science, should be frankly awarded the high commendation of patient, honest, and judicious observation of facts and phenomena. A measure of the same commendation is due to some of his followers. The great principle of the modern science is the principle or law of Evolution. This is first propounded as "a working hypothesis." Thus used, it is found to account for certain phenomena, and it is asserted that because it accounts for them, and no other principle does, therefore it is true. Now this seems to bear a considerable resemblance to a well-known and universally accepted method of geometrical proof. But the resemblance is only apparent. The geometer proves that of all *possible* hypotheses, all

and the country is in the humour for it, I am perfectly willing to see a . . . Bill introduced."

We do not know whether the leader of the opposite party could give a much better reason for *not* having introduced the measure in question.

but one lead to consequences inconsistent with demonstrated or self-evident truths, and concludes that that one is true. The modern scientist finds that of all the principles *which he knows*, only one is consistent with observed phenomena, and concludes that that one is true. But he overlooks the fact that he knows neither all the phenomena nor all the relevant principles; that there may be phenomena unknown to him which his principle would fail to solve, and there may be a principle unknown to him which would account alike for the phenomena which he knows and for those hitherto unobserved. Precisely thus might Ptolemy, or any Ptolemaist before Newton, have *proved* the Ptolemaic system, and till a much later time might an optician have *proved* the emanatory theory of light.

If the evolutionary theory be inept in the region of material science, it is suicidal in the science of mind. Let us look at the matter. The principle of evolution, as we understand it, may be stated thus. There is a tendency in all objects in material nature to rise in the scale of being. Thus the inorganic is transformed into the organic; as inorganic earth, and the inorganic juices of the soil and the gases of the atmosphere are by natural processes transmuted into organic vegetables. The vegetable is by the process of digestion changed into animal substance, and that by the same process into higher orders of animals, culminating in man. Now, what is the material result of this process? In time the whole inorganic world must become organised, the whole vegetable world animal, and the whole animal world human. The inorganic, the vegetable, and the lower animal must cease to exist, and without these the human cannot subsist. Now, how is this catastrophe to be averted? Why has it not occurred ere now, since the assumption of unlimited time is an essential element of the hypothesis? The power that preserves the races in being is *death.* By means of it the organic, whether in its vegetable, animal, or human stage, is disorganised, and so the materials are provided for an ever-beginning, ever-recurring cycle of ascent. This exposition of evolution—though we are not aware that any of its advocates would care to expound it thus—bears a remarkable analogy to another natural phenomenon. Evaporation is evolution. The ocean, with its "dark unfathomed caves" and "the innumerable laughter of its billows," is by a slow but certain process evolved into vapour; the vapour takes the form of clouds, and so at last there should be no more ocean, consequently no more vapour, and by further consequence no more cloud. But, lo, the cloud breaks in rain, and as the rain descends the bow appears in the cloud—the bow of promise, the bow of covenant; the waters through the channels of brooks and rivers return to the ocean whence they came, and the blessed process goes on unceasingly. But the hypothesis is wholly inapplicable to non-material nature. We have all that is necessary for the rise, and nothing to take the place of the kindly death which saves the material world from extinction, and the genial rain which keeps the ocean in being. Begin with your protoplasm; ascend through how many steps you will, till there be evolved the instinct of the ant and the bee; let that instinct in the course of countless generations become the rudimentary reason of the dog and the elephant. Evolve the highest animal instinct into the lowest human, and the lowest human into the highest, and what have you done? You have sucked all the element of intelligence out of your protoplasm and your bee and your elephant and your Hottentot and your men of lower intellect, and have left but one inconceivably monstrous intellect, monarch indeed of all he surveys, that *all* being a mindless universe! Be it not said that there may be a way unknown to us whereby intellect or its seed is restored to the protoplasm whence it sprang, and so start afresh on its long ascent. On the evolutionist lies the *onus* of proving that there *is* such a way, else his hypothesis is not "a working hypothesis" in this department.

We are not afraid of evolution. We adore Jehovah as our Creator, although we know that many ancestors have intervened betwixt us and that creative act by which Adam became a living soul. And we should adore Him as our Creator still, were it proved that the creative art consisted in the informing the protoplasm called into being in long preceding æons with the inevitable necessity of evolving Adam out of itself in due time. Between evolution and theism there is no incompatibility. When evolution is proved, it will be time enough to consider whether it is consistent with Christianity.

As the inscription over the gate of the *Academy* or school of Plato was (or perhaps was not, see p. 6) a prohibition to the non-geometrician to enter, so that over the *Museum* or school and library of Alexandria was ψυχῆς ἰατρεῖον, the dispensary, or, in transatlantic phrase, the drug-store, of the soul—

"And as Æneas . . .
Did from the flames of Troy upon his shoulders
The old Anchises bear";—

so from the flames of the Museum store did some stalwart Arab warrior, not on his shoulders, but in his pocket, or what in the uniform of Omar's host corresponded to a pocket, bear a phial labelled *The Elements*, with what results we have partly seen.

An English novelist of our younger days turned the idea of the drug-store to good account. Thus Lord Lytton ended a long, ingenious, and amusing chapter on the "Hygienic Chemistry of Books":—

"But, continued my father, more gravely, when some one sorrow, that is yet reparable, gets hold of your mind like a monomania, when you think, because Heaven has denied you this or that, on which you had set your heart, that all your life must be a blank, oh! then diet yourself well on biography, the biography of good and great men, see how little a space one sorrow really makes in life, see scarce a page, perhaps, given to some grief similar to your own; and how triumphantly the life sails on beyond it. You thought the wing was broken. Tut! tut! it was but a bruised feather. See what life leaves behind it when all is done! a summary of positive facts far out of the reach of sorrow and suffering, linking themselves together with the being of the world. Yes, biography is the medium here. Roland, you said you would try my prescriptiom. Here it is. And my father took up a book and reached it to the captain. My uncle looked over it. It was the LIFE OF THE REVEREND ROBERT HALL." — *The Caxtons*, by the Right Hon. Lord Lytton.

The disease for which the good Mr. Caxton prescribed so judiciously was of the heart and the feelings. That with which we have now to deal is of the head and the intellect. It is an epidemic, and we have little hope of the cure of those on whom it has laid hold. Our treatment is of the kind which physicians would call prophylactic. As they strive to "stamp out" smallpox by vaccination, so would we seek to impregnate the mental systems of our youth with the belief in truth and the love of it which are embodied in geometrical study. Thus would we render them "immune" from the attacks of cynical indifference and cold agnosticism and death-dealing materialism.

Often when in private conversation with our friends, and when we read the leading articles and the "Letters to the Editor" in our newspapers, and when we read or hear the speeches of our politicians in Parliament or on platform, yea, sometimes when we hear the noblest of themes expounded from the pulpit, we are tempted to wish that writer and speaker and preacher had been inoculated with the Euclidean lymph. The temptation we try to resist and overcome by the consideration that our reasoning may seem to them as defective as does theirs to us.

But, after all, the reasoning faculty is not the highest or the best portion of man. The head is important, but of vastly less importance than the heart. We know no reason why the purification of the heart and its affections should not go on simultaneously with the development of the intellect and the rectification of its processes; nor do we deny that there is an action and reaction between them. Still it is certain that the connection between them is not such but that the rectification of either may be effected to a very great extent, while that the other is effected only in a very moderate degree. Amongst the grandest intellects we have had some whose tempers and whose lives showed no purity or nobleness of heart. We trust these are exceptional cases. Happily the converse cases are not exceptional. We have had, and, thanks be to God! we do have, thousands and tens of thousands of men and women, ay, and of young men and maidens, with very moderately cultivated intellects, but with hearts aflame with love of God and love of man, and lives ennobled by the constant practice of every virtue and the exercise of every grace. Yet is the formation of this character closely connected with the apprehension of truth. Its generation and development is the fulfilment of the great Master's aspiration for His disciples, "Sanctify them by Thy truth. Thy word is truth."

It is not without interest to reflect that the diffusion of this truth through the world was very closely connected with this same Alexandrian library. (We might continue the metaphor of the pharmacy, but prefer returning to the literality of the library.) The same Ptolemy who brought Euclid to Alexandria, employed seventy scribes to translate the Jewish Scriptures into Greek, apparently for no higher end than that his library might contain all the classical books of the world. By means of this Septuagint version the knowledge of God's truth was kept alive among the Hellenist Jews, who in the time of Christ were probably not less numerous, and were certainly far more intelligent and more widely influential, than their brethren in Palestine. These Hellenists alone are mentioned as the subjects of the marvellously mighty Pentecostal movement. They continued for a time under the apostles' teaching, and it is of them that it is said that they went everywhere preaching the word. And the word which they taught they found in Ptolemy's version. It is certainly not impossible — perhaps not even very improbable — that the first mathematical teacher in the Museum, as he rested in the library between his hours of teaching, may have occasionally taken up this strange book in his own language. We can imagine how he would be attracted by the enunciations of so many propositions in the writings of Solomon, not less terse nor less demonstrable than his own. Just as probable is it that Ptolemy, an astronomical teacher in the same Museum, may have read in the same way the rapturous exclamations of Solomon's father, "When I consider Thy heavens, the work of Thy fingers, the moon and the stars which Thou hast ordained; what is man that Thou art mindful of him, and the son of man that Thou visitest him?" "Praise ye Him, sun and moon; praise Him, all ye stars of light. . . . Let them praise the name of the Lord; for He commanded, and they were created. He hath also stablished them for ever and ever; He hath made a decree which shall not pass."

This is mere conjecture of what may possibly have been. But it is matter of certainty that two theological teachers in this same Museum, Origen and Clement, made this book the subject of their daily teaching and of their endless pondering, and by such pondering and such teaching did much to bring the truth of God into contact with the souls to which that truth was the appointed instrument of their sanctification. In view of all this, may not King Ptolemy himself, "albeit he meant not" the results of what he did, be ranked amongst those whom the Supreme Ruler has deigned to constitute epoch-making men?

As a specimen of the application of the evolutionary system to historical and literary criticism, we quote a paragraph from a book published a few days ago. We select it, not as being better or worse than a great deal of what we have had occasion to read, but simply as being the latest that we have met. The paragraph is entitled, THE EVOLUTION THEORY IN BIBLICAL CRITICISM, and is as follows:—

"Now in ritual, as in everything else, the more developed must be later than the less developed, out of which, on the principle of evolution, it has gradually grown. The progress of Biblical Criticism, especially in recent years, has really been due to the application of the Evolution theory to the problems of Israel's development. The effects of Darwinism have been by no means confined to the realm of Science. There are a thousand and one facts to prove that Jewish ritual did change from age to age, but it will be sufficient to compare Ezek. xlvi. 11 with Num. xxvii. 11–14, both of which passages treat of the sacrifices offered on the regular feast-days. In Ezekiel the very same cereal offering is prescribed for a bullock as for a ram, namely, an ephah of fine flour, while for a lamb each worshipper has to give a cereal offering according to his means. In Numbers, however, the prescribed cereal offering for a bullock is $\frac{5}{10}$ of an ephah, for a ram $\frac{2}{10}$, and for a lamb $\frac{1}{10}$. Furthermore, while no mention is made in Ezekiel of any accompanying drink offering, according to Numbers the drink offering presented with each bullock was ½ hin of wine, with each ram ⅓, and with each lamb ¼ of a hin. The ritual in Numbers is undeniably the more developed. Consequently it must be the later. The further conclusion is therefore inevitable, that the Book of Numbers, which prescribes the later ritual, must have been written after Ezekiel's time. The conclusion, however, could not possibly occur to any one who believed in the Mosaic authorship of the Pentateuch. Hence, accomplished casuists though the Jewish Rabbis were, they could not but acknowledge these discrepancies between Ezekiel and the recognised lawbook of their nation. They were therefore at their wits' end to reconcile them."

In commenting on this extract, it is difficult to know where to begin. It will probably be more difficult to know where to end. The fulness of edible and nutrient matter proverbially ascribed to the egg is, to our thinking, mere emptiness in comparison with the repletion of this paragraph with absurdity. In the first place, the chapter in Ezekiel referred to does not contain a description of any ritual existent in the prophet's time or any previous time, nor any prescription of a ritual to be observed in times then at hand, but an ideal ritual to be observed in some future time, a time which, some 2500 years after the prophet's day, has not yet arrived. The legislative prescriptions in Numbers required to be definite and precise. Such precision would have been wholly out of place in the record of a vision. Then, is it so that "development" or amplification is the process through which a system of ritual or whatever else must necessarily pass? Is there no such thing as condensation or simplification? In the paragraph quoted it is said, "The ritual in Numbers is undeniably the more developed, consequently it must be the later." Now, if we should say, "The ritual in Ezekiel is undeniably the more simplified, consequently it must be the later," the two consequences would be equally legitimate, since each would be absolutely illegitimate as consequences. The Ptolemaic astronomy, with its cycles and epicycles, necessitated by its geocentric basis, is "undeniably more developed" than the Copernican and especially the Newtonian astronomy with its heliocentric basis, "consequently" the *Almagest* is later than the *Principia.* A Roman physician would have prescribed for our author a course of Hellebore. Our recipe is Euclid, to be taken undiluted!

For reasons which it is unnecessary to state, we have referred to this as a specimen of historical and literary, rather than, as the author describes it, of Biblical Criticism. But the matter becomes awfully serious when it is considered that that criticism professes to set aside a history and a literature with which are bound up the faith and the hope of millions of our race.

Essentially the same method of slipshod argument —reasoning it cannot with any propriety be called— finds a large place in our political and social discussions. It may be that in these departments, especially in the latter, the results are not so disastrous, because the conclusions are frequently sound, being formed by intuition and common sense, however faulty the reasonings by which they are supported,—very much as we have seen that Simson's corollary deduced from Euc. I. 5 is absolutely true, although the deduction of it *by him* from Prop. 5 is absolutely false. But then, unhappily, error has ever its own Nemesis. Sound conclusions may be accepted by some who do not detect the unsoundness of the processes by which they are reached; and this is well, because the conclusions, and not the processes, are practically important. But then others, perceiving the fallacy of the processes, will almost certainly be led to set aside the conclusions. As in morals so in intellectuals, a good end cannot justify wrong means.

In advocating geometrical study as a corrective of prevalent false reasoning, some may think that we state its merits extravagantly and too exclusively. Now we do not mean to assert that the habit of sound reasoning may not be formed, in greater or less degree, by well-directed study in other departments. But our position is that there is no department of study so available for the accomplishment of this end as is geometry. But it must not go alone. The term *a mere mathematician* has become a term of reproach, and deservedly so. In our next section we shall have to do with the part to be assigned to geometry, in comparison with other branches of mathematics, in training the *mathematician.* At present our concern is with the part of geometry, in comparison with non-mathematical studies, in training the *man.* In the one case it is paramount; in the other we have attempted to show that it is highly important as one of many branches of important studies. It is desirable that we have *mathematicians;* it is indispensable that we have *men.* We have been somewhat unsparing, it may be, in setting forth the evil of the defect of mathematical training. We have no wish to blink the fact that there is an evil also in its excess, with the explanation only that what would be excess to many is not excess to some. We have known some, happily living among ourselves, who carry loads of mathematics which would crush the manhood of ordinary men. They not only "carry their honours meekly," but their mathematics most manfully. But others have staggered and stumbled under a smaller load. The first Napoleon displayed characteristic sagacity when he excused himself for not putting Laplace into high administrative position, on the ground that he carried the method of infinitely divisibles into everything. We have given an instance of the defect in the department of Biblical Criticism. We shall give an instance of the excess in the same department. In a book to which we have constantly referred, and from which we have received invaluable aid in the composition of this volume, we read the following sentence: "A small piece of evidence which tends to show that the Jews had not paid much attention to it (geometry), is to be found in the mistake made in their sacred books, where it it is stated that the circumference of a circle is three times its diameter." The reference is to 1 Kings vii. 23, and the corresponding passage 2 Chron. iv. 2. The former passage reads thus: "And he made a molten sea, ten cubits from the one brim to the other . . . and a line of thirty cubits did compass it about." Now we do not say that the Tyrian artist who made the molten sea knew precisely the proportion between the diameter of a circle and its circumference; very likely he did not. But we should like to ask Mr. Ball whether he, not *specifying* for the brazier's use the dimensions of a bath to be made, but *describing* in a book for popular use the aspect of a bath already made, would have changed the last clause of the quoted sentence into, "a line of 31·4159, etc.

cubits did compass it about." He would have seen that this statement, save in respect of the indefinite *etc.*, was inaccurate as really, though not to the same extent, as the other, and that no statement would have been really accurate except "a line of 10 π cubits did compass it about"! Would he have made either of these statements? Assuredly not. It is so probable as to be virtually certain that the Jews had not estimated the value of π with any approach to the measure of accuracy of modern approximations. Without decimal notation and decimal arithmetic they could not. But none the less is the inference that "the Jews had not paid much attention to geometry" illegitimately drawn from the fact that a historian makes use of somewhat broad approximations on a matter which even now could only be stated approximately, and where any closer approximation than that employed would even now be the perfection of finical pedantry.

XI

SUCH of our readers as accept the definition of a straight line as the shortest distance between two points, will, we fear, have difficulty in conceding to us that it has been by such a line that we have reached our present position. We admit that our course has been somewhat devious; yet we have pretty steadily held on our way, and have now to enter on the "last departure" of our allotted voyage. The question that lies before us is as to the future of Mathematical Study. Is Euclid to be deposed from the throne which he has occupied so long and so worthily? Or is the time come for his retiral from that position of distinguished honour and noble work?

In order to answer these questions aright, we must keep steadily in view the twofold use which we have regarded mathematical study as serving, according as it is viewed as a mental discipline, or as the means of acquiring important and useful knowledge. We frankly admit that, viewed in the latter aspect, the Euclidean geometry must take second place to the Cartesian in the technical or professional education of the mathematician. As an instrument of scientific investigation and discovery, the modern analysis is in ordinary hands far more potent than the ancient. While it is scarcely possible to conceive that Newton could have done more by the former than he actually did by the latter, we are not entitled to say that what he so nobly did, he might not have done still better. This it were difficult to believe. We have his own strong testimony that *he* believed the contrary. But certainly what he did he would have done with much more ease to himself. And then it is to be considered that it is not certain that Newton did not make more or less use of the modern analysis in his investigations, while in his teaching he rigidly confines himself to the ancient. Some have supposed that he did so; but we are not aware that they have any ground for the supposition beyond the difficulty of conceiving the possibility of *any* man's accomplishing such an end by such means. But it is to be borne in mind that Newton was not *any* man. As there may have been "mute inglorious Miltons," so there may have been mute inglorious Newtons; but in all the ages there has been but one vocal and glorious Newton; and none can tell whether there shall ever be another.

In forecasting the future, we must bear in mind that in Newton's day the Euclidean geometry was mature, while the Cartesian was in its infancy—*animosus infans,* indeed, but quite immature. Since those days the former has indeed made progress, but, as might have been expected, the latter has made much more. The future successor of Newton, then, if he shall ever have a successor, will have but a slightly better instrument than Newton had; while the follower of Laplace will have ready to his hand a much better than was available to Laplace. The modern analysis will then—it cannot be doubted—be the staple of the modern mathematics as an instrument of scientific investigation. But it is to be earnestly hoped that the modern geometers will not cast aside the old love, however they may be under the paramount power of the new. There is no incompatibility between the two. Rather the new will gain in potency as the old is cultivated. It is not wholly with us a matter of theory, albeit our practice is little worth mention; that the use of the new is greatly enhanced when it is brought constantly into contact with the old, and translated into its language. Thus, and only thus, the student is able to estimate the progress which he has made, and to ascertain precisely the position which at any time he occupies. The modern analyst has as much need as any other of the Platonic caution against the neglect of geometry.

While it is freely admitted, then, that in the technical education of the mathematician of the future, and of all who are to be engaged professionally or otherwise in the Sciences of applied Mathematics, the non-Euclidean geometry must occupy a large place, we trust it shall never hold an exclusive one. The non-Euclidean is safe only in the hand of him who has drunk in the spirit of the Euclidean. The former is the motive power, potent and irresistible; the latter is the guiding principle, potent also and salutary. "Behold also the ships, which though they be so great, and are driven of fierce winds, yet are they turned about with a small helm, whithersoever the governor listeth."

Briefly, then, the matter stands thus. In all our primary schools we would have arithmetic, the only branch of mathematics that can be taught in these schools, made an indispensable subject of teaching; and we would take steps for having it well taught, much better taught than it usually was in the early days of men still living. We suppose that already there is a great improvement; but in a matter of such importance every present improvement should be only an occasion for earnest inquiry whether further improvement be not possible.

In our secondary schools we would have the six books of the *Elements* thoroughly taught, and numerous exercises prescribed. We would also have algebra taught, but only as an extension of arithmetic. We would also treat plane trigonometry (geometrically).

In the non-technical, corresponding roughly with the undergraduate, department of our universities, the course should begin with a thorough revisal of the six books and plane trigonometry; then Books XI. and XII. of the *Elements*, or perhaps only parts of these books; then spherical trigonometry. This, with algebra treated somewhat more scientifically, would probably occupy the first session. The second we would devote to the conic sections, which we would treat both geometrically and analytically, and to analytical trigonometry.

In the technical schools of mathematics, and the technical departments of the universities, the modern analysis, as we have already hinted, must be paramount. In these schools the object of the teacher should not be so much to instruct his students as to lead them in the path of study, and initiate them in the lifelong work of instructing themselves.

It were not for us, comparatively inexperienced as we are, to dogmatise on the details of a course of mathematical study. But we believe that such a course as we have outlined would enable us to get all the good out of mathematical instruction that it is capable of yielding, both as a mental discipline and as a utilitarian study. Practical teachers and professors, however, are alone able to form a definite scheme; and while, if such a scheme were devised, it must in its general principles be regarded in a national system as imperative, the greatest amount of freedom that may be found possible should be allowed for modifications. Of all men, teachers should be men, not machines, and should be treated accordingly.

XII

Taking it for granted—has it not been proved?—that the Euclidean geometry should form a part of every system of general education, we now proceed to consider whether Euclid's method must be retained in all its detail. Innumerable efforts have been made to effect improvements, none, in our judgment, with any great measure of success. The only one that has obtained extensive acceptance is that of Legendre. Regarding Legendre's system not as anti-Euclidean, but as essentially Euclidean with important modifications, we must repeat the statement which we have already made in substance, that in our judgment the modifications are not improvements. Having tasted the new wine, we acknowledge it to be wine; and we do not say that it is not good wine, but we say that "the old is better." We brush aside multitudinous attempts to condense and simplify Euclid's demonstrations. These all proceed by deteriorating their rigidity, and so rob them of the main part of their value. Let us give an example of these simplifications. Rightly or wrongly, Euc. I. 5 has been regarded by many as presenting difficulties too great for the student at so early a stage in his course, whence it has derived the unenviable and opprobrious name by which it has long been designated. Now, such difficulty as there is might be easily got over. If we bisected the vertical angle of the isosceles triangle, we should divide the triangle into two triangles, having one side in each equal to one side in the other, one side common to both, and an equal angle contained by these equal sides. It would at once follow, Prop. i. 4, that these two triangles are equal in every respect, and so that the angles at the base of an isosceles triangle are equal. Then the second part of the proposition relating to the angles at the other side of the base would follow from the first part, if we assumed as an axiom the almost axiomatic Prop. i. 13. Now all this Euclid knew just as well as his improvers. Why did he not adopt their method, which would certainly have resulted in a demonstration somewhat simpler than his? Manifestly because at that stage he had geometrically no cognisance of the half of a given angle. He knew that every angle has two halves. Had he not known that, he would not have sought to find the half of a given angle, as he does in Prop. I. 9. Till he had solved that problem, the half of a given angle was a thing unknown to him. It may be noticed that he proceeds on precisely the same principle with reference to straight lines. From his definition of a circle he knew that all radii of the same circle are equal. That was at the outset his only criterion of the equality of two lines, and upon this he proceeds in his first three propositions; whereas Legendre in his first proposition sets out by simply taking a line equal to a given line, that is, practically assuming as a postulate Euc. I. 2. While we are confident that Euclid's demonstration of I. 5 is preferable to that proposed to be substituted for it, we are of opinion that the Euclidean rigidity might be united with more than the Euclidean simplicity by another method of proof. It will be sufficiently intelligible without a figure. Take an isosceles triangle whose base is AB and vertical angle C. To us it seems legitimate to compare the triangle ACB with the triangle BCA, and to infer from Prop. 4 that the angles A and B are equal. No doubt the triangles ACB and BCA are one and the same. But it does not seem to contravene the geometrical instinct to suppose them two *pro hac vice.* When the method of proof is put in the form of supraposing the triangle on itself with the sides reversed, Mr. Dodgson wittily compares it to the Irish feat of a man's jumping down his own throat! The wit is wholly commendable; its applicability to the matter in hand is questionable. At all events it is not applicable to the form in which we would put the demonstration.

Thus far, the matter stands thus. The systems of geometry, exclusive of the so-called systems which are not systems at all, are three: the Euclidean, the Cartesian, and the Legendrian. The first and second are essentially different, and are to be studied apart. We cannot afford, on utilitarian grounds, to discard the Cartesian; as little can we afford, on intellectual and therefore ultimately utilitarian grounds, to neglect the Euclidean. The Legendrian method, as a modification of the Euclidean, we regard as not an improvement, but the reverse. There is no reason why both should be studied by the mathematician, for in matter they are identical, while in method it may be freely admitted that Euclid has not *always* the advantage. But as an educational text-book, we hope that our own countrymen will never abandon Euclid; we can scarcely hope that our French neighbours will ever abandon Legendre. Thus we must agree to differ. While we regard our own as "the more excellent way," we do not regard the difference as vital. When from time to time a great Legendrian geometer appears, we seek not to disparage his greatness. We rather argue that, great as he is, he might possibly have been greater still, if he had had the good fortune to be trained after the stricter fashion of the grand old man of Alexandria.

Attempts innumerable have been made to improve Euclid. These are handled with a degree of acuteness which is simply marvellous by Mr. Dodgson,[1] who has shown that, with the exception of two or three emendations, and those of little or no consequence, Euclid still holds the field. While this is so, there is a very common—we might almost say a universal—impression that Euclid, with all his excellence and his superiority over all his rivals, is not absolutely perfect. Undeterred by the fate of so many who have failed, we have a lingering hope that success is possible, and that success might possibly fall even to our lot. Sustained above all by the consideration that "in great attempts 'tis glorious even to fail,"[2] we venture to present ourselves, not as a rival, nor even as a humble editor, but as the offerer of some suggestions which may be worthy of consideration on the part of future editors.

If there be any defects in Euclid, they are in his definitions and axioms. In order to judge of these, we must have a clear apprehension of the real nature of definitions and axioms. This is all the more necessary

[1] *Euclid and his Rivals.*
[2] *Magnis excidere ausis laudabile.*

to be insisted on, as we find a great amount of vagueness in quarters where definiteness might have been expected to be found.

Demonstration or proof consists generally in the manifestation of the dependence of that which is to be proved on something which has already been proved. This necessitates that there must be some truths incapable of proof. The proposition C is demonstrated when it is shown that it is a necessary consequence of the proposition B, if B has been shown to be a necessary consequence of A. Ultimately we must come to a proposition which has no one going before it on which it might be shown to be consequent. The first proposition in every detailed train of reasoning must therefore, from the very nature of the case, be not proved but assumed. Each link in the chain of reasoning is supported by a link until we come to the first, which can have no link to support it, but must hang on an assumption. Now, every inquirer is free to assume what he will, and to construct a system of deductions from that assumption. According to the truth or falsehood of the assumption will be the truth or falsehood of the system consequentially deduced from it. If the inquirer assume as true a proposition of which he has no doubt, and deduce by accurate process a series of propositions which will constitute a system, then he will have no doubt of the truth of that system. But others may doubt it, either on the ground of their doubting the truth of the assumption or the validity of the reasoning. With the latter we have, for the present, nothing to do. But if the investigator's system is to be accepted by others, its initial assumption must be believed or admitted by others, and no system can get universal acceptance unless it is deduced from assumptions which all must admit. Such propositions are called axioms. They are not necessarily, as some would have it, incapable of demonstration, for it is possible that one axiom may be a consequence of another. It is enough that they are such as must be universally believed.

Thus, from the necessity of the case, there can be no demonstration without initial assumptions, and we cannot regard Col. Thompson's *Geometry without Axioms* but as an attempt to accomplish an impossibility. We have been surprised at the tolerance with which it was received by mathematicians. In point of fact, the method consists in the substitution of disguised or concealed for expressly stated assumptions. Yet its author was a mathematician of great power, and we can only regret the wasting of these powers. And, indeed, the powers were not wholly wasted. For although the attempt did not—as it could not—succeed, yet the attempt called forth admirable ingenuity.

As it is with demonstration, so it is also with definition. To define an object or a concept means, in the general, to express it in terms that are known. We may define a *man*, for example, to be an *animal*, with certain peculiarities, as hands, the power of speech, and the power of reason. But in order that this definition may have any value, we must have a definition both of an animal—the *genus*, and of these characteristics, the *differentia*. We define an *animal* as an object having *life*, and *speech* by its difference from the *roaring* of the lion, and the *barking* of the dog, and the *singing* of the nightingale, and the *squeak* of the mouse. So we distinguish reason from instinct.

We may stop there, or we may attempt to go a step further in a definition of *life*, and of *uttered sound*, and of *immaterial processes*. But sooner or later we must come to our limit. We must come to a concept so simple, that we cannot refer it to anything simpler. This has been represented as a defect in geometry. You boast, it is said, of the certainty of your science, yet you are obliged to admit that you cannot give a strict definition of the simplest objects with which your science is conversant. Now this is not a defect chargeable on geometry more than on any other science. Yea, it is not a defect at all, but a simple result of the very nature of definition. The first demonstration must set out from an assumption of something which cannot be demonstrated, the first definition from a concept of something which cannot be defined.

The geometer's first definitions, then,—those of the *point*, the *line*, the *surface*, and the *angle*,—are not and cannot be definitions in the rigid sense of the term. But they are very useful, and indeed necessary, *explanations* of the sense in which the terms are used, or perhaps rather of the sense in which they are *not* used; for the more important portion of the explanation is not the positive but the negative part. Thus, when we say that "a point is that which has position but not magnitude," we know, quite as well as do the deriders of geometry, that in order to make this a definition, we must first define *position* and *magnitude*, and we must use a substantive instead of the pronominal *that*. But it is important that it be explained that the point with which we have to deal is not a sword's point, or a needle's point, or the point of a bee's sting. These, indeed, have small magnitude,[1] but the points with which the geometer has to deal have none. This explanation is all that the so-called definition is meant to be. And so with that of the line. It assumes that there is such a thing as a line, and that every man must have an idea of what it is; but it excludes the inaccuracy of that idea. It explains that the line with which the geometer has to do is not a rope or a cord, or a spider's thread, or a "line" drawn by the sharpest material point, but as it were a line drawn by a geometrical point, and therefore without breadth, as that is without magnitude. The analogy is indeed perfect. As a so-called point made on paper by a material point is to the geometrical point, so is the so-called line which yet is a surface, or rather a solid with length and breadth and thickness, to the line of the geometer. It is traced, and therefore has length, but traced by a point, and therefore has no breadth, and on a surface, and therefore has no thickness.

A *point* and a *line* are incapable of definition because they are embodiments of simple or primary concepts. A *straight line* is not such. It is a species of the genus *line*, and can so far be defined, just as a man can be rightly defined as a species of animal. But the *differentia*, the straightness, is a primary concept. It cannot be defined. It does not need definition, because all have an apprehension of what it is. But the entrant on the study of geometry, who has all his days had a conception of straightness but has never considered what the concept is, needs to

[1] It is interesting to note that the negative portion, the exclusion of *magnitude*, is the whole of the explanation as given by Euclid. The ascription of *position* is a subsequent addition.

have the matter brought under his notice more distinctly than it has ever been before. This end would be accomplished by a definition, if such were possible. But it is not possible. The next best is accomplished by the statement of some characteristic or property which the straight line possesses in virtue of its straightness. We cannot think that Euclid was happy in his selection of a characteristic for this purpose. He defines a straight line as one that "lies evenly between its extremities." Now to us this lying *evenly* (ἐξ ἴσου) means, and can mean, nothing else than lying *straightly*, and so the definition amounts to this, "a straight line is a line that is straight!" Multitudes of attempts have been made to improve this definition. Perhaps the favourite ones in recent and present times are those which introduce the term *direction*. Thus Sir John Leslie says: "The uniform description of a line which through its whole extent stretches in the same direction gives the idea of a straight line." Were it not more accurate to say that the idea of a straight line gives rise to the idea of direction? We think so. And we are confirmed in the thought by the fact that in our subsequent geometrical studies we are never called to judge of the straightness of a line by its direction, but always of the direction of a line at any point by reference to a straight line passing through that point. Thus we are taught to consider that the direction of a curve at any point is the straight line which touches it at that point.[1] In fact, our idea of direction is derived from our idea of a straight line, not the latter from the former. Pro-

[1] As, for example, in the mechanical and astronomical problems regarding centrifugal force.

fessor Playfair, in his edition of Euclid, substituted the following for Euclid's definition: "If there be two straight lines such that they cannot coincide in part without coinciding altogether, each of them is called a straight line."[1] Now it is quite true that two straight lines do so coincide, and that no two lines which are not both straight lines can so coincide. But it is not true that this is *the* idea which, either primarily or ultimately, we have of a straight line. Apparently Playfair thought of a straight line as coinciding with the edge of a ruler which is assumed to be straight. To be consistent with himself, Playfair should have changed Euclid's postulate into something like this: "That from any point to any other point two lines coincident throughout may be drawn." But this would not do, for it is not two lines we want to define, but one line, and that a straight line. Playfair's definition might, properly enough, have been deduced as a corollary from an apt definition of a straight line, but as a definition it is wholly inept. Long before the days of Playfair or Leslie, or of our *alma mater*, on which they shed a brilliant lustre, Archimedes had given a definition of a straight line which is adopted by Legendre and most of the French mathematicians, and which, regarding it not as a rigid definition, which we hold to be impossible, but as the statement of a characteristic and exclusive property, we regard as superior to all others. "A straight line is the shortest distance between two points." We do not say that this is our

[1] We have not a copy of Playfair's Euclid before us, and cannot vouch for the verbal accuracy of our quotation. But as "Playfair" was the fountain from which we first drew geometrical waters, we are certain as to its substantial accuracy.

primary conception of a straight line. Of the two propositions, *This line is a straight line, therefore it is the shortest possible between its extremities*, and, *This line is the shortest possible between its extremities, therefore it is a straight line*, we admit that the former is more than the latter in the order of the conception. But believing that straightness cannot be rigidly defined, we think that the fact that the straight line is the shortest between two points is the characteristic by which it can be best described. In point of fact, this is the only property that is generally available to us for judging of the straightness, or approach to straightness, of any line. The ordinary time-tables tell us that of the two railway lines between Edinburgh and Glasgow the respective lengths are 46¼ and 47¼ miles; and thus, and thus only, does the Midlothian or the Lanarkshire peasant judge that the one is straighter than the other; thus and thus only does the Edinburgh or Glasgow mathematician conclude that the one deviates less than the other from a straight line. If peasant or mathematician could be assured that one is not only the shorter of actual railways but the shortest of possible lines, he would at once conclude that that one is straight.

We must admit that this definition has not obtained among English-writing geometers the acceptance to which we think it is entitled. Thus in an American work[1] of much merit we read: "We often see, for example, as a definition, '*A straight line is the shortest distance between two points.*' Now, in the

[1] *The Teaching of Elementary Mathematics*, by David Eugene Smith, Principal of the State Normal School at Brockport, New York. New York, 1900.

first place, this is absurd, because a *line* is not *distance;* distance is measured on a line, and usually on a curved line. Furthermore, the statement merely gives one property of a straight line; it is a theorem, and by no means an easy one to prove. A definition should be stated in terms more simple than the term defined, but distance is one of the most difficult of the elementary concepts to define. Mathematicians have long since abandoned the statement."

This onslaught is more truculent than it is to us convincing. Let us examine it in a little detail. First, its form reminds us of the reasons said to have been given by a native of an island not very distant from our own, for the absence of a friend from a court to which he had been summoned as a juror, "In the first place, he's dead." The judge is said to have ruled that it was not necessary to give other reasons. Now we should have supposed that absurdity is to a definition very much what death is to a man. But is the definition absurd? Long before we read Mr. Smith's attack, we had been in the habit of quoting the definition as "the shortest line between two points." Surely Mr. Smith might have made some such substitution, as he could not but know that in this connection *distance* could not possibly mean aught else than *line*. "Furthermore," says our author, "the statement gives merely one property of a straight line." Of course it does. So does every good definition give but one property—supposed to be the most obvious—of the object defined. Would Mr. Smith have had Euclid include the whole of his Third Book in the definition of a circle? "It is a theorem," says our author, "and by no means an easy one to prove." It is *not* a theorem, and it is impossible to prove it. It is simply a statement that to the shortest of the innumerable lines which may be drawn from one point to another, we give the name of a "straight line." "Distance is one of the most difficult of the elementary concepts to define." The difficulty of defining any elementary concept amounts to impossibility, and of this there are no degrees.

If we have not succeeded in answering Mr. Smith, we submit that he has answered himself. The paragraph immediately following that from which we have quoted opens with this sentence: "The fact is, the concept straight line is elementary; it is not capable of satisfactory definition, and hence it should be given merely some brief explanation." Precisely so; this is just what we have been contending for. What Mr. Smith calls an *elementary*, we have called a *primary* concept; what he calls a *satisfactory*, we have called a *rigid* definition; and we have represented shortness as nothing more than a *brief explanation* of straightness.

Colonel Thompson's definition of a straight line is highly ingenious. It is this: "From one of two assigned points to the other may be described a line, which, being turned about its extreme points, every point in it shall be without change of place. Such a line is called a straight line." The idea consists essentially in regarding the two points as poles of a sphere. Innumerable lines can be drawn between these poles. If the sphere be made to rotate, all these lines will describe solids (spheroids) within the sphere, with the exception of one line, the axis, every point of which will retain its original position when the sphere revolves. Then the material sphere is regarded as being removed and the geometrical line as continuing to rotate; and inasmuch as that line has no breadth or thickness, every point in it will be without change of place. In other words, while every point in every one of the lines joining the poles will describe a circle, the circle described by every point in the axis will have the diameter *nothing*, that is, it will be a point. We have said that Colonel Thompson's definition is ingenious. It is highly so. But it is manifestly unfit to be used as a definition in an elementary treatise. We need not, therefore, say more about it.

Before leaving this interesting subject,—interesting to us, and we hope to a few, though probably not many, readers,—it ought in fairness to be pointed out that the definition of Archimedes, whether as originally given or with our modification, involves the assumption that there *is* a shortest line between two points, that is, that there cannot be two, or more, shorter than all others, but equal in length to one another. That there cannot be two such lines cannot, we think, be legitimately inferred from any definition of the straight line except Playfair's, which, we suspect, he framed mainly with the view of evading this difficulty, but which, notwithstanding this advantage, on other grounds we are compelled to reject. But it is surely axiomatic. We would therefore include it under Euclid's first postulate, which we would read thus: "That a straight line, *and only one*, can be drawn from any one point to any other point."

Many other exceptions have been taken to others of Euclid's definitions. These may generally be set aside by slight verbal modifications of the definitions. For example, it has been argued that the definition of a plane rectilineal angle is applicable only to an angle less than two right angles. It is so. But the omission of the last clause (*but which are not in the same straight line*) of this definition would include under it two right angles, while the substitution of the word *divergence* for *inclination* would bring all other angles within its scope. To us it seems that this would be an unexceptionable definition: "A plane rectilineal angle is the divergence of two straight lines which meet one another." It is quite true that this is little more than the identical proposition that an angle is an angle, for *angle* being a primary concept, cannot be rigidly defined. But what we have suggested may perhaps be accepted as a useful explanation of it.

The definitions of the rectilineal figures, such as the triangle, the square, the rectangle, or oblong, etc.,[1] have been objected to; and not, it must be admitted, without some reason. But the objections, while valid, are not important. For example, it is said that the definition of a square as "a four-sided figure which has all its sides equal and all its angles right angles," assumes that there is or can be such a figure, which is not altogether an axiomatic truth; and that he might just as well have defined an equilateral triangle as a three-sided figure which has all its sides equal and all its angles right angles. It happens, indeed, that the square as defined *does* exist, and the triangle as so defined does *not* exist. But that circumstance

[1] At p. 156 we charged upon Dr. Simson the error of defining the isosceles triangle as having *only* two equal sides, and then in Prop. 5 regarding the equilateral triangle as isosceles. We are afraid that the error was Euclid's own, and that Simson is only chargeable with not having detected and corrected it.

does not make the one definition more legitimate than the other. It would, we think, have been better if he had put the definitions of parallel lines and of parallelograms before the definitions of the rectilineal figures. So he could have given unquestionable definitions of the square, the rectangle, and the rhombus as species of the genus parallelogram.

Some of the terms defined by Euclid have passed out of the language of geometry; as the rhomboid, the trapezium, and the trapezoid. The first of these survives only in the older editions of Euclid, and in the exquisite St. Andrews joke, which, we doubt not, was as amusing to the venerable and genial divine who was the butt of it, as it was to the clever young scapegrace who was its author. The story runs thus: At a school examination at which the gentleman referred to presided, an exercise was given. It might have been to prove that the lines bisecting the sides of any quadrilateral form a parallelogram. After making a figure on the blackboard, the demonstrant began with preternatural gravity, "Let AKHB be a rum-Boyd." We do not vouch for the truth of the story. It ought to be true!

But the definition of parallel straight lines as being "such as are in the same plane, and which, being produced ever so far both ways, do not meet," is the field on which the trumpet has blared and the clang of deadly weapons has rung for generations, and may probably ring for generations to come. It is objected to the definition, first of all, that it is only negative. It tells us what parallel lines are not, but gives us no hint of what they are. To our thinking the objection is valid, but not vital; for while a positive definition or one partly positive and partly negative may be *better* than one wholly negative, it does not follow that the latter is *bad*. Further, the definition is vitiated by the introduction of infinity into it. It may be *proved* that certain lines will not meet, though infinitely produced; but it ought not to be *assumed*, as it is in this definition. But there is an objection to the definition which we see no way of getting over, which, we are confident, cannot be got over. Whatever other qualities may be essential to a good definition, it ought to be such that we ought to be able to start at once from it, and to deduce from it the special qualities of the thing defined. Thus it is, for example, with Euclid's definition of a circle. With the definition that all radii of the same circle are equal, and by means of that property alone, he is ready to tackle the problems which are Props. 1, 2, 3, 12, 22 of Book I., and in due course all the Props. in Books III. and IV. It is quite otherwise with the definition of parallels. Instead of being able to start from the fact that two given lines do not meet, however far they be produced, and to deduce from this propositions as to the equality of the alternate angles made with these lines and a line intersecting them, he is obliged to follow precisely the opposite course, and to prove that if a line intersecting two other lines makes the alternate angles equal, these two lines cannot meet. Having done this, he is unable to prove the converse, namely, that the alternate angles are equal when the lines are according to his definition parallel, without the assumption of an axiom which scarcely any geometer regards as axiomatic. Even Simson, the first article of whose creed is Euclid's infallibility, admits that "it seems not to be properly placed among the axioms, as indeed it is not self-evident, but it may be demonstrated thus"; the *thus* being by means of two definitions, one axiom, and five theorems! Mr. Dodgson, too, whose faith in Euclid is nearly as strong as Simson's, introduces Euclid into one of his witty dialogues as defending this axiom, thus: "It is not axiomatic until Prop. 28 has been proved. What is an axiom at one stage of our knowledge is often anything but an axiom at an earlier stage." Now we humbly submit that this is an abuse of language. We do not like to introduce a reference to the Omniscient One into such a discussion as this, else we might say that to Him all truth is axiomatic. We shall only say that just as well Euclid might have made it an axiom that "all the exterior angles of any rectilineal figure are together equal to four right angles." That is certainly "not axiomatic until" Prop. 32, Cor. 1, "has been proved." It is a very obvious corollary from that corollary, but we must pervert the term *axiom* before we call it axiomatic. There is a legend that in a northern city a tax was imposed, and was to continue until "Union Street should become a street." In due time the question arose when a street becomes a street, and the answer of experts is said to have been that if a street is not a street before it becomes a street, it cannot be a street after it becomes a street! The phraseology is a little mixed, but the idea underlying it is applicable to the term *axiom*. An axiom *is*; it does not *become*. The statement as to the necessary intersection of lines on which a line falling makes the interior angles less than two right angles, must be removed from the list of axioms. It may be introduced as a deduction or corollary from Prop. 28,—though we do not think so,[1]—or it may be proved as a separate proposition, or, as is done by Simson and others, as the last of a chain of propositions.

From the fact, then, that it cannot be proved that two lines which never meet make angles together equal to two right angles with a line that falls on them, whereas it can be proved that two lines cannot meet which make with a line falling on them angles together equal to two right angles, we hold it to be an indisputable inference that the latter condition is a characteristic of parallelism prior to non-intersection. And this seems to be in accord with what we may call our pre-geometrical conception of the matter. We have two lines set before us, and we are told that they are parallel. If we have a smattering of Greek we understand that they are so called because they lie *παρ' ἀλλήλων*, alongside of each other. Whether we have Greek or no, we do not trace their course along a thousand or ten thousand miles, and conclude that they are called parallels because they would not meet

[1] Prop. 28 is: "If a straight line falling on two other straight lines makes the exterior angle equal to the interior and opposite upon the same side of the line, the two straight lines shall be parallel to one another," that is, according to Euclid's definition, shall never meet. Ax. 12 is in substance: "If a straight line falling upon two other straight lines makes the exterior angle less than the interior and opposite on the same side of the line, these two lines shall meet." We do not see that the latter of these propositions is a logical deduction from the former. To make it such we must introduce what is commonly known as Playfair's axiom, that two straight lines which are parallel to the same straight line cannot pass through the same point. But this is not an axiom, but is Euclid's Prop. 30.

though they were produced to these, or ten thousand times these lengths; but we see that they are everywhere just so far apart that the wheels of a railway carriage run upon them, or that we can run a board, or a book, or a ruler between them. We submit, then, that equidistance is our primary idea of parallelism, and that it is so because it is in nature the primary characteristic. Some of the geometers whom Mr. Dodgson calls Euclid's rivals have accordingly adopted equidistance as their definitive criterion of parallelism, and others have introduced what they call *direction*, and would define parallel lines to be those whose direction is the same. We have but a vague notion of what they mean by this, and are not without suspicion that their own notion of it is not quite distinct. We think of two intersecting lines as having the same direction, namely, a direction towards the point of intersection; but of two non-intersectional[1] lines we should say that they have *not* the same direction. How then can the geometers of whom we are speaking say the very opposite of this, namely, that non-intersectional lines have the same direction, and intersectional lines not the same direction? Their idea seems to be this: two men going due south are going in the same direction, and they will never meet. But two men going south are not going in the same direction unless they set out from the same point, or the one sets out due south of the point of that from which the other sets out, otherwise each travels in the

[1] We borrow the term from Dodgson. It expresses the idea not only of lines not intersecting within any length of which we have cognisance, but of lines incapable of intersecting within any length whatever; in fact, of parallel lines as defined by Euclid.

direction which is southward *from his own starting-point.* So that no parallel lines have the same direction; and when it is said that parallel lines have the same direction, either the statement is false, or its meaning can only be that they have parallel direction, and the vaunted definition—for it has been spoken of and written of in glowing terms—amounts to this, that parallel lines are those whose directions are parallel! We do not, in forensic language, take much by this motion.

Other geometers have just reversed the order that Euclid takes in his demonstration, taking his Prop. 27 as their definition, and showing that parallels as thus defined will never meet; in other words, that their parallels and Euclid's parallels are identical. After this, Euclid's propositions require but slight modification.

Of these three methods we hold the second to be simply vicious, and can only wonder that any mathematician should have employed it. In charity we withhold the names of those who have. The first and third methods are nearly identical, at least either can be very easily converted into the other. On the whole, we like the third form better than the first. But we would make a modification of it which in our judgment would materially improve it. If any of our readers should now or at any future time aspire to the noble distinction of being the author of an improved edition of the *Elements*, we would very earnestly commend not to his acceptance, but to his most serious consideration, this definition: *Parallel straight lines are such as have a common perpendicular*, or some other definition embodying this idea and no other. In showing how we should proceed with this definition, we shall be a good deal hampered by the want of figures,[1] but we hope to make our process sufficiently clear.

We presume that all admirers of the *Elements* must regard the 16th Proposition as

> "a blot
> Which so much beauty would do well to purge."

Its existence is very closely connected with Euclid's definition of parallels.

In order to treat the doctrine of parallels directly, define we parallels as we may, we cannot dispense with what is Euclid's 32nd Proposition. But that proposition he could not demonstrate without drawing a line parallel to another; and this, from his faulty definition of parallels, he could not do without Prop. 32. He thus found himself shut up to the necessity of reasoning in a circle, unless he could devise a way of escape. His device was clever but clumsy. He proved in Prop. 16 that the exterior angle of every triangle is greater than either of the interior opposite angles. This sufficed to enable him to demonstrate Prop. 27, and that enabled him to draw a straight line parallel to a given straight line. And so he was able in Prop. 32 to prove that the exterior angle of a triangle is equal to the sum of the two interior opposite angles. Thus it appears that Prop. 16 is only a scaffolding for the erection of Prop. 32. When the builder of material structures has erected his

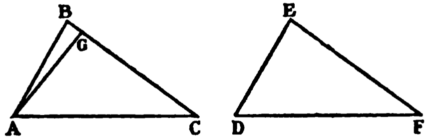
[1] Through the kindness of the publishers, this hindrance has been removed.

prescribed building, he removes his scaffolding and there is an end of it. But intellectual scaffolding cannot be treated thus. Having served its purpose, it remains a thing of no-beauty and of no-joy for ever. Now, if we could erect the structure without the scaffolding, we should render a service which, we venture to think, would be acceptable to all who are gifted with geometrico-architectural taste. We shall attempt now to show how we think this may be done. Let it be distinctly understood that our immediate task is to substitute Prop. 32 for Prop. 16.

It has been said innumerable times that a triangle has six particulars, namely, three sides and three angles, of which, if three be given, and one of them be a side, the other three are determined. This assertion ignores the fact that Euclid deals not with six, but with seven particulars, namely, the three sides, the three angles, *and the area* of the triangle, so that in I. 4, 8, 26, he shows that if three, of which one is a side, be given, the other two being either two sides or an angle and a side, the other four are determined. But while in the propositions referred to he introducees the *area* into the conclusion, he nowhere introduces it into the data; nor, so far as we know, does any one of his followers. Such introduction of it would, in our judgment, be eminently beneficial. We would therefore interpolate between Props. 4 and 5 the following Prop.:—

If two triangles have one side of the one equal to one side of the other, and an angle adjacent to the equal side in the one equal to an angle adjacent to the equal side in the other, and have likewise *their areas equal, the triangles are equal in every respect.*

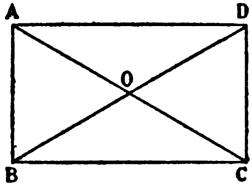
Let ABC, DEF, be two triangles, having the side AC equal to the side DF, the angle C equal to the



angle F, and the area of the one triangle equal to the area of the other, these triangles are equal in every respect.

Let the triangle DEF be applied to the triangle ABC, so that DF shall coincide with AC. Then because the angle F is equal to the angle C, FE will lie along CB, and the point E will coincide with B. For if not, let E coincide with any other point G in CB, so that CG = FE. Join AG. Then the two triangles AGC, DEF, have the sides AC, CG, equal respectively to the sides DF, FE, and have likewise the angles contained by these sides equal. Therefore (I. 4) the area of AGC is equal to the area of DEF. But the area of DEF is by hypothesis equal to the area of ABC, therefore the area of AGC is equal to the area of ABC, the less to the greater, or the greater to the less, if G be taken in CB produced, which is absurd. Therefore the point E coincides with B, and the triangles ABC, DEF, are equal in all respects. *Q.E.D.* We shall refer to this theorem as 4*.

We are to place immediately after Euc. I. 15 a proposition which is essentially Euc. I. 32. *The three internal angles of every triangle are together* *equal to two right angles.* Our process then would be after this fashion—

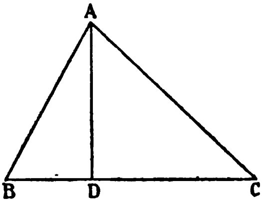


Case 1. Let ABC be a *right-angled* triangle. Bisect the hypotenuse AC in O. Join BO and produce it to D, making OD = BO. The triangles AOB, COD, have AO = CO, BO = DO, and (I. 15) the angle AOB = COD ∴ (I. 4) these triangles are equal in every respect, and have the angle BAC = DCA, and ABD = BDC, and the side AB = CD, and the area of AOB = the area of COD. In like manner the triangles BOC, DOA, are equal in every respect, and have the angle DBC = ADB, and ACB = DAC, and the side AD = CB, and the area of BOC = the area of DOA.

By adding equals to equals, it is proved that the angle ADC = ABC, and is therefore a right angle, also that the angles BAD, DCB, are equal to one another. Also that the areas of the four triangles ABC, BCD, CDA, DAB, are all equal, and that each of these triangles is half of the quadrilateral ABCD.

Thus the triangles ABC, BAD, have a side AD = CB, and an angle DAC adjacent to AD = angle ACB adjacent to CB, and have also their areas equal, therefore (4*) they are equal in every respect ∴ AC = BD, and BAD is a right angle. Thus the angle BAC, CAD, make up a right angle. But CAD has been proved equal to ACB, therefore BAC and ACB are equal to a right angle, and ABC+BAC+ACB=two right angles. Now BAC is a right-angled triangle, therefore the angles of every right-angled triangle are together equal to two right angles.

Case 2. Let ABC be any triangle. Let BAC be one of its angles which is not less than either of the others. Draw AD perpendicular to BC. Then by Case 1 the angles BAD + B + BDA = two right angles.



Also CAD + C + CDA = two right angles.

∴ BAC + B + C + BDA + CDA = *four right* angles. But BDA and CDA are right angles ∴ BAC + B + C = two right angles. Therefore the three angles of every triangle are together equal to two right angles. *Q.E.D.*

Cor. 1. As any interior angle, together with its exterior, are equal to two right angles, and as any interior angle, together with the two other interiors, have been proved to be also equal to two right angles, therefore any exterior angle is equal to the two opposite interiors.

Cor. 2. If two triangles have two angles of one equal to two angles of the other, either each to each or in the aggregate, the third angle of the one is equal to the third angle of the other.

Euclid's Cor. 2 would be our Cor. 3.

Our proposition with Corollary 1 makes up Euc. I.

32. Our Cor. 2 includes I. 26, along with a useful extension of that proposition. We believe that our proof is perfectly rigid, while it might easily be made more elegant than as we have given it. We submit, then, that gain of no small amount would be effected if our proposition and its corollaries, with Euclid's corollaries from his I. 32, were put immediately after his I. 15, and his I. 16, I. 17, and I. 26 were dispensed with. Thus should be removed from the *Elements* a blot which is very visible, but which, like certain blood-stains we in Scotland wot of, has not hitherto been erased. It will be perceived that while our construction is identical with that of Euc. I. 16, the reason that we have proved what he could not prove is that the rectangle has the property which no other parallelogram has, that its diagonals are equal.

As to the other propositions between I. 16 and I. 27, it appears that if the definition of a straight line which we have suggested were adopted, I. 20 might be made a corollary from that definition. Perhaps it were as well to retain it as a proposition, but its proof would be extremely simple.

Thus far we have been concerned only with what all, we should think, will admit to be a blot or inelegance. We have now to grapple with a positive flaw, a defect, or rather two defects, in rigidity. Let it be distinctly understood what these defects are. First, Euclid ought, according to the principles of right demonstration, to have been able to start with his definition of parallels, and to show what properties, besides those which his definition assigns to them, such lines possess. This he has not done, and could not do. Instead of this, he starts with lines having certain properties, and shows that these lines have also the property which he has taken as distinctive of parallel lines. This is the first defect. The second is that, while he has proved that these lines are parallel, he has not proved that none others are parallel. It does not follow that because lines are parallel with which a line intersecting both makes the alternate angles equal, therefore lines which make these angles unequal are not parallel. This he required to prove, and he could not do so by his method of treating parallels, but assumed it as his 11th (or, in some copies, his 12th) Axiom. Now it is not an axiom, but a proposition which he ought to have proved.

Our position is that these defects, in one form or other, are inseparable from the retention of Euclid's definition. We therefore bluntly propose to replace that definition by this: *Two straight lines are parallel which are in the same plane, and which are both at right angles to a third line.* We must now give a general sketch of the course we would pursue. Our first proposition would be that parallel lines, as so defined, will not meet, that is, that our parallels are Euclid's parallels. The proof of this would simply be that if they met, the three lines would form a triangle with two of its angles right angles, which has been shown to be impossible (Euc. I. 17, or our 16.

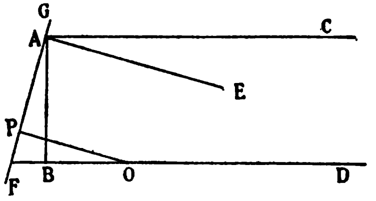
We should next have to show that if one straight line is at right angles to two straight lines, then every straight line which is at right angles to one of these two is at right angles to the other. This also is virtually proved already; for it has been shown in the proof of Case 1 above, that when a quadrilateral figure has three of its angles right angles, the fourth is also a right angle.

What is commonly known as Playfair's axiom, which is in substance that intersecting lines cannot be parallel to any third line, follows from this, for the contrary would involve that from the same point there would be a straight line drawn at right angles to each of the intersecting lines, which is a direct contradiction of the definition of a right angle.

Then would come the proof of Euc. 27, 28, and their converse 29. This also has been to a large extent anticipated; for we have shown above that the angle DAC=ACB and ADB=DBC. But if the alternate angles are equal, it follows at once that the exterior is equal to the interior and opposite on the same side, and the two interior on the same side are equal to two right angles. Thus Euc. I. 29 is demonstrated without the help of his 11th Axiom. Props. 27 and 28 follow, of course, from Prop. 29.

We have seen that all that Euclid proves regarding parallels can be proved without having recourse to his 11th Axiom. But that so-called axiom is an important proposition, and is worthy of independent proof. Many proofs have been given of it satisfactory enough, excepting in that they all employ, under more or less flimsy disguises, the doctrine of proportion and of similarity of triangles, which cannot be legitimately employed at this stage. We venture, though with diffidence, to add another.

Let AC and BD be at right angles to AB and therefore parallel to one another, and let the angle BAE be less than a right angle. Then AE will meet BD. Draw AF at right angles to AE and produce it towards A. Take any point O in BD, and draw OP perpendicular to AF. In like manner perpendiculars to AF may be drawn from every point in BD, and will all be parallel to AE, being all at right angles to FG; and as they will fall on every point in FG, one of them



must fall on it at the point A, and will be coincident with AE. Therefore AE may be produced to meet BD. We could not show where the point of intersection is without having recourse to the method of disguised proportion, which we have repudiated. But we are not required to do this. It is enough if we have shown that in every case there is such a point of intersection. Our proof refers only to the case in which one of the angles is a right angle and the other less than a right angle; but the general case, in which the sum of the two is less than two right angles, is easily reduced to this.

We think, then, that with these simple propositions we are prepared to enter with Euclid on the discussion of parallelograms. But if our proof of Euclid's axiom come short of satisfying the appetite for rigidity of any geometer, we would remind him that we were not required to prove it, as we have nowhere made use of it.

Some slight modifications would be required in the demonstration of some of the propositions concerning parallelograms, if our suggestion were adopted, to define a parallelogram as a quadrilateral having its opposite sides parallel, a rectangle as a parallelogram having one of its angles a right angle, a square as a rectangle having two of its adjacent sides equal; and a rhombus (if there be any need to define it at all, since Euclid never, so far as we remember, uses the term) as a parallelogram having two of its adjacent sides equal. It is of consequence that the learner should very clearly apprehend that what is proved of the parallelogram is thereby proved of the rectangle, the square, and the rhombus; whatever of the rectangle is thereby proved of the square; but not conversely, as, for example, while the rectangle has no properties which the square has not, the square has many which no other rectangle has.

We have only one thing more to say on this part of our subject. It is desirable on many accounts that the principal propositions should retain the numbering which they have had for twenty-one centuries. We are afraid that the 32nd must be given up, as its proper place is immediately after the 15th, but "by hook or crook" *the* 47th should be the 47th. This might be effected by the introduction of some propositions which are very useful, but which Euclid did not introduce, because they are not links in the chain which led him to the 47th and 48th. Such, for example, are various propositions as to the common points of intersection of certain lines in a triangle. This, however, should not be overdone, as we think it has sometimes been.

Books II., III., and IV. may not be absolutely perfect; but only trifling amendments are required to make them so. It is otherwise with Book V., which, in point of fact, is generally abandoned by teachers. Our late excellent friend, Professor Kelland, strenuously protested against this abandonment, and insisted on the retention of the book as Euclid left it. As in the case of parallels, the difficulty can be got over only by the change of a definition. Ratio is essentially a matter of number, and enters into the field of geometry only by transference from that of arithmetic. We have seen that arithmetic was in an extremely unsatisfactory condition in the days of Euclid, and continued so till a much later time. We know not whether it has been pointed out by others, but we have long thought that one great defect in the ancient arithmetic consisted in its regarding the particular numbers 1, 2, 3, etc., as the *only* numbers. This gave rise to the idea of certain numbers being commensurable, and of others being incommensurable. Thus 4 and 6 were commensurable numbers, because they can both be divided by 2, or have 2 as a common measure; but 3 and 5 were incommensurable, because there is no number greater than unity—which, by the way, Euclid does not regard as a number—by which they can be divided; or they have no common measure. Now we hold that every number can be divided by every other. Thus, if we divide 5 by 4 we get the quotient $1\frac{1}{4}$, or 1·25, or $\frac{5}{4}$. Now, these are just as much real numbers as any others, as 2, 3, or 4. So also the square roots of 4, 9, 16 are said to be numbers, but the square roots of 2, 3, 8 are not recognised as numbers. We can only express them severally as $\sqrt{2}$, $\sqrt{3}$, and $2\sqrt{2}$; yet these have as real a value as the 2, 3, and 8 have. In fact, we have constantly to deal with them as numbers, and why should we scruple to give them the name? Now, this notion of incommensurables constituted a difficulty with which Euclid had to contend, and with which he grappled with great ingenuity, so that his definition of proportion has never been improved, and will never be improved while the notion of incommensurables continues to exist. Yet Euclid's definition, however ingenious, we are sure he did not contemplate with any measure of the satisfaction with which he must have regarded the 47th Proposition. We know that the diagonal of a square whose side is a has the same ratio to a that the diagonal of a square whose side is $2a$ has to $2a$, and why should we not say at once that each diagonal is $\sqrt{2}$ times the side of the square of which it is the diagonal? In practice we say this; and we say that the edge of a cube which is double of another is $\sqrt[3]{2}$ times the edge of the other, and we say that the circumference of every circle is π times its diameter. So we call the base of the Neperian logarithms e, and the modulus of the decimal logarithms m; but we are told that these are not numbers, because we cannot express them in a finite number of Arabic numerals. Yet we deal with them as numbers, and a correct result follows. It may be objected that we deal in the same way with imaginary quantities, and with equally correct results. True; but the treatment of the imaginary quantity, as $\sqrt{-1}$, is strictly analogous to the method of proof known as *reductio ad absurdum*. We extract the square root of -9 and find it to be $3\sqrt{-1}$. That result means simply that there is no number, positive or negative, which being multiplied by itself will produce the negative number -9 or any other negative number.

The reverse process is more difficult to explain. When we multiply $3\sqrt{-1}$ by itself, we find the product to be -9. Therefore the product of two imaginary numbers may be a real number. This anomaly is explained by the consideration that $a\sqrt{-b}$ is not *any* imaginary quantity. It is *the* number which being multiplied by itself produces the real number, $-a^2b$. True, there is no such number, but this imaginary number differs from other imaginary numbers precisely as one real number differs from another. $\sqrt{-4}$ is double $\sqrt{-1}$, precisely as $\sqrt{4}$ is double $\sqrt{1}$. On the whole, while we use the term *number* as designating only what we call *whole* numbers, we indicate by the very use of the term that there are other numbers which are not whole but fractional or irrational. Why should we not then acknowledge in theory that fractions and surds and interminable decimals are numbers, as real as the numbers denoted by the Arabic numerals, when, by the very designation of the latter as *whole* numbers, we admit that the former are numbers too? But this acknowledgment would sweep away the notion of incommensurability.

The idea is overpowering. Abolish incommensurables! To abolish monarchy, to abolish the Bank of England, to put down suicide, as we remember a London magistrate declared his determination to do, to abolish Punch and Judy, were astounding undertakings. But what were any one of them, or all of them together, in comparison of this?

We have had occasion to advert repeatedly to three problems about which the minds of geometers in old time were much exercised, and which they did not succeed in solving—the trisection of an angle, the duplication of a cube, and the rectification and quadrature of a circle. There is, we suspect, very prevalent misconception of the nature of these problems, of what it is that constitutes their difficulty, and of the position which they now occupy. We have therefore thought that it would be germane to our subject to devote a few paragraphs to the elucidation of these points, for the information of intelligent but non-mathematical readers.

That an angle, like every other magnitude, has a third part, as well as a half, a fourth part, etc., is, of course, unquestionable. No geometrical problem is easier than the division of *any* angle into two, four, eight, sixteen, etc., equal parts. Scarcely more difficult is the division of *a right angle* into three, six, twelve, etc., equal parts.[1] But the trisection of *any* angle by plane-geometric means has not been found to be possible. What we mean precisely by plane-geometric means will be made clear by reference to the method which Euclid adopts for the bisection of any angle. In Euc. I. 9 he makes the given angle the vertical angle of an isosceles triangle, and on the opposite side of the base of that triangle he describes an equilateral triangle. Then he shows that the line joining the vertices of these triangles bisects the given angle. Now no one has been able to show that the position of the two lines which trisect the angle can be ascertained by means of the rule and compasses. Modern analysis has shown that their position cannot be so ascertained. But the Euclidean plane geometry does not show this, and so the student, apart from the modern analysis, is left to strive to do what he finds that he cannot do,

[1] In our own time, Gauss has shown that four right angles can be divided into seventeen equal parts, and consequently one right angle into 34, 68, 136, etc., equal parts.

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without being unable to prove that, from the nature of the case, it cannot possibly be done. What makes the problem all the more tantalising is, that it can be reduced to another which bears such an aspect that it is impossible to avoid thinking that it ought to be very easy, but which yet contains under disguise the whole difficulty of the original problem. In old days the problem was solved otherwise than by means of the rule and compasses, by the help of various curves, to one of which was given, for this reason, the name of the *trisector*. The operation of trisection is easily performed mechanically, and many neat instruments have been invented for the purpose.[1]

The difficulty of the duplication of the cube has considerable resemblance to that of the trisection of the angle. There must be a cube which is the double of any given cube, and its edge must bear some ratio to the edge of that cube. It might be assumed—or it could easily be proved—that that ratio is a constant one, that is, that the edge of any cube has to the edge of the cube which is its double as the edge of any other cube to the edge of its double. The object of the duplication of the cube is to ascertain that ratio geometrically. Euc. I. 47 enables us to find very easily the side of a square which is any multiple—two, three, four, ten, a hundred, a thousand times any given square. Also we can easily determine the edge of a cube which is 8, 27, 64, 125, etc., times a given cube. But the geometrical determination of the edge of the cube which is double of a given

[1] A description of such an instrument was very lately sent us by Mr. J. N. Miller, Portobello. It struck us as admirably fitted to serve all the purposes of the draughtsman.

cube, has not been effected. It resolves itself into the solution of the algebraic equation $x^2=2a^3$, or, which is the same thing, to find x so that $a : x=x^2 : 2a^2$. This looks as if it should not be difficult. And, algebraically, or rather arithmetically, it is easy enough, since $x=a\sqrt[3]{2}$. The artificer who should have to solve the Delian problem practically could easily approximate its solution to any amount of nearness; but absolute accuracy could only be attained by the geometric solution of the problem. This has not been effected, and probably will never be. This problem does not appear to be so fascinating as that of the trisection of an angle or the quadrature of a circle; probably because solid geometry is less studied than plane geometry. Whatever be the cause, we have seen many supposed solutions of the others, while we do not remember to have seen any of this.

The difficulty of the rectification and quadrature of the circle has also much resemblance to that of the other two problems. Thus the circumference of every circle is equal to some straight line. The length of that line in every case is dependent on the length of the diameter; and it may be regarded as axiomatic that the ratio of the circumference to the diameter is constant. If, then, we could find that ratio in the case of any one circle, we might fairly conclude that that is the ratio in all circles. But the ratio has not been determined geometrically, that is, it has not been found that that ratio is the same with the ratio of any two straight lines which can be drawn by geometrical construction. A rough approximation to the ascertainment can be very easily made in various ways. As, for example, the side of a regular hexagon inscribed in a circle is equal to the radius of that circle. Its perimeter is therefore equal to six times the radius, or three times the diameter. But the circumference of the circle is evidently greater than this perimeter, therefore it is more than three times the diameter. Also the side of a square circumscribing a circle is equal to the diameter. Its perimeter is therefore four times the diameter. But it is greater than the circumference. Therefore the circumference is less than four times the diameter. A first approximation, therefore, is that the length of the circumference is more than three times, and less than four times, that of the diameter. If we go on constantly doubling the number of sides of the inscribed and circumscribed regular polygon, their perimeters will constantly approach to equality with the circumference; those of the inscribed figures being always less, and those of the circumscribed always greater, than the circumference, the defect in the one case and the excess in the other becoming less with every doubling of the number of sides. By the help of Euc. I. 47 the lengths of these perimeters can be easily, though laboriously, determined numerically to any degree of accuracy. Enormous labour has been expended on the determination of this ratio, which mathematicians have agreed to denominate π. We have before us such a determination carried out to 240 decimal places, and we think we have heard or read that it has been carried still further. This means that if we have a circle with an inch diameter, we cannot err in the estimate of its circumference by so much as one millionth part of a millionth part[1] . . . of an inch. So much for the

[1] The hiatus is to be filled up by the repetition of "of a millionth part" forty times.

rectification of the circle, or the ascertainment of the value of π, or ratio of the circumference to the diameter. The quadrature of the circle follows immediately from this. Just as the area of a triangle is half the rectangle contained by its base and its altitude, so the area of a circle is half the rectangle contained by its radius and a line equal to its circumference. If, then, the radius be called r, the circumference will be $2\pi r$, and the area πr^2, or $\frac{1}{4}\pi d^2$.

With all this marvellous approximation to the numerical value of π, no step has been taken towards its geometrical ascertainment. Does this mean that arithmetic succeeds where geometry fails? No. It means that when geometry fails,—as it does only when, from the nature of the case, success is impossible,—it calls in the aid of arithmetic, and, with that aid, achieves an approach to success. The method of "exhaustion" is a geometrical method none the less because arithmetic is necessary to make it available for practical uses.

Our task is done. We have found a man who in his lifetime possessed not even in a limited measure any of the qualities which make for popularity,—whose only recorded utterance is the epigrammatic sentence, "There is no royal road to geometry,"—who took no part in any of the matters which ordinarily influence the lives of men, religion, statesmanship, jurisprudence, war, commerce, economics,—who was the author of a few books, which never were of much value, and now are of none,—and the author in part, and in great part only the compiler, of a book which at the best is but a school-book,—whose very name was unknown for many centuries, and came to be known again only through the possession of that school-book by some soldier or camp-follower of a ruthless army;—we have found that man at this day, two millennia after his death, taking part, by means of that small book, in the development of the mind and formation of the character of all the ingenuous youth of all civilised lands; we have found that on that small book of his as a foundation has been reared the magnifical fabric of our science and art and enterprise and commerce. How has this been? It is because this man did with his might what his hand found to do, and because the Supreme Ruler has decreed that not by might or by power, but by His own Spirit, His mighty ends are to be accomplished; and because for the accomplishment of these ends He ofttimes uses as His instruments not the mighty or the powerful among men, nor the might nor the power of those whom He does employ, but only their quiet faithfulness, and their forgetfulness of themselves in accomplishing the task, however humble, which He assigns to them. So faithful and so self-forgetful was Euclid, and thus he ranks high this day as one of THE WORLD'S EPOCH-MAKERS.

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The Pythagorean triangles of hypothenuse less than 200

cos θ				θ	
·710	119	120	169	45.2	1.2
·724	20	21	29	46.4	1.5
·742	65	72	97	47.9	1.0
·753	48	55	73	48.9	1.1
·766	88	105	137	50.0	3.1
·800	3	[illegible]	[illegible]	53.1	2.7
·827	104	153	185	55.8	·8
·835	60	91	109	56.6	·6
·841	85	132	157	57.2	·9
·849	28	45	53	58.1	1.4
·862	33	56	65	59.5	1.0
·870	95	168	193	60.5	1.4
·882	8	15	17	61.9	2.1
·899	39	80	89	64.0	1.0
·906	36	77	85	65.0	2.4
·923	5	12	13	67.4	2.0
·936	44	117	125	69.4	0.5
·939	51	140	149	69.9	1.2
·946	12	35	37	71.1	0.9
·951	57	176	185	72.0	0.6
·954	52	165	173	72.6	0.3
·960	7	24	25	72.9	2.8
·969	16	63	65	75.7	1.7
·976	9	40	41	77.4	1.1
·980	20	99	101	78.5	1.2
·984	11	60	61	79.7	0.7
·986	24	143	145	80.4	0.7
·988	13	84	85	81.1	0.8
·990	28	195	197	81.9	0.4
·991	15	112	113	82.3	0.9
·993	17	144	145	83.2	0.5
[illegible]	19	180	181	83.7	

The Pythagorean triangles of hypothenuse < 200

3	$2^2=4$	5
5	12	13
15	8	17
7	24	$25=5^2$
21	20	29
35	12	37
$3^2=9$	40	41
45	28	53
11	60	61
33	56	65 }
63	$4^2=16$	65 }
55	48	73
13	84	85 }
77	$6^2=36$	85 }
39	80	89
65	72	97
99	20	101
91	60	109
15	112	113
117	44	125
105	88	137
17	$12^2=144$	145 }
143	24	145 }
51	140	149
85	132	157
119	120	$169=13^2$
165	52	173
19	180	181
57	176	185 }
153	104	185 }
95	168	193
195	28	197