TEXT-BOOK

OF

ELEMENTARY PLANE GEOMETRY

BY

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INTRODUCTION.

1. Every object which occurs in nature has numerous properties; to facilitate the investigation of which, one preliminarily endeavours to consider one at a time, independently of the others; the different investigations of the same kind are then collected, and thus the different sciences, *natural sciences*, are formed. Thus, if we examine a piece of chalk, we might ask about its origin and occurrence, its specific gravity and colour, the combination of its elements, its form, &c., and these questions will be answered respectively by Geology, Physics, Chemistry, and Geometry.

2. Geometry treats of the *form* without regard to the substance; when we speak of a sphere, we do not consider of what it is made, but only of the space which it occupies; every object occupies a space, which has extension in all directions; this is a geometrical *body*.

The boundary between a body and the surrounding space is called a *superficies* or the surface of the body; a superficies has no thickness.

If a part of a superficies be cut off from the surrounding part, the boundary is called a *line*; a line has no breadth. If a part of a line be cut off, the limit is termed a *point*; it has no magnitude, but only denotes position.

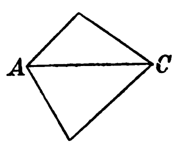
3. When a point moves, it produces a line, similarly a superficies may be imagined as being produced through the

movement of a line, a body through the movement of a superficies.

4. Let us imagine a body revolving round two fixed points, that is, moving so that two points in it do not change their places in space, we could then suppose a line to be drawn through these two points and through all the other points in the body, which during the movement do not change their places. Such a line is called a *straight line*. The straight line, therefore, is determined by two points, that is, that *through two given points there can only be drawn one straight line.* From this it follows again, that *two different straight lines can only have one point in common (a point of intersection)*, for if they had two points in common, they must coincide. A straight line can be imagined to be infinitely produced by the addition of other straight lines, each having two points in common with the preceding ones. Where it cannot be misunderstood line is often used for straight line.

A line of which no part is straight is called *curved*, a line composed of several straight lines is called a „*broken line*".

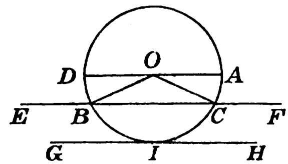
5. A plane superficies or *plane* is one in which every straight line lies wholly, when two of its points lie in it; the position of a plane is determined, when it contains three given points which are not in a straight line. A straight line and a plane are supposed infinitely produced, when the opposite is not stated; a straight line may, without being changed, be imagined to be made to slide along itself; a plane may, without being changed, be made to slide along itself or turned in itself round one of its points.



6. A *figure* is a limited part of a plane, its boundary is called *Perimeter*, when it is broken, *Periphery*, when it is curved. A *triangle* is contained by three straight lines (sides), a quadrilateral by four, a multilateral figure or polygon by an indefinite number; a line joining the extremities of two sides of a polygon, *without itself* being a side, is called a *diagonal* (AC). A polygon

is called *convex*, when the prolongations of the sides fall outside the polygon; in the opposite case it is said to be *not convex*.

7. When a straight line of fixed length (OA) revolves in a plane round one of its extremities, its other extremity describes a closed curved line called a *circle*; the fixed point O is called the centre and OA the *radius*; *all radii of the same circle are equal*; a line joining two points of the circumference is called a *chord* (BC); a *diameter* (DA) is a chord through the centre; *all diameters are equal*; a chord produced is called a *secant* (EF); it lies partly inside and partly outside the circle and cuts it in two points; if the secant be moved, so that the two points coincide in one point (I), this is called a *point of contact* and the line a *tangent* (GH); the tangent has only one point in common with the circle and lies wholly outside of it. A part of the circumference is called an *arc* ($\smile AC$). A part of the circle contained by a chord and an arc (BIC) is called a *segment*; a part contained by two radii and an arc (BOC) is called a *sector*.



A figure is said to be *inscribed* in a circle, when its sides are chords, *circumscribed*, when they are tangents.

8. Two figures are said to be *congruent*, when they are the same in everything, but only lying in different places; two congruent figures may be supposed to be placed on each other, so that they cover one another; the parts which then coincide are called *corresponding*. The sign for congruence is $\cong$. Circles with equal radii are congruent.

9. That which is given in a proposition is called *Hypothesis*, that which is to be proved, *Thesis*.

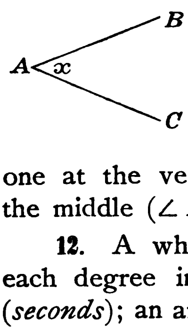
10. *Plane geometry* only treats of figures in the same plane, chiefly the straight line and the circle; Stereometry treats of lines, superficies, and bodies in general.

I.

THE SITUATION OF STRAIGHT LINES.

I. THE MUTUAL DEPENDENCE OF ANGLES.

11. When a straight line revolves round one of its points, until it again arrives at the position from whence it started, it is said to have completed a whole revolution; if it has not revolved so much, its position is determined by stating through how great a part of the revolution it has turned. We say that the line forms a certain *angle* with its former position, and *the angle between two lines is thus that part of a revolution, which one line must perform in order to cover the other.* The two lines are called the *legs* or *sides* of the angle, their point of intersection its *vertex*. The symbol for angle is $\angle$; where it cannot be misunderstood, an angle is denoted by a single letter at the vertex ($\angle x$ or $\angle A$); otherwise by three letters, one at the vertex and one at each of the legs, the first in the middle ($\angle BAC$).

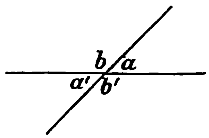


12. A whole revolution is divided into 360° (*degrees*), each degree into 60′ (*minutes*), and each minute into 60″ (*seconds*); an angle of 180°, corresponding to half a revolution, is called an *angle of continuation*; *its legs lie in a straight line*; an angle of 90°, corresponding to $\frac{1}{4}$ revolution, is called a *right angle* (symbol R.); angles less than 90° are called *acute*, between 90° and 180° *obtuse*. Angles which are not right angles are called *oblique*. Two lines which form a right angle are said to be at *right angles* or *perpendicular* to one another. The symbol for „perpendicular“ is $\perp$.

13. One angle is said to be the *complement* of another angle, when their sum is equal to 1R., and the *supplement*, when their sum equals 2R. Thus, when an angle is a°, its complement is $90^\circ - a^\circ$, its supplement $180^\circ - a^\circ$.

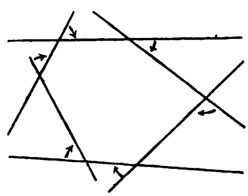
Two angles must therefore be equal, when their complements or supplements are equal.

14. *Adjacent angles* are those which have one leg in common and whose other legs are continuations of each other (b and a); two such angles are together equal to 180°, and are therefore supplementary angles.



15. *Vertical angles* are those whose legs are continuations of each other; *they are equal*, as the revolution which makes the legs of the one angle coincide also makes the legs of its vertical angle coincide; the angles formed by the intersection of two lines can therefore be found, when the one is known; for example, if $a = 45^\circ$, then also $a' = 45^\circ$, $b = b' = 135^\circ$.

16. *The sum of the exterior angles of a polygon is 4R.* For if a line be imagined to be placed on one of the sides of the polygon, from thence turned on to the next side and so on, till it again falls on the first side, it will by degrees have turned through all the exterior angles of the polygon; but it has thereby performed a whole revolution, that is 4R.*).



17. *The sum of the angles of a polygon is found by taking as many times 2R., as the polygon has sides, and from that subtracting 4R.* ($2n$R. — 4R., when it has n sides). For the sum of two adjacent angles is 2R.; therefore the angles

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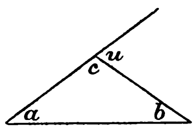
*) If some angular points of the polygon turn inwards, the proofs in 16 and 17 still hold good, when the revolution in backward direction is reckoned as negative.

We say in 16 that the line has performed a whole revolution, although it has arrived back on itself in a different way than in 11; we really hereby supplement the definition of the plane; the same reasoning could, for example, *not* be applied to arcs lying on the surface of a sphere.

of the polygon together with their adjacent angles will make as many times 2R., as the figure has sides (2*n*R.); from this must be subtracted the sum of the adjacent angles, which is 4R.

The sum of the angles of a triangle will be 2R., of a quadrilateral 4R., of a seventeen sided figure 30R., &c.

18. From 17 it follows that only one of the angles in a triangle can be a right angle or obtuse; in the *right-angled triangle* the side opposite to the right angle is called the *hypothenuse.*



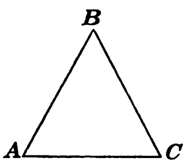
19. *The angle adjacent to one angle of a triangle equals the sum of the other two.*

For $u = 180° - c$

and $a + b = 180° - c$

therefore $u = a + b.$

20. A triangle which has two sides equal is termed *isosceles*; the intersection of the equal sides is termed the vertex, the third side the *base*, and its opposite angle the vertical angle.



The angles at the base of an isosceles triangle are equal. For if the triangle be lifted up and placed on itself in an inverted position, so that $\angle B$ covers itself, *BC* falls along *BA*, and *BA* along *BC*, then *A* must fall in *C*, and *C* in *A*, and consequently *AC* coincides with *CA*; as the angles *A* and *C* thereby coincide they are equal.

21. *When two angles of a triangle are equal, the triangle is isosceles.* This is proved in a similar way; when *AC* covers *CA*, *AB* must fall along *CB*, because the angles are equal, and *CB* along *AB*; now as the sides fall on each other, *B* must coincide with *B*, so that the sides coincide.

22. When all the sides of a triangle are equal, it is termed *equilateral*; the angles of this must all be equal (20), therefore *each 60°, and conversely.*

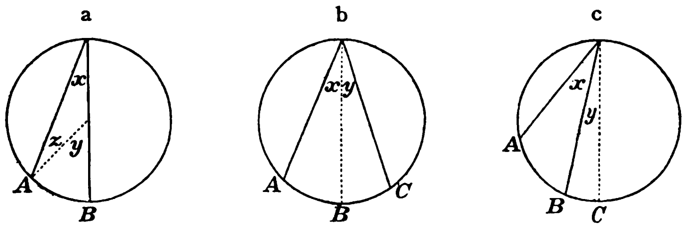
23. When one of the angles of an isosceles triangle is known, the others can be found; if one of the angles at the base is g°, the other will be the same, and the vertical angle $180^\circ - 2g^\circ$; if the vertical angle is t°, the others will together be $180^\circ - t^\circ$, therefore each $90^\circ - \frac{1}{2}t^\circ$.

II. THE DEPENDENCE OF ANGLES AND ARCS.

24. An angle having its vertex at the centre of a circle, and its legs being radii, is called an *angle at the centre*; by superposition it may be shewn *that equal angles at the centre of the same circle or of equal circles stand on equal arcs* and conversely, and *that equal arcs are subtended by equal chords.*

The circle is like the whole revolution divided into 360°, so that an angle at the centre contains the same number of degrees as the arc on which it stands; we therefore say that *the angle at the centre is measured by its arc.*

An angle at the circumference of a circle has its vertex in the circumference, and its sides are chords; *it contains half as many degrees as the arc on which it stands.*



a) If the one leg of the angle passes through the centre, a radius is drawn to the extremity of the other leg; we then have

$$y = x + z \quad \text{. . . . . (19)}$$

or, as

$$x = z \quad \text{. . . . . (20)}$$

$$y = 2x, \text{ therefore } x = \tfrac{1}{2}y.$$

As y is measured by the arc AB, x will be measured by half the arc.

b) If one leg lies at each side of the centre, this case is made to refer to the preceding one by drawing a diameter from the vertex; we then have

$$x = \tfrac{1}{2}\smile AB; \quad y = \tfrac{1}{2}\smile BC, \; . \; . \; . \; . \; . \; (24, a)$$

therefore

$$x + y = \tfrac{1}{2}\smile AB + \tfrac{1}{2}\smile BC = \tfrac{1}{2}\smile AC$$

c) In the same manner we get, if both legs lie at the same side of the centre,

$$x + y = \tfrac{1}{2}\smile AC; \quad y = \tfrac{1}{2}\smile BC, \; . \; . \; . \; . \; (24, a)$$

therefore by subtraction

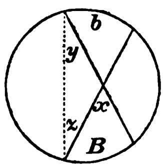
$$x = \tfrac{1}{2}\smile AB.$$

Thus the proposition holds good in all cases.

25. As the proposition in 24 holds good however small the one chord becomes, it must also hold when the chord grows infinitely small, that is, when it (produced) becomes a tangent. *Therefore an angle contained by a chord and a tangent is measured by half the arc which the chord cuts off.*

26. From these propositions it follows that *an angle at the circumference, which stretches over a diameter, is a right angle*, for the arc on which it stands is 180°, and that *the opposite angles of an inscribed quadrilateral are supplementary angles* as they together stand on the whole circumference, which is 360°.

27. *An angle, having its vertex inside the circle, is measured by half the sum of the arcs, which it and its vertical angle intercept.*



For $$x = y + z \; . \; . \; . \; . \; . \; . \; . \; . \; (19)$$
and $$y = \tfrac{1}{2}\smile B; \quad z = \tfrac{1}{2}\smile b, \; . \; . \; . \; . \; (24)$$
therefore

$$x = \tfrac{1}{2}B + \tfrac{1}{2}b = \tfrac{1}{2}(B + b).$$

28. *An angle, having its vertex outside the circle, is measured by half the difference of the arcs, which it intercepts.*

The legs of the angle can be two secants, a secant and a tangent, or two tangents.

In all cases we have

$$y = x + z, \quad \text{or} \quad x = y - z; \quad \ldots\ldots\ldots (19)$$

but

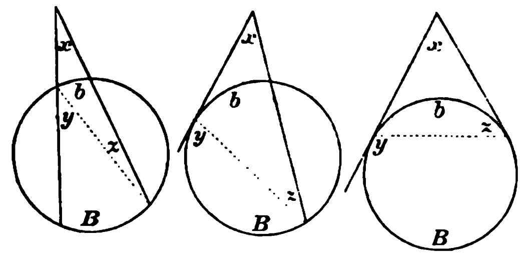
$$y = \tfrac{1}{2}B; \quad z = \tfrac{1}{2}b, \quad \ldots\ldots (24 \text{ and } 25)$$

therefore

$$x = \tfrac{1}{2}(B - b).$$

In the last case, instead of B, we can put $360° - b$, from which we get

$$x = \tfrac{1}{2}(360° - 2b) = 180° - b.$$



Therefore an angle contained by two tangents is measured by subtracting the number of degrees in the smaller arc from 180°.

From 27 and 28 it follows that of any angles standing on an arc AB, those which have their vertices outside the circle are less, and those having their vertices inside, greater, than an angle at the circumference standing on AB.

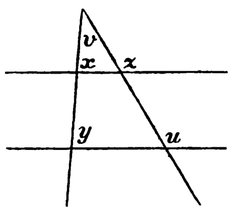
Note. If the legs of an angle C pass, the one through A, the other through B, a circle can always be described through A and B, so that C shall lie within this circle. Describe a semicircle on AB; if $\angle C >$ R., C will lie within this semicircle; if not then with the middle point of the circumference of the semicircle as centre, describe a circle through A and B; the angles at the circumference of this circle, standing on AB, are $\frac{1}{2}$R.; if $\angle C > \frac{1}{2}$R., then C will lie within this second circle; but if not we proceed in this manner, and shall in succession get circles, the angles at the circumferences of which are $\frac{1}{4}$R., $\frac{1}{8}$R., &c., we must then at last get a circle in which the angles at the circumference standing on AB are less than $\angle C$, and this circle will pass outside C.

29. *Two tangents, drawn from any one point to a circle, are equal, reckoned from this point to the points of contact.* For the triangle formed by joining the points of contact is isosceles, as the angles at the base are measured by the same arc (25). (See the last figure to 28).

30. *A tangent is perpendicular to the radius to the point of contact,* for the angle they contain is a right angle, as the arc is 180°. (25).

III. PARALLEL LINES.

31. If a straight line cuts two other straight lines and makes the exterior angle equal to the interior, opposite angle, on the same side of the line (as x and y), the two lines are said to be *parallel.* „Parallel to" is denoted thus $\neq$. *When the angles formed by the intersection of one straight line with the parallel lines are equal, they will be equal for all straight lines.*



For as
$$\left.\begin{aligned} z &= x + v, \\ u &= y + v, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (19)$$
then
$$z = u, \text{ when } x = y.$$

32. At each of the points of intersection there are four angles; when the lines are parallel, each of the acute angles in the one group is equal to each of the acute angles in the other group, but is the supplement of each of the obtuse angles. Conversely, when one of these conditions is fulfilled, the lines are parallel.

33. *Parallel lines can never cut each other.*

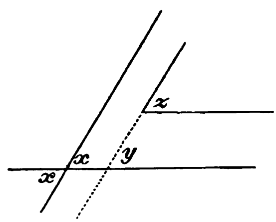
For if they did, we would, by drawing a line cutting them both, get a triangle, in which the angle adjacent to one angle of the triangle would be equal to one of the other angles of the triangle, which is in opposition to 19.

Note. Conversely, we see that the lines must meet if x and y are not equal; for let A be a point in the one line, B a point in the other, and AB the line joining them; we can then always (28, Note) in the line through A, within a certain finite circle find a point C, so that $\angle ACB = y - x$ (or $x - y$); CB must therefore just be the line through B, as its angle with AB becomes y; that is, the two lines meet in a point C at a finite distance.

34. By observing the angles formed by an intersecting line, it is evident that *through one given point, there can only be drawn one line parallel to a given line*, and that

When two lines are parallel to the same third line, they are parallel to one another.

35. *Angles with parallel legs are equal, when the legs both extend in the same direction or both in opposite directions.*



If one of the legs of the one angle be produced till it cuts a leg of the other angle, we get

$$\left.\begin{aligned} x &= y, \\ y &= z, \end{aligned}\right\} \quad \ldots\ldots (31)$$

therefore $\qquad z = x.$

EXAMPLES.

1. In a triangle one angle is 75°, another 42°; how great is the third?
2. In a triangle the two angles are each 42° 12′ 42″; how great is the third?
3. In a triangle one angle is A, the other B; how great is the third?
4. In an isosceles triangle the vertical angle is 60°; how great is the angle at the base?
5. How great is the angle between perpendiculars on two of the sides of an equilateral triangle?
6. O is a point within a triangle ABC; prove that $\angle AOB > \angle ACB$.

7. The vertical angle of an isosceles triangle is 40°; prove that the line bisecting its adjacent angle is parallel to the base.
8. *How great is the angle between the lines bisecting two adjacent angles?*
9. In a triangle the two angles are 40° and 80°; bisect the third angle, and from its vertex draw a perpendicular to the opposite side; how great is the angle between this and the bisecting line?
10. In a right-angled triangle one af the acute angles is v, how great is the other and how great are the angles in which the right angle is divided by a perpendicular from it to the hypothenuse?
11. Prove that the perpendicular from one end of the base of an isosceles triangle to the opposite side, forms an angle with the base, half as great as the vertical angle.
12. In a right-angled triangle the one angle is 60°; this is bisected, and from the vertex of the right angle a perpendicular is drawn to the hypothenuse. Prove that one of the triangles, thus formed, is equilateral.
13. *The legs of one angle are perpendicular to the legs of another angle; prove that the two angles are equal or supplementary.*
14. One of the angles at the base of an isosceles triangle, whose vertical angle is 36°, is bisected; prove that the two small triangles will be isosceles, and find the lengths of all the lines in the figure, when the sides of the given triangle are r and the base t.
15. In a triangle one angle is v; how great is the angle between the lines which bisect the two other angles?
16. Prove that an angle of a triangle is a right angle, when a line from its vertex to the middle point of the opposite side is half as great as this.
17. *Prove that, when an angle of a right-angled triangle is 30°, then the lesser of the sides containing it is half as great as the hypothenuse.*

18. The side AB of a triangle ABC is produced to D, so that $BD = BC$. Prove that the line bisecting the angle B is parallel to the one joining DC.
19. Prove that in a right-angled isosceles triangle, the perpendicular from the vertex of the right angle to the hypothenuse is half as great as this.
20. The exterior angles of a triangle are bisected, thereby three triangles are formed, each having a side in common with the given triangle. Prove that the three triangles contain the same angles.
21. On one leg of any angle ABC, mark off any part AB, thereupon draw a line $AD \neq BC$, and make $AD = AB$.
    Prove that the line BD bisects the given angle.
22. In a triangle ABC the lines bisecting the angles A and B intersect in O. Through O is drawn $DE \neq AB$. Prove that $DE = AD + BE$.
23. The demonstration of the proposition in 31 cannot be applied, if the two lines xy and zu cannot by prolongation be made to cut each other. How may it be demonstrated in this case?
24. Two angles have parallel legs; shew that the lines bisecting them are parallel or perpendicular to each other.
25. *How many diagonals can be drawn in a polygon with n sides?* $\left(\text{Ans. } \frac{n(n-3)}{2}\right)$
26. In a quadrilateral the opposite sides of which are parallel the one angle is v; how great are the others? How great must the angle v be, in order that a circle can be described about the quadrilateral, and where will the centre of this circle lie?
27. Prove that the chord of an arc of $60°$ equals the radius.
28. In a circle two chords issue from the same point, and cut off arcs of $120°$ and $80°$. How great is the angle between the two chords?
29. In a circle a diameter AB is drawn, and a chord CD equal to the radius. How great is the angle between AC and BD, and between AD and BC?

30. In a triangle ABC, $BC < BA$. With B as centre and BC as radius a circle is described, cutting CA in E, BA in D. Shew that $\angle DEA = \frac{1}{2}B$.

31. In a triangle ACB, $\angle C = R$. With centre C describe a circle through the middle point of AB, cutting the hypothenuse or its prolongation the second time in D. Shew that the acute angle which CD makes with the hypothenuse is double of one of the angles of the triangle.

32. A circle has its centre on the circumference of another circle; the two circles cut one another in A and B; prove that the one arc AB contains half as many degrees as one of the other arcs AB.

33. In a circle two equal angles at the circumference BAC and DEF are drawn; shew that $BF \not\equiv CD$.

34. A triangle ABC is inscribed in a circle with centre O; D is the middle point of the arc BC. Shew that the angle ADO is half as great as the difference between B and C.

35. Two chords EA and EB in a circle are produced to C and D, so that CD is parallel to the tangent at E. Prove that the opposite angles of $ABCD$ are supplementary.

36. A is one of the points of intersection of two circles, of which the one passes through a point B, the other through a point C. Through A a line is drawn, cutting the first circle in D, the other in E. Prove that the angle between the lines BD and CE is constant (the same, in whatever way the line through A is drawn).

37. A circle touches one side of a triangle and the two other sides produced. Prove that the distances from the points of contact with the latter to their point of intersection equals half the perimeter of the triangle.

38. A triangle ABC, the angles of whieh are known, is inscribed in a circle. How great is the angle which the tangent touching the circle at A makes with BC?

39. Three small triangles are cut off from a triangle by tangents to its inscribed circle. Shew that the perimeters

of the three triangles are together equal to the perimeter of the given triangle.

40. Prove that the four lines bisecting the angles of any quadrilateral bound a quadrilateral, the opposite angles of which are supplementary.

41. Two circles cut each other in A and B. From A the diameters AC and AD are drawn; shew that C, B, and D lie in a straight line.

42. An angle has its vertex outside the circumference; one of its legs passes through the centre, and the part outside the circle of the other one equals the radius. Prove that the greater of the intercepted arcs is three times as great as the lesser.

43. Through each of the points of intersection of two circles a straight line is drawn. Prove that the chords, joining the other points of intersection of these with the circles, are parallel.

44. Two circles with equal radii cut each other; shew that they divide each other into arcs which are respectively equal.

45. With one of the points of intersection of two equal circles as centre describe a circle cutting the given circles. Shew that two and two of the four points of intersection are in a straight line with the other point of intersection of the given circles.

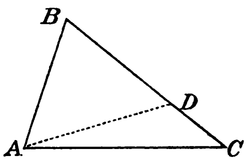
46. Two equal circles cut each other in A and B. Through A a line is drawn cutting the circles in D and E. Shew that DE is bisected by a circle on AB as diameter.

47. Between the circumferences of two circles, two lines are drawn through one of the points of intersection of the circles. Prove that the chords joining the extremities of the lines make the same angle in whatever way the two lines are drawn.

IV. RELATIONS BETWEEN THE LENGTH OF STRAIGHT LINES.

36. The length of a straight line is measured, by stating how many times it contains another straight line, *the unit*, the length of which is supposed known; the foot it used as unit; it is divided into 12 inches, the inch into 12 lines (Duodecimal measure) or the inch into 10 parts (Decimal measure). 5 feet 7 inches 3 lines is written $5' \, 7'' \, 3'''$.

37. *The greater side of a triangle has the greater angle opposite to it.*



From the greater side BC cut off $BD = BA$.

Then $\angle DAB = \angle BDA$,

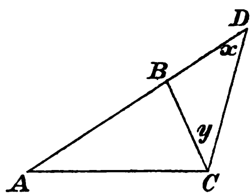
but $\angle A > \angle BAD$,

$\angle BDA > \angle C$ . . . . . (19),

therefore $\angle A > \angle C$.

38. *A greater angle of a triangle is subtended by a greater side.*

When $\angle A > \angle C$, then $BC > BA$; for if not, then either $BC = BA$ or $BC < BA$, but from the first case it would follow that $\angle A = \angle C$, and from the second that $\angle A < \angle C$, and both are in opposition to what was given.

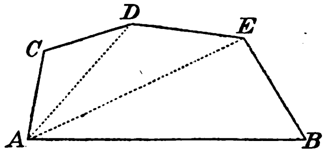


39. *One side of a triangle is less than the sum of the other two.*

Make $BD = BC$, then $x = y$, therefore $\angle ACD > \angle x$, but from that it follows (38) that $AD > AC$, or $AB + BC > AC$.

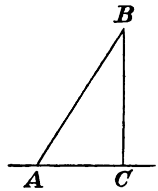
This result may also be written $AB > AC - BC$, so that *in any triangle one side is greater than the difference of the other two.*

40. *A straight line is shorter than any broken line joining its extremities.*



$ACDEB > ADEB > AEB$
$> AB$ . . . . . (39)

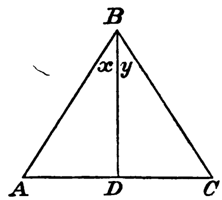
41. *From a point there can only be dropped one perpendicular to a line, and this is shorter than any other line from the point to the line.*



When $BC \perp AC$, then we cannot have $BA \perp AC$, as there cannot be two right angles in a triangle (18); further by 38, $BA > BC$ as $\angle C > \angle A$.

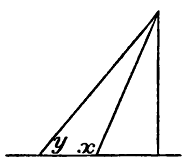
The point C, where the perpendicular from B meets the line AC, is called the *projection* of B on AC. If D is the projection of another point E, CD is called the projection of the line BE.

42. *Two lines which diverge equally from the perpendicular are equal.*



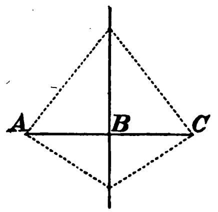
That AB and BC diverge equally either signifies, that $\angle x = \angle y$ or that $AD = DC$. In both cases the one side of the figure must coincide with the other, when it is turned round the perpendicular.

43. *When two lines diverge unequally from the perpendicular, that is greatest which diverges most.*



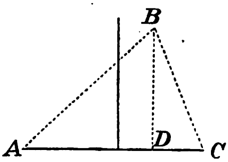
If both lines lie at the same side of the perpendicular, the proposition follows from 38, as $\angle x > \angle y$; if they lie one on each side, the one is turned round to the same side as the other.

44. *Every point in the perpendicular on the middle point of a line is equidistant from the extremities of the line.*



The proposition follows from 42, as $AB = BC$.

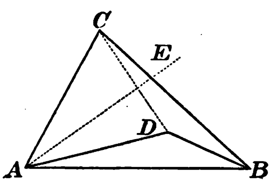
45. *A point, lying outside the perpendicular on the middle point of a line, is nearest to that extremity of the line, which is on the same side.*



For if the perpendicular BD be drawn, then $AD > DC$, therefore $AB > BC$ (43).

The perpendicular on the middle point of a line therefore just contains all the points equidistant from the extremities of the line, and no others.

46. *If two triangles have two sides equal, but the angles contained by them unequal, the third side is greatest in the triangle which has the greatest angle.*

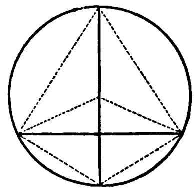


Let ACB and ADB be the triangles; they are placed, so that the equal sides AB coincide, further $AD = AC$; a perpendicular on the middle of CD will pass through A (45, last piece); therefore $CB > DB$. (45).

47. The *altitude* of a triangle is a line, drawn from an angular point, perpendicular to the opposite side, which is then called the *base*. The altitude must coincide with one of the sides, if one of the angles at the base is a right angle, and fall outside the triangle, if one of the angles is obtuse; besides this line, two others of particular interest issue from each angular point, namely, the *median line*, going to the middle of the opposite side, and the line bisecting the angle.

In the isosceles triangle the altitude, median line and the line bisecting the vertical angle, coincide in one line; for if the altitude did not bisect the vertical angle or the base, the sides of the triangle would be unequal (43).

48. *The middle point of a chord, the middle points of the corresponding arcs, and the centre all lie in a line, perpendicular on the middle point of the chord.*



For all the points lie equidistant from the extremities of the chord.

EXAMPLES.

48. Prove that the diameter of a circle is the greatest chord.
49. Prove that the arcs between two parallel chords in a circle are equal.
50. Which is the greatest and which the least line, that can be drawn from a point to the circumference of a circle?
51. From a point O within a triangle ABC lines are drawn to B and C. Prove that $OB + OC < AB + AC$.
52. *Prove that in a circle, when* $\smile AB < \smile AC$, *then also* $AB < AC$. *(The arcs are both supposed less than 180°).* (46).
53. Prove that each side of a triangle is less than half the perimeter.
54. *Prove that in a circumscribed quadrilateral, the sum of the one pair of opposite sides is equal to the sum of the other pair.* (29).
55. With a point A outside a circle as centre, a circle is described through the centre O of the given circle, and two chords are drawn from O, equal to the diameter of the given circle. Shew that these chords cut the given circle in the points of contact of the tangents from A.
56. Shew that any triangle, which lies wholly inside another triangle, has less perimeter than this.
57. From a point within a triangle lines are drawn to the three angular points; shew that the sum of these three lines is greater than half the perimeter of the triangle, but less than the whole perimeter.

58. From the angular points of a triangle three lines are drawn in such a manner, that any two of them are legs of an isosceles triangle, of which one of the sides is the base. Prove that the three lines intersect in the same point. (Suppose that the three lines form a triangle and express the sides of this in terms of the legs of the isosceles triangles).

II.

I. CONSTRUCTION, CONGRUENCE AND SYMMETRY.

49. For the construction of figures with certain given properties are used only the *ruler*, by which a straight line can be drawn through two given points, and the *compasses*, by which a circle, with given centre and given radius, can be described. All constructions must therefore be a combination of these two. In order that a point may be determined, there must be given two conditions, which it must fulfil; if there is only one condition given, there will be an infinite number of points, which satisfy the problem, but these will all lie in a certain straight or curved line, called the *locus* of the point; thus we have from the preceding:

The locus of the points which are at a given distance from a given point is a circle, with the given point as centre and the given distance as radius.

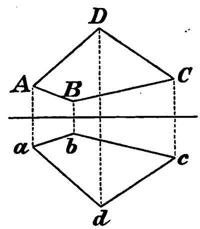
The locus of the points which are equidistant from two given points is a straight line, perpendicular on the middle point of the line joining the two given points. (45).

More loci will be mentioned hereafter.

Now if the problem is to find a point, we examine, which two loci correspond to the two conditions, which the point, according to its position, must fulfil; if the two loci can be constructed by compasses and ruler, the point can be found, for, as it is to lie in them both, it must lie where they intersect; the problem has therefore as many solutions, as the loci have points of intersection.

50. *When certain parts only in one way can be put together to make a figure, then two figures, both containing these parts, must be congruent,* for if they were not, the parts would be put together in two ways; but if the parts can be put together in different ways, then two figures containing these parts do not require to be congruent.

51. Two figures are said to be *symmetrical* with regard to a straight line, when to each point in the one figure, there is a corresponding point in the other, so placed, that it is found by dropping a perpendicular from the first point on to the line, and producing it equally far on the other side; thus A corresponds to a, B to b, &c. *Symmetrical figures are congruent,* as the one, by being turned over, round the straight line (axis of symmetry), will coincide with the other; conversely, two figures, which, by being turned over, round a straight line, would coincide, must lie symmetrically with regard to it. As examples of symmetrical figures, we may mention that the altitude from the vertex divides an isosceles triangle symmetrically, that every diameter divides a circle symmetrically, &c.

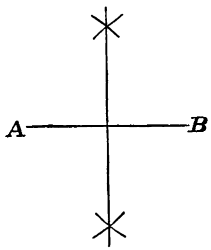


When two pair of figures lie symmetrically with regard to the same line, their points of intersection must also lie symmetrically, for when the figures, by being turned, coincide, their points of intersection must also coincide.

In constructing a figure, we often get two symmetrical solutions, which therefore are congruent.

52. *To draw a straight line, perpendicular on the middle point of a given straight line.*

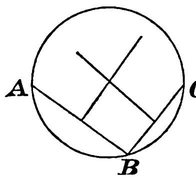
With the extremities A and B of the line given as centres, describe arcs with equal radii, then their points of intersection must lie in the line required (45); in this manner two points in the required line are found, which is then drawn through these.



The radii must be taken greater than half AB, so that the arcs can cut each other.

By the same construction a line may be bisected.

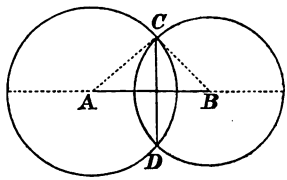
53. *To describe a circle, passing through three given points A, B and C.*



Find the centre; it must be equidistant from A and B, therefore in a perpendicular on the middle of AB, likewise in a perpendicular on the middle of BC, and must therefore lie where these intersect; as there is only one point of intersection, there can only be described one circle passing through three given points. If the three points lie in a straight line, the two perpendiculars will be parallel and have no point of intersection, so that the problem is impossible; we usually say, that the parallel lines cut one another at an infinite distance, and that therefore, through three given points lying in a straight line, there can be described a circle, of which the centre is infinitely distant and the radius of which therefore is infinitely great; by this is only meant, that by taking the centre sufficiently far away, a circle can be got passing through the two points, and as near as we please, to the third point.

If the centre of a given arc is to be found, we employ the same construction, by taking any three points in the arc. From this it follows, that two circles only can cut each other in two points, for if they had three points in common, they must coincide. The line joining the centres is called the *line of centres*; if it is produced, it divides both circles symmetrically, *and therefore is perpendicular on the middle of the line joining the points of intersection of the circles.* (51).

If radii be drawn to one of the points of intersection, they will be sides of a triangle, of which the line of centres is the third side, *and this must therefore be less than the sum of the radii and greater than their difference, when the* *circles cut one another* (39), whilst it is *greater than the sum of the radii, when the circles lie outside each other, and less than their difference, when one circle lies within the other.*



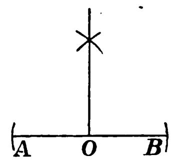
If the chord, which the circles have in common, becomes infinitely small, so that the points of intersection coincide in one point, this is called a *point of contact*; *it must lie in the line of centres*, for, so long as there is one common point on the one side of the line of centres, there must also be one symmetrically with this on the other side.

From this we see, *that the line of centres is equal to the sum of the radii, when the circles touch one another externally, and equal to their difference, when they touch internally.*

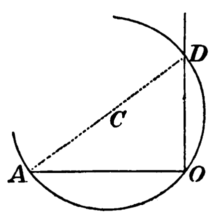
Conversely, *if the line of centres equals the sum or difference of the radii, the circles must touch one another.*

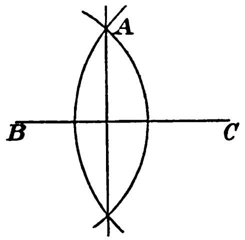
54. *To raise a perpendicular on a given line at a given point.*

a) From the given point O cut off the equal parts OA and OB. Next determine a point equidistant from A and B, this must lie in the required line, which now can be drawn.

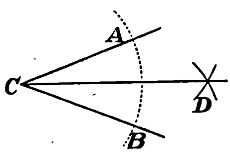


b) If the given point is the extremity of the line, and this cannot well be produced, a circle is described with any centre C passing through the given point O; from the other point of intersection of the line with the circle draw the diameter AD; DO will then be the perpendicular. For $\angle O$ is a right angle, as it stands on a diameter (26).



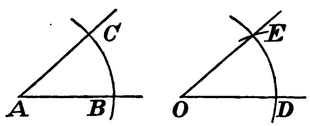


55. *From a given point A to draw a perpendicular to a given line BC.* Take any two points in the line as centres, and describe circles through the given point; the line joining the points of intersection of the circles will be the required line (53).



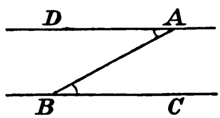
56. *To bisect a given angle or arc.*

With the vertex of the angle C as centre, describe the arc AB; next find a point D equidistant from A and B. Then CD bisects both the angle and the arc, as it by construction is perpendicular on the middle point of the chord AB. *Any* angle cannot be divided by compasses and ruler into equal parts in any other way than by continued bisection, therefore only into 2, 4, 8, 16, &c. equal parts.



57. *Through a given point in a line to draw a line, making an angle with the given line equal to a given angle.*

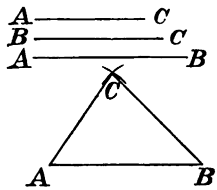
Let O be the given point and A the given angle, with O and A as centres describe the arcs BC and DE with equal radii; now when $\smile DE$ is made equal to $\smile BC$, then $\angle O = \angle A$. (24).



58. *Through a given point to draw a line parallel to a given line.* From the given point A draw any line AB, cutting the given line BC; then if we make $\angle A = \angle B$, then $DA \not\mathrel{\#} BC$. (31).

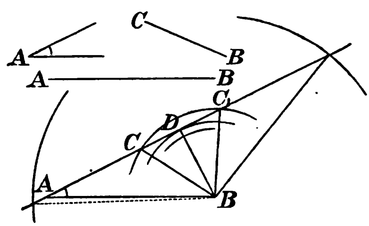
59. *To construct a triangle, having given the three sides.*

Mark off the one side AB; the vertex of the triangle will then be the point of intersection of two circles, with centres A and B, and with the other given sides as radii. The problem is possible, when the circles cut each other, therefore, when the one side is less than the sum of the other two and greater than their difference (53). The circles cut each other in two points, but the two solutions are symmetrical, so that three lines only in one way can be put together to form a triangle. From this it follows, *that two triangles are congruent, when they have the three sides respectively equal.*



60. *To construct a triangle, having given two sides and the angle contained by them.* Mark off the given angle, and mark off parts on its legs, equal to the given sides. As the problem only has one solution, *two triangles are congruent, when they have two sides and the angle contained by them respectively equal.*

61. *To construct a triangle, having given an angle, an adjacent side, and the opposite side.*



Mark off the given angle A, and on one of its legs, the adjacent side AB. The point C will be determined by a circle with centre B and radius BC; different cases can now occur:

a) If BC is less than the perpendicular BD, the circle will not cut AD, and the problem is impossible.

b) If $BC = BD$, the circle will touch AD at D, and there will be one solution, namely, the right-angled triangle ADB.

c) If $BC > BD$ but $< BA$, the circle will cut AD in two points, so that the problem has two solutions.

d) If $BC = BA$, the one point of intersection falls in A, so that the one solution gives no triangle.

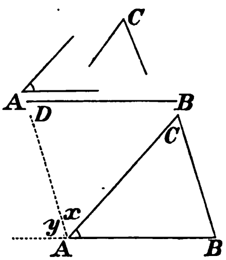
e) If $BC > BA$, the one point of intersection falls on the opposite side of A, and the corresponding triangle does not contain the given angle A, but its adjacent angle; there is therefore in this case only one solution.

From this it follows, that *two triangles, having one angle, an adjacent side, and the opposite side respectively equal, are congruent, if the opposite side be greater than or equal to the adjacent side, but that, if this is not known, it is not certain that the triangles are congruent.*

62. *To construct a triangle, having given a side and the angles at the side.*

Mark off the given side and on it the given angles, and produce their legs till they intersect.

The problem is always possible, when the sum of the given angles is less than $2R$. As there only is one solution, *two triangles are congruent, when they have one side and the angles at that side respectively equal.*



63. *To construct a triangle, having given a side, an angle at the side, and the opposite angle.*

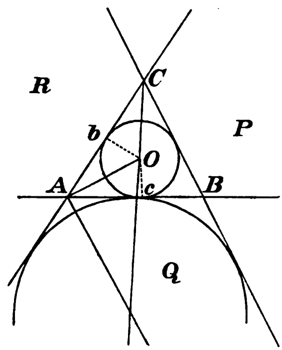
Mark off the given side AB and on it the given $\angle A$; thereupon make $\angle x = \angle C$; then $\angle y$ equals $\angle B$, as the sum of the angles is $2R$. Then draw BC parallel to AD.

The conditions of possibility for the construction of this is the same as for the preceding one. As there only is one solution, *two triangles are congruent, when they have one side, an angle at the side, and the opposite angle equal respectively.*

An isosceles triangle can be constructed, having given the base and the vertical angle. Draw any isosceles triangle with the given vertical angle D, and on the base AB of this triangle mark off AC equal to the given base; thereupon through C draw a parallel to BD.

64. *To describe a circle, touching three given straight lines.*

If O is the centre, Oc and Ob radii to the points of contact, then $\triangle AbO \cong \triangle AcO$ (61, e; for we have $Ob = Oc$; $AO = AO$; $\angle b = \angle c = R$. and $AO > Ob$), so that the centre must *lie in the line* bisecting $\angle A$. Similarly the centre must lie



in the line bisecting $\angle C$, therefore in the point of intersection of these lines; the line bisecting the third angle also passes through this point. This circle is called the *inscribed* circle of the triangle; besides this one there can, in a similar way, be constructed three other circles, touching the three lines, which will lie in the spaces P, Q and R. These are called the *escribed* circles of the triangle.

The length of the distances, between the points of contact and the angular points of one of the lines, may be expressed in a simple manner by the sides of the triangle.

If $AB = c$, $AC = b$, $BC = a$, and if we for the sake of brevity call the perimeter $2s$, and $Ac = x$, $Bc = y$, $Cb = z$, we have (29),

$$x + y = c; \quad y + z = a; \quad z + x = b,$$

wherefore by addition and division by 2,

$$x + y + z = s,$$

therefore

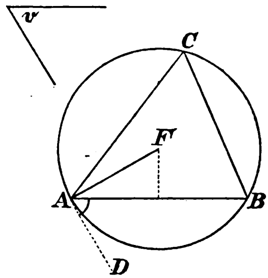
$$x = s - a; \quad y = s - b; \quad z = s - c.$$

Of the other distances may be mentioned the one from C to the points of contact of the escribed circle with CA or CB. These two distances, it will be seen, are together equal to the perimeter of the triangle; therefore each equal to s. (29).

The determination of O above also shews, that *the locus of the points which are equidistant from two given lines is two lines, which bisect the angles between the given lines,* and therefore are perpendicular to one another.

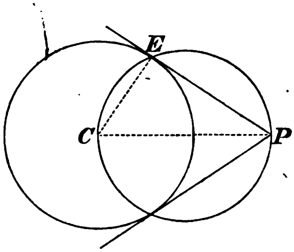
65. *On a given line as chord to describe an arc, so that all the angles, standing on the chord and having their vertices in the arc, shall be equal to a given angle.*

Let AB be the given chord and v the given angle; as the angles at the circumference, standing on AB, are to be equal to v, the same must be the case with the angle, formed by



the chord and the tangent at A, as it is measured by the same arc; therefore make $\angle BAD = \angle v$, and the centre may then easily be found. For as AD must be a tangent, the centre must lie in the perpendicular AF on AD, and therefore lie where this cuts a perpendicular on the middle point of the chord. If the given angle is a right angle, the arc will be a semicircle.

We say that the arc or segment *contains* the given angle, and that *the arc is the locus of the points, at which the line* AB *subtends the given angle*, or from which AB *is seen under the given angle.*



66. *To draw tangents from a given point P to a circle.*

As the tangent must be perpendicular to the radius and therefore $\angle CEP$ be a right angle, the points of contact must lie in a circle on CP as diameter. If the given point P lies in the circumference, a perpendicular is drawn on the radius to the point; if it lies within the circle, the problem is impossible.

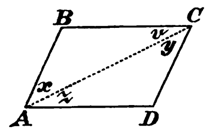
II. POLYGONS.

67. The propositions regarding the congruence of triangles are often applied to prove that two parts of a figure are equal. We then endeavour to find, or by the help of lines to form, two triangles, in which these parts occur as corresponding parts; if we can prove that the two triangles are congruent, the problem is solved; in the following, when we state that two triangles are congruent, we will place the letters in such an order, that the corresponding parts come in the same place; thus, if we write $\triangle ABC \cong \triangle abc$, then $\angle A = a$, $B = b$, $C = c$, $AB = ab$, $AC = ac$, $BC = bc$.

68. A *trapezium* is a quadrilateral, which has a pair of sides parallel.

A *parallelogram* is a quadrilateral, which has both pair of sides parallel.

69. *The opposite sides and angles of a parallelogram are equal.*



$AB \neq CD$, therefore $\angle x = y$;

$AD \neq CB$, therefore $\angle v = z$;

therefore

$$\triangle ABC \cong CDA \quad .\ .\ .\ .\ (62)$$

from this it follows

$$AB = CD;\quad BC = DA;\quad \angle B = \angle D.$$

From this proposition it follows that parallels all over are at the same distance from each other, and that *the locus of the points which are at a given distance from a given line is two lines parallel to the given line, at the given distance.*

70. *A quadrilateral, having the opposite sides equal, is a parallelogram.*

We have
$$AB = CD,$$
$$BC = DA,$$
$$AC = AC,$$
therefore
$$\triangle ABC \cong \triangle CDA,$$
from which it follows
$$\angle x = y;\quad z = v$$
or
$$AB \neq CD;\quad BC \neq AD.$$

71. *A quadrilateral, having a pair of opposite sides equal and parallel, is a parallelogram.*

Here we also get $\triangle ABC \cong \triangle CDA$. (60).

From this proposition it follows *that two lines are parallel, when two points in the one line are at the same distance from the other line (and lie at the same side).*

72. *A quadrilateral, having the opposite angles equal, is a parallelogram.*

We have
$$\angle A + B + C + D = 4\,\text{R.},$$
but
$$A = C;\quad B = D;\quad (\text{Hyp.}),$$

therefore $2A + 2B = 4\text{R.},$

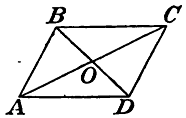
$A + B = 2\text{R.},$

from which it follows $BC \# AD.$

Similarly $A + D = 2\text{R.},$

therefore $AD \# CD.$

73. *The diagonals of a parallelogram bisect each other.*



For we have

$$\triangle BOC \cong \triangle DOA$$

therefore

$$BO = DO;\quad OC = OA.$$

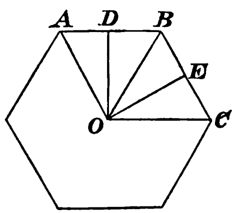
74. Of special parallelograms may be mentioned the *rectangle*, containing right angles, and the *rhombus*, the sides of which are all equal. *A square* is both rhombus and rectangle.

75. *A regular nrayed figure* is one, which coincides with itself, when it revolves through an angle equal to $\frac{1}{n}$ of 4 R. round a certain point, *the centre.*

Corresponding points, lines, &c. are such, which, by the said revolution, take each others places.

If we join the corresponding points in order, a *regular polygon* is formed, which thus is an *n* rayed *n* sided figure; the regular polygon is convex; the *n* rayed polygon with re-entrant angles is called a star shaped polygon.

76. *All the sides and angles of a regular polygon are equal;* for they all by degrees coincide by being turned round the centre. For the same reason the lines from the centre to the angular points, *the greatest radii* (OA), are equal, and similarly the perpendiculars from the centre on the sides, *the least radii* (OD). The greatest radii divide the polygon into congruent, isosceles triangles, *central triangles* (AOB); the angles of these at the centre are called *central angles* ($\angle AOB$).



If n congruent, isosceles triangles, the vertical angles of which are $\frac{4}{n}$R., be laid side by side, with their vertices at one point, a regular n sided figure is formed; having obtained this, we can draw another regular n sided figure, with sides equal to a given line (63, last piece).

77. *The angles of a regular* n *sided figure are each* $2R. - \frac{4}{n}R.$

This is found by dividing the sum of the angles $2n$ R. — 4 R. by their number.

78. *A polygon is regular, when all its sides and all its angles are equal.*

For if it be compared to a regular polygon (76, last piece), the sides of which are of the same length, and which has the same number of sides, the two polygons will be seen to be congruent, as they also have the angles equal (77), and but one polygon can be constructed of these sides and angles.

Of regular figures we have already mentioned the equilateral triangle and the square.

79. *If a circumference be divided into a certain number, n, equal parts, and the points of division be joined by chords, a regular inscribed* n *sided figure is formed; if tangents be drawn to all the points of division, a regular circumscribed* n *sided figure is formed.* For in both cases the figure will coincide with itself, by being turned $\frac{1}{n}$ revolution round the centre.

EXAMPLES.

59. In a square $ABCD$ a diagonal AC is drawn, and on this marked off $AE = AB$, at E a perpendicular is raised cutting BC in F. Prove that $BF = FE = EC$.
60. *Prove that the diagonals of a rectangle are equal.*
61. *Prove that the diagonals of a rhombus are perpendicular to each other.*
62. Prove that a rectangle is formed by joining the middle points of the sides of a rhombus.
63. Prove that a rhombus is formed by joining the middle points of the sides of a rectangle.

64. The distance between two parallels is a; any line AB is drawn between the parallels, and the two supplementary angles A and B are bisected; how great is the altitude of the right-angled triangle, formed thereby?
65. In a right-angled triangle, the right angle is bisected, from the point where the bisecting line cuts the hypothenuse, lines are drawn parallel to the two sides containing the right angle. Prove that the figure contained by these two lines and the two sides is a square.
66. The point C bisects the line AB. The three points are projected on to any line in A_1, B_1 and C_1; shew that C_1 bisects A_1B_1.
67. Two points A and B lie on the same side of a straight line. From A a line is drawn to the point C, which lies symmetrically with B. This line cuts the given line in a point D. Prove that AD and BD make equal angles with the given line.
68. A line is parallel to the diagonal of a parallelogram; shew that the one pair of opposite sides cut off a part of the line, equal to the part cut off by the other pair.
69. Prove that the opposite sides of a square cut off a part of any line, equal to the part cut off a line perpendicular to this, by the other two opposite sides.
70. In a right-angled triangle squares are described on the sides containing the right angle, and from the outermost angular points of these, perpendiculars are dropped on the hypothenuse; prove that the sum of the perpendiculars equals the hypothenuse.
71. From a point in the base of an isosceles triangle, perpendiculars are drawn to both the sides. Prove that the sum of these perpendiculars equals the altitude from an extremity of the base.
72. Which parallelograms can be inscribed in a circle, and which can be circumscribed?
73. *Prove that a polygon can be constructed, when all its parts are known, except three, which follow after each*

other (1 side and 2 angles or 2 sides and 1 angle). *Which proposition on the congruence of polygons follows from this?*

74. In a sector of 60° a circle is inscribed; prove that its radius is a third of the radius of the circle to which the sector belongs.
75. Which is the longest line that can be drawn between the circumferences of two circles through their one point of intersection.
76. Two given lines intersect in *O*. In each of the lines any point, *A* and *B*, is chosen, and a triangle *ABC* with given angles is constructed. The angle at *C* is equal to the angle at *O* and lies to the same side of *AB*. Shew that *C* falls in the same line through *O*, wherever *A* and *B* are chosen.
77. From a point a line is drawn to the centre of a circle, and on this line as diameter, another circle is described. Prove that this circle bisects all chords passing through the given point.
78. In a right-angled triangle a circle touches one of the sides containing the right angle and also the other side at its extremity; prove that the part which the circle cuts off from the hypothenuse is equal to twice the altitude of the triangle.
79. In a parallelogram a diagonal is drawn, and through a point in this, lines parallel to the sides of the parallelogram. The parallelogram is thereby divided into four smaller parallelograms. Prove that the two of these, through which the diagonal does not pass, are equal.
80. Prove that the one part of the altitude of a triangle, reckoned from the point of intersection of the altitudes to the foot, is equal to the prolongation of the altitude to the circumference of the circumscribed circle.
81. A triangle is divided into two other triangles by a line from an angular point to the point of contact of the opposite side with the inscribed circle. Shew that the

circles inscribed in the smaller triangles touch the side, which the triangles have in common, at the same point.

82. Through the vertex of an angle and a given point in the line bisecting the angle any circle is described; prove that the sum of the two chords, which the circle cuts off from the legs of the angle, is constant.

83. On the three sides of a triangle arcs are described inwardly, containing angles of 120°; shew that the three arcs pass through the same point.

84. On the three sides of a triangle equilateral triangles are described outwardly. Shew that the three lines joining the outermost angular points of the equilateral triangles with the opposite angular points of the given triangle 1) are equal, 2) cut each other at angles of 120°, and 3) pass through the same point.

85. *Prove that a quadrilateral can be inscribed in a circle, when its opposite angles are supplementary.*

86. On the three sides of a triangle squares are described outwardly. Prove that the three lines, joining the extremities of the outermost sides of the three squares, are twice as great as the median lines of the triangle, and perpendicular to these.

87. Shew that the altitudes of a triangle bisect the angles of another triangle, having its angular points at the feet of the altitudes. (Find in the figure systems of four points lying in the circumference of the same circle). Prove by the help of this proposition, that the three altitudes intersect in one point.

88. Any tangent is drawn to a circle with centre O. Let it cut a fixed line through O in M and mark off on the tangent $MP = MO$. What is the locus of P?

89. In a given circle a triangle is inscribed, the two angular points of which are fixed, whilst the third moves on the arc. Which are the loci of the point of intersection of the altitudes of the triangle, and of the centre of the inscribed circle?

90. AB is a fixed chord, C a moveable point of the circumference. AC is produced to D, so that $CD = CB$. Find the locus of D.

91. From any point in the circumference of a circle lines are drawn to the angular points of an inscribed equilateral triangle; shew that one of the three lines equals the sum of the other two.

92. Two men, both living in the neighbourhood of a lake, jointly own a boat. Where should this be placed on the shore, so as to be equally distant from them both?

93. Construct a triangle, having given an angle, the opposite side, and the median line to one of the other sides. (Seek the middle point of this side, when the given side is fixed).

94. Describe a circle with given radius, passing through two given points or touching two given straight lines.

95. Construct a triangle, having given a side, the altitude and median line to this.

96. Construct a triangle, having given a side, the altitude to it, and the opposite angle.

97. Construct a quadrilateral $ABCD$, having given AB, BC, AC, BD and $\angle D$.

98. Construct a quadrilateral $ABCD$, which can be inscribed in a circle, having given $\angle A$, AB, AC and BD.

99. Given a line and in it the point A, and outside the line the point P. Find in the given line a point X, such that $AX + XP$ equals a given line.

100. In a given line determine a point, such that lines from it to two given points on the same side of the line make equal angles with it. (Ex. 67).

101. In a given sector to inscribe a circle.

102. In a given circle to draw a chord equal and parallel to a given line.

103. Construct a right-angled triangle, having given a point in each of the sides containing the right angle, two points in the hypothenuse, and the length of the altitude to the hypothenuse.

104. Construct a triangle, having given the centres of its escribed circles (Ex. 87).

105. Each side of a square is divided into two parts m and n, so that two equal parts nowhere meet together. Prove that the quadrilateral, having its angular points in each of the four points of division, is a square. (Turn $\frac{1}{4}$ revolution).

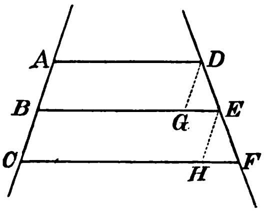
106. In a regular pentagon all the diagonals are drawn. Prove that a new regular pentagon is formed thereby.

107. Prove that two diagonals in a regular hexagon are parallel, and that a third diagonal is perpendicular to the two preceding ones, and parallel to two of the sides of the hexagon, and that three of the diagonals form an equilateral triangle.

III.

I. SIMILAR FIGURES.

80. *When several parallels cut off equal parts of a straight line, the parts which they cut off from any other straight lines, will also be equal.*



If $AB = BC$ and if DG and EH be drawn parallel to AC, we have $DG = AB$ and $EH = BC$ (69).
therefore

$$DG = EH.$$

Further

$$\angle D = \angle E \quad \ldots \ldots (31)$$

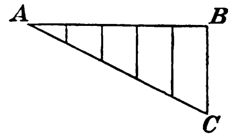
and $$\angle G = \angle H \quad \ldots \ldots (35),$$

therefore $\triangle DGE \cong EHF$,

and consequently

$$DE = EF.$$

81. *To divide a straight line AB into a given number, for example, 5, equal parts.*



From A draw any line, and along it mark off 5 equal parts. Join the extremity C of the last part with B, and through the other points draw lines parallel to CB. The parts of AB will then be equal (80).

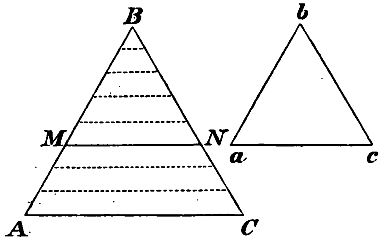
82. Two triangles containing the same angles are said to be *similar*. „Similar to" is denoted thus $\sim$.

The letters at the vertices of equal angles are put in the same place. *Corresponding* sides are such which subtend the equal angles; the letters by which they are denoted are placed in the same order, for example, when

$$\triangle ABC \sim abc,$$

then $\angle A = a$; $B = b$; $C = c$, and AB, BC, and AC correspond respectively to ab, bc and ac.

83. *In similar triangles the ratios between two pair of corresponding sides are equal.* (The sides are *proportional*).



Place the triangle abc on ABC, so that $\angle b$ coincides with $\angle B$, a falls in M, and c in N; we then have $MN \nparallel AC$, as $\angle M = \angle a = \angle A$. Now BM and BA must either have a common measure or not.

a) If they have a common measure (are commensurable), that is, if there is a small line which is contained in each of them a certain number of times exactly, let this be marked off along them, and be contained, for example, p times in BA and q times in BM; we then have

$$\frac{BA}{BM} = \frac{p}{q}.$$

Now if lines be drawn through the points of division parallel to AC, these will divide BC and BN respectively into p and q equal parts (80); therefore we have

$$\frac{BC}{BN} = \frac{p}{q},$$

and consequently

$$\frac{BA}{BM} = \frac{BC}{BN},$$

or

$$\frac{BA}{ba} = \frac{BC}{bc}.$$

β) If the lines have no common measure (are incommensurable), divide BM into an indefinite number for example q equal parts, and mark off these further; the point A will then fall between two points of division for example the p^{th} and $(p + 1)^{\text{th}}$; we then have

$$\frac{p}{q} < \frac{BA}{BM} < \frac{p+1}{q}.$$

If we draw parallels as before, we also get

$$\frac{p}{q} < \frac{BC}{BN} < \frac{p+1}{q}.$$

The two ratios $\frac{BA}{BM}$ and $\frac{BC}{BN}$ therefore both lie between the fractions $\frac{p}{q}$ and $\frac{p+1}{q}$, the difference of which is $\frac{1}{q}$. The difference between the ratios must therefore be less than $\frac{1}{q}$; but this fraction can be made as small as we please, for q can be taken as great as we please; but when the difference between the ratios is less than any ever so small quantity, it must be zero, and the ratios therefore be equal*).

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*) This proof does not hold good here alone, but shews in general, that when two kinds of quantities are proportional when the ratios are commensurable, they will also be proportional when the ratios are incommensurable.

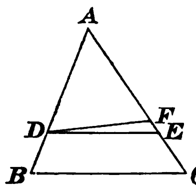
In the same way we get, by placing c in C,

$$\frac{BC}{bc} = \frac{CA}{ca},$$

therefore
$$\frac{AB}{ab} = \frac{BC}{bc} = \frac{CA}{ca}.$$

84. We have also shewn by this, that *a line parallel to one side of a triangle cuts off proportional parts of the two other sides.*

Conversely: *When a line cuts off parts of two sides of a triangle, which are proportional to the sides, then the line will be parallel to the third side.*



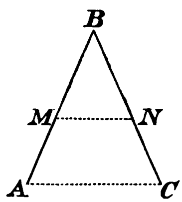
Hyp. $\frac{AD}{AB} = \frac{AE}{AC}$.

Th. $DE \neq BC$.

For if DE were not parallel to BC, we would be able to draw another line for example, $DF \neq BC$; but from that would follow
$$\frac{AD}{AB} = \frac{AF}{AC},$$
which is in opposition to what was given.

According to a proposition in proportion, the proportion
$$\frac{AD}{AB} = \frac{AE}{AC}$$
can also be written
$$\frac{AD}{AB - AD} = \frac{AE}{AC - AE}$$
or
$$\frac{AD}{DB} = \frac{AE}{EC},$$
so that this proportion holds, when $DE \neq BC$, and conversely.

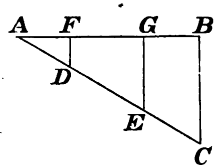
85. *To construct a fourth proportional to three given lines*, that is, a line which is the fourth term in a proportion, in which the three other terms are the given lines.



On the legs of any angle B mark off BM, BA, and BN equal to the given lines; if thereupon MN be joined, and AC drawn parallel to MN, then BC will be the required line. If both the mean terms are equal, the required line is called the *third proportional* to the two given lines and is constructed in the same way. If the given lines are a, b, and c, the required line x, we must have $\frac{a}{b} = \frac{c}{x}$. Here it is of no consequence, whether the letters represent concrete or abstract numbers, expressing the lines measured by the same unit $\left(\text{as, } \frac{5 \text{ feet}}{7 \text{ feet}} = \frac{5}{7}\right)$; but if the equation is written $x = \frac{bc}{a}$, it can only be understood in the latter way, as there would be no meaning in multiplying two concrete numbers.

Hereby we can again construct $x = \frac{abc}{de}$, by first constructing $y = \frac{ab}{d}$, and thereupon $x = \frac{yc}{e}$; it is apparent how this may be expanded to $x = \frac{abcd \ldots.}{efg \ldots.}$, when there is one line more in the numerator than in the denominator.

86. *To divide a given line into parts, which are to one another as given lines or numbers.*



AB is to be divided into parts, which are to each other as the given lines *AD*, *DE* and *EC*, which are marked off on a line from *A*; join *BC*, and draw *EG* and *DF* parallel to *BC*; we then have

$$\frac{BG}{CE} = \frac{GA}{EA} = \frac{GF}{ED} = \frac{FA}{DA} \quad (84).$$

If numbers are given, any line must be chosen as unit, and this must be marked off as many times as the numbers indicate.

87. *A line bisecting an angle of a triangle divides the opposite side into two parts, which are to each other as the sides containing the angle.*

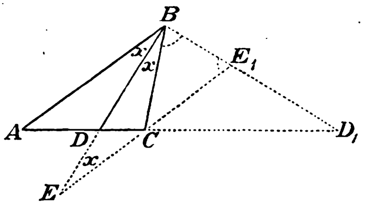
If we draw $CE \neq AB$, we have

$$\triangle ABD \sim \triangle CED,$$

therefore
$$\frac{AB}{CE} = \frac{AD}{CD},$$

but $CE = CB$. . . (21);

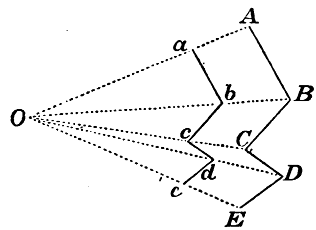
therefore $\frac{AB}{CB} = \frac{AD}{DC}$.



The line, bisecting the angle adjacent to B, cuts AC (produced) in a point D_1. By putting D_1 instead of D, all over in the demonstration above, we prove that $AB : CB = AD_1 : D_1C$. We say that D_1 divides AC (externally) in the same ratio as D divides AC (internally), or that AC is divided *harmonically* by D and D_1 in the ratio $AB : BC$.

88. Two figures (systems of points) are said to be *similar in the ratio m*, when there to every point in the one figure is a corresponding point in the other figure, and the distance between two points in the one figure all over is m times the distance between the corresponding points in the other figure. When $m = 1$, the figures are congruent.

That there always is a figure $abcd$.... similar to a given figure $ABCD$.... in a given ratio is shewn thus:



From any point O draw lines to the angular points of the given figure, and on these mark off $Oa = m \cdot OA$; $Ob = m \cdot OB$, &c. a, b, c, d will then be the angular points of the required figure. O is called the *centre of similitude*, the lines through O *rays of similitude*. The figures are said to be similarly situated. Of such figures it holds that:

Corresponding points lie in the same ray by construction.

Corresponding lines (lines joining corresponding points) are *parallel*, for example, $ab \neq AB$ . . . . (84)

For
$$\frac{Oa}{OA} = \frac{Ob}{OB} = m.$$

Corresponding lines are proportional in the ratio $m : 1$, for example,
$$\frac{ab}{AB} = \frac{Oa}{OA} = m \text{ . . . . (83)}$$

This also holds for broken lines, for when each separate line becomes m times greater, their sum must also become m times greater.

Corresponding angles are equal (35).

The figure we get, will be the same wherever we choose the centre of similitude, for if we had, by choosing it somewhere else, constructed a figure $a_1b_1c_1d_1\ldots$, then this and $abcd\ldots$ must have all the corresponding lines and angles respectively equal, and consequently be congruent.

When certain points in the one figure lie in a straight line, the corresponding points in the other figure also lie in a straight line.

This will be apparent, when we imagine the centre of similitude chosen in the straight line.

The points in the corresponding lines are therefore jointly corresponding.

When certain points in the one figure lie in a circle, the same must be the case with the corresponding points in the other figure.

This is apparent, when we imagine the centre of similitude chosen in the centre of the circle.

We have termed triangles, containing the same angles, similar; as such triangles can be placed so that they are similarly situated (see fig. to 83), they are also similar in the general sense of the word.

89. When the cases in which two figures are congruent are known, we can therefrom deduct the cases in which they are similar; we can namely in the demonstrated propositions on congruence read „*similar*“ instead of „congruent“, when we simultaneously instead of „sides equal“ read „*sides proportional*“; we will apply this to triangles.

90. Of the proposition:

Two triangles are congruent, when they have all three sides respectively equal,

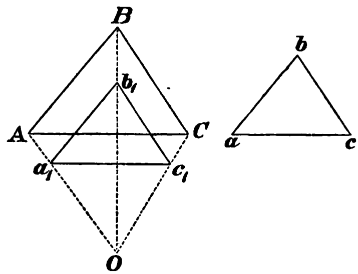
the new proposition is formed:

Two triangles are similar, when they have all three sides proportional.

$$\text{Hyp.}\quad \frac{ab}{AB} = \frac{bc}{BC} = \frac{ac}{AC} = m.$$

We choose any centre of similitude O, and construct $a_1b_1c_1 \backsim ABC$ in the ratio m, when m is the value of the equal ratios; we then have

$$\frac{a_1b_1}{AB} = \frac{b_1c_1}{BC} = \frac{a_1c_1}{AC} = m.$$



These ratios are equal to the given ones, and as the consequents respectively are the same, the same must be the case with the antecedents, therefore

$a_1b_1 = ab$; $b_1c_1 = bc$; $a_1c_1 = ac$.

The small triangles are therefore congruent, and as one of them is similar to the large one, the other must also be.

91. Of the proposition:

Two triangles are congruent, when they have two sides and the angle contained by them respectively equal,

the new proposition is formed:

Two triangles are similar, when they have two sides proportional and the angle contained by them equal.

$$\text{Hyp.}\quad \angle A = \angle a;\quad \frac{ab}{AB} = \frac{ac}{AC} = m.$$

By the same construction as before, we have

$$\angle a_1 = \angle A;\quad \frac{a_1b_1}{AB} = \frac{a_1c_1}{AC} = m,$$

wherefrom, by comparison with what is given

$$\angle a = a_1;\quad ab = a_1b_1;\quad ac = a_1c_1,$$

therefore $\triangle abc \cong \triangle a_1b_1c_1 \backsim \triangle ABC.$

92. Of the proposition:

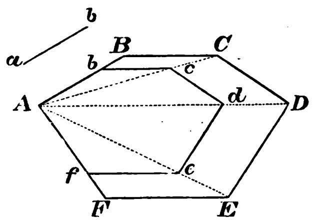
Two triangles, having one angle, an adjacent side, and the opposite side respectively equal, are congruent, if the opposite side be greater than or equal to the adjacent side, the new proposition is formed:

Two triangles, having one angle equal, and an adjacent side and the opposite side proportional, are similar, if the opposite side be greater than or equal to the adjacent side.

The demonstration is precisely the same as for the preceding proposition.

93. In the same way it is demonstrated, that *two figures, having all their corresponding angles equal and sides proportional, can be placed so as to be similarly situated*; such figures can therefore, by corresponding diagonals, be divided into similar triangles. If the condition for the congruence of polygons only involves the equality of one side in each, this condition is omitted in the condition of similarity. Thus all regular polygons with the same number of sides are similar.

94. *To draw a figure similar to a given figure, so that one of its lines has a given length.*



Let $ABCDEF$ be the given figure and ab the given line, which in the required figure is to correspond to AB; mark off $Ab = ab$, from A draw lines to the angular points of the figure, therupon draw $bc \neq BC$, $cd \neq CD$, &c. $Abcdef$ is then the required figure, as it is similarly situated with $ABCDEF$, A being the centre of similitude.

Hereby we can draw a regular n sided figure with a given side, if we can draw any regular n sided figure. (Compare 76).

EXAMPLES.

108. From the centre of a circle perpendiculars are drawn on two of the sides of an inscribed triangle. Prove that the line, joining the feet of the perpendiculars, is half as great as the third side of the triangle.

109. In a parallelogram $ABCD$, E is the middle point of AB, and F of CD. Shew that DE and BF divide AC into three equal parts.

110. The middle points of the four sides of a quadrilateral are joined. Prove that the quadrilateral thus formed is a parallelogram. A new-proposition can be formed from this, by considering the diagonals as sides and a pair of opposite sides as diagonals.

111. Prove that the altitudes of a triangle are reciprocally proportional to the corresponding bases.

112. In a triangle ABC two altitudes Aa and Bb, cut each other in O; prove that $Cb \cdot CA = Ca \cdot CB$.

113. Prove that $BO \cdot Ob = AO \cdot Oa$.

114. Prove that $Bb \cdot Ob = Ab \cdot bC$.

AOB can be regarded as the given triangle; C then becomes the point of intersection of the altitudes; thereby new propositions are formed of 112, 113, and 114.

115. A diameter AB is drawn in a circle, and also tangents AC and BD. E is a point of the circumference, in which BC and AD intersect. Shew that the diameter is a mean proportional between BD and AC.

116. Shew that a chord is a mean proportional between the diameter and the perpendicular from one extremity of the chord to the tangent at its other extremity.

117. In an inscribed triangle ABC, AD is drawn parallel to the tangent at B. Shew that $\overline{AB}^2 = BD \cdot BC$.

118. A square is inscribed in a right-angled triangle, so that one of its sides lies in the hypothenuse. Shew that the one part of the hypothenuse is a mean proportional between the two others.

119. In an isosceles triangle each of the equal sides is divided into three equal parts, and a line is drawn through the upper point of division of the one side and through the lower point of division of the other side, till it cuts the base produced. Prove that the two parts of the line are equal, and that the prolongation of the base is a third part of the whole base.

120. In a triangle ABC mark off on AC two points D and E, so that $AE = AB$, and so that AE is a mean proportional between AD and AC. Shew that BE bisects the angle DBC.

121. *Prove that the median lines of a triangle divide each other into parts, which are to each other as 1 : 2.*

122. *Prove that the line in a trapezium, joining the middle points of the sides which are not parallel, equals half the sum of the parallel sides.*

123. The parallel sides of a trapezium are produced in opposite directions, so that each of the prolongations equals the opposite side. Shew that the line, joining the extremities of the prolongations, bisects one of the diagonals of the trapezium, and cuts the other in a point lying equally distant from one extremity of the diagonal as the point of intersection of the diagonals from the other extremity.

124. From a given point O to a given line, any line OA is drawn, and on this a point a is marked off, the product $OA . Oa$ being given. Shew that a describes a circle through O, when A traverses the given line.

125. In a triangle BAC the angle A is a right angle, and CD bisects the angle C. Prove that $AB . AD = AC . (BC - AC)$.

126. Two circles touch each other externally, shew that the part of their common tangent between the points of contact is a mean proportional between the diameters.

127. In a triangle ABC the median line AD is drawn, and through A a line $AE \mathrel{\not\stackrel{\scriptstyle\smile}{=}} BC$. Any line through D cuts AE in E, AB in F, and AC in G; prove that

$$DF : DG = EF : EG.$$

128. Through the point of intersection of the diagonals of a trapezium, a line is drawn parallel to the parallel sides. Prove that these sides are to each other as the parts in which the diagonal divides the not parallel sides.

129. AD is a diameter of a circle and AB, BC, and CD tangents. Shew that both diagonals in $ABCD$ bisect the perpendicular from the point of contact of BC to the diameter.

130. Two given circles touch each other in O. Through O draw any line, cutting the circles in A and B. Shew

that the radius of a circle, touching the one circle in A and passing through B, is constant.

131. A circle touches another internally at A. BC is a chord in the large circle, touching the lesser one at D. Shew that AD bisects the angle BAC.

132. From a point in the circumference of a circle, perpendiculars are drawn to two tangents, and to the chord, joining their points of contact; prove that the last perpendicular is a mean proportional between the two first.

133. *In a triangle ABC a line is drawn, cutting AB in c, AC in b, and BC produced in a. Prove that*

$$Ba \cdot Ac \cdot Cb = Ab \cdot Ca \cdot Bc.$$

Draw $Cm \mathrel{\mp} AB$, and employ $bmC \sim bcA$, $maC \sim caB$, and eliminate Cm from the two equations obtained thus.

134. *From the angular points of a triangle ABC, the lines Aa, Bb, and Cc are drawn to the opposite sides and cutting each other in the same point O; prove that*

$$Ba \cdot Ac \cdot Cb = Ab \cdot Ca \cdot Bc.$$

Apply ex. 133 to $\triangle ABb$, cut by Cc, and to BbC, cut by Aa; thereupon eliminate AC from the two equations.

135. Demonstrate the proposition in 87, by the help of perpendiculars drawn from the extremities of the divided side to the bisecting line.

136. *In two given circles with centres C and c draw any two parallel radii CA and ca. Shew that Aa cuts the line of centres in a fixed point (the exterior centre of similitude), when the two radii go in the same direction, and in another fixed point (the interior centre of similitude), when the radii go in opposite directions. What construction of the common tangents to the circles can be deducted from this?*

137. The points a and b divide the line AB harmonically; prove that A and B also divide ab harmonically.

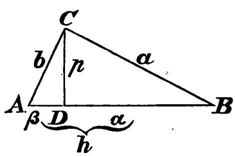
When A and B are fixed, and a moves from A to B, how does b move simultaneously? When $AB = k$, $Aa = r_1$, $Ab = r_2$, then prove that.

$$\frac{2}{k} = \frac{1}{r_1} + \frac{1}{r_2}.$$

138. Shew that OA (Ex. 137) is a mean proportional between Oa and Ob, when O is the middle point of AB.

II. THE RIGHT-ANGLED TRIANGLE.

95. *The altitude on the hypothenuse of a right-angled triangle, divides it into two triangles, which both are similar to the whole triangle, and therefore similar to one another;* for we have



$$\angle A = A,$$
$$\angle D = C = R.$$
$$\triangle ADC \sim \triangle ACB.$$

In the same way we get

$$\triangle CDB \sim \triangle ACB$$

and therefore also

$$\triangle ADC \sim \triangle CDB.$$

From this different proportions are deducted, of which we will mention the most important.

From $ADC \sim CDB$

we get $\frac{AD}{CD} = \frac{CD}{BD}$ or $p^2 = \alpha\beta$

which shews, that

a) *the altitude is a mean proportional between the parts of the hypothenuse.*

From $ADC \sim ACB$

we get

$$\frac{AD}{AC} = \frac{AC}{AB} = \frac{DC}{CB}$$

or $b^2 = \beta h$ and $ab = ph$,

from which is seen, that

b) *one of the sides containing the right angle is a mean proportional between its projection on the hypothenuse and the whole hypothenuse*, and that

c) *the product of the sides equals the product of the altitude and the hypothenuse.*

The most important proposition of the right-angled triangle is the one discovered by Pythagoras in the 6th century *B. C.*

d) *The square on the hypothenuse equals the sum of the squares on the sides.*

This is found by adding

$$b^2 = \beta h$$

and the analogous $a^2 = \alpha h$,

from which $a^2 + b^2 = (\alpha + \beta) h = h^2$.

This proposition shews that the one side of a right-angled triangle can be calculated, when the two others are known; we find

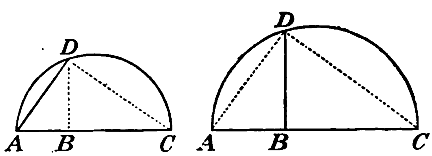
$$a = \sqrt{h^2 - b^2}; \quad b = \sqrt{h^2 - a^2}; \quad h = \sqrt{a^2 + b^2}.$$

By the help of the propositions concerning the right-angled triangle, when two of the parts a, b, h, α, β, and p are known, the others can be found; some of these problems, however, lead to equations of the second degree.

If $\angle C$ is obtuse, and the opposite side is c, and the sides containing it a and b, then $c > \sqrt{a^2 + b^2}$; if $\angle C$ is acute, then $c < \sqrt{a^2 + b^2}$; for if the triangle be compared to the right-angled triangle with the sides a and b, then in the first case $c > h$, in the second $c < h$. (46).

96. *To construct a mean proportional between two given lines, a and b.*

a) On the greater line b mark off $a (AB)$, and on b as diameter describe a semicircle; at the extremity of a raise a perpendicular, and draw the chord $x (AD)$; x will be the required line (95, b).



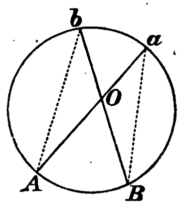
b) On $a + b$ as diameter describe a semicircle; where a and b meet together raise a perpendicular $x (BD)$, which then will be the required line (95, a).

As $\frac{a}{x} = \frac{x}{b}$, then $x = \sqrt{ab}$. A line which can be expressed by known lines in this form is therefore constructed in one of the methods indicated; if, for example, $x = a\sqrt{5} = \sqrt{5a \cdot a}$, then x will be a mean proportional between a and $5a$. We could also put $x = \sqrt{(2a)^2 + a^2}$, and therefore construct x as hypothenuse of a right-angled triangle, in which the one side is a and the other $2a$.

$z = \sqrt{ab + cd}$ is constructed by putting $ab = x^2$, $cd = y^2$; x and y are then found by 96; thereupon z is found as hypothenuse of a right-angled triangle with sides x and y.

III. THE POTENSE OF A POINT WITH REGARD TO A CIRCLE.

97. a) *When two chords of a circle intersect, the product of the two parts of the one chord equals the product of the parts of the other chord.*



Draw the lines Ab and Ba; then

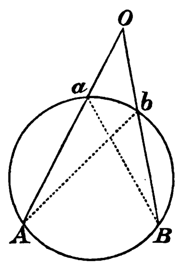
$$\angle A = B, \qquad \angle b = a \qquad \ldots\ldots\ldots \; 24$$

therefore $\triangle AbO \sim BaO$,

from which it follows that

$$\frac{OA}{OB} = \frac{Ob}{Oa}$$

or $$OA \cdot Oa = OB \cdot Ob.$$



b) *When two secants cut each other, the product of the one secant and the part outside the circle equals the product of the other secant and the part of it outside the circle.*

The former demonstration also applies to this case. As the proposition holds good for every position of the secants, it also holds good in the case in which b

and B coincide; here both Ob and OB are equal to the tangent; so that the proposition will be:

The tangent is a mean proportional between the whole secant and the part outside the circle.

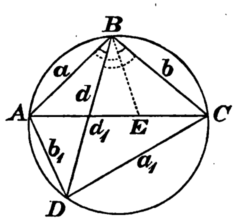
These three propositions may be expressed under one, thus:

When a circle cuts several straight lines, meeting at the same point, the product of the two parts cut off from each line (both reckoned from the fixed point to one of the points of intersection of the circle with the line) *is the same for all the lines.* This product is called the *potense* of the point with regard to the circle; by drawing the line through the centre, we see that the potense of a point, of which the distance from the centre is a (radius r), is $a^2 - r^2$ when $a > r$, and $r^2 - a^2$ when $r > a$.

IV. PTOLEMY'S PROPOSITION.

98. *In an inscribed quadrilateral the product of the diagonals equals the sum of the products of the opposite sides.*

Make	$\angle CBE = \angle DBA$;
we then have	$CBE \sim DBA$,
whence	$EC : b_1 = b : d$;
	$EBA \sim CBD$,
whence	$AE : a_1 = a : d$;
but from	$b \cdot b_1 = d \cdot EC$,
	$a \cdot a_1 = d \cdot AE$



we get by addition

$$aa_1 + bb_1 = dd_1.$$

If the inscribed quadrilateral is a rectangle, the proposition gives Pythagoras' proposition (95) as a particular case of Ptolemy's.

EXAMPLES.

139. In a right-angled triangle the sides containing the right angle are a and b, their projections on the hypothenuse α and β, the altitude p, and the hypothenuse h; how great are the other parts, when
   1) $\alpha = 5.4$; $\beta = 9.6$?
   2) $\alpha = 3.6$; $a = 6$?
   3) $a = 0.5$; $b = 1.2$?
   4) $h = 6$; $a = 4.8$?

140. Construct $\sqrt{a^2+b^2+4c^2}$, $\sqrt{\dfrac{abc^2}{(a+b)(b+c)}}$, $\sqrt{a^2+b^2+ab}$, $\sqrt{\dfrac{a^3+b^3}{3a-2b}}$, when a, b, and c are given lines.

141. In a triangle, the sides of which are
$$a = 7;\quad b = 7.7;\quad c = 8.4,$$
a transversal, $d = 1.68$, is drawn parallel to the side c; how great are the parts in which it divides the sides a and b?

142. In a rectangle the one side is 6 feet, and the diagonal is 2 feet longer than the other side. How great is this?

143. In a triangle the one altitude is 15, and the parts of the base 6 and 10. At what distance over the base is the point of intersection of the altitudes?

144. How great is the diagonal of a square with the side a?

145. In a right-angled triangle the hypothenuse is $m^2 + n^2$ and the one side $2mn$. Find the other side.

146. How great is the altitude of an equilateral triangle with the side a?

147. In a quadrilateral the diagonals are at right angles to each other. Prove that the sum of the squares on the one pair of opposite sides is equal to the sum of the squares on the other pair.

148. A circle passes through the centre C of another circle and touches it at A. A line perpendicular to AC cuts the first circle in D, the second in E.

   Shew that $\overline{AE}^2 = 2\overline{AD}^2$.

149. From a point E in the circumference of a circle a perpendicular EP is dropped on a diameter. D is the middle point of the semicircle with radius r. Shew that $\overline{PD}^2 + \overline{PE}^2 = 2r^2$.

150. In a circle with radius r an angle of 45° is drawn at the centre. On the one leg of the angle a perpendicular CE is raised, cutting the other leg in D and the circle in E. Shew that $\overline{CD}^2 + \overline{CE}^2 = r^2$.

151. From a point to a circle a secant is drawn, both parts of which equal a. How great is the tangent from the same point?

152. In a right-angled triangle with the sides containing the right angle equal to 1.41 and 1.88, a perpendicular is raised on the middle of the hypothenuse. How great are the parts in which it divides the one side?

153. In a semicircle there is over each half r of the diameter described other semicircles. How great is the radius of the circle touching the three semicircles?

154. *In a triangle with sides* a, b, *and* c, *a perpendicular is dropped on* c *from the opposite vertex. Prove that* $a^2 - b^2 = 2cl$, *when* l *is the distance from the foot of the perpendicular to the middle point of* c, *and when* $a > b$. *Prove that* $a^2 + b^2 = \frac{1}{2}c^2 + 2m^2$. *(*$m$*, the median line).*

155. A and B are two given points in a diameter, equidistant from the centre, P is any point in the circumference. Shew that $\overline{AP}^2 + \overline{BP}^2$ is constant.

156. Through A (Ex. 155) any chord is drawn, the extremities of which are joined to B. Shew that the sum of the squares on the sides of the triangle thus formed is constant.

157. In a triangle with sides a, b, and c, the angle between the two first is bisected. How great are the parts in which the bisecting line divides the third side?

158. *BC* is a diameter and *A* and *F* two points in the circumference. $FD \perp BC$ cuts *BA* in *E* and *CA* in *G*. Shew that *DF* is a mean proportional between *DE* and *DG*.

159. Shew that any point in the circumference of one of two concentric circles has the same potense with regard to the other circle, as any point in the circumference of the second circle has with regard to the first.

160. Prove that the line bisecting the angle between two opposite sides of an inscribed quadrilateral divides the two other sides into parts which form a proportion.

161. In a triangle with sides 13, 20, and 21, the altitude is drawn to the latter side; how great are the parts of the base, and how great is the altitude?

162. Given two sides of a triangle and one of the parts in which the altitude divides the third side; how great is the third side?

163. *a* and *b* are two parallel chords, and the distance between them *d*; how great is the radius of the circle?

164. A tangent at *O* cuts two parallel tangents in *A* and *B*. Shew that $OA \cdot OB$ is constant.

165. A right-angled triangle is divided by the altitude into two other triangles. Find an equation between the radii of the circles inscribed in the three triangles.

166. How long a shadow is cast by an object 2 inches high, when a candle 8 inches high is placed at a distance of 6 inches?

167. Two circles have radii *R* and *r* and the line of centres *c*. At what distance from the centres is the line of centres cut by a line touching both the circles?

168. In a quadrilateral with diagonals *d* and *D* a rhombus is inscribed, having its sides parallel to the diagonals; how great is the side of the rhombus?

169. *A* and *B* are the extremities of a line, *C* and *D* two points in a line perpendicular to *AB*. Prove that

$$\overline{CA}^2 - \overline{CB}^2 = \overline{DA}^2 - \overline{DB}^2.$$

170. In a right-angled triangle the one angle is 30°, the opposite side a. Find the other sides, the perpendicular, and the parts of the hypothenuse.

171. Prove that the sum of the squares on the four parts of two chords, which are perpendicular to each other, equals the square on the diameter.

172. a, b, and c are three chords, the arcs of which are together 180°. By what equation is the diameter of the circle determined?

173. The four sides of a trapezium are given; find the sum of the squares on the diagonals.

174. A circle has with regard to two points A and B respectively the potenses p_1 and p_2, whilst P is its potense with regard to a point which divides AB into the parts α and β. Shew that

$$(P+\alpha\beta)(\alpha+\beta) = \beta p_1^2 + \alpha p_2^2.$$

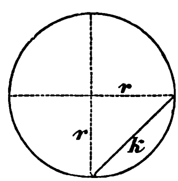
175. M is the middle point of the one side of a triangle, the line bisecting the opposite angle cuts the side in K, the altitude cuts it in H, and the inscribed circle touches it at N. Shew that

$$MN.HN = MH.NK.$$

176. On a line AD as diameter a semicircle is described, and from a point B in this is drawn $BF \perp AD$. From D draw the chord $DC = DB - AB$ and draw $CE \perp AD$. Shew that $AE = 2BF$.

V. THE DIVISION OF THE CIRCUMFERENCE OF THE CIRCLE.

99. *A circle is divided into 4 equal parts by two diameters perpendicular to each other.*



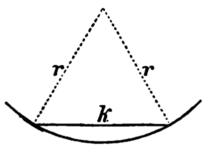
If the radius be denoted by r and the chord of $\frac{1}{n}$ of the circumference by k_n (the side of an n sided figure), then

$$k_4^2 = r^2 + r^2 = 2r^2$$

therefore $$k_4 = r\sqrt{2}.$$

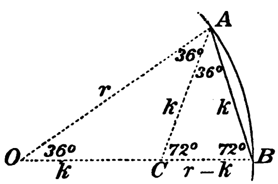
By continued bisection of the arcs, the circle can be divided into 8, 16, 32 . . . . 2^n equal parts.

100. *k_6 or the chord of an arc of 60° equals the radius.* When the arc is 60°, the angle at the centre is also 60°, and as the triangle is isosceles, the two other angles will also each be 60°, the triangle therefore is equilateral, and consequently the chord equals the radius. The circumference, therefore, is divided into 6 equal parts by marking off the radius 6 times as chord; it can be divided into 3 equal parts by missing out every other point of division; by continued bisection of the arcs, it can be divided into 12, 24, 48 . . . 3.2^n equal parts.



101. *k_{10} or the chord of 36° equals $\frac{r}{2}(\sqrt{5}-1)$.*

The required chord is base of an isosceles triangle, the sides of which are radii, and whereof the vertical angle is 36° and the angles at the base therefore each 72°. If one of these be bisected by the line AC, then $\triangle CAB$ is isosceles, for $\angle B = \angle C = 72°$; therefore $AC = k$; also $\triangle ACO$ is isosceles, therefore $OC = k$ and $CB = r - k$. Now the angles shew that $\triangle OAB \sim ACB$, therefore (see also 87)



$$\frac{r}{k} = \frac{k}{r-k}; \quad \text{or} \quad k^2 + rk = r^2.$$

Then $$k^2 + rk + \left(\frac{r}{2}\right)^2 = r^2 + \left(\frac{r}{2}\right)^2;$$

but as $$k^2 + rk + \left(\frac{r}{2}\right)^2 = \left(k + \frac{r}{2}\right)^2,$$

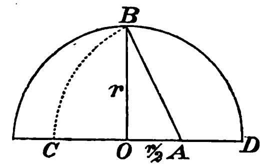
we get by extraction of the square root

$$k + \frac{r}{2} = \sqrt{r^2 + \left(\frac{r}{2}\right)^2},$$

or $$k = \sqrt{r^2 + \left(\frac{r}{2}\right)^2} - \frac{r}{2} = \frac{r}{2}\sqrt{5} - \frac{r}{2} = \frac{r}{2}(\sqrt{5} - 1).$$

The construction of k_{10} by dividing the radius in extreme and mean ratio.

The first expression for k above shews how the side of a decagon may be constructed. $\sqrt{r^2 + \left(\frac{r}{2}\right)^2}$ is namely the hypothenuse of a right-angled triangle, the sides of which are r and $\frac{r}{2}$.



The figure shews the construction; bisect the radius OD, make $OB \perp OD$ and join AB; thereupon make $AC = AB$, then OC will be the side of the decagon, for we have $OC = AC - AO$ $= AB - AO = \sqrt{r^2 + \left(\frac{r}{2}\right)^2} - \frac{r}{2}$.

A line is said to be divided into *extreme and mean ratio*, when it is divided into two parts, so that the greatest of these is a mean proportional between the least and the whole line; the proportion we got for the determination of k therefore shews that *the side of the decagon is the greatest part of the radius thus divided.*

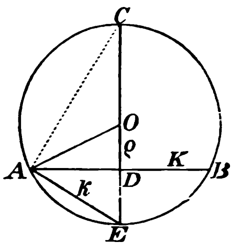
When the circumference is divided into 10 equal parts, we can, by doubling or continually bisecting the arcs, divide it into 5, 20, 40, 80 . . . . 5.2^n equal parts.

102. If an arc of 36° be subtracted from an arc of 60°, we get an arc of 24°, and thus, therefore, a circle can be divided into 15 and, by continued bisection of the arcs, into 30, 60 . . . . 15.2^n equal parts. The expression for k_{15} will be found below (Ex. 183).

103. When a chord is calculated, its distance from the centre ρ is found by 95, d; thus if the chord is k, radius r, we get

$$\rho = \sqrt{r^2 - \tfrac{1}{4}k^2} = \frac{r}{2}\sqrt{4 - \left(\frac{k}{r}\right)^2}.$$

104. When the radius is given, and the chord of a certain arc has been calculated, we can from that again calculate *the chord of half and the chord of double the arc.*



Let $AB = K$, $AE = k$, $OD \perp AB$, therefore $\smile AB = 2 \smile AE$; we then have (103)

$$OD = \frac{r}{2}\sqrt{4-\left(\frac{K}{r}\right)^2}$$

and from this

$$DE = r - OD = r - \frac{r}{2}\sqrt{4-\left(\frac{K}{r}\right)^2}.$$

As now $\angle CAE = 90°$, then AE or k is a mean proportional between DE and the diameter CE, therefore

$$k = \sqrt{2r \,.\, DE}$$

or

$$k = r\sqrt{2-\sqrt{4-\left(\frac{K}{r}\right)^2}}.$$

Example. From $k_6 = r$ we hereby find $k_{12} = r\sqrt{2-\sqrt{3}}$;

$$k_{24} = r\sqrt{2-\sqrt{2+\sqrt{3}}};$$

from $k_4 = r\sqrt{2}$ we find $k_8 = r\sqrt{2-\sqrt{2}}$.

From k to calculate K, we remark that

$$2r \,.\, AD = k \,.\, AC \quad (95, \text{c})$$

or

$$rK = k \,.\, \sqrt{4r^2 - k^2},$$

from which

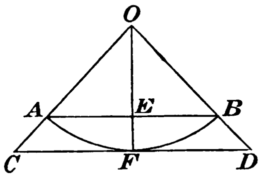
$$K = k\sqrt{4-\left(\frac{k}{r}\right)^2}.$$

Example. From $k_{10} = \frac{r}{2}(\sqrt{5}-1)$ we find

$$k = \frac{r}{2}(\sqrt{5}-1)\sqrt{4-\frac{(\sqrt{5}-1)^2}{4}} = \frac{r}{2}(\sqrt{5}-1)\sqrt{\frac{10+2\sqrt{5}}{4}}$$

$$= \frac{r}{2}\sqrt{10-2\sqrt{5}}.$$

105. When the side and the least radius of an inscribed regular polygon are calculated, we can calculate the side and the greatest radius of the circumscribed regular polygon with the same number of sides.



Let AB be the side of the inscribed polygon, from the centre O draw a perpendicular OF to AB; if thereupon the tangent CD be drawn, it becomes the side, and OC the greatest radius of the circumscribed polygon, for we have $AB \neq CD$, as they both are perpendicular to OF, and therefore the triangles AOB and COD are similar. From this it follows

$$\frac{CD}{AB} = \frac{CO}{AO} = \frac{FO}{EO}$$

but if we put $CD = t$, $AB = k$, $OC = R$, $OE = \rho$, then

$$\frac{t}{k} = \frac{R}{r} = \frac{r}{\rho} \quad \text{or} \quad t = \frac{rk}{\rho}; \quad R = \frac{r^2}{\rho}.$$

EXAMPLES.

177. How great are k_3 and ρ_8, t_6 and R_6, ρ_8 and t_8?
178. What is the product $k_{12}\rho_{12}$?
179. What is t_3, and what is the least radius of an equilateral triangle with the side t?
180. Prove that the line joining BC on the fig. to 101 is the side of the inscribed pentagon.
181. Find the mean proportional between the chord of 30° and the chord of 150°.
182. Find k_{64}, k_{96}, ρ_{64}, and ρ_{96}.
183. *a, b, and c are sides of an inscribed triangle; find c, when a, b, and r are given.*

Draw the diameter from the point of intersection of a and b, and lines from the extremities of the diameter to the extremities of a and b. These lines become $\sqrt{4r^2 - a^2}$ and $\sqrt{4r^2 - b^2}$, and we then have (98)

$$2rc = a\sqrt{4r^2 - b^2} \pm b\sqrt{4r^2 - a^2},$$

the upper sign when c is the chord of the sum of the arcs, and the lower sign when c is the chord of the difference of the arcs, of which a and b are the chords $(a > b)$.

Hereby we get k_{15}, when $a = r$, $b = \frac{r}{2}(\sqrt{5} - 1)$ (102),

$$k_{15} = \frac{r}{4}\left(\sqrt{10 + 2\sqrt{5}} - \sqrt{3}(\sqrt{5} - 1)\right).$$

VI. THE LENGTH OF THE CIRCUMFERENCE OF THE CIRCLE.

106. A curved line cannot be measured by a straight line, for the unit must be of the same kind as that which is to be measured; we must therefore more closely explain what is meant when we speak of the length of the circumference.

It is evident that an inscribed regular $2n$ sided figure has greater perimeter than an inscribed n sided figure, whilst the opposite is the case with the circumscribed figures. If we therefore continue to double the number of sides of an inscribed and a circumscribed regular figure, the perimeter will in the first case continually increase and in the second continually decrease; as now the latter always is greater than the former, they must approach nearer and nearer to each other, and we can prove that the difference between them can be made less than any assigned quantity. The perimeter of the inscribed n sided figure with the side k is nk and of the circumscribed figure nt; the difference is therefore, as $k = \frac{\rho t}{r}$ (105),

$$nt - \frac{n\rho t}{r} = nt\frac{r - \rho}{r} = nt\frac{(r^2 - \rho^2)}{r(r + \rho)},$$

but
$$r^2 - \rho^2 = \frac{k^2}{4},$$

therefore the difference

$$f = nt\frac{k^2}{4r(r + \rho)};$$

but here the first factor nt is the perimeter of the circumscribed figure and is accordingly less than the perimeter we had before doubling the number of sides, whilst the second factor can be made as small as we please, by doubling the number of sides till k is sufficiently small.

The lengths of the perimeters of the inscribed and circumscribed figures therefore approach nearer and nearer to each other, so that they have a common limit, a value to which they both approach. *It is this limit that is meant, when we speak of the length of the circumference of a circle.*

The limit will be the same, whichever regular polygon we take.

Let the circumscribed and inscribed polygons with several sides have the perimeters P and p, whilst G is the limit. By continuing to double the number of sides, we can make $P - G$ and $G - p$ less than any ever so small quantity. Let P_1, p_1, and G_1 be the perimeters and the limit, when we take another regular polygon. As p_1 lies within P and p within P_1 we can (Ex. 56 expanded) put

$$P - p_1 = \alpha; \quad P_1 - p = \beta$$

where α and β are certain positive quantities; therefore

$$P_1 - p_1 + P - p = \alpha + \beta.$$

As here $P_1 - p_1$ and $P - p$ by continued doubling become less than any ever so small given quantity, the same must be the case with the positive quantities α and β. P_1 must therefore have the same limit as p, or G_1 must be equal to G.

107. *The lengths of the circumferences of two circles are to each other as the radii.*

For the perimeters of two regular figures with the same number of sides are to each other as the greatest radii, and this proposition continues to hold, how ever often the number of sides of the polygon is doubled; therefore it must also hold good for circles.

If we denote the circumferences by P and p, the radii by R and r, and the diameters by D and d, we therefore have

$$\frac{P}{p} = \frac{D}{d} = \frac{R}{r}, \quad \text{or} \quad \frac{P}{D} = \frac{p}{d},$$

which shews that the ratio between the circumference and diameter is the same for all circles (is constant); this ratio, which is an abstract number, is denoted by the letter π, so that

$$P = \pi D = 2\pi R.$$

It therefore depends upon, once for all, to calculate this number π; when it is known, we can always of the three quantities P, D, and R calculate the two, when we know the third. As the ratio is the same for all circles, we take radius equal to 1, therefore the diameter equal to 2, and calculate the perimeter of an inscribed polygon with several sides; if the number of sides be n, we have

$$nk_n < p < nt_n$$

therefore

$$\frac{nk_n}{2} < \pi < \frac{nt_n}{2}.$$

We have $k_6 = 1$ and found from this (104) $k_{12} = \sqrt{2 - \sqrt{3}} = 0.517638090\ldots$; from this we find by degrees k_{24}, k_{48} . . . for example, to k_{768}; by multiplying the value found thus by 768, we get the perimeter of the 768 sided figure (nk_n) equal to 6.283160...; after division by 2, we therefore have

$$3.141580 < \pi.$$

From k_{768} we thereupon calculate ρ_{768} (103) and from this again t_{768}; if this be multiplied by 768, we get the perimeter of the circumscribed 768 sided figure (nt_n); for this we find the numbet 6.283212..., and get by dividing by 2

$$\pi < 3.141606.$$

As π lies between the values found thus, we have with 4 correct decimals

$$\pi = 3.1416.$$

By easier methods, which cannot be shewn here, π has been found with several hundred decimals; the first are

$$\pi = 3.14159\,26535\ldots.$$

The number is incommensurable, but is nearly expressed by the fractions $\frac{22}{7}$ and $\frac{355}{113}$, of which the first already was *found* by *Archimedes*.

108. As an arc of 1° is $\frac{1}{360}$ of the circumference, its length will be $\frac{\pi r}{180}$, and consequently the length b of an arc of $g°$

$$b = \frac{\pi r g}{180}.$$

By the help of this equation, when two of the three quantities b, r, and g are known, we can calculate the third. If g be expressed in minutes or seconds, the denominator must be multiplied by 60 or 60^2 respectively.

EXAMPLES.

184. How great is the circumference of a circle, when the radius is 2′ 4″, and how great is the radius, when the circumference is 3′ 6″?
185. A circle touches another circle internally in A and passes through its centre. Through this centre a line is drawn cutting the circles in B and C; shew that $\smile AB = \smile AC$.
186. Two circles with radii R and r touch a circle with radius $R + r$ internally respectively in A and B, whilst C is one of their points of intersection. Prove that $\smile AB = \smile AC + \smile CB$.
187. How long is an arc of 12° 13′ 20″, when the radius is 1′ 8″?
188. How many degrees are contained in an arc, the length of which is 3″, when the radius is 2″?
189. A row of semicircles is so placed that the diameters are prolongations of each other. Prove that the sum of the lengths of the circumferences of all the semicircles is equal to the length of a semicircle, the diameter of which is the sum of the given diameters.
190. *Find the number of degrees contained in an arc, the length of which equals the radius.*
191. How great is the radius of the earth, when one degree of latitude is 60 geographical miles?
192. O is the centre of a circle and A a point, the distance of which from O is n times the radius. On OA as di-

ameter a circle is described. Two lines from A intercept arcs on the first circle, having the lengths b_1 and b_2; how long is the arc intercepted by the two lines on the second circle?

193. An undulating line is formed by dividing a line a into n equal parts, and on each of the parts as chord describing an arc of 60°, alternately upwards and downwards. How long is the undulating line? What does this length become, and what becomes of the undulating line, when n grows infinitely, whilst a remains unchanged?

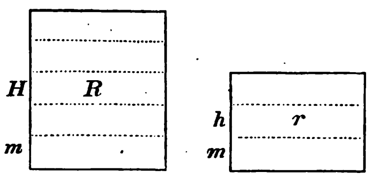
IV.

AREA.

109. The area of a figure can be measured by the area of a square, the side of which is the unit of length.

According as this is foot, inch, &c., the unit of surface is called square foot, square inch, &c.; the designations are the same as for lineal measure, but with the addition of the sign □.

110. *Rectangles with equal bases are to each other as their altitudes.*



If a certain line m is contained exactly in both altitudes, for example, $H = pm$; $h = qm$, then

$$\frac{H}{h} = \frac{p}{q}.$$

Parallels through the points of division divide the rectangles respectively into p and q equal rectangles, therefore

$$\frac{R}{r} = \frac{p}{q}$$

and consequently
$$\frac{R}{r} = \frac{H}{h}.$$

If the altitudes are incommensurable, the proposition also holds good and is proved as β in 83.

111. *Rectangles with equal altitudes are to each other as their bases.*

This proposition is the same as the preceding one, for any of the sides we please can be taken as base.

112. *The ratio between any two rectangles is the product of the ratio between the altitudes and the ratio between the bases.*

Let the rectangles be R, with altitude H and base G, and r, with altitude h and base g; let us then imagine a third rectangle P, with altitude H and base g.

Then
$$\frac{R}{P} = \frac{G}{g} \quad \text{and} \quad \frac{P}{r} = \frac{H}{h},$$
from which by multiplication
$$\frac{R}{r} = \frac{H}{h} \cdot \frac{G}{g}.$$

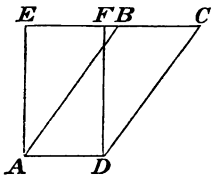
If r denote the unit of surface, therefore h and g the units of length, this equation shews that:

The number of units of surface in a rectangle is found by multiplying the number of units of length in the altitude by the number of units of length in the base; this is written
$$R = H.G,$$
where R, H, and G denote abstract numbers, and we ourselves must remember the denominations; to prevent this, we have adopted the phrase, that foot multiplied by foot gives square foot &c., by which we are enabled to calculate with concrete numbers; we therefore say, that

The area of a rectangle equals the product of the altitude and the base.

The area of a square is therefore the square of the side; we therefore have $1' \square = 144'' \square$ and $1'' \square = 144''' \square$ Duodecimal measure, $1'' \square = 100''' \square$ Decimal measure.

113. *A parallelogram is equal to a rectangle with the same altitude and the same base.*



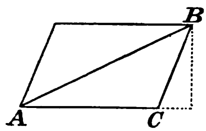
For we have

$$\triangle EAB \cong \triangle FDC.$$

By subtracting these triangles one at a time from the whole figure, the parallelogram AC and the rectangle AF will be left respectively; they are therefore equal.

From this it follows that *the area of a parallelogram is the product of the altitude and the base.*

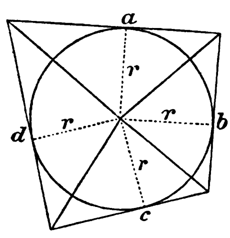
114. *The area of a triangle is half the product of the altitude and the base.*



For the triangle ABC is half of the parallelogram AB, which has the same base and the same altitude. *The area of a right-angled triangle is half the product of the sides containing the right angle.*

In an equilateral triangle with the side s the altitude is $\frac{s}{2}\sqrt{3}$, and the area therefore $\frac{s^2\sqrt{3}}{4}$.

115. *The area of a figure in which a circle can be inscribed is half the product of the radius of the circle and the perimeter of the figure.*



From the centre draw lines to the angular points; thereby the figure is divided into triangles, which all have the radius r of the circle for their altitude, and the bases of which are the sides of the figure a, b, c ..., we therefore have the area of the figure

$$A = \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc \ldots . = \tfrac{1}{2}r(a+b+c\ldots) = \tfrac{1}{2}rp,$$

when p is the perimeter.

The radius of a circle inscribed in a triangle is therefore equal to the area divided by half the perimeter.

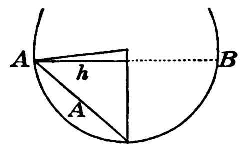
116. *The area of a circle with radius r is πr^2.*

According to the former proposition the area of a circumscribed polygon is $\frac{1}{2}rp$; if the number of sides in this be made infinitely great, it will coincide with the circle, and p becomes $2\pi r$, therefore the area of the circle $C = \pi r^2$.

117. *The area of a sector of a circle of g° is $\frac{\pi r^2 g}{360}$.*

For a sector of 1° is $\frac{1}{360}$ of the circle and therefore its area is $\frac{\pi r^2}{360}$; a sector of g° therefore $\frac{\pi r^2 g}{360}$.

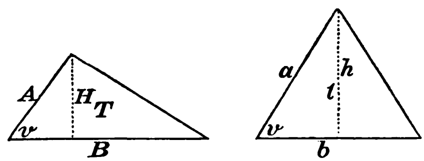
118. *The area of a segment A* is the difference between the areas of the corresponding sector and the triangle; for the calculation of the latter, one of the radii is usually taken as base; the altitude h is then half the chord AB of double the arc of the segment, and this is often known.



119. *The ratio between two triangles, having one angle equal, is the product of the ratios of the sides containing the angle.*

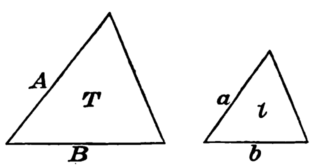
Let the triangles be denoted by T and t, the altitudes by H and h, then we have (114)

$$\frac{T}{t} = \frac{H}{h} \cdot \frac{G}{g}$$



but A and H, a and h are corresponding sides of similar triangles, so that $\frac{H}{h} = \frac{A}{a}$, and therefore $\frac{T}{t} = \frac{A}{a} \cdot \frac{B}{b}$.

120. *The ratio between two similar triangles is the duplicate of the ratio between a pair of corresponding sides.*



We have (119) $\frac{T}{t} = \frac{A}{a} \cdot \frac{B}{b}$

but $\frac{A}{a} = \frac{B}{b}$, therefore $\frac{T}{t} = \left(\frac{A}{a}\right)^2$.

5*

121. *The ratio between the areas of any two similar figures is the duplicate of the ratio between a pair of corresponding lines or of the linear ratio.*

For two similar figures are divided by corresponding diagonals into similar triangles (93); if these be respectively denoted be T_1, T_2, $T_3 \ldots$, t_1, t_2, $t_3 \ldots$, and the ratio beween any two corresponding lines or the linear ratio by f, we have

$$\frac{T_1}{t_1} = \frac{T_2}{t_2} = \frac{T_3}{t_3} \ldots = f^2 \ldots (120),$$

from which, by a well-known proposition in proportion,

$$\frac{T_1 + T_2 + T_3 \ldots\ldots}{t_1 + t_2 + t_2 \ldots\ldots} = f^2,$$

which was to be proved.

As the proposition holds good independently of the number of sides, it also holds when this is infinite, therefore also of figures contained by curved lines.

122. *If similar figures be constructed on the sides of a right-angled triangle, so that these are corresponding lines in the figures, then the figure on the hypothenuse equals the sum of the figures on the sides containing the right angle.*

If the sides be denoted by a, b, h, the corresponding similar figures by A, B, H, we have by 121:

$$\frac{H}{h^2} = \frac{A}{a^2} = \frac{B}{b^2} = \frac{A+B}{a^2+b^2},$$

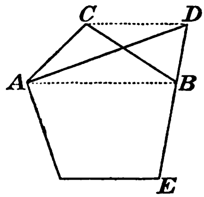
but, as $h^2 = a^2 + b^2$, then also $H = A + B$.

By the help of this proposition and of 94, it is easy to solve this problem, *to draw a figure, the area of which shall be equal to the sum or difference of two given similar figures, and which shall be similar to these.*

Specially must be noticed the problem, to draw a square equal to the sum or difference of two given squares.

123. *To change a given polygon to a square.*

From the polygon cut off a triangle ACB by the diagonal AB; if CD be drawn parallel to AB, every triangle, having



its vertex in this line, and the base of which is AB, will have the same area as ACB, and therefore could be placed instead of this; if the vertex be taken in D, which is found by producing BE, then the new polygon has one side less than the given one.

We continue in this way till the polygon is changed to a triangle; if we call the altitude of this h, its base g, the side of the required square x, we must have

$$x^2 = \tfrac{1}{2}hg,$$

which shews that x is a mean proportional between $\frac{1}{2}h$ and g or between $\frac{1}{2}g$ and h, it can therefore be constructed by one of the methods in 96.

124. Triangles can be changed to other triangles satisfying certain given conditions; *if thereby one of the sides is to remain unchanged, the altitude on this must also remain unchanged*, and the vertex therefore fall in a known line parallel to the base; one more condition must then be given, which can give one more locus of the vertex. *If one of the angles is to remain unchanged, the product of the sides containing it must also remain unchanged* (119); now if one of the sides be given, the other is easily determined as fourth proportional to this side and the sides containing the angle in the given triangle.

125. *To draw a figure similar to a given figure and n times as great.*

If one side of the given figure be a, the corresponding side in the required figure x,

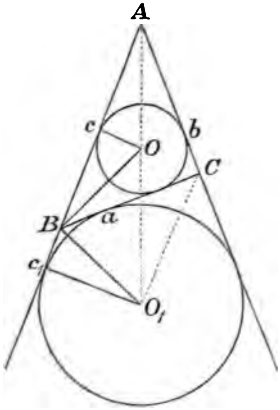
we have $$\frac{x^2}{a^2} = \frac{n}{1} \quad (121)$$

or $$x^2 = na^2$$

x is then constructed as mean proportional between a and na, and thereupon apply 94.

126. *The area of the triangle and the radii of its inscribed and escribed circles.*

Let the area of the triangle be T, its angles A, B, C, and the sides opposite to these a, b, c; let r be the radius of the inscribed circle, and r_a the radius of the escribed circle, touching the side a and the prolongations of b and c; we then have



$$T = \triangle O_1BA + \triangle O_1CA - \triangle O_1BC$$

or $$T = \tfrac{1}{2}r_a c + \tfrac{1}{2}r_a b - \tfrac{1}{2}r_a a = r_a(s-a) \quad (64).$$

We have thus (115)

$$T = rs = r_a(s-a) = r_b(s-b) = r_c(s-c).$$

From the two first of these equations we get

$$T^2 = r r_a s(s-a),$$

but from $\triangle BOc \sim \triangle O_1Bc_1$ we get, as

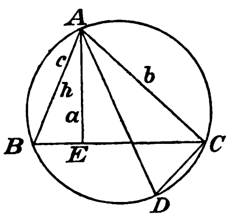
$$Bc = s - b \quad (64),$$

$$Bc_1 = Ac_1 - c = s - c \quad (64),$$

$$r r_a = (s-b)(s-c),$$

therefore $$T = \sqrt{s(s-a)(s-b)(s-c)}.$$

127. *The diameter $2R$ of the circumscribed circle equals the product of two sides, divided by the altitude on the third side.*



Draw the diameter AD and join DC; from $\triangle BAE \sim \triangle DAC$ we get

$$\frac{c}{2R} = \frac{h_a}{b} \quad \text{or} \quad R = \frac{bc}{2h_a}.$$

When the numerator and denominator of the fraction are multiplied by a, the equation assumes the form

$$R = \frac{abc}{4T} \quad \text{or} \quad abc = 4RT.$$

EXAMPLES.

194. Prove that the area of a rhombus equals half the product of the diagonals.
195. How great is the area of a right-angled, isosceles triangle, the hypothenuse of which is a?

196. *Prove that the area of a trapezium is measured by the altitude, multiplied by half the sum of the parallel sides.*

197. From a point within a regular polygon perpendiculars are dropped on all the sides; prove that the sum of the perpendiculars is constant.

198. In a triangle one angle is 30° and the sides containing it a and b; how great is the area?

199. The area of an equilateral triangle is 1 □′ 8 □″; how great is the radius of the circumscribed circle?

200. Prove that a triangle is divided into three equal parts by lines from the point of intersection of the median lines to the angular points.

201. By a line, parallel to the side of a triangle, cut off another triangle equal to $\frac{1}{3}$ of the given one (120 and 96).

202. To divide a triangle into two equal parts by a line issuing from a point in the one side (119).

203. What is the ratio between the side of a regular hexagon and of an equally great equilateral triangle?

204. A triangle is divided by two transversals, parallel to the one side, into three parts, the areas of which are A, B, and C. Find the ratio between the distances between the parallels.

205. Prove that a quadrilateral is twice as great as the parallelogram, formed by joining the middle points of the sides.

206. The area of a quadrilateral is known and also the segments of the diagonals; find the areas of the four triangles, into which the diagonals divide the quadrilateral.

207. How large is the greatest triangle that can be cut out of a square piece of paper with the side a?

208. Prove that the area of a regular $2n$ sided figure inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed n sided figure.

209. How great is the radius of a circle, the area of which is 1 □′?

210. How great is the radius of a circle, in which a sector of $7^\circ\ 12'$ has an area of $2\,\square''$?
211. How great is the area of a segment of a circle, the arc of which is 90°, when the radius is r?
212. To divide a circle into 3 equal parts, by the help of other circles, concentric with the given one.
213. How great is the area of a segment of a circle, the arc of which is 30°, when the radius is r?
214. In a circle with radius r two parallel chords are drawn, of which the arcs are 60° and 120°; how great is the area between them?
215. Two equal circles with radius r pass through the centre of each other; how great is the area of the figure they have in common.
216. Three equal circles with radius r touch each other. How great is the area between them?
217. A circle is inscribed in a sector of 90°; what is the ratio between the areas of the two figures?
218. In a circle two radii AB and AC are drawn perpendicular to each other; on each of these as diameter a semicircle is described; the two semicircles cut one another in D. Find the ratio between the areas of the figures AD and DBC.
219. On the sides of a right-angled triangle any similar curves are described, so that the one, corresponding to the hypothenuse, turns to the same side as those, corresponding to the other sides of the triangle. Prove that the area of the figure bounded by the three curves equals the area of the triangle; as a special case may be mentioned that, where the similar figures are semicircles.
220. A figure is bounded by three straight lines and two arcs. Draw another figure similar to this and three times less.
221. Find R, r, r_a, and r_b for an isosceles triangle with the base a and the sides b.
222. Find the area of a triangle with sides 6, 7, and 9; thereupon find R, r, r_a, r_b, and r_c for the same triangle.
223. Prove that $T^2 = r r_a r_b r_c$.

224. Shew that

$$r_a + r_b + r_c = 4R + r.$$

225. Prove that the proposition in 119 also holds good, when the two angles, instead of being equal, are supplementary.

226. Prove that two quadrilaterals, the diagonals of which make the same angle with each other, are to each other as the products of the diagonals.

227. A circle has its centre O on the circumference of another circle. Draw any tangent to the first circle cutting the second circle in A and B. Shew that $OA \, . \, OB$ is constant (127).

228. *An inscribed quadrilateral has the sides* a, b, α, *and* β, *the diagonals* d *and* δ (d joins the point of intersection of a and b to the point of intersection of α and β). Prove that

$$\frac{d}{\delta} = \frac{ab + \alpha\beta}{a\beta + \alpha b}.$$

Employ the formula in 127 for expressing the area in a double manner.