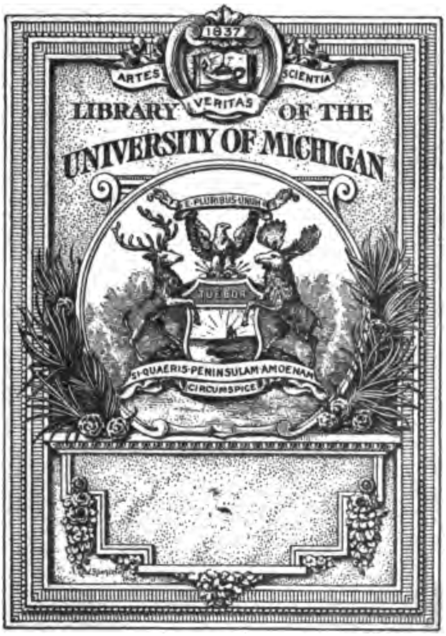
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# THE FOUNDATIONS

OF THE

# EUCLIDIAN GEOMETRY

AS VIEWED FROM THE

# STANDPOINT OF KINEMATICS

DISSERTATION
SUBMITTED TO THE
BOARD OF UNIVERSITY STUDIES
OF THE
JOHNS HOPKINS UNIVERSITY
IN CONFORMITY WITH THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

BY

ISRAEL EUCLID RABINOVITCH

APRIL 29, 1901

NEW YORK:
PUBLISHED BY THE AUTHOR
1903

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ERRATA.

Page x, next to bottom line read "Bolyai" instead of Bolyi.
Page xi, under Lobatchevski, close the quotation with ".
Page 26, last line, read "Matematiche" instead of Mathematiche.

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PRESS OF
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LANCASTER, PA.

Dedicated

TO

FABIAN FRANKLIN, PH.D.,

FORMERLY PROFESSOR OF MATHEMATICS IN THE JOHNS HOPKINS UNIVERSITY,

In Grateful Acknowledgment

OF BENEFITS CONFERRED UPON THE AUTHOR DURING HIS RESIDENCE IN BALTIMORE AS A GRADUATE STUDENT OF THE JOHNS HOPKINS UNIVERSITY.

# PREFACE.

THE present work is the result of long meditations and of an earnest search for truth. A conviction that truth in mathematics must be absolute, not admitting of any compromises, and an inmost feeling that Nature is not deceiving us and that She reveals Herself to us in Her true appearance, unmutilated by false logic, have guided me in my endeavors to solve one of the hardest mathematical problems, so intimately connected with the problem of the origin of our ideas, — namely, the problem of the Foundations of Geometry. These meditations were finally written up, in April, 1899, at the prompting of my excellent and highly esteemed friend and benefactor, Dr. Fabian Franklin, formerly Professor of Mathematics in the Johns Hopkins University, to whom I have availed myself of the present opportunity of expressing my gratitude, by inscribing this work to him.

Another name I ought to mention with gratitude is that of Dr. Alexander S. Chessin, Professor of Mathematics in Washington University, who was the first to appreciate the value of this work and to talk to me unreservedly about it, and also to urge me to use it as a thesis for the Ph.D. degree.

And, finally, I owe a duty of gratefulness to my distinguished professor, Dr. Frank Morley, for his guidance in the work of reading up the literature of the subject, for discussing with me points of difficulty in the literature, and also for allowing me to present some of my theorems before the conference of the mathematical seminary of the university, where the general discussion by the audience helped me in improving the mode of presentation of these theorems. To this discussion I owe, in particular, the analytical presentation of my proof that, *with the point as its element*, space must be three-dimensional. The whole of the Introduction was undertaken and carried out, during the academic year of 1900–1901, at the suggestion and under the direction of Professor Morley, and it is intended as a critical review of some of the most important results obtained by modern mathematicians in the subject (Riemann, Beltrami, Lie and Poincaré), so that in the light of these an adequate estimate of the results achieved in this Dissertation might become possible.

And last, but not least, my thanks are due to Professor Edwin R. A. Seligman and Professor Felix Adler of Columbia University for the generous interest they have taken in the publication of my work in full; and also to Mr. Isador Goetz, A.B., of New York City, for his valuable assistance in the revision of the proof-sheets, and to the gentlemen of The New Era Printing Company for the care and efficiency with which the printing of this volume has been executed.

130 Henry Street, New York City,
November, 1903.

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# INTRODUCTION.

## A SURVEY OF THE MOST IMPORTANT VIEWS OF MODERN MATHEMATICIANS ON THE FOUNDATIONS OF GEOMETRY.

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Both mathematicians and philosophers at present agree that — although the science of mathematics as a whole is undoubtedly the most exact of sciences, one of her most important branches, at once the oldest and the most fascinating, namely geometry, has to some extent lost in its prestige and can no longer be quoted by epistemologists as the prototype of purely deductive, *a priori* science, and as a proof of the existence of certain innate ideas, having a purely transcendental origin, wholly independent of experience and partly conditioning it. This change of view upon geometry has taken place during the past century of all-pervading doubt and criticism, and, strange to say, it did not originate with men outside the profession of mathematics, but with those who had the greatest interest in preserving her sanctity, in keeping up the halo of her alleged transcendental origin. The very priests who worship at her shrine, the greatest mathematicians who contributed most to her fabulous growth and development in the nineteenth century, — Gauss, Riemann, Helmholtz, Beltrami, and Clifford among the immortal dead, and many prominent names still among us, — have done most to cause this change. At present, it is almost regarded as a heresy to attempt to restore some of the old prestige to the science which was considered by the Greeks to be the prototype of all science and all philosophy.

All this change of view has occurred, of course, not with regard to the method employed by geometry, the soundness and legitimacy of which have never been seriously doubted, but with regard to the very foundations upon which geometry rests. It is the *body of axioms and postulates, the definitions and common notions*, — propositions and assumptions, both implicit and explicit, which had, for a very long time, been considered as self-evident, intuitive, and independent of all elaborate proof, — propositions that need only be stated in order to elicit unconditional consent, — it is this body of, so-called, self-evident truths which at present are questioned and doubted, and by many relegated to the realm of empiricism, true only with a certain degree of approximation, and capable of being modified in an infinity of ways and, hence, of giving rise to a corresponding multitude of geometries, each consistent in itself but in contradiction with the others, each as perfect in theory as any other, and all *very nearly* agreeing with our limited experience.* It is, however, admitted on all sides, that the old system, namely, the Euclidian system, more than all others, seems to agree with the results of our experience, as far as this last goes; and if we were able to extend our experience considerably beyond the limits of the fixed stars, and if even then we should find its norms to remain unaltered and not needing revision, it would to a certain extent prove the physical reality of the Euclidian geometry and the unreality of the other systems, although the others would still be theoretically admissible and would form a body of imaginary geometries.

Now, this is rather a peculiar state of the geometrical science, singular in its kind. For, while in other branches of science two contradictory systems of thought would hardly be allowed to stand side by side, both claiming to represent the truth simultaneously, — while, for instance, the Ptolemaic and Copernican systems of astronomy could not avowedly coexist — the latter having superseded the former as soon as it was found to agree better with astronomical observations and with the abstract laws of mechanics, — while no quarter was given to the Aristotelian theory of the fixity of species by the new evolutionary systems of Lamarck and Darwin, or to the ancient doctrine of the Four Elements by modern chemistry and physics, — the contradictory systems of geometry, according to some mathematicians of note, could be allowed to stand together, side by side, and be of equal theoretical (if not practical) value and importance. So, for instance, F. Klein, in his memoirs in the

* Professor F. Klein in many places in his memoirs on the non-Euclidian geometry, *Math. Ann.*, Bd. IV, VI, XXXVII, and in his "Nicht-Euklidische Geometrie," lithographed impression, Gött., 1893, forcibly presents and defends this opinion. See lithogr. lectures, I, pp. 298–365; *Math. Ann.*, XXXVII, p. 570.

*Mathematische Annalen*, vols. 4, 6, 7 and 37, and in his "Nicht-Euklidische Geometrie," second impression, Göttingen, 1893, develops from the projective point of view three systems of geometry, — the Elliptic, the Hyperbolic, and the Parabolic systems (which in broad features had been drawn already by Riemann), corresponding to the three possible hypotheses which can be made concerning our space, namely, as possessing constant positive, negative, or zero, curvature.

Giving no theoretical preference to any of these systems, he even goes so far as to think that the question, whether one of these systems is to be preferred as expressing the real relations of our space, is unanswerable, since by allowing the radius of curvature to be sufficiently great, the elliptic or hyperbolic geometry would give results approximating, with as great a degree of accuracy as we please, to the results obtainable by the most exact measurements, performed with the most powerful telescopes upon distances such as are involved in the determination of the annual parallax of a fixed star. He prefers the parabolic geometry, however, on account of its presenting the simplest hypothesis in the theory of measurement. So he says in his lectures on the non-Euclidian geometry, referred to above, first part, page 277:* "There is, however, on the other hand, no lack of enthusiasts, who do not answer the question in the way we have done, by asserting that to our conception and experience of space could with sufficient precision correspond alike the hyperbolic or the elliptic, as well as the parabolic system of measurement, and that we decide in favor of the parabolic, solely on account of its offering the simplest hypothesis (just as in physics, among hypotheses of equal probability, the simplest is always allowed to prevail)." Each of these geometries, according to Klein, in another place in the same work (p. 295), admits of an infinity of different space-forms, — in which respect he differs from Killing, who thinks that only in case of the elliptic geometry an infinite variety of space-forms — *Raumformen* — is possible. He says: "We thus expressly contradict the remark of Killing that in case of the hyperbolic or parabolic metrics there exists the possibility only of one space-form; we say, on the contrary, that also in these cases there exists an infinity of space-forms." — It has come to

***See also pp. 161–170 of same work, where this idea is presented with especial force and elegance.**

pass, indeed, that some mathematicians are vying with one another in devising new space-forms, as they call them, for which new systems of geometry are supposed to hold. Clifford, Klein, Lindemann, Killing, and others have contributed much to this field of investigation. The enumeration and description of some of these geometries would lead us too far, and the reader interested is referred to the numerous memoirs in the *Math. Ann.* and in *Crelle's Journal* and to separate books and reprints from mathematical and philosophical periodicals, which have appeared since the beginning of the seventies up to the end of the past century.

It may, however, be said without exaggeration, that most of these space-forms impress the reader rather with the ingenuity of their inventors than with their actual value in bearing upon the question of the foundations of geometry. There seems to be rather too much license given to the imaginative faculty of the human mind; and while the origin of almost all investigations of this nature is to be sought in the impetus given to the non-Euclidian geometry by the deep-searching criticism of Riemann's inaugural dissertation on the foundations of geometry,*—where, it may be said, he formulated questions without giving final answers to some of them,—the new systems devised hardly ever carry conviction with them, and it may be stated as a certainty, that many of them would not stand a scrutinizing criticism and would have to be relegated to the realm of fancy rather than be classed with such an exact science as mathematics. To quote an instance, the elliptic space, *i. e.*, one of positive curvature with two geodesics meeting only in one point, described by Klein, Lindemann † and Killing,‡ is one of such systems. Beltrami in his "Teoria fondamentale degli spazii di curvatura costante" § makes the express statement that any two geodesics in a space of positive curvature meet in two antipodal points, through which a whole pencil of (an

* "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen," Math. Werke, pp. 254–269. It was not intended for publication by the author in the form it appeared after his death in the Göttinger Abhandlungen. See "Nicht-Euklid.," I, p. 206; Lie, "Transformationsgruppen," III, pp. 485–486.

† Clebsch, Vorlesungen über Geometrie, t. II.

‡ Crelle, t. 86, p. 72.

§ *Annali di Matematica*, ser. 2, II, 1868 (The French translation of this work of Beltrami and also of his famous "Saggio," by Hoüel, appeared in the *Jour. de l'Ecole Normale*, t. VI).

infinity of) similar geodesics must pass; and to this statement Klein takes exception in his "Nicht-Euklid. Geometrie," pp. 240–242 and in other places of the same work, and in the *Math. Ann.*, t. 6, p. 125 and t. 37, p. 554 *et seq.* Another instance is the spiral space-form, in which a rotation of a rigid body is accompanied by an increase or decrease in its volume, so that by a continuous rotation about a fixed point, any arbitrarily chosen body can be made to enclose any given point of space, and by an inverse rotation the body can be made to shrink down to an arbitrarily small portion of space around the fixed point. (Killing, "Ueber die Grundlagen der Geometrie," *Crelle's Jour.*, t. 109, 1891, pp. 185–266).

There are, however, other mathematicians, who,—agreeing in the main that the foundations of geometry have thus far not been laid down with any degree of certitude and that they are, therefore, open to considerable differences of opinion, — think, nevertheless, that there ought to be some objective truth concerning the nature of these foundations, and that it is not at all unlikely that some day a satisfactory body of axioms and postulates may be found, which will prove undebatable. The question is only in finding the minimum of simple truths, derived from experience as an original source and formulated by abstraction into a body of definitions and propositions which, — on account of their incontestable efficiency as a basis for geometry, on the one hand, and by their unquestionable reality, on the other, as well as by their being irreducible to a smaller number with equal efficiency, — should carry conviction into the mind of the mathematician, whose taste is especially fastidious in this relation, and should satisfy him that the basis is a unified whole, without leaks, and that it is capable of standing the test of a scientific scepticism (of course, not a metaphysical scepticism putting questions of the nature of whether space and time or even matter and mind, the ego, the universe, and so on, have any reality, objective or subjective, phenomenal or noumenal, etc.). Among these mathematicians is especially to be mentioned Sophus Lie, who, it seems, has contributed more than any contemporary mathematician to sound views in this matter, by treating the so-called Riemann-Helmholtz problem in a masterly way, which won for him the Lobatchevski Prize in 1897. So Lie says in his "Transformationsgruppen," Vol. III, p. 398, "We wish, however, to express the opinion, which is *a conviction with us* (wollen wir doch als unsere Ueberzeugung die Auffassung aussprechen), *that it is in no wise impossible to establish a system of geometrical axioms which at once shall be sufficient and shall contain nothing superfluous.* It is distressingly certain, however, that there are very few investigations which have actually furthered the problem as to the foundations of geometry."

In another place in the same work (p. 536) he says: "Geometry in its different stages ought, as much as possible, to be founded on a purely geometrical basis; this is a demand with which everybody undoubtedly will agree.

"For the first stage of geometry are necessary, in the first place, certain fundamental conceptions, like space, curve, point, and surface; second come certain axioms concerning, for instance, the properties of the right line, the existence of a sphere, and so on. Every new conceivable stage is characterized by the introduction of new axioms,—one stage, for instance, by the axiom of parallels, another by the Cantor-axiom."—In my own work, this last axiom, *i. e.*, that the straight line is a number-manifold, will be proved to be one of the fundamental properties of the straight line, from which its construction and all its other properties are obtained.—"Upon this axiom it is possible to establish rationally the conceptions of *area*, *length of arc*, etc., while Euclid virtually needs a separate axiom in each case.

"The great question is now, what axioms in each stage are not only sufficient but also necessary, in other words, are indispensable. In the answer to this question the whole problem of the foundations of geometry would find its solution.

And then further (p. 537) Lie sketches a programme for the mathematician who would undertake to establish the necessary axioms, which perhaps might prove fewer in number than those which Lie assumed provisionally for the purpose of solving the Riemann-Helmholtz problem.—"First one would have to establish certain fundamental conceptions, like space, point, curve, surface, and also the conception of motion. . . .

"As a first axiom one would have to establish the following: If a point $P$ is fixed, every other point can still describe a surface which does not pass through the point $P$. In this may be found the reason that two points in all *rigid motions* (Bewegungen) remain separated." [These two conceptions are certainly also connected in my treatment (see pp. 63–64, definition of distance and the proof following of the existence of a sphere), except that the second is coming first, as the simplest one, and following at once from the conception of rigidity.]

"As a second axiom it would be necessary to assume that, — *When two points $P_1$ and $P_2$ are fixed*, there are still an infinity of other points which remain fixed simultaneously, and these points form one and only one line passing through $P_1$ and $P_2$."— I think Lie would certainly not object to a proof of this proposition, which at once makes space a number-manifold and, thus, supplies also the Cantor-axiom. He says that these propositions are not yet sufficient for the first stage of geometry. Now, I think, that by means of only one additional axiom (*viz.* axiom 1, p. 63), which, according to my mind, happens to coincide with Lie's fundamental notion of a continuous group of displacements, I have succeeded in establishing all that is necessary for the first stage together with the most important postulate of the second stage, namely, the postulate of parallels, —in a sense, however, that does not exclude the spherical and pseudospherical geometries, proving only the necessity of a plane geometry in the Euclidian sense.

Lie himself treated this subject from the point of view of continuous group-transformations. Starting with the Riemann-Helmholtz postulate that space is a manifold of three dimensions, in which the position of the single element, the point, is determined by three coördinates, and adding a few very simple postulates, characterizing the group of continuous motions of rigid bodies (*i. e.*, transformations in which every two points have one essential invariant), he showed that there remains only the possibility of the Euclidian and the two non-Euclidian systems of motions.* The latter two are such as leave invariant respectively the imaginary surface $x_1^2 + x_2^2 + x_3^2 + 1 = 0$ (Riemannian group), or the real surface $x_1^2 + x_2^2 + x_3^2 - 1 = 0$ (Lobatchevski group of motions). The group of Euclidian motions together with the group of transformations by similar figures are characterized by their leaving invariant the *absolute*,

$$x^2 + y^2 + z^2 = 0, \quad t = 0.\dagger$$

* Transformationsgruppen, III, pp. 464–479. The whole of the fifth chapter is devoted to the Riemann-Helmholtz problem. See also *Leipziger Berichte*, 1890, pp. 356–418, 284–321, and 1892, pp. 297–305.

† Transformationsgruppen, III, p. 218.

The following are the axioms * which Lie considers sufficient for his investigation:

I. Space is a number-manifold of three dimensions, $R_3$.

II. The displacements or motions of $R_3$ form a real continuous group of point-transformations.

III. If any real point of general position, $y_1^0, y_2^0, y_3^0$, is held fixed, all the real points $x_1, x_2, x_3$ into which another real point $x_1^0, x_2^0, x_3^0$ can be moved, satisfy an equation with real coefficients, of the form

$$W(y_1^0, y_2^0, y_3^0;\ x_1^0, x_2^0, x_3^0;\ x_1, x_2, x_3) = 0,$$

which is not fulfilled for $x_1 = y_1^0$, $x_2 = y_2^0$, $x_3 = y_3^0$, and which, in general, represents a real surface passing through $x_1^0, x_2^0, x_3^0$. (A synthetic proof of this proposition is given in Theorem 1, p. 64, of my Dissertation.)

IV. About the point $y_1^0, y_2^0, y_3^0$ a finite triply-extended region may be so bounded that, after fixing the point $y_1^0, y_2^0, y_3^0$, every other real point $x_1^0, x_2^0, x_3^0$ of the region can still pass by continuous motion into the position of every other real point of the region which satisfies the equation, $W = 0$, and which is joined to the point $y_1^0, y_2^0, y_3^0$ by an irreducible continuous series of points. (This proposition is also proved in the theorem referred to above.)

In the *Leipziger Berichte*, 1890, pp. 357–358, he characterizes the axioms necessary and sufficient to define the Euclidian and the two kinds of non-Euclidian motions in a somewhat different way, which, however, amounts to the same thing. The analytical formulæ are interesting, and I repeat them here. The assumptions are :—

An infinite aggregate of real transformations of the points of $R_3$ $(x, y, z)$ is given by the equations :—

$$x_1 = f(x, y, z, a_1, a_2, \cdots), \quad y_1 = \phi(x, y, z, a_1, a_2, \cdots),$$
$$z_1 = \psi(x, y, z, a_1, a_2, \cdots).$$

These equations have to satisfy the following conditions:

A. The functions $f$, $\phi$, $\psi$ are analytical functions of the coordinates $x, y, z$ and of the parameters $a_1, a_2, a_3, \cdots$

---

* *Ibid.*, pp. 506–507.

B. Any two points $x_1, y_1, z_1; x_2, y_2, z_2$ have one essential invariant under all transformations of the group, of the form

$$\Omega(x_1, y_1, z_1; x_2, y_2, z_2) = \text{const.}$$

Hence,

$$\Omega(x_1, y_1, z_1; x_2, y_2, z_2) = \Omega(x_1', y_1', z_1'; x_2', y_2', z_2'),$$

where $x_1', y_1', z_1'; x_2', y_2', z_2'$ are the new positions of the original pair of points $x_1, y_1, z_1; x_2, y_2, z_2$, which these obtain in virtue of any transformation of the group.

C. The group is transitive, *i. e.*, any point of the $R_3$ can be transformed into any other point. If, however, one point $x_1, y_1, z_1$ is fixed, every other point $x_2, y_2, z_2$ can assume $\infty^2$ different positions, which are defined by the equation:

$$\Omega(x_1, y_1, z_1; x_2', y_2', z_2') = \Omega(x_1, y_1, z_1; x_2, y_2, z_2).$$

If two points $x_1, y_1, z_1$ and $x_2, y_2, z_2$ are fixed, any third point of general position, $x_3, y_3, z_3$, can assume $\infty^1$ different positions $x_3', y_3', z_3'$ defined by the equations:

$$\Omega(x_1, y_1, z_1; x_3', y_3', z_3') = \Omega(x_1, y_1, z_1; x_3, y_3, z_3)$$
$$\Omega(x_2, y_2, z_2; x_3', y_3', z_3') = \Omega(x_2, y_2, z_2; x_3, y_3, z_3).$$

[The point $x_3, y_3, z_3$ must be of general position, in order to be able to assume $\infty^1$ different positions, since there exists a singly-infinite number of points defined by the last two equations $\Omega$, such that $x_3', y_3', z_3' \equiv x_3, y_3, z_3$, namely the $\infty^1$ points collinear with $x_1, y_1, z_1; x_2, y_2, z_2$.] If three points $x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3$ are fixed, all points of space remain fixed, and three similar equations will be satisfied only for $x_4' = x_4, y_4' = y_4, z_4' = z_4$.

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The interpretation of the results obtained by Lie with regard to the possibility of the two groups of non-Euclidian motions, according to my mind, represents still an unsolved mathematical problem, alongside with the interpretation of similar results obtained by him for $n$-dimensional manifolds. A concrete interpretation of these must be found in our empirical space, for which the Euclidian axioms hold, in order to appreciate their full geometrical significance.

Not considering myself competent at present to undertake even a partial solution of the problem, I wish, however, to indicate that for two-dimensional manifolds an interpretation seems to be near at hand, not at all in conflict with our Euclidian conceptions of space. The non-Euclidian groups of transformations, assuming an invariant between two elements, only of the most general kind, and presenting projective relations to certain special forms of the fundamental quadric, must represent the real metric relations of our space, as imaged by projection upon certain surfaces of the second degree. In such a projective image the anharmonic ratio of four points, or some function of it, will remain unaltered, which will have to be taken for a definition of distance or angle in these transformed metrics; so that these metrics will coincide with the generalized Cayleyan metrics, developed by Klein. For an interpretation of this nature, we may refer to Poincaré's paper on the foundations of geometry in the *Bull. de la Soc. Math. de France*, t. 16, Nov., 1887. Another interpretation for these groups has been found in the metrics upon surfaces of positive and negative curvature, when the straight lines are replaced by geodesics on these surfaces, — an interpretation which has been fully justified by the works of Beltrami, to which reference will be made later. Klein, in his "Nicht-Euklid. Geometrie," has shown, it seems to me, conclusively (although he intended his procedure to illustrate and to show in a concrete manner the possibility of the plane having elliptic or hyperbolic metrics) that what he calls an elliptic plane is actually the central projection upon a Euclidian plane of the metrics upon a sphere,* and what he calls the hyperbolic plane is the orthographic projection of a system of metrics upon a sphere touching the plane, in which the straight lines are represented by circles orthogonal to the equator of the sphere which is parallel to the plane. At any rate, this also shows that by certain processes of projection and by making certain conventions as to the meaning of "distance" and "angle," we may be able to account for the two non-Euclidian groups of motions, as well as for the non-Euclidian metrics deduced by Klein from the Cayleyan metrics. The non-Euclidian groups of displacements for three dimensions,

* The radius of the sphere is taken to be $= 2k$, and $\pm 1/4k^2$ is the measure of curvature of the elliptic or hyperbolic plane, resp. See "Nicht-Euklid.," pp. 94–97 and pp. 220–237.

however, as well as the Euclidian and the non-Euclidian groups of displacements for manifolds of a higher number of dimensions than three, which Lie derives from postulates similar to those he assumed for $R_3$, still need an interpretation, and this interpretation seems to lie in the change of element from a point to a figure depending upon any number of parameters, which is Plücker's idea of making our space $n$-dimensional. (To this I shall yet have occasion to refer later.)

Another eminent thinker on the subject who, in the main, holds the same opinions and who has written some expositions of the ideas of Lie and of his own views on the subject, is the illustrious French mathematician Poincaré. The first publication of his on this subject is the paper in the *Bull. de la Soc. Math. de France,* quoted above; then papers of his on the same subject appeared in the *Revue Générale des Sciences Pures et Appliquées,* t. III, 1892, and in the *Revue de Métaphysique et de Morale.* Another paper, in which he further develops and complements his views in the previous papers, appeared in the *Monist,* Vol. IX, 1898, translated into English by McCormack. I quote extensively from this paper, as I find in it so many points of agreement with some of my own views — to which I have arrived independently — about the relation of experience and pure reasoning to the formation of our geometrical notions, also in relation to the number of dimensions of space, and other points.

He begins thus: "Our sensations cannot give us the notion of space. That notion is built up by the mind from elements preëxisting in it, and external experience is simply the occasion for its exercising this power . . . ."

He maintains further that variations in our sensations give rise to our notions of space. We observe two kinds of changes in our impressions, which we thus separate into two classes:

1) External changes, independent of our will, and

2) Internal changes, accompanied by voluntary muscular exertions.

The external changes again fall into two subdivisions:

1) *Displacements,* capable of being corrected by an internal change, and

2) *Alterations,* or physical changes not having this property.

*Only Displacements are the Object of Geometry.*

An identical displacement can be repeated a number of times. Hence the introduction of number.

The ensemble or aggregate of displacements form a group, since the combination of any number of these is one of the aggregate.

The notion of group could not be formed by *a priori* reasoning, but by experience together with reasoning. We *abstract from the concrete alterations* which may accompany displacements, so that *geometry is safe from all revision.*

"When experience teaches us that a certain phenomenon does not correspond to the laws of the group, we strike it from the list of displacements. When it obeys these laws only approximately, we consider the change by an artificial convention as the resultant of two compound changes. One is regarded as a displacement, rigorously satisfying the laws of the group, while the second is regarded as a qualitative alteration. Thus we say that solids undergo not only great changes of position, but also small thermal alterations." (Compare with this Postulate 1 and Scholium to Definition 8 in my Dissertation.)

"The fact that the displacements form a group contains in a germ a host of important consequences. Space must be *homogeneous*; that is, all points are capable of playing the same part . . .

"Being homogeneous, it will be unlimited, for a category that is limited cannot be homogeneous, seeing that the boundaries cannot play the same part as the center. But this does not say that it is infinite, for a sphere is an unlimited surface, and yet it is finite."

[To this reasoning I should object, for I should ask: Is not a sphere a bounded body, and therefore non-homogeneous in the third dimension? If space were finite, it would not be homogeneous in some dimension; but as all dimensions belong to space, it would be non-homogeneous in some of its own dimensions; hence, we could not say without limitation that any two displacements form a new displacement.

In my proof of the infinite extent of the straight line, I do not, however, assume the infinity of space. I only postulate that "each point is capable of playing the same part as any other," which is postulated by Poincaré also, as being involved in the notion of the group. Hence, from any point we can describe a sphere with a distance actually given by a previous construction, if it is possible to do it for one point and for one given distance. (See Theorem 2, p. 67.)]

After postulating continuity for the group of displacements, and defining what is meant by subgroups, isomorphism, invariant subgroups, etc., into which I cannot go in detail without transcribing the paper as a whole, Poincaré goes on to say that by experience combined with abstraction we arrive at the notion of a *rotative subgroup*, or the ensemble of displacements which conserve a certain system of sensations. Then he says:

"By new experiences, always very crude, it is then shown:

"1. That any two rotative subgroups have common displacements.

"2. That these common displacements, all interchangeable among one another, form a sheaf, which may be called a rotative sheaf (rotations about a fixed axis).

"3. That any rotative sheaf forms part not only of two rotative subgroups but of an infinity of them. There is the origin of the notion of the straight line, as the rotative subgroup was the origin of the notion of the point."

[In these few sentences one may discover a somewhat crude empirical statement of the facts of which I availed myself in constructing the straight line in Theorem 2 (see pp. 67–83).]

He then goes on to say that *the existence of an invariant subgroup, namely, the subgroup of translations*, in which all displacements are interchangeable, is the only fact "*that determines our choice in favor of the geometry of Euclid, as against that of Lobatchevski, because the group that corresponds to the geometry of Lobatchevski does not contain such an invariant subgroup.*"

When he comes to the discussion of *dimensions*, Poincaré points out the distinction, from the point of view of the theory of groups, between the order $k$ and the degree $n$ of a group, and states that the order $k$ is the more important characteristic of a group. So that two groups can be isomorphic (*i. e.*, their operations obey the same laws of combination and hence have the same number of subgroups etc.), and still be of different degree, provided their order $k$ is the same. In continuous groups, in general, and in the group of displacements, in particular, the object of operations "is the ensemble of a certain number $n$ of quantities susceptible of being varied in a continuous manner, which quantities are called coördinates." — "Then, every infinitesimal operation of the group can be decomposed into $k$ other operations belonging to $k$ given sheaves. The number $n$ of the coördinates (or of the dimensions) is then the *degree*, and the number $k$ of the components of an infinitesimal operation is the *order.*" — "The degree is an element relatively material and secondary, and the order a formal element." — The study of the group is mainly the study of its formal properties. The order $k$ corresponds to the number of essential parameters of a group of transformations. — "The group of displacements is of the sixth order." — The order $k$ in case of $R_3$ is *6*, since $R_3$ can have $\infty^6$ displacements. — As to the degree $n$, it depends upon the choice of the element.

If we choose the different transformations of a rotative subgroup, we get a triple infinity of elements. "The degree of the group is three. We have chosen the point as the element of space and given to space three dimensions.

"Choosing the different transformations of a helicoidal subgroup, we obtain a quadruple infinity of elements. We have chosen the straight line as the element of space, — which gives to space four dimensions.

"Suppose, finally, that we choose the different transformations of a rotative sheaf. The degree would then be five. We have chosen as the element of space the figure formed by a straight line and a point on that straight line. Space would have five dimensions.

"The introduction of a group more or less complicated, appears to be absolutely necessary. Every purely *statical theory of the number of dimensions* will give rise to many difficulties, and it will always be necessary to fall back upon a dynamical theory."

[I wish to observe here that the deduction of the number of dimensions in my Dissertation is based upon kinematical principles.]

"When *I pronounce the word 'length,'* a word which we frequently do not think necessary to define, *I implicitly assume that the figure formed by two points is not always superposable upon that which is formed by two other points;* for, otherwise, any two lengths whatever would be equal to each other. Now, this is an important property of our group.

"I implicitly enunciate a similar hypothesis when I pronounce the word 'angle.'"

I have still to quote his ideas concerning contradictions in geometry, as I think they are of cardinal importance. Here is what he says:

"In following up all the consequences of the different geometrical axioms are we never led to contradictions? . . . The axioms are conventions. Is it certain that all these conventions are compatible?

"These conventions, it is true, have all been suggested to us by experience, but by crude experience. We discover that certain laws are approximately verified, and we decompose the observed phenomenon conventionally into two others: a purely geometrical phenomenon, which exactly obeys these laws; and a very minute disturbing phenomenon.

"Is it certain that this decomposition is always permissible? It is certain that these laws are approximately compatible, for experience shows that they are all approximately realized at one and the same time in nature. But is it certain that they would be compatible if they were absolutely rigorous?

"*For us the question is no longer doubtful. Analytical geometry has been securely established, and all the axioms have been introduced into the equations which serve us as its point of departure; we could not have written these equations if the axioms had been contradictory. Now that the equations are written, they can be combined in all possible manners; analysis is the guarantee that contradictions shall not be introduced.*"

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We see thus that both Lie and Poincaré are of the opinion, that the question about the foundations of geometry represents a more concrete and, therefore, more easily manageable problem than Klein and Killing and some others are willing to grant. Neither of the former mathematicians allows an infinity of contradictory geometries, and Poincaré even gives some reasons for our choice in favor of the Euclidian geometry. He thinks only that this is solely due to the mode of experience we have of space, and when he speaks of hypothetical beings, whose experience might have led them to a predilection for the geometry of Lobatchevski, he certainly is right, in the sense that our geometrical notions, *such as they are*, are not altogether independent of the mode of experience we have, and the nature of the universe we live in. In Poincaré's own words:

"*It is our mind* that furnishes a category for nature. But *this category is not a bed of Procrustes into which we violently force nature, mutilating her as our needs require.* We offer to nature a choice of beds, among which we choose the couch best suited to her stature."

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Now, this view of the matter is certainly more encouraging to the one who would venture to find, by methods more elementary and, consequently, more legitimate for the given purpose than those based upon the laws of continuous groups, exactly which couch is the most suitable for nature's stature, and even under what conditions the other couches may become just as suitable.

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Let us now turn to Riemann, whose paper on the foundations of geometry seems to have been the occasion (apparently unintended by the author),* of many a misconception sanctioned by his name. Riemann himself, as well as can be gathered from the fact that he tried to find the laws of free mobility in manifolds of $n$ dimensions, *considering space as a special case of such manifolds, where* $n = 3$, seems to have been of the opinion that geometry, as a science of *space and spacial magnitudes alone*, must be one and only one, although he did not decide in favor of any of the three possible systems. He thought, at any rate, *that the discovery of the truth concerning the nature of the geometry of our space represents a concrete and not unsolvable scientific problem.* He stated expressly that the solution of this problem was not to be found in investigations of such a general character as was his own about manifolds in general — a thing which his followers have not always heeded sufficiently — since space was for him a manifold *of special character*, whose science alone he called geometry, as distinguished from the science of the general laws of manifolds, which, according to him, belongs to analysis and the theory of functions. So he says on p. 258 of his "Math. Werke": "These magnitudinal relations" (of multiply-extended manifolds in general) "admit of investigation *only in terms of abstract quantity, their natural connection being representable by formulæ; under certain assumptions they can, however, be decomposed into relations, which, taken separately, are capable of geometrical representation, and through this it becomes possible to express geometrically the*

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* See above, note to p. 4.

*result of the calculation.* So that, although an abstract investigation, by means of formulæ, remains unavoidable, it will still be possible to invest its final results in geometrical attire." And on page 256 he says, "More frequent occasions for creating and developing these conceptions (of multiply-extended manifolds) we find only in the higher mathematics."

His own investigation he considered useful only in so much as it threw light upon the extent and nature of the *implicit assumptions of geometry and upon the many questions of measurement in the wider region of multiply-extended manifolds, upon which these assumptions touch,* and which, apparently, have escaped the attention of his predecessors. He says on p. 268: "Such investigations which, like the one here carried through, start from general conceptions, can serve only the end that this work (the investigation of the real facts underlying our notions of space and of its magnitudinal relations) shall not be hampered by too narrow conceptions, and that the progress of discovery of the connection of things should not be impeded by the burden of inherited prejudice."—He looked, however, for the solution of this problem in the wrong direction, when he thought that some physical hypothesis which may in time prove necessary, to account for certain, as yet unexplained, physical phenomena in the realm of the infinitely small, might also throw some light upon the true nature of geometry. He certainly erred in respect of this physical hypothesis of the geometrical properties of our space.* They could lead to no better results than astronomical observations upon the stellar parallaxes, instituted with the purpose of finding some testimony in the immensely large triangles concerning the amount by which the sum of the three angles of a triangle is less than two right angles,—as is very evident from the truly philosophical treatment of this subject by Poincaré. But be this as it may, the fact still remains that Riemann, in the first place, regarded space as an unbounded manifold of *three dimensions,* and spoke of it as being an empirical certainty greater than any other we have, and, secondly, thought that the problem as to the admissibility of the propositions of the Euclidian geometry beyond the

* "It must be, therefore, either that the realities which lie at the basis of space form a discrete manifold, or that the foundation of its magnitudinal relations ought to be looked for outside, in the binding forces working upon it." (p. 268).

bounds of observation was still an unsolved, but *not unsolvable, scientific problem.*

Some of the results arrived at by Riemann are:

1. The simplest form of the linear element in any $n$-fold-extended manifold admitting of measurement, is

$$ds = \sqrt{\Sigma a_{ik} dx_i dx_k},$$

where the $a$'s are continuous functions of the $x$'s; of these, $n$ functions can be taken arbitrarily, and $n(n-1)/2$ are fixed by the nature of the manifold. In space, for instance, even if it were curved, three of the $a$'s could be taken $= 0$, each, and the rest would have to take their chances, which would depend upon the nature of the curvature. At this stage of his investigation, Riemann seems to assume that space and the plane are flat manifolds, so that their linear elements can be brought to the form of $\sqrt{\Sigma dx^2}$ (see p. 200, Werke). It would seem, therefore, that when later he speaks of the possibility of space-curvature, and of a physical investigation in the realm of the infinitely small, he means rather that, since he does not see any logical, *a priori* necessity of the necessary and sufficient assumptions of the Euclidian geometry, which he establishes in § 1 of Art. III, p. 205, he hopes to find an explanation of their necessity in physics, as he does not hope to find light on this subject in geometry proper, her realm being only the finite.

2. In an $n$-manifold we can construct at each point $\infty^{n-1}$ geodesics; then a surface-element is determined by any two of these given by their linear elements, when these are prolonged until they become finite geodesics. In other words, as Klein puts it in his "Nicht-Euklid. Geom.," p. 211, we have to con sider the collectivity of geodesics whose linear elements

$$ds_i = \lambda' d' s_i + \lambda'' d'' s_i,$$

or such whose initial directions are in the same linear manifold with the two given ones. Each of the surfaces thus obtained will have its own initial Gaussian curvature, which Riemann defines as the curvature of the $n$-manifold at the given point in the given surface-direction.

3. A manifold of constant curvature is such as has the Gaussian curvature in its surface-element the same at all points and in all surface-directions. But the nature of the manifold at a

point will be completely determined as soon as the surface-curvature is given in $n(n-1)/2$ surface-directions.

4. Only a manifold of constant curvature allows free mobility of figures, and if the Gaussian curvature be denoted by $\alpha$, the linear element of such a manifold can be reduced to the form of

$$\frac{1}{1+\frac{\alpha}{4}\Sigma x^2}\sqrt{\Sigma dx^2}.$$

5. All metrical relations of the manifold depend upon the value of the curvature. The number of ways in which an $n$-manifold can move in itself without deformation is

$$\frac{n(n+1)}{2}=n^2-\frac{n(n-1)}{2}=$$

number of coördinates minus number of distances between $n$ points.

Riemann then gives three possible forms of the conditions necessary and sufficient to determine the measure-relations of space, as distinguished from all other three-dimensional manifolds admitting of measurements and flat in their smallest parts, *i. e.*, such whose line-length is independent of position and in which the linear element is expressible as the square root of a positive differential expression of the second degree.

1°. The Gaussian curvature in three surface-directions is zero at each point; or, otherwise, the metric relations of space are completely determined, if the sum of the three angles of a triangle is always equal to two right angles.

2°. Besides the independence of line-length from position, we may assume with Euclid the existence of rigid bodies, independent of position,—which is equivalent to postulating constant curvature. The sum of the three angles in all triangles is then determined, when it is known in one triangle.

3°. We may assume not only the independence of line-length from position, but also the independence of length and direction of lines from position.

Each of these three alternatives adds something to the properties of a manifold, flat in its smallest portions. The first and last lead at once to the Euclidian geometry; the middle one allows the possibility of all three different geometries, according as the sum of the angles of a triangle is greater than, equal to, or less than, two right angles.*

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Now, then, the position and the opinions of this second category of mathematicians (Riemann, Lie, and Poincaré), and especially those of Lie, seem to indicate that the task of establishing an efficient system of axioms is not perfectly hopeless. And if the reader admit with Lie that but few investigations have materially furthered the problem concerning the foundations of geometry, he may find it of interest to read through the present memoir, even if its purpose is disclosed at the very beginning to be—the establishment of such a system and the restoration of a goodly part of the old prestige to the origin and foundations of the geometrical science.

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It may not be amiss here, in this connection, to remind the generously disposed and impartial reader that this problem, besides its philosophic interest, is also of importance from a purely mathematical point of view. The fact which will substantiate my statement is, indeed, very well known to mathematicians who have familiarized themselves, at least superficially, with the non-Euclidian geometry, although little stress is put upon its bearings by those mathematical writers on the subject, who, having satisfied themselves that within the bounds of our limited experience the Euclidian geometry holds, concluded that beyond these limits actual deviations of the metrical relations of space may take place, of which we are not bound to take heed in our analytical geometry, as long as we intend to avail ourselves of its results in actual practice only. Already the earliest non-Euclidians, and among them the two great founders of the hyperbolic geometry, Lobatchevski and Bolyai, have made it clear that the theory of proportion and similar figures is based upon the parabolic system of measurement, and that it has no meaning when the Euclidian postulate of parallels does not hold. In fact, they have both given formulæ for the solution of rectilinear triangles, perfectly analogous to those of the spherical trigonometry.† Further, Bolyai has shown

*See Lie, "Transformationsgruppen," Vol. III, p. 497, where he finds this paragraph in Riemann's paper (§ 1 of art. III) not clear.

†See Lobatchevski, "Theory of Parallels," translated by Halstead, pp. 35–45. Also, "Urkunden zur Geschichte der Nicht-Euklid. Geom." F. Engel, pp. 216–235.

that, by assuming the hyperbolic geometry to be true, the problem of the squaring of the circle presents no difficulty.* It is, therefore, evident that the establishment of the Euclidian geometry on a basis more rational than mere empiricism even if very accurate, still remains a desideratum.

It will, however, become incumbent upon me to explain my own point of view in this matter, and give in outline the results at which I have arrived, and also to throw some light upon the methods pursued in this dissertation, as well as to present the reasons which, according to the best of my judgment, can be assigned to the final success with which these methods have been rewarded.

To give my own views *upon the rôle of experience and reason in the formation of our geometrical conceptions* would, I think, only be a repetition of what is stated more or less explicitly in my introductory chapter on dimensions, as well as a repetition of many excellent remarks of Poincaré in his paper in the *Monist*, which I have allowed myself to quote so extensively. In a few words these views may, however, be summarized thus:—

As in all pure sciences, *our fundamental conceptions in geometry are formed by experience helped on by pure reasoning, which abstracts from certain unessential irregularities in the rough data of experience, by reducing certain general norms to ideal forms, not admitting of exception.* The exceptions, indeed, are purposely eliminated by ascribing to them some other causes, which are not the subject of the given investigation. So, for instance, in mechanics, the fact that no body in actual experience, possessing a certain momentum, can go on and move forever, does not bother the physicist, who postulates the first law of Newton, and ascribes the stopping of the body or the retardation of its motion to external causes, like frictional resistance, etc. Similarly, if ideal solids are postulated in geometry, the deformation which natural solids undergo in motion is ascribed to physical causes, and not to properties of space.†

Further, I think that we cannot start simply with an axiom — that space is a number-manifold, *i. e.*, each point in it can

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* See "Science Absolute of Space" by John Bolyai, Halstead's translation, p. 47.

† See postulate 1, Definition 6, and Scholium to Definition 8, of the introductory chapter of my Dissertation (pp. 40, 41).

be determined by three coördinates, or three numbers which can be made to vary continuously. I think rather that space ought to be proved capable of being made a number-manifold, and the best starting point in this direction is, according to my opinion, to be found in the simple physical facts of impenetrability, rigidity, and divisibility of bodies, each in geometry being idealized. The bulk of a given portion of space, bounded on all sides, can certainly be represented by a number, showing how many times it will contain a smaller bulk of definite shape. By considering the smaller bulk as rigid, or such in which internal motion or rearrangement of parts is excluded, and making this smaller bulk take up all possible positions within the larger bounded portion of space, we observe that *no matter where it be placed* within the larger one, *it always occupies or fills up the same numerical portion of the larger bulk.* The number of other bulks like the smaller, necessary to fill up the larger completely, besides the smaller one itself, or the number of places the smaller can be made to occupy within the larger, such that no two have any portion in common, is always the same. And this is true also when the smaller bulk is broken up into infinitely small portions, free to change position with respect to one another, but still capable of filling up completely the same space; or, in other words, when the smaller bulk is allowed to change its form in all possible ways, so as to retain only impenetrability. Equal bulks are then measured by equal spaces of same shape, which they are capable of filling.*

We postulate that this be true for any bounded space and for any small bulk which is placed in the larger one, in any position, — that there should always be the same numerical relation between the smaller and the larger bulk as soon as these are given, as a rigid solid, on the one hand, and a bounded vacuum in which the first is to lie in any position, on the other hand. We arrive at the notion of *congruent* portions within the bounded space, meaning such which the same solid fills up to the exclusion of others, — and by considering, besides, very small portions of the smaller bulk, their number and disposition with respect to one another are seen not to change as long as rigidity of form is postulated for the whole. So that, after

* For a complete and rigorous treatment of this question, the reader is referred to the introductory chapter, Scholium to Definition 8, pp. 41–44. Here is possible only a short indication of the procedure.

we have worked ourselves up to the notion of surface, line, and point, as done in the introductory chapter, rigidity is seen to imply that no two points, separated in one position of a rigid body, become ever coincident on account of change of position of the solid. And, moreover, the same continuous series of separated points of the solid must be capable of being constructed between any two given points of the solid in any one of its positions as in any other.

But this is only a starting point. We must further make clear to ourselves what we understand when we say that space is a three-dimensional manifold, considering a point as its element, and whether there is any sense in looking for a fourth dimension, not directly given by experience. It will appear from the treatment of the question in the introductory chapter that the tridimensionality of space is actually postulated by the definition of a point, and that, therefore, to look for a fourth dimension, without changing its element from that which lies at the basis of the metrical geometry of Euclid to some other geometrical object (which is, in fact, a figure in the Euclidian sense, depending upon a certain number of parameters), as is done, for instance, in Plücker's line-geometry, — is a contradiction in terms. Finally, from the same principle of rigidity, *the notion of distance as an invariable relation between two points in rigid connection or in fixed space, is easily derived by a definition which makes use of the principle of superposition.*

Next, continuity must be postulated,* and then we must show how distances can be added and subtracted, and whether there is a line, or a one-dimensional manifold, in space, capable of representing by the actual distances of its points all possible distances arrived at by addition and subtraction, and whether this can be done in a unique way. It appears, that from this property alone a notion of the straight line can be deduced, which will have all other properties of the straight line, commonly postulated for it in the Euclidian geometry; the construction, moreover, based upon this property, will make it a number-manifold † of infinite extent, such as can in no way be mixed up with a geodesic returning into itself at a finite distance. This, in a

* See axiom 1, p. 63.

† The construction makes the straight line a number-manifold in the Cantor-sense, since, as we can construct all possible sums of all possible rational numbers, we can construct the irrational numbers by sequences, in the way it is done by Cantor for pure numbers. *Math. Ann.*, t. 5, pp. 123–128.

certain way, disposes of the so-called elliptic geometry of the straight line. Further, the line so constructed will prove to be both an axis of rotation and an axis of helicoidal motion, or, in Poincaré and Lie's language, admitting either the transformations of a rotative sheaf, or those of a helicoidal subgroup, according as one point, at least, upon the line is taken as an invariant point, or none, at a finite distance, is taken as an invariant point — the line being able to slide upon itself, while all points in rigid connection with the line, but outside it, being free only to twist, *i. e.*, to move with a screw-like motion.

The notion of distance, as thus defined, is, of course, of a very abstract nature, and does not depend upon the paths which either of two non-coincident points can be made to pass *in a given surface* from its own position to the position of the second, nor upon any given position of the point-couple in space, but only upon the relative position of the two points themselves. It is simply a fact of experience that a pair of points in a solid are capable of coincidence only *with certain determinate couples* of points in other solids or vacant space, and this fact of congruence or non-congruence is the only factor determining equality or non-equality of distances. It is only after it has been proved that this geometrical magnitude is representable uniquely and perfectly by some line (*a priori*, a surface or a volume might perhaps have been found more capable to represent this magnitude, as happens, for instance, with the angular magnitude, which is equally well represented by a portion of a circular arc as by the area of a sector of the circle of radius unity, and as, for instance, the solid angle, considered as a geometrical magnitude, may be measured equally by the area of a spherical surface of radius unity whose center is the vertex of the angle, or by the corresponding spherical sector; so that it is only an accident, having, of course, its reasons in the nature of things *a posteriori*, that distance as a geometrical magnitude has for its representation a line), it is only after this fact has been established, that any curve can be broken up into linear elements $ds$, each of which is comparable with the elements of three given straight lines $dx$, $dy$, $dz$, and can be expressed in terms of these. So that any complicated expression for a linear element of some curve in space, in terms of $dx$, $dy$, $dz$, say, $ds = f(dx, dy, dz)$, must necessarily be based, in the first instance, upon a certain relation, which could be considered the simplest and which would be formed exactly in the same way as a finite distance is expressed in terms of three other finite distances, which must be taken as parameters in the case of tridimensional space.

I wish here to call attention to the fact that the deduction of the existence of the straight line in space, not as a line in the plane, as far as I am aware, seems never to have constituted a serious problem with mathematicians. This is, perhaps, the only reason why the straight line has always been regarded as a geodesic which is determined by two points *in the surface to which it belongs, i. e., as a geodesic in some plane.* The plane is *postulated* or constructed before the straight line, and angles and triangles and circles are regarded not only as plane figures, *that is, such as can lie in a plane,* but also as *figures constructed in a plane.* The distinction, according to my mind, is not at all trivial, since the existence of a plane can be proved with perfect rigor only after a number of theorems concerning angles, triangles, and circles, have been established for these figures in space. Then only, according to my opinion, we ought to prove that these simple figures, as well as the straight line, are plane figures.

*The plane, as constructed from a certain origin, must then be shown to be capable of moving upon itself in a triply-infinite number of ways, and also of coincidence with itself when its two sides are interchanged.* The first will establish the legitimacy of what is called in analytic geometry change of origin and change of axes; the second, a revolution of the plane through an angle $\pi$, which appears in the theory of groups to need special subgroups of displacements. (See Lie, "Continuierliche Gruppen," p. 101. The first kind of displacements form the group of congruent figures, the second, the group of symmetrical figures.*) In the Euclidian geometry both processes are invariably used in superposition of figures for demonstrations. In spherical and pseudospherical geometry this is also practised, with the understanding that bending without stretching is postulated. In spherical geometry bending is necessary only for making the inner side of a portion of a spherical surface coincide with the outer side of the same surface, or for the purpose of applying

* Group of congruent figures,—— $x_1 = x \cos \alpha - y \sin \alpha + a$, $y_1 = x \sin \alpha + y \cos \alpha + b$; group of symmetrical figures,—— $x_1 = x \cos \alpha + y \sin \alpha + a$, $y_1 = x \sin \alpha - y \cos \alpha + b$.

its figures to figures formed upon other surfaces of equal constant curvature; in case of pseudospherical geometry, *i. e.*, surfaces of constant negative curvature, no superposition of parts of different regions, even on the same side of the same surface, would be possible without bending. And it was just by abstracting from rigidity, in so much as bending without stretching was allowed, that Beltrami, in his "Saggio"* and in his "Teoria fondamentale," was able to prove that Lobatchevski's geometry holds good, in our Euclidian space, upon surfaces of constant negative curvature, which he first named pseudospherical surfaces.—"The fundamental criterion of the demonstrations in the elementary (Euclidian) geometry," Beltrami begins his investigation in his "Saggio," "consists in superposition of figures. The criterion is applicable not only to the plane, but also to all surfaces upon which there can exist equal figures in different positions, that is to say, to all surfaces whose any portion can by means of simple flexion be applied to any other portion of the same surface. We see, in fact, that the rigidity of the surfaces upon which the figures are traced, is not an essential condition for the application of this criterion; for instance, the exactitude of the plane Euclidian geometry would not become deteriorated, if we should begin by conceiving the figures traced upon the surface of a cylinder or a cone, instead of a plane."

Stating then that the surfaces whose figures have a structure independent of position, and hence allowing the principle of superposition without restriction, are those of constant curvature only, he goes on to say:

"The most important element of figures is the straight line. The specific characteristic of this line is that it is completely determined by two of its points, so that two straight lines cannot pass through two points in space without their coinciding in all their extent. *In plane geometry, however, this principle is used only in the following form:*

"*In making coincide two planes, in each of which there is a straight line, it is sufficient that the two lines coincide in two points, in order that they coincide in the whole of their extent.*

"*Now, this property the plane has in common with all surfaces of constant curvature, where, instead of the straight lines, we take*

* "Saggio di interpretazione della geometria non-Euclidea," *Giornale di Mathematiche*, 1868, t. VI (see note above, p. 4).

*the geodesics. . . . If we make coincide two surfaces of constant and equal curvature, so that two of their geodesic lines have two points in common, these lines will coincide in all their extent.*

*" It follows that, excluding the cases where this property is subject to exceptions, the theorems of planimetry which are proved by the principle of superposition and the postulate of the straight line for plane figures, are true also for figures formed in an analogous way, upon surfaces of constant curvature, by geodesic lines.*

*" Upon this are based the many analogies between the geometry on a plane and that on a sphere, the straight lines corresponding to geodesics, i. e., to arcs of great circles. For a sphere, however, there exist exceptions, for any two points diametrically opposite, or antipodal points, do not determine a geodesic without ambiguity, since through such points an infinity of great circles will pass. This is a reason why certain theorems in plane geometry are not true for the sphere, as, for instance, the theorem that two perpendiculars to the same line do not meet."*

Beltrami further makes clear *that the basis of investigation in plane geometry is too general, if, as usually done, the only facts lying at this basis are taken to be the principle of superposition and the postulate of the straight line. The results of the demonstrations must exist whenever this principle and this postulate are true. They must, evidently, be true for surfaces of constant curvature, in which the postulate of the straight line holds without exception.*

*Now, the purpose of Beltrami's investigation was precisely to show that this postulate does not admit of exceptions in case of surfaces of constant negative curvature. And, in his own words,— " If we can prove that such exceptions do not exist for these surfaces, it becomes evident that the theorems of the non-Euclidian planimetry hold without restriction upon such surfaces. And then certain results which seem incompatible with the hypothesis of a plane, may become conceivable upon such a surface and obtain thereby an explanation, not less simple than satisfactory. At the same time the determinations which produce the transition from the non-Euclidian to the Euclidian planimetry, are shown to be identical with those which specify the surfaces of zero curvature in the series of surfaces of constant negative curvature."*

Lobatchevski and Bolyai, who, together with Legendre, were aware of the defects of the Euclidian geometry in respect to all the postulates regarding the straight line and the plane, have both tried to construct them and to deduce their properties from the construction.* But, as it seems to me, *they did not succeed in doing it with sufficient rigor; and, besides, neither of them freed himself from the idea that a plane is given in conception previous to a straight line, and, therefore, they constructed first the plane and then the straight line in it.* The straight line, therefore, again has the same properties as a geodesic upon a surface, which will coincide with another geodesic in a similar surface, of same constant curvature, as soon as the two surfaces are superposed so that two congruent pairs of points in the two geodesics are made to coincide. *At least, neither of the two mathematicians separated the straight line from the plane sufficiently, to come to the clear idea, that figures of straight lines in space can be considered without considering the planes in which they lie. This is one of the reasons why they could not prove the postulate of parallels, which, in fact, distinguishes the plane from all other surfaces of constant curvature.* For, it is evident, that since the curvature of a surface is an extrinsic property of the surface, *i. e.*, it is a parameter by varying which we can obtain all surfaces of constant positive and negative curvature, the limit between the two being the plane (of zero curvature)†, —*those properties of the geodesics which depend upon any particular value of the curvature, could not be found, except by leaving the surface and going out into space.* Of course, the expression of the linear element can, certainly, give the true metrical properties of the corresponding surface, and hence, also of the plane. *But this very expression, as Riemann has shown, depends upon the curvature.* (In fact, for a surface of Gaussian curvature, $1/k^2 = a$, the linear element is reducible to the form

$$\frac{1}{1+\frac{a}{4}\Sigma x^2}\sqrt{\Sigma dx^2},$$

so that in case of positive curvature, $a$ is $+$, of negative curvature, $a$ is $-$, *and in case of the plane, if the "parallel postulate"*

*See "Urkunden zur Geschichte der Nicht-Euklidischen Geometrie, Nikolaj Iwanowitsch Lobatschefskij," by F. Engel, 1898 (pp. 93–109), and Frischauf, "Elemente der absoluten Geometrie," 1876, pp. 8–18.

† $\pm 1/k^2$ gives curvature of spheres and pseudospheres, with all their varieties resulting from bending, and $k = \infty$ gives the plane. See Riemann, Ueber die Hypothesen, etc.," II, 5.

*holds, $\alpha = 0$, and, conversely, if $\alpha = 0$, then the postulate of parallels holds.)*

*Hence, as long as we remain in the surface itself, and supposing we do not know the form of the linear element, we could in no way prove or disprove the "parallel postulate." And this is exactly the reasoning of those who think that the "parallel postulate" cannot be proved.* I could not do better than refer again to the last quotations from Beltrami, who, although he, as far as seen from his "Saggio" and "Teoria fondamentale," may never have expressed himself directly on the possibility or impossibility of proving the postulate of parallels, showed, however, the reason why Lobatchevski and Bolyai had arrived at a geometry for the plane which is actually true only for pseudospheres. In another place in his "Saggio" he says: "We see, then, that two points of the surface (pseudospherical surface), chosen in any manner whatever, always determine uniquely a geodesic line. . . . Thus, surfaces of constant negative curvature are not subject to exceptions which, in this respect, happen in the case of surfaces of positive curvature, and, therefore, we can apply to them the theorems of the non-Euclidian planimetry. *Moreover, these theorems, in their greatest part, are not susceptible of a concrete interpretation, unless they are referred precisely to these surfaces, instead of the plane, as we are going to prove presently in detail.*"

It is interesting to compare this sober view upon the non-Euclidian geometry of the great Italian mathematician, whose works more than those of any other succeeded in putting this geometry upon a respectable footing, with a few quotations from Bianchi, "Lezioni di geometria differenziale," German translation, 1898, t. II, p. 434. The quotations referred to are headed by the superscription, "*A Review of the Non-Euclidian Geometry,*" and follow an excellent account of pseudospherical geometry treated by the method of conform representation, and they read as follows: —

"In the principal theorems of the pseudospherical geometry, which we have deduced in the preceding paragraphs, a close analogy is observable to the propositions of the plane and the spherical geometry. The basis for these analogies, as well as for the differences, we can foresee *a priori*. *If, in fact, we examine the axioms and the fundamental postulates of the plane geometry, as laid down by Euclid in his first book, and if, in case* *of pseudospherical surfaces, we replace the straight line by the geodesic line, we see that when we leave out of consideration the XII postulate, concerning parallels, all others will hold without change in the pseudospherical geometry. This is, in particular, the case with the principle of congruence and also with the principle that a geodesic line is uniquely determined by two of its points. Those propositions of plane geometry which do not depend upon the parallel postulate hold, therefore, for the pseudospherical geometry; the others undergo such changes as to become identical with the old ones, as soon as the radius R of the pseudospherical surface is made infinite.*

"*The above considerations show readily the* USELESSNESS OF ALL ATTEMPTS MADE TO PROVE THE PARALLEL POSTULATE. *If this proof could be logically deduced from the other principles,* IT WOULD HAVE TO HOLD EQUALLY FOR THE PSEUDOSPHERICAL SURFACES IN EUCLIDIAN SPACE.

"And, in fact, if in the plane geometry we drop the Euclidian postulate, we are led to the so-called abstract or non-Euclidian geometry, whose foundations were laid down by Bolyai and Lobatchevski, and which (the straight line being taken as unlimited) perfectly coincides with the pseudospherical geometry."

*Now, this objection is certainly valid, if we were bound to consider plane figures only. Fortunately, we can have constructions involving distances, or straight lines, and angles, not bound to lie in any particular surface at all. The conception of a quadrilateral without fixed area, consisting of four fixed distances, which are equal by pairs (opposite sides), and whose angles are variable, and equal by pairs, is an important conception in elementary geometry.* A triangle is fixed as soon as its sides are fixed: *it cannot "rack"*; its *area* and its *angles are fixed* with its *sides*; it is necessarily a *plane figure.* And if the proposition is true that *the area of a geodesic triangle = a const. times the excess or deficiency of the sum of the angles of the triangle over* $\pi$, *i. e.*, $\Delta = \pm K^2(A + B + C - \pi)$, where $\Delta$ is always positive and $K$ is the radius of curvature, *we have no means of varying this area, when the geodesic triangle is conceived as bound up with the surface. But,* the 'immaterial quadrilateral,' which consists always of two triangles, is not bound up with any fixed*

*See Definition 10 of my Dissertation (p. 44).

*area and is not bound to move in any particular surface, and its angles, under certain imposed conditions, being in certain determinate relations among themselves, are still variable,—and, by considering the continuous series of deformations given in Theorem 18,* we arrive at the conclusion that there must exist a quadrilateral with equal opposite sides, whose two angles adjacent to the same side are equal to two right angles, not knowing, however, whether the quadrilateral is a plane quadrilateral or not. And then only, with the aid of the lemma, Theorem 19, which proves that the sum of any two face-angles of a triedral angle is greater than the third, we find either that such a quadrilateral must be a plane quadrilateral, or that the sum of the three angles is greater than two right angles. But the second alternative,* which holds for surfaces of positive curvature, *was proved to be impossible in the case of a plane, by Theorems 14–17; hence, only the first alternative remains,* FROM WHICH IT IMMEDIATELY FOLLOWS THAT THE SUM OF THE THREE ANGLES OF A RECTILINEAR TRIANGLE IS EQUAL TO TWO RIGHT ANGLES, OR, *in other words,* THE GEOMETRY OF THE PLANE IS PARABOLIC.

*Now, it so happens that this proof has been formulated in such a manner as to take in consideration the existence of both the spherical and the pseudospherical geometries. It actually leaves room for the existence of surfaces of constant curvature—such where the sum of the three angles of a geodesic triangle is greater than two right angles, and such where this sum is less than two right angles.* The first prove to be finite in extent, all their normals meeting in a point at a finite distance,—as follows immediately from Theorems 14 and 17, since these theorems assume the impossibility for two geodesics in a plane to intersect in two points, and the infinite extent of the plane, (both of which assumptions have been proved in Theorems 2–5 and 8); while the second class of surfaces have their normals non-coplanar, or meeting at an imaginary point. Hence, the plane appears to be the limiting case between the two, being infinite in extent and having its normals meeting in a point at $\infty$. *The considerations of Bianchi and others, which they put forward against the possibility of proving the Euclidian postulate, on the ground that if it could be proved for the plane, it would have to hold for the pseudosphere (or for the sphere), evidently have no value against such a proof where both of these cases have been con-*

* For the pages corresponding to the references given on this page see Table of Contents, above.

*sidered, and where the proof first establishes the existence of a kind of metrics somewhere in space, that afterwards is proved to be possible only for the surface of zero curvature which is the plane. Then only follows the proof of Euclid's postulate proper, for the plane, which by its metrics is now singled out among all surfaces of constant curvature, as the only one for which the sum of the angles of a triangle equals two right angles.*

If we examine in detail the axioms which have been made use of in my Dissertation, *we find that they are actually those which are necessary for defining the group of continuous motions in general, without, however, postulating that a point in space is definable by three coördinates.* Of the postulates that Helmholtz enumerates in his work "Ueber die Thatsachen, die der Geometrie zum Grunde liegen," Kön. Ges. der Wis. zu Göttingen, 1868, I assume only the first and the third, stated as follows:

1) Continuity in motion — defined exactly in Axiom 1, on page 63 of my memoir.

2) Free mobility of rigid bodies and consequent independence from position of spacial figures of all kinds.

To these, whose order, of course, is reversed, I add impenetrability and infinite divisibility. *The number of dimensions I do not assume, but deduce from the postulates.*

A word still with regard to the non-Euclidian geometry. What is its significance and the place it has to occupy in the general science of geometry?

I shall here, even at the risk of repetition, recapitulate and supplement in a concise way what I have said with reference to the two non-Euclidian groups of motions obtained by Lie from the Riemann-Helmholtz axioms as modified by him. I think that in this regard the position of Beltrami in his "Saggio" and "Teoria fondamentale" is, in general, the only tenable one. In our point-space of three dimensions the Lobatchevskian and Riemannian geometries for two dimensions are realized respectively on the pseudospherical surfaces and on the surfaces of constant positive curvature, and they can have no other concrete interpretation. With special conventions, however, as to the meaning of "distance" and "angle," we get the Poincaré interpretation by means of what he calls the quadratic geometries, *i. e.*, geometries on certain quadric surfaces. Professor Klein's interpretation of the generalized metrics can in no way make the plane become either an elliptic or a hyperbolic plane, and if Cayley's generalized metrics can be turned into account for obtaining the elliptic, hyperbolic, and parabolic geometries, it is only for different surfaces actually existing in the Euclidian space that this interpretation can be of any value. *The very arbitrariness of these different kinds of metrics, which depends upon the arbitrary value of the constant c in the formula** $(x, y) = c \log [x, y, O, O']$, where $O, O'$ are the points of intersection of the range $xy$ with the fundamental quadric, *shows that the different metrics must refer to a whole series of different two-dimensional manifolds, differing in curvature and constituting the elements $M_2$ of some $R_3$.* The aggregate of all these, however, *will, in our case, constitute a flat manifold of three dimensions, namely, the point-space of our experience,*—just as the aggregate of all possible plane curves of different curvature passing through a point in a plane, constitute a plane manifold of two dimensions.

As to the non-Euclidian metrics in three dimensions, I cannot see any interpretation for this, unless the space to which these metrics refer, be a derivative manifold contained in a higher manifold of four dimensions, since again the very parameter of the curvature suggests only a particular case out of an infinity of possibilities, arrived at by giving the parameter all possible values, ranging from $-\infty$ to $+\infty$, *and the aggregate of all these would then plainly make a manifold of four dimensions.* Here, again, Beltrami seems to have hit the truth with regard to this interpretation of the geometries of Riemann and Lobatchevski for space of three dimensions. Thus, at the beginning of his "Saggio," Beltrami says: "We have sought to render account to ourselves of the results to which this new doctrine (the geometry of Lobatchevski) leads; *and following a process which seemed to us actually to conform to the good traditions of scientific investigation, we have tried to find a real basis to these results. We think we have found one for the planimetric part, but it seems to us impossible to find one for the case of three dimensions.*" And then, at the end of the same work, after having proved the interpretability of Lobatchevski's planimetry, in every particular, by the geometry upon pseudospherical surfaces, Beltrami goes on to say: "From the very nature of the interpretation, we can easily foresee that *there can exist no analogous interpretation,* EQUALLY REAL, *for the non-Euclidian stereometry.* In fact, in order to obtain the interpretation

* *Math. Ann.*, Bd. 6.

which we have just given, it was necessary to substitute for the plane a surface which cannot be reduced to a plane, that is to say, whose linear element can in no way be reduced to the form $\sqrt{dx^2 + dy^2}$, which essentially characterizes the plane itself. *Consequently, if we were lacking the notion of surfaces non-applicable to a plane, it would be impossible for us to attribute a veritable geometric meaning to the constructions which we have developed up to this point.* Now, the analogy leads one naturally to think that *if there can exist a similar interpretation for the non-Euclidian stereometry, this interpretation must be deducible from the consideration of a space whose linear element is not reducible to the form* $\sqrt{dx^2 + dy^2 + dz^2}$,*—a form which essentially characterizes the Euclidian space. And since up till now, as it seems to us, we have been wanting the notion of a space different from the Euclidian, or, at least, such a space is beyond the domain of ordinary geometry, it is reasonable to suppose that, even if the analytical considerations upon which the preceding constructions are based, were susceptible of being extended and carried over from the region of two variables into that of three variables, nevertheless the results obtained in this last case could not be interpreted by the ordinary geometry.* This conjecture acquires a degree of probability bounding very closely on certainty, when one undertakes to extend the preceding analysis to the case of three variables.

"Putting

$$18 \begin{cases} ds^2 = \dfrac{R^2}{(a^2 - t^2 - u^2 - v^2)^2}\left[(a^2 - u^2 - v^2)dt^2 + (a^2 - v^2 - t^2)du^2\right. \\ \left. + (a^2 - t^2 - u^2)dv^2 + 2uvdudv + 2vtdvdt + 2tudtdu\right], \end{cases}$$

which takes the place of (1) in two dimensions,* — it is easy to assure oneself that the analytic deductions obtained from formula (1) subsist integrally for the new expression, *and that the value of ds given by this last is effectually the value of the linear element of a space in which the non-Euclidian (Loba-*

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* The formula (1), here referred to, is the one which Beltrami gives at the beginning of his work, for the linear element of a surface of constant negative curvature,—— $ds^2 = R^2 \dfrac{(a^2 - v^2)du^2 + 2uvdudv + (a^2 - u^2)dv^2}{(a^2 - u^2 - v^2)^2}$; this serves as the basis for his interpretation.

*tchevskian) stereometry finds an interpretation, as complete,* SPEAKING ANALYTICALLY, *as that given for the planimetry.*

"But putting

$$t = r\cos\rho_1,\ u = r\sin\rho_1\cos\rho_2,\ v = r\sin\rho_1\sin\rho_2,$$

and

$$\frac{Radr}{a^2 - r^2} = d\rho,$$

we get

$$ds^2 = d\rho^2 + \left(R\sinh\frac{\rho}{R}\right)^2(d\rho_1^2 + \sin^2\rho_1 d\rho_2^2),$$

a formula which shows that $\rho$, $\rho_1$, $\rho_2$ are the orthogonal curvilinear coördinates of the space considered.

"Now, M. Lamé has proved that, taking as curvilinear coordinates of points in space the parameters $\rho$, $\rho_1$, $\rho_2$, of three families of orthogonal surfaces — in which case the distance between two infinitely near points is represented by an expression of the form

$$ds^2 = H^2 d\rho^2 + H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2,—$$

the three $H$'s, as functions of the $\rho$'s, must satisfy two distinct systems, each consisting of three partial differential equations of the types,

$$\frac{\partial^2 H}{\partial\rho_1\partial\rho_2} = \frac{1}{H_1}\cdot\frac{\partial H}{\partial\rho_1}\cdot\frac{\partial H_1}{\partial\rho_2} + \frac{1}{H_2}\cdot\frac{\partial H}{\partial\rho_2}\cdot\frac{\partial H_2}{\partial\rho_1},$$

and

$$\frac{\partial}{\partial\rho_1}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial\rho_1}\right) + \frac{\partial}{\partial\rho_2}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial\rho_2}\right) + \frac{1}{H^2}\frac{\partial H_1}{\partial\rho}\frac{\partial H_2}{\partial\rho} = 0$$

(Leçons sur les coordinées curvilignes, pp. 76 and 78).

"In our case

$$H = 1,\ H_1 = R\sinh\frac{\rho}{R},\ H_2 = R\sinh\frac{\rho}{R}\sin\rho_1,$$

and for these values, the first system is evidently satisfied; but the second system is satisfied only for $R = \infty$. Hence the expression (18) cannot belong to the linear element of the ordinary Euclidian space, *and the formulæ founded upon this expres-*

*sion cannot be constructed by means of figures given us by the ordinary geometry.*"

And, again, in his "Teoria fondamentale degli spazii di curvatura costante," Beltrami says:

"Thus all conceptions of the non-Euclidian geometry find a perfect correspondence in the geometry of a space of constant negative curvature. It is only necessary to observe *that while the conceptions of the planimetry obtain a true and proper interpretation, since they can be constructed upon a real surface, those, on the contrary, which refer to three dimensions, are susceptible only of an analytic representation, since the space in which such a representation could be realized is different from that to which we ordinarily give the name of space.* At least, it does not seem that experience could be brought into agreement with the results of this more general geometry, unless we suppose $R$ to be infinitely great, that is, the curvature of the space to be zero. This circumstance might, of course, also be due only to the smallness of the triangles which we can measure, or to the smallness of the region to which our observations extend themselves."

In his "Teoria fondamentale," Beltrami shows, from a general discussion of $n$-dimensional manifolds, that the linear element in the Riemannian geometry of three dimensions may be taken to be the same as the linear element upon a hypersphere in a space of four dimensions. The equation of the hypersphere, with center at origin, will be

$$t^2 + u^2 + v^2 + w^2 = a^2,$$

and hence,

$$d\sigma^2 = dt^2 + du^2 + dv^2 + dw^2$$

is at once the representation of the linear element upon the hypersphere of radius $a$, and in a Riemannian space of curvature $1/a^2$.

To obtain the linear element of the three-dimensional space of Lobatchevski, he substitutes $ds = -Rd\sigma/w$, and by eliminating $w$ he gets (18). The curved Lobatchevskian space, of infinite extent, is then imaged upon the interior of the sphere of Euclidian space,

$$t^2 + u^2 + v^2 = a^2, —$$

the geodesics of that space being represented by chords of the sphere. *Every geodesic has two distinct real points at ∞*, which are imaged upon the representative sphere by the two ends of the corresponding chord, so that the spherical surface itself corresponds to the locus of all points at ∞ in the Lobatchevskian space.

But since we proved * that to assume a four-dimensional point-space is to commit a logical error, and since Beltrami's results have certainly given a *conclusive analytical proof* that we could obtain Lobatchevski's geometry for three dimensions, *if we could actually construct a curved three-dimensional manifold, contained in a four-dimensional plane manifold*, — we may surmise that the only way to obtain *a concrete and true interpretation* of the Lobatchevskian (as well as the Riemannian) stereometry is to be found in Plücker's idea that our space becomes a manifold of a higher number of dimensions, when, instead of the point, we take as its element *a figure* depending upon $n$ parameters, making space a manifold of $n$ dimensions. Therefore, it would seem that one of the simplest ways to look for such a concrete interpretation would be to start with line geometry, which makes space an $R_4$ — the next simplest after the point-space, which is an $R_3$ — and seek what in this geometry would be meant by the terms: "distance," "angle," "linear element," "curvature," "parallel," "perpendicular," and other metrical terms; and see whether the results thus arrived at, — by considering in it the special three-dimensional manifolds (complexes) possessed of "curvature," (since by excluding the postulate of parallels, which is now proved for a flat manifold, such as our point-space must undoubtedly be, we actually obtain a manifold of constant curvature), — whether these results do agree with those obtained in the Lobatchevskian and Riemannian geometries, respectively. But such an investigation would go far beyond the limits of the present Dissertation.

* In the introductory chapter of the Dissertation.

# ON THE FOUNDATIONS OF THE EUCLIDIAN GEOMETRY.

## CHAPTER I.

### SPACE AND ITS DIMENSIONS.

Definition 1. — *Geometry is the science which treats of spacial forms and magnitudes and their mutual relations.* Dealing with *magnitudes,* and with spacial forms only in so far as these are determined by their *magnitudinal* relations, *Geometry is a branch of the general science of quantity — Mathematics.*

A few introductory remarks are necessary in the way of more accurately specifying the subject of geometry, which I prefer to put in the form of definitions. These definitions, however, will not be only nominal; most of them prove the actual existence of the objects they define.

Definition 2. — *Space is that in which all bodies exist. It is the condition sine qua non of material objects.* This truth is expressed in physics by the assertion that *matter,* or the *substance of which all bodies consist, has extension,* or, in other words, *material objects occupy space.*

Definition 3. — Experience teaches us that *matter is also impenetrable, i. e., that every material object occupies a definite portion of space, which is fixed by certain limits or boundaries and which cannot at the same time be occupied by any other material object. The portion of space that is for a time exclusively occupied by a certain material object is called the place of that object.* For the sake of accurate terminology I propose to call it *the geometrical place.*

Definition 4. — Experience further teaches us that the resources of space with regard to its capacity of containing material objects, or of affording place to the material substance, are absolutely limitless. Thus, *notwithstanding the impenetrability of material substance,* explained above, *beside any occupied space there is always room enough for the existence of other material objects, and any vacant space is always conceived of only as susceptible of being filled up with matter.* Moreover, any material object is conceived of as capable of being divided into any number of portions, and these again subdivided into lesser ones, and so on, *ad infinitum.* In this respect, *extended substance* and, hence, also *space, follow perfectly the nature of abstract quantity,* ranging both ways — from the finite to the indefinitely small, on the one hand, and to the indefinitely large, on the other.

Definition 5. — The *geometrical place of a body, being that portion of space which is occupied by that body to the exclusion of any other body, has the same spacial form and dimensions as the body which fills it up.* We mean by this, that whatever measurements in regard to extension the whole body, or its several parts, may have, the same are attributed to its geometrical place, and whatever arrangement of extended parts makes up the form of the body, the same belongs likewise to the geometrical place, and *vice versa,* — so that in these regards the geometrical place may be substituted for the body, and conversely. The *geometrical place alone,* apart from all other physical properties, or, in fact, apart from the matter filling it up, is dealt with in geometry, — and is regarded by this science :

1°. As a *magnitude* — that is, not only as something that can be *greater or less,* the reason for which is given in Definition 4, but as something *that can be measured,* that is *accurately compared,* with a view of an exact quantitative determination, *with a standard magnitude* of the same kind, which is arbitrarily taken as a *unit,* and can be repeated any number of times, or divided into a certain number of equal parts, thereby becoming either equal to, or greater, or smaller than the magnitude in question; and

2°. As a *form,* consisting of *a definite arrangement of parts* according to some law, which can also be *expressed by numbers.*

Definition 6. — The *geometrical place* of a body is called a *solid* in geometry, meaning by it, that it is *mentally represented as preserving a fixed form and dimensions.* The geometrical solid is *a mere ideal abstraction* and has nothing to do with physical solidity, from which, however, it is originally derived. Geometry does not, therefore, treat it as impenetrable. It is, indeed, only the impression left by a body in surrounding space conceived of as capable of preserving the impression after the body itself has been removed. As a magnitude or thing to be measured and expressed in numbers, without regard to its form or outer appearance, it is called *volume.*

Postulate 1.—*The geometrical solid or body may be mentally imagined as moving about in fixed space, or changing its position with respect to other bodies, whether physical or geometrical, without distortion or change of form. The solid is said to possess geometrical rigidity,* meaning by it that *the disposition of the parts with respect to each other is fixed and unchangeable, or, that there is no internal motion.* This idea of geometry is derived from the fact that *space is conceived of as affording a mere passive capacity of being filled up with matter, all changes of form being referred to the active principle of the material substance proper, i. e., to physical causes alone* (cf. Definition 4). It is also based upon the undoubted fact of universal experience that, in so far as can be ascertained by observation and experiment (measurements—astronomical, physical, and geodesic), *no real, or physical, rigid body, moving about in space, has ever been known to undergo any alteration in form or dimensions on account only of change of position in space,* without regard to physical causes which, in most cases, have been found quite adequate to account for such alterations. And even if the contrary were the truth in the case of real bodies, the *Euclidian geometry* would still have nothing to do with such alterations, as *it considers only ideal rigidity,* where change of form or dimensions as depending upon position, is purposely eliminated for the sake of simplicity, and may be left to other branches of the mathematical sciences to consider (Kinematics, for instance, may very properly consider such questions as a special kind of *liaisons*—constraint—depending upon any number of parameters, those of position included). But the Euclidian geometry considers only the simplest case, even if it were only an idealized abstraction.

Definition 7.—*A body is said to be equal to another geometrically, when their geometrical places can be made to fill each other without remainder of any parts of the one, not filled by corresponding parts of the other.* When the geometrical places thus fill each other, we say that the geometrical bodies coincide, —coincidence, as thus defined, being a proof of equality, invariably resorted to in geometry.

When the coincidence can take place only with the change of form resulting from a mere rearrangement of parts, these last preserving separately their respective magnitudes and forms, the bodies are said to be of equal volume, though not equal in form.

Definition 8. — *One body is said to be greater than another,* when some of the parts of the one can be made to coincide with all the parts of the other, while there still remain some parts of the first, having no corresponding parts of the second to coincide with. The other body is then called the less.

Scholium. — Having firmly established the empirical and rational basis of the notions contained in the previous definitions, it may not be amiss to give a more compact and abstract form to the logical process by which they are obtained, which is free from all cavils on the part of those who think that spacial forms and magnitudes may, for all we know, be certain functions of absolute position, which we shall never be able to ascertain or disprove. Starting with the notions of space, matter or extended substance in general, position, and change of position or motion, and with the abstract notion of quantity, we may assume, for the sake of abstraction, the existence of a hypothetical impenetrable material substance, infinitely divisible, — *i. e.,* possessed of the following properties: —

1. Impenetrability. — *Every determinate portion or quantity of this substance occupies or fills up a corresponding portion of space which cannot at the same time be occupied by any other portion of the same substance. Any two portions of space thus filled up by the same quantity of the substance at different times, are said to be equal in capacity, and any two portions of the substance which can fill up the same portion of space at different times, or different portions of space of equal capacity at the same time, are said to be of the same bulk.* So that to each bulk, which measures the quantity of the hypothetical substance, there is a corresponding capacity of the space which is occupied by it at any moment, to the exclusion of any other portion of the same substance; to a greater portion of the substance, there corresponds a greater capacity of the space occupied by it, to a double or multiple bulk, a double or equimultiple capacity of the space, and to any part of a given bulk, a corresponding part of the capacity of the space occupied. *The generic term for bulk or capacity alike is volume,* so that the quantity of the *substance* (bulk) and *the space filled up by it,* to the exclusion of any more of the same substance (capacity), are said to be equal in volume.

2. Infinite divisibility. — *If a portion of the substance is divided into n portions, such that they can fill up spaces of equal capacity each, that portion is said to be divided into n equal parts.*

*The property of infinite divisibility* now becomes perfectly comprehensible, and *is possessed,* according to hypothesis, *both by the hypothetical substance, and by the space giving position to this substance.*

3. Form; rigidity, or plasticity. — *If two equal portions of the substance (of equal bulk or volume) are made to fill up successively the same fixed portion of space* (not merely spaces of equal capacity), *then in these two portions of the substance we observe not only equality in bulk of the whole, but also of corresponding parts, filling up* in the two cases *the same corresponding parts of space;* that is, the two portions of the substance, *while each fills up in turn the space considered, have a similar arrangement of parts that are equal in bulk in the two cases, no matter in what manner the division is made, and however small the parts considered. The two equal portions of the substance, in their successive positions, are, therefore, said to be equal not only in volume, but also in form — equal form thus meaning an equal arrangement of equal parts.* When a portion of the substance leaves a certain position, passing into another position, it may change its form (*i. e.,* the arrangement of parts may change, so that an arrangement denoted by $a, b, c, \cdots k$, may now have to be represented by $\cdots e \cdots k \cdots f \cdots g \cdots a \cdots h \cdots$, where $a, b, c, e, f, g, h, k$, etc., denote unequal portions). Moreover, even if as a whole it does not change its position, that is, if some one part of it, at least, preserves its old position, *the form of the whole may still change, and actually does change, whenever the remaining parts change their relative positions to this fixed part and to each other;* and, provided the space filled up by the whole is still continuous, that is, it is still filled up compactly and represents one concrete whole, without interruptions of vacant or unoccupied portions intervening, we say, the space occupied by the whole has not changed its volume, but has changed its form, and so has the substance filling it up. The distinction between two portions of space of equal volume and two portions of both equal volume and equal form is now clear and unambiguous. In fact, we have seen that, when the substance as a whole does not leave its original position, *i. e.,* when at least one of its portions preserves its position, a change in form is possible only in virtue of the change of position of the remaining parts with respect to the stationary part and with respect to one another, — in other words, change of form is caused by mo-

tion of parts of the whole with respect to one another, that is, by internal motion. The same is, therefore, true with respect to a substance which has left its original position entirely, filling now up a portion of space which has not the smallest part in common with the original position. If there has been internal motion or a rearrangement of unequal parts besides, then the substance has also changed its form; if there has been no internal motion, the original arrangement of parts having been preserved, the substance has only changed its position, but not its form. If a substance resists a change of form, as just defined, whether the whole is at rest or in motion, we may say that the parts are held fixed to one another, and we call this state of the substance *the rigid state;* the whole substance is then said to form a *solid or a rigid body.* The portions of space which represent any two successive positions of a solid in motion, are said to be equal to each other, in volume and form, just as two solids that can be made to fill up successively the same space, are themselves said to be equal in volume and form. When two equal solids are made successively to fill up the same space, then, by abstracting from time, that is, disregarding the fact that the filling up can take place only at different moments, we simply say that the two solids are made to fill up the same space, or, they are made to coincide with each other — coincidence being a test of equality in volume and form. If, on the contrary, the substance does not resist a rearrangement of parts, these parts are not held rigidly to one another, and change of form is possible without change of volume. Such a portion of the substance is said to possess *plasticity.*

We see now that these notions, though having a firm empirical basis, are not absolutely dependent upon the actual condition of things. *The hypothetical substance, of absolute impenetrability, need not actually exist, but as an abstraction, agreeing, in general, with our experience, it may serve as a starting point for the only possible science of measurement of extension;* since the notions based on its assumption are clear and unequivocal, and absolutely necessary to make the investigation of the laws of spacial forms and magnitudes possible. Moreover, we must agree to class all actual phenomena, in so far as they conform to the laws deducible from these notions and from those that follow in this introduction, as geometrical phenomena, that is, such as depend upon the essence of extension only; and, in so far as they deviate from these laws, they must be explained by physical causes, and any attempt to confuse these two (as very able geometers, like Clifford and others, have done) would only tend to raise a dust of endless discussion, which would never permit us to see the real foundations of geometry.

Definition 9. — A rigid physical body is said to be surrounded by vacant space on all sides, when it can be moved in all directions: forwards and backwards, to the right or to the left, and so on, in all intermediate directions.* *When some other body is posited beyond the vacant space, in any part of it, the two bodies are said to be at a distance from each other*, that admits, either of the position of some third body between them, or of the motion of one towards the other. In the latter case, the bodies are said to approach each other, the distance between them becoming less and less, until it vanishes altogether, admitting of no further approach towards each other, the bodies then being in contact. These ideas of *distance* and *contact* are *transferred upon geometrical solids, or the geometrical places of the bodies corresponding to the positions of the physical bodies just mentioned.*

Definition 10. — When two rigid physical bodies are brought into close contact with each other, so that no further motion of one toward the other is possible, they are said to have reached the *limits* or boundaries of each other, and if these limits are to some extent *continuous*, — *i. e.*, when they touch each other in many parts, the touchings being uninterrupted by intermediate vacant space, which happens when the bodies fit each other,— the limits are then called surfaces. A physical surface is, accordingly, the continuous boundary where the rigidity of a body just begins, or where the physical property of impenetrability just begins to act. If the body is surrounded by vacant space, the surface of the body is the boundary separating the impenetrable matter of the body from the capacious space. But it is neither the one nor the other, since the smallest part of the body has some of its parts removed from the boundary by the interposed rigidity of other parts of itself. No other rigid body could possibly have access to those concealed parts without overcoming rigidity, and, therefore, no part of the body, how-

* The word *direction* is used here in the common acceptance of its meaning, *viz.:* some course, but it is really vague. The scientific meaning of the word will be given in another place in this work.

ever small, can belong to its surface, of which the essential characteristic is *its being in contact with some other body, or, with vacant space* and, hence, capable of being brought into contact with some other body. Surface, therefore, has no magnitude of the same kind as a body; in other words, it has no bulk or volume, and it can never amount to any part of volume. But, as is shown in the following definition, it nevertheless has magnitude and form of its own; in other words, it is a thing that can be measured and expressed in numbers — these numbers being in determinate relations to those expressing the volume of the body bounded by the surface. One of the tasks of geometry is, in fact, the discovery and determination of these relations. The idea of surface is also transferred from physical bodies to geometrical solids, and the *geometrical surface* may be said to *represent the geometrical place of a physical surface.* Geometry regards it as a separate entity, capable of existing by itself and moving about in space, or changing its position with regard to other bodies and surfaces, whether physical or geometrical, without distortion or change of form.

Definition 11. — Since a rigid body, immersed in unoccupied space, or in any plastic material substance, displaces a portion of the material, or occupies a portion of the void, to the exclusion of other matter, equivalent to its own volume, and since this rigid body exposes only its surface — the interior parts not coming into play at all in the act of this displacement (the interior might as well be imagined hollow or devoid of matter in this connection), — it is evident that, in general, surface ought to be a function of volume, increasing with the increase of the last; that is, to a large volume there must in general correspond a large surface, although the converse is not a necessary consequence. At any rate, it is quite inconceivable how a rigid impenetrable body could take up space to the exclusion of other material substance, which, on account of its capability of motion or change of position, *can be prevented from occupying the same space as the body considered, only by the boundaries of the last, — were it not that these boundaries are in themselves an extended magnitude, standing in some functional relation to volume and form.* Accordingly, surface must have portions, all of which may be exposed to vacant space, or in contact with surfaces of other bodies, or some exposed and the others covered by corresponding surfaces of other bodies. In this last case, the exposed surfaces can again be brought into contact with surfaces of other bodies. Two surfaces will then be equal, if they can lie upon each other and mutually cover all their parts; and one is greater than the other, when a part of the first can cover the whole of the second, while another part of the first will remain exposed, or covered by a third surface.

Scholium. Any part of a body may be regarded as a separate body (in the geometrical sense of the term) from the remaining part, since each can be imagined to move about in space independently of the other, and without distortion or change of magnitude; and, while the two constitute the parts of the same solid, the limit common to both is a surface of some definite shape and magnitude.

Corollary. Surfaces coincide with one another when the bodies limited by them coincide and, conversely, bodies limited by surfaces that can be made to coincide with one another, must themselves be capable of coincidence, since when the surfaces are brought into actual coincidence, none of the bodies can help being everywhere within and nowhere beyond their coinciding limits.

Definition 12. — Both experience and reasoning lead us to the conclusion that, while *volume, or unspecified space — space in all possible directions, wherever motion is possible — is homogeneous*, surface, or that which limits a body, may be of very different kinds, having almost nothing in common, except that an indefinitely small, or infinitesimal, part of any surface may be imagined to move towards another infinitesimal part of the same surface by a continuous infinity of paths in the surface itself, as will be shown later. But this common property is not sufficient to make surface a magnitude, always definable with mathematical precision, and capable of being expressed in a voluntarily chosen unit. The indefinite size of the small part that is capable of congruence would make the computation of areas with mathematical precision impossible, *unless there be ways of reducing these to surfaces capable of coincidence in finite portions.* Volume, as a magnitude, which, in fact, is only the capacity of space to contain matter of a constant ideal impenetrability, is everywhere the same. Any part of volume is capable of coincidence with any other corresponding part of volume; this coincidence is given directly in the fundamental idea of motion together with the idea of rigidity of the moving bodies. Volume, therefore, or space unspecified, is homogeneous; whereas surface, as having an infinite variety of forms, is not so. And while, for instance, a smaller body, being posited within, or surrounded by, a larger one, invariably occupies a part of the volume of the larger,—the limits of the two unequal bodies may be, and, in fact very frequently are, incapable of coincidence in any of their finite parts. In order to make surface *a mathematical magnitude* (*i. e.*, definable with precision), there must be found at least one homogeneous surface, of which any finite part is capable of coincidence with any other corresponding part of the same, and, after taking such a surface as standard, there must be found rules how to reduce other surfaces, with any desirable degree of precision, to this standard. Such homogeneous surfaces, the essential characteristic of which is that any part of them can be imagined to slide upon the whole, remaining always in coincidence in all its parts with corresponding portions of the whole, do really exist; and their existence is likewise a matter both of experience and of mathematical deduction. Any of these homogeneous surfaces might be taken as a standard of measurement; but one, as affording the greatest advantages for computation, and being capable of indefinite extension, is accepted as the standard, and all others are always reduced to this single standard. (It is needless to remark that a difference of choice of the standard surface would, like different systems of numeration, lead only to different methods of computation, but not to different results.)

To sum up:—surfaces are multiform and are, therefore, seldom capable of coincidence in their finite portions. The nature of measurement, however, requires a homogeneous standard, to which all magnitudes of the same kind that are to be measured, can easily be reduced. Surfaces answering this description of homogeneity and, hence, capable of serving as a standard of measurement of area and form, actually exist—of different species and infinite in number—the essential characteristic of all of which is the capability of any portion of such a surface to slide along the whole, remaining always in coincidence with different corresponding portions of the whole. The simplest of these is chosen as the norm; it will be shown that it possesses the additional properties of being indefinitely extended, beyond any arbitrary limit, and of its side towards the interior of the body which is limited by it, fitting upon the opposite exposed side, so that the two can be made to coincide (plane).

Definition 13. — If two bodies are in partial contact of their surfaces with each other, the boundary separating the part of surface in contact from the part not in contact, in either of the bodies, is the limit of either the covered or the exposed portion of the surface. For simplicity, let us imagine the surfaces to be homogeneous. Any finite portion of the uncovered surface can be imagined to be in contact with the surface of some third body (see Def. 11), whose form at a finite distance is immaterial, and which moves upon the rest of the uncovered surface, along any path in it, until it reaches the limit of the covered surface, where it is checked in its motion by the rigidity of the surface of the covering body, and can only move so, that, while a portion of its surface touches the covered, another portion touches the covering, body. Motion along the boundary separating the covered from the uncovered surface, must still be possible for an indefinitely small portion of a finite body, whose contact with the two bodies, in the exposed portions of their respective surfaces, blends along the boundary for the infinitesimal element; for, as this finite body can be conceived to move along the two, remaining always in partial contact with each, the infinitesimal element, touching the two simultaneously, must, of necessity, find a region of motion along their common boundary. This boundary will, therefore, have parts of its own, *viz.*, the specializations of position of the infinitesimal touching element considered; but these parts will not be of the same kind as the parts of a surface; neither can the whole be a part of surface. In fact, it cannot be a part of the covered surface, since it must likewise belong to the uncovered surface; but, however small, a portion of the covered surface, if not infinitesimal, will always have some of its parts removed from the exposed region by intervening parts of the covered region. Similarly, it cannot be a part of the exposed surface. But we have proved that it must have parts of its own. The boundary between two portions of surface is, therefore, a new magnitude; it is a line, and its parts are different from those of a solid or a surface, but like them expressible in numbers having some determinate relation to the numbers expressing the magnitudes of volume and surface. (The same reasoning holds also in the case of non-homogeneous surfaces, provided a certain amount of plasticity of form is allowed to the touching parts of the moving body).

Scholium. — The analysis of the last Definition can be made more concrete by the following résumé, which also puts its result in a somewhat different light:—

When three bodies touch one another in their surfaces and they also fit one another, so that no vacant space is left between the parts of the touchings, the same part of surface can belong only to two of the touching bodies at once; while the boundary separating the surface belonging to any pair simultaneously, from the surface belonging to either of the pair and the remaining third body, belongs to all three bodies simultaneously and, hence, is not a surface, but a line. It is the continuous boundary of rigidity of three bodies that have come into contact with one another, and any indefinitely small part of it can be conceived to pass to the position of any other of the same only by two different courses, in case the boundary is completed and the line returns into itself, and only by one course, if the boundary is not completed.

Corollary. — From this follows immediately the statement made in Definition 12, that an infinitesimal part of a surface can move towards another infinitesimal part of the same surface, by a continuous infinity of different paths in the very surface.

The idea of a line is also transferred from a physical surface upon a geometrical one, and the geometrical line may be said to represent the geometrical place of a physical line. Geometry regards it as a separate entity, which can be conceived to move about in space, or upon a suitable surface, without change of form or magnitude.

Definition 14. — Any body may be regarded as divided into two definite parts, having one part of their surfaces — namely, that created by the section — in mutual contact, while the surface of the original whole body is now also cut into two parts, each belonging to one of the two bodies now taking the place of the original one. The boundary separating the common surface from the distinct parts will, according to Definition 13, be a line on the surface of the original body,— thus being, from one side at least, exposed to space. Any part of this line may be brought into contact with some other body. When the whole line is in contact with other bodies, leaving no part of it adjacent to vacant space, it can be regarded as covered by another, the duplicate of the first in form and magnitude, which is traced upon the surfaces of the touching bodies,— and no body, besides, can be in contact with the line. The line and its duplicate are then said to coincide — coincidence in this case also being a proof of equality in both form and magnitude. If we form, in a similar way, the duplicate of only a portion of the line, this duplicate will evidently be less in magnitude than the whole line.

Corollary. — Lines coincide with one another, when the surfaces limited by them coincide, since none of the lines limiting these surfaces can be within or without the limited surfaces. The converse, however, is not necessarily true (except in the case of the plane, as will be shown later).

Scholium. — Lines, like surfaces, are multiform; but there exist also homogeneous lines, of which any part is capable of coincidence with any corresponding part of the same. One of such homogeneous lines, capable of indefinite extension and uniquely determined by any two of its elements, is accepted as the standard of line-measurement (straight line).

Definition 15. — When the surfaces of two bodies are in partial contact, the bounding line being, in its turn, brought into partial contact with a corresponding line upon the surface of a third body, or, in other words, a part of the line being covered,— any indefinitely small part of the remaining uncovered line may again be covered with a corresponding infinitesimal line upon the surface of still another body; then the parts of the last body, immediately adjacent to the line in question, may be imagined to move along the uncovered part of the line, only by two opposite paths, until they reach the limit of the covered part of the line, where further motion is checked altogether by the rigidity of the surface of the body which has effected the first partial covering of the line. This limit, therefore, separating the covered part of the line from the uncovered, has no parts at all, since an infinitesimal element, coming up to it, finds no extension to move upon. (There would be no advantage in this case in starting with a finite part of the line moving upon the uncovered part, by allowing plasticity of form in case of non-homogeneity, since — a line being a region of motion, only for an infinitesimal part of a body, touching two surfaces at once — the motion upon it must be a filing in, or successional motion, all along the line; that is, the motion of a row of individual members, where there is a unicursal succession of each member into the place of the one immediately preceding, going on either indefinitely, or returning to the original starting place, — the members being the infinitesimal portions of the line, and all of them belonging to the same series.) The proof that the limit we have found is a part neither of the covered, nor of the uncovered, portion of the line, is perfectly similar to the proofs given for the limits of a solid and a surface, and need not be repeated again. The boundary of a line is no magnitude. It is called a point in geometry, which regards only its position or geometrical place. We say its motion generates a line, meaning by this that a line represents a field of motion for it, or, otherwise, the path of a moving point. It is regarded in geometry as a separate entity, which can move about in space independently. It is neither homogeneous nor heterogeneous, since it has no parts. The geometrical places of two points always coincide with each other, as soon as they are brought into the same position in space, within a body, upon a surface, or a line. It is, therefore, regarded as the element of space.

Corollary.—When two lines coincide, their ends, or the points limiting them, coincide also, *i. e.*, these ends have the same positions, two by two.

To sum up what has been stated in the foregoing definitions:—

The boundary separating impenetrable substance from capacious space, or the region where no motion is possible, from the region where motion is wholly unimpeded, is a surface, and admits only of motion in contact along finite regions, with the condition of plasticity of the touching surface of the moving body for the case of non-homogeneity.

The boundary separating uncovered surface from covered surface, or the region where *motion in contact* is possible, from the region where motion in contact for finite bodies is also impossible, is a line, and belonging neither to the first nor to the second, it admits motion in contact for an indefinitely small portion of a body.

The limit separating an uncovered portion of a line from a covered portion, or the region where motion in contact is possible for an infinitesimal portion of a body, from the region of absolute exclusion of motion, is a point, which is, thus, the position of an infinitely small portion of a body at rest; it has, therefore, no dimensions and only position.

Space, in its totality, being the repository of extended substance which is capable of motion (change of position) and endowed with the properties of impenetrability, rigidity, and infinite divisibility, limiting and bounding vacant space in certain definite ways,—gives rise to three different kinds of spacial magnitude, so connected that one is the limit of the other and is limited by the third. The point is the result of limitless divisibility of any of the three kinds of extended spacial magnitude.

Space is, therefore, a tridimensional manifoldness, only because of its three chief attributes, giving three different kinds of specializations of position, limiting each other and so connected that there is always a certain determinate relation between the units of one kind of space and the units of the other kinds. It was shown in Definition 11, that the rigid surface by means of which the property of impenetrability makes itself effective, must be some function of the volume; reasons were also given why a surface should be a field of motion for finite regions in contact, and a line, a field of motion for infinitesimal regions of contact; from which will follow at once, that a line is a differential element of a surface, and a point, an element of a line. Limitless space, in its totality, would be a one-dimensional magnitude, ranging from zero to infinity, in terms of volume alone,—were it not for the invariable relations between the units of volume and those of the two subcategories of specified space, resulting from the fact that they always limit one another. In order, therefore, to suppose that space has more than three dimensions, we must conceive that a point has dimensions, since we began our analysis from free unlimited space, and found it, in itself, without considering its limits, to be one-dimensional; and only in relation to its limits does it become tridimensional, since its first two limits (but not the third) are also magnitudes and bear certain fixed relations to unspecified space as a magnitude. Since $u$, $du$, $d^2u$, where $u$ represents a piece of unspecified space, *i. e.* volume, are all variables, but not so $d^3u$, it follows that, if $u$ is a function of $x$, it must be of the third order: $u = ax^3$ may do for the simplest representation of such a function, where $dx$ will represent the differential of a line; $dx$ may, of course, be a homogeneous function of the first degree, of a number of differentials $dx_1$, $dx_2$, $dx_3$, $\cdots$ $dx_n$; but then there must be $n - 3$ linear relations between the $dx$'s, reducing the number of independent ones to three. It is absurd, by reasoning in a reversed order, to infer by a kind of induction, that, just as a point in moving generates a line, a line, in moving out of its regions, generates a surface, and a surface generates a body, so tridimensional space, in moving out of itself, will produce a new kind of space. *In the first place, we would have to prove in general, that if our reasoning holds for $n$, it will hold for $n + 1$, as we always do in mathematics in such a kind of induction.* Riemann's construction of an $(n + 1)$-fold variability, out of an $n$-fold one and a variability of one dimension, is based on the assumption that the $n$-fold variability passes over into another one, *entirely different*, in a determinate way, so that *each point of the first passes over into a definite point of the other, which is not at the same time a point of the first, — an assumption that must be proved in each particular case.* In our case, this assumption is actually equivalent to assuming, that space can move out of space, which is absurd by the very definition of space, *viz.*, space is that which gives place to material objects, whether at rest or in motion; so that wherever motion is at all possible for a tri-dimensional piece of space, there is space again. In the second place, even if we admit the possibility of space moving out of itself into some other region, we have not admitted any new property of space which might be the object of measurement, since we tacitly assume that the new region is not space, so that space would remain again tridimensional. And even if we should admit a new unknown property of space, we would still have to prove that tridimensional space is its limit; and that its units bear a fixed mathematical relation to the units of space we have already considered. Indeed, it is not sufficient for a phenomenon to have a certain number of properties, in order to consider that phenomenon of as many dimensions as the number of its properties. The properties must be the limits of each other, and their units must stand in a certain invariable mathematical relation to one another.

The following is an analytical deduction of the number of dimensions of space considered as a point-manifold, which was written up and added, as a supplement to the introductory chapter of the main body of the Dissertation, two years after the latter had been completed. The purpose of this analysis is to prove the *a-priori* necessity of three dimensions, when the point, as usually defined in elementary geometry, is taken as the element of space. It will follow that to make space a four-dimensional manifold, without changing its element to some other geometrical entity, will involve a contradiction in terms. The discussion is divided into 13 paragraphs. —

1) *Let the whole original manifold be S,* which we suppose to be *continuous, i. e.,* any two different positions in it can be reached from each other *only through an unbroken series of other positions, all in S,* whose number is infinite.

2) Let an invariable piece of it — *U,* endowed with impenetrability, rigidity, and infinite divisibility, be imagined as capable, as a whole, of changing its position in *S.* The invariability of *U* is characterized by the fact that the mutual disposition, or arrangement, and the relation, of the parts of *U,* obtained in a determinate way by any arbitrary subdivision, is to remain unaltered with respect to one another and with respect to the whole of *U,* considered as an entire manifold in itself. We say that internal motion, of parts of *U* within *U,* is excluded, and any two parts that have been separated from each other by certain continuous series of other parts, in one position of *U* within *S,* will remain so in any other position. We thus arrive at the notion of Bulk or Volume. Further, this notion is made more precise by postulating, that, when such a rigid piece has once occupied a definite portion of vacant or unoccupied *S,* to the exclusion of any other impenetrable piece, it will always be brought into coincidence with that portion again, as soon as a finite part of it, no matter how small, will be brought into coincidence with the corresponding part of *S* it has occupied originally; any other small portion of *U* will then also have to occupy its original position within *S.* (According to this postulate, it is perfectly indifferent whether *U* is conceived to move within *S,* or *U* is conceived as stationary and *S* as changing its relation with respect to *U,* in giving it position in different portions of itself — all these portions being, of necessity, equal in bulk or volume to one another and to *U.*) If we should measure only bulk or volume, we would get only one dimension. The property of impenetrability of the movable pieces, however, leads us to distinguish a *new category of manifold,* subordinate to the category *S* and contained in it, in the following way: —

3) Suppose *U* fixed, and another piece *V,* of same nature, moving up to it, reaches the boundary of the latter. It will

then be prevented from occupying the same place as $U$ by the impenetrability of the two, exhibited in their boundaries, so that a certain kind of motion of $V$ is checked when the latter comes *into contact with U.* Here we have the notion *of the boundary of U separating it from vacant S, or from other pieces V of same nature as itself.*

4) This first *derivative boundary* of $U$, which we may denote by $U'$, is neither a portion of $U$, nor of $V$ that has come into contact with $U$, nor of the vacant $S$, in which $U$ is posited; it may, or may not, have portions of its own. Example:—the limits of a period of time, no matter how great, have no parts, since there is no possibility for an invariable piece of time to change its position with respect to other invariable pieces, and move up to them to come in contact with their boundaries. In our case, however, $U'$ must have parts of its own, as will become evident from the following considerations:

5) By infinite divisibility of a rigid piece $V$, we may arrive at the notion of a plastic substance, each infinitesimal portion of which occupies a corresponding infinitesimal portion of the manifold $S$, and the whole retaining only the property of impenetrability, having lost, however, rigidity. It will represent, in fact, a plastic substance, the smallest portions of which are easily capable of separation and change of position with respect to one another and with respect to the whole of their aggregate. The rigid piece $U$ considered, if immersed in this plastic substance, will displace a portion of it equivalent to the bulk of the portion immersed. To a smaller portion of $U$ immersed will correspond a smaller bulk of the plastic material displaced, and to a greater portion immersed will correspond a greater bulk displaced. Now this displacement is necessarily effected only by the boundary of the immersed portion of $U$, and by that part of it alone which has, before immersion, been exposed to vacant $S$, as distinguished from the remaining part which—separating the immersed portion of $U$ from the non-immersed—could have no effect in the act of displacement considered. It follows, therefore, that to a greater bulk displaced—and hence to a greater portion of $U$ immersed—corresponds a greater portion of the boundary $U'$, which we may call $U'_i$ ($i \equiv$ immersed), as distinguished from the remaining portion of $U'$, which, as belonging to the exposed portion of $U$ only, we may call $U'_e$ ($e \equiv$ exposed). By changing con-

tinuously the portion of $U$ immersed, from an infinitesimal bulk to the whole bulk of $U$, we arrive at the notion of a continuously increasing boundary $U'_i$, from an infinitesimal to the whole of $U'$, and a correspondingly decreasing $U'_e$, from the whole of $U'$ to an infinitesimal of $U'_e$, and then to zero.

6) The limit separating the two portions of $U'$ (*i. e.*, $U'_i$ from $U'_e$), at each and every stage of the process, is evidently a new kind of boundary — $U''$, — and the infinitesimal portion of $U'$, between two very near $U''$'s corresponding to two very nearly equal bulks of $U$ immersed, in the continuous process described above, may be denoted by $dU'$, meaning an infinitesimal of $U'$, and the aggregate of all these, $\sum_k dU'_k$, may be taken to equal the whole of $U'$ belonging to the whole of $U$, just as $U = \sum_k dU_k$. (In the synthetic discussion of the dimensions, above (Definition 12), allusion was made to the geometrically indispensable notion of a homogeneous $U'$, which might serve as a standard of measurement for different $U''$s, and later the actual existence and construction of such a $U'$ will be rigorously proved.) A homogeneous $U'$ may be considered a field of motion for finite portions of $U'$, covered by corresponding portions of $V'$ belonging to some moving rigid $V$, which, in the process of motion, touches $U$ in the variable portions of $U'$ considered. $U'$ in itself, without the piece $U$ which is bounded by it, has an *independent existence, only as an abstraction* (like, for instance, force without matter). In fact, *we can speak of it as moving about,* either in $S$ or in its own region, *only in virtue of corresponding motions of* $U$ (*or* $V$) to which $U'$ (or $V'$) belongs.

7) Corollary. — It follows also, that $U'$, as a boundary between $U$ and unoccupied adjoining portions of $S$, or between $U$ and $V$, must be considered as having the property of impenetrability. In fact, $U'$ is conceived of as that which prevents any portion of $V$, no matter how small, from penetrating into the region occupied by any portion of $U$, and *vice versa.* But, *as a region of motion for portions of itself, i. e., as a manifold in itself, it cannot possess the property of impenetrability, and must, in fact, be of the same nature with respect to portions of* $V'$ *or* $U'$ *moving in the whole, as vacant* $S$ *is with respect to impenetrable* $U$ *or* $V$. This is a reason why the boundary between $U'_i$ and $U'_e$ can be obtained only through the medium of an auxiliary $V$, which may also be considered rigid, and portions

of whose boundary — $V'_e$, very near $V'_c (= U'_i,\ c \equiv$ covered), will then serve as a check to a third rigid piece $W$, of same nature as $U$ and $V$, a finite portion of whose boundary — $W'_c$ is conceived as covering corresponding finite portions of $U'_{e_v}$ (exposed with respect to $V$ only) and moving in the manifold $U'$, until a finite portion of $W'_e$ comes into coincidence with a corresponding portion of $V'_e$, which thus becomes $V'_{c_w} = W'_{c_v}$.

8) It is important to observe that, when a piece of the original manifold $V$ comes into coincidence with another piece of the same kind $U$, in general only finite portions of their boundaries coincide and thus become $U'_c = V'_c$; the remaining portions of their boundaries combine in forming a new combined boundary of the piece $(U+V)$, taken as a whole, — this combined boundary being now represented by $(U+V)' = U'_e + V'_e$.* It is, therefore, evident that, when $W$, considered in No. 7, moves up to $V$, the two portions of its boundary — $W'_{c_u}$ and $W'_{c_v}$ — become now combined into the boundary separating $W$ from the combined piece $(U+V)$, *viz.*,

$$(W'_{c_u} + W'_{c_v}) = (U+V)'_{c_w};$$

and, in general, for the same reason as above, there will yet remain a finite portion of $(U+V)'$ exposed which we may denote by $(U+V)'_e$.

9) We see now that $U''$, — originally obtained as the limit separating $U'_i$ from $U'_e$, when $V$ was considered as the plastic substance in which a portion of $U$ was immersed, and then identified with the boundary separating $U'_c (= V'_c)$ from $U'_e$, and also from $V'_e$, in case $V$ is considered rigid and a portion of its boundary $V'_c$ covering an equal portion of $U'$, — by means of the process considered in No. 7 becomes broken up into two portions: one lying in the region of $(U+V)'_e$ exclusively, and separating $U'_{e_w}$ from $V'_{e_w}$, or each of these from the same corresponding portion of $U'_c (= V'_c)$, and the other lying exclusively in the region of $(U+V)'_{c_w} = (W'_{c_u} + W'_{c_v})$, and separating $U'_{c_w}$ from $V'_{c_w}$. Let the first be called $U''_{e_w}$ and the second $U''_{c_w}$.

10) By introducing another piece $T$, which we make to play the rôle of $W$ for the breaking up of $U''_{e_w}$ into $U''_{e(w,t)}$ and

* This combined boundary is in no way different in character from $U'$, being like it continuous, and having finite parts represented by $\Delta(U+V)' = \Delta U'_e + \Delta V'_e$.

$U''_{e_w c_t}$, and then still another piece $X$ for the breaking up of $U''_{e(w,\,t)}$ into two pieces, and so on, we see that $U''$ consists of as many parts as we please. Moreover, by making $W$ move up to $T$, so that $(W'_{c_u} + W'_{c_v})$, — conceived as changing its position continuously and as changing its form and magnitude if need be,* — shall always remain during this process in coincidence with an equivalent variable $(U + V)'_{c_w}$, we shall convince ourselves that an infinitesimal $d(W'_{c_u} + W'_{c_v})$, in the neighborhood of $U''$, remaining always in coincidence with a variable $d(U + V)'_{c_w}$, will displace itself and find a region of motion along the element $d(U + V)'$ taken in the neighborhood of $U''$ and conceived, *in toto*, as a locus for the different positions of the moving infinitesimal portion of the first derivative boundary. $U''$ itself, therefore, as a whole, will prove to be a *locus in quo* for $U''_{c_w} = \lim\limits_{0} d(U + V)'_{c_w}$. When $W$ comes up to $T$, it is there checked, — and we arrive at the conception of $U'''$, separating the region of motion for $U''_{c_w}$ from the region where such a motion is impossible. It is, of course, the same boundary as that which separated $U''_{e_w c_t}$ from $U''_{c(w,\,t)}$, obtained at the beginning of this paragraph.

11) In summing up, we see that, while $S$ is a region of motion for pieces like $U$, $V$, etc., the first derivative boundary $U'$ is the limit of $dU$, and that, not being a portion of $U$, it has portions of its own, being a region of motion for corresponding finite portions of the boundaries of a movable covering piece $V$. $U''$ is the limit of $dU'$ as $dU' \doteq 0$, and, not being a part of $U'$, *on account of its separating the region of motion in contact for a finite piece W from the region where such a motion is impossible*, it still has portions of its own. For, although $W$ cannot find upon it a region of motion, even *for a finite portion of its boundary*, it can find upon it a region of motion for the limit of an infinitesimal portion of its boundary, namely for $\lim\limits_{0} dW'$ $(= W'')$. For, as it was shown, $W$ can be made to move so, that, while two finite portions of its boundary, distinct but contiguous in $W''$, move each upon a corresponding portion of $U'$ and $V'$ respectively, an infinitesimal of its boundary $dW'$, contiguous to $W''$, and taken as near $W''$ as we please,

* This was previously shown to be possible, by supposing the first derivative boundary to be homogeneous, for simplicity, or by allowing sufficient plasticity to $W'$.

on either side of it, will move and remain in coincidence with corresponding variable infinitesimal portions of both $U'$ and $V'$. In passing as near to the limit as we please, we come to the conception of portions of the limit $W''$ itself, moving in the whole of it as in a *locus in quo*.

The third derivative boundary $U'''$ separates the region of motion for the *limit of an indefinitely small portion of the boundary of the first order* from the region where motion is impossible even for the limit of an infinitesimal portion of such a boundary. An infinitesimal portion of $U''$, very near $U'''$, on both sides of it, namely $dU''$, may be considered an element of $U''$, so that we have $\Sigma dU'' = U''$.

12) And now, if we agree that any manifold derived from $S$ in the manner indicated, will have to be capable of giving place to impenetrable substance, or *to its boundary* which is *directly connected with the substance*, we find that the last derived boundary, $U'''$, which does not, even near its limit, afford a region of motion to an infinitesimal of the first derivative boundary $dU'$ of the impenetrable substance, has only position but no dimensions. So that we have three categories of space, corresponding to three properties of bodies — rigidity, impenetrability, and infinite divisibility, — each limiting the preceding and limited by the following.

These are: —

$S$, a region of motion for pieces of the impenetrable substance itself — $U$, $V$, etc., and their boundaries, both finite and infinitesimal;

$U'$, the first derivative boundary of impenetrable substance, a region *of motion in contact* of finite pieces like $U$, $V$, *i. e.*, *a region of motion for a finite portion of the boundary of a piece of impenetrable substance*, and, lastly,

$U''$, the second derivative boundary, *a region of motion for the limit of an infinitesimal portion of the first derivative boundary* $dU'$.

The boundary, $U'''$, between two portions of this last region of motion, because of its limiting the region of motion for an infinitesimal portion of the boundary belonging to impenetrable substance, from the region of no motion even for an infinitesimal portion of the boundary, must be of zero dimensions, but still capable of having definite position, being the primary irreducible element of space. It follows, therefore, since each of the boundaries is capable, near the corresponding limit, of being considered an infinitesimal element of the category which is bounded by it, the last category in order derived, namely $U''$, has one dimension; the one preceding, two dimensions; and the original one $U$, or a piece of vacant $S$ as measured by $U$, three dimensions.

13) * By the very process of the deduction of the three qualitatively different categories of space (regions of different kinds of motion, which, however, are the boundaries of each other, in a series), we have arrived at the notion of a manifold of two dimensions, objectively not independent from the main category, but, none the less, having a true abstract reality, and which, by its very nature, as a manifold in itself and not as a boundary, is devoid of the property of impenetrability (see No. 7). In this derived manifold, therefore, boundaries of portions can be established only by means of a piece of the higher manifold, having the property of impenetrability in the region of the dimensions of the derived manifold considered, and, for this very reason, suggesting a dimension over and above those of the derived manifold. It is no wonder, then, that the general reasoning, applicable to the original manifold, is not applicable to the derived manifold. The figure given as an objection, instead of disproving the reasoning, is only another proof of its validity. Mark, that in order to attribute impenetrability to the limits of the circle, you must postulate it to be infinitely thin, and $V$ an infinitely thin film, — which, of course, is equivalent to postulating a third dimension, but in a very disguised form.

Now, this infinite thinness is already capable of being increased indefinitely. In other words, the assumption of impenetrability, by the reasoning employed above, would involve a third dimension, outside of the given manifold of two dimensions, leaving this last unchanged. A reference to Nos. 7, 8, 9, 10 will fully justify this assertion.

And still otherwise, — perhaps this way of looking at the thing may be more satisfactory: — To such intelligent beings

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*** 13) is a reply to an objection raised during a conference when this was presented: Why by the same general reasoning we are unable to prove a third dimension even in the domain of an admittedly two dimensional manifold, as is, for instance, the first derivative manifold. The nature of the objection will be understood by the reader from the reply, which alone is given here explicitly.**

as would have no sense of the third dimension, if such beings were at all possible, the speculation about a third dimension would not only involve no logical contradiction, but, on the contrary, would be a perfectly logical and necessary generalization. For, they would have to postulate some dimension — not directly given in experience — as a medium for the continuous passage of $W$ to $T$, remaining always in contact with both $U$ and $V$, which, even in two dimensions are not essentially separated, since they are both on one and the same side of $U'$ and $V'$. But the speculation about a fourth dimension for such beings as

$U$, $V$, $W$, $T$ are supposed to be two dimensional regions of motion to startwith.

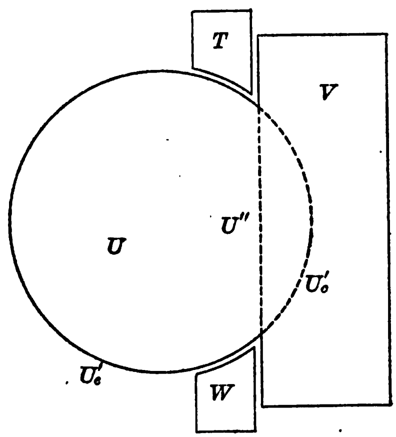


Fig. A.

have already risen to the empirical verification of the abstract deduction of three dimensions, would certainly involve a logical inconsistency. For, starting with the manifold $S$ as defined above, in its most general aspect, without boundaries at all, and, for all we know, having $n$ dimensions, where $n$ may be any entire positive number, we were led, by a simple analysis of its definition, to three different manifolds, containing each other in series, and the third derivative boundary which is the boundary of the lowest category in this series, was found to be something that can have, at most, position, but no dimensions. If preferred, you may say that we have proved that the maximum number of dimensions of space as defined by the properties of $S$, is three, and that, having logically arrived at the maximum, we find it in perfect agreement with our intuitive experience, which, of course, also served us as a starting basis in defining the manifold $S$, at the beginning of the present discussion.

I deem it necessary to repeat at the conclusion that I acknowledge the fruitfulness of the idea of making space a manifold of a higher number of dimensions, by dropping the property of impenetrability in the physical sense, and assuming a figure depending on $n$ parameters, as the element of space.

## Chapter II.

### THE SPHERE, THE CIRCLE, THE STRAIGHT LINE, THE ANGLE, THE TRIANGLE, THE PLANE, ETC.

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Definition I. — *A pair of fixed points in space, or any two points in rigid connection which can be made to coincide with these, are said to be at an invariable distance from each other.*

Corollary I. — *If a pair of points, $A$ and $B$, in rigid connection, are capable of coincidence, one by one, with another pair, $C$ and $D$, likewise in rigid connection between themselves, the two pairs will be at equal distances, each point from its pair, i. e., distance of $B$ from $A$ = distance of $C$ from $D$ or of $D$ from $C$.*

Corollary II. — *Two pairs of fixed points in space, $A$, $B$, and $C$, $D$, both of which are capable of coincidence successively with the same freely movable pair in rigid connection, $E$ and $F$, are respectively at equal distances.*

For shortness we shall call a pair of points in rigid connection, free to move as a whole, simply *a rigid pair of points,* and will denote them thus ($AB$).

Axiom 1. — If any surface or line can be made to coincide with another surface or line, it can do so only by passing from one position to the other by a continuous path, consisting of an infinity of such positions, every position in which is a surface or line, respectively congruent with the moving one.

Lemma 1. — From the principle of rigidity and the definitions of body, surface, line, and point, it follows that a body is absolutely fixed in space when, and only when, a finite portion of the surface limiting it, or in any way rigidly fixed in it, is fixed in space.

For, whenever a body is fixed in space, the whole of its surface, limiting it from all space around, and hence every finite portion of this surface, is fixed in space; and whenever the body is moved, the whole of its surface, and hence every finite portion of it, no matter how small, changes its position. We cannot say, however, the same of a point in the surface, since a point, having no magnitude, but only position, does not limit the surface to which it belongs and which may be conceived to change its position as a whole, and hence to interchange the positions of all congruent lines in it that are drawn from a common point limiting them all and remaining fixed. Hence, a body and, therefore, also any surface or line, rigidly fixed in it, may be conceived to move when only one point in the body, the surface, or the line, is held fixed in space. Such a motion of the body is called rotation, revolution, or turning, about the fixed point.

Theorem 1. — If $(AB)$ denote a rigid pair of points, of which $A$ is fixed and $(B)$ is made to assume all possible positions compatible with the rigidity of the pair and the fixity of $A$, then $(B)$ will describe *a homogeneous surface called a spherical surface, limiting a body called a sphere. A is called the center of the sphere, or of the spherical surface, and the invariable distance* $\overline{AB}$, from the center to any point in the surface, is called *the radius of the sphere.*

In fact, the moving point $(B)$ can pass from any fixed point $B$ with which it initially coincided to any other point $B'$, $B''$, and so on, by a continuous infinity of paths crossing and recrossing one another in all conceivable ways, provided all these points are at the same distance from $A$ as $B$ (preceding Lemma and Axiom 1). Let $(B)$ pass from $B$ to $B'$ by a continuous path of some determinate rigid form, $BB'$; while doing so, any rigid line connecting $A$ and $(B)$ will describe a portion of a surface. This surface, in its turn, conceived as a rigid form, can (according to Lemma) be imagined so to move while $A$ is fixed, that every one of its points describe a line not already contained in the original position of the surface itself, — thus describing a body. The line $(BB')$, conceived rigid and always limiting the moving surface $(ABB')$, will then be dragged along with it in its motion and will describe a surface, since each of its points describes a line not coincident with the original position of the moving line; in other words, each point of the line $(BB')$ passes to another point not already contained in the original position of the line. Moreover, since the surface $(ABB')$ can be conceived to sweep through all that portion of space around $A$ whose aggregate of points is characterized by the property that their distances from $A$ are correspondingly equal to those of the aggregate of points contained in $(ABB')$ in its original position, it follows that the surface described by the line $(BB')$ will contain the whole aggregate of points which are at the same distances from $A$ as $(B)$ in its original position. It is also evident from the mode the body described by the surface $(ABB')$ is generated, that it is a continuous body, *i. e.*, the aggregate of points composing it is a continuous aggregate, which allows to pass from any point in the body to any other through any third point belonging to the body, by a continuous path lying wholly in the body (*i. e.*, every point of which belongs to the body). The surface, therefore, is also continuous, in the same sense, — namely, that we can pass by continuous motion from any point in it to any other, through any third, by a path every point of which is in the surface. Moreover, the moving point will during such a motion remain at the given distance from $A$, hence, on the surface of an imagined fixed sphere; and if the moving point is conceived to be fixed in a sphere, of same center $A$, but which is dragged along with $(B)$ in its motion, we see that every portion of a spherical surface is congruent with every other of same limits; that is to say, *the spherical surface is a homogeneous surface.*

Corollary I.—Since the spherical surface separates a continuous portion of space from all other space, that is, all points that can be reached by one continuous path wholly contained in the body, from all points of space that can not be reached by a continuous motion from the center, unless the boundary of the body is crossed by a path of which a portion, at least, does not belong to the body, — it follows that this surface is also a closed surface.

Definition II. — *All points in the sphere* (body limited by the spherical surface) generated by the given radius $\overline{AB}$, from the given center $A$, *which can be reached by continuous motion from the center, without crossing the surface at all, or after crossing and recrossing it an even number of times, are said to be within the surface or inside the sphere, and the distances of all these points from the center* (excluding those of the surface itself) *are said to be smaller than the distance* $\overline{AB}$. *All points that can be reached by continuous motion from the center, only after crossing the surface once, or crossing and recrossing it an odd number of times, are said to be without the surface or outside the sphere, and their distances from A are said to be greater than the distance* $\overline{AB}$.

Corollary II. — When a sphere is conceived to move within the boundaries of a fixed spherical surface which is always in coincidence with the surface of the moving sphere, its center (or centers, if there can be more than one), remaining at a constant distance (or constant distances) from each and every one of the same fixed aggregate of points belonging to the same surface, will remain fixed in position. But since, when the sphere moves around one fixed center, from which it is conceived to be generated, every other point in it, at a distance from the center, *moves* upon the surface of a sphere (Theorem 1), *it follows that, given a spherical surface as a whole, its center is uniquely determined. A complete spherical surface is, therefore, said to be the locus of all points equidistant from a unique fixed point, called the center.*

Corollary III.— If two spheres coincide in any finite portion of their surfaces, they coincide throughout, and, hence, have the same center and radius.

For, by holding the finite common portion of each spherical surface fixed, each of the surfaces remains fixed (Lemma 1). But if either of the spheres is conceived as moving within its fixed boundaries, around its fixed centre, every point *at a distance* from the fixed centre, moves upon a corresponding sphere, and passes into the position of every possible point on the last, and of no other point. Hence, there is only one center common to both, since the portion of each movable spherical surface, which is initially in coincidence with the corresponding common portion of the two fixed surfaces, can, without leaving either of the spherical surfaces, pass over into any congruent portion of either.

It follows from what has preceded, that a distance is a geometrical magnitude. No two points at different distances from a given point, can be connected with the given point by the same line, which might serve as a path of motion from one endpoint of each distance to the other. If we take, to fix ideas, as the connection between $A$ and $B$, a line of some determinate shape, lying wholly within the sphere described by $(AB)$, a concentric sphere, described by a distance $AC$ less than $AB$, will cut every line $(AB)$ into two parts, one of which will lie within the smaller sphere, and the other, outside. Thus we see that to a smaller distance corresponds a smaller path of motion from one end of the distance to the other. This suggests the idea of representing the distance-magnitude by a line. But, then, we must find a line such that to a fixed distance, from one fixed point on it to another, should correspond a unique position of the line, and to a smaller portion of the line should also correspond a smaller distance, — which is not always the case with any line. Is there such a path between $A$ and $B$? I say, that if we eliminate all paths between $A$ and $B$ which can assume different positions while $A$ and $B$ remain fixed, — since their distance is unaltered, — there must still be left a path between them which is unique for this fixed position of the ends, and which will also satisfy the other requirement,— as I shall proceed to prove with the utmost rigor in the following two propositions and their corollaries.

*Measurement of Distances From a Fixed Origin; Addition and Subtraction of Given Distances.*

Theorem 2. — Let $S$ represent a sphere described by radius $a$ from origin $O$. $\overline{OA} = a$, as soon as $A$ is on its surface. Describe from $A$ a sphere $S'$ with a radius $b$, where $b < a$. Then $O$ is outside $S'$ (by Defin. 2), and, hence, some portions of $S$ are outside $S'$; for if the whole of $S$ were inside $S'$, the center of $S$, which is separated from its own outside by some portions of $S$, would be inside $S'$ *a fortiori*. Similarly, some portions of $S'$ are outside $S$; otherwise $A$ would have to be within the surface of $S$, and not on it. But some portions of $S$ are also inside $S'$; for otherwise it would be impossible to reach by continuous motion from $A$ any point belonging to $S$, before crossing the surface of $S'$ once, — which is absurd, since $A$ itself is on the surface of $S$, and hence belongs to $S$; a portion of $S$ is, therefore, inside $S'$. Also some portions of $S'$ are inside $S$; for if $S'$ were wholly outside $S$, it could not contain in its interior any point belonging to $S$ — contrary to what has just been proved to be the case. Hence, the spheres penetrate each other, — having one portion of space in common, limited by the portions of their surfaces which they have inside each other, and two other portions of space, enclosed each between the interior side of the one and the exterior of the other. The two surfaces must intersect along a line every point of which is at equal distances from $O$ and from $A$, respectively. Hence, a rigid connection of any point $C$ on this line, with both centers $O$ and $A$, will be capable of displacing itself in such a manner that, while $O$ and $A$ remain fixed, $C$ shall take all different positions on the line. Describe now from $O$ a sphere $S''$ concentric with $S$, with a radius equal to the distance from $O$ of a point lying in the interior of both $S$ and $S'$. Its surface will pass through this point, hence will cut this space into two portions, of which only one will lie inside $S''$. This new sphere will still have a portion of its surface inside $S'$ and the other portion outside, since it encloses centre $O$, which lies outside $S'$. Describing again a concentric sphere from $O$, with a radius whose end-point passes now through a point lying in the interior of $S'$ and $S''$, we shall still more reduce the portion of volume enclosed in the interior of $S'$ and $S''$, to that only which is left inside this new sphere $S'''$ and $S'$. (The portion of the surface of $S'$ exposed is continually increasing during this process of variation of $S$, $S''$, $S''' \cdots$ etc., since some portions of it enclosed by a greater sphere described from $O$ are exterior to any of the smaller spheres.) Evidently, by continuing this process far enough and in a suitable manner, the variable portion of volume can be made less than any assignable small volume. This will happen when the two spheres, the constant one $S'$, described from $A$, and the variable $S^{(n)}$, described from $O$, touch each other in one or several points, or in one or several lines; for they cannot touch in any portion of surface without their coinciding in every part (Cor. 3 to Theorem 1). Now, they cannot touch in several distinct lines or distinct points; for any line of fixed form, connecting $O$ and $A$ with a point of the contact, could be displaced so, that, while $O$ and $A$ remain fixed, the point on the line which has been in coincidence with the touching point originally taken, shall coincide with any other point in the common touching parts, which must be at the same distances from $O$ and from $A$, respectively. Hence, since by the principle of continuous congruence in motion (Axiom 1), the figure described by the displaced line is a continuous one, — the figure described by a determinate point in it is also a continuous one. It must, therefore, be either one line, — any portion of which can, by a continuous motion along itself, pass without deformation into any other of same limits, — or one point.

Scholium. — By continually decreasing the radius of the variable sphere, and dropping the condition of its having to pass through a point within $S'$, we shall, at last, arrive to a series of spheres which have all their volume outside $S'$, and have not one point in common with the constant sphere. It is evident, therefore, that in this process of passing continuously from spheres having some portions inside $S'$, to such whose volume and surface are wholly exterior to $S'$, — so that there is an appreciable distance from each and every point belonging to the region of these exterior spheres, and each and every point belonging to the region of $S'$, — we must on the way encounter a sphere, such that any sphere described by a radius greater than its own, has some portions of volume inside $S'$, and any sphere described by a smaller radius, has every point of its region at some appreciable distance from the region of $S'$. And in crossing from the series of greater to the series of smaller spheres, we obviously can choose two, one from either series, whose radii differ as little as we please from each other and from the radius of the bounding sphere. Evidently the minimum sphere of the first series and the maximum of the second series, both tend towards the same limit — the bounding sphere, which, we say, touches the constant sphere. This bounding sphere must have at least one point in common with the constant sphere, and may have even a continuous line of contact, until otherwise shown, — which is done in what follows. Similar remarks hold in the next supposition of the present demonstration, *viz.*, when we have to add a distance to a given one, instead of subtracting (see p. 70).

Now, the contact cannot be a line. For this would imply that the surface of the sphere $S^{(n)}$, lying wholly outside $S'$ according to hypothesis, is divided by this line into two parts: one,* which together with the exposed portion of $S'$ that has been continually increasing during the process of variation of $S^{(n)}$, makes up a closed surface, separating the interior of both spheres from the whole of the exterior space; and the second part, which, as derived from the portion of the variable surface that has remained inside the combined closed surface of both spheres during all the process of variation, must still be-

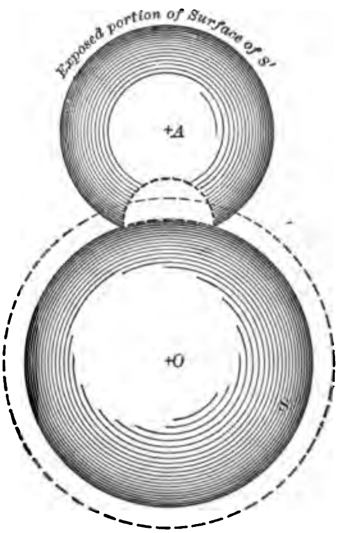

FIG. 1, *a*

* Fig. 1*a* and 1*b* on p. 70 illustrate this in two ways.

long to the interior and, hence, must be enclosed between the outside of $S'$ and the interior side of the part of surface of $S^{(n)}$ previously considered. Some volume must, therefore, be enclosed between the outside of $S'$ and the outside of the portion of surface of $S^{(n)}$, last considered. Describing now a sphere from $O$ with a radius of some point in this portion of volume, which, consequently, is greater than the radius of the touching sphere $S^{(n)}$, we see that the new spherical surface must enclose every point on $S^{(n)}$ and, hence, also the line of contact. It must, therefore, enter into the interior of $S'$ along a line lying on the covered portion of $S'$, and again come out from there by another line, lying on the exposed portion of same, — which is possible.

Hence, the contact must be in a single point.

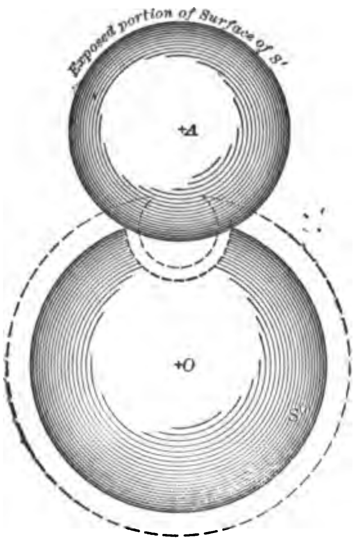


FIG. 1, b

We say that the radius of $S^{(n)}$, described from $O$ and touching $S'$, is $= (a - b)$, and the distance from $O$ to any point in $S^{(n)}$ and, in particular, to the point of contact, is $(a - b)$; the distance of this last point from $A$ being equal to $b$, the distance from $O$ to $A = (a - b) + b = a$ — in perfect agreement with original hypothesis.

If, instead of diminishing the radius of the sphere $S$, we should continuously increase it, we would, by a reasoning perfectly similar to the preceding, come to the conclusion that the space between the exterior portion of the surface of $S^{(n)}$ and the interior of $S'$, would continuously diminish (and likewise the exposed portion of the surface of $S'$) and could be made less than any assignable volume, — when the two surfaces would have to touch, either along a single line, or in a single point. Now, it cannot be along a line; for, then, — since $S^{(n)}$ in the limit will have to enclose in its interior both the portion of $S'$ which has been interior to $S$ at the outset and which has kept on increasing during the process of variation of $S^{(n)}$, and also the remaining portion of

$S'$ which has been exterior to $S^{(n)}$ and has been decreasing during the process, — there must be, in the limit, a portion of volume, contained between the portion of the surface of $S'$ considered last, on the outside of it, and that portion of the surface of $S^{(n)}$ which is now separated from the first portion of $S'$ by the surface of the other portion of $S'$. The two portions of surface, enclosing this volume and belonging to the two spheres respectively, are separated by the line of contact. Now, if we describe a sphere from $O$ by the distance from it to any point within this space, this sphere will lie wholly within $S^{(n)}$, and must be one of the spheres which have cut $S'$ before reaching the limit. It must, therefore, first emerge from that space; but as it cannot cut $S^{(n)}$, it must cut $S'$ in two lines, one on each of the portions of its surface, separated from each other by the line of contact of $S'$ and $S^{(n)}$, — which is impossible. Hence, the contact $S'$ and $S^{(n)}$ must be in a single point.

We then say that the radius of the sphere $S^{(n)}$ described from $O$, enclosing $S'$ and touching it in one point, is equal to $a + b$. The distance from $O$ to any point in $S^{(n)}$, and, in particular, to the point of contact, is $= (a + b)$; the distance of this last point from $A$ being equal to $b$, the distance from $O$ to $A$ is $(a + b) - b = a$, in perfect agreement with the original hypothesis.

Suppose now we have to add to $\overline{OA}$ a distance $\overline{AB}$ greater than $\overline{OA}$. We describe then from $A$ a sphere $S'$, with radius $\overline{AB}$; it will enclose $O$ and, hence, at least a portion $S$; it may, however, enclose it all. Describe now a sphere $S_1$ from $O$, with radius equal to $\overline{AB}$; it will enclose $A$, and hence $S_1$ and $S'$ must have common portions; but neither can enclose the whole of the other, since, as one cannot be greater than the other, they would have to coincide, which is impossible — the centres being distinct. Hence, they will intersect as previously. Increasing now the radius of the sphere described from $O$, we can prove, as in the last case, that some $S^{(n)}$ will come to touch $S'$ in one point $B$, and enclose it all, so that $\overline{OB}$ will be equal to $\overline{OA} + \overline{AB}$, as previously. Also, by diminishing the radius of $S_1$ continuously, we shall get another point of contact, whose distance from $O$ we shall call $\overline{OA} - \overline{AB} = a - b < 0$. Such a point of contact as this last can never be obtained from the above rules of addition and subtraction by adding any positive distance, either greater or less than $\overline{OA}$, nor by subtracting from $\overline{OA}$ any positive distance less than $\overline{OA}$. Hence, points corresponding to negative distances from $O$, where $\overline{OA}$ is positive, are distinct not only from one another, but also from all points marking positive distances from $O$ under same hypothesis. We see now, that we can add and subtract distances as abstract quantities.

Scholium. — The rules for addition and subtraction given above are based upon the fact that every two rigid pairs of points connected at one end give rise to an infinity of distances between the free ends, having a perfectly determinate higher and lower limit (maximum and minimum), which we have defined, respectively, as the sum and the difference of the two given distances represented by the rigid pairs themselves. An additional reason for singling out the maximum and minimum distances from the host of all the aggregate, is found in the further fact that, when the ends of one of the two connected rigid pairs are fixed in position, the position of the remaining free end is not fixed for any one of the derived distances, with the exception of these two limiting distances. It is also evident, that one of the essential conditions which the addition and subtraction of measurable quantities must satisfy, namely, that the sum increase with the increase of each of the terms, and that any two quantities differing in value should always give a *determinate* difference (see Weber, "Traité D'Algèbre Supérieure," t. 1, p. 9), is perfectly satisfied by our rules for all the three cases considered in the theorem. In order, however, that these rules be perfectly consistent, and should lead to no contradictions in their application, it is further necessary to prove that, in the first place, the operation of addition obeys the *associative and commutative* laws of addition of abstract quantities, and, in the second place, the rule for subtraction is actually the inverse of the rule for addition. This can be done by the aid of the following two lemmas:

Lemma 1, Theorem 3. — If the point $B$ has been so determined by our rule of addition, from the fixed points $O$ and $A$, that $\overline{OB} = \overline{OA} + \overline{AB}$, then, taking $B$ as origin and a movable rigid pair $(AO')$ congruent with $(AO)$, the fixed point $O_1$ determined so that $\overline{BO_1} = \overline{BA} + \overline{AO'}$ coincides with the original starting point $O$.

Demonstration. — Let $O$, $A$, $B$ be the triplet of points of the original operation; $S_0'$, $S_1''$, $S_0''$, the corresponding spheres. Then, by construction, if $C$ is any point on $S_1''$ not coincident with $B$, we have $\overline{OB} > \overline{OC}$. The spheres $S_0'$ and $S_0''$ have same centre $O$ and pass through $A$ and $B$, respectively, and the sphere $S_1''$, with center $A$, is wholly interior to $S_0''$, with the exception of point $B$, at which the two spheres touch. Let now the sphere $(S_0'')$, conceived rigid, rotate about the fixed point $A$; every point $(P)$ belonging to the rotating sphere, will remain upon a sphere described about $A$ as a center, with radius $(AP)$. Hence, the center of the moving sphere, $(O)$, will move upon the surface of $S_1^0$ described from $A$ with radius $(AO)$; and, since $(B)$ is the only point of the surface of $(S_0'')$ which has, at the beginning of the motion, been at a distance $(AB)$ from $A$, this point $(B)$ will be the only point in the moving surface $(S_0'')$ which will, during all the rotation, remain upon the surface of $S_1''$. All other points in the moving surface $(S_0'')$, having been at the beginning of the rotation outside $S_1''$, will remain outside during any moment of the rotation. Hence, the moving sphere $(S_0'')$ will, during all the rotation, remain in interior contact with the stationary sphere $S_1''$, touching it at any moment in some point $B'$, which marks the corresponding position in space, at that moment, of the moving point $(B)$ rigidly fixed in the moving surface of $(S_0'')$. Let $O'$ be the corresponding position, at the moment considered, of the centre $(O)$. It is now evident that the rotation of $(S_0'')$ can be so arranged that its centre $(O)$ describe the whole of the spherical surface $S_1^0$. (It is sufficient for this purpose, that the radius $(AO)$ describing $S_1^0$, be rigidly fixed in the sphere $(S_0'')$, which moves along with the radius about $A$ fixed.) Simi-

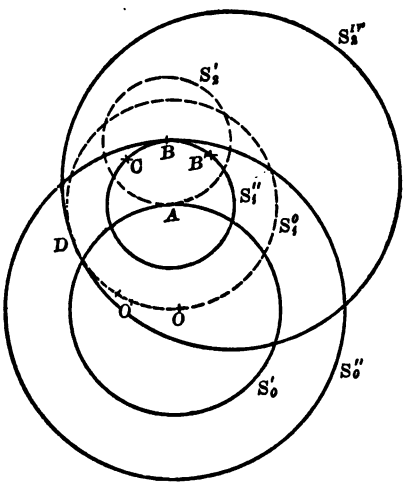

FIG. 2.

larly, $(S''_0)$ can rotate so, that $(B)$ describe the whole of the surface $S''_1$. Our original construction of the triplet of points $O$, $A$, and $B$, will, in either case, at any moment be presented by a congruent triplet $O'$, $A$, $B'$; hence, $\overline{O'B'} = \overline{O'A} + \overline{AB'}$. We conclude, therefore, that:—

*First.* To any point $O'$ on the surface of $S^0_1$ corresponds one, and only one, point $B'$ on the concentric surface $S''_1$, such that $\overline{O'B'} = \overline{O'A} + \overline{AB'} = \overline{OB}$, and, at the same time, $\overline{O'A} = \overline{OA}$ and $\overline{AB'} = \overline{AB}$, and, further, $\overline{O'B'}$ is the maximum distance between the ends of the connected rigid pairs $(O'A)$ and $(AB')$.

*Second.* To any point $B'$ on the surface $S''_1$ corresponds some point $O'$ on the surface $S^0_1$, such that our original construction, repeated from $O'$ and $A$, instead of $O$ and $A$, will lead us exactly to the point $B'$. We cannot say, however, until proven so, that to $B'$ corresponds only one point $O'$. For, although we know that there is only one point $B'$ on $S''_1$, corresponding to a fixed point $O'$ on $S^0_1$, such that $\overline{O'B'} > \overline{O'C}$—$C$ being any other point on $S''_1$, not coincident with $B'$, we cannot affirm that, conversely, $\overline{B'O'}$ is also the maximum distance from fixed $B'$ to any point on the surface $S^0_1$, or, in other words, that $\overline{B'O'} > \overline{B'D}$, where $D$ is any point on $S^0_1$ not coincident with $O'$; and if it is not the maximum distance, then we know from the preceding, that a spherical surface described from $B'$ with radius $= \overline{B'O'}$, will cut the surface $S^0_1$ in a closed line, every point of which is exactly at the same distances from $B'$ and from $A$ as $O'$, and can, therefore, replace $O'$ in our construction.

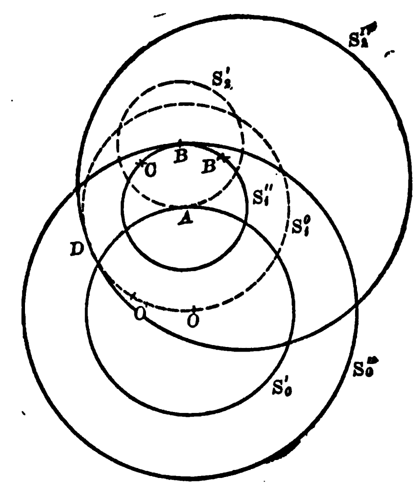

Fig. 2.

Let us now start with the fixed points $B$ and $A$, as in the theorem, and let the movable rigid pair $(AO')$, congruent with

$(AO)$, be put in the position $AO_1$, where $\overline{BO_1} = \overline{BA} + \overline{AO_1}$, in the sense of the definition of addition given in Theorem 2. I say, that $O_1$, which is by construction unique on $S_1^0$, and which satisfies the inequality $\overline{BO_1} > \overline{BO'}$, where $O'$ is any other point on $S_1^0$ not coincident with $O_1$, cannot be any other point than the original starting point $O$. For, let it be some other point $D$; $D$ is unique on $S_1^0$, and $\overline{BD} > \overline{BO}$. In this new construction, $B$ being the starting point and $D$, the final point in the operation, we know that, when $(B)$ moves upon $S_1''$, its corresponding unique point $(D)$ moves upon $S_1^0$. (It is hardly necessary to explain that the sphere rotating about $A$ fixed in the new construction, and corresponding to $S_0''$ in the old one, is $(S_2^{iv})$ touching internally in $D$ the stationary sphere $S_1^0$, and supposed, at the beginning of the motion, to have its center at $B$.) We know, further, that to every point $(D)$ on $S_1^0$, there is at least one point $(B)$ (and maybe an infinity of such points) on $S_1''$, such that the construction repeated from the point $(B)$, or from any one of such points, and from $A$, will lead exactly to $(D)$. Let now $(D)$ move up to $O$; none of its corresponding points $(B)$ can, for this new position of $(D)$, coincide with $B$, since, by construction, $\overline{BD} > \overline{BO}$. Hence, the point corresponding to $O$ in the new construction, must be some other point $C$ on $S_1''$, — and we have $\overline{CO} = \overline{BD} > \overline{BO}$ — which is in contradiction with the result obtained from the first construction, namely, $\overline{OB} > \overline{OC}$. Hence $O_1$ marking that position of the end of the rigid pair $(AO')$ in the new construction, which corresponds to the maximum distance $\overline{BO'}$ from $B$ to any point on $S_1^0$, cannot coincide with any point on $S_1^0$ except $O$. We conclude hence, that if $\overline{OB} = \overline{OA} + \overline{AB}$, then $\overline{BO} = \overline{BA} + \overline{AO}$ — in the sense of our rule of addition, — and if the distances are measured from the same origin $O$, then $b + a = a + b$.

Q. E. D.

Lemma 2, Theorem 4. — If $\overline{OB} = \overline{OA} + \overline{AB}$, then a sphere described from $B$ with radius $\overline{BA}$, will touch externally the sphere described from $O$ with radius $\overline{OA}$. In other words, if from the rule of addition $\overline{OA} = \overline{OB} - \overline{AB}$, then $\overline{OA}$ is the minimum distance between the extreme ends, of the fixed rigid pair $(OB)$ and the movable rigid pair $(BA)$ jointed to the first at $B$; or, otherwise still, the rule for subtraction is an actual inverse of the rule for addition.

Demonstration. — Let the theorem not be true. Then a construction of a triplet of points, $O$, $A$, and $B$, is possible, like that in the figure, where the sphere $S''_3$ — touching internally at $B$ the sphere $S''_0$ of center $O$, and having its center $A$ on the spherical surface $S'_0$, likewise described from $O$ — has a radius $\overline{AB}$ such, that the sphere $S'_2$ described from $B$ with this radius, cuts $S'_0$ in a closed line $AA'$. Then, we can describe from $B$ as center another sphere $S'''_2$, such as will touch externally the sphere $S'_0$ in the point $C$, and, consequently, whose radius $\overline{BC} < \overline{BA}$. Hence, the sphere $S^{iv}_3$ described from $C$ with radius $\overline{CD} = \overline{AB}$, having $B$ and some space in the neighborhood, in its interior, cuts $S''_0$ or encloses it wholly in its interior. By conceiving now the spherical shell contained between the concentric surfaces $S'_0$ and $S''_0$, to be rigid and to rotate about the stationary sphere $S'_0$, dragging along ($S^{iv}_3$) rigidly fixed to this shell, ($C$) will come to $A$, and $S^{iv}_3$ will now coincide with a congruent sphere $S''_1$ described from $A$ with radius $\overline{CD} = \overline{AB}$ and cutting $S''_0$ in a closed line or enclosing it wholly in its interior; but this is contrary to hypothesis, according to which the sphere described from $A$ with radius $\overline{AB}$, is $S''_3$, tangent internally to $S''_0$. Our theorem, therefore, can never be untrue, but must be true without exception. Hence, the rule for subtraction is a real consequence of the rule for addition.

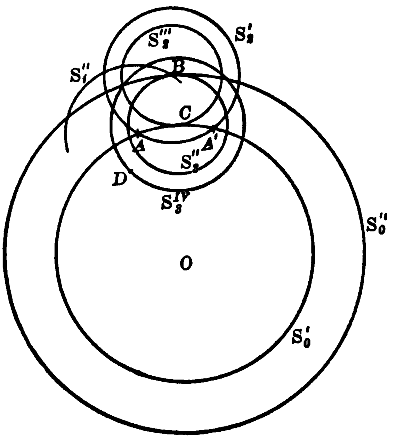

Fig. 3.

Q. E. D.

Scholium. — Now we can prove that the operation of addition obeys the associative and commutative laws in their full generality. We have, however, to remark : — first, that from the very sense of the rule for addition follows the equality : $a + b + c = (a + b) + c$, and, in general, any number of terms following the first, written separately with signs $+$, may be incorporated in a parenthesis enclosing the first, and hence such a parenthesis may be dropped; and second, that in any parenthesis, by Lemma 1, we may interchange the order of the first two terms, if they are both positive.

Theorem 5. — The operation of addition obeys the associative and commutative laws, so that any number of terms, beginning from, and ending with, any term we please, may be enclosed in a parenthesis, and the result added to the preceding terms, — and any term may be transposed forwards and backwards, through any number of other terms.

Demonstration. — We have by lemma 2:

$$b + c - c - b = b - b = 0,$$

and also $$b + c - (b + c) = (b + c) - (b + c) = 0;$$

$$\therefore -(b + c) = -c - b.$$

Therefore, $$a + b + c - (b + c) = a + b + c - c - b$$

$$= a + b - b = a.$$

$$\therefore a + b + c - (b + c) + (b + c) = a + (b + c),$$

or $$a + b + c = a + (b + c),$$
and still otherwise,
$$(a + b) + c = a + (b + c).$$

Put now $$b = d + e + \cdots,$$

$$c = (f + g) + \cdots + (q + \cdots + r) + \cdots + t;$$

then we get

$$a + (d + e + \cdots) + [(f + g) + \cdots + (q + \cdots + r) + \cdots + t]$$

$$= [a + (d + e + \cdots)] + [(f + g) + \cdots + \{(q + \cdots + r) + \cdots + t\}]$$

$$= [a + d + e + \cdots] + [f + g + \cdots + \{q + \cdots + r + \cdots + t\}]$$

$$= (a + d + e + \cdots) + (f + g + \cdots + q + \cdots + r + \cdots + t)$$

$=a+d+e+\cdots+f+g+\cdots+q+\cdots+r+\cdots+t$

$=(a+d)+(e+\cdots+f+g)+(\cdots+q+\cdots+r+\cdots+t)$

$=$etc.

Suppose, now, we have the sum,

$$a+b+c+d+e+f+g+\cdots,$$

and we wish to transpose $b+c$ over the terms

$$d+e+f.$$

We write then,

$$a+b+c+d+e+f$$

$$=a+[(b+c)+(d+e+f)]$$

$$=a+[(d+e+f)+(b+c)]=a+(d+e+f+b+c)$$

$$=a+d+e+f+b+c.$$

We add then the remaining terms to both sides of the equation, and get the desired transposition. Q. E. D.

Scholium. — The process of obtaining any integral multiple of a given distance, and also, of obtaining the ratio (commensurable or incommensurable) of two given distances, ought no longer to detain us, and we shall only remark that this process is a direct consequence of the rules for addition and subtraction and of the postulate of continuity, which says that between any two positions of any geometrical entity (a point included) there is always an infinity of other such positions, all in space, and that at least one infinite series of such positions must be passed through, to reach either of the two given positions from the other. By the associative law of addition and its extension to subtraction in considering this operation as the addition of negative terms, we can get a given series of points, either by adding and subtracting separate terms, each to, or from the preceding sum, or by adding to the same given term a series of distances equivalent to one, two, three $\cdots$, $n\cdots$, of the added and subtracted terms taken in any order we please. This last process, in assuming that the term to be added (in a positive and a negative sense) increases continuously, leads us to the important conception of a homogeneous line, completely determined by two points in it, which serves as the basis of all line-measurements, and is fully described in the following definitions and succeeding corollaries.

Definition III.—By conceiving the sphere $S'$ described from $A$ (in Theorem 2), to vary continuously beginning from one whose radius is indefinitely small, and, passing all imaginable distances, to become indefinitely large, we shall get all possible distances $\overline{OB}$, positive and negative, whose general expression is $(a+x)$. To each such distance from $O$ corresponds, by construction, one, and only one, point, and the aggregate of all these points will, evidently, form a continuous line, which, satisfying fully the conditions necessary for a line suitable to represent the distance-magnitude, may be called the *distance-line* or the *straight line.*

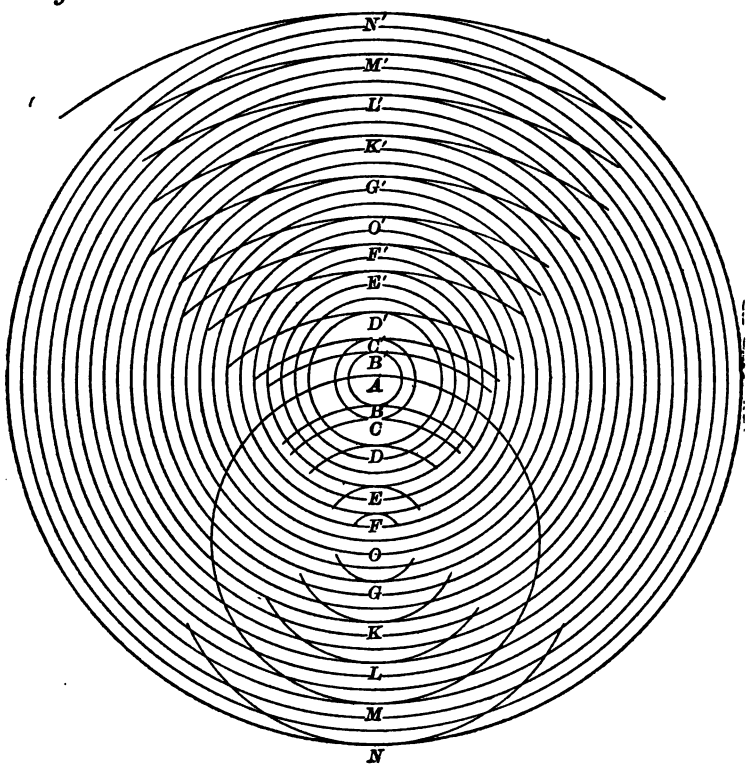


FIG. 4.

Corollary I. — To every distance on the line, as measured from $O$, there will correspond a point $B$ at a corresponding distance from $A$, such that no other point in space can have the same distances from $O$ and $A$, respectively.

For, if we describe two spheres from $O$ and $A$ with the corresponding distances as radii, these two spheres will have only one point in common, namely, the point on the distance line; but if there were another point in space at the same distances from $O$ and $A$, this last point would also be on both these spheres, which is contrary to construction.

Corollary II. — If $A$ and $B$, and $A'$ and $B'$, are two pairs of points, at equal distances from each other singly, then the straight lines constructed from each pair as from $O$ and $A$ originally, will coincide with each other along the whole of their extent as soon as the congruent pairs ($AB$) and ($A'B'$) are made to coincide. This follows immediately from the fact that the distance-line is unique when constructed from two points of fixed position.

Corollary III.— Every point not on the distance-line between $A$ and $B$ or its prolongation, is such that we can find a continuous series of points of which this is one, which have the same distances from $A$ and $B$, respectively, as the given one.

For, since the given point is not on the distance-line constructed from $A$ and $B$, it follows that the two spheres through the point, described from $A$ and $B$ as centers, will not touch in this point, but cut along a line which is the locus of points having the same distances from $A$ and $B$ as the given one; hence our corollary.

Corollary IV. — If we imagine a rigid body to be placed so, that two of its points coincide with $A$ and $B$ considered previously, and to be fixed there, then all the points of the body fall in either of the following two categories: —

(1) A continuous series of them coincide with corresponding points on the distance-line through $A$ and $B$, which is fixed in space as long as $A$ and $B$ are fixed;

(2) The remaining points coincide, each with a point in space whose distances from $A$ and $B$, respectively, it has in common with every point on the intersection of the two spheres described from $A$ and from $B$ through this point.

It is evident, therefore, that any point in the body, of the second category, can be displaced along the line of intersection of the two spheres, on which it originally fell, while $(A)$ and $(B)$ are still fixed in $A$ and $B$, letting all other points in the body take care of themselves, or, rather, the rigidity of the body—which consists in preserving the relative distances of the points of the body—take care of their individual positions during this displacement. We find then that every other point of the second category will have, in all its displacements, to remain on a corresponding line of intersection of two spheres from $A$ and $B$. Every point in the first category, however, will have to remain stationary, since there is no other point in space having the same distances from $A$ and $B$, besides the one with which it coincided originally. These conditions are seen also to be perfectly compatible with the relative distances of the individual points of the body from one another, as soon as a continuous series of spheres, described from $A$ and $B$ as centers, is imagined, together with their mutual intersections; for, as the points of the body move along these lines of intersection, these lines, together with their spheres, can be conceived to glide upon themselves, never changing their form, since a sphere is a homogeneous surface; hence, the mutual distances of the moving points are defined alike throughout their motion. Such a motion of a rigid body, about a stationary straight line (axis), we call *rotation*, and the body moving with such a motion, is said *to rotate* or *to revolve* about the *axis* $AB$, or, simply, about the two fixed points $A$ and $B$.

Corollary V. — Since in the rotation of the solid, considered in the preceding corollary, any two points in the solid, $(C)$ and $(D)$, which lie on the axis $(AB)$, will remain in coincidence with a pair of congruent points $C$ and $D$ fixed upon the distance-line $AB$, and since no other points in the solid, besides those lying on the axis, remain fixed,—it follows that if a solid is moved so, that a given pair of points in it, $(C)$ and $(D)$, remain fixed in space, then all the points in the moving solid fall in two categories: — *such as remain on a fixed axis determined by $C$ and $D$ and constructed from any two points in it, $A$ and $B$*, and such as move constantly upon corresponding intersections of two systems of spheres of centers $A$ and $B$, respectively.

For, rotating upon the fixed axis constructed from, and therefore determined by, $C$ and $D$, no other points in the solid besides those lying on the axis can remain fixed in space. Any two points $(A)$ and $(B)$ in the axis, however, remaining fixed, the solid rotates also about the distance-line constructed from $A$ and $B$, and this must coincide with that determined by $C$ and $D$.

Corollary VI. — It follows,* that there is no difference in the form of the distance-line when constructed from any pair of points in space, and that any two such lines will coincide with each other throughout the whole of their extent as soon as two congruent points in both are made to coincide.

Corollary VII. — Any portion of a straight line will, by the preceding corollary, coincide with any other portion of the same straight line, as soon as their limits coincide. In other words, a straight line is homogeneous, *i. e.*, any portion can move upon the whole without deformation.

Corollary VIII. — If we move up the segment $(AB)$ of a straight line upon itself a distance $AA'$, so that the position of $(AB)$ at the end of the motion will be $A'BB'$, then the whole line $AA'BB'$ is also a straight line.


FIG. 5.

In fact, $AA'B$ and $A'BB'$, separately considered, being, respectively, the original and final position of the same segment $)AB)$, are two segments of a straight line, having the portion $A'B$ in common. Hence, by Corollary VI, they must coincide, each with a corresponding segment upon the unique distance-line determined by $A'B$, *i. e.*, $AA'BB'$ is likewise a segment of a straight line.

It follows, that in this way we can prolong a segment of a straight line indefinitely far, solely by shoving it along itself and its successive prolongations.

Corollary IX. — A straight line cannot have more than one branch on each side of a point belonging to it. It cannot, for instance, have the branch $AB$ on one side of the point $B$, and $BC$ and $BD$ on the other side of it.

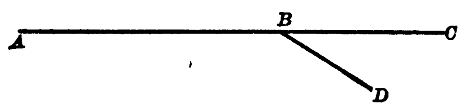


FIG. 6.

* This corollary follows also very readily from the associative law of addition of distances.

For, otherwise, by revolving a solid containing both branch $ABD$ and branch $ABC$, about $ABD$ fixed, $BC$ would be displaced — which is impossible if $ABC$ is a straight line.

Another proof is obtained thus: — If the line $ABD$ is constructed from $AB$, then the points between $B$ and $A$ and between $B$ and $D$ are obtained by continuously increasing the distance $x$ — respectively to be subtracted from and added to $AB$ — from zero to infinity. But, in doing so, the radius of the corresponding sphere $S'$ described from $B$ as center, on whose surface the corresponding points lie, increases continuously, *i. e.*, these points recede more and more from the center $B$, both ways, and can never come back to it without crossing the intermediate spherical surface, or, which is the same thing, without retracing their steps backwards. But as this is not permissible in the continuous description of the distance-line, we can never come back to $B$. Hence, it is impossible that the additional branch $BC$ be ever described.

Theorem 6. — A straight line $AB$, issuing from a fixed point $A$ and prolonged indefinitely on the other side of $B$, will not return to $A$ again, after any number of prolongations, each equal to $AB$, however large that number.

For, if this were possible, then taking just half of the whole extent, whose end let be $C$, we would have two distinct straight lines between $A$ and $C$, namely, $ABC$ and $CB'A$, since $CB'A$ is supposed to be different from $ABC$, as the point $B$ is not supposed to retrace its steps backwards beginning from $C$.

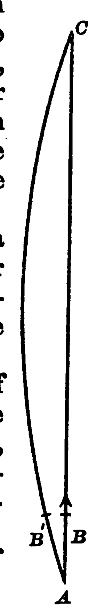

FIG. 7.

Another proof is exactly like the second proof of Corollary IX to Theorem 5, where the impossibility of returning back to a point in the distance-line is deduced from the fact that, as we continually move away from it both ways, all the points passed in either sense, become separated from those not yet reached, by a series of closed spherical surfaces, which, by the definition of an increasing distance, can never be crossed backwards.

Corollary.—From this theorem and from the preceding one it follows, that two distinct straight lines can have no more than one point in common, like $AOB$ and $COD$. As soon as, besides $O$, another point $E$ in $COD$, is made to coincide with

$E'$ in $AOB$, the two lines must coincide up to the ends of the smaller segments of either, on both sides of $O$, and their prolongations must coincide as far as we please to take them.

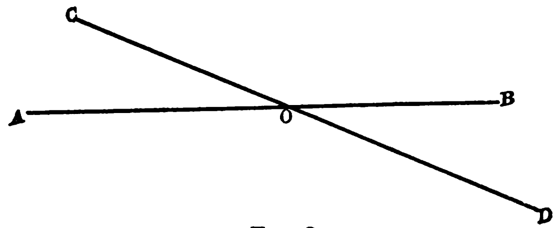


Fig. 8.

Definition IV. — A pair of straight lines having only one point in common are said to *diverge* and *form an angle* at the point $O$, which is the *vertex of the angle*, and the straight lines themselves are *the sides of the angle.* Such a pair of straight lines is called *a crossing pair of straight lines.*

Scholium. — It is evident, since an angle has four segments, having six combinations in pairs, if we leave out the two pairs belonging to the same straight line each, we get only four — the number of angles as defined above. (We shall learn later an extension of the definition, which will give an indefinite number of angles.)

We shall consider at present one angle, made only by two segments of distinct straight lines issuing only on one side of the vertex $O$, like $AOB$. We imagine this to be a rigid figure, in which any two points $k$ and $l$, each on one side, preserve always a constant distance between each other.

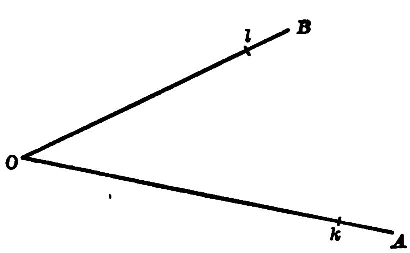


Fig. 9.

The vertex and one side of any other angle $A'O'B'$ can, evidently, be applied to the vertex and either side of $AOB$. *If*, at the same time, *a point in the remaining side $O'B'$ can be made to coincide with a point equidistant from $O$ on $OB$* — the corresponding free side of the angle $AOB$ — without breaking the rigidity of either of the figures as explained above, the whole side $O'B'$ will coincide with $OB$, and *the two angles will be said to be equal; if on the contrary, this is not possible, the angles are unequal.*

Let us see whether there is a way of measuring angles.

Theorem 7. — If two straight lines, $AOC$ and $BOD$, intersect so that they form two adjacent angles $AOB$ and $BOC$ (such as have one side in common) equal to each other, all four angles are equal.

Demonstration. — For, put an exactly equal movable figure, which we denote by small letters upon the given one, so that $\angle aOb$ of the movable figure shall coincide with $BOC$ of the

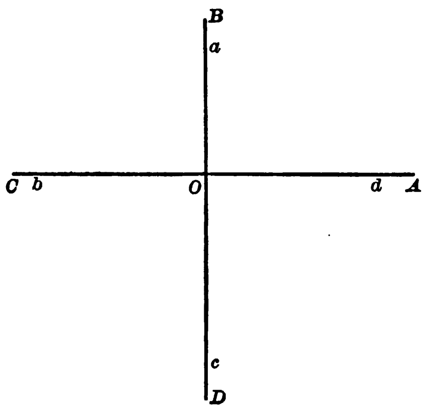


FIG. 10.

original figure; then $Oc$, the prolongation of $aO$, will coincide with $OD$, the prolongation of $BO$, and $Od$, the prolongation of $bO$, will coincide with $OA$, the prolongation of $CO$.

Hence $\angle BOC = \angle aOb = \angle bOc = \angle COD = \angle cOd = \angle DOA = \angle dOa = \angle AOB$; that is, all four angles are equal.

Q. E. D.

Definition V. — *Each of the four equal angles formed by the two intersecting lines, is called a right angle, and the lines are said to be perpendicular to each other.*

Theorem 8. — Let the straight lines $AOA'$ and $BOB'$ intersect each other at right angles. Let one of these be fixed, and the other turn round $O$ in space, — $AOA'$, say, taking up all possible positions compatible with the rigidity of the figure. Then $OA$ will generate a surface, which is indefinite in extent if $OA$ is indefinite in extent, and which is capable of revolving upon itself around $O$ without deformation, *i. e.*, any portion of it inclosed between any two positions of $OA$, like $OA_1$ and $OA_2$, will be capable of coincidence with any other portion inclosed between two other positions of $OA$ making an angle equal to $A_1OA_2$; moreover, if $BOB'$ is turned over about the fixed point $O$, so that the segment $OB'$ come to coincide with the original position of $OB$, and $OB$ with that of $OB'$, the whole surface will, without deformation, turn about $O$ and come into coincidence with the trace of its original position, upside down, as soon as $BOB'$ comes into coincidence with $B'OB$.

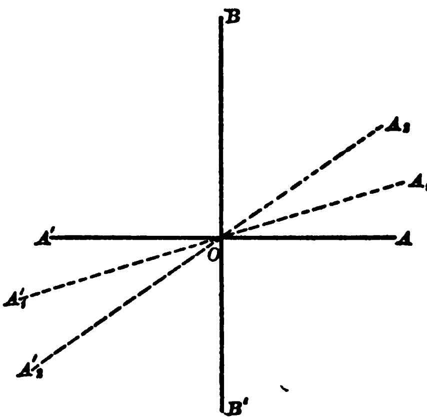


Fig. 11.

Demonstration.— The first half of the proposition follows immediately from the principle of continuity of congruence in motion (Axiom 1) and from the property of a straight line, combined with the fact that in the rotation of a rigid body any point of the second category moves upon a closed line — the intersection of two spheres described from any two points in the fixed axis as centers and passing through the point in question (Corollary IV. to Definition 3)* The second half

* If the two points on the axis are taken equidistant from $O$, the plane represents the system of concentric circles which served for Lobatchevski as a definition of a plane in his work (see "Urkunden zur Geschichte der Nicht-Eukl. Geom.," Engel, 1898, pp. 93–109).

follows from the equality of the angles $AOB$ and $AOB'$. For we can imagine from the start, that a duplicate of the figure $BOAOB'$ has been brought into coincidence with $B'OAOB$; so that $BOA$ and the duplicate of $B'OA$ generate the same surface on one side, as the duplicate of $BOA$ and $B'OA$ *itself*, on the other side. Hence the theorem.

Definition VI. — The surface is called *a plane*, $OA$ — *the generator*, and, in any of its fixed positions, *a half-ray*, which together with its prolongation on the other side of $O$ composes a complete ray or *element*. The point $O$ is *the origin;* the *axis* $B'OB$ is said to be *normal to the plane* $AOA_1$.

Corollary I. — It follows, that we can revolve the plane about any element or ray passing through $O$, until its upper side comes into coincidence with the original lower side, and

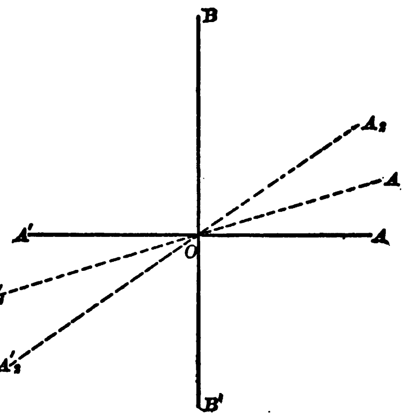


FIG. 11.

*vice versa* — the ray itself remaining fixed; since, in this motion, the ray will always coincide with itself while the *normal* is displaced and *revolves around O into coincidence with its own reversed position.*

Corollary II. — It follows also, that a plane divides all space into two equal portions, which become coincident with each other as soon as the normal coincides with its reversed position.

Corollary III. — The plane described by the segment $OA$, using the perpendicular $OB$ as an axis, is identically the same as that described by the prolongation of $OA$, *viz.*, $OA'$; also this plane is unique as long as $BOB'$ is fixed.

Corollary IV. — Given the plane $AOA_1$ whose normal is $OB$, and a straight line $OC$ not lying in this plane, then $\angle BOC$ is not a right angle.

For, joining by a straight line, $b$ and $a$, a point on the normal and some point in the plane, not $O$, and then revolving $(BbOa)$ around $BbO$ fixed, $(ba)$ must somewhere in its motion intercept, in some point $c$, the straight line $OC$, which is supposed to remain stationary until this occurs; since, after $(Oa)$ comes round back to its original position, $(bOa)$ will have described a closed surface, separating a finite portion of space from all other space. This surface will consist, partly of the portion of the plane described by the segment $(Oa)$ of the element $OA_2$ in which $a$ is situated, and partly of the lateral surface described by $(ab)$, every point of which will be at some appreciable distance from $O$. Hence, a sphere described by the smallest distance from $O$ of any point in the lateral surface,

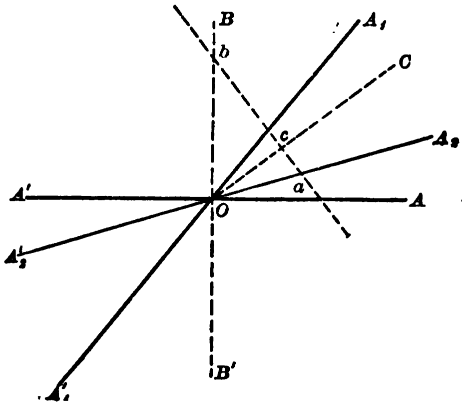


FIG. 12.

will enclose an appreciable portion of $OC$; while a sphere described by a distance greater than that of the point in the lateral surface farthest from $O$ (only the finite segment $(ab)$ is considered), will enclose in its interior a portion of $OC$ having some points exterior to the closed surface. Hence, a variable point on $OC$, in moving continuously from a position on the interior portion towards a position on the exterior portion, must cross the closed surface. But as $OC$ can meet neither an element of the plane $AOa$, nor the normal $OB$, in any other point than $O$, it must pierce the lateral surface described by $(ba)$ in some point $c$. Thus, the generator of the lateral surface $(ba)$ in the position of $bca$ intercepts $OC$ in the point $c$. Let then the element $(OA_2)$, starting from its new position, describe its own plane anew, dragging along $OC$ in its motion. Then, this last will describe a surface which will lie wholly on that side of the plane $AOA_1$ where the half of the normal $OB$ is situated. Reversing now the plane, the surface generated by $OC$ will lie wholly on the side where $OB'$ was originally, and will be separated from its original position by double the space between its original position and the plane $AOA_1$. Therefore, the surface generated by $OC$ in its rotation around the axis $BOB'$, is not a plane (Corollary II), and hence $\angle BOC$ is not a right angle.

Corollary V. — It follows, that all planes coincide with one another as soon as their normals and origins coincide.

Theorem 9. — All right angles are equal.

Demonstration. — If the vertex of any right angle is made to coincide with the origin $O$ of our plane, and one of its sides, with the normal $OB$ of the plane, the other side must coincide with some one of the elements of the plane, say $OA_1$, by preceding corollary. Hence, all right angles can be made to coincide, or are equal. Q. E. D.

The preceding theorems about the properties of a plane and of right angles will suffice to render more concrete our notions about angles in general as geometrical magnitudes. First, we observe that we have found a natural unit for measuring angles, namely, the right angle; and, secondly, we shall soon see how the plane will afford us a means of comparing all possible angles with our standard unit, the right angle. The following preliminary remarks are necessary.

Scholium I. — Any pair of crossing lines will be capable of being applied to the plane so, that the vertex shall lie on the origin, and the lines themselves shall lie wholly in the plane.

In fact, one of the two lines can always be brought into coincidence with any ray in the plane so, that the vertex fall on the origin. If, now, the other line falls upon another ray, the proposition is proved for the case under consideration. If not, this other line will have pierced the plane (which is admissible during the process of applying; we may imagine, if necessary, a portion of the plane removed during the application and then restored back to its original position; see also definition of rigidity), and will have only one point in common with the plane, *viz.*, the origin, like $OB$ or $OC$ in Fig. 12. If, now, we revolve the plane about the fixed ray with which the first side is in coincidence, it will sweep through all space during a revolution that brings the normal into its reversed position. Hence, some time during this revolution, it will have to intercept another point on the line which has not been in it originally and which is supposed to have remained fixed in position; but just as this interception occurs, this line will have two points in common with some ray in the plane, and will, hence, coincide with it along the whole of its extent — *i. e.*, both lines of the crossing pair will lie in the plane. Hence, all pairs of crossing lines can be made to coincide, each with a pair of crossing rays in the plane. In other words, all possible angles are to be found among the different angles between the different rays of any plane.

Scholium II. — If we apply any two perpendicular lines of indefinite extent to a corresponding pair of rays in the plane, we see that the whole extent of the plane can, from any initial position of a ray, $OA$ say, be divided into four congruent parts, each of which will be enclosed by a right angle. The end of any fixed distance from $O$, measured along a ray, will generate in its motion around $O$ a homogeneous curve every point of which is equidistant from $O$. The curve is called the *circumference* of a circle, and $O$ is its center. A portion of the plane enclosed between any two rays measured one way from center to circumference and termed *radii* is called a *sector*, that is, a part of the whole portion of the plane enclosed by the circumference; such a sector is congruent with any other sector enclosed between two radii making the same angle with each other. It is now evident that, since, whenever the angle between two fixed radii is equal to the angle between two other fixed radii, both the sectors of the circle and the *segments of the circumference*, or *arcs*, enclosed by the respective pairs of radii, are equal each to each — the angles can be measured either by the corresponding sectors or by the arcs enclosed between their sides, so that a greater angle corresponds to a greater sector or arc, and a multiple or part of an angle corresponds to the same multiple or part of the corresponding sector or arc. Since every possible position of two intersecting straight lines, with respect to each other, has been proved to find its analogue in some position of two intersecting rays of a plane, it is sufficient to investigate these last. Now, we can conceive the whole aggregate of displacements possible for a ray in the plane (Fig. 13) with respect to a fixed ray $A'a'OaA$, to be bound up with the corresponding displacements of a circle rigidly fixed to the moving ray and revolving with it about the center, so that the circumference moves in its own trace. Then we see that, if $(Oc)$, one segment of the moving ray, makes in any of its positions an angle $AOc$ with $OA$, measured by the arc—passed over by the

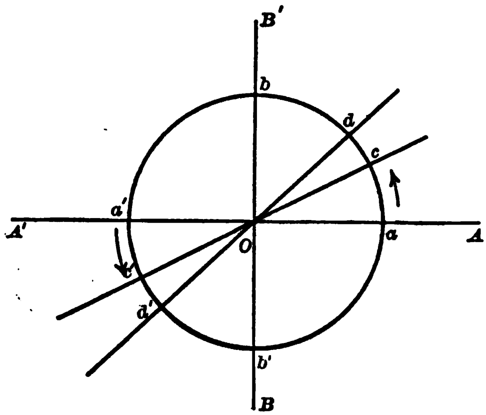

FIG. 13.

moving point in the circumference—from $a$ to $c$, then the prolongation of $(Oc)$ must have been displaced just as much on the other side of $OA'$—the prolongation of $OA$—and must form an equal angle $A'OC'$—as measured from the fixed segment $OA'$ or, along the circumference of the circle, from $a'$ to $c'$, in the same sense as $ac$. For, any fixed point in the moving circumference must have been displaced an equal arc with any other. So we see that the two half-rays of the same ray lie on opposite sides of the two half-rays of any other ray not coincident with it—since any fixed ray can be taken as the initial ray—and the angle made by the prolongations of any two half-rays will be exactly equal to the angle made by the half-rays themselves. Also, that any point in the moving ray will be displaced a quarter, a half, three quarters, and a whole circumference, when the ray is displaced one, two, three, or four, right angles; further, that the half-ray will fall upon its own prolongation — for a displacement of two right angles, and will come to its original position — for a displacement of four right angles. It follows, then, that a half-ray, in any of its positions, makes with the two half-rays of another ray two angles, one of which is just as much less than a right angle, as the other is greater, —

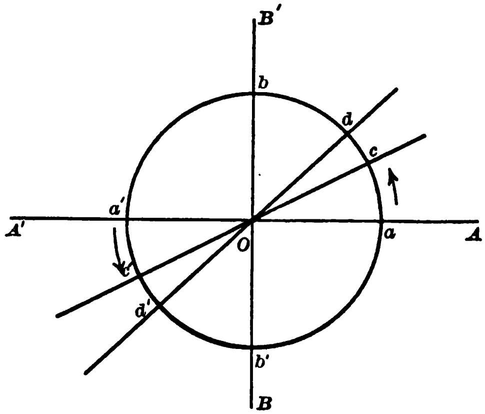


FIG. 13.

their sum being equal to two right angles (*acute and obtuse angles*).

The following theorem, concerning angles in space, becomes at once evident by the application of the angles to the plane.

Theorem 10. — Two adjacent * angles whose two non-common sides form the opposite prolongations from the vertex of the same straight line, are together equal to two right angles; the two non-adjacent angles, made by two straight lines and their prolongations, respectively, are equal; and all the four angles taken together are equal to four right angles.

Definition VII. — A figure bounded by three straight lines intersecting, two by two, in three points and making three angles, each less than two right angles, is called a *triangle*. We must not, at the beginning, consider the triangle as containing any surface that may be limited by the sides of the triangle.

* Adjacent angles are defined as usual.

Scholium. — It is evident from the way we have constructed the straight line, that any three points at fixed distances from each other are capable of congruence with any other triplet of points of the same fixed distances; since, if in any such a triad we revolve one of the points around the straight line connecting the remaining pair, we get the locus of all the points which are at the same two distances from the fixed pair as the given third one, and no point outside this locus can have the same distances from the fixed pair, — as shown in Theorem 2–5 and corollaries. Hence, when the corresponding points of the two equidistant pairs in the two triads are brought into coincidence, the third ones, in each triad, will be capable of coincidence.

Corollary. — Since as soon as the ends of two equal distances coincide the straight lines representing these distances coincide, it follows that any two triangles whose sides are respectively equal to each other are congruent.

Theorem 11. — Two triangles are equal when two sides and the included angle of one are respectively equal to two sides and the included angle of the second.

The demonstration given by Euclid in his Elements for this theorem, holds here word for word, and need not be repeated.

Corollary. — In every *isosceles* triangle the angles at the base (opposite the equal sides) are equal. For, its duplicate can be applied to it so, that the equal sides be interchanged by turning over; the base and, hence, the angles interchanged, will still coincide with the corresponding ones in the original triangle.

These two theorems concerning congruent angles and triangles are sufficient to deduce a most fundamental property of the plane—namely, that the origin may be transposed to any point in the plane, and hence, any straight line having only two points in common with the plane, anywhere, will lie wholly in the plane; hence, the plane itself is capable of translation or rotation upon itself without deformation.

Theorem 12. — A straight line having two points in common with a plane, will lie wholly in that plane.

Demonstration. — If the two points are upon the same ray, the straight line coincides with the ray, *i. e.,* lies in the plane. Suppose, however, that the two points lie upon different rays; we then can prove that any other point of the straight line lies upon some other ray of the plane.

Let $BOB'$, where $OB = OB'$, be the normal of the plane $AOC$, and let the straight line $X'X$ cut the rays $OA$ and $OC$ in the points $A$ and $C$, respectively; we have to prove that any point $D$ of the line $X'X$ lies upon a ray $OD$. Connecting $OD$, we join $B$ and $B'$, respectively, with $A$, $C$, and $D$.

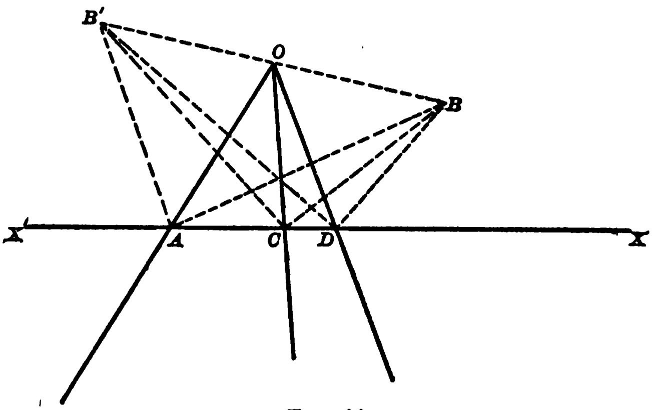


FIG. 14.

Then $\Delta BOC = \Delta B'OC$ and $\Delta BOA = \Delta B'OA$, by last proposition.

$\therefore$ $BC = B'C$, $BA = B'A$, whence $\Delta ABC = \Delta AB'C$, by corollary to Definition VII.

Whence, $\angle BAD = \angle B'AD$;

$\therefore$ $\Delta BAD = \Delta B'AD$, and $BD = B'D$;

whence, again, $\Delta BOD = \Delta B'OD$.

$\therefore$ $\angle BOD =$ right angle by Theorem 10.

$OD$ is, therefore, a ray. Now, since this is true for any point $D$ in the line $X'X$, the whole of its extent lies in the plane.

Q. E. D.

Definition VIII. — A plane can now be re-defined as the surface in which every straight line lies wholly if it passes through two of its points.

Corollary. — It immediately follows, that any angle can be made to lie anywhere in the plane; and so also a triangle, — for one side of the triangle will lie wholly in the plane as soon as its two ends lie in the plane; now, if also the third vertex is brought into coincidence with some point in the plane, by revolving the triangle around the side held fixed in the plane, the other two sides will likewise come wholly into the plane.

Scholium.—When a movable half-ray in a plane describes a positive (continuously increasing) angle, and in doing so it slides upon a fixed straight line termed *transversal* and intersecting its initial position in a point other than the vertex, the segments which it cuts off upon this transversal—measured from the fixed point of intersection towards the corresponding positions of the variable point of intersection with the moving half-ray—will continuously increase as long as the moving half-ray meets the transversal; that is, until a segment is reached which exceeds in length any arbitrarily given finite segment, no matter how great. For, any point of the transversal, that is at a finite distance greater than the given length from the origin of the segments, can readily be joined with the vertex of the variable angle, and will therefore form the limit of one of the segments greater than the given one.

Definition IX.—The normal is said to be *perpendicular to the plane*, because it is perpendicular to every straight line passing through it and lying wholly in the plane.

Theorem 13.—A straight line perpendicular to any two intersecting straight lines at their point of intersection, is perpendicular to the plane in which these straight lines are situated. The straight lines need not be the rays of the original construction of the plane, but any two straight lines intersecting these and, hence, lying in their plane.

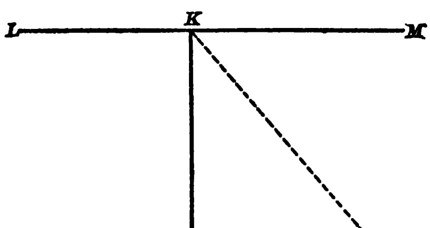

Fig. 15.

The proof is word for word that given by Euclid in Book XI, proposition IV, of his Elements.

Corollary I.—An immediate consequence is that at every point of a plane we can erect a perpendicular to the plane.

For, if $K$ is such a point, not the origin of the original construction, and $LKM$ and $KN$ are any two perpendicular straight lines through $K$ in the same plane, then, fixing the line $LKM$ and revolving ($NK$) about it the amount of one quadrant, into the position of $N'K$, this last is now perpendicular to $KM$ and $KN$; hence, it is perpendicular to the plane in which they are situated. In other words, $KN'$ may now be regarded as a normal, and all straight lines in the given plane through $K$, as the rays.

Corollary II.—Since any two planes coincide as soon as their origins and normals coincide, it follows that a plane will coincide with itself when the origin is displaced, in any manner whatever, to any other point in the same plane, provided the normal at the origin is made to coincide with the normal at the point to which the origin is shifted; also, that the origin or, indeed, any point in a plane, treated as such, may displace itself continuously, describing any curve in the plane — the whole plane remaining unaltered in shape or position; and, further, that the plane may be turned upside down, shifted upon itself in any manner whatever, without altering its position or shape as a whole (Leibnitz).

At this stage of our investigation our elementary figures, *viz.*, the angle, the triangle, and the circle, can be made more concrete. Since each of these figures can be made to lie wholly in a plane, we may suppose their boundaries to limit corresponding portions of a plane. This is the reason of their being called plane figures, in contradistinction to those which cannot be made to lie wholly in a plane. Thus, a network of straight lines and circles may be conceived to cover these plane figures, just like the plane itself; crossing and recrossing one another in all possible ways, since every point in a plane may be conceived as an origin of rays and angles and as a center for the circumferences of the circles described by the fixed points in these rays. Geometry invariably makes use of this conception of the plane, in the way of a *fiat:* "Describe a circle, from such and such a point as centre, and with such and such a distance as radius," etc., — just as it does with respect to homogeneous space in general.

# CHAPTER III

## THE QUADRILATERAL AND THE IMMATERIAL QUADRILATERAL. PARALLEL STRAIGHT LINES

The following propositions of the first book of Euclid's Elements can now be proved very rigorously, either by Euclid's demonstrations, or in a more elegant way — like the one used by Legendre and his followers, — since none of the hypotheses, whether tacit or explicit, which Euclid assumes in the shape of postulates, definitions, and axioms, in these demonstrations, are now wanting a solid basis. I omit the demonstrations, referring the reader to Euclid or Legendre, whose treatment of these propositions is admitted to be rigorous and faultless, once you grant the postulates and axioms upon which their proofs rest, and which have now been established with the utmost rigor. The propositions referred to are:

IX*, X, XI, XII, XIV, the converse of XV, XVI, XVII, (for the last two I prefer to give my own proofs, which will throw some light on the nature of parallels); then XVIII, XIX,† XXI, XXII, XXIII, XXIV, XXV, XXVI. Now we come to Euclid's treatment of parallels, which has been acknowledged to be the weakest spot in the Euclidian geometry. I propose to treat this subject in an entirely different manner, using again, as in the rest of this treatise, the kinematic method, which is much more powerful than the static method adopted by Euclid, and which, I flatter myself with the belief, will establish on a foundation firmer than ever before, the most im-

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* The propositions left out have been proved by us explicitly or implicitly, excepting VII, which is unnecessary, and VI, which is a consequence of XXVI, case 1, since the duplicate of a triangle with two equal angles can be turned over and applied to the original triangle, with which it will coincide.

† The Twentieth of Euclid and also the more general proposition that a straight line is the shortest path between two points (not the shortest distance, since there is only one distance between two fixed points) can be proved immediately by the consideration, — that any point of the second category with respect to two fixed points, lies upon the intersection of two spheres described from the points with radii whose sum is greater than the sum of the radii of the two spheres in contact described from the same fixed points, whose point of contact is a point of the first category, on the straight line between the fixed points, by the very construction.

portant truths of similarity and proportion, on which rests all the grand superstructure of our actual mathematics, both pure and applied.

I proceed to prove XVI and XVII of Euclid's first book.

Theorem 14. — The sum of any two interior angles of a triangle is less than two right angles.

$$a + b < 2 \text{ rt. } \angle\text{'s}.$$

Demonstration. — If $a + b \geqq 2$ rt. $\angle$'s, then, because $\angle ABE + b = 2$ rt. $\angle$'s (Theorem 10), it follows $a + b \geqq \angle ABE + b$, and $a \geqq \angle ABE$. For a similar reason $b \geqq \angle DAB$. Now, applying the lower part of the diagram, namely $DABE$, to the upper $ABC$ so, that $A$ shall fall upon $B$ and $B$ upon $A$, and the angle $a$, being $\geqq \angle EBA$, shall coincide with or inclose the latter, and $b$ shall coincide with or inclose $\angle DAB$, — $AD$ will lie, either on the side $BC$, or within the angle $b$, and $BE$, either on $AC$, or within the angle $a$. In either case $AD$ and $BE$ will intersect — namely, at $C$, the vertex of the triangle, or in some point within the triangle; and when returned to their original positions, they —while making the prolongations of $CA$ and $CB$, respectively — will intersect at some other point $C'$, below $AB$. Or, in other words, the two lines $CAD$ and $CBE$ will intersect in two points $C$ and $C'$, — which is impossible (Theorems 2, 5 and, corollaries). Therefore, it is impossible that in the triangle $ABC$, $a + b \geqq 2$ rt. $\angle$'s. Q. E. D.

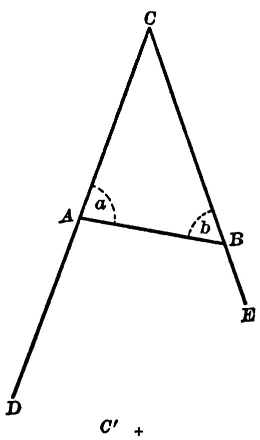

FIG. 16.

Corollary I. — If in the formula $a + b \geqq 2$ rt. $\angle$'s, we separate the case of equality, namely, $a + b = 2$ rt. $\angle$'s, we shall have $a = \angle ABE$ and $b = \angle DAB$; and the two straight lines making such angles with the *secant*, cannot meet either below or above the secant, — for, in either case, they would have to meet simultaneously also on the other side of the secant, which is impossible. Therefore, if two straight lines intersected by a third one make with it two interior angles on the same side of the secant, equal to two right angles, they cannot meet, even if produced indefinitely both ways.

Corollary II. — Any exterior angle of a triangle is greater than either of the interior and opposite angles.

Because any of the interior angles with its adjacent exterior angle equals two right angles (Theorem 10); while with eithe of the opposite interior angles it is less than two right angles; hence, each of these last ones is less than the exterior opposite to it.

Corollary III. — If one of the interior angles of a triangle is a right or an obtuse angle, either one of the remaining two is less than a right angle; therefore, in any triangle there can be no more than one right or obtuse angle, and not less than two acute angles.

Corollary IV. — A perpendicular is the shortest line from a point to a straight line (Proposition XVIII of Euclid's Elements). A perpendicular is, therefore, assumed to represent the distance from a point to a straight line.

Definition X. — A *quadrilateral* is a figure bounded by four sides containing four interior angles. A *rectilinear quadrilateral* is one bounded by the segments of four straight lines — of fixed length each — between four points which are the vertices of the quadrilateral, — each vertex being connected only with two adjacent ones. A *plane quadrilateral* is one that can be placed wholly in a plane. We shall have occasion to use quadrilaterals of fixed sides only, but not of fixed angles. Of course, such are, so to speak, non-material ones, *i. e.*, bounding no fixed plane area, and in such, the relative distances of the four vertices are fixed only for the four (out of all six possible) pairs, constituting the ends of the four sides respectively. While the three distances, $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, must necessarily fix all distance-relations between three points — a distance being a relation between one pair (number of combinations of three different things taken two at a time), only *six* distances will be sufficient to fix uniquely the distance-relations between four points (number of combinations of four different things taken two at a time); hence, four distances are insufficient in the case of a quadrilateral. If, however, the quadrilateral is restricted to lie in a plane, we have an additional relation,—and only one additional distance, like one diagonal (the distance between either pair of the opposite vertices), will be sufficient to determine the complete form of the quadrilateral. This property of a quadrilateral is expressed by saying that a quadrilateral can "rack." We shall call such a quadrilateral with variable angles *an immaterial quadrilateral.*

Theorem 15.—In any rectilinear quadrilateral, whether plane or not, whose opposite sides are equal, the opposite angles are equal.

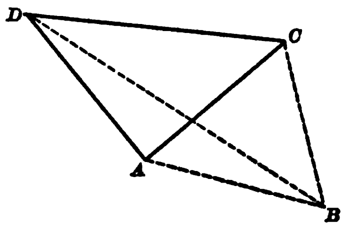

FIG. 17.

Demonstration.—Let $AB = DC$, $AD = BC$. Then joining $AC$ and $DB$, we get

$$\triangle ABC = \triangle CDA$$

and

$$\triangle ABD = \triangle CDB;$$

$$\therefore \angle ABC = \angle CDA, \quad \angle BAD = \angle DCB.$$

Q. E. D.

Corollary I.—It follows, that $\angle ABD = \angle BDC$ and $\angle ADB = \angle CBD$; similarly $\angle CAB = \angle ACD$ and $\angle ACB = \angle CAD$,—that is, the angles made by the same diagonal with the two opposite sides, are equal.

Corollary II.—It also follows, that in a plane quadrilateral, with equal opposite sides, two non-opposite sides and the angle enclosed by them are sufficient to determine the whole quadrilateral.

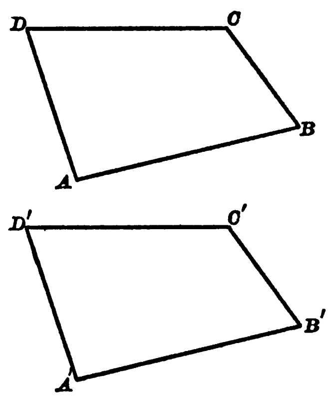

FIG. 18.

Theorem 16.—Two plane quadrilaterals $ABCD$ and $A'B'C'D'$, having three sides and two included angles in the one equal respectively to the corresponding three sides and included angles in the other, are equal to each other.

Demonstration.—If

$$DA = D'A',$$

$$AB = A'B',$$

$$BC = B'C',$$

and also,

$$\angle DAB = \angle D'A'B',$$

$$\angle ABC = \angle A'B'C',$$

then, by superposition, we see that the two quadrilaterals coincide with each other. Q. E. D.

Theorem 17. — In a plane quadrilateral, two of whose opposite sides are equal and in which the interior angles made by these with one of the remaining sides, are supplementary, this last side cannot be greater than the side opposite to it.

Let $OA = O_1A_1$, $\angle AOO_1 + \angle OO_1A_1 = 2$ rt. $\angle$'s;

then $OO_1 \not> AA_1$.

For, let $OO_1 > AA_1$. Then since $OA = O_1A_1$ and $\angle AOO_1 = \angle A_1O_1O_2$, the supplement of $\angle OO_1A_1$, we can apply the lower side of the quadrilateral to its upper side $O_1A_1$, and we get $A_1A_2 < O_1O_2 = OO_1$.

Now, suppose $OO_1 - AA_1 = l$, some length; then, every time $O$ is transposed a distance $= OO_1$ along the straight line $XX'$, $A$ is transposed a distance $= OO_1 - l$. Let $OA = ml + l_1$, where $m$ is a positive integer, and $l_1$ is either zero or a length less than $l$. Then, applying this quadrilateral to itself, along the same straight line $XX'$, $2(m + 1)$ times, we get, —

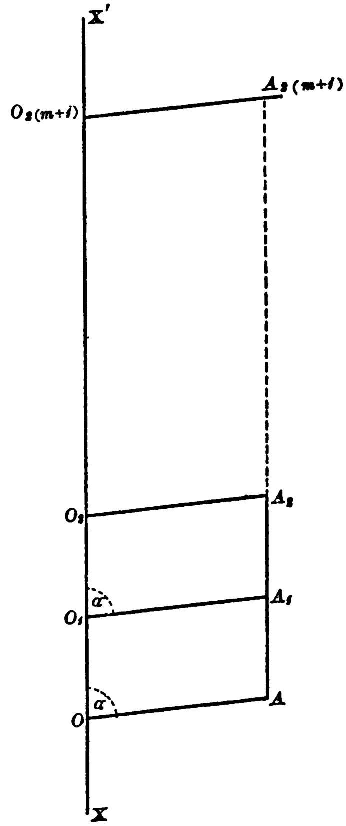


FIG. 19.

$$OO_{2(m+1)} = 2(m+1)\,OO_1 = 2(m+1)\,AA_1 + 2(m+1)\,l > 2(m+1)\,AA_1 + 2ml + 2l_1 = AA_1A_2 \cdots A_{2(m+1)} + OA + O_{2(m+1)}A_{2(m+1)};$$

that is, the straight line between $O$ and $O_{2(n+1)}$ is greater than the broken line between these same two points — which is absurd * (Euclid XX, Book 1). Hence, $OO_1 \not> AA_1$.

Q. E. D.

Corollary. — The sum of the three angles of a triangle cannot be greater than two right angles.

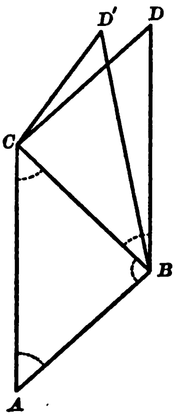


FIG. 20.

For, if $ABC$ is such a triangle, then applying to it an equal triangle $BCD$, where $BD = AC$ and $CD = AB$, we get $\angle DBC + \angle CBA + \angle BAC = \angle C + \angle B + \angle A > 2$ rt. $\angle$'s; therefore, if $\angle D'BC + \angle B + \angle A = 2$ rt. $\angle$'s, and $D'B = DB$, we have $\angle D'BC < \angle DBC$, whence $CD' < CD$ (Euclid I, XV); we have also, $D'B = CA$ and $\angle D'BA + \angle BAC = 2$ rt. $\angle$'s, *i. e.*, $D'BAC$ is just such a plane quadrilateral as has been discussed in the theorem, and $CD' < AB$, — which is impossible. Hence, the sum of the three angles of a triangle cannot be greater than two right angles.

Q. E. D.

Theorem 18. — An immaterial quadrilateral whose opposite sides are equal, can so "rack" or change its angles that, while one of them is passing through all possible magnitudes, the middle points of one pair of opposite sides shall always remain at a distance from each other, equal to each of the other pair of opposite sides.

Demonstration.—Let $ABCD$ be a quadrilateral whose opposite sides are equal, and $E$ and $F$, the mid-points of $AD$ and $BC$, respectively.

We observe first, that, $ABCD$ being an *immaterial quadrilateral*, the only restriction imposed upon the relative position of its sides is that resulting from the equality of either pair of its opposite sides. In any deformation or "racking" of the quadrilateral, the sides need not stay in one plane, if this is required by the conditions of the relative motion to which they are subjected, so that any pair may be in a different plane from the remaining pair. ($B$), for instance, is free to move in some path or other, situated in a spherical surface around ($C$), this

* See note, p. 116.

last moving in a sphere about $(D)$, and this one about $(A)$—provided the four distances do not change during these motions. Initially the distance $\overline{EF}$ is not given; it is, however, capable of determination. At least in one of the infinite number of relative positions which can be assumed by the points $(A)$, $(B)$, $(C)$, and $(D)$, as just defined, the distance $\overline{EF}$ will be equal to each of the sides $(AB)$, $(CD)$; this will, evidently, be the case

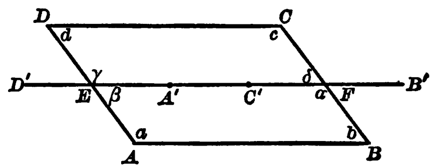


FIG. 21.

when all four points $(B)$, $(C)$ $(A)$, $(D)$ will fall upon the same straight line, say, upon $B'$, $C'$, $A'$, $D'$, respectively, where $B'C' = 2BF = 2CF = A'D' = 2AE = 2ED$, and $A'B' = AB = DC = D'C'$.

For $\angle B = \angle D$—always (Theorem 15); hence, when the first becomes zero, the second must also become zero; that is, when $(BC)$ falls upon $(AB)$, $(AD)$ falls upon $(DC)$; hence, $(A'B')$ will be in the same straight line with $(B'C')$ when and only when $(A'D')$ is in the same straight line with $(D'C')$, where the primed letters denote some particular position of the sides; in other words, $B'$ will be in the same straight line with $A'C'$, and $D'$ will be in the same straight line with these. Hence, $F$ and $E$ are the middle points of $B'C'$ and $A'D'$, respectively. We have, then,

$$EF = EA' + A'B' - FB' = \tfrac{1}{2}A'D' + A'B' - \tfrac{1}{2}C'B' = A'B' = AB = CD,$$

since $$A'D'/2 = C'B'/2.$$

In this position, then, let $(EF)$ be hinged to the two ends of a straight line of fixed length, or let $E$ and $F$ become fixed, so that $(BC)$ singly may revolve around $F$ in a sphere, and $(AD)$ singly, around $E$ in a sphere. I say that the quadrilateral may be brought out of coincidence with a single straight line and may assume all values for any one of its angles, provided the equality of opposite angles is preserved, not only in the whole quadrilateral, but also in the partial quadrilaterals. For, if not, the reason of this must be sought only in the mutual geometrical relations of the distances, which are the only parts of our immaterial quadrilateral given, including now the additional distance $\overline{EF}$ between the mid-points of $(AD)$ and $(BC)$; that is to say, these distances must be incompatible with an angle $b$ other than zero, so that an attempt to change this value would involve a contradiction in theory and, hence, a break in the distance-relations in practice. Calling the interior angles of the two partial quadrilaterals as in the figure, we get six independent relations between the angles, which, on the supposition that the motion sought is possible, must hold for all values of the angle $b$, the value $b = 0$ included. We have also, in addition, two independent relations, derived from the supposition that $(BFC)$ and $(AED)$ are to remain straight lines in the whole course of the motion. We have thus at first appearance relations enough, and just enough, to determine each angle, if such a determination is at all involved in the mutual relations of the angles and distances. Accordingly, if the motion postulated were impossible, these relations would have to lead us to a contradiction for any value of $b$ other than $b = 0$. Now, instead of leading to such a contradiction, the first six relations give us, as an immediate consequence, a new relation verified by the remaining two, — showing that our supposition does not involve any contradiction, and that such a motion can be realized. The relations referred to are:

$$\left.\begin{array}{l} 1)\ \alpha = a \\ 2)\ \gamma = c \\ 3)\ a = c \end{array}\right\} \therefore \alpha = \gamma,\ \text{I}; \qquad \left.\begin{array}{l} 4)\ \beta = b \\ 5)\ \delta = d \\ 6)\ b = d \end{array}\right\} \therefore \beta = \delta,\ \text{II}.$$

$$\text{Besides} \qquad \left\{\begin{array}{l} 7)\ \alpha + \delta = 2 \text{ rt. } \angle\text{'s} \\ 8)\ \beta + \gamma = 2 \text{ rt. } \angle\text{'s}. \end{array}\right.$$

Combining I and II, we get,

$$\alpha + \delta = \beta + \gamma,\ \text{III}, \text{—}$$

which agrees with (7) and (8). Of course, I and II can be realized without the relations (7) and (8), when $(BFC)$ and $(AED)$ are not restricted to remain straight lines. But the argument is thereby not invalidated — that the motion presupposed in the theorem cannot be impossible under the only conditions given. It may, of course, be impossible under certain

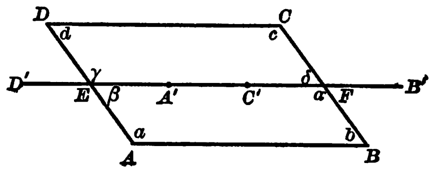


FIG. 21.

additional conditions, like the restriction of the lines to slide on certain surfaces; but such a restriction is excluded in the given conditions, where only abstract distances are given. The motion itself may involve certain positions as possible and others as impossible, — which circumstance we shall presently proceed to investigate only in so far as will be necessary for our main purpose, which is the establishment of the theory of parallels.

*Remark.* — Such a motion would also be possible, even in the case of spherical arcs of great circles, provided these arcs are at liberty to move out of the surface to which they belong initially, — that is, provided an additional restriction is not imposed upon them that every point of all five must remain, during the motion, at a constant distance from the center of the sphere to which they belong when the spherical angle $b$ is zero. In other words, in order that such a motion be realized for arcs of great circles of the same sphere, the planes determined by the arcs in motion can all have a common point of intersection only when the spherical angle of two of these linked arcs is zero or $\pi$ — *i. e.*, when they coincide with one and the same great circle; for all other values of the angles these planes cannot intersect in the same point, — in other words, the arcs must leave the common surface to which they originally belong, as can easily be proved when we come to a maturer stage of the science of geometry, to which the treatment of spherical geometry properly belongs. A similar remark holds with respect to geodesics upon a pseudosphere. It seems, that proceeding from these considerations, it ought to be possible to prove that, whenever the normals to a surface-element of a homogeneous surface meet in a point, or, what is the same thing, whenever the surface has constant positive curvature, there must be an excess of the sum of the three angles of a triangle over two right angles; and whenever the normals lie in different planes, or the homogeneous surface has constant negative curvature, there is a deficiency of the sum of the three angles of a triangle from two right angles.

Analytically, the proof of the possibility of such a motion as defined in the theorem, is obtained also by showing that, if we add to the above eight relations an arbitrary relation $b = \theta$, the determinant of the nine linear equations in nine variables — when the equations are made homogeneous — vanishes identically, proving that $b$ can have any value. The determinant in question is — if $a$, $\alpha$, $b$, $\beta$, $c$, $\gamma$, $d$, $\delta$, 1 is the order in which the variables are written — as follows:

$$\begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -\pi \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -\pi \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\theta \end{vmatrix} \equiv 0.$$

Q. E. D.

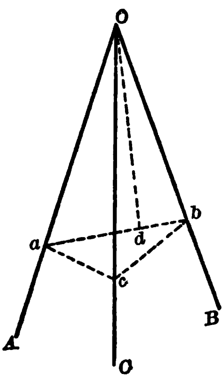

Fig. 22.

Theorem 19. Lemma. — Given an angle and a straight line through its vertex, not in the plane of the angle, the sum of the angles made by this line with the sides of the angle, is greater than the angle.

The theorem needs proof only for the supposition that $\angle COA$ and $\angle COB$, taken singly, are each less than $\angle AOB$. Connect any two points, $a$ and $b$, on the sides of the given angle $AOB$, by a straight line $ab$; this last will be in the plane of the angle (Theorem 12), and hence $OC$, the line through $O$ outside the plane, cannot meet $ab$. Pass now a line $Od$ in the plane $AOB$, at an angle $aOd$ equal to $\angle AOC$ and cutting $ab$ in some point $d$—which it must do, since it cannot cut either $Ob$ or $Oa$ again (Theorem 6 and corollary), and because the straight line from $O$ will pierce a sphere inclosing the whole triangle $abO$ (Theorems 2–5 and corollaries). Lay off on $OC$ a segment $Oc$ equal to $Od$, and join $ca$, $cb$; then $ca = da$ (Theorem 11), and in the triangle $acb$, $ac + cb > ad + db$ (Euclid I, XX *); hence, $\angle cOb > \angle dOb$ (Euclid I, XXV). $\therefore \angle aOc + \angle cOb > \angle aOd + \angle dOb$, or $\angle AOC + \angle COB > \angle AOB$.

Q. E. D.

Theorem 20.—In the motion considered in Theorem 18, the four sides of each quadrilateral will, at any instant, be in one plane.

For from I, $a = \gamma$, or from II, $\beta = \delta$, combined with (7), $a + \delta = 2$ rt. $\angle$'s, it follows that $\gamma + \delta = 2$ rt. $\angle$'s. Now, joining $FD$, we find that if $CF$ is always in the plane of $\triangle DFE$, then $CD$, $FB$, $EA$, and $AB$ are in the same plane, and the theorem is conceded. But if $CF$ is sometimes out of the plane

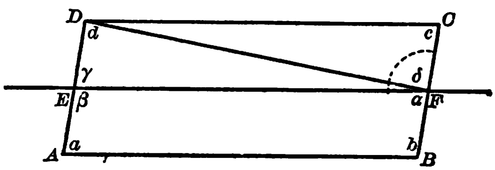


FIG. 23.

$DFE$, then $\angle CFE$ determines a plane different from plane $DFE$—that is, $DF$ is not in the plane of $\angle CFE$; hence, by preceding lemma, $\angle CFD + \angle DFE > \angle CFE$, or $\angle CFD + \angle FDC > \delta$ (Theorem 15); and since $c = \gamma$, we get $c + \angle CFD + \angle FDC > \gamma + \delta$, or $\angle DCF + \angle CFD + \angle FDC > 2$ rt. $\angle$'s—which is impossible (cor. to Theorem 17). Hence, $CF$ is always in the plane $DFE$, and the theorem is proved as before.

Q. E. D.

Corollary I.—It follows, that any *immaterial quadrilateral with equal opposite sides, can move in a plane so, that one of its angles assume any value we please, and that the distance between the middle points of one pair of opposite sides constantly remain equal to each of the remaining pair of opposite sides.*

* See also note, p. 97.

Corollary II. — It is evident that the sum of the interior angles and, hence, also of the exterior angles, adjacent to the same side in any of the positions of such a moving quadrilateral, is equal to two right angles, since 2 rt. $\angle$'s $= \alpha + \beta = \gamma + \delta = d + c = a + b = b + c$, etc.

Theorem 21. — The sum of the interior angles adjacent to the same side of any plane quadrilateral with equal opposite sides, is equal to two right angles.

For, an immaterial quadrilateral having sides of equal length with the given one, can assume such a position in a plane, that the angle enclosed between any two of its non-opposite sides, be equal to the corresponding angle in the given quadrilateral. And as the sum of the interior angles adjacent to the same side of this immaterial quadrilateral in the particular position, is equal to two right angles, it follows by Corollary II to Theorem 15, that the sum of the interior angles adjacent to the same side in our given quadrilateral, is likewise equal to two right angles. Q. E. D.

Corollary I. — The sum of the three interior angles of any triangle is equal to two right angles.

For, if in the triangle $ABC$ we produce $AB$ to $E$, $\angle CBE > C$; hence, making $\angle CBD = \angle BCA$, $BD$ falls within angle $CBE$ (Theorem 14, Cor. II). Taking $BD = AC$, and joining $CD$, we have, $\triangle CBD = \triangle BCA$, and $CD = AB$ (Theorem 11), hence, we have a plane quadrilateral with equal opposite sides therefore, $\angle CAB + \angle ABD =$ 2 rt. $\angle$'s, or $\angle A + \angle B + \angle C$ = 2 rt. $\angle$'s.

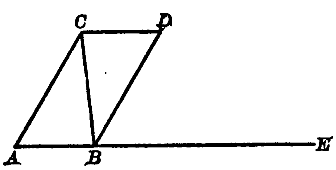

FIG. 24.

Corollary II. — The exterior angle of a triangle is equal to the two interior and opposite angles.

Corollary III. — The four angles in any plane quadrilateral equal 4 rt. $\angle$'s, since it can be divided into two triangles having their six angles coincident with the four given ones.

Theorem 22. — Two straight lines perpendicular to a third one in the same plane with it, —

1) Will both be perpendicular to any other perpendicular drawn from any point in the one to the other;

2) Will both have the same inclination towards any secant to both;

3) Will be everywhere equidistant from each other, *i. e.*, any perpendicular from one to the other will be of the same length with any other.

Demonstration. — 1) Let *AB*, *CD* be both perpendicular to the same straight line *KL* in the same plane; and draw any other perpendicular *MN*, from any other point *M* in *AB* to *CD*; and join *LM*. The sums of the interior angles of each of the two triangles are equal to two right angles singly, and to-

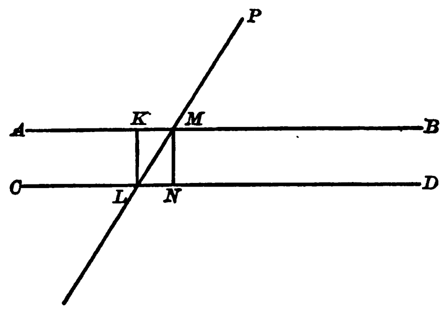


FIG. 25.

gether they are equal to four right angles. But the six angles of both triangles make up together the four angles of the quadrilateral *KLMN*. Of these last, the angles *K*, *L*, and *N* are each a right angle (by construction); therefore, $\angle M$ also is a right angle, and hence any perpendicular to *CD*, from any point in *AB*, will also be perpendicular to *AB*. For the same reason, any perpendicular to *AB*, from any point in *CD*, will also be perpendicular to *CD*.

2) In the same diagram, if *PML* is a secant to *AB* and *CD*, which are both perpendicular to a third line in the same plane, then draw from *L* a perpendicular *LK* to *AB*, and from *M*, a perpendicular *MN* to *CD*. The first will also be perpendicular to *CD*, and the second, to *AB* (section 1 of our proposition). Now, in the triangle *LMN* we have $\angle LMN + \angle NLM = 2$ rt. $\angle$'s $-$ $\angle N =$ rt. $\angle$ (corollary to Theorem 21); besides, $\angle LMN + \angle LMK = M =$ rt. $\angle$. Hence, $\angle LMN + \angle NLM = \angle LMN + \angle LMK$; $\angle NLM = \angle LMK$, or $\angle PLD = \angle PMB$ (Theorem 10). That is, both *AB* and *CD* have the same inclination owards any common secant.

3) If $KL$, $MN$ are any two perpendiculars to both $AB$ and $CD$, which are situated in the same plane, and $ML$ joins the opposite angles $M$ and $L$, then $\angle NLM = \angle LMK$; $\angle LMN = \angle MLK$ (section 2 of our proposition), and $ML$ is common to both triangles; therefore, $\triangle LMN = \triangle MLK$, and $MN = KL$, — and as the same reasoning applies equally to any other two perpendiculars to both, all the points of either line $AB$ or $CD$, are at equal distances from the other.

Definition XI. — Two straight lines perpendicular to a third in the same plane, as having the same inclination towards any common secant and being everywhere equidistant from each other, are said to be parallel to each other, meaning — beside each other, or going in the same direction and being everywhere at the same distance from each other.

Corollary I. — Any perpendicular to one of a pair of parallel lines, in the same plane with both, if produced indefinitely must meet the other at right angles.

For, if from the point of intersection with the first a perpendicular be drawn to the other, it must also be perpendicular to the first (section 1) and, therefore, coincide with the perpendicular to the same, previously drawn.

Corollary II. — From any given point without a given straight line only one parallel can be drawn, namely, that line which is at right angles to the perpendicular from the point to the given line.

Corollary III. — Any two points at the same distance from a given straight line, in the same plane with, and on the same side of it, determine another straight line parallel to the first. For, joining these points and drawing the perpendiculars to the given straight line, we get a plane quadrilateral which is congruent with the duplicate turned over, so that the equal perpendiculars become interchanged in position; hence, the angles opposite the given right angles are equal, and as their sum equals two right angles (Theorem 19, corollary III), each is equal to one right angle; that is, the line connecting the points, and the given line, are both perpendicular to the same straight line in the same plane with these. Hence, they are parallel.

Corollary IV. Any two straight lines lying in the same plane and having same inclinations towards a common secant are parallel; for if a parallel to one of the given lines is constructed through the intersection of the other with the secant, the corollary becomes evident (section 2).

Theorem 23.—Not more than two points of equal distances from a given straight line, can be situated in another straight line which is in a different plane from that passing through the given line and one of these points.

Demonstration.—Let $AB$, $CE$ be the two straight lines, of which $EC$ is not in the plane $BAC$ passing through the line $AB$ and the point $C$; and let the distances of the points $C$ and $E$ from $AB$ (*i. e.*, the perpendiculars $EB$ and $CA$ drawn from these points to $AB$) be equal. Then no other point on $EC$ can be of the same distance from $AB$.

For, produce the plane $BAC$ indefinitely beyond $BC$, and draw in it $CD \perp AC$, $\therefore \parallel AB$ (Definition XI), and $BD \perp AB$. $BD$ will meet $CD$, and $BD = AC = BE$ (Theorem 22, Cor. I and Definition XI). Join $AD$, $BC$, $EA$. The right-angled triangles $BAE$, $ABD$, and $BAC$ are equal (Theorem 11); therefore, $EA = DA = CB$, and $\angle EAB = \angle DAB = \angle CBA$. Also, since $EA$ is not in the plane $CAB$, $\angle CAE + \angle EAB > \angle CAB$ (Theorem 19), or $\angle CAE + \angle EAB > \angle CAD + \angle DAB$; hence, $\angle CAE > \angle CAD$. Now the triangle $EAC$ must have $\angle ECA <$ rt. $\angle$. For, if we imagine it removed from its original position and applied to $\triangle DAC$ in such a way that $EA$ shall coincide with its equal $DA$—the other side, inclosing the greater angle $EAC$, will fall without the smaller angle $DAC$ and will take the position $AC'$, as in the diagram; and joining $CC'$, we obtain an isosceles triangle $ACC'$, of which the angles $C$ and $C'$, at the base, are equal, and each less than a right angle (Cor. to Theorem 11 and Cor. III to Theorem 14). Then, $DC$ and $CC'$ form an angle $DCC' < 2$ rt. $\angle$'s, having its opening towards $A$; that is, the prolongation of $C'C$ is separated from $A$ and from any point on $AD$ by the half-ray $CD$. Therefore the half-ray $C'C$, including its prolongation, in passing continuously, not through $A$, to the position $C'D$ containing one point of the segment $AD$, must first pass through some point on the prolongation of $AD$ before reaching its final position. Hence $\angle AC'D$ is less than some angle which is less than $\angle AC'C$ (Scholium to Theorem 12); or $\angle AC'D < \angle AC'C <$

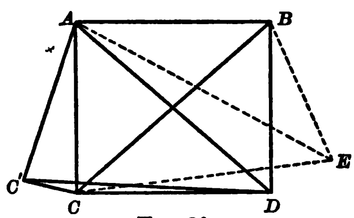

Fig. 26.

rt. ∠. Now, any point on the prolongation of $EC$ to the left, must also be without the plane $DCA$, since one part of $EC$ cannot be without, and the other within, the plane $DCA$ (Theorem 12); and any straight line connecting $C$ and another point of the same distance from $AB$, without the plane $DCA$, makes with $CA$ an angle < rt. ∠ and, consequently, cannot be the prolongation of $EC$, since such a prolongation forms with $CA$ an angle > rt. ∠ (Theorem 10). Therefore, no point of the same distance from $AB$ as $C$, can be on the prolongation of $EC$ to the left. For a similar reason, no point of the same distance from $AB$ as the two given points, can be on the prolongation of $CE$ to the right, because $CE$ is not in the plane $AEB$ passing through $AB$ and the point $E$. Neither can a point of an equal distance from $AB$ be situated on $CE$, between $C$ and $E$, since then, either $E$ or $C$ would be on the prolongation of a straight line connecting two points of equal distances from another straight line—one of the points connected being in a different plane from that passing through the other point and the other straight line, — which has just been demonstrated to be impossible. Therefore, no other point besides $E$ and $C$, on the same straight line with them, or on its prolongations, is possible, of the same distance from $AB$ as $E$ and $C$.

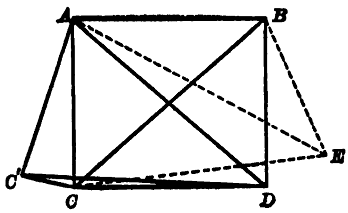


FIG. 26.

Q. E. D.

Corollary I. — Hence, any straight line in space which connects three points at equal distances from another straight line, must lie wholly in the plane passing through the other line and one of its own points, — and, therefore, is parallel to the other; and all that is said in Theorem 22, with reference to a parallel straight line, is also true of any straight line in space having, at least, three points at equal distances from a given straight line.

Corollary II. — It also follows, that if two equal lines intersect two others in four points, and three of these lines are at right angles to one another, all four are in the same plane, and, therefore, are parallel, each pair singly.

Corollary III. — *A parallel can also be defined as the locus* *of all points equidistant from a straight line, and collinear with two given points.*

The definition of alternate interior and exterior angles of a line intersecting two others in the same plane, is as usually given. If the definition of parallels is taken provisionally in the sense in which we have defined them, we see that we have proved propositions XXVII, XXVIII, XXIX of Euclid, also XXXII and some others, from which a number of corollaries can be drawn, regarding the conditions of parallelism of straight lines in space, which we will not give here. We are now in a position to prove propositions XXX, XXXIII, and XXXIV of Euclid, using his proofs word for word. This will now enable us to deduce Euclid's famous *Eleventh Axiom*, and thereby extend the definition of parallelism to any two lines situated in the same plane and not meeting each other at a finite distance.

Theorem 24.— Two straight lines in the same plane, of which one is perpendicular to a third one, and the other makes an acute angle with it, if sufficiently produced on that side of the secant where the acute angle is situated, must meet each other somewhere.

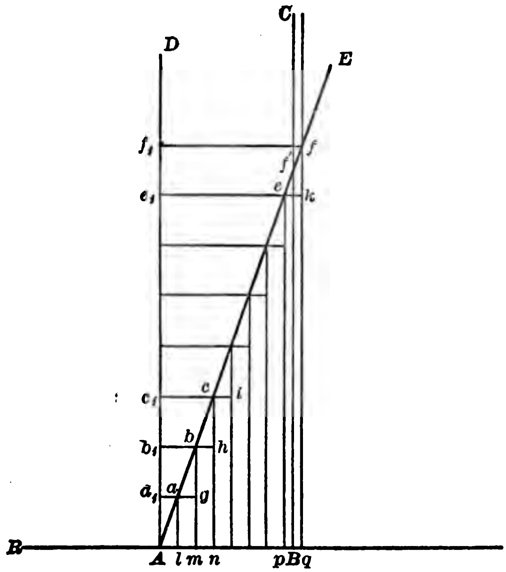


Fig. 27.

Let $BC$ be perpendicular to $AB$, and $AE$ make an acute angle with $AB$ at $A$, then $AE$ and $BC$, if produced towards the opening of the acute angle, will meet in some point $f$.

Demonstration. Take some point $a$ on $AE$, draw a perpendicular from it to $AB$ (Euclid, prop. XI). This perpendicular will cut $AB$ in some point $l$ between $A$ and $B$, and not on its prolongation $AR$; otherwise $\angle BAE$, being according to supposition an acute angle, would be greater than a right angle (Theorem 14, Cor. II) — which is absurd. Let now $\overline{Al}$ be contained in $\overline{AB}$ $m$ times ($m$ being a whole number), or $m$ times with some remainder less than $\overline{Al}$. Take upon $AE$, beginning from $A$ and proceeding towards $E$, $m+1$ parts equal to $\overline{Aa}$, namely $\overline{Aa}$, $\overline{ab}$, $\overline{bc}\cdots$, and let $f$ be the end of the last part.

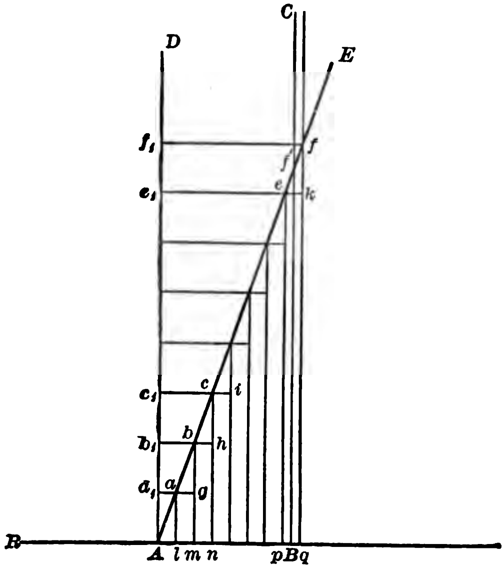


FIG. 27.

Draw now perpendiculars $aa_1$, $bb_1$, $cc_1\cdots ff_1$ to $AD$ which is made perpendicular to $AB$ (Eucl., prop. XI); they will also be parallel to $AB$ and to one another (Defin. XI). Draw also $bm$, $cn$, $\cdots fq$, parallel to $BC$, $AD$, $al$, — which will all meet at right angles all the perpendiculars to $AD$ (Cor. 1 to Theorem 22). Let the vertices of these right angles be $g, h, i, \cdots f$. All these perpendiculars will represent two sets of parallels. The triangles $Aal$, $abg$, $\cdots efk$ will be equal to one another because they have one side and two adjacent angles equal respectively, in all of them — those adjacent angles being alternate angles (Theorem

22, section 2, and Euclid, prop. XXVI). Therefore $Al = ag = bh \cdots ek$, and therefore also — according to Euclid, proposition XXXIII—$Al = lm = mn \cdots = pq$. For the same reason $a_1a = Al$, $b_1b = Am$, $c_1c = An \cdots f_1f = Aq$. But $Aq$, containing $m + 1$ times $Al$ is greater than $AB$; hence $f_1f > AB$. In other words, the distance of $f$ from $AD$ is greater than the distance between the parallels $BC$, $AD$, which is everywhere the same and equal to $AB$; hence, $f$ must lie beyond the space inclosed between both the parallels — and since it is a point on $AE$, $AE$ must intersect $BC$ in some point of $f'$ below $f$.

Q. E. D.

Corollary I. — Two straight lines in the same plane, which cannot meet how far soever produced both ways, are parallel to each other. For, any perpendicular to one of them, drawn from any point in the other, cannot make an oblique angle with the latter — otherwise they would meet on the side of the acute angle; it must, therefore, be at right angles to both — or both the lines must be parallel (Theorem 22, Defin. XI).

Q. E. D.

Corollary II. — Any straight line intersecting one of a pair of parallel lines, and in one plane with the other, if produced sufficiently far, must meet the latter.

For, being in one plane with both, were it not to meet the second somewhere, it would be parallel to it (preceding Corollary) and hence also to the first (Euclid, prop. XXX); but it is not, since it intersects the first. Therefore it must likewise intersect the other, somewhere at a finite distance. Q. E. D.

* * *

With this, the theory of parallels, as well as that of the elements of geometrical measurement — distance, straight line, plane, angle, circle, etc., — is firmly established. We see, then, that THE FOUNDATIONS OF THE EUCLIDIAN GEOMETRY rest on a much firmer basis than mere arbitrary assumptions verified by experience to a very great degree of approximation. They are, rather, implanted in the very nature of our logic, being to a great degree what the Kantists call *a priori*, and are empirical only in so much as all our conceptions of quantity, form, and motion depend upon experience.