Geometry without Axioms.

&c.

Geometry without Axioms.

or

THE FIRST BOOK

OF

EUCLID'S ELEMENTS.

WITH ALTERATIONS AND FAMILIAR NOTES;

AND AN

INTERCALARY BOOK

IN WHICH THE STRAIGHT LINE AND PLANE ARE DERIVED FROM PROPERTIES OF THE SPHERE.

Being an attempt to get rid of Axioms and Postulates; and particularly to establish the Theory of Parallel Lines without recourse to any principle not grounded on previous demonstration. In the present Edition the part relating to Parallel Lines is reduced in bulk one half.

TO WHICH IS ADDED AN APPENDIX

Containing Notices of Methods at different times proposed for getting over the difficulty in the Twelfth Axiom of Euclid.

By Col. Thompson.

By a Member of the University of Cambridge.

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FOURTH EDITION.

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PREFACE TO THE FOURTH EDITION.

IN the preceding Editions endeavour had been made to get rid of Axioms, and particularly to establish the Theory of Parallel Lines without recourse to any principle not grounded on previous demonstration.

On showing the results to some of the leading mathematicians at Cambridge, they replied by the remark, that they had always felt something to be more urgently wanted for the emendation of Geometry,—which was, information on the nature and construction of the straight line and plane.

It had been stated, about the time when the circumstances were engrossing the attention of the public, that NAPOLÉON on his voyage from Egypt amused himself and staff with *circular geometry*. What circular geometry might be, could only be collected from the tradition, that the problem given by the future Emperor was 'to divide the circumference of a circle into four equal parts by means of circles only.' But this sufficed to indicate, that the idea which had passed through the mind of that eminent practical geometer, was that in the properties of the circle, or still more probably of the sphere, might be discovered the elements of geometrical organization.

The author had in consequence been led at different times to attempt the collecting of the conditions, under which figures of various kinds may be turned about certain points and be what may be termed *introgyrant*, or turn upon their own ground without change of place. And on receipt of the remark mentioned above, this track was pursued with renewed vigour, and the results are presented in the sequel.

For the use of those who are not disposed to go into details, the substance may be stated as follows. *A solid may be described, all the points in whose surface shall be equidistant from a given point within; such a solid is called a* sphere. *A sphere may be turned in any manner whatsoever about its centre, without change of place. Consequences deducible from this are, that if two spheres touch one another externally, they touch only in a point; and if they are turned as one body about the two centres which remain at rest, the point of contact remains unmoved. Hence if about two assigned points be described a succession of spheres touching one another, any number of intermediate points may be determined that shall be desired, which, on the whole being turned about the two centres, shall be without change of place; and if this be extended to imagining one sphere to increase continuously in magnitude and the other to decrease, the line described by their point of contact will be without change of place throughout; such a line is called a* straight *line. If two equal spheres be placed touching one another externally, and about the centre of each a sphere be described passing through the centre of the other, and a straight line of unlimited length be drawn from the point of contact of the two smaller spheres to any point in the intersection of the others; this straight line, on the whole being turned about the two centres of the spheres, will describe a surface in which any two points being taken, the straight line between them, with its prolongation either way, may be proved to lie wholly in that surface. A surface of this kind is called a* plane. *From these, all the relations of straight lines and planes may be inferred.* If in this there is any novelty and truth, it is surprising that a property which was the foundation of the Platonic notion of the perfection of the sphere (*Vide Plat. De Anim. Mund.*) should not have been sooner carried into its consequences.

These results, together with the proofs offered of what have usually been termed the Axioms and Postulates, have been formed into an Intercalary Book, with a view to facilitate their postponement in the case of students beginning geometry for the first time. And for the further convenience of this class, a Recapitulation of the principal contents of the Intercalary Book has been given at page 46; thereby placing the beginner in the same situation as by the ordinary proceeding. The best time for commencing the Intercalary Book would probably be after having gone once through as much of the Elements of Euclid as is usually read; not, of course, that the contents are in any degree dependent upon what follows, but to take advantage of the habits of reasoning that may be thus acquired, before attempting what must be characterized as, in some parts, at least equal in complexity to anything in the succeeding Books.

If this process is objected to as irregular, it may be a great irregularity that nature should not have framed the elements of geometry so as to present a concinnous whole with the easiest parts always foremost and the Planes in the Eleventh Book. But if she has not, or till somebody can establish that she has, there seems to be no cause why bad reasoning should be admitted for the sake of a conventional arrangement. If the sphere is the simplest of figures and the properties of all others are derivable from it, it is more reasonable to be thankful for the knowledge, than to quarrel with the dispensation. The same argument appears to apply to objections against the introduction of motion. If motion is introduced without utility, it is superfluous; and all superfluity is an evil. There may be places where it would be possible to evade the recognition of continuous motion, by a forced and affected substitution of a succession of positions instead, with the effect of greatly reducing the distinctness of the whole; as for instance in the conclusions drawn with respect to the sphere and straight line from turning them about certain points. But there are others where the reasoning depends altogether on the supposition of a continuity of motion; as, notably, in the very pinch and nip of the argument on Parallel Lines in Prop.

XXVIII *D* in the First Book, where the conclusion rests entirely on the impossibility of a certain line ceasing to cut a series of other lines during a continuous motion, in a way incapable of being supplied by any succession of insulated positions; as likewise in part of Prop. XII Cor. 8 of the Intercalary Book, and elsewhere. On the whole therefore it may be an interesting question in what place a geometer would be warranted in saying, 'I could have proved these preliminaries, but it would have been necessary to disturb the order which directs that lines be treated of first and solids afterwards, and to introduce motion; for which reason it was considered better that they should be adopted without proof.'

In labouring to get rid of Axioms, the object has been to assail the belief in the existence of such things as self-evident truths. Nothing is self-evident, except perhaps an identical proposition. There may be things of which the evidence is continually before the senses; but these are not self-evident, but proved by the continual evidence of the senses. There may be things whose connexion with other things is so constantly impressed upon us by experience, that few people ever think of inquiring into the cause; but for that very reason there is often considerable difficulty in clearly explaining the cause, and among this class of things the admirers of axioms have found their greatest crop of self-evident truths. In arguments on the general affairs of life, the place where every man is most to be suspected, is in what he starts from as 'what nobody *can* deny.' It was therefore of evil example, that science of any kind should be supposed to be founded on axioms; and it is no answer to say, that in a particular case they were true. The Second Book of Euclid would be true, if the First existed only in the shape of the heads of the Propositions under the title of Axioms; but this would make a most lame and imperfect specimen of reasoning. The ways in which the Axioms and their kindred the Postulates have been disposed of, will be easily traced. Some have been demonstrated as Theorems, or executed as Problems; others (as the Axiom on coincidence) resolved into the mere declaration of the matter to which a certain Nomenclature is assigned; and others into Corollaries from the rest. The Axiom which declared the whole to be greater than its part, has been omitted as amounting, after the explanation of the terms 'greater' and 'less' introduced from Euclid's Book of Data, to no more than the proposition that 'the greatest is greatest.'

In the present Edition, the part relating to the disputed principle of Parallel Lines has been reduced in bulk to one half, and in substance to the following. *If a tessera* [or quadrilateral rectilinear plane figure of which two of the opposite sides are equal to one another and make equal interior angles with a side between them which shall be called the base] *has the angles at the base less than right angles, the angles opposite to the base cannot be right angles. And this because, if a number of such figures are placed side by side, a straight line of unlimited length which shall travel continuously from one of the angular points in the series of bases, along the straight line formed by the junction of two sides, keeping ever at right angles to it, cannot cease to cut the bases and make angles at the point of section greater than given angles, before it has reached the series made by the sides opposite to the bases; which is inconsistent with those sides forming one straight line as must be the case if the angles opposite to the bases in the tesseras were right angles. Whence may be inferred that the three angles of every triangle are not less than two right angles; and Euclid's Axiom.* The formula of *generalization* (see p. 5) has been extended and rigidly adhered to, for the express purpose of protesting against the unfortunate use which has been made of the term *abstract* as applied to what ought to have been called *universal* propositions; than which nothing has given a greater handle to the supporters of practical mal-reasoning. With a view to further resisting the notion of self-evident truths, no Corollary has been inserted without being followed by the reason; a practice which, like the preceding, it may not be necessary to carry through the whole range of the sciences, but which will be found very useful at the threshold. *Nomenclature* has been substituted for Definitions, as being more closely accordant with the principle of LAVOISIER, '*de ne procéder jamais que du* *connu à l'inconnu.'* The difference may at first sight appear to be not much; but if all men would agree to treat of nothing that had not previously been laid out for naming after the manner of the anatomists, the consequences in the aggregate would be considerable. The latest innovation has been the assertion, that an angle (or the thing spoken of under that term by geometers whether they knew it or not,) is *a plane surface;* an alteration which will probably be considered as among the most violent in the book, and for which reference must be made to the text (see p. 50 and elsewhere). No Proposition or Corollary has been admitted throughout, which is not subsequently cited; with the exception of the concluding one, and of a small number the application of which in the following Books of Euclid, or other reason for insertion, is pointed out in the Notes.

In all this the object has been to do something towards giving Geometry a right to the denomination of an *exact science;* a title, to which, after allowing for its superior capabilities, it has in fact had less claim than several other branches of knowledge. Its value, even as an imperfect model of reasoning, has been admitted by the fear at different times displayed of it by the allies of general obscuration. Not the least significant of the processes understood to have been in operation in France during the temporary restoration of the Bourbons, was the discouragement of geometrical lectures in the institutions under the influence of the government, and the attempt to substitute a series of philosophical recreations in the form of experiments. The hint is too good to be lost. Can nobody write a book of geometry that should be *prohibited in Austria?*

T. Perronet Thompson,
Queen's Coll. Camb.

May 15, 1833.

FIRST BOOK.

(In the marginal references, the Roman numerals express the Book; and the Arabic ones next following, the Proposition. The Intercalary Book is expressed by INTERC.)

NOMENCLATURE.

I. After a thing, or a class of things, has been rendered obvious to the senses, and either by speech, writing, pictorial representation, or examination in its proper substance, has been made sufficiently the subject of knowledge to be distinguished from other things; the word appointed to signify it in future, is called its *name*. The giving of names for the purposes of science, is called *Nomenclature*.

See Note.

II. Anything that can be made the object of touch, is called a *body*.

III. A body whose particles are immoveable among themselves, at least by any force there is question of employing; is called a *hard* body.

IV. That which has length, breadth, and thickness, is called a *solid*.

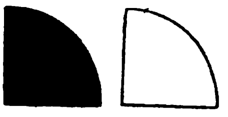
A solid may either be presented by the external parts of some body, or by the internal. The first may be called a *substantial* solid; the other, a *hollow* one. Thus a block of marble in the form of a die, is a substantial cube; a room or apartment of corresponding form, is a hollow cube.

If nothing else be specified, the solid is understood to be substantial.

V. That which bounds a solid, is called a *surface*.

A surface, consequently, has length and breadth, but not thickness. For if it had thickness, it would not be the boundary, but part of the solid.

VI. That which bounds a surface, is called a *line*.



A line, consequently, has length, but not thickness or breadth. For the surface itself has no thickness; wherefore its boundary can have none. And if the so-called boundary had breadth, it would not be the boundary, but part of the surface.

Thus the boundaries of the black surface at the side, are lines; which manifestly have neither breadth nor thickness. And if to save trouble the same surface is designated by black scrawls instead, it is in strictness not the scrawl, but the edge of the scrawl, that is the line. See Note.

VII. The extremity of a line, is called a *point*.

A point, consequently, has position, but not dimensions of any kind. For the line itself has neither thickness nor breadth; wherefore its extremity can have neither. And if the so-called extremity had length, it would not be the extremity, but part of the line.

Thus the extremity of a line in the black surface above, is a point; which manifestly has no dimensions of any kind. And if the lines are represented by black scrawls, the point is in strictness the extremity of one of their edges.

When a point is directed to be taken in some part of a line which is not the extremity, it may be imagined to be determined by causing the line to terminate at that point. See Note.

See Note. VIII. Anything that has boundaries which are fixed, is called a *figure*.

Figures are *solid*, *superficial*, or *linear*, according as the subject is a solid, surface, or line. To which may hereafter be added *angular*, or those of which the subject is an angle.

Figures of all kinds, lines, and points, will always be considered as exhibited on a hard body of some kind, which causes the position of the several parts or points to be fixed with relation to one another; and will, on occasion, be supposed to be turned about an assigned point or points, in any manner that can be shown to be practicable with the hard body on which they are understood to be represented.

Nevertheless the application of one object to another will, when required, be imagined to take place without bar of corporeal substance;—that is to say, without impediment from the existence of other parts than those it is desired to compare. Which, though avowedly only an act of the imagination, is sufficient for obtaining the results the geometer desires.

IX. Things which occupy the same place, are said to *coincide.*

The only way in which coincidence can actually be brought to pass, is when the things touch one another or are in contact. Thus there is coincidence between the surface of a cast and the surface of the mould which contains it. And in like manner two lines or points may actually touch one another and coincide.

But besides this, there is an imaginary coincidence frequently appealed to by geometers; which is, when it can be shown that coincidence *would* take place, if the objects in question could be applied to one another without bar of corporeal substance as intimated above. And in the same manner that *two* objects may be thus imagined to coincide, may *three*, or any other number.

X. Points which do not coincide, are said to be *distant.*

XI. Two points A and B are said to be *equally distant with* two others C and D, or to be *at a distance equal to* the distance of C and D; when if A and C were applied to one another, B and D might also be applied to one another at the same time.

XII. Two or more points, as B, C, D, are said to be *equidistant* from another point A, when B and A, C and A, D and A, are equally distant each pair with the other.

XIII. Figures in general, as being things capable of being compared in point of greatness with objects of their own kind, [*that is to say, solids with solids, surfaces with surfaces, &c.*], are called *magnitudes.*

XIV. Magnitudes which if their boundaries were applied to one another, would coincide; or might be made capable of doing so, by a different arrangement of parts; are called *equal.*

Equal magnitudes may be divided into,

1. Those whose boundaries can actually be applied together and be in contact in every part at once; as a cast to its mould, a stamp to its impression, or one side of an indenture to the other. Which are called *reciprocals.*
2. Those whose boundaries it can be shown *would* coincide, if they could be applied to one another without bar of corporeal substance,—though they cannot be actually applied, by reason of such bar. Which are called *fac-similes.*
3. Such as might be reduced to reciprocals or fac-similes, by a different arrangement of parts. Which are called *equivalents.*

XV. A magnitude is said to be *greater* than another by a certain magnitude, when this last-mentioned magnitude being taken from the first, the remainder is equal to the second. And a magnitude is said to be *less* than another by a certain magnitude, when this last-mentioned magnitude being added to the first, the sum is equal to the second.

See Note.

XVI. Magnitudes are said to be *given*, when equals to them can be assigned.

See Note.

XVII. The science which treats of the relations and properties of magnitudes, is named *Geometry*.

XVIII. An assertion which it is proposed to show to be true, is called a *Theorem*. An operation which it is proposed to show how to perform, is called a *Problem*. And both are called by the title of *Propositions*.

XIX. A Proposition dependent on some other that has been previously established; but which, by reason of the simplicity of the steps required to connect it with the former, or other causes, it is convenient to state without the formalities of a separate Proposition; is called a *Corollary*.

See Note.

XX. A Proposition of which the establishment is necessary to the establishment of some other; but which, by reason of its being taken from a different branch of science, or other causes, it is convenient to distinguish by a separate title, rather than to number among the Propositions in the ordinary way; is called a *Lemma*.

See Note.

XXI. A remark or observation connected with something that has gone before, is called a *Scholium*.

XXII. When a thing is said to be so and so *by the hypothesis*, the meaning is, that its being so and so, makes part of the original supposition or statement of the question in hand.

Thus, if the proposition is, that if two magnitudes be equal, some particular result shall follow, as, for instance, that the sum of them shall be equal to some third magnitude; one of these magnitudes may at any period of the operations in hand be avouched to be equal to the other *by the hypothesis*. For that they are equal, is the original supposition or groundwork on which the whole question proceeds.

XXIII. When a thing is said to be so and so *by construction*, the meaning is, that it has at some previous period of the operations in hand, been made to be what it is said to be.

Thus, if in the course of the operations in hand, some magnitude has been cut off equal to another magnitude; these two magnitudes may at any time afterwards be avouched to be equal *by construction*. For one of them has been specially constituted and constructed equal to the other.

XXIV. When a thing is said to be so and so *by parity of reasoning*, the meaning is, that what has been shown to be true in some previous instance, may by taking the same steps be shown to be true in this.

Thus, if it has been shown that because two particular magnitudes are each equal to a third magnitude they are equal to one another; and if the same steps can be applied in any other instance; it may be avouched that *by parity of reasoning* any other two magnitudes which are equal to a third, are equal to one another. For it is open to be demonstrated by the same steps.

When the parity of reasoning is extended to all instances to which the proposition is capable of being applied, [or in other words, to all which come under its terms], the proposition is said to be established *universally*. And a proposition so established is called a *universal* proposition.

XXV. A conclusion the truth of which is shown to be so connected with the truth of some preceding position or statement, that the preceding cannot be true, without the other being true also; is called a *consequence*.

XXVI. When such preceding position is not known to be true, but is only assumed to be true for the purpose of trying its truth or falsehood by examination of the consequences, it is called an *assumption*. And because the assumption cannot be true without the consequence being true also; if the assumption is found to involve a false or impossible consequence, the assumption cannot be true. See Note.

This is the *rationale* [or reasonable principle] of the mode of proof improperly called *reductio ad absurdum*, 'reduction to an absurdity.'

XXVII. The establishment of some universal proposition, is called a *demonstration*.

The peculiar object of a demonstration, is to display the reasons which make a certain proposition necessarily true in *all* the instances that can by possibility come under its terms; in contradistinction to such propositions as are only known to be true in the instances in which experiment has actually been made.

For example, a cooper knows that in every instance where he has tried it, the distance that went exactly round the rim of his cask at six times, was the distance to be taken in his compasses in order to describe the head that would fit. But he does not know the reasons why this will necessarily be the case, not only in the instances which he *has* tried, but in all which he *has not* tried also. And to supply these reasons, is the object of Euclid's 15th Proposition of the Fourth Book.

XXVIII. The conduct of a regularly-ordered demonstration in geometry, divides itself into five parts, which succeed one another, and are named, as follows :—

The first conveys a statement of the universal proposition to be finally established. Which is called the *enunciation.*

The second presents to the eye, or to the imagination, a particular instance, in respect of which the truth of the proposition is to be established; with an understanding always, that nothing shall be done in respect of this particular instance, which would not be equally applicable to any other particular instance that could be substituted. Which is called the *specification.*

The third performs, or supposes to be performed, such operations [*as, for example, drawing or dividing lines, describing figures, &c.*] as are to be made use of in the further progress of the demonstration. Which is called the *construction.* Sometimes the construction, or part of it, precedes the specification. And sometimes no construction is required.

The fourth derives from all that has preceded, the establishment of the proposition in the instance presented in the specification. Which is called the *proof.*

The fifth extends the conclusion to all instances that come under the terms conveyed in the enunciation. Which is called the *generalization.*

XXIX. When all the instances to which a proposition may be applied, cannot be included under one specification or one construction, the proposition is said to divide itself into two or more *Cases;* which may in fact be considered as so many distinct propositions, each of which has, or is capable of having, its separate enunciation, specification, construction, &c., the which, taken together, amount to the establishment of the universal proposition.

XXX. What is called the *converse* of a proposition is, when the premises and the conclusion are made to change places, and the proposition so arising is presented as a new proposition.

For example, if the original proposition is, that magnitudes which are equal to the same, are equal to one another; the converse of this proposition is, that magnitudes which are equal to one another, are equal to the same.

XXXI. What is called the *negative* of a proposition is, when a negation is inserted both in the premises and the conclusion, and the proposition so arising is presented as a new proposition.

For example, if the original proposition is, that if of equals one be greater than some thing else, the rest are severally greater than the same; the negative of this proposition is, that if of equals one be *not* greater than some thing else, the rest are severally *not* greater than the same.

XXXII. What is called the *contrary* of a proposition is, when both the premises and the conclusion are altered, not merely by the insertion of a negation, but by being changed into something of a positively contrary kind.

For example, if the original proposition is as in the last article; the contrary of this proposition is, that if of equals one be *less* than some thing else, the rest are severally *less* than the same.

Scholium. Neither the *converse*, the *negative*, nor the *contrary* of any proposition, is to be admitted to be true, till it has been demonstrated as a distinct proposition. For till this be done, it is impossible to know whether it is true or not. For example, it is shown in the sequel, that if one angle of a triangle is greater than a right angle, the other two are necessarily less than right angles. The converse of which is, that if two angles of a triangle are less than right angles, the other is necessarily greater than a right angle. The negative is, that if one angle of a triangle is *not* greater than a right angle, the other two are *not* less than right angles. The contrary is, that if one angle of a triangle is less than a right angle, the other two are greater than right angles. Every one of which is totally and absolutely false.

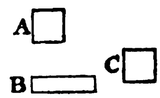
(The Nomenclature of the First Book is continued without interruption of numbers in page 50, after the Intercalary Book.)

INTERCALARY BOOK.

PROPOSITION I.

Theorem.—*Magnitudes which are equal to the same, are equal to one another.*

Let A and B be two magnitudes, each of which is equal to C. A and B are equal to one another.



* I. Nomenclature 14. For because A is equal* to C, if their boundaries were applied to one another A and C would coincide; or else might be made capable of doing so, by a different arrangement of parts. And because B is equal to C, in like manner would B and C. But because A and B would each coincide with C; if the boundaries of both could be applied to those of C at once, A and B would coincide with one another; wherefore A and

† I. Nom. 14. B are equal †. And in the same manner if the magnitudes equal to C were more than two.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, magnitudes which are equal to the same, are equal to one another. Which was to be demonstrated.

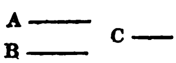
Corollary 1. If of equals, one be equal to some thing else, the rest are severally equal to the same.

Let A and B be equal, and let B be equal to C. A shall also be equal to C.

For A is equal to B, and C is equal to B; therefore (by Prop. I. above) A and C are equal.

Cor. 2. If of equals, one be greater, or less, than some thing else; the rest are severally greater, or less, than the same. Or if some thing be greater, or less, than one; it is greater, or less, than each of the others also.

Let A and B be equal, and let B be greater than C. A shall also be greater than C.

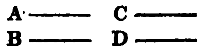


‡ I. Nom. 15. For since B is greater‡ than C, a certain magnitude may be taken from B, and the remainder be equal to C; or B is equal to the sum of the magnitude equal to C, and of a certain magnitude besides. But because A and B are equal, A (by Cor. 1) is also equal to the sum of the magnitude equal to C, and of the certain magnitude besides. Therefore a certain magnitude may be taken

* I. Nom. 15. from A, and the remainder be equal to C; or A is greater* than C. And in a similar manner if one were less.

Or if C is less than B, it is less than A also. For B is greater than C; therefore (as has been proved above) A also is greater than C; that is, C is less than A. And in a similar manner if C were greater than B.

Cor. 3. Magnitudes which are equal to equals, are equal to one another.

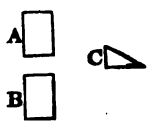


Let A be equal to B, and C to D; and let B and D be equal. A and C shall also be equal.

For A is equal to B, and D is equal to B; therefore (by Prop. I.) A is equal to D. But C is also equal to D; therefore (by Prop. I.) A and C are equal.

Cor. 4. Of magnitudes, if to equals be added the same, the sums are equal.

Let A and B be equal magnitudes, to each of which, one after the other, is added another magnitude C. The sum of A and C is equal to the sum of B and C.



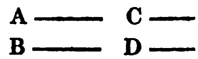
First Case; If A and B are such that on their boundaries being applied to one another they would coincide, and C be added to each in such manner as would then and there likewise coincide; the whole magnitude which is the sum of A and C, if its boundaries were applied to those of the magnitude which is the sum of B and C, would coincide with it, and therefore is equal* to it.

•I. Nom. 14.

Second Case; If either C be added in some other manner than as above; or if A and B would not coincide with one another, but are only such as by a different arrangement of parts might be made to do so; the magnitudes which are the sums, may be made capable of coinciding by merely altering the arrangement of parts, and therefore they are equal †.

†I. Nom. 14.

Cor. 5. If equals be added to equals, the sums are equal.



Let A be equal to B, and C to D. The sum of A and C is equal to the sum of B and D.

For because A is equal to B, the sum of A and C (by Cor. 4) is equal to the sum of B and C. And because C is equal to D, the sum of B and C (by Cor. 4) is equal to the sum of B and D. Therefore (by Cor. 1) the sum of A and C is equal to the sum of B and D. And in like manner if other equal magnitudes be added to these.

Cor. 6. If unequals be added to equals, the sums are unequal. And that sum is greatest, in which the unequal was greatest.

For, of the unequals, one is greatest; that is, it is equal to the other and a certain magnitude besides. But if to the equals were added *equals*, the sums (by Cor. 5) would be equal; therefore because to one is added a certain magnitude besides, that sum is made greater.

So also [*with slight verbal alterations*] if unequals be added to *the same*.

COR. 7. If equals be taken from equals, the remainders are equal.

For, if this be disputed, let it be assumed that one remainder is greater than the other. Add each to the equals that were before taken away; and because to equals unequals are added, that sum in which the unequal was greatest, (by Cor. 6) must be greater than the other. Which is impossible; for the things by the hypothesis were equal to begin with. The assumption*, therefore, which involves this impossible consequence, cannot be true; or one remainder cannot be greater than the other. And because one is not greater than the other, they are equal.

*I. Nom. 26.

So also [*with slight verbal alterations*] if equals be taken from *the same*, or *the same* from equals.

COR. 8. If equals be taken from unequals, the remainders are unequal. And that remainder is greatest, in which the unequal was greatest.

For, of the unequals, one is equal to the other and a certain magnitude besides. But if the equals were taken from *equals*, the remainders (by Cor. 7) would be equal; therefore, because to one of the objects from which subtraction is made, is added a certain magnitude besides, that remainder is made greater.

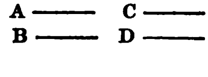
So also [*with slight verbal alterations*] if from unequals be taken *the same*.

COR. 9. If unequals be taken from equals, the remainders are unequal. And that remainder is least, in which the unequal was greatest.

For if, instead of the unequals, magnitudes equal to the *smallest* of them were taken from the equals, the remainders (by Cor. 7) would be equal. But because from one is taken a certain magnitude besides, that remainder is made less.

So also [*with slight verbal alterations*] if unequals be taken from *the same*.

COR. 10. Magnitudes which are *double* of the same or of equal magnitudes, are equal to one another. And so if, instead of the *double*, they are the *treble*, *quadruple*, or any other equimultiples.



Let A and B be equal magnitudes. The double of A is equal to the double of B.

For, to take the double of A, is to add to it a magnitude C that is equal to A; and to take the double of B, is to add to it a magnitude D that is equal to B. But because A and B are equal to one another, (by Cor. 3) C and D are equal to one another. Therefore (by Cor. 5) the sum of A and C is equal to the sum of B and D.

And in like manner if to the sum of A and C be added another magnitude equal to A, and to the sum of B and D another magnitude equal to B. And so on.

COR. 11. The *double* of a greater magnitude is greater than the *double* of a less. And so of any other equimultiples.

If A be greater than B, the double of A is greater than the double of B.

A ——— C ———
B —— D ——

For, let there be taken a magnitude C, equal to A; and a magnitude D, equal to B. If to C and D severally, be added B, the sum of B and C (by Cor. 6) shall be greater than the sum of B and D. And if to A and B severally, be added C, the sum of A and C shall (by Cor. 6) be greater than the sum of B and C; still more shall it be greater than the sum of B and D. But the sum of A and C is the double of A; and the sum of B and D is the double of B. Wherefore the double of A is greater than the double of B. And in like manner if to the double of A be added a *third* magnitude equal to A, and to the double of B a *third* magnitude equal to B. And so on.

COR. 12. Magnitudes which are *half* of the same or of equal magnitudes, are equal to one another. And so if, instead of the *half*, they are the *third*, *fourth*, or any other equipartites.

For, if this be disputed, let it be assumed that one is greater than the other. But if it be greater, its *double* must (by Cor. 11) be greater than the *double* of the other; which is impossible, for it is
I.Nom.26. equal. The assumption, therefore, cannot be true; or the one magnitude, that was doubled, is not greater than the other; and because one is not greater than the other, they are equal. And in like manner in respect of magnitudes which are the *third*, *fourth*, &c.

COR. 13. The *half* of a greater magnitude is greater than the *half* of a less. And so of any other equipartites.

For, if it be not greater, it must either be equal or less. It cannot be equal, for then (by Cor. 10) its *double* would be equal to the *double* of the other; and its *double* is not equal, for it is greater. And it cannot be less, for then (by Cor. 11) its *double* would be less than the *double* of the other; and its *double* is not less, for it is greater. But because it is neither equal nor less, it is greater. And in like manner in respect of the *third*, *fourth*, &c.

COR. 14. If the *doubles* of two or more magnitudes be added together, the amount is *double* of the sum of the magni-
See Note. tudes. And so of any other equimultiples.

Let A and B be two magnitudes. The *double* of A, added to the *double* of B, is *double* of the sum of A and B.

A ——— C ———
B —— D ——

For, let C be another magnitude equal to A, and D to B. Because A is equal to C, the sum of A and C is equal to the *double* of A; and for the like reason, the sum of B and D is equal to the *double* of B. Therefore (by Cor. 5), the *double* of A, added to the *double* of B, is equal to the sum of A, B, C, and D. But (by Cor. 5), the sum of A and B is equal to the sum of C and D ; therefore the sum of A, B, C, and D, is equal to *double* the sum of A and B. Wherefore (by Cor. 1), the *double* of A, added to the *double* of B, is equal to *double* the sum of A and B.

If the magnitudes are more than two, then to each of them is to be taken an equal, as before.

If the same is to be proved of the *trebles*, *quadruples*, &c., then to each magnitude are to be taken *two, three*, &c. magnitudes, equal to it.

Cor. 15. If there be two unequal magnitudes, and from the *double* of the greater be taken the *double* of the less; the remainder is *double* of the difference of the magnitudes. And so of any other equimultiples.

See Note.

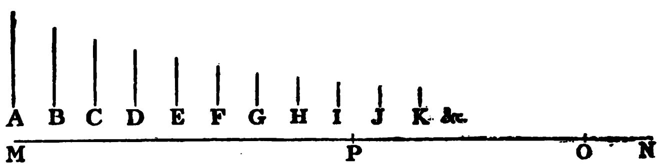
Let A and B be two magnitudes; of which A is the greater. If from the *double* of A be taken the *double* of B, the remainder is *double* the difference of A and B.

For A the greater, is equal to the sum of B the less, and of the difference between A and B. Therefore (by Cor. 14) the *double* of A, is equal to the *double* of B, added to the *double* of the difference between A and B. And if from each of these equals be taken the *double* of B, (by Cor. 7) the *double* of A diminished by the *double* of B, is equal to the *double* of the difference between A and B.

And in like manner if instead of the *doubles*, were taken the *trebles*, *quadruples*, &c.

Cor. 16. If there be magnitudes which, being added together to any number however great, cannot surpass a given magnitude; these magnitudes cannot be all equal to one another.

See Note.



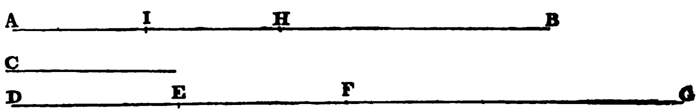
For, let A, B, C, D, E, F, G, H, I, &c. be magnitudes which being added one to another to any number however great, cannot surpass a given magnitude MN. But if so, either MN is the smallest magnitude which they cannot pass, or it is not. And if it is not, then there is some magnitude which may be cut off from it, and the remainder be a magnitude which they cannot pass. Wherefore there will be some remainder MO such, that it is a limit which they cannot pass, but if any smaller magnitude be substituted they *shall* pass it; for if they would not pass such smaller magnitude, the difference might be cut off. And because A, B, C, D, E, F, G, H, I, &c. may be taken till they surpass any magnitude that is

smaller than MO, they may be taken till they surpass the half of MO. Let, then, a certain number (as for instance A, B, C, D, and E) be the magnitudes which are together greater than half MO; or of which the sum MP is greater than PO the remainder of MO. Because A, B, C, D, and E are together greater than PO, they are together greater than the sum of all the remaining magnitudes F, G, H, I, &c. however many, that may be taken afterwards; still more are they greater than the sum of only an equal number of those magnitudes. But if the magnitudes were all equal to one another, the magnitudes A, B, C, D, and E would (by Cor. 10) be together equal to the sum of an equal number of the magnitudes which should be taken afterwards; and they are *not* equal to this,
I.Nom.26. for they are greater. The assumption therefore cannot be true; or the magnitudes A, B, C, D, E, F, G, H, I, &c. cannot be all equal to one another.

COR. 17. Any given magnitude may be multiplied [*that is to say, magnitudes equal to it may be added one to another*], so as at length to become greater than any other given magnitude of the same kind which shall have been specified.

For if not, there would be a magnitude which they cannot pass; and consequently (by Cor. 16) the magnitudes would not be all equal to
†I.Nom. 26. one another. Which cannot be, for they *are* equal. The assumption† therefore cannot be true; or there is not a magnitude which they cannot pass. That is, they may be added one to another so as at length to become greater than any other given magnitude of the same kind which shall have been specified.

COR. 18. If from the greater of two proposed magnitudes be taken not less than its half, and from the remainder not less than its half, and so on; there shall at length remain a magnitude less than the least of the proposed magnitudes.



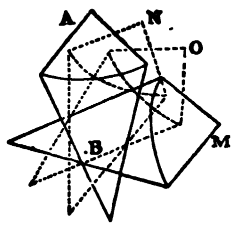
For if AB and C be the two proposed magnitudes, of which AB is the
See Note. greater, then because (by Cor. 17) C may be multiplied [*that is to say, magnitudes equal to it may be added one to another*] so as at length to become greater than AB, C may be doubled successively [*that is to say, over and over*] so as at length to become greater than AB; for on each doubling is added a portion as great or greater than the portion added by the simple addition of C. Let C be doubled successively till the result, as DG, is greater than AB. If then from AB be taken its half BH, and from the remainder HA be

taken its half HI, and so on till the half be taken successively as many times as C was doubled successively to make DG; the last remainder AI will be less than C. For as many times as C is contained in DG, so many times is AI contained in AB; but AB is less than DG; therefore AI (by Cor. 13) is less than C. And if from AB or any of the remainders were taken *more* than the half; only the more would the last remainder AI be less than C.

PROPOSITION II.

THEOREM.—*A hard body may be turned about any one point, or about any two points, in it; such point or points remaining unmoved.*

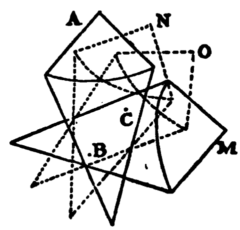
Let A be a hard body.



First Case; the body A may be turned about any *one* point in it, as B, such point remaining unmoved.

For the body may be placed in another situation, as M, such that the point B occupies the same place in fixed space as it did when the body was in the situation A. Also between the situations A and M, the body may be placed in other situations as N and O, in all of which the point B shall occupy the same place as when the body was in the situation A.

And because the number of such situations, with their nearness to one another, may be increased without limit; it may be increased till the body is moved continuously from the situation A to the situation M, the point B preserving ever the same place, that is to say, remaining unmoved. And in like manner, the body may be moved continuously to any other situation which is such that the point B preserves the same place; or about any other point in it than B.



Second Case; the body A may be turned about any *two* points in it, as B and C, both these points remaining unmoved.

For because the body is* a hard† body, the distance from the point B in it to the point C will be unaltered in every situation of the body. Wherefore the body may be placed in another

* By the Hypothesis.
† I. Nom. 3.

situation, as M, such that the points B and C respectively occupy the same places in fixed space as they did when the body was in the situation A. And for the same reason, between the situations A and M the body may be placed in other situations as N and O, in all of which the points B and C shall respectively occupy the same places as when the body was in the situation A.

And because the number of such situations, with their nearness to one another, may be increased without limit; it may be increased till the body is moved continuously from the situation A to the situation M, the points B and C preserving ever the same places, that is to say, remaining unmoved. And in like manner, the body may be moved continuously to any other situation which is such that the points B and C preserve the same places respectively; or about any other two points in it than B and C.

And by parity of reasoning, the like may be proved of every other hard body. Wherefore, universally, a hard body may be
See Note. turned &c. Which was to be demonstrated.

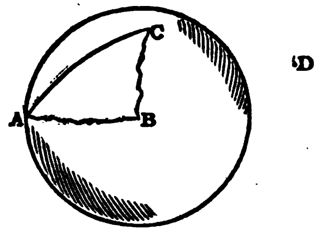
Cor. Any solid, surface, line, or figure, may be turned about any one point, or about any two points, in it; such point or points remaining unmoved.

For it is supposed to be represented on a hard body which may be so turned.

PROPOSITION III.

Problem.—*To describe a solid, all the points in whose surface shall be equidistant from an assigned point within, and at a distance equal to the distance of any two points that have been assigned.*

Let A and B be the two assigned points, in a hard body of any kind; and let B be the point from which all the points in the surface of the required solid are to be equidistant and at a distance equal to the distance of A and B.



The point B remaining at rest, let A describe a line of any kind AC by the hard body in
Interc. 2. which are the points A and B being turned about B; and let the line AC be expressed on a hard body in which the point B is also situate. By the turning of this last-mentioned hard body about B, let the line AC be made to turn about the fixed point B; and

let the turning be continued in various directions as called for, until the line AC shall have been applied to the entire surface of a solid [as would be done if it were employed to scoop out a hollow figure from the interior of some yielding substance like plaster]. There shall be described the solid required.

* By Construction. For, the point B remaining at rest, the point A can* be applied to every point in the line AC, in any of its situations.

† Constr. Also every point in the surface of the described solid, has† had some point in the line AC applied to it. Wherefore the point A can be applied to every point in the described surface, the point B remaining at rest; consequently each point in the surface

‡ 1.Nom.11. and B, are equally distant‡ with A and B, or all the points in

* 1.Nom.12. the surface are equidistant* from B. Which was to be done.

And by parity of reasoning, the like may be done in every other instance.

Cor. 1. Instead of being described about B, the solid, if required, may be described about any other assigned point as D.

For the hard body in which are the points A and B, may be moved till the point B is applied to D; and the solid described as before.

Nomenclature.—A solid figure of which all the points in the surface are equidistant from a certain point within, is called a *sphere*. And such point within, is called the *centre* of the sphere; and the distance from it to every point in the surface, is called the *central distance*. Spheres are said to *touch* one another, which meet but do not cut one another. Spheres described about the same centre, are called *concentric*.

Cor. 2. A sphere may be described about any centre, and with a central distance equal to the distance of any two points that shall have been assigned.

For it may be done as in Cor. 1 above.

Cor. 3. When a sphere is described as above, the interior sphere, and the hollow one formed by the substance out of which

† 1.Nom.14. it may be supposed to be taken, are reciprocals†.

For their boundaries are in contact in every part at once.

Cor. 4. The surfaces of concentric spheres either coincide throughout or not at all.

‡ Constr. For if they coincide in one point, they will‡ coincide throughout.

Cor. 5. If the centres of two spheres coincide, and their central distances are equal; their surfaces will coincide throughout. And if their central distances are not equal, their surfaces will not coincide at all, but one be interior to the other.

For (by Cor. 4 above) their surfaces either coincide throughout or not at all. But if their central distances are equal, then two points on the surfaces may be made to coincide, and the surfaces (by Cor. 4 above) will coincide throughout.

And if their central distances are not equal, no two points in the surfaces can coincide; for if they did, the central distances would be equal.

Cor. 6. Spheres having equal central distances, are equal. And equal spheres, have equal central distances.

For if spheres having equal central distances, have their centres applied together, their surfaces (by Cor. 5 above) will coincide throughout; wherefore the spheres are equal*.

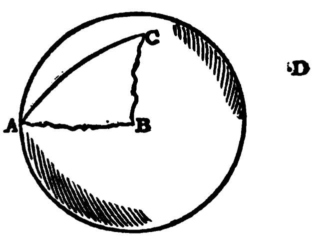
*I.Nom.14.

And if equal spheres have not equal central distances, they must have unequal; wherefore, if their centres be applied together, their surfaces (by Cor. 5) will not coincide at all, but one be interior to the other; that is to say, one sphere will be greater† than the other; which is impossible, for they are equal. The central distances, therefore, cannot be unequal; that is, they are equal.

†I.Nom.15.

PROPOSITION IV.

Theorem.—*If a sphere be turned in any manner whatsoever about the centre which remains at rest, the sphere will be without change of place.*



Let the substantial sphere whose centre is B, be‡ turned in any manner whatsoever about the centre B which remains at rest. The sphere shall be without change of place.

‡ Interc. 2.

For if its* reciprocal be supposed to remain unmoved, any particular point in the surface of the substantial sphere as A, will† at all times be coincident with some point or other in the reciprocal; because they are equally distant from the centre. And in like

* Interc. 3. Cor. 3.

† Constr.

manner every other point in the surface of either sphere, will at all times be coincident with some point in the surface of the other; wherefore the two surfaces will be always everywhere coincident. And because the surface of the substantial sphere which is turned, is always everywhere coincident with that of the reciprocal which is without change of place; the substantial sphere is without change of place. So also of the hollow sphere, if it be turned while the substantial one which is its reciprocal remains unmoved.

And by parity of reasoning, the like may be proved of every other sphere. Wherefore, universally, if a sphere be turned &c. Which was to be demonstrated.

PROPOSITION V.

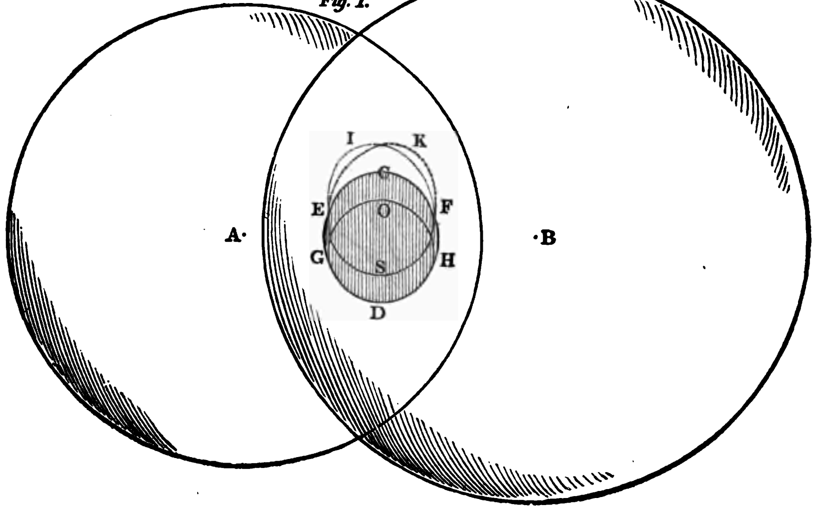
THEOREM.—*If two spheres touch one another externally, they touch only in a point.*

Let there be two spheres whose centres are A and B, touching one another externally. The one cannot touch the other in more than a single point at once.

For if this be disputed,

First Case; let it be assumed that they coincide in the surface CEGDHF in Fig. 1 below, and not elsewhere. But if so,

Fig. 1.

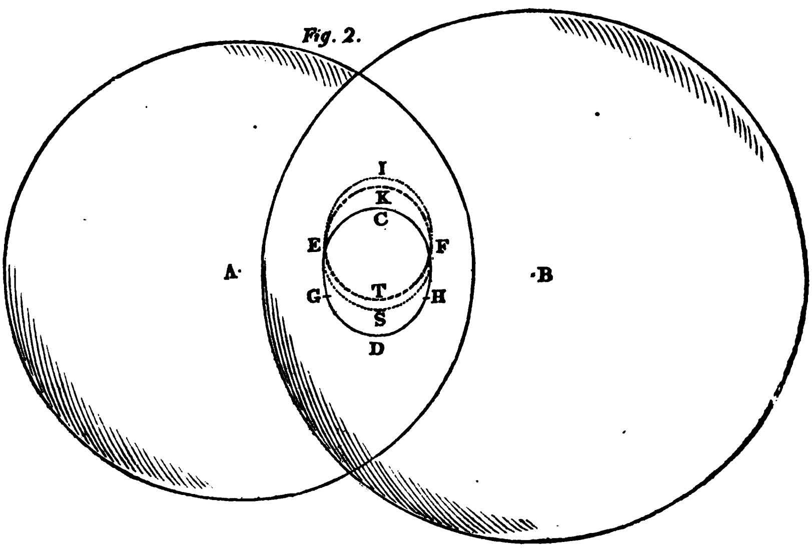


either or both of the spheres may be turned in any manner whatsoever about its centre, and they must still always coincide in the surface in fixed space CEGDHF and not elsewhere. For each of them will be* without change of place; wherefore they must at all times coincide in that surface and not elsewhere; for if they do not, one or both must have suffered change of place. Let then the sphere whose centre is A, be turned† about A, till the portion of its surface which was originally in the situation CEGDHF, is brought into the situation IESF; and because the spheres will still coincide in CEGDHF and not elsewhere, the portion CESF of IESF will coincide with the sphere whose centre is B, and the portion IECF will not. Let now the surface IESF be returned into its original situation (by turning the sphere back again about its centre A); and let the portion of CEGDHF with which CESF thereupon coincides, be OGDH. And after this, let the sphere be turned about its centre again, till the portion of its surface OGDH coincides with CESF as before; and let the sphere whose centre is B be turned about its centre till the portion of its surface OGDH coincides also with CESF, and let the remainder of its surface which formerly coincided with the surface of the other sphere in CEGDHF, be KECF. Wherefore the two surfaces which in the situation CEGDHF coincided entirely with one another, do now coincide in the portion CESF as before, but in their remaining portions IECF and KECF they do not. Which is impossible. For if the two surfaces are made to coincide entirely as in the situation CEGDHF, the portions IECF and KECF can in no way be made to cease coinciding and be separate while the remainders continue to coincide, other than by their particles, or some of them, being moved among themselves; which cannot be, for the bodies on which the surfaces are exhibited are hard‡ bodies. The assumption*, therefore, which involves this impossible consequence, cannot be true; or the two spheres cannot coincide in the surface CEGDHF. And in like manner may be shown that they cannot coincide in any other surface.

* INTERC. 4.
† INTERC. 2.
‡ I. Nom. 3.
* I. Nom. 26.
See Note.

Second Case; let it be assumed that they coincide in the self-rejoining line CEGDHF in Fig. 2. below; and not elsewhere. But if so, it may be shown as before, that they must continue to coincide in the same line in fixed space CEGDHF and not elsewhere, in whatsoever manner they may be turned about their respective centres. Let then the sphere whose centre is A, be turned about A, till the line on its surface which originally coincided with CEGDHF, is brought into the situation IESF; and on the sphere being returned to its former situation, let the points E and F in IESF fall on G and H. And after this let the sphere be turned round its centre again, till the points on its surface G and H coincide with E and F as before; and let the sphere whose centre is B be likewise turned about its centre till the points on its surface G and H coincide also with E and F, the line on its surface which originally coincided with CEGDHF being thereby brought into the situation KETF. Wherefore the two lines on the surfaces of the different spheres, which in the situation CEGDHF coincided entirely with each other, do now coincide in the points E and F only, and their portions which are on different sides of those points are so posited, that if the portion EKF of the one line lie *above* the portion EIF of the other line, the portion ETF of the first also lies above the portion ESF of

Fig. 2.



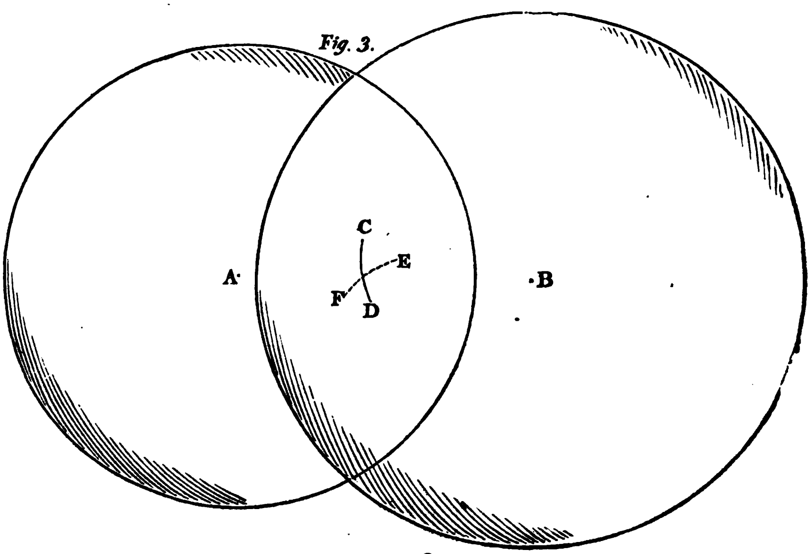
the second, and *vice versâ*. Which is impossible. For if the two lines IESF and KETF are made to coincide entirely as in the situation CEGDHF, the two portions EKF and ETF of KETF can in no way be turned about the points E and F so as to be *both of them above or both below* the line IESF or its portions, other than by their particles, or some of them, being moved among themselves; which cannot be, for the bodies on which the lines are exhibited, are hard* bodies. The assumption†, therefore, which involves the impossible consequence, cannot be true; or the two spheres cannot coincide in the self-rejoining line CEGDHF. And in like manner may be shown that they cannot coincide in any other self-rejoining line.

* I. Nom. 3.

† I. Nom. 26.

Third Case; let it be assumed that they coincide in the line CD in Fig. 3. below, which is not a self-rejoining line; and not elsewhere. But if so, let the two spheres be united as one body [as may be supposed to be done by their being imbedded in one inclosing body of hard, and for convenience, transparent matter], and let this body be turned always in the same direction about the points A and B in it which remain at rest, till it returns into the situation from which it was first moved. And because one face of the line CD, as for instance

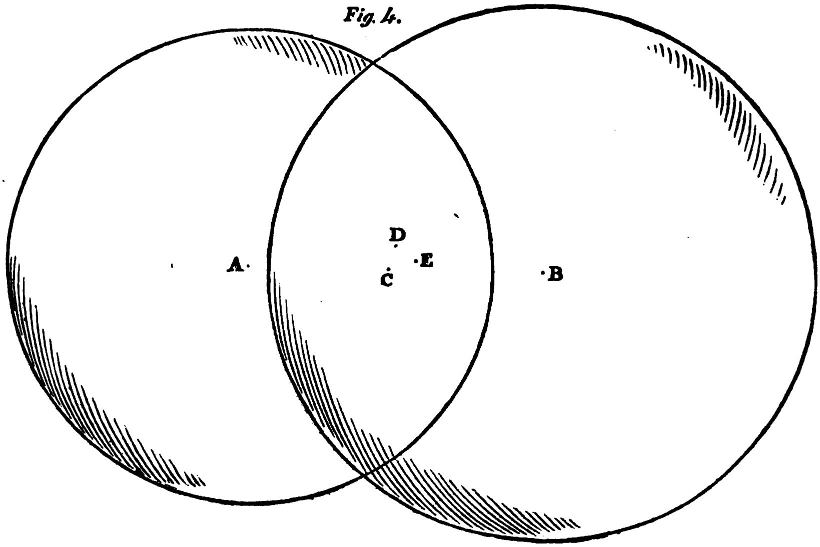
Fig. 3.



the face towards A, will be turned to all sides in succession; the line CD, which is not a self-rejoining line, must change its place; for it cannot have its face turned to all sides in succession and be without change of place. Let it therefore, at some period during the turning, be in the situation EF. But if the spheres do at first coincide in CD, it may be shown as has been done before, that during the turning they must always continue to coincide in CD and not elsewhere. But they also, at one period during the turning, coincide in EF.
* I.Nom.26. Which is impossible. The assumption*, therefore, which involves this impossible consequence, cannot be true; or the two spheres cannot coincide in the line CD. And in like manner may be shown that they cannot coincide in any other line which is not a self-rejoining line.

Fourth Case; let it be assumed that they coincide in insulated points more than one, as C and D in Fig. 4. below; and not elsewhere. But if so, let the two spheres be united as one body, and turned about the points A and B, as before. But if during such turning, one of the points C and D remains at rest, then the other must revolve round it and change its place; and if neither of them remains at rest, then both must change their places. Let one of them, therefore, at some period during

Fig. 4.



the turning be in the new situation E. But if the spheres do at first coincide in C and D, it may be shown as has been done before, that during the turning they must always continue to coincide in C and D, and not elsewhere. But they also, at one period during the turning, coincide in E. Which is
* I. Nom.26. impossible. The assumption*, therefore, which involves the impossible consequence, cannot be true; or the two spheres cannot coincide in the insulated points C and D. And in like manner may be shown that they cannot coincide in any other insulated points more than one at a time.

But if the two spheres cannot coincide in any surface, nor in any line, nor in any insulated points more than one at a time; they can touch only in a point.

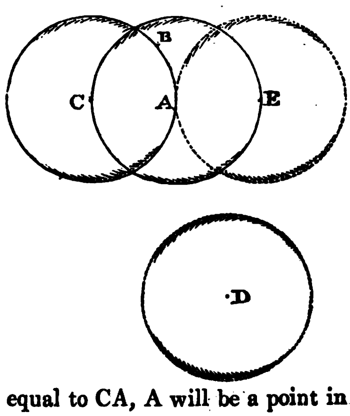
And by parity of reasoning, the like may be proved of any other spheres. Wherefore, universally, if two spheres touch one another externally, they touch only in a point. Which was to be demonstrated.

Cor. Two spheres may be applied to one another externally, so that they shall touch in a point assigned on the surface of one, or of each severally.

For in whatever manner the spheres are made to touch externally, (by Prop. V. above) they will touch in a point. Whereupon either or each of them may be turned about its centre
† Interc. 4. without† change of place, till any assigned point on its surface be made to coincide with the point of contact.

PROPOSITION VI.

Theorem.—*A sphere cannot have more than one centre.*



Let there be a sphere described about the centre A. No other point, as B, can also be a centre [*that is, can also be equidistant from all the points in the surface of the sphere.*]

In the surface of the sphere described about A, take any point as C; and about C as a centre, with a central dis-
‡ Interc. 3. Cor. 2. tance equal to CA, describe‡ another sphere. Because the central distance of this sphere is equal to CA, A will be a point in

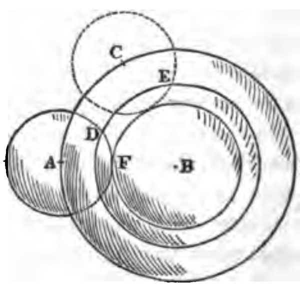
its surface; for if not, its central distance would not be equal to CA. If now the point B is not in the surface of the sphere whose centre
* INTERC. 2. is C, let the sphere whose centre is A be turned* about A till the
† INTERC. 3. Cor. 2. point B is in that surface. About any centre as D, describe† another sphere with a central distance equal to AC; and apply this sphere
‡ INTERC. 5. Cor. to the sphere whose centre is C, in such manner‡ that they shall touch in the point A. Whereupon the point D in this last-described sphere will be found in the surface of the sphere whose centre is A; for if not, their central distances would not be equal. Let it be found then in E. But because B is in the surface of the sphere whose centre is C, and the two spheres whose centres are C and
INTERC. 5. E touch only in the point A, B is not in the surface of the sphere whose centre is E; wherefore the distance BE is not
† Constr. equal to the distance AE, or AC. But BC is† equal to AC; wherefore BC is not equal to BE; for if they were equal, BE
‡ INTERC. 1. Cor. 1. would also be‡ equal to AC, and it is not. And because BC is not equal to BE, the point B is not equidistant from all the points in the surface of the sphere. And in like manner may be shown that any other point which is not A, is not equidistant from all the points in the surface of the sphere.

And by parity of reasoning, the like may be proved of every other sphere. Wherefore, universally, a sphere cannot have &c. Which was to be demonstrated.

PROPOSITION VII.

PROBLEM.—*A point being assigned outside a given sphere, to describe about it a sphere which shall touch the given sphere externally.*

[N.B. For brevity, a sphere may be named by the letter which is at its centre alone, when no obscurity arises therefrom.]



Let the sphere whose centre is A, be the given sphere; and B the assigned point outside. It is required to describe about B, a sphere which shall touch the sphere whose centre is A externally.

The given sphere and the point B being supposed imbedded in one continuous hard body, let the
INTERC. 2. whole be turned about B, in such sort that the sphere A shall be moved to a new situation having no

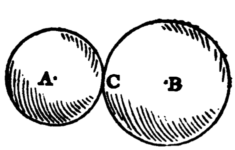
part in common with the old, as for instance to the situation of the sphere whose centre is C; and let the line then described by the point A from A to C, be afterwards turned about B in various
* INTERC. 3. directions as called for, till a sphere is described* with a central distance equal to BA. In the surface of the sphere A, in that portion of it which is within the sphere so described about B, let any point be taken, as D. But when the point A (in the process of describing a sphere about B as above) was made to describe a line, the point D at the same time described a line of some kind from D to some other point E, inasmuch as every point in the sphere whose centre is A was moved from its place; and by the after turning of this line about B, (if the turning was continued in various directions as called for,) was described a sphere with a central distance equal to BD; and in the same manner with any other point in the sphere A. By the motion of every point therefore in the sphere A, was described a sphere about the centre B. And because the concentric spheres so described either
† INTERC. 3. Cor. 4. coincide† throughout their whole surfaces or not at all, there is some one (whether traced by the motion of a single point or of more) which has none other of the concentric spheres within it either totally or in part. [As would be mechanically exhibited, if the space about B were filled with some yielding substance like plaster, out of which a sphere was to be formed through the scraping away of the greatest possible quantity of matter by the turning of the sphere A in different directions about B.] Let then this sphere, which has none other of the concentric spheres within it, be the sphere BF. It touches the sphere whose centre is A, as required.

For the sphere A, during its turning about B, is or may be applied to every point in the surface of the sphere BF (as by Prop. III); wherefore they meet. Also it does at no time cut the sphere BF; for if any point in the sphere A ever fell within the sphere BF, there would be traced by it a concentric sphere interior to the sphere BF, and there is not. Wherefore because it always meets
‡ INTERC. 3. Nom. the sphere BF but does not cut it, it always touches‡ it. And because the spheres A and BF touch one another in every situation of the sphere A, they touch one another when the sphere A is returned to the situation from which it was first moved. Wherefore there has been described a sphere BF, touching the given sphere externally. Which was to be done.

And by parity of reasoning, the like may be done in every other instance.

PROPOSITION VIII.

THEOREM.—*If two spheres which touch one another externally, be turned as one body about the two points which are the centres and which remain at rest, the point of contact shall remain unmoved.*



Let the two spheres whose centres are A and B, touch one another externally in the point C; and in this situation let them be united as one body [by means of an inclosing body of hard matter, as be-
INTERC. 2. fore], and then be turned about the two points A and B which remain at rest. Their point of contact C shall remain unmoved.

Because the centre of each sphere remains at rest, each sphere
† INTERC. 4. is turned about its own centre; wherefore each sphere will† be without change of place. Hence their point of contact will at all times during the turning, coincide with the point in fixed space, in which the spheres touched one another before the turning began. For if it did not, the surfaces of the spheres would at one instant coincide in some point in fixed space and both of them pass through it, and at another instant they would not; which cannot be without one or both of the spheres having suffered change of place. And because the point of contact always coincides with one and the same point in fixed space, it remains unmoved.

And by parity of reasoning, the like may be proved of any other spheres. Wherefore, universally, if two spheres which touch one another externally, be turned as one body about the two points which are the centres and which remain at rest, the point of contact shall remain unmoved. Which was to be demonstrated.

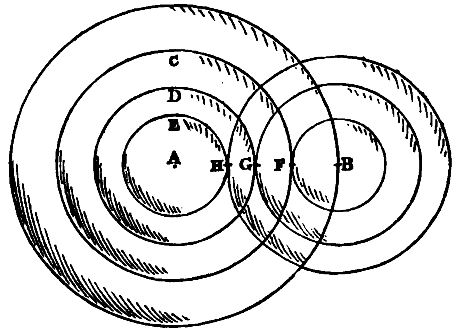
COR. If along with the substantial spheres, the inclosing body of hard matter is turned about the two centres; this inclosing body will also be without change of place, as respects the parts bounded by the hollow spherical surfaces.

For these hollow surfaces are always in contact with the surfaces of the substantial spheres, which are without change of place. Wherefore these hollow surfaces are also without change of place.

PROPOSITION IX.

PROBLEM.—*From one of two assigned points to the other, to describe a line, which being turned about its extreme points, every point in it shall be without change of place.*

Let the two assigned points be A and B. It is required to describe a line from B to A, which being turned about its extreme points B and A, every point in it shall be without change of place.



* INTERC. 3. Cor. 2. About the centre A, with the central distance AB, describe* a sphere; and about the same centre, with a central distance equal to the distance from A to any point C which is within the sphere last described, describe another sphere AC; and about the centre B,
† INTERC. 7. describe† a sphere touching the sphere AC externally, and let the
‡ INTERC. 5. single point‡ in which it touches it, be F. If then the hard body
* INTERC. 2. in which are all the spheres, be turned* about the points A and
† INTERC. 8. B; the point of contact F will† remain unmoved. And in like manner if about the centre A, in the same hard body be described other spheres successively less than the sphere AC and than each other, as AD, AE; and if about the centre B be described spheres respectively touching these, as in the points G, H; on the hard body in which are all the spheres being turned about A and B, the points of contact G, H, will also severally remain unmoved.

In this manner, therefore, between A and B may be determined any number of points that shall be desired, which on the hard body in which they are all situate being turned about A and B, will remain unmoved. Wherefore if one of the spheres that touch one another, as BF, be imagined to increase in magnitude and the other to decrease, till the sphere BF meets the point A, and *vice versâ*, (the spheres during such process remaining ever in contact); their point of contact will describe a line, every point in which, on the hard body in which the line is situate being turned about B and A, will remain unmoved, or be without change of place. Which was to be done.

And by parity of reasoning, the like may be done in every other instance.

NOMENCLATURE.—A line, which being turned about its extreme points, every point in it is without change of place, is called a *straight* line. A body or figure which is turned about two points in it that are also the extremities of a straight line, (inasmuch as the whole of the straight line remains without change of place) is said to be *turned round such straight line.* When from any point to any other point, a straight line is described or made to pass; the two points are said to be *joined.* If to a straight line addition is made at either end, in such manner that the whole continues to be a straight line, the original straight line is said to be *prolonged,* and the part added is called its *prolongation.* Any straight line drawn from the centre of a sphere to the surface, is called a *radius* of the sphere. A straight line drawn through the centre and terminated both ways by the surface, is called a *diameter* of the sphere.

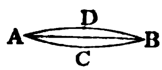
See Note.

COR. A straight line may be described or made to pass from any one point to any other point.

For it may be done by Prop. IX. above.

PROPOSITION X.

THEOREM.—*Between two points there cannot be more than one straight line.*



From the point A to the point B, let AB be the straight line described as by Prop. IX. No other line between the points A and B, can be a straight line.

For, if this be disputed, let ACB be some other line.

Because AB is the straight line described as by Prop. IX, if the body in which it is situate be turned* about the extreme points A and B, every point in the line AB will remain without change of place. Wherefore the line ACB will be turned round† the straight line AB and moved into some other situation, as ADB. And because the line ACB on being turned about its extreme points A and B is not without change of place, it is not a straight line; for if it was a‡ straight line, every point in it would be without change of place. And in like manner may be shown that any other line from A to B which is not AB, is not a straight line.

* INTERC. 2.

† INTERC. 9. Nom.

‡ INTERC. 9. Nom.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, between two points there cannot be more than one straight line. Which was to be demonstrated.

COR. 1. Two straight lines cannot inclose a space.

For if they did, between two points there would be two different straight lines. Which (by the Proposition above) cannot be.

COR. 2. Any portion of a straight line is also a straight line.

For inasmuch as when it is turned about its extreme points, every point in it is without change of place; every portion of it is at the same time turned about its own extreme points without change of place, and consequently is a* straight line.

* INTERC. 9. Nom.

COR. 3. The straight lines between equidistant points, are equal. And the extremities of equal straight lines are equidistant.

For if the points are equidistant†, they can be applied to one another; and if they are applied to one another, the straight lines between them (by Cor. 1 above) will coincide; and because they coincide, they are equal‡.

† I. Nom. 12.
‡ I. Nom. 14.

And if two straight lines are equal, their extremities may be made to coincide. For if when one extremity of each are made to coincide, the other extremity of the one cannot be also made to coincide with the other extremity of the other; then the extremity of one of them may be made to coincide with a point in the other which is not the extremity, and one of the straight lines is greater than the other; which cannot be, for they are equal. And because their extremities can be made to coincide, these extremities are equidistant.

COR. 4. All the radii of the same or equal spheres are equal. And spheres that have equal radii, are equal.

For all the points in the surface are* equidistant from the centre; wherefore (by Cor. 3 above) the straight lines from the centre to any points in the surface are equal.

* Constr.

And if two spheres have equal radii, the extremities of the radii (by Cor. 3 above) are equidistant. Wherefore the spheres have equal central distances; and consequently† are equal.

† INTERC. 3. Cor. 6.

COR. 5. The sphere that has the greater radius, is the greater.

For if *(See the Figure to Prop. IX)* a straight line be described from the point B in the surface of a sphere, to the centre A, no point in that straight line, between B and A, as I, can also be in the surface of such sphere, but is necessarily within it; because every such point is (by the Construction in Prop. IX) situate in the surface of some other sphere concentric with the first and within it. Therefore the concentric sphere that cuts AB in any point that is between B and A (that is to say, which has a radius less than AB) is less than the sphere whose radius is AB; or the sphere whose radius is AB, is the greater.

Cor. 6. A sphere may be described about any centre, and with a radius of any length that shall have been assigned.

Interc. 3. Cor. 2. For it has been shown how it may be described with a central distance equal to the distance between any two points; therefore it may be described with a central distance equal to the distance between the extremities of the proposed radius.

Cor. 7. If two spheres touch one another externally, the straight line which joins their centres shall pass through the point of contact.

For it is one of the points through which the straight line †Interc. 10. described as by Prop. IX, passes. And there can be† no other straight line between the centres than this.

Cor. 8. If the surfaces of two spheres pass through a point in the straight line which joins their centres; the spheres shall touch one another externally in that point, and no other point in the straight line shall be in the surface of either of the spheres.

For it is one of the points in which the various spheres by which the straight line is described as by Prop. IX, touch one another. And no other sphere can be described about either of ‡Interc. 3. Cor. 4. the same centres, whose surface shall pass through‡ the same point; nor any other point in the straight line (as described by means of a succession of concentric spheres in Prop. IX) be in the surface of either of the touching spheres.

Cor. 9. A straight line from the centre of a sphere to a point outside, coincides with the surface only in a point.

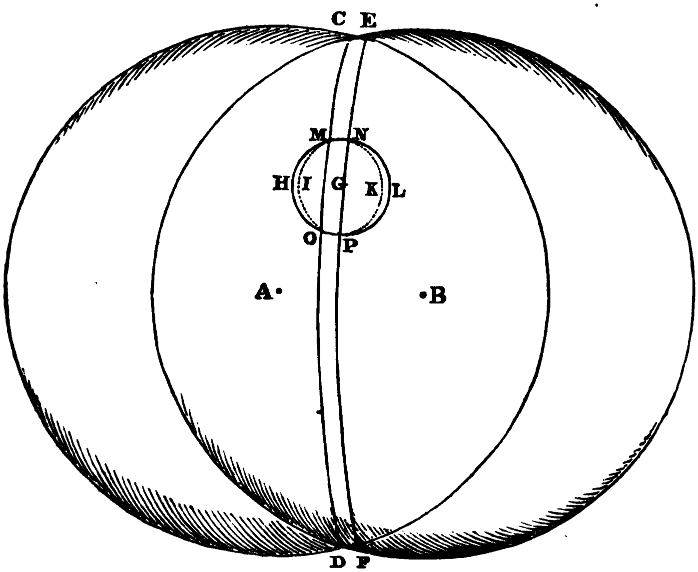
Interc. 7. For if about the point outside, another sphere be described touching the first; the straight line (by Cor. 7 above) will pass through the point of contact, and (by Cor. 8 above) no other point in the straight line will be in the surface of either of the spheres.

Scholium.—Henceforward, mention will never be made of *distances*, but always of *straight lines*; nor of *central distances*, but always of *radii*. And if one point be said to be *farther* or *nearer* than some other, it shall always be understood that the straight line drawn to it is *greater* or *less* than the straight line drawn to the other. See Note.

PROPOSITION XI.

THEOREM.—*If two equal spheres cut one another, their surfaces coincide only in one self-rejoining line.*

Let two equal spheres whose centres are A and B, cut one another. Their surfaces coincide only in one self-rejoining line.



Because the spheres cut one another, the coincidence of their surfaces is not in a point only; for if it was in a point only, the spheres would meet but not cut one another. And because the coincidence of the surfaces is not in a point, it is in a figure or figures of some kind, either surface or line. Also the figure or figures will be such, that if the two spheres are united as one
* INTERC. 2. body and turned* about the two points A and B which are the centres, the figure, or figures severally, shall be without change of place. For because each sphere is turned about its own
† INTERC. 4. centre, and is consequently† without change of place, the surfaces will at all times during the turning coincide in the whole of such figure or figures and not elsewhere. For if they did not, they would at one instant coincide in some point in fixed space and both of them pass through it, and at another instant they would not; which cannot be without one or both of the spheres having

suffered change of place. Such figure therefore (or each such, if there can be more than one) will be either a self-rejoining line or a belt of surface, which on the united body being turned about A and B, revolves on its own ground in fixed space without change of place. Take then any point C in which the surfaces of the two spheres at any time coincide; and on the united spheres being turned as before till the united body returns to the situation from which it set out, the point C, whether it be in a line or in a belt of surface, will describe a self-rejoining line as CD. [The other half of CD is supposed to be on the other side the spheres, and consequently is not represented.] The surfaces of the spheres shall coincide no where but in CD.

For if this be disputed,

First Case; let it be assumed that they coincide in a belt of surface having breadth, as that contained between the lines CD and EF; and not elsewhere. In EF take any point as G; and on the other side of CD take any point as H, in the surface of the hollow sphere whose centre is A; and by the turning of the substantial sphere which is its reciprocal, about the points A and G while the hollow sphere remains at rest, let the point H trace on the surface of the substantial sphere the line HOPLNM cutting EF in N. In like manner let the substantial sphere whose centre is B, be turned about the points B and G; and let the point N in its reciprocal which remains at rest, trace on its surface the line IOPKNM. There are therefore two surfaces HOPLNM and IOPKNM, which coincide with one another in the portion MOPGN, but in their remaining portions MOH, MOI, and NPL, NPK, they do not coincide. Let now the sphere whose centre is B, be turned about the point G, till its centre B is applied to the centre A of the other sphere (which can be done inasmuch as the radii of these equal spheres are* equal); whereupon the surfaces of the two spheres will† coincide throughout. And because the lines HOPLNM and IOPKNM are now traced on a common surface by a common point N, these lines will coincide with one another, and the surfaces inclosed by them will coincide throughout. Also if on the one sphere the points M, N, O, P, do not coincide with the same points on the other respectively, they may be made to do so by turning one of the spheres about the point G

*Interc. 10. Cor. 4.
†Interc. 3. Cor. 5.

and the common centre A, and the whole surfaces HOPLNM and IOPKNM will continue to coincide throughout. Which having been done, let the spheres be returned to their original situation. Wherefore the two surfaces HOPLNM and IOPKNM which, when the centres of the spheres were applied together, coincided entirely with one another, do now coincide in the portion MOPGN as before, but in their remaining portions MOH, MOI, and NPL, NPK, they do not. Which is impossible. For if the two surfaces coincide entirely in the one situation, the portions MOH, MOI, and NPL, NPK, can in no way be made to cease coinciding and be separate while the remainders continue to coincide, other than by their particles, or some of them, being moved among themselves; which cannot be, for the bodies on which the surfaces are exhibited are hard* bodies. The assumption†, therefore, which involves this impossible consequence, cannot be true; or the two spheres cannot coincide in the belt CDFE. And in like manner may be shown that they cannot coincide in any other belt; and this equally whether CD be one of its boundaries or not.

* I. Nom. 3.
† I. Nom. 26.

Second Case; let it be assumed that besides coinciding in CD they also coincide in some other self-rejoining line as EF, but not in the surface between. Whereupon may be shown by the same process as in the preceding case, that the two lines HOPLNM and IOPKNM which when the centres of the spheres are applied together coincide entirely with one another, do afterwards coincide in the points M, N, O, P as before, but in the rest of their extent they do not coincide. Which is impossible. For if the two lines coincide entirely in the one situation, the portions MHO, MIO, and NLP, NKP, can in no way be made to cease coinciding and be separate while in the points M, N, O, P the lines continue to coincide, other than by their particles, or some of them, being moved among themselves; which cannot be, for the bodies on which the lines are exhibited are hard‡ bodies. The assumption*, therefore, which involves this impossible consequence, cannot be true; or the two spheres cannot coincide in the self-rejoining line EF in addition to CD. And in like manner may be shown that they cannot coincide in any other self-rejoining line in addition to CD.

‡ I. Nom. 3.
* I. Nom. 26.

Third Case; let it be assumed that besides coinciding in the line CD, they coincide in some insulated point, or in some surface or line which is not a belt or a self-rejoining line. Then, if the two spheres be united as one body and turned about the points A and B as before, their surfaces must coincide in all the belt or self-rejoining line which will be described by such surface, line, or point during the turning; as was shown of the point C during the construction of the figure. But it has been shown that they cannot coincide in any belt or self-rejoining line in addition to CD. Wherefore they cannot coincide in any point, surface, or line as assumed.

But if the surfaces of the two spheres cannot coincide in any surface, line, or point, in addition to the line CD; they coincide only in CD.

And by parity of reasoning, the like may be proved of any other equal spheres. Wherefore, universally, if two equal spheres cut one another, &c. Which was to be demonstrated.

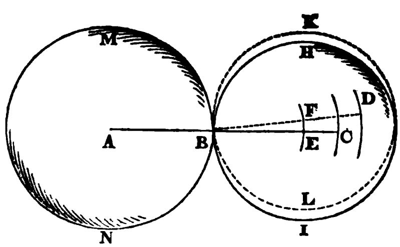
Cor. If two equal spheres with different centres, have a point in their surfaces in common, they will cut one another in one self-rejoining line passing through that point, or else touch one another externally in that point only.

For their surfaces cannot coincide throughout; because then
Interc. 6. they could not have different centres. And if they cut, their surfaces (by Prop. XI above) will coincide only in one self-rejoining line; wherefore any point which they have in common, must be in that line. And if they do not cut, but still have a
†Interc. 5. point in common; they touch in that point, and† that only.

PROPOSITION XII.

See Note. Theorem.—*Two straight lines, which are not in one and the same line, cannot have a common segment.*

Let ABC be a straight line. No other line (as ABD), whereof a segment AB is common to ABC and the remainder BD is not, can also be a straight line.



For if this be disputed, let it be assumed that ABD is also a straight line. About B as a centre, with the radii BC and BD
INTERC.10. respectively, describe concentric spheres. In BC take any point
Cor. 6. as E, between B and the surfaces of both these concentric spheres on the side of B which is towards C; and about B as a centre, with the radius BE, describe a sphere, whose surface (because it lies within the surfaces which pass through the extremities C and
†INTERC.10. D) will cut BC and BD. Let it cut them in the† points E and F;
Cor. 9. ‡INTERC.10. wherefore BE, BF are‡ equal. About E as a centre, with the
Cor. 4. radius EB, describe a sphere; and about A as a centre, with the radius AB, describe another sphere.

* Constr. Because the surfaces of these two spheres pass* through the point B, which is a point in the straight line joining their centres;
† INTERC.10. the spheres touch† one another externally in the point B. And if
Cor. 8. the straight line AC, together with the two substantial spheres whose centres are A and E, be turned about the points A and C;
* INTERC. 9. because AC is a* straight line, every point in it, and among
Nom. others the point E, will be without change of place. Wherefore the substantial spheres whose centres are A and E and which touch one another externally in the point B, are turned about their centres which remain at rest, and consequently the spheres
‡INTERC. 4. are‡ without change of place; and if these spheres are supposed imbedded in one inclosed body of hard matter which is turned
INTERC. 9. along with them round the straight line AC, this inclosing
Nom. † INTERC. 8. body will† also be without change of place, as respects the parts
Cor. bounded by the hollow spherical surfaces. And because the inclosing body being turned round the straight line AC is without change of place, its solid parts which lie in any one direction from the point B, as for instance towards M and H, are capable of coin-
‡ I. Nom. 9. ciding‡ with, and consequently are equal* to, its solid parts which
* I. Nom. 14. lie in any other direction from B, as for instance towards N and I; for in the course of the turning, one set of these is made to occupy the place of the other. If then ABD is also a straight line; about F as a centre, with a radius equal to FB or EB, de-
† INTERC.10. scribe† a sphere. Whereupon may be shown as before, that if
Cor. 6. ABD is a straight line, the sphere so described will touch the sphere whose centre is A externally in the point B, and if the body of hard matter inclosing the two spheres whose centres are A and F be turned round the line ABD, this body also will be without change of place. Which is impossible. For since the spheres described

about F and G have different centres, they cannot coincide; because if they coincided, they would make one sphere having two centres, which cannot* be. And because the spheres described about F and G are† equal but do not coincide, and the point B is in the surface of both; they will‡ cut one another in a self-rejoining line passing through B, or else touch in B only. But whichever of these they do, (inasmuch as the sphere whose centre is G must necessarily be exterior to the sphere whose centre is E, for it touches it externally), addition is made to the inclosing body last-mentioned in the parts which lie in one direction from B as for instance on the side of L, and subtraction at the same time made from it in the parts which lie in some other direction from B as for instance on the side of K. Wherefore the augmented parts NBL cannot upon turning occupy the place of the parts MBH as before, for they have been made greater; still more they cannot occupy the place of the parts MBK which have been made less. And if they cannot occupy the same place, the body during the turning cannot be without change of place. The assumption*, therefore, which involves the impossible consequence, cannot be true; or ABD cannot also be a straight line. And in like manner may be shown that any other line, of which the segment AB is common to ABC and the remainder is not in one and the same line with ABC, cannot be a straight line. But if no other line having the segment AB common as aforesaid, can be a straight line; two straight lines, which are not in one and the same line, cannot have the common segment AB.

* INTERC. 6.
† INTERC. 3. Cor. 6.
‡ INTERC. 11. Cor.
* I. Nom. 26.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, two straight lines, which are not in one and the same line, cannot have a common segment. Which was to be demonstrated.

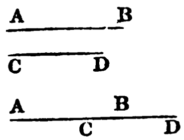
COR. 1. If any two points in one straight line are made to coincide with two points in another, the two straight lines shall coincide with one another to the extent of the length that is common to both, and be one straight line throughout.

For, *First;* If they fail to coincide *between* the points, two straight lines must inclose a space; which is† impossible.

† INTERC. 10. Cor. 1.

Secondly; If they afterwards fail to coincide *beyond* the points, to the extent of the length common to both; two straight lines, which are not in one and the same line, must have a common segment. Which (by Prop. XII above) is impossible.

Thirdly; If there be two straight lines as AB and CD, and their extremities B and C are made to coincide each with a several point between the extremities of the other line; the whole AD shall be a straight line.



For if the line ACBD be turned about the points C and B which remain unmoved; because CB, which is a portion of CD, is* a straight line, no point in it will change its place. And because CBD is a straight line, no point in BD will change its place; for if it did, there would be two straight lines, not in one and the same line, having a common segment CB; which is impossible. And in like manner, because BCA is a straight line, no point in CA will change its place. Wherefore no point in all the line AD will change its place; and because no point in it changes its place, the extreme points A and D remain unmoved, and the line is turned about its extreme points and every point in it is without change of place. Therefore AD is a† straight line.

* INTERC. 9. Cor. 2.

† INTERC. 10. Nom.

COR. 2. If two straight lines cut one another, they coincide only in a point.

For if they coincided in two points, (by Cor. 1 above) they must coincide throughout.

COR. 3. Any straight line may be applied to any other, so that they shall coincide to the extent of the length common to both.

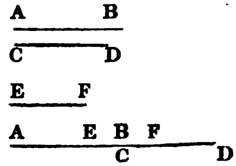
For two points in the one straight line may be made to coincide with two points in the other.

COR. 4. Any straight line CD [*see the last figure*] may be united to any other AB, so that part of CD shall coincide with part of AB, and the remainder be in the same straight line.

For a point between C and D may be made to coincide with B, and at the same time C with some point between B and A.

COR. 5. Any straight line CD may be added to any other straight line AB, so that their sum shall be one straight line.

For any third straight line, as EF, may (by Cor. 4 above) be united to AB, so that a part of it EB shall coincide with part of AB, and the remainder BF be in the same straight line. And afterwards the extremity C of CD may be made to coincide with the point B in AEBF, and some other point in CD with some other point in BF; whereupon (by Cor. 1 above) AD will be one straight line; and it is the sum of AB and CD.



Cor. 6. A terminated straight line may be prolonged to any length in a straight line.

For (by Cor. 5 above) it may be multiplied [*that is to say, straight lines equal to it may be added one to another*] any number of times, and the whole be one straight line. Wherefore it may be thus multiplied, till* the straight line which is the result is greater than the straight line between any two points which shall have been specified.

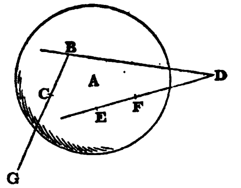
* Interc. 1. Cor. 17.

Nomenclature.—When a straight line is said to be *of unlimited length*, the meaning is, that no point is assigned at which it shall be held to be terminated, but on the contrary it shall without further notice be considered as prolonged to any extent a motive may ever arise for desiring.

Cor. 7. Any three points (which are not in the same straight line) being joined, there shall be formed a three-sided figure; and no point in any one side, shall coincide with any point in either of the other sides, except the points which were joined.

For if any points other than these, should coincide; the whole of the straight lines must coincide, and the three points be in the same straight line.

Cor. 8. Any straight line from a point within a sphere, being prolonged shall cut the surface.



For if BC be any straight line within the sphere whose centre is A, from any point outside as D let a straight line of unlimited length be made to pass through B; and because the sphere may be† turned about its centre without change of place, D will during any such turning continue to be outside the sphere, and a straight line may at any time be‡ made to pass from D to B, which shall cut the surface of the sphere. Let then the sphere be turned about its centre (the straight line of unlimited length DB, at the same time turning about the point D so as always to pass through B), till the point C be made to coincide with some point in DB as F, the point B at the same time occupying the situation E. After which let the sphere together with the straight line EFD be turned again about the centre A, till the points E and F are returned to their original situations B and C, and the straight line EFD is brought into the situation BCG. And because the sphere during all these turnings

† Interc. 4.

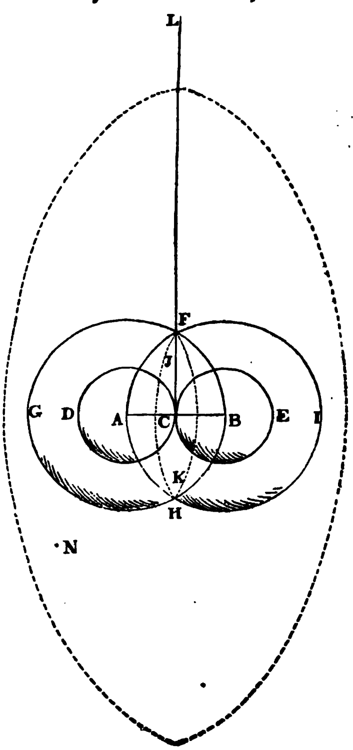
‡ Interc. 9. Cor.

INTERC. 4. about the centre is without change of place, the straight line BCG which cuts the surface of the sphere after the last turning, cuts also the surface as it was before the first turning. That is to say, the straight line BC being prolonged before the first turning, cuts the surface.

PROPOSITION XIII.

PROBLEM.—*To describe a surface in which any two points being taken, the straight line between them lies wholly in that surface.*

†INTERC. 10. Cor. 6. About any centre as A, and with any radius as AD, describe† a sphere; and about any other centre, and with a radius equal to AD, describe another sphere, which ‡INTERC. 10. Cor. 4. will‡ be equal to the first; and let these two equal spheres be placed touching *INTERC. 5. one another in any point* †INTERC. 9. Cor. as C. Join† the centres A and B, and the straight ‡INTERC. 10. Cor. 7. line AB will‡ pass through the point of contact C. About A as a centre, with the radius AB, describe a sphere BFGH; and about B as a centre, with the radius BA, describe a sphere *INTERC. 10. Cor. 4. AFIH, which will* be equal to the sphere BFGH. Because the spheres BFGH and AFIH pass through each other's centres, the surface of each will be partly within and partly without the surface of the other; therefore they cut one another. And because the spheres are equal and cut one another,



Interc. 11. their surfaces coincide only in one self-rejoining line. Let them coincide in FJHK; in which take any point as F, and join CF,
†Interc. 12. Cor. 6. and let CF be† prolonged to an unlimited length on the side of L. Let then the spheres BFGH and AFIH be united as one
‡Interc. 9. Nom. body, and together with the straight line CL be turned round‡ the straight line AB which remains at rest. The straight line of unlimited length CL, shall describe a surface as required.

Because the united spheres BFGH, AFIH are turned about
Interc. 4. their respective centres, each of them will be without change of place. Whence the intersection of their surfaces, which is FJHK, will also be without change of place; for if it was not, the surfaces would at one instant intersect one another in some point or points in fixed space and both of them pass through the same, and at another instant they would not; which cannot be without one or both of the spheres having suffered change of place. Also because the intersection is without change of place, the point F, which is a point in it before the turning begins, will at all times during the turning be found in some part of FJHK; for if it was not, the intersection would at one instant coincide every where with FJHK, and at another it would not; which cannot be, if the intersection is without change of place. And if the body formed by these united spheres be transposed, in such manner that the points A and B shall change places; the straight line BA after the transposition will coincide with and occupy the same place as AB did before, for if not, between the same two points there would be
†Interc. 10. two different straight lines, which cannot† be; and because the radii
‡ Constr. of the two spheres are‡ equal, the spheres are* equal, and each will
*Interc. 10. Cor. 4. coincide with and occupy the place formerly held by the other, and consequently their point of contact C will occupy the same place as before, and their intersection will coincide with their intersection before the transposition, and, like it, will be without change of place during any turning of the united spheres round the straight line AB; also the point F in the straight line CL will at all times during the turning be found in the same line in fixed space FJHK as before, and the united spheres may be turned round the straight line BA till the point F occupies any place in the line in fixed space FJHK which was occupied by it before the transposition. Whence THE SURFACES DESCRIBED BY THE STRAIGHT LINE CL BEFORE AND AFTER THE TRANSPOSITION, SHALL COINCIDE THROUGHOUT; for if there be any point N in the one surface, which

does not coincide with some point in the other surface, then let the straight line CFL be brought into the situation in which it passes through the point N, and after transposition of the united spheres let them be turned round BA till the point F in CFL occupies the same place it occupied when CFL passed through N; and because the point N is not in the second surface, the straight line CFL after the transposition cannot pass through N, for if it did, the transposed surface would pass through N, and it does not; and because in one situation of the straight line CFL the points C and F coincide with the points C and F in the other situation of it, but one of these straight lines passes through the point N and the other does not, the two straight lines CFL coincide in two points C and F, but do not coincide to the extent of the
INTERC. 12. Cor. 1. length common to both, which is impossible. And in the same manner may be shown that there is no other point in the one surface, either within or without the line FJHK, which does not after transposition of the united spheres coincide with some point in the other surface. **Next then,** in the straight line of unlimited length CL *(See Fig. 2 in the next page)* take any point other than F, as M; and if about the centres A and B, with the radii AM and BM, spheres be described, THE POINT M, AT ALL TIMES DURING THE REVOLUTION OF CL, SHALL BE FOUND IN THE INTERSECTION OF THESE SPHERES. For, join AM, BM; and afterwards let the united spheres be transposed, and turned round BA till M be in the same place as before. Because the extremities A and M occupy the places previously occupied by the extremities B and M, AM and BM are equal; wherefore, if about the centres A and B, with the equal radii AM and BM, be described two spheres, these
†INTERC. 10. Cor. 4. will† be equal to one another; and because they are equal but have different centres A and B, and have the point M in their surfaces in
‡INTERC. 11. Cor. common, they will‡ cut one another in one self-rejoining line passing through M, or else touch externally in the point M only; but they cannot touch externally in M, for then the straight line AB
INTERC. 10. Cor. 7. which joins their centres must pass through M, and it does not; wherefore they will cut one another in one self-rejoining line passing through M. Let this line be MQOP; and if afterwards the united spheres together with the straight line CL be turned round AB, M shall always be found in some part of the line MQOP and no where else; for because AM and BM are the radii of the two equal spheres that were described, the point M is always in the sur-

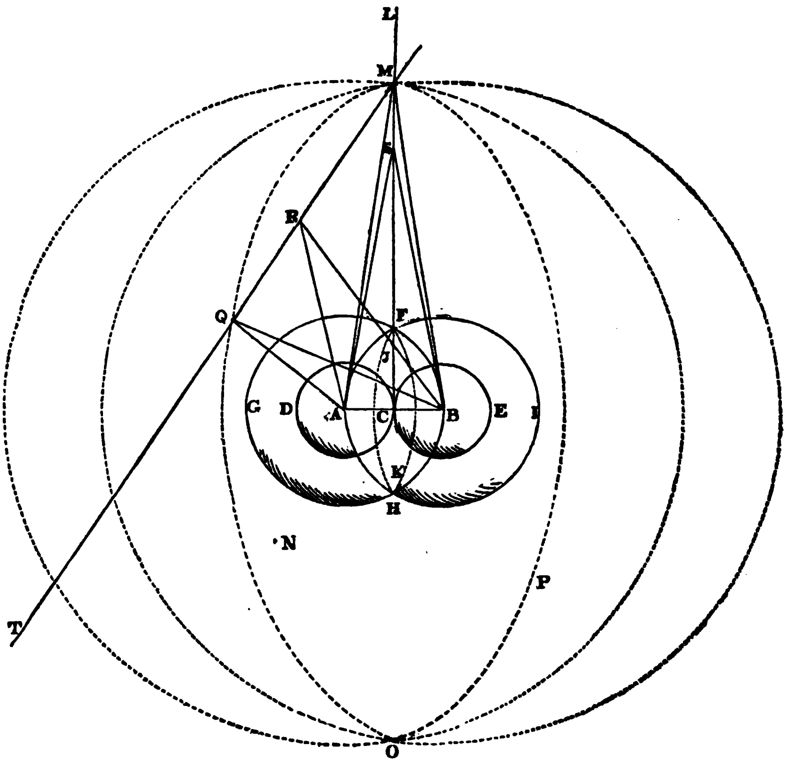
face of both; whence, if it be not always in some part of MQOP, the surfaces of these spheres must coincide elsewhere than in the one self-rejoining line MQOP, which cannot* be. And in the same way of every other point in CL, or in the surface described by it.

*Interc.11.

†Interc.9. Cor.

Whereto next, in MQOP take any second point as Q, and join† MQ; THE STRAIGHT LINE BETWEEN M AND Q, with its prolongation either way, SHALL LIE WHOLLY IN THE SURFACE DESCRIBED BY CL. For if MQ pass through the point C, it will without further question lie wholly in the surface described by CL, inasmuch as each half of it will coincide with part of CL as it was found at some instant of the revolution by which the surface was described. But if MQ do not pass through C, join AM, AQ, BM, BQ, and these straight lines will be‡ all equal one to another, because they are radii of the same or equal spheres; where-

*Interc.10. Cor. 4.



fore if about the centres M and Q, and with the radii MA and QA, were described two spheres, it might be shown as has been done in like circumstances before, that the surfaces of these would cut one another in one self-rejoining line in which is the point A, and that if the figure MBQ was turned round the straight line MQ, the point B would always be found in the self-rejoining line which passes through A, and consequently the turning might be continued till the point B coincided with A, every point in the straight line MQ remaining without change of place; whence, if in MQ had been taken any point as R, and RA and RB had been joined, when the point B was made to coincide with the point A the straight
INTERC.12 Cor. 1. line RB would at the same time have coincided with the straight line RA, and therefore RA and RB are equal. About the centres A and B, with the radii AR and BR, describe two spheres [omitted in the figure]; which, it may be shown as has been done before, will cut one another in one self-rejoining line. Because AR and BR are equal, they are necessarily each greater than AC or BC (unless the point R coincides with C, in which event it is without further question in the surface described by CL); for if they were *equal* to AC and BC, they could meet in no point but C (inasmuch as the spheres AC and BC meet no where but in that point), and if they were *less* they could not meet at all. Wherefore the sphere described about the centre A with the radius AR, will
†INTERC.10. Cor. 5. be† greater than the sphere described about the same centre with the radius AC; and its surface will not coincide at all with the surface of the other, but be exterior; for if it coincided at
‡INTERC. 3. Cor. 4. all, the surfaces would‡ coincide throughout and the spheres be equal; and if it did not coincide but was interior, the sphere would be the less, which cannot be, for it is the greater. And because C is a point within the sphere described with the radius
INTERC.12. Cor. 8. AR, the straight line of unlimited length CL drawn from it will cut the surface of such sphere and be cut by it. Let it be met in S; and
See Note 1. join AS, BS. Because S is a point in CL, it may be shown as has been done before, that AS and BS are equal; wherefore the point S shall be in the self-rejoining line which is the intersection of the spheres described with the radii AR and BR, for if not, then AS would be equal to AR, but BS would not be equal to BR which is
†INTERC. 1. Cor. 3. equal to AR; which is† impossible, for AS and BS are equal. And because, as has been shown before, the point S will at all times during the revolution of CL be found in some part of the inter-

section of the spheres which pass through S, which intersection also passess through R; the point S in CL, will by the revolution of CL be made to pass through R; wherefore R is in the surface described by the revolution of CL. And in like manner may be shown of every other point in the straight line MQ or in its prolongation either way, that some point in the straight line of unlimited length CL passes through it on the united spheres being turned round AB; wherefore every point in MQ or in its prolongation either way, is in the surface described by CL. And if from M be drawn a straight line to any other point in MQOP, and prolonged to an unlimited length, in like manner may be shown that every point in this straight line or in its prolongation either way, is in the surface described by CL. Wherefore IF A STRAIGHT LINE OF UNLIMITED LENGTH AS MT, BE MADE TO TURN CONTINUOUSLY ABOUT M AND PASS ALWAYS THROUGH SOME SECOND POINT IN MQOP, this straight line (both the portion which may lie within MQOP, and its prolongation either way), will at all times lie wholly in the surface described by CL. And because MT may be turned till it passes through any point in the surface which is described by CL *in* or *within* the self-rejoining line MQOP; the straight line from M to any such point, as also its prolongation either way may be shown to lie wholly in the surface described by CL. But every point in the surface described by CL shall be *in* or *within* the self-rejoining line MQOP, which is not farther from the centres A and B than the point M. For if N be any point in the surface described by CL, it may be shown as before, that N is in the intersection of the spheres that should be described with the equal radii AN and BN; wherefore if AN and BN were respectively *equal to* AM and BM, N would be *in* MQOP, for the spheres respectively would* coincide, and consequently their intersections would coincide; and if AN and BN are respectively *less than* AM and BM, the sphere with the radius AN will† be less than the sphere with the radius AM, and its surface will‡ be interior to the surface of the other throughout, and in the same manner the surface of the sphere with the radius BN will be interior to the surface of the sphere with the radius BM; and because the surfaces of the two spheres are respectively interior to the surfaces of the two others, their intersection, in which is N, will not coincide with the intersection of the others but be *within* it. Wherefore the straight line from M TO ANY POINT IN THE SUR-

See Note 2.

*INTERC. 10. Cor. 4.

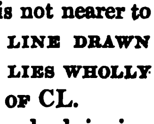
†INTERC. 10. Cor. 5.

‡INTERC. 3. Cor. 4.

FACE DESCRIBED BY CL WHICH IS NOT FARTHER FROM THE CENTRES A AND B THAN THE POINT M, as also its prolongation either way, may be shown to lie wholly in that surface. Lastly then, if in the surface described by the revolution of CL be taken any other two points whatsoever; in the same way may be shown that from one of them (viz. either of them that is not nearer to the centres A and B than the other) A STRAIGHT LINE DRAWN TO THE OTHER, as also its prolongation either way, LIES WHOLLY IN THE SURFACE DESCRIBED BY THE REVOLUTION OF CL.

And by parity of reasoning, the like may be done by bringing into contact any other two equal spheres.

NOMENCLATURE.—A surface in which any two points being taken, the straight line between them lies wholly in that surface, is called a *plane surface*. The same when no particular boundaries to it are intended to be specified, is called a *plane*. If to a plane surface addition is made in any direction, in such manner that the whole continues to be a plane surface, the original plane surface is said to be *prolonged*, and the part added is called its *prolongation*. A figure which lies wholly in one plane is called a *plane figure*. The whole plane surface within the boundaries of a plane figure which is bounded on all sides, is called the *area* of the figure; and its whole linear boundary, of whatever kind or composition, is called the *perimeter*. If a given straight line in a plane be turned in that plane about one of its extremities which remains at rest, till the straight line is returned to the situation from which it set out, the plane figure described by such straight line is called a *circle*, and its boundary the *circumference*. The point in which one extremity of the straight line remains at rest, is called the *centre* of the circle. Any straight line drawn from the centre of a circle to the circumference, is called a *radius* of the circle; and any straight line drawn through the centre and terminated both ways by the circumference, is called a *diameter* of the circle.



When a circle is said to be described about the centre A with the radius AB, the meaning is, that it is described by the revolution of the given straight line AB about the extremity A.

COR. 1. A plane surface may be prolonged to any extent in a plane.

INTERC.12. Cor. 6. For the straight line CL, by the revolution of which it is described, may be prolonged to any length.

Cor. 2. If a straight line in a plane be prolonged, its prolongation lies wholly in that same plane.

For it was shown (in Prop. XIII above) that if any two points in a plane be joined, both the straight line between them and its prolongation either way, shall lie wholly in that plane.

Cor. 3. A circle may be described about any centre, and with any radius.

For a plane of unlimited extent (as by Cor. 1 above) may be made to pass through the proposed centre, and turned till it also pass through the other extremity of the proposed radius; by the revolution of the radius in which plane, a circle will be described.

Cor. 4. All the radii of the same circle are equal. And circles that have equal radii, are equal.

For in the same circle, the straight line by the revolution of which the circle was described can be made to coincide with all
Interc. 1. of the radii respectively; wherefore the radii are all equal.

And if two circles have equal radii, the two extremities of any radius of the one may be applied to the two extremities of any radius of the other; and because these two straight lines coincide, the figures described by their revolution about the same point will
†I.Nom.14. coincide also, and coinciding will be† equal.

Cor. 5. A straight line from the centre of a circle to a point outside, coincides with the circumference only in a point.

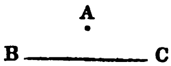
‡Interc.10. Cor. 6. For if about the centre of the circle be‡ described a sphere with a radius equal to the radius of the circle, the straight line from
Interc.10. Cor. 9. the centre coincides with the surface only in a point. And because the circle (as being described with the same radius) lies in the surface of the sphere and no where else; the straight line which coincides with the surface only in one point, can coincide with the circle only in that same point.

PROPOSITION XIV.

Theorem.—*Any three points are in the same plane. [That is to say, one plane may be made to pass through them all.]*

First Case. If the three points are in one straight line, any
†Interc.13. Nom. plane† surface that is made to pass through the two extreme, will also pass through the other; for the whole of the straight line will lie in that surface.

Second Case; where the three points are not in one straight line, but in some other situation, as A, B, C.



If then a plane be made to pass through two of the points as *Interc. 2. Cor. B and C, it may be* turned about them till it also passes through the other point A.

And by parity of reasoning, the like may be done in every other instance. Wherefore, universally, through any three points, one plane may be made to pass. Which was to be demonstrated.

Cor. 1. Any three points which are not in the same straight †Interc. 12. Cor. 7. line being joined, the straight lines which are the sides of the† three-sided figure that is formed lie all in one plane.

For (by Prop. XIV. above) a plane may be made to pass through the three points. And the straight lines joining them will lie wholly in that plane.

Cor. 2. Any two straight lines which proceed from the same point, lie wholly in one plane.

For (by Cor. 1 above) a plane may be made to pass through the point in which they meet, and also through their two other extreme points. And the straight lines between these points, will lie wholly in such plane.

Cor. 3. If three points in one plane (which are not in the same straight line) are made to coincide with three points in another plane, the planes shall coincide throughout, to any extent to which they may be prolonged.

For because the three points are in each of the two planes, the ‡Interc. 12. Cor. 7. straight lines which join them and form‡ a three-sided figure will lie wholly in each of the planes; and consequently the planes will coincide throughout the straight lines which form the sides of such figure. But if the planes do not also coincide in all that is *within* the figure, then from two points [among the points in which the planes coincide], may be drawn two straight lines [one in each of the planes, in the parts where they do not coincide], which shall inclose a space. And if they do not further coincide in all that is *without* the figure, then two straight lines may be drawn [one in each of the planes, in the parts where they do not coincide], which shall have a common segment [in the part wherein the two planes coincide]. Each of which is impossible.

END OF THE INTERCALARY BOOK.

RECAPITULATION

of matters and propositions which have been usually received without proof under the title of Axioms, Postulates, &c., or have not been proved where required. With references to the places where they are demonstrated in the Intercalary Book.

(Intended to enable such as postpone the reading of the Intercalary Book, to proceed as if the same had been prefixed under the title of Axioms, Postulates, &c.)

Intercalary Book.	
Prop. 1.	*Magnitudes which are equal to the same, are equal to one another.*
Prop. 1. Cor. 1.	*If of equals, one be equal to some thing else, the rest are severally equal to the same.*
Prop. 1. Cor. 2.	*If of equals, one be greater, or less, than some thing else; the rest are severally greater, or less, than the same. Or if some thing be greater, or less, than one; it is greater, or less, than each of the others also.*
Prop. 1. Cor. 3.	*Magnitudes which are equal to equals, are equal to one another.*
Prop. 1. Cor. 4.	*Of magnitudes, if to equals be added the same, the sums are equal.*
Prop. 1. Cor. 5.	*If equals be added to equals, the sums are equal.*
Prop. 1. Cor. 6.	*If unequals be added to equals, the sums are unequal. And that sum is greatest, in which the unequal was greatest.*
Prop. 1. Cor. 7.	*If equals be taken from equals, the remainders are equal.*
Prop. 1. Cor. 8.	*If equals be taken from unequals, the remainders are unequal. And that remainder is greatest, in which the unequal was greatest.*

Prop. 1. Cor. 9. *If unequals be taken from equals, the remainders are unequal. And that remainder is least, in which the unequal was greatest.*

Prop. 1. Cor. 10. *Magnitudes which are* double *of the same or of equal magnitudes, are equal to one another. And so if, instead of the* double, *they are the* treble, quadruple, *or any other equimultiples.*

Prop. 1. Cor. 11. *The* double *of a greater magnitude is greater than the* double *of a less. And so of any other equimultiples.*

Prop. 1. Cor. 12. *Magnitudes which are* half *of the same or of equal magnitudes, are equal to one another. And so if, instead of the* half, *they are the* third, fourth, *or any other equipartites.*

Prop. 1. Cor. 13. *The* half *of a greater magnitude is greater than the* half *of a less. And so of any other equipartites.*

Prop. 1. Cor. 14. *If the* doubles *of two or more magnitudes be added together, the amount is* double *of the sum of the magnitudes. And so of any other equimultiples.*

See Note.

Prop. 1. Cor. 15. *If there be two unequal magnitudes, and from the* double *of the greater be taken the* double *of the less; the remainder is* double *of the difference of the magnitudes. And so of any other equimultiples.*

See Note.

Prop. 1. Cor. 17. *Any given magnitude may be multiplied,* [that is to say, magnitudes equal to it may be added one to another], *so as at length to become greater than any other given magnitude of the same kind which shall have been specified.*

Prop. 2. Cor. *Any solid, surface, line, or figure, may be turned about any one point or about any two points, in it; such point or points remaining unmoved.*

Prop. 9. *From one of two assigned points to the other, to describe a line, which being turned about its extreme points, every point in it shall be without change of place. Such a line is called a* straight *line.*

A body or figure which is turned about two points in it that are also the extremities of a straight line, (inasmuch as the whole of the straight line remains without change of place) is said to be turned round such straight line. *When from any point to any other point, a straight line is described or made to pass; the two points are said to be* joined. *If to a straight line addition is made at either end, in such manner that the whole continues to be a straight line, the original straight line is said to be* prolonged, *and the part added is called its* prolongation.

See Note.

Prop. 9. Cor. *A straight line may be described or made to pass from any one point to any other point.*

Prop. 10. *Between two points there cannot be more than one straight line.*

Prop. 10. Cor. 1. *Two straight lines cannot inclose a space.*

Prop. 10. Cor. 2. *Any portion of a straight line is also a straight line.*

Prop. 12. *See Note.* *Two straight lines, which are not in one and the same line, cannot have a common segment.*

Prop. 12. Cor. 1. *If any two points in one straight line coincide with two points in another, the two straight lines shall coincide with one another to the extent of the length that is common to both, and be one and the same straight line throughout.*

Prop. 12. Cor. 2. *If two straight lines cut one another, they coincide only in a point.*

Prop. 12. Cor. 3. *Any straight line may be applied to any other, so that they shall coincide to the extent of the length common to both.*

Prop. 12. Cor. 6. *A terminated straight line may be prolonged to any length in a straight line.*

> *When a straight line is said to be* of unlimited length, *the meaning is, that no point is assigned at which it shall be held to be terminated, but on the contrary it shall without further notice be considered as prolonged to any extent a motive may ever arise for desiring.*

Prop. 12. Cor. 7. *Any three points (which are not in the same straight line) being joined, there shall be formed a three-sided figure; and no point in any one side, shall coincide with any point in either of the other sides, except the points which were joined.*

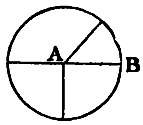
Prop. 13. *To describe a surface in which any two points being taken, the straight line between them lies wholly in that surface. Such a surface is called a* plane surface; *or when no particular boundaries to it are intended to be specified, a* plane.

> *If to a plane surface addition is made in any direction, in such manner that the whole continues to be a plane surface, the original plane surface is said to be* prolonged, *and the part added is called its* prolongation.
>
> *A figure which lies wholly in one plane, is called a* plane figure. *The whole plane surface within the boundaries of a plane figure which is bounded on all sides, is called the* area *of the figure; and its whole linear boundary, of whatever kind or composition, is called the* perimeter.

Prop.13. Cor.1. *A plane surface may be prolonged to any extent in a plane.*

Prop.13. Cor.2. *If a straight line in a plane be prolonged, its prolongation lies wholly in that same plane.*

Prop. 13. Nom. *If a given straight line in a plane be turned in that plane about one of its extremities which remains at rest, till the straight line is returned to the situation from which it set out, the plane figure described by such straight line is called a* circle, *and its boundary the* circumference. *The point in which the extremity of the straight line remains at rest, is called the* centre *of the circle. Any straight line drawn from the centre of a circle to the circumference, is called a* radius *of the circle; and any straight line drawn through the centre and terminated both ways by the circumference, is called a* diameter *of the circle.*



When a circle is said to be described about the centre A with the radius AB, the meaning is, that it is described by the revolution of the given straight line AB about the extremity A.

Prop.13. Cor.3. *A circle may be described about any centre and with any radius.*

Prop.13. Cor. 4. *All the radii of the same circle are equal. And circles that have equal radii, are equal.*

Prop.13. Cor.5. *A straight line from the centre of a circle to a point outside, coincides with the circumference only in a point.*

Prop. 14. *Any three points are in the same plane. [That is to say, one plane may be made to pass through them all.]*

Prop.14. Cor.1. *Any three points which are not in the same straight line being joined, the straight lines which are the sides of the three-sided figure that is formed lie all in one plane.*

Prop.14. Cor.2. *Any two straight lines which proceed from the same point, lie wholly in one plane.*

Prop.14. Cor.3. *If three points in one plane (which are not in the same straight line) are made to coincide with three points in another plane; the planes shall coincide throughout, to any extent to which they may be prolonged.*

(The above Recapitulation contains the principal matters likely to be referred to. But should reference be made to any thing that is not found in it, recourse is to be had to the Intercalary Book.)

FIRST BOOK *continued from page 6.*

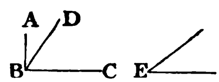
NOMENCLATURE.

XXXIII. Lines formed by the junction of several straight lines which are not in one and the same straight line, and surfaces formed by the junction of several plane surfaces which are not in one and the same plane, are called *compound*. Lines other than straight lines, and surfaces other than plane surfaces, [*that is to say, which are neither such, nor compounded of such,*] are called *curved*.

XXXIV. Straight lines which proceed from the same point but do not afterwards coincide, are said to be *divergent*.

XXXV. If through two divergent straight lines of unlimited
INTERC.14. Cor. 2. length a plane be made or supposed to pass, and another straight line of unlimited length be turned about the point from which the two divergent straight lines proceed, continuing ever in the same plane with them, and so travel from the place of one to the place of the other; such travelling straight line is called the *radius vectus*.

XXXVI. The plane surface (of unlimited extent in some directions but limited in others) passed over by the *radius vectus* in
See Note. travelling from one of the divergent straight lines to the other, is called the *angle* between them.



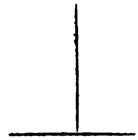
Hence angles are compared together by their extension sideways only; without reference to the greater or smaller length of the straight lines between which they lie. Thus the angle between the straight lines BC and BA, is greater than that between BC and BD, or that between BD and BA; and is in fact equal to their sum. Also if the *radius vectus* instead of moving by the shortest road from BC to BA, should go round by the contrary way, the plane surface so passed over is likewise an *angle*. Such an angle may be called *circuitous*; and the other where the *radius vectus* goes by the nearest road, *direct*.

When several direct angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, [*that is, at the point from which the straight lines that make the angle, proceed*], is put in the middle, and one of the remaining letters is somewhere upon one of those straight lines, and the other upon the other. Thus the angle between the straight lines BA and BC, is named the angle ABC, or CBA; that between BA and BD, is na ed the angle ABD, or DBA; and that between

BD and BC, is named the angle DBC, or CBD. But if there be only one such angle at a point, it may be named from a letter placed at that point; as the angle at E, or more briefly still, the angle E.

If the angle intended is the *circuitous* one, it must be expressed by the use of the term, or something equivalent. But whenever the contrary is not expressed, it is always the *direct* angle that is meant.

XXXVII. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a *right* angle. And the straight line which stands on the other is called a *perpendicular* to it; and is also said to be *at right angles* to it.

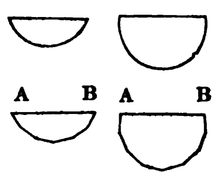


XXXVIII. An angle greater than a right angle, is called *obtuse*.

XXXIX. An angle less than a right angle, is called *acute*.

XL. Angles greater or less than right angles, are called by the common title of *oblique*.

XLI. A straight line joining the extremities of any portion of the circumference of a circle, is called a *chord*.



The same name is applied to a straight line joining the extremities of any series of straight lines that approaches to the like form. The points A, B, where the chord meets the ends of the series, are called the *cusps*; and the angles A and B, the *angles at the cusps*.

XLII. Figures which are bounded by straight lines, are called *rectilinear*.

Linear figures of all kinds are understood to lie wholly in one plane, when the contrary is not expressed.

XLIII. Of rectilinear figures, such as are contained by three straight lines, are called *triangles*.

XLIV. Those contained by four straight lines, are called *quadrilateral*.

XLV. Those contained by more than four, are called *polygons*.

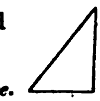
Figures in which a number of sides is specified or intimated, are always understood to be rectilinear, when the contrary is not expressed.

XLVI. Of triangles, such as have two sides equal, are called *isoskeles*.

XLVII. A triangle which has all its three sides equal, is called *equilateral*.

Hence all equilateral triangles are at the same time isoskeles; but isoskeles triangles are not all equilateral.

XLVIII. A triangle which has a right angle, is called *right-angled*.



The side opposite to the right angle is called the *hypotenuse*. The other two sides are sometimes called the *base* and *perpendicular*.

XLIX. A triangle which has an obtuse angle, is called *obtuse-angled*.

L. A triangle which has all its angles acute, is called *acute-angled*.

LI. A triangle which has all its angles oblique, is called *oblique-angled*.

LII. The nomenclature of the various kinds of quadrilateral figures cannot with propriety be given, till these figures have been shown to be capable of possessing certain properties from which their distinctions are derived. It is therefore to be found in the places where such properties are demonstrated. (*See the Nomenclature at the end of Propositions* XXVIII *A*, XXXIII, and XXXIV *bis*, of the First Book.

LIII. In any quadrilateral figure, a straight line joining two of the opposite angular points is called a *diagonal*.

For brevity, quadrilateral figures may be named by the letters at two of their opposite angles, when no obscurity arises therefrom.

LIV. Of polygons, such as have five, six, seven, eight, nine, ten, eleven, twelve, and fifteen sides respectively, are called a *pentagon*, *hexagon*, *heptagon*, *oktagon*, *enneagon*, *dekagon*, *hendekagon*, *dodekagon*, *pendekagon*.

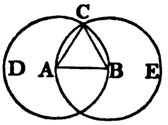
For polygons with other numbers of sides, names might probably be found, or be framed from the Greek; but they are not in common use.

SCHOLIUM.—Henceforward all lines, angles, and figures linear or superficial, whether single or formed by the junction of many, will be understood to lie wholly in one plane, viz. the plane of the paper on which they are represented; when the contrary is not expressed.

PROPOSITION I.

See Note. PROBLEM.—*To describe an equilateral triangle upon a given straight line.*

Let AB be the given straight line. It is required to describe an equilateral triangle upon it.



INTERC. 13. Cor. 3. About the centre A, with the radius AB, describe the circle BCD; and about the centre B, with the radius BA, describe the circle ACE. Because the circles ACE and BCD pass through each other's centres, each will cross the other in two places and be crossed by it. From a point in which the circles meet (as for

†INTERC. 9. Cor. instance C) draw† the straight lines CA, CB, to the points A and B. ABC shall be an equilateral triangle.

Because the point A is the centre of the circle BCD, and C

‡INTERC. 13. Cor. 4. and B are points in the circumference, AC is‡ equal to AB. And because the point B is the centre of the circle ACE, and C and A are points in the circumference, BC is equal to AB. But it has been shown that AC is equal to AB; therefore AC and BC are each of them equal to AB. And things which are equal to the

INTERC. 1. same, are equal to one another; therefore AC is equal to BC. Wherefore AC, BC, AB are equal to one another, and the triangle ABC is equilateral; and it is described upon the given straight line AB. Which was to be done.

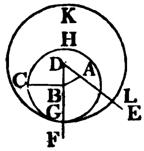
And by parity of reasoning, the like may be done in every other instance.

SCHOLIUM.—It has not yet been proved that the place where the two circles cross one another is only a point. There might, therefore, for all that has yet been proved, be more equilateral triangles than one, describable on the same side of AB. Which if it were possible (though it will hereafter be shown that it is not), would in no way affect the accuracy of the assertion that it has been shown how to construct an equilateral triangle upon AB.—*Referred back to, in the Scholium at the end of Prop. VII of the First Book.*

PROPOSITION II.

See Note. PROBLEM.—*From a point assigned, to draw a straight line equal to a given straight line.*

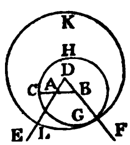
First Case. Let A be the point assigned, and BC the given straight line; where the point is not situate in any part of the given line. It is required to draw from the point A a straight line equal to BC.



INTERC. 9. Cor. From the point A to B draw the straight line
† I. 1. AB, and upon it describe† the equilateral triangle
‡INTERC. 12. Cor. 6. DAB, and prolong‡ the straight lines DA and DB, to E and F;
INTERC. 13. Cor. 2. which will be in the same plane with the rest. About the centre B, with the radius BC, describe† the circle CGH; and about the
†INTERC. 13. Cor. 3. centre D, with the radius DG, describe the circle GLK, cutting
‡INTERC. 13. Cor. 5. DE in the‡ point L. AL shall be equal to BC.

INTERC. 13. Cor. 4. Because the point B is the centre of the circle CGH, BC is equal to BG; and because D is the centre of the circle GLK,
† Constr. DL is equal to DG. And DA, DB, parts of them, are† equal;
‡ INTERC. 1. Cor. 7. therefore the remainder AL is‡ equal to the remainder BG. But it has been shown that BC is equal to BG; wherefore AL and BC are each of them equal to BG. And things which are
INTERC. 1. equal to the same, are equal to one another; therefore AL is equal to BC. Wherefore from the point A a straight line AL has been drawn, equal to BC.

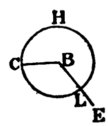
Second Case; where A the point assigned is situate in some part of the given straight line BC, but not in one of its extremities.



Upon either of the portions AB, AC, (as, for
† I. 1. instance, upon AB), describe† an equilateral
‡INTERC. 12. Cor. 6. triangle DAB, and prolong‡ the straight lines DA and DB, to E
INTERC. 13. Cor. 3. and F. About the centre B, with the radius BC, describe the circle CGH; and about the centre D, with the radius DG, describe the circle GLK, cutting DE in L. AL shall be equal to BC.

Because the point B is the centre of the circle CGH, &c. [*the remainder of the proof is word for word as in the last Case.*]

Third Case; where the point assigned is situate in one of the extremities of the given straight line BC; as, for instance, in B.



†INTERC. 13. Cor. 3. About the centre B, with the radius BC, describe† the circle CLH.

If from B any straight line, as BE, be drawn to a point without the circle, and L be a point where it is cut by the circumference, then because the point B is the centre of the circle CLH, and

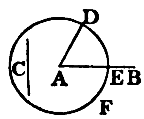
Interc.13. Cor. 4. L and C are points in the circumference, BL is equal to BC. Wherefore from the point B a straight line has been drawn, equal to BC.

In all the possible Cases, therefore, has been shown how from an assigned point to draw a straight line equal to a given straight line BC. Which was to be done. And by parity of reasoning, the like may be done in every other instance.

Scholium.—It has not yet been proved, that there might not be more equilateral triangles than one, described upon AB on the same side, and that the prolongations of these might not make straight lines from B other than BF. Which if it were possible (though it will hereafter be shown that it is not), would in no way affect the accuracy of the assertion that it has been shown how from A to cut off a straight line equal to BC.—*Referred back to, in the Scholium at the end of Prop. VII of the First Book.*

PROPOSITION III.

Problem.—*From the greater of two given straight lines, to cut off a part equal to the less.*



Let AB and C be the two given straight lines; whereof AB is the greater. It is required to cut off from AB, a part equal to C.

† I. 2. From either of the ends of AB (as, for instance, from A), draw† a straight line AD equal to C; [*which straight line will fall in some direction that will be determined by the operations described in the preceding Problem, and will not, except by accident, coincide with the given straight line* AB, *from which a part equal to* C *is desired to be cut off*].

‡Interc.13. Cor. 3. About the centre A, with the radius AD, describe‡ the circle *Interc.13. Cor. 5. DEF, cutting AB in the* point E.

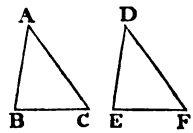
†Interc.13. Cor. 4. Because A is the centre of the circle DEF, and E and D are points in the circumference, AE is† equal to AD. But the straight line C is likewise equal to AD; whence AE and C are ‡Interc. 1. each of them equal to AD. Wherefore AE is‡ equal to C, and from AB the greater of the two straight lines, a part AE has been cut off, equal to C the less. Which was to be done.

And by parity of reasoning, the like may be done in every other instance.

PROPOSITION IV.

THEOREM.—*If two triangles have two sides of the one, equal to two sides of the other respectively; and have also the angles between those sides, equal to one another; they shall have their third sides equal; and the two triangles shall be equal; and their other angles shall be equal respectively, viz. those to which equal sides are opposite.*

See Note.



Let BAC, EDF be two triangles, which have the two sides AB, AC of the one, equal to the two sides DE, DF of the other respectively, viz. AB to DE, and AC to DF; and the angle BAC equal to the angle EDF. The third side BC shall be equal to the third side EF, and the triangle BAC to the triangle EDF; and the other angles, to which the equal sides are opposite, shall be equal respectively, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

For, if the triangle BAC was applied to the triangle EDF, so that the point A was on D, and the straight line AB on* DE; the point B would coincide with the point E, because AB is† equal to DE. And, AB so coinciding with DE, if the triangle BAC was turned round the straight line AB till the point C was made to be in the plane in which is the triangle DEF, the planes in which are the two triangles would‡ coincide throughout. And because the angle BAC is* equal to the angle EDF, the straight line AC would coincide with the straight line DF, to the extent of the length common to both; for if it did not, the angle BAC would be either greater or less than EDF, which is impossible, for they are† equal. And because the straight line AC is‡ equal to DF, the point C would coincide with the point F. But the point B coincides with the point E; wherefore (the point B coinciding with E, and C with F,) the straight line BC would* coincide with the straight lineEF; and coinciding, they would be equal†. Wherefore the whole triangle BAC would coincide with the whole triangle EDF, and be equal to it; and the remaining angles of the one would coincide with the remaining angles of the other respectively, and be equal to them, viz. the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

*Interc. 12. Cor. 3.
† Hyp.
‡ Interc. 14. Cor. 3.
* Hyp.
† Hyp.
‡ Hyp.
* Interc. 12. Cor. 1.
† I. Nom. 14.

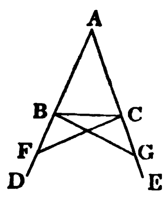
And by parity of reasoning, the like may be proved of all other triangles under the same conditions; that is to say, which have two sides of the one equal to two sides of the other respectively, and also the angles between those sides equal to one another. Wherefore, universally, if two triangles have two sides of the one, equal to two sides of the other respectively; &c. Which was to be demonstrated.

PROPOSITION V.

THEOREM.—*In any isoskeles triangle, the angles opposite to two equal sides are equal to one another.*

Let BAC be an isoskeles triangle, of which the side AB is equal to AC. The angles ACB and ABC shall be equal to one another.

INTERC.12. Cor. 6. † I. 3. Prolong AB and AC to any lengths, as for instance to D and E. In BD take any point F, and from AE cut off† AG equal to AF, and join BG, CF.



‡ Constr. * Hyp. † I. 4. In the triangles FAC, GAB, because AF is‡ equal to AG, and AC to* AB, the two sides AF, AC are equal to the two AG, AB respectively; and the angle at A is common to the two triangles FAC, GAB; therefore the third side FC is† equal to the third side GB, and the triangle FAC to the triangle GAB; and the remaining angles of the one are equal to the remaining angles of the other respectively, to which the equal sides are opposite; viz. the angle AFC to the angle AGB, and the angle ACF to the angle ABG. Also, because the whole AF is equal to the whole AG, of which the parts AB, AC are equal, the remainder ‡INTERC. 1. Cor. 7. BF will be‡ equal to the remainder CG. And FC was shown to be equal to GB; therefore in the triangles BFC, CGB, the two sides FB, FC are equal to the two GC, GB respectively; and moreover the angle BFC was shown to be equal to the angle CGB; * I. 4. wherefore the triangles are* equal, and the remaining angles of the one are equal to the remaining angles of the other respectively, viz. those to which the equal sides are opposite; therefore the angle FBC is equal to the angle GCB, and the angle BCF to CBG. And since it has been demonstrated that the whole angle ACF is equal to the whole angle ABG, parts of which, the angles

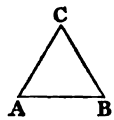
BCF and CBG, are also equal; it follows that the remainders
Interc. 1. Cor. 7. are equal, viz. the angle ACB equal to the angle ABC.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, in an isoskeles triangle the angles opposite to two equal sides are equal to one another.
See Note. Which was to be demonstrated.

Cor. 1. If two equal sides of any isoskeles triangle be prolonged, the two angles exterior to the third side shall be equal.

For the angle FBC was shown to be equal to GCB.

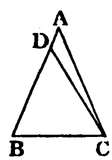
Cor. 2. Every equilateral triangle is also equiangular.



For if ABC be an equilateral triangle, then because the sides CB and CA are equal, the angle A (by Prop. V above) is equal to the angle B. Again, because the sides AB and AC are equal, the angle C is equal to the angle B. But the angle A also was
†Interc. 1. equal to the angle B; therefore the angles A and C are† equal to one another. And since the angles A and C are also equal to B, the angles A, B, C are all equal to one another.

PROPOSITION VI.

Theorem.—*If two angles of a triangle are equal to one another, the sides opposite to the two equal angles are equal to one another.*



Let BAC be a triangle having the angle ACB equal to the angle ABC; the side BA is equal to the side CA.

For if BA be not equal to CA, one of them must be greater than the other. Let BA be
‡ I. 3. assumed to be the greater, and from it cut off‡ BD equal to CA, and join CD.

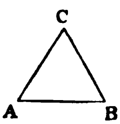
Because in the triangles DBC and ACB, BD is equal to CA, and BC common to both, the two sides BD and BC must be equal to the
* Hyp. two CA, CB respectively. And the angle DBC is* equal to the
† I. 4. angle ACB. Therefore the triangle DBC must be† equal to the triangle ACB, the less to the greater; which is impossible.
‡ I.Nom.26. The assumption‡, therefore, which involves this impossible consequence, cannot be true; or BA is not greater than CA. And in the same way may be shown, that CA is not greater than

BA. But because neither is greater than the other, they are equal.

And by parity of reasoning, the like may be proved in every other triangle whereof two angles are equal to one another. Wherefore, universally, if two angles &c. Which was to be demonstrated.

See Note. COR. Every equiangular triangle is also equilateral.

For if ABC be a triangle of which all the angles are equal to one another, then because the angles A and B are equal, the side CB (by Prop. VI above) is equal to the side CA. Again, because the angles C and B are equal, the side AB is equal to the side CA. Wherefore the sides CB and AB are* equal to one another. And since CB and AB are also equal to CA, the sides CB, CA, AB are all equal to one another.



*INTERC. 1.

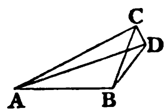
PROPOSITION VII.

THEOREM.—*Upon the same given straight line and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of it equal to one another, and also those which are terminated in the other extremity.*

See Note.

For if this be disputed, let it be assumed that there are two triangles ACB, ADB, upon the same given straight line AB and on the same side of it, which have their sides AC, AD, that are terminated in the extremity A, equal to one another, and also their sides BC, BD, that are terminated in the other extremity B.

First Case; where the vertex of each of the triangles is *without* [that is, on the outside of] the other triangle. Join CD.

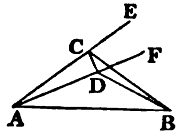


Because AD is equal to AC, the angle ACD is† equal to ADC. But the angle ACD is greater than BCD; therefore the angle ADC (which is equal to ACD) is also‡ greater than BCD; still more is the angle BDC (which is greater than ADC) greater than BCD. Again, because BC is equal to BD, the angle BDC must be equal to BCD; but it has been shown to be greater; and that it should

† I. 5.

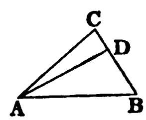
‡ INTERC. 1. Cor. 2.

I.Nom. 26. be both equal and greater, is impossible. The assumption, therefore, that AC and AD are equal to one another, and also BC and BD equal to one another, cannot be true.



Second Case; where one of the vertices, as D, is *within* the other triangle ACB. Join CD; †Interc. 12. Cor. 6. and prolong† AC and AD, to E and F.

Because AD is equal to AC in the triangle CAD, the angles ECD and FDC on the other ‡I.5. Cor. 1. side of CD are‡ equal to one another. But the angle ECD is greater than BCD; therefore the angle FDC (which is equal *Interc. 1. Cor. 2. to ECD) is also* greater than BCD; still more is the angle BDC (which is greater than FDC) greater than BCD. Again, because BC is equal to BD, the angle BDC must be equal to BCD; but it has been shown to be greater; and that it should †I.Nom. 26. be both equal and greater, is impossible. The assumption†, therefore, that AC and AD are equal to one another, and also BC and BD equal to one another, cannot be true.



Third Case; where one of the vertices, as D, is upon a side of the other triangle.

In this case BC and BD are not equal to one another; for BC is the greater.

And by parity of reasoning, the like may, in any of the Cases, be proved of all other triangles under the same conditions. Wherefore, universally, upon the same given straight line &c. Which was to be demonstrated.

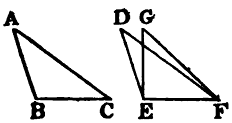
Cor. If about two centres be described two circles that cut one another, their circumferences shall coincide only in two points, one on each side of the straight line that joins the centres.

For if they coincided in more, then upon the given straight line which is between the centres and on the same side of it, might be two triangles having their sides which are terminated in one extremity of it equal to one another, and also those which are terminated in the other extremity.

Scholium.—From this it follows that the two circles described in Prop. I of the First Book, could cut one another only in one point on each side of the straight line on which an equilateral triangle was required to be described; and consequently no more equilateral triangles than one could possibly be described on each side.—*See Scholia to Prop. I and II of the First Book.*

PROPOSITION VIII.

THEOREM.—*If two triangles have two sides of the one equal to two sides of the other respectively, and have also their third sides equal; the angle which is between the two sides of the one, shall be equal to the angle which is between the two sides equal to them, of the other; and the two triangles shall be equal; and their other angles shall be equal respectively, viz. those to which equal sides are opposite.*



Let BAC, EDF be two triangles, having the two sides BA, AC, equal to the two sides DE, DF respectively, viz. AB to DE, and AC to DF; and also the third side BC equal to the third side EF. The angle BAC shall be equal to the angle EDF; and the triangle BAC to the triangle EDF; and the other angles, to which the equal sides are opposite, shall be equal respectively, viz. the angle ABC to DEF, and ACB to DFE.

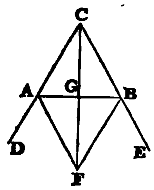
INTERC. 12. Cor. 3. For, let the triangle BAC be applied to the triangle EDF, so that the point B may be on E, and the straight line BC on EF.

† Hyp. Because BC is† equal to EF, the point C will coincide with the point F. And, BC coinciding with EF, BA and CA will coincide with ED and FD. For, if BC coincides with EF, but the sides BA, CA do not coincide with the sides ED, FD, but have any different situation as EG and FG or otherwise; then upon the same given straight line EF and on the same side of it, there will be two triangles which have their sides that are terminated in one extremity of it equal to one another, and also ‡ I. 7. their sides that are terminated in the other extremity; which is‡ impossible. Therefore the sides BA, CA cannot but coincide with the sides ED, FD; wherefore the angle BAC will coincide with the angle EDF, and be equal to it; and the whole triangle BAC will coincide with the whole triangle EDF, and be equal to it; and the remaining angles of the one will coincide with the remaining angles of the other respectively, and be equal to them, viz. the angle ABC to DEF, and ACB to DFE.

And by parity of reasoning, the like may be proved of any other triangles under the same conditions. Wherefore, universally, if two triangles have two sides of the one equal to two sides of the other respectively, and have also their third sides equal; &c. Which was to be demonstrated.

PROPOSITION IX.

See Note. PROBLEM.—*To bisect a given straight line.* [*That is, to divide it into two equal parts.*]



Let AB be the given straight line. It is required to bisect it.

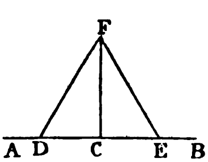
* I. 1. Describe* upon it an equilateral triangle ACB, † INTERC. 12. Cor. 6. and prolong† CA, CB, to D and E. Upon AB and on the other side of it, describe another equilateral triangle AFB. Because the angles ‡ I. 5. Cor. 1. BAD and ABE are‡ equal to one another, and * I. 5. Cor. 2. the anles BAF and ABF are* equal to one another, and AF and BF meet in F, the angles BAF and ABF are respectively less than BAD and ABE; for if they were respectively equal, AF and BF would never meet on the side of F; still less if they were greater; and because they are neither equal nor greater, they are less; and because they are less, the point F which is in both AF and BF lies *between* the straight lines AD and BE. Join CF, and let CF cut AB in G. AB is bisected in G.

† Constr. Because in the triangles CAF, CBF, CA is† equal to CB, and ‡ Constr. CF common, and AF equal‡ to BF; the angle ACF is* equal to * I. 8. † Constr. BCF. And because in the triangles ACG, BCG, GA is† equal to CB, and CG is common, and the angle ACG (or ACF) has been shown to be equal to BCG (or BCF); the third side AG ‡ I. 4. is‡ equal to the third side BG; or AB is bisected in G. Which was to be done.

And by parity of reasoning, in like manner may every other given straight line be bisected.

PROPOSITION X.

PROBLEM.—*To draw a straight line at right angles to an assigned straight line, from an assigned point in the same.*



Let AB be the assigned straight line, and C the assigned point in it. It is required to draw a straight line from C, at right angles to AB.

* I. 3. Take any point D in AC, and make* CE equal to CD, and upon DE describe† † I. 1. the equilateral triangle

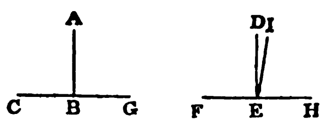
INTERC. 9. Cor. DFE, and join CF. The straight line CF drawn from the assigned point C, is at right angles to AB.

† Constr. Because CD is† equal to CE and CF common to the two triangles DCF and ECF, the two sides DC, CF are equal to the
‡ Constr. two EC, CF respectively. Also the third side DF is‡ equal to
* I. 8. the third side EF; therefore the angle DCF is* equal to ECF; and they are adjacent angles. But when the adjacent angles which one straight line makes with another straight line are
†I. Nom.37. equal to one another, each of them is† a right angle; therefore each of the angles DCF, ECF, is a right angle. Wherefore, from the point C in the straight line AB, CF has been drawn at right angles to AB. Which was to be done.

And by parity of reasoning, the like may be done in every other instance.

PROPOSITION XI.

See Note. THEOREM.—*All right angles are equal to one another.*



Let ABC, DEF be two right angles. They are equal to one another.

‡INTERC.12. Cor. 6. Prolong‡ CB to G, and FE to H; and because ABC is* a right angle, ABC and ABG are† equal
* Hyp. †I. Nom.37. to one another; and in like manner the angles DEF and DEH are equal to one another. Let now the straight line CBG be applied to the straight line FEH, so that the point B shall be
‡INTERC.12. Cor.3. on E, and the straight line CBG on‡ FEH; and let the whole figure CBGA be* turned round the straight line CG, till the
* INTERC. 9. Nom. point A be placed in the same plane in which are the points F, H, and D.

If then BA does not coincide with ED, let it be assumed that it falls on the side of it which is towards H, as EI. But because the angle IEF is equal to ABC, and IEH to ABG, and ABC
†INTERC. 1. Cor. 3. and ABG are equal to one another, IEF and IEH must be† equal to one another. Wherefore the angles DEF and DEH must be equal to one another, and also the angles IEF and IEH equal to one another; which is impossible. For if DEF be equal to DEH, then because DEH is greater than IEH,
‡INTERC. 1. Cor. 2 DEF must be greater‡ than IEH. The assumption*, therefore,
*I.Nom.26. which involves the impossible consequence, cannot be true; or

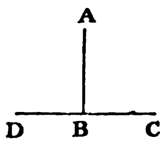
BA cannot fall on the side of ED which is towards H. And in the same way may be shown, that it cannot fall on the side of ED which is towards F; and because it does not fall on either side of ED, it falls upon it, and the angle ABC coincides with the angle DEF, and is equal to it.

And by parity of reasoning, the like may be proved in every other instance, Wherefore, universally, right angles are equal to one another. Which was to be demonstrated.

PROPOSITION XII.

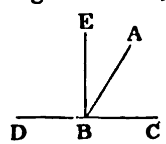
THEOREM.—*The angles which one straight line makes with another on the one side of it, are together equal to two right angles.*

Let the straight line AB make with CD, on the one side of it, the angles ABC, ABD. These are together equal to two right angles.



First Case. If the angles ABC, ABD are
I.Nom.37. equal to one another, each of them is a right angle, and consequently they are together equal to two right angles.

Second Case. If the angles ABC, ABD are not equal to one
† I. 10. another, from the point B draw† BE at right angles to CD; therefore the angles EBD, EBC are two right angles.



Because the angle ABD is equal to the angles ABE and EBD together, if to each of these
‡ INTERC. 1. Cor. 4. equals be added the angle ABC, the sums will‡ be equal; that is to say, the sum of the angles ABC and ABD, will be equal to the sum of the angles ABC, ABE, EBD. Again, because the angle EBC is equal to the angles ABC and ABE together, if to each of these equals be added the angle EBD, the sums will be equal; that is to say, the sum of the angles EBC, and EBD, will be equal to the sum of the angles ABC, ABE, EBD. But the sum of ABC and ABD was shown to be also equal to the sum of ABC, ABE, EBD; therefore the sum of ABC and
INTERC. 1. ABD is equal to the sum of EBC and EBD. But EBC, EBD
† Constr. are† two right angles; therefore the sum of the angles ABC and
‡ INTERC. 1. Cor. 1. ABD is‡ also equal to two right angles.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, the angles which one straight line makes with another on the one side of it, &c. Which was to be demonstrated.

Cor. 1. All the angles, however many, on the one side of a straight line, at the same point and in the same plane, are together equal to two right angles.

For their sum is equal to the sum of the two angles made by any one of the straight lines that fall upon the other, on the one side.

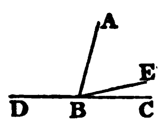
Cor. 2. All the angles made by any number of straight lines proceeding from one point and in the same plane, are together equal to four right angles.

For if any one of the straight lines be prolonged on the other side of the angular point, the angles on each side of this line will (by Cor. 1 above) be together equal to two right angles, and those on both sides will be equal to four.

PROPOSITION XIII.

Theorem.—*If, at a point in a straight line, two other straight lines, on the opposite sides of it, make the adjacent angles together equal to two right angles; those two other straight lines shall be in one and the same straight line.*

At the point B in the straight line AB, let the straight lines BD, BC, on the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles. BC is in the same straight line with DB.



Interc.12. Cor.6. Prolong DB on the side of B. If then it be disputed that the straight line so added shall coincide with BC, let it be assumed that it falls on one side of it, as BE.

Therefore, because the straight line AB makes angles with the straight line DBE on the one side of it, the angles ABD, ABE
† I. 12. must be together† equal to two right angles. But the angles ABD,
‡ Hyp. ABC are together‡ equal to two right angles; wherefore the
Interc. 1. angles ABD, ABE are together equal to ABD, ABC. Take away the common angle ABD, and the remaining angle ABE
†Interc. 1. Cor. 7. must be† equal to the remaining angle ABC, the less to the greater; which is impossible. DB prolonged, therefore, cannot fall on the side of BC which is towards A; and in the same way may be shown, that it cannot fall on the other side; and because it does

not fall on either side of it, it falls upon it, and consequently BC is in one and the same straight line with DB.

And by parity of reasoning, the like may be proved of all other straight lines under the same conditions. Wherefore, universally, if, at a point in a straight line, &c. Which was to be demonstrated.

Cor. 1. Any straight line making with another an angle less or greater than the sum of two right angles, is not in the same straight line with it.

For let BD, BE be two straight lines making an angle DBE less than the sum of two right angles. From B draw any straight line BA between BD and BE; and prolong* DB to C. Because ABD, ABC (by Prop. XIII above) will be together equal to two right angles, and ABD, ABE (which are equal to DBE) are together† less than two right angles, ABD and ABC will be‡ together greater than ABD and ABE, and by taking away the common angle ABD, the angle ABC will be* greater than ABE; consequently BE does not coincide with BC. And because BC is† in the same straight line with DB, and BE does not coincide with it, BE is not in the same straight line with DB. Also if BD, BE make an angle *greater* than two right angles; because all the angles at the point B are‡ together equal to four right angles, and the angle on one side of BD and BE is *greater* than two right angles, the angle on the other side is *less*; wherefore, as before, BD and BE are not in the same straight line.

*Interc. 12. Cor. 6.
† Hyp.
‡ Interc. 1. Cor. 2.
* Interc. 1. Cor. 8.
† Constr.
‡ I. 12. Cor. 2.

Cor. 2. Any two straight lines which meet and make an angle DBE less than the sum of two right angles, being prolonged shall cut one another.

For because the angle ABC (as in Cor. 1 above) is greater than ABE, BC (which is the prolongation of DB) will fall on the side of BE which is most remote from A. And in like manner may be shown that EB being prolonged will fall on the side of BD which is most remote from A. That is, DB and EB being prolonged will cut one another.

Cor. 3. If from any point (as B) in a straight line DC, be drawn another straight line BE making with BD an angle that is less than the sum of two right angles; and two other points in these straight lines (as D, E,) be joined; there shall be formed a triangle, on that side of DC on which is the angle that is less than the sum of two right angles; and no point in the straight line drawn from D to E shall coincide with any point in BD or BE, except the points D and E which were joined.

For because BE makes with BD an angle less than the sum of two right angles, (by Cor. 1 above) it is not in the same straight line with it. Therefore if the points D and E be joined, there shall be formed* a three-sided figure or triangle; and no point in the straight line from D to E shall coincide with any point in BD or BE, except the points D and E which were joined. Also the straight line from D to E cannot pass on the side of DC which is remote from E; for then, to arrive at E, it must pass through DC or its prolongation, and two straight lines would inclose a space. Therefore it shall pass on the side on which is E; that is, on which is the angle that is less than the sum of two right angles.

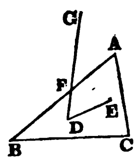
*Interc. 12. Cor. 7.

Cor. 4. If any number of straight lines be added one to another in succession, so that at each of the points of junction a straight line drawn to such point makes the adjacent angles together equal to two right angles, all the first-mentioned straight lines shall be in one and the same straight line.

For (by Prop. XIII above) the second is in one straight line with the first; and in like manner the third with the straight line which is the sum of the second and first; and so on.

Cor. 5. Any straight line within a triangle, as DE, being prolonged shall cut the perimeter.

For if the point D and any point in the perimeter of the triangle, as F, be joined; because the angles DFB and DFA are† together equal to two right angles, DFB is less than two right angles, and (by Cor. 2 above) DF and BF being prolonged will cut one another. And because the same may be shown of any other point in the perimeter, if a straight line of unlimited length DG be turned about D so as to pass through every point in the perimeter in succession, it will at all times cut the perimeter; wherefore if it be moved till it also pass through the point E, the straight line DE will‡ coincide with such unlimited straight line to the extent of the length that is common to both, and its prolongation will also be in the same straight line, and consequently will cut the perimeter.

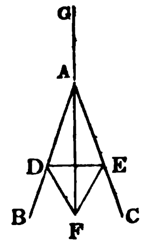


† I. 12.

‡Interc. 12. Cor. 1.

PROPOSITION XIV.

See Note. PROBLEM.—*To bisect a given angle.* [*That is, to divide it into two equal angles.*]



Let BAC be the given angle. It is required to bisect it.

First Case; if BAC is less than the sum of two right angles. Take any point D in AB, and from
* I. 3. AC cut off* AE equal to AD. Join DE, and
† I. 1. upon it describe† an equilateral triangle DFE on the side remote from A; whereupon may be shown (as in Prop. IX) that the point F shall lie *between* the straight lines DB and EC. Then join AF; the straight line AF shall bisect the angle BAC.

‡ Constr. Because AD is‡ equal to AE, and AF is common to the two triangles DAF, EAF, the two sides AD, AF are equal to the
* Constr. two sides AE, AF respectively. Also the third side DF is* equal to
† I. 8. the third side EF; therefore the angle DAF is† equal to the angle EAF. Wherefore the given angle BAC is bisected by the straight line AF.

Second Case; if the angle BAC is equal to the sum of two
‡ I. 13. right angles, BAC will be‡ in one and the same straight line, and DE be bisected by the point A, and AF be perpendicular to BAC and bisect the angle BAC by dividing it into two right angles.

Third Case; if the angle BAC is greater than the sum of two right angles (*or the angle intended is that formed by the turning of the* radius vectus *circuitously by the way of* G), bisect the angle between AB and AC which is less than two right angles,
* INTERC. 12. Cor. 6. by AF as before; and prolong* FA to G. The angle BAG is equal to the angle CAG; or the circuitous angle BAC is bisected.

† I. 12. For because the angles BAF, BAG, are together† equal to two right angles, and the angles CAF, CAG, are together equal
‡ INTERC. 1. to two right angles; BAF and BAG are together‡ equal to CAF
* Constr. and CAG. But the angle BAF is* equal to CAF; wherefore the
† INTERC. 1. Cor. 7. remaining angle BAG is† equal to the remaining angle CAG.

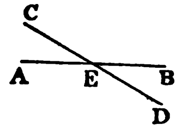
And by parity of reasoning, in like manner may every other given angle be bisected.

COR. Only one straight line, can bisect a given angle.

For any other would divide the angle BAC into two angles, one greater than the half and the other less; which cannot be equal.

PROPOSITION XV.

THEOREM.—*If two straight lines cut one another, the vertical angles shall be equal.*



Let the two straight lines AB, CD, cut *Interc. 12. Cor. 2. one another in the point* E. The angle AEC shall be equal to the angle DEB, and CEB to AED.

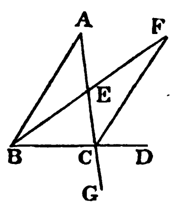
Because the straight line AE makes with CD the angles † I. 12. AEC and AED, these angles are together† equal to two right angles. Again, because the straight line DE makes with AB the angles AED and DEB, these also are together equal to two ‡ Interc. 1. right angles. Wherefore the angles AEC and AED are‡ together equal to the angles AED and DEB. Take away the common angle * Interc. 1. Cor. 7. AED, and the remaining angle AEC is* equal to the remaining angle DEB. In the same way may be shown, that the angles CEB, AED are equal.

And by parity of reasoning, the like may be proved of all other straight lines under the same conditions. Wherefore, universally, if two straight lines cut one another, &c. Which was to be demonstrated.

PROPOSITION XVI.

THEOREM.—*If one side of a triangle be prolonged, the exterior angle is greater than either of the interior opposite angles.*

Let ABC be a triangle, and let its side BC be prolonged to D. The exterior angle ACD is greater than either of the interior opposite angles CAB, CBA.



† I. 9. Bisect† CA, in E; join BE, and prolong ‡ I. 3. it to F; and make‡ EF equal to EB. Join also FC; and prolong AC to G.

* Constr. Because EA is* equal to EC, and EB to* EF, EA and EB are equal to EC and EF † I. 15. respectively. Also the angle AEB is† equal to CEF, because they are vertical angles; ‡ I. 4. therefore the third side AB is‡ equal to the third side CF, and

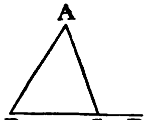
the remaining angles of the triangle AEB are equal to the remaining angles of the triangle CEF respectively, viz. those to which the equal sides are opposite. Wherefore the angle ECF is equal to the angle EAB. But the angle ACD is greater than ECF; therefore it is also* greater than EAB which is equal to ECF. In the same manner if the side BC be bisected, may be shown that the angle BCG is greater than CBA. But the angle BCG is† equal to ACD, for they are vertical angles; therefore the angle ACD is also‡ greater than CBA. And in the same way may be shown that any other exterior angle made by prolonging a side of the triangle ABC, is greater than either of the interior opposite angles.

* Interc. 1. Cor. 2.
† I. 15.
‡ Interc. 1. Cor. 2.

And by parity of reasoning, the like may be proved in every other triangle. Wherefore, universally, if one side of a triangle be prolonged, &c. Which was to be demonstrated.

PROPOSITION XVII.

THEOREM.—*Any two angles of a triangle are together less than two right angles.*



Let ABC be a triangle. Any two of its angles are together less than two right angles.

Prolong* BC to D.

Because ACD is the exterior angle of the triangle ABC, it is† greater than the interior opposite angle ABC. To each of these add the angle ACB; and the angles ACD, ACB are together‡ greater than ABC, ACB. But the angles ACD, ACB are together* equal to two right angles; therefore the angles ABC, ACB are together† less than two right angles. In the same manner may be shown that the angles BAC, ACB are together less than two right angles; and by prolonging a side adjacent to another angle, as for instance AB, may be shown that the angles ABC, BAC are together less than two right angles.

* Interc. 12. Cor. 6.
† I. 16.
‡ Interc. 1. Cor. 6.
* I. 12.
† Interc. 1. Cor. 2.

And by parity of reasoning, the like may be proved in every other triangle. Wherefore, universally, any two angles of a triangle are together less than two right angles. Which was to be demonstrated.

COR. 1. A triangle cannot have more than one right angle;

nor more than one obtuse angle; neither can it have one right angle and also one obtuse angle.

For if it had any of these, two of its angles would be together not less than two right angles. Which (by the Prop. above) cannot be.

Cor. 2. A right or an obtuse angle in any triangle, is greater than either of the remaining angles of the triangle.

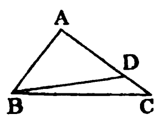
For if it was not, there would be two angles which are together not less than two right angles.

Cor. 3. Every triangle has at the least *two* acute angles.

For if it had not, it must either have two right angles, or two obtuse, or one right angle and one obtuse. Which (by Cor. 1 above) cannot be.

PROPOSITION XVIII.

Theorem.—*The greater side in any triangle has the greater angle opposite to it.*



Let ABC be a triangle, of which the side AC is greater than the side AB. The angle ABC is greater than the angle ACB.

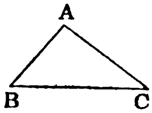
* I. 3. Since AC is greater than AB, make* AD equal to AB; and join BD.

Because ADB is the exterior angle of the triangle BDC, it
† I. 16. is† greater than the interior opposite angle DCB. But because
‡ Constr. AD is‡ equal to AB, the angle ABD is* equal to ADB; there-
* I. 5. fore the angle ABD is† greater than DCB; still more is the
† Interc. 1. Cor. 2. angle ABC (which is greater than ABD) greater than DCB.

And by parity of reasoning, the like may be proved in every other instance. Wherefore the greater side in any triangle &c. Which was to be demonstrated.

PROPOSITION XIX.

Theorem.—*The greater angle in any triangle has the greater side opposite to it.*



Let ABC be a triangle, of which the angle B is greater than the angle C. The side AC is greater than the side AB.

For, if it be not greater, AC must either be equal to AB, or less. It is not equal; be-

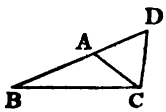
* I. 5. cause then the angle B would be* equal to the angle C; but
† Hyp. the angle B is not equal to the angle C, for it is† greater.
‡ I. 18. Neither is it less; because then the angle B would be‡ less than
* Hyp. the angle C; but it is not less, for it is* greater. And because AC is neither equal to AB nor less, it is greater.

And by parity of reasoning, the like may be proved in every other instance. Wherefore the greater angle in any triangle &c. Which was to be demonstrated.

Cor. In any right-angled or obtuse-angled triangle, the side opposite to the right or obtuse angle is the greatest.
†I.17.Cor.2. For the angle is† the greatest.

PROPOSITION XX.

Theorem.—*Any two sides of a triangle are together greater than the third side.*



Let ABC be a triangle. Any two sides of it are together greater than the third side; viz. the sides BA and AC, greater than BC; and AC and CB, greater than BA; and CB and BA, greater than AC.

‡Interc.12. Cor. 6. Prolong‡ BA; and make* AD equal to AC. And join DC.
* I. 3. Because AD is equal to AC, the angle ACD is† equal to ADC.
† I. 5. But the angle BCD is greater than ACD; therefore it is also‡
‡Interc. 1. Cor. 2. greater than ADC. And because in the triangle BCD the angle
* I. 19. BCD is greater than the angle BDC, the side BD is* greater than the side CB. But BD is equal to BA and AD, or to BA and AC;
†Interc. 1. Cor. 2. therefore the sides BA and AC are† together greater than CB. In the same way by prolonging AC, may be shown that the sides AC and CB are together greater than BA; and by prolonging CB, may be shown that CB and BA are together greater than AC.

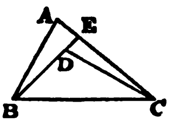
And by parity of reasoning, the like may be proved in every other triangle. Wherefore, universally, any two sides of a triangle &c. Which was to be demonstrated.

Cor. Three straight lines, any two of which are not together greater than the third, cannot form a triangle.

For if they did, there would be two of its sides together not greater than the third.

PROPOSITION XXI.

See Note. THEOREM.—*If from the ends of the side of a triangle, there be drawn two straight lines to a point within the triangle, these shall together be less than the two sides of the first triangle, but shall make a greater angle.*



From B and C the ends of any side of a triangle ABC, let the two straight lines BD, CD be drawn to the point D within it. BD and CD are together less than BA and CA; but make an angle BDC greater than the angle BAC.

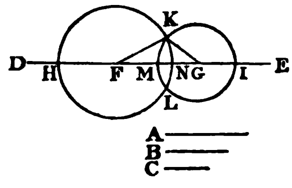
INTERC. 12. Cor. 6. Prolong BD till it meets the side CA in E.

†I. 20. Because two sides of a triangle are together† greater than the third side, BA and AE are together greater than BE. To each of ‡INTERC. 1. Cor. 6. these add EC; therefore BA, AE, EC are together‡ greater than BE, EC. But AE, EC are equal to AC; therefore BA, AC are together greater than BE, EC. Again, because two sides of a triangle are together greater than the third side, EC and ED are together greater than CD. To each of these add DB; therefore EC, ED, DB are together greater than CD, DB. But ED, DB are equal to BE; therefore BE, EC are together greater than CD, DB. Still more, then, are BA, AC (which were shown to be together greater than BE, EC) together greater than CD, DB. Again, because the exterior angle of a *I. 16. triangle is* greater than the interior opposite angle, the exterior angle DEC of the triangle EAB is greater than the angle EAB. And for the same reason the exterior angle BDC of the triangle DEC is greater than the angle DEC. Still more, then, is it greater than the angle EAB, which was shown to be less than DEC.

And by parity of reasoning, the like may be proved in every other triangle. Wherefore, universally, if from the ends of the side of a triangle, &c. Which was to be demonstrated.

PROPOSITION XXII.

PROBLEM.—*To describe a triangle of which the sides shall be equal to three given straight lines respectively; provided always, that any two of these be together greater than the third.*



Let A, B, C be the three given straight lines; of which any two are together greater than the third; (for if they are not, it has been shown* that they cannot form a triangle). It is required to make a triangle, of which the sides shall be equal to A, B, C, respectively.

*I. 20. Cor.

In a straight line DE, make† FG equal to any one of the straight lines as A; FH equal to another as B; and GI equal to C. About the centre F, with the radius FH, describe‡ a circle; and let it cut FE in the* point N. About the centre G, with the radius GI, describe another circle; and because FH and GI are together† greater than FG, this last circle will cut GD in a point (as M) nearer to D than is the point N. For if M and N should coincide, FH and GI must be together *equal to* FG; and if M should lie on the other side of N, they must be together *less than* FG; neither of which is possible, for they are‡ *greater*. Also because FN (which is equal to FH) is* less than the sum of FG and GI, the point N will fall on the side of I which is towards D; and because GM (which is equal to GI) is† less than the sum of GF and FH) the point M will fall on the side of H which is towards E. Wherefore the circumference of each of the circles will pass through that of the other; and in passing from the outside of it to the inside, and from the inside to the outside again, will necessarily cut it in two points‡. Let one of these be K. Join KF, KG. The triangle FKG has its sides equal to the three straight lines A, B, C.

† I. 3. ‡ INTERC. 13. Cor. 3. * INTERC. 13. Cor. 5. † Hyp. ‡ Hyp. * Hyp. † Hyp. ‡ I. 7. Cor.

Because the point F is the centre of the circle HKL, FK is* equal to FH. But FH is† equal to the straight line B; therefore FK is‡ equal to B. Again, because G is the centre of the circle IKL, GK is equal to GI. But GI is equal to the straight line C; therefore GK is equal to C. Also FG is equal to A; therefore the three straight lines FG, FK, GK (which are the sides of the triangle FKG) are respectively equal to the three straight lines A, B, C. Which was to be done.

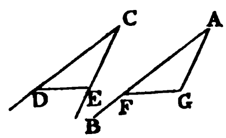
* INTERC. 13. Cor. 4. † Constr. ‡ INTERC. 1. Cor. 1.

And by parity of reasoning, the like may be done in every other instance.

COR. 1. In this manner may be described a triangle of which the sides shall be equal to those of a given triangle respectively. And if it is required that one particular side of the triangle shall be on an assigned straight line DE, and terminated at an assigned point F in it, and that another particular side shall likewise be terminated at F; this also may be done.

For it may be effected by making FG equal to the side which is to be on DE, and FH equal to the other side which is to be terminated at F, and GI equal to the remaining side.

COR. 2. By the help of this, on an assigned straight line and at an assigned point in it, may be made an angle equal to a given angle.



For if AB be the assigned straight line, A the point in it, and C the given angle; in the two straight lines between which is the angle C, take any points D, E, and join DE; and (by Prop. XXII above) describe the triangle AFG, in which the side AF shall be equal to CD, and AG to CE, and FG to DE. Because AF, AG are equal to CD, CE respectively, and the third side FG is
* I. 8. equal to the third side DE, the angle FAG will be* equal to the angle DCE.

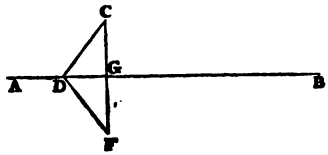
COR. 3. Hence also on an assigned straight line and at an assigned point in it, may be described a rectilinear figure having its sides and angles respectively equal to those of a given rectilinear figure of any number of sides; and the two rectilinear figures shall be equal.

For the given rectilinear figure may be divided into triangles, by straight lines drawn from some of its angular points to others. And triangles having their sides equal to the sides of these respectively, may (by Cor. 1 above) be constructed in succession, and in the corresponding positions. And because in the corresponding triangles
† I. 8. angles the angles will be† equal respectively, viz. those to which equal sides are opposite; by adding equal angles to equal, the
‡ INTERC. 1. angles of one of the rectilinear figures will be‡ equal to the angles
Cor. 5. of the other respectively. And because the several triangles are equal respectively, the sums, which are the rectilinear figures, are equal.

PROPOSITION XXIII.

PROBLEM.—*To draw a straight line perpendicular to an assigned straight line of unlimited length, from an assigned point without it.* See Note.

Let AB be the assigned straight line, which may be prolonged to any length that is desired, both ways; and let C be the point without it. It is required to draw from the point C, a straight line which shall be perpendicular to AB.



In AB take any point, as D; and join CD. If CDB is a right angle, then there is drawn a straight line CD perpendicular to AB as required. But if CDB is not a right angle, then on the other side of AB make* an angle BDF equal to BDC, and make† DF equal to DC, and join CF. Because the equal angles CDB and FDB are either greater or less than right angles, and all the angles at the point D are‡ together equal to four right angles; DC and DF make an angle less than two right angles, either on one side or the other. Wherefore CF will* form a triangle with them, on that side on which is the angle CDF that is less than the sum of two right angles. Bisect† CF, in G; and join DG. Because in the triangles CDG, FDG, DC is‡ equal to DF, and CG to* FG, and DG is common; the triangles are† equal, and the angle CDG is equal to FDG, and DGC to DGF, and because they are adjacent angles they are right‡ angles. And because the angle CDF has been shown to be bisected by the straight line DG, and was also* bisected by the straight line DB, DG and DB are† in one straight line; wherefore EB shall pass through G, and cut‡ CF. And because CGD and CGB are* together equal to two right angles, and CGD is a right angle, CGB is also a right angle, and the straight line CG drawn from the point C is† perpendicular to AB. Which was to be done.

*I.22.Cor.2. †I. 3. ‡I.12.Cor.2. *I.13.Cor.3. †I. 9. ‡Constr. * Constr. † I. 8. ‡I.Nom.37. * Constr. † I. 9. Cor. ‡I.13.Cor.2. * I. 12. †I.Nom.37.

And by parity of reasoning, the like may be done in every other instance.

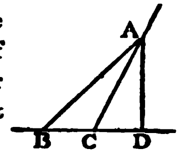
COR. 1. Of all the straight lines that can be drawn to a straight line from a point without it, the perpendicular is the shortest.

For any other will be the‡ hypotenuse of a right-angled triangle, and therefore* greater than the perpendicular.

‡I.Nom.48. *I. 19. Cor.

COR. 2. If two straight lines make an acute angle ACD; and from a point A in one of them, a straight line be drawn* perpendicular to the other; the perpendicular shall fall on that side of C, on which is the acute angle.

* I. 23.



† Hyp.

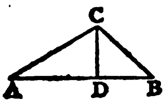
For the perpendicular cannot fall upon the point C; because ACD is† not a right angle but an acute angle. Neither can it fall on the side of C on which is the obtuse angle, as for instance upon the point B; for since ACD (which is the exterior angle of the triangle ACB) is less than a right angle, still more is ABC (which is‡ less than ACD) less than a right angle. Therefore, because it neither falls upon C, nor on the side of it towards B, it falls on the side towards D; that is, on which is the acute angle.

‡ I. 16.

COR. 3. If in any triangle a side AB be taken which lies between* two acute angles, and from the opposite angular point a perpendicular CD be drawn† to the side; the perpendicular shall fall *between* the extremities of the side.

* I.17.Cor.3.

† I. 23.

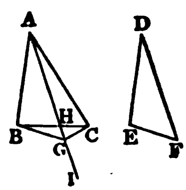


For because BAC is an acute angle, CD (by Cor. 2 above) falls on the side of A which is towards B. And because ABC is an acute angle, CD falls on the side of B which is towards A. Therefore it falls between A and B.

PROPOSITION XXIV.

THEOREM.—*If two triangles have two sides of the one, equal to two sides of the other respectively, but the angle between the two sides of the one triangle is greater than the angle between the two sides which are equal to them in the other triangle; the remaining side of that which has the greater angle, shall be greater than the remaining side of the other.*

See Note.



Let ABC, EDF be two triangles which have the two sides AB, AC of the one, equal to the two DE, DF of the other respectively, viz. AB equal to DE, and AC to DF; but the angle BAC is greater than the angle EDF. The remaining side BC is greater than the remaining side EF.

Of the two sides AB, AC, let AB be one that is not greater than the other; and at the point A in the straight line
I.22.Cor.2. AB, make the angle BAI equal to the angle EDF. Because AB is not greater than AC, the angle ACB is not greater than
† I. 19. ABC; for if it was greater, AB the side opposite to it would be† greater than AC the side opposite to the other. Again, the
‡ I. 16. exterior angle AHC is‡ greater than the interior opposite angle ABH or ABC; and because the angle ACB is not greater than ABC, the angle AHC is greater than ACB or ACH. Therefore in the triangle AHC, the side AC which is opposite to
* I. 19. the angle AHC the greater, is* greater than AH which is opposite to the angle ACH the less. Since, therefore, AH is less than
† I. 3. AC, make† AG equal to AC, and the point G will be on the different side of BC from that on which is the point A. Join BG, GC.

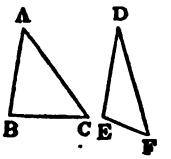
‡ Hyp. *Constr. †Constr. ‡ I. 4. *Constr. † I. 5. Because AB is‡ equal to DE, and AG to* AC or to DF, and the angle BAG is† equal to the angle EDF, BG is‡ equal to EF. And because AC is* equal to AG, the angle AGC is† equal to ACG. But the angle ACG is greater than BCG; therefore the angle AGC (which is equal to ACG)
‡INTERC. 1. Cor. 2. is‡ greater than BCG. Still more is the angle BGC (which is greater than AGC) greater than BCG. Wherefore in the triangle BGC, the side BC (which is opposite to the angle
* I. 19. BGC the greater) is* greater than BG (which is opposite to the angle BCG the less). But BG has been shown to be
†INTERC. 1. Cor. 2. equal to EF; therefore BC (which is greater than BG) is† greater than EF.

And by parity of reasoning, the like may be proved in all other triangles under the same conditions. Wherefore, universally, if two triangles have two sides &c. Which was to be demonstrated.

PROPOSITION XXV.

THEOREM.—*If two triangles have two sides of the one, equal to two sides of the other respectively, but the third side of the one is greater than the third side of the other; the angle between the sides of that which has its third side the greater, shall be greater than the angle between the sides equal to them, of the other.*

Let ABC, DEF be two triangles which have the two sides AB, AC equal to the two DE, DF respectively, viz. AB to DE, and AC to DF; but the third side BC is greater than the third side EF. The angle A is greater than the angle D.



For, if it be not greater, it must either be equal or less. But the angle A cannot be equal to the angle D; because then the
* I. 4. third side BC would be* equal to the third side EF; and it is not
† Hyp. equal, for it is† greater. Neither can it be less; because then the
‡ I. 24. third side BC would be‡ less than the third side EF; and it is not less, for it is greater. But because the angle A is neither equal to the angle D, nor less, it is greater.

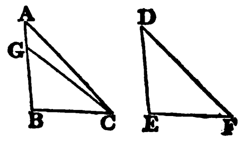
And by parity of reasoning, the like may be proved in all other triangles under the same conditions. Wherefore, universally, if two triangles have two sides of the one, &c. Which was to be demonstrated.

PROPOSITION XXVI.

THEOREM.—*If two triangles have two angles of the one, equal to two angles of the other respectively; and have moreover one side of the one triangle equal to one side of the other triangle, viz. either the sides which lie between the equal angles, or sides opposite to equal angles in each; then shall the remaining angle of one triangle be equal to the remaining angle of the other; and the two triangles shall be equal; and their other sides shall be equal respectively, viz. those which are opposite to equal angles.*

Let ABC, DEF be two triangles which have the angles ABC, ACB equal to the angles DEF, DFE respectively, viz. ABC to DEF, and ACB to DFE; and also one side in the triangle ABC equal to one side in the triangle DEF.

First Case; where the side BC which lies *between* the angles ABC and ACB in the triangle ABC, is equal to EF which lies between the angles DEF and DFE in the triangle DEF.

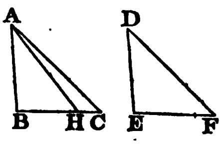


The remaining angle BAC shall be equal to the remaining angle EDF; and the triangle ABC to the triangle DEF; and their other sides, which are opposite to equal angles in the two triangles, shall be equal respectively, viz. AB to DE, and AC to DF.

For, if BA be not equal to ED, one of them must be the
* I. 3. greater. Let BA be assumed to be the greater; and make* BG equal to ED, and join CG.

† Hyp. Because BC is† equal to EF, and BG to ED, BC and BG are
‡ Hyp. equal to EF and ED respectively. Also the angle CBG is‡ equal
* I. 4. to the angle FED; therefore the triangles CBG, FED must be* equal, and the angle GCB, which is opposite to BG, must be equal to the angle DFE which is opposite to ED. But the angle DFE
† Hyp. is† equal to the angle ACB; therefore the angle GCB (which is
‡ Interc. 1. Cor. 1. equal to DFE) must be‡ equal to ACB, the less to the greater, which is impossible. It cannot be, therefore, that BA is greater than ED; and in the same manner may be shown that ED cannot be the greater; and because neither of them is greater than the other, they are equal. Wherefore, BA being shown to be equal to ED,
* Hyp. and BC being* equal to EF, BA and BC are equal to ED and EF
† Hyp. respectively; and also the angle ABC is† equal to the angle DEF;
‡ I. 4. therefore AC is‡ equal to DF, and the triangle ABC to the triangle DEF, and the remaining angle BAC to the remaining angle EDF.

Second Case; where a side *opposite to* one of the angles, as AB in the triangle ABC, is equal to DE which is opposite to the equal angle in the triangle DEF. The remaining angle BAC shall be equal to the remaining angle EDF, and the triangle ABC to the triangle DEF: and their other sides, which are opposite to equal angles in the two triangles, shall be equal respectively, viz. AC to DF, and BC to EF.

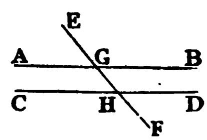


For if BC be not equal to EF, let BC be assumed to be the
I.3. greater; and make BH equal to EF, and join AH.
†Hyp. Because BH is equal to EF, and BA to† ED, BH and BA
‡Hyp. are equal to EF and ED respectively. Also the angle ABH is‡ equal to the angle DEF; therefore the triangles ABH, DEF
I. 4. must be equal, and the angle AHB which is opposite to BA must be equal to the angle DFE which is opposite to ED. But the
†Hyp. angle DFE is† equal to the angle ACB or ACH; therefore the
‡Interc. 1. Cor. 1. angle AHB (which was shown to be equal to DFE) must be‡ equal to ACH, the exterior angle equal to the interior and op-
I. 16. posite, which is impossible. It cannot be, therefore, that BC is greater than EF; and in the same manner may be shown that EF cannot be the greater; and because neither of them is greater than the other, they are equal. Wherefore, BC being
†Hyp. shown equal to EF, and BA being† equal to ED, BC and BA are
‡Hyp. equal to EF and ED respectively; and also the angle ABC is‡
I. 4. equal to the angle DEF; therefore AC is equal to DF, and the triangle ABC to the triangle DEF, and the remaining angle
See Note. BAC to the remaining angle EDF.

And by parity of reasoning, the like may be proved in all other triangles under the same conditions. Wherefore, universally, if two triangles have two angles of the one, equal to two angles of the other respectively; &c. Which was to be demonstrated.

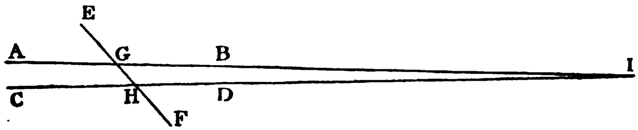
PROPOSITION XXVII.

Theorem.—*If a straight line falling upon two other straight lines makes the alternate angles equal to one another, those two straight lines being prolonged ever so far both ways, shall not*
See Note. *meet.*



First Case; where the two straight lines are both in the same plane. Let the straight line EF, falling upon the two straight lines AB, CD, which are both in the same plane, make the alternate angles GHC, HGB, equal to one another. AB and CD, being prolonged ever so far both ways, shall not meet.

For if this be disputed, let it be assumed that on being prolonged they meet on the side of B and D, in the point I.



Therefore GHI must be a triangle, and its exterior angle GHC greater* than the interior opposite angle HGI. Which is impossible, for it is† equal to it. It cannot be, therefore, that AB and CD being prolonged, will meet on the side of B and D. And in the same manner may be shown that they cannot meet on the side of A and C. So also if HGA, GHD had been the alternate angles taken to be equal.

* I. 16.
† Hyp.

Second Case; if the two straight lines are not in the same plane, they can never meet though prolonged ever so far both ways, whether the alternate angles they make with the other straight line are equal or not.

For if they ever meet, they must‡ lie wholly in one plane. But they are *not* in one plane; therefore it cannot be true that they ever meet.

‡ INTERC. 14. Cor. 2.

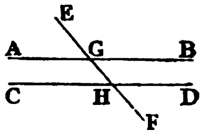
And by parity of reasoning, the like may, in either of the Cases, be proved of all other straight lines under the same conditions. Wherefore, universally, if a straight line falling upon two other straight lines &c. Which was to be demonstrated.

NOMENCLATURE.—Straight lines which are in the same plane, and which being prolonged ever so far both ways do not meet, are called *parallel.*

COR. 1. If a straight line falling upon two other straight lines which are both in the same plane, makes the alternate angles equal to one another, those two straight lines shall be parallel.

For it has been shown in the Proposition above, that being prolonged ever so far both ways they will not meet.

COR. 2. If a straight line falling upon two other straight lines which are both in the same plane, makes the exterior angle equal to the interior and opposite on the same side of the line; or if it makes the two interior angles on the same side, together equal to two right angles; those two straight lines shall be parallel.



For *First;* where the exterior angle EGA is equal to the interior and opposite angle GHC. Because the angles EGA, HGB are vertical angles, they are* equal to one another. And because the angle EGA is† equal to GHC, the angle HGB (which is equal to EGA) is‡ equal to GHC. And they are alternate angles; therefore (by Cor. 1 above) AB and CD are parallel.

* I. 15.
† Hyp.
‡ INTERC. 1. Cor. 1.

Secondly; where HGB and GHD the interior angles on the same side of the line, are together equal to two right angles. Because the angles HGB, GHD are together* equal to two right angles, and the angles GHC, GHD are also together† equal to two right angles; the angles HGB, GHD are together‡ equal to GHC, GHD. Take away the common angle GHD; therefore the remaining angle HGB is* equal to the remaining angle GHC. And they are alternate angles; therefore AB and CD (by Cor. 1 above) are parallel.

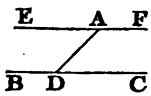
* Hyp.
† I. 12.
‡ INTERC. 1.
* INTERC. 1. Cor. 7.

COR. 3. If the exterior angle EGA is *less than* the interior and opposite angle GHC, GA and HC being prolonged ever so far, will not meet on the side of A and C.

For (by Cor. 2 above) when the angle GHC is equal to EGA, GA being prolonged ever so far will not meet HC. Still less can it meet a straight line drawn from H and lying on the other side of HC; as will be the straight line that makes with HG an angle greater than GHC or than its equal EGA.

PROPOSITION XXVIII.

See Note. PROBLEM.—*Through an assigned point to draw a straight line, parallel to an assigned straight line.*



Let A be the assigned point, and BC the assigned straight line. Through the point A it is required to draw a straight line, parallel to BC.

Interc. 9. Cor. In BC take any point as D, and join AD; and through the
†Interc.14. pointsB,C,A, let a plane be† made to pass, which (because BC is a straight line) will also have the point D in it, and consequently also the straight line DA. In this plane, on the straight line AD
‡I.22.Cor.2. and at the point A in it, make‡ the angle DAE equal to the angle
Interc.12. Cor. 6. ADC; and prolong EA to F.

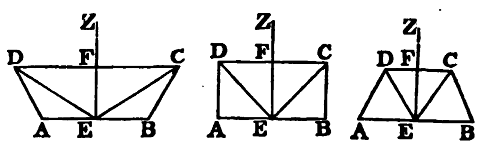
Because the straight line AD meets the two straight lines BC and EF which are both in the same plane, and the alternate
† Constr. angles EAD and ADC are† equal to one another, EF and BC are‡
‡I.27.Cor.1. parallel. Therefore through the point A is drawn a straight line EAF, parallel to BC. Which was to be done.

And by parity of reasoning, the like may be done in every other instance.

PROPOSITION XXVIII *A*.

Theorem.—*If from the ends of a given straight line* [which shall for distinction be called the base], *two straight lines equal to one another be drawn towards the same side, making equal interior angles at the base, each less than the sum of two right angles, but the two equal straight lines do not meet; and the extremities of the two equal straight lines be joined; there shall be formed a quadrilateral figure, on that side of the base on which are the angles each less than the sum of two right angles; of which the angles opposite to the base shall be equal to one another, and each less than the sum of two right angles.*

Let AB be the given straight line, and from its ends A and B let there be drawn the two equal straight lines AD and BC towards the same side, making equal interior angles BAD and ABC with AB the base,



each less than the sum of two right angles, but AD and BC do not meet (as might happen where the angles BAD

INTERC. 9. Cor. and ABC are less than right angles); and let DC be joined. There shall be formed a quadrilateral figure ABCD, on the side of AB on which are the angles BAD and ABC; of which the angles ADC, BCD, which are opposite to AB, shall be equal to one another, and each less than the sum of two right angles.

† I. 9. ‡ I. 10. Bisect† AB, in E; and from E draw‡ a straight line EZ of unlimited length, at right angles to AB, on the side of it on which are the angles BAD and ABC. Join also ED, EC.

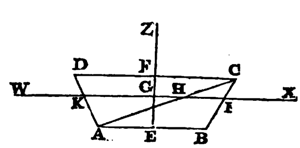
Because at the point B the straight lines BE and BC make an
* Hyp. angle EBC that is* less than the sum of two right angles, and E and C which are points in these two straight lines are joined;
†I.13.Cor.3. there will be formed† a triangle EBC, on the side on which is the angle EBC that is less than the sum of two right angles. In the same way EAD will be a triangle. And because the
‡I.12.Cor.1. angles BEC, CED, AED are‡ together equal to two right angles, CED is less than the sum of two right angles. Therefore
I.13.Cor.3. the straight line DC, which joins the points D and C, will form a triangle with the straight lines ED and EC on the side on which is the angle CED. Wherefore DC will lie wholly on the side of AB on which are the angles BAD and ABC that are each less than the sum of two right angles; and ABCD will be a quadrilateral figure on the same side. And because in the tri-
† Constr. ‡ Hyp. * Hyp. †I. 4. angles EDA and ECB, AE† and AD‡ are equal to BE and BC respectively, and the angle EAD is* equal to the angle EBC, ED is† equal to EC, and the angle AED to BEC, and the angle EDA to ECB. And because BEC, CED, AED are together equal to two right angles, BEC and AED are together less than two right angles; and because they are also equal to one another, they are each less than one right angle, and less than AEZ or BEZ; wherefore EC, ED will fall on different sides of EZ. And because the
‡I. 11. angle AEZ is‡ equal to BEZ, and AED to BEC, the remaining
INTERC. 1. Cor. 7. angle DEZ is equal to the remaining angle CEZ. Wherefore the angle CED is bisected by the straight line EZ; and because in the angles DEF, CEF, the sides ED and EF are equal to the sides EC and EF respectively, and the angle DEF has been shown to
† I. 4. be equal to the angle CEF, FD is† equal to FC; and the angle EFD is equal to EFC, and because they are adjacent angles
‡I.Nom.37. they are‡ right angles; and the angle EDF is equal to ECF. But the angle EDA has been shown to be equal to ECB; therefore
INTERC. 1. Cor. 5. the sum of the angles EDF and EDA is equal to the sum

of the angles ECF and ECB; or the angle ADC is equal to the angle BCD. Also these are each less than the sum of two right angles. For if they were each equal to two right angles,
I.13.Cor.4. ADCB would be one straight line, and two straight lines ADCB
†Interc.10. Cor. 1. and AB would inclose a space, which is† impossible; and if they were yet greater [*that is, if* DA *and* CB *made with* DC *equal angles each less than two right angles on the side towards* Z] the straight line joining the extremities A and B must (as before shown) lie on the side of DC which is towards Z; and it does not, for it is on the other side. And since they are each neither *equal* to the sum of two right angles nor *greater*, they are *less*.

And by parity of reasoning, the like may be proved of all other straight lines under the same conditions. Wherefore, universally, if from the ends of a given straight line &c. Which was to be demonstrated.

Cor. 1. The side of the quadrilateral figure which is opposite to the base, is bisected at right angles by the perpendicular drawn from the middle of the base; and is parallel to the base.

For it has been shown that EF bisects DC at right angles.
‡Constr. Wherefore, because EFC is a right angle, and AEF also is‡ a
I. 11. right angle, they are equal to one another. And they are
†I.27.Cor.1. alternate angles; therefore DC and AB are† parallel.



Cor. 2. If through any point in EF as G, a straight line of unlimited length both ways (as WX) be drawn at right angles to EF; it shall cut the straight lines AD and BC between their extremities, and make with them angles GKA, GIB on the side towards AB the base, each greater than the angle ACB which is between the straight lines drawn from C to the two extremities of the base.

For WX cannot meet either AB or CD; because, since the alternate angles DFG, FGX, and WGE, GEB, are right angles,
‡I. 11. and consequently equal‡ to one another, it will be* parallel to them
*I.27.Cor.1. both. Therefore it will cut AD and BC between their extremities; for if it did not, it must meet either AB or CD. Let it cut them in I and K; and because GIB is the exterior angle of
†I. 16. the triangle HIC, it is† greater than the interior and opposite HCI or ACB.

COR. 3. If a straight line WX of unlimited length both ways, be made to pass through the point E at right angles to EZ, and afterwards be moved along EZ keeping ever at right angles to it, till it pass through the point F; it will never cease to cut the straight lines AD, BC, but will always cut them and make with each of them, on the side towards AB, an angle greater than ACB.

For (by Cor. 2 above) it will make such angles with them, in all and every of the situations through which it moves.

NOMENCLATURE.—A quadrilateral figure as above, viz. of which two opposite sides are equal, and the two interior angles made by them with a side between them are equal; is called a *tessera*.

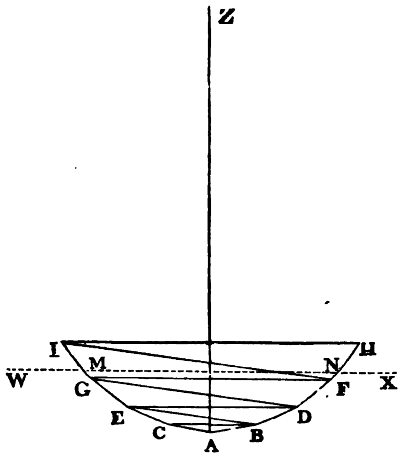
PROPOSITION XXVIII *B*.

THEOREM.—*If two equal straight lines make an angle less than the sum of two right angles, and at the outward extremity of each be added another straight line equal to the first, having its extremity terminated in the extremity of the other, and making with it an angle equal to the first-mentioned angle and on the same side; and at the outward extremity of each of these last, be added another equal straight line as before, and so on continually; and the extremities of every two equal straight lines so successively added at one and the same time at the two ends of the series, be joined by a straight line or chord; each of these chords shall make the angles at the two cusps, where it meets the series, equal to one another; and the several chords shall in succession make greater and greater angles at the cusp, each than the preceding.*

Let AB, AC be two equal straight lines, making an angle BAC less than the sum of two right angles; and at B, C, let there be added the straight lines BD, CE, each equal to AB or AC, and making the angles ABD, ACE, each equal to BAC; and let BC, DE be* joined. And in like manner let there be added DF and EG, each equal to AB or AC, and

Article Parallels in Penny Cyclopædia, by De Morgan.
The author makes the assumption that the second pair of straight lines will not always meet. The proposition that they do not always meet is an Axiom.

making the angles BDF, CEG, each equal to BAC, and let FG be joined; and so on successively. The angles ABC, ACB, which are at the two ends of the straight line BC, shall be equal to one another; and the angles BDE, CED, which are at the two cusps of the figure DBACE, shall also be equal to one another; and the angles DFG, EGF, which are at the two cusps of the figure FDBACEG, shall



be equal to one another; &c. Also the angle BDE shall be greater than the angle ABC, and DFG than BDE; and so on.

Interc. 9. Cor. Join BE, DG, FI, &c.

†Hyp. ‡I.13.Cor.3. Because BAC is† less than the sum of two right angles, the straight line which joins B and C will‡ form a triangle. And because in the triangle BAC the sides AC and AB are equal, the

* I. 5. † Hyp. angle ABC is* equal to ACB. From the equal† angles ABD, ACE (which are each less than the sum of two right angles), take

‡Interc. 1. Cor. 7. away the equal angles ABC, ACB, and there remain‡ the equal angles CBD, BCE, each less than the sum of two right angles.

* Hyp. Also, the sides BD, CE are* equal to one another; wherefore

†I. 28 *A*. Nom. DBCE is† a tessera, and the angles BDE, CED equal‡ to one

‡I. 28 *A*. another. And in the same way may be shown that FDEG, HFGI, &c. are likewise tesseras. Again, because at the ends of the

* Hyp. base AC are drawn* two equal straight lines AB, CE, making with

† Hyp. AC the equal† angles CAB, ACE, each of which is less than the sum of two right angles; BACE is a tessera, and the angles ABE, CEB are equal to one another. And because the angles ABE, CEB are equal to one another, and the angles ABD, CEG

‡ Hyp. are‡ equal to one another, the remaining angle EBD will be*

*Interc. 1. Cor. 7. equal to the remaining angle BEG; wherefore, because BD, EG

† Hyp. are† equal to one another, DBEG is also a tessera; and in the same manner may be shown that FDGI, &c. are likewise tesseras. And because the angle ABE is greater than ABC, and

the angle CEB has been shown to be equal to ABE, the angle
Interc. 1. Cor. 2. CEB is greater than ABC; still more is the angle CED (which is greater than CEB) greater than ABC. And because the angle BDG is greater than BDE, the angle EGD (which is equal to BDG) is greater than BDE; still more is the angle EGF (which is greater than EGD) greater than BDE. And the angle EGF is equal to DFG; therefore DFG is greater than BDE. And so on.

And by parity of reasoning, the like may be proved of every other series of equal straight lines that make equal angles with one another as supposed. Wherefore, universally, if two equal straight lines &c. Which was to be demonstrated.

Nomenclature.—If the angle BAC be bisected by a straight line AZ of unlimited length, such straight line is called the *axis* of the series; the point A is called the *vertex;* and the angle CAZ or BAZ, the *angle with the axis.*

Cor. 1. If the angular points of any one of the tesseras be joined diagonally as I and F, the angle GIF is greater than the angle ABC or ACB, which are the angles made by joining the extremities of the first two straight lines of the series.

For the angles ABC, CEB, BDE, EGD, DFG, GIF, &c. have been shown to be successively greater each than the preceding.

Cor. 2. If a figure be made by joining any two whatsoever of the points A, B, C, D, E, F, &c.; the angles at each of the two cusps of such figure shall be equal to one another.

For it may be shown by the same process as in Prop. XXVIII *B* above; beginning from the middle angular point if the number of angular points is odd, and from the two middle ones if even.

Cor. 3. The chords BC, DE, FG, &c. are bisected at right angles by the axis.

† Hyp. For because AB is† equal to AC, and the portion of the axis intercepted between A and the intersection of BC with the axis is
‡ I. 4. common, and the angle BAZ equal to CAZ; BC is‡ bisected by the axis, and the angles at the intersection are equal, and conse-
I. Nom. 37. quently right angles. And because the axis thus bisects the base of
†I. 28 *A*. Cor. 1. the tessera DBCE at right angles, it† bisects the opposite side DE at right angles. And so on.

Cor. 4. If a straight line WX of unlimited length both ways, be made to pass through the vertex A at right angles to AZ, and afterwards be moved along AZ keeping ever at right angles to it; it will at all times (after passing through B and C) make with such parts of the series as it cuts, interior angles on the side towards A each greater than ACB.

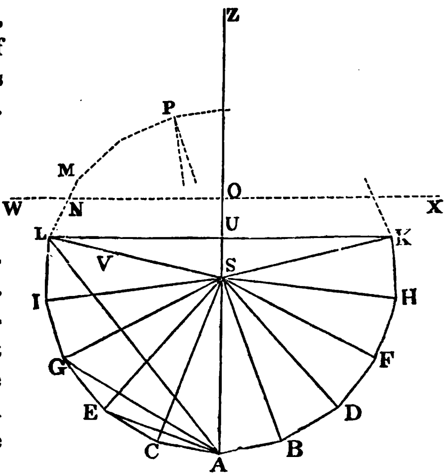
For when it passes through any of the chords, (which, by Cor. 3 above, are at right angles to the axis), as GF, it makes (by Prop. XXVIII *B* above) angles EGF and DFG, each greater than ACB. And when it passes through any other points as M and
* I. 28 *A*. Cor. 2. N, it makes* angles GMX and FNW, each greater than GIF, which is (by Cor. 1 above) greater than ACB.

SCHOLIUM.—Though it has been shown that the angle at the cusp goes on continually increasing, this is not sufficient to prove that it will ever arrive at a certain specified greatness; as, for instance, that it will ever be so great as a right angle, or as CAZ, or even as twice the angle ACB. Nevertheless it is allowable to establish what will be the results *if* the angle at the cusp is ever found to have arrived at a certain specified greatness. Which is what is done in the Proposition next following.

PROPOSITION XXVIII *C*.

THEOREM.—*In a series of equal straight lines as in the last Proposition, if the angle at the cusp be ever greater than the angle with the axis; the straight lines bisecting the several angles of the series (being prolonged) shall have met in the axis, in the part that is between the chord and the vertex.*

Let AB, AC, BD, CE, &c. be a series of equal straight lines as in the last Proposition. If the angle at the cusp, as ILK, be ever greater than the angle with the axis CAZ; the straight lines bisecting the angles ACE, CEG, &c. of the series, being prolonged, shall have met in the axis AZ, in the part which is between the chord at U and the vertex A.



†I.22.Cor.2. ‡Hyp. *INTERC. 1. Cor. 2. At the point L in the straight line LI, make† an angle ILV equal to CAZ; and because the angle ILK is‡ greater than CAZ, the angle ILV (which is equal to CAZ) will be* less than

*I. 28 *B.* Cor. 2. ILK. Also because the angle ILA is* equal to CAL, and CAL
†INTERC. 1. Cor. 2. is less than CAZ; ILA is† less than CAZ, and than ILV which is equal to CAZ. Wherefore the straight line LV will lie *between* the two straight lines LU, LA, and being prolonged will
‡I.13.Cor.5. cut‡ the perimeter of the triangle LUA; and because it cannot cut it in LU or LA (for then two straight lines would inclose a space), it will cut it in UA, that is to say, it will cut AZ the axis in some point between U and A. Let it cut it in S; and join SC, SE, SG, &c. These shall be the straight lines bisecting the angles ACE, CEG, EGI, &c. of the series.

Constr. Because the angle CAS is equal to ILS, and CAL to ILA,
†INTERC. 1. Cor. 7. the remaining angle SAL is† equal to SLA; wherefore in the
‡ I. 6. triangle ASL, the side SL is‡ equal to SA. If then SC be not also equal to SA, it must either be greater or less. And first, let it be assumed that it is greater. But if SC be greater than SA,
* I. 18. the angle SCA, which is opposite to SA the less, must be *less than SAC, which is opposite to SC the greater; and the angle SAC is
†INTERC. 1. Cor. 12. equal to half the angle BAC or to† half the angle ACE; therefore
‡INTERC. 1. Cor. 2. the angle SCA must be‡ less than half the angle ACE, and the angle SCE must be greater than half the angle ACE, and consequently greater than the angle SCA which is less than half. Therefore in the triangles SCA, SCE, because the sides SC, CA are equal to the sides SC, CE respectively, but the angle SCE is
I. 24. greater than SCA, the third side SE must be greater than the third side SA. Again, join AE, and because SE is greater than
† I. 18. SA, the angle SEA must be† less than SAE. Add to each
‡ I. 28 *B.* the equal‡ angles CEA, CAE, and the whole angle SEC must be*
*INTERC. 1. Cor. 6. less than the whole angle SAC. But SAC is equal to half the
†INTERC. 1. Cor. 12. angle BAC, or to† half the angle CEG; therefore SEC must be‡
‡INTERC. 1. Cor. 2. less than half the angle CEG; and SEG must be greater than half the angle CEG, and consequently greater than the angle SEC which is less than half. Therefore in the triangles SEC, SEG, because the sides SE, EC are equal to the sides SE, EG respectively, but the angle SEG is greater than SEC, the third
* I. 24. side SG must be* greater than the third side SC. But SC was greater than SA; still more therefore must SG be greater than SA. In like manner, by joining AG, may be shown that because SG is greater than SA, the angle SGA must be less than SAG, and the angle SGE less than SAC, and consequently less than half EGI; whence the angle SGI must be greater than half EGI, and conse-

quently greater than SGE which is less than half; wherefore the third side SI must be greater than the third side SE, and consequently greater than SA which has been shown to be less than SE; and so on, with each of the other straight lines in succession. Wherefore it will follow that SL must be greater than SA; which is impossible, for it is equal to it. The assumption*, therefore, which involves this impossible consequence, cannot be true; or SC is not greater than SA. And in the same way may be shown, that it cannot be less. But because SC is neither greater than SA nor less, it is equal to it. And because SA is equal to SC, the angle SCA is† equal to SAC. And the angle SAC is‡ half the angle BAC, therefore SCA is* equal to half the angle BAC, and consequently equal† to half the angle ACE; whence the remainder SCE is equal to half the angle ACE, and the angle ACE is bisected by the straight line CS. And because SC, CE are equal to SC, CA respectively, and the angle SCE is equal to the angle SCA, the third side SE is‡ equal to the third side SA, and the angle SEC to the angle SAC; wherefore the angle SEC is equal to half the angle BAC, or to* half the angle CEG, and the angle CEG is bisected by the straight line ES. And in the same way may be shown in succession, that all the other angles EGI, &c. of the series are severally bisected by the straight lines drawn from the angular points G, &c. to S. But because† only one straight line can bisect a given angle; straight lines which should have been drawn bisecting the several angles of the series, if sufficiently prolonged, would have necessarily met in S.

*I. Nom. 26. † I. 5. ‡ Hyp. * Interc. 1. Cor. 1. † Interc. 1. Cor. 12. ‡ I. 4. * Interc. 1. Cor. 12. † I. 14. Cor.

And by parity of reasoning, the like may be proved of every other series of straight lines under the same conditions. Wherefore, universally, in a series of equal straight lines &c. Which was to be demonstrated.

Cor. 1. If the angle at the cusp be ever *equal to* the angle with the axis, the straight lines bisecting the angles of the series shall in like manner have met in the axis, at its intersection with the chord.

For the angle ILV will be equal to ILK, and the point S coincide with the point U.

Cor. 2. If a straight line of unlimited length WX moving along the axis as before, shall at any time when it cuts the series but not in any of its angular points, make with it on the side remote from the vertex an angle MNX *equal to* or *less than* the angle with the axis CAZ, the straight lines bisecting the angles of the series shall in like manner have met in the axis, at a point between O which is the intersection of the axis with WX, and the vertex A.

For if the angle MNX is *equal to* CAZ, it is equal to NLV which is* equal to CAZ, and LV though prolonged ever so far will† never meet NX ; still less‡, if the angle MNX is *less than* NLV. Whereon may in either case be shown as has been done before, that LV being prolonged will cut the axis in some point between O and A, as S; and that the straight lines bisecting the angles of the series will have met in S.

* Constr.
† I.27.Cor.2.
‡ I.27.Cor.3.

Cor. 3. If the series ever *re-meets* the axis, the straight lines bisecting the angles of the series shall have met in the axis, at a point nearer to the vertex.

For if P be the last of the angular points before the series re-meets the axis, it may be shown as was done with the angular point L, that if P and the vertex A be joined, the straight line bisecting the angle of the series at P will lie on the side of such joining straight line which is towards the axis; and consequently the line bisecting the angle of the series at P will cut the axis in some point as S, between the vertex and the point in which the series re-meets the axis. Whereon may be shown as before, that the straight lines bisecting the angles of the series will have met in S.

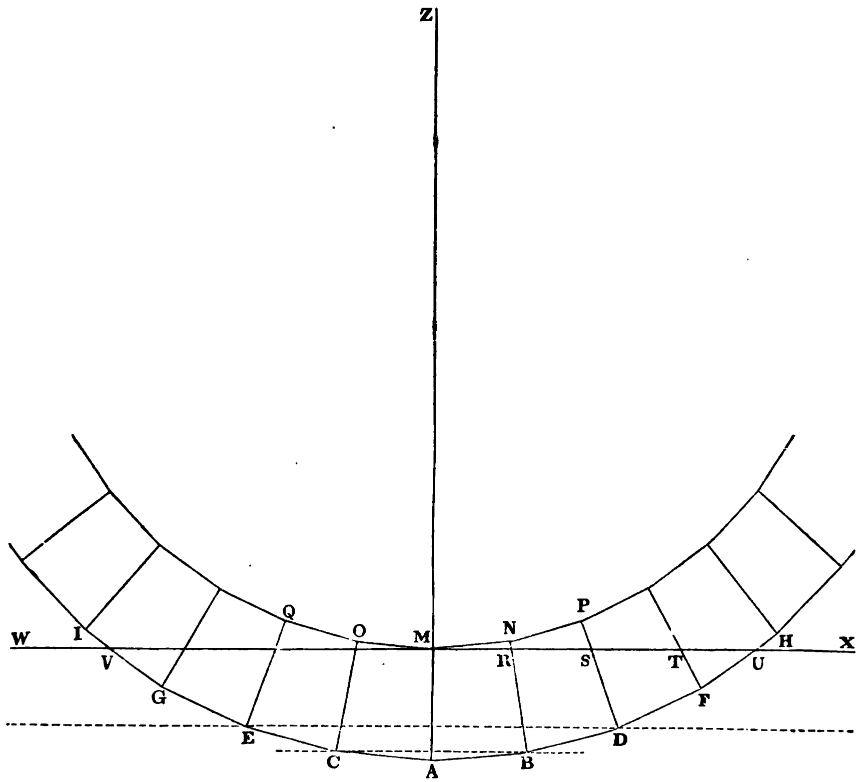
Scholium.—It would be easy to collect, that under the conditions stated in the Proposition above and its Corollaries, all the angular points of the series lie in the circumference of a circle whose centre is S; and consequently the series being continued will meet the axis, if it has not done so already. But as this is no where distinctly appealed to in anything that follows, it will be no further noticed.

PROPOSITION XXVIII *D*.

Theorem.—*If the angles at the base of a tessera be less than right angles, the angles opposite to the base cannot be right angles.*

Let ABNM be a tessera, in which the angles BAM, ABN at the base AB are less than right angles. AMN, BNM which are opposite to the base, cannot be right angles.

Interc. 12. Cor. 6. Prolong AM to an unlimited length on the side of *Z*. At the
†I.22.Cor.3. point A in the straight line AM describe† the rectilinear figure



ACOM, equal in all respects to ABNM, and having its base AC (at which are the angles that are equal to BAM, ABN), adjacent to the base AB. In like manner at the points B and C, make the rectilinear figures BP and CQ, equal in all respects to ABNM or ACOM; and so on continually on both sides, for as many as shall at any time be desired.

‡Constr. Because the angles MAB, NBA, MAC, OCA, &c. are‡ equal to one another and each less than a right angle, and the straight lines AB, AC, BD, CE, &c. are equal to one another; these equal straight lines form a series as in Prop. XXVIII *B*; of

which series AZ is the axis. Let then a straight line WX of unlimited length both ways, be made to move along the axis, from the vertex A towards Z, keeping ever at right angles to the axis; till it cuts the axis in M. But it has been shown that on the side towards A, it will at all times during such motion (after passing through BC) make with such parts of the series as it may cut, an interior angle each way, greater* than a given angle ABC. And on the side towards Z, it will at all times make an interior angle each way, greater than a given angle BAZ. For when WX arrives at any of the angular points (as D, E), the next of the equal straight lines of the series (as DF) cannot fail to make with WX an angle EDF greater than BAZ which is half BAC, unless the angle BDE at the cusp be equal to or greater than half BAC; which last cannot be, because if BDE the angle at the cusp was either equal to† or greater than‡ half BAC, the straight lines bisecting the angles of the series must have met in the axis, in the part between the point where WX in passing through D and E cuts the axis, and the vertex; and they have not. And for a like reason, it cannot at any time during its passing from one pair of angular points to another, have* failed to make an interior angle on the side towards Z, greater than BAZ; neither can it have† re-met the axis. Wherefore (because WX can in no way have previously ceased to cut the series and make with it on both sides of WX an interior angle greater than a given angle) it will cut the series when it arrives at the point M; as for instance in U and V. Whence the angles opposite to the base of the tessera ABNM cannot be right angles. For if this be disputed, let it be assumed that they are right angles. But if so, then in the tessera whose base is AB, because AMX is a right angle, the side opposite to the base must be in the same straight line with MX; and the angle MRB must be‡ equal to AMR and consequently a right angle, and BR equal to AM; and because MRB, BRX are* together equal to the sum of two right angles and MRB is a right angle, BRX must be a right angle; and because in the tessera whose base is BD, the side which falls on the extremity of the base B is† equal to AM, and BR is equal to AM, such side must be‡ equal to BR and coincide with it; and for like reasons the sides opposite to the bases BD, DF, &c. of the tesseras which follow in order, must coincide with further portions RS, ST, &c. of the straight line

* I. 28 *B*. Cor. 4.

† I. 28 *C*. Cor. 1.

‡ I. 28 *C*.

* I. 28 *C*. Cor. 2.

† I. 28 *C*. Cor. 3.

‡ I. 28 *A*.

* I. 12.

† Constr.

‡ Interc. 1.

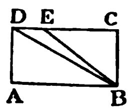
WX; wherefore, because WX cuts the series, some or other of these sides must meet its base, as FH in U; which is impossible,
* I. 28 *A*. Cor. 1. because every one of these sides is* parallel to its base. The as-
† I. Nom. 26. sumption†, therefore, which involves this impossible consequence, cannot be true; or the angles opposite to the base of the tessera ABNM, cannot be right angles.

And by parity of reasoning, the like may be proved of every other tessera where the angles at the base are less than right angles. Wherefore, universally, if the angles at the base of a tessera be less than right angles, &c. Which was to be demonstrated.

Cor. 1. If the angles at the base of a tessera be right angles, the angles opposite to the base cannot be less than right angles.

For if they were, then (by considering the opposite side as a base) there would be a tessera having the angles at the base less than right angles, and the angles opposite to the base right angles; which (by the Prop. above) is impossible.

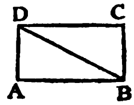
Cor. 2. If the angles at the base AB of a tessera be right angles, the side opposite to the base is not greater than the base.



‡ I. 3. For if CD be greater than AB, from CD cut off‡ CE equal to AB; and join BD, BE. Because DEB is the exterior angle of the triangle ECB
* I. 16. it is* greater than the interior and opposite angle
† Interc. 1. Cor. 2. C; and consequently greater† than the angle EDA which is‡
‡ I. 28 *A*. equal to C, and still more than the angle EDB which is less than
* I. 19. EDA; wherefore BE the side opposite to EDB is* less than BD the side opposite to DEB. But if CE is equal to AB, then
† Hyp. because AD is† equal to CB, and AB to CE, the triangles BAD, ECB must have the sides AD, AB equal to the sides CB, CE
‡ Hyp. respectively. And because the angle BAD is‡ a right angle, and the angle BCE (by Cor. 1 above) is not less than a right angle, the angle BCE is not less than the angle BAD; for if it was less than BAD, it must be less than a right angle, and it is not. Whence the third side BE must be not less than the third side
* I. 25. BD; because if it was less, the angle BCE must be* less than the angle BAD, and it is not. Which is impossible, for BE has been shown to be less than BD. It is not true, therefore, that CE, which is a part of CD, is equal to AB; that is, that CD is greater than AB.

PROPOSITION XXVIII *E*.

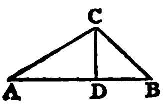
THEOREM.—*In every right-angled triangle the two acute angles are together not less than a right angle.*



Let DAB be a triangle, of which the angle
I.17.Cor.3. DAB is a right angle. The two acute angles
ADB, DBA, are together not less than a right
angle.
†I. 10. From the point B draw† a straight line BC at right angles to
‡I. 3. BA; and make‡ BC equal to AD. And join CD.
Constr. Because DABC is a tessera having the angles at the base AB
†I. 28 *D*. Cor. 2. right angles, the side CD is† not greater than AB. Whence in the triangles DAB, BCD, the sides DA, DB are equal to the sides BC, BD respectively, and the third side AB is not less than the third side CD; wherefore the angle ADB is not less than the
‡I. 24. angle CBD; for if it was less, the third side AB must be‡ less than the third side CD, and it is not. And because the angle ADB is not less than CBD, add to each the angle DBA, and the sum of ADB and DBA will be not less than the sum of CBD and DBA; for if it was less, then by taking DBA from each, the
INTERC. 1. Cor. 8. remaining angle ADB must be less than the remaining angle CBD, and it is not. But the sum of the angles CBD and DBA
†Constr. is the angle CBA which is† a right angle; therefore the sum of ADB and DBA is not less than a right angle.

And by parity of reasoning, the like may be proved in every other right-angled triangle. Wherefore, universally, in a right-angled triangle the two acute angles are together not less than a right angle. Which was to be demonstrated.

COR. 1. The three angles of every triangle are together not less than two right angles.

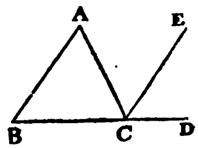


For because every triangle has at the least
‡I.17.Cor.3. two‡ acute angles, let AB be a side of the
triangle which lies between two such. From
I. 23. the opposite angular point draw a straight
line CD perpendicular to AB; and because the side AB lies be-
†I.23.Cor.3. tween two acute angles, CD will† fall between the extremities of AB. And because the triangle ADC is right-angled at D, the angles DAC, ACD (by the Proposition above) are together not less than a right angle. And for the like reason the angles BCD, DBC are together not less than a right angle. Wherefore the

angles DAC, ACD, BCD, DBC are together not less than two right angles. But the angles ACD, BCD are together equal to the angle ACB; therefore the angles DAC, ACB, DBC are together not less than two right angles.

COR. 2. If one side of a triangle be prolonged, the two interior opposite angles are together not less than the exterior angle.

For if BC be prolonged to D, the angles ACB and ACD are* together equal to two right angles. And the angles ACB, ABC, BAC are (by Cor. 1 above) together not less than two right angles. Whence ACB, ABC, BAC, are together not less than ACB and ACD; for if they were less, they must also be† less than two right angles, and they are not. And taking away the common angle ACB, the angles ABC and BAC are together not less than ACD; for if they were less, then by adding ACB to each, the angles ACB, ABC, BAC must be‡ less than ACB and ACD, and they are not.

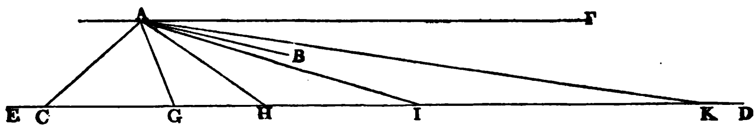


* I. 12.
† INTERC. 1 Cor. 2.
‡ INTERC. 1. Cor. 6.

PROPOSITION XXVIII *F.*

THEOREM.—*If two straight lines which are in the same plane are met by a third, and the two interior angles on the one side of this third straight line are together less than two right angles; the two first-mentioned straight lines being continually prolonged, shall at length meet on that side of the other straight line on which are the angles which are together less than two right angles.*

Let the two straight lines AB and CD which are in the same plane, be met by the straight line AC; and let the two interior



angles CAB, ACD on the one side of AC, be together less than two right angles. AB and CD being continually prolonged, shall at length meet on the side of B and D.

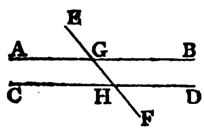
INTERC. 12. Cor. 6. Prolong DC to E; and at A make† the angle CAF equal to
†I. 22. Cor. 2. ACE. In CD take‡ CG equal to CA; and join GA.
‡ I. 3. Because the angle CAF is equal to ACE, the angles CAF and
INTERC. 1. Cor. 4. ACD are together equal to ACE and ACD; but ACE and
†I. 12. ACD are† together equal to two right angles; wherefore CAF

* Interc. 1. Cor. 1. and ACD are* together equal to two right angles. And because
† Hyp. CAB and ACD are† together less than two right angles, they are‡
‡ Interc. 1. Cor. 2. less than CAF and ACD. Wherefore, by taking away the
* Interc. 1. common angle ACD, the angle CAB is* less than CAF.
Cor. 8. And because CG is† equal to CA, the angle CAG is‡ equal
† Constr. ‡ I. 5. to CGA; and because these angles CAG, CGA are also* together
* I. 28 *E.* not less than the exterior angle ACE, CAG is not less than half
Cor. 2. † Interc. 1. the angle ACE, or than half the angle CAF which is† equal to it.
Cor. 12. In GD take‡ GH equal to GA, and join HA; and it may be shown
‡ I. 3. in like manner that the angle GAH is not less than half the angle AGE. But the angle AGE is equal to CAG, which is not less than half CAF and consequently not less than the remaining angle GAF; wherefore the angle AGE is not less than GAF, and GAH (which is not less than half AGE) is not less than half GAF. And in like manner take HI equal to HA, and join IA; and so on. Wherefore, because from the angle CAF is cut off not less than its half, and from the remainder not
* Interc. 1. Cor. 18. less than its half, and so on; there will* at length remain an
† I. 13. Cor. 5 angle, as KAF, less than FAB, and AB being prolonged will† cut the perimeter of the triangle CAK, in some point between C and K; that is to say, it will meet CD.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, if two straight lines which are in the same plane are met by a third, and the two interior angles &c. Which was to be demonstrated.

PROPOSITION XXIX.

Theorem.—*If a straight line falls upon two parallel straight lines, it makes the alternate angles equal to one another.*



Let the straight line EF fall upon the parallel straight lines AB, CD; the alternate angles GHC, HGB are equal to one another.

For, if the angle GHC be not equal to HGB, one of them must be the less. Let HGB be assumed to be the less. Wherefore, since HGB is less than GHC, if each be added to the
‡ Interc. 1. Cor. 6. angle GHD, the angles HGB and GHD must be‡ together less
* I. 12. than the angles GHC and GHD. But GHC and GHD are*

together equal to two right angles; therefore HGB and GHD
INTERC. 1. Cor. 2. must be together less than two right angles; wherefore the straight lines AGB, CHD, being continually prolonged must
†I. 28 *F.* meet† on the side of B and D. Which cannot be, for they are‡
‡Hyp. *I. Nom. 26. parallel. The assumption*, therefore, which involves the impossible consequence, cannot be true; or the angle HGB is not less than GHC. In the same way may be shown, that GHC is not less than HGB. But because neither is less than the other, they are equal. And in the same way might be shown that the angles GHD, HGA are equal to one another.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, if a straight line falls upon two parallel straight lines, &c. Which was to be demonstrated.

COR. 1. If a straight line falls upon two parallel straight lines, it makes the exterior angle equal to the interior and opposite on the same side of the line; and makes also the two interior angles on the same side, together equal to two right angles.

For *First;* because (by the Prop. above) it makes the alternate
† I. 15. angles GHC, HGB equal to one another, and HGB is† equal to
‡INTERC. 1. Cor. 1. EGA, the angle EGA is‡ equal to GHC. And in the same way, because the alternate angles GHD, HGA are equal, might be shown that EGB and GHD are equal.

Secondly; because the angle GHC is equal to HGB, add to
INTERC. 1. Cor. 4. each the angle GHD, and the angles GHC and GHD will be
†I. 12. together equal to HGB and GHD. But GHC and GHD are† together equal to two right angles; therefore HGB and GHD
‡INTERC. 1. Cor. I. are‡ together equal to two right angles. And in the same way, by adding to GHD and HGA the angle GHC, might be shown that HGA and GHC are together equal to two right angles.

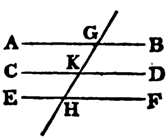
COR. 2. Through a point in one of two parallel straight lines, no other straight line can be drawn in the same plane, which (all the straight lines being continually prolonged both ways) shall not meet the other of the two parallel straight lines.

For if through any point in CD, as H, any straight line other than CD be drawn, it will make with a line drawn from H to any point G in AB, the two interior angles on one side or on the other together less than two right angles; therefore (on the lines being
*I. 28 *F.* prolonged) it will* meet AB.

PROPOSITION XXX.

THEOREM.—*Straight lines which are parallel to the same straight line (the whole being in the same plane), are parallel to one another.*

Let AB, CD be each of them parallel to EF; all the three being in the same plane. AB shall be parallel
See Note. to CD.



Let the straight line GKH cut AB, CD, EF.

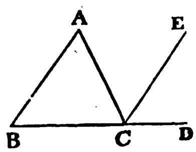
Because GKH cuts the parallel straight lines AB and EF, the alternate angles
I. 29. AGH, GHF are equal to one another. Again, because the straight line GKH cuts the parallel straight lines
†I.29.Cor.1. CD and EF, the exterior angle GKD is† equal to the interior and opposite on the same side, GHF. And it was shown that the angle
‡INTERC. 1. AGH is equal to GHF; therefore the angle AGH is‡ equal to
I.27.Cor.1. GKD; and they are alternate angles; therefore AB is parallel to CD.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, straight lines which are parallel to the same straight line &c. Which was to be demonstrated.

SCHOLIUM.—The same is true when the straight lines are not all in the same plane. But this is not demonstrated by Euclid till Book XI.

PROPOSITION XXXI.

THEOREM.—*If a side of any triangle be prolonged, the exterior angle is equal to the sum of the two interior and opposite angles.*



Let ABC be a triangle, and let one of its sides, as BC, be produced to D. The exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC.

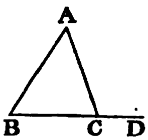
†I. 28. Through C draw† CE parallel to BA.

Because CE is parallel to BA, and CA
‡I. 29. falls upon them, the alternate angles ACE, CAB are‡ equal. Also,
I.29.Cor.1. because DB falls upon them, the exterior angle ECD is equal to ABC the interior and opposite on the same side of the
†INTERC. 1. Cor. 5. line. Therefore the sum of ACE and ECD is† equal to the sum of CAB and ABC. But the sum of ACE and ECD is the angle
‡INTERC. 1. Cor. 1. ACD; therefore ACD is‡ equal to the sum of CAB and ABC.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, if a side of a triangle be prolonged, &c. Which was to be demonstrated.

PROPOSITION XXXII.

THEOREM.—*The three interior angles of every triangle are equal to two right angles.*



Let ABC be a triangle. The three interior angles CAB, ABC, and BCA, are together equal to two right angles.

INTERC. 12. Cor. 6. Prolong one side of the triangle; as BC.

† I. 31. Because the exterior angle ACD is† equal to the sum of the two interior and opposite angles ABC and CAB;

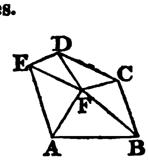
‡ INTERC. 1. Cor. 4. add to each the angle BCA, and the sum of ACD and BCA is‡ equal to the sum of the three angles ABC, CAB, and BCA.

* I. 12. But the sum of ACD and BCA is* equal to two right angles,

† INTERC. 1. Cor. 1. Therefore the sum of the three angles ABC, CAB, and BCA is† equal to two right angles.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, the three interior angles of a triangle are equal to two right angles. Which was to be demonstrated.

COR. 1. All the interior angles of any plane rectilinear figure that incloses a space, are equal to twice as many right angles as the figure has sides, diminished by four right angles.



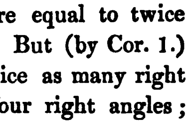
For *First;* if the figure is one, as ABCDE, that can be divided into as many triangles as the figure has sides, by drawing straight lines from one point F within the figure to each of its angular points. All the angles of these triangles (by the preceding Proposition) are together equal to twice as many right angles as there are triangles; that is, to twice as many right angles as the figure has sides. But the angles at the point F are

‡ I. 12. Cor. 2. together‡ equal to four right angles. Therefore the other angles of the triangles, (which are together equal to all the interior

* INTERC. 1. Cor. 7. angles of the figure), are together* equal to twice as many right angles as the figure has sides, diminished by four right angles.

Secondly; if the figure be such as cannot be divided into triangles by lines drawn from one point within the figure, it can be divided into smaller rectilinear figures each severally capable of being so divided. And because the interior angles of each of such smaller figures are equal to twice as many right angles as the figure has sides, diminished by four right angles, (or, which is the same thing, to twice as many right angles as the sides of the figure with the subtraction of two sides); the interior angles of all these smaller figures are together equal to twice as many right angles as is the number of the sides of all those figures after subtraction of two sides from each. And this number after subtraction, is equal to the number of sides of the whole or composite figure with the subtraction of two; for every straight line that makes division between the smaller figures, constituted and was counted as a side of two of those figures, and the number of the divisions is less by one than the number of the figures. Wherefore the interior angles of all the smaller figures, are* together equal to twice as many right angles as is the number of sides of the whole or composite figure with the subtraction of two; or, which is the same thing, to twice as many right angles as the whole or composite figure has sides, diminished by four right angles.

*Interc. 1. Cor. 10.



Cor. 2. In any plane rectilinear figure inclosing a space, of which each of the interior angles is less than two right angles; if the sides are prolonged consecutively, the exterior angles so made are together equal to four right angles.

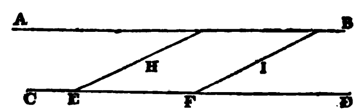
See Note.

For because every interior angle ABC, with its adjacent exterior ABD, is† equal to two right angles; all the interior together with all the exterior angles of the figure, are equal to twice as many right angles as the figure has sides. But (by Cor. 1.) all the interior angles are together equal to twice as many right angles as the figure has sides, diminished by four right angles; and (by adding four right angles to each), all the interior angles together with four right angles, are‡ equal to twice as many right angles as the figure has sides. Therefore all the interior together with all the exterior angles of the figure, are* equal to all the interior angles of the figure together with four right angles; and by taking away the interior angles from both, all the exterior angles of the figure are† equal to four right angles.

†I. 12.
‡Interc. 1. Cor. 4.
*Interc. 1.
†Interc. 1. Cor. 7.

PROPOSITION XXXII *bis*.

THEOREM.—*If in one of two parallel straight lines be taken two points, and through these be drawn two other straight lines parallel to one another; on all the straight lines being continually prolonged both ways, there shall be formed a quadrilateral figure, of which the opposite sides are parallel.*



Let AB and CD be two parallel straight lines, and E and F two points taken in one of them. Any two other straight lines (as EH, FI) drawn through E and F parallel to one another, on all the lines being continually prolonged both ways shall meet AB and form a quadrilateral figure, of which the opposite sides are parallel.

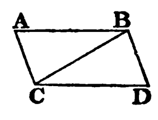
Because through the point E is drawn a straight line EH other
I.29.Cor.2. than ED, EH and AB (being prolonged) will meet. And because FI, EH are parallel and DC meets them, the exterior angle
†I.29.Cor.1. DFI will† be equal to the interior and opposite FEH; wherefore FI will be a straight line other than FD, and FI and AB (being
‡I.29.Cor.2. prolonged) will‡ meet, and there will be formed a quadrilateral
* Hyp. figure; and its opposite sides are* parallel.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, if in one of two parallel straight lines &c. Which was to be demonstrated.

NOMENCLATURE.—A quadrilateral rectilinear figure, formed (as by the Proposition above) by four lines of which two and two are parallel, is called a *parallelogram*.

PROPOSITION XXXIII.

See Note. THEOREM.—*The opposite sides and angles of every parallelogram are equal to one another; and a diagonal bisects it [that is, divides it into two equal parts].*



Let ACDB be a parallelogram, of which BC is a diagonal. The opposite sides and angles of the figure shall be equal to one another, and BC shall bisect it.

Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are* equal to one another. And because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal to one another. Wherefore the two triangles ABC, DCB have two angles ABC, ACB in the one, equal to two angles BCD, CBD in the other respectively, and one side BC common to the two triangles, and lying between the equal angles in each; therefore their other sides are† equal respectively, viz. the side AB equal to the side CD, and AC to BD; and the angle BAC is equal to the angle BDC; and the triangle ABC is equal to the triangle DCB, and the parallelogram is bisected by BC. And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB, the whole angle ABD is‡ equal to the whole ACD. And in the same way, if the angular points A and D were joined, might be shown that the opposite sides and angles of the parallelogram were equal, and that the diagonal from A to D likewise bisects the parallelogram.

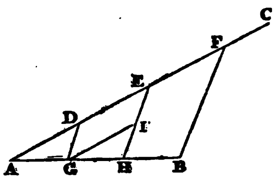
* I. 29.
† I. 26.
‡ Interc. 1. Cor. 5.

And by parity of reasoning, the like may be proved in every other parallelogram. Wherefore, universally, the opposite sides and angles of every parallelogram &c. Which was to be demonstrated.

Cor. 1. By the help of this Proposition, may be shown how to divide a given straight line AB into any assigned number of equal parts.

See Note.

For, from one of its extremities A, draw a straight line AC of unlimited length, making with AB any angle less than the sum of two right angles. From the point A in the straight line AC cut off any part as AD; and in the same straight line cut off* DE, EF successively equal to AD, till the number of equal parts taken in AC is the same as the number into which it is required to divide AB. Join the extremity of the last (as for instance F) with the extremity B of AB; and at D, E, make† the angles ADG, AEH, respectively equal to the angle AFB; wherefore DG, EH will be‡ parallel to FB and to one another, and meet AB as in G, H. Because ADG and GDE are* together equal to two right angles, and AGD and DGH are together equal to two right angles, the



* I. 3.
† I. 22. Cor. 2.
‡ I. 27. Cor. 2.
* I. 12.

INTERC. 1. Cor. 5. † I. 17. four are together equal to four right angles. But ADG and AGD are together† less than two right angles; wherefore GDE and DGH are together greater than two right angles. At G ‡I.22.Cor.2. then make‡ an angle DGI equal to GDA; and because DGI (or GDA) and GDE are together equal to two right angles, and DGH and GDE are together greater than two right angles, *INTERC. 1. Cor. 8. DGH is* greater than DGI; and GI, which is† parallel to DE †I.27.Cor.1. and also lies between GH and GD, being prolonged will‡ meet EH ‡I.13.Cor.5. between E and H, as in I. Because DGIE is* a parallelogram, * Constr. † I. 33. GI is† equal to DE, and consequently‡ to AD. Also because GI ‡ INTERC. 1. Cor. 1. and DE are parallel, and BA meets them; the angles IGH and DAG *I.29.Cor.1. are* equal. And because EH and DG are parallel, and AH meets them; the angles DGA and IHG are equal. Wherefore the triangles IGH and DAG have the angles IGH and IHG equal to the angles DAG and DGA respectively, and also the side IG † I. 8. equal to the side DA; consequently the triangles are† equal, and the side GH equal to the side AG. And if GI be prolonged till it meets FB, and from H a straight line be drawn making with HE an angle equal to HEA; in the same way may be shown that HB is equal to GH; and so of any others.

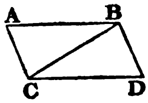
COR. 2. All the perpendiculars drawn from one of two parallel straight lines towards the other, being prolonged will meet it and be perpendicular to it, and be equal to one another.

For because any two such perpendiculars make the two interior angles on the same side of the line (at the points where they meet the straight line from which they are drawn) right angles, they ‡I.27.Cor.2. are‡ parallel, and the whole of the lines being prolonged will* * I.32 *bis.* form a parallelogram; wherefore (by Prop. XXXIII above) its opposite sides, among which are the two perpendiculars, are equal, and its opposite angles are also equal. And because each of the angles which the perpendiculars make with the straight line towards which they are drawn is opposite to a right angle, it is a right angle.

PROPOSITION XXXIV.

THEOREM.—*The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are themselves equal and parallel.*

Let AB, CD be two equal and parallel straight lines and joined towards the same parts [*that is to say, not cross-wise, as would be if* A *were joined with* D, *and* B *with* C] by the straight lines AC, BD. AC, BD are also equal and parallel.



Join BC.

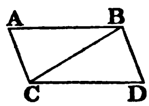
Because AB is parallel to CD, and BC meets them, the alternate
* I. 29. † Hyp. angles ABC, BCD are* equal. And because AB is† equal to CD, and BC common to the two triangles ABC, DCB, the two sides BA, BC are equal to the two CD, CB respectively; and the angle ABC has been shown to be equal to DCB; wherefore the
‡ I. 4. third side AC is‡ equal to the third side DB, and the angle ACB to the angle DBC. And because the straight line BC meets the two straight lines AC, BD, and the alternate angles ACB, DBC
I.27.Cor.1. are equal to one another; AC is parallel to BD. And it has been shown to be also equal to it.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, the straight lines which join the extremities of two equal and parallel straight lines, &c. Which was to be demonstrated.

PROPOSITION XXXIV *bis*.

See Note. THEOREM.—*Every quadrilateral rectilinear figure, of which the opposite sides are equal to one another, is a parallelogram.*

Let ACDB be a quadrilateral rectilinear figure, of which the opposite sides AB, CD are equal to one another, and also the opposite sides AC, BD are equal to one another. ACDB shall be a parallelogram.



Because in the triangles ABC and DCB, AB is equal to CD, and AC to DB, and BC is common, the sides AC and CB of the one are equal to the sides DB and BC of the other respectively, and also the third side AB of the one is equal to the third side DC
† I. 8. of the other; wherefore the angle ACB of the one is† equal to the angle DBC of the other, and the angle ABC to the angle DCB. But because the angles ACB, DBC are equal to one
‡I.27.Cor.1. another, and they are alternate angles, AC is‡ parallel to BD; and because the angles ABC, DCB are equal to one another, and they are alternate angles, AB is parallel to CD; wherefore
*I. 32 *bis*. Nom. ACDB is* a parallelogram.

And by parity of reasoning the like may be proved of every other quadrilateral rectilinear figure, of which the opposite sides are equal to one another. Wherefore, universally, a quadrilateral rectilinear figure, of which &c. Which was to be demonstrated.

Cor. 1. In any quadrilateral rectilinear figure of which the opposite sides are equal, if one of the angles be a right angle, all the angles shall be right angles.

For, because the quadrilateral figure (by the Prop. above) is a
I. 33. parallelogram, the opposite angles are equal to one another; wherefore if one of the angles be a right angle, the angle opposite to it will also be a right angle, and the two together will be equal to two right angles. And because all the angles of every quadri-
†I.32.Cor.1. lateral rectilinear figure are† together equal to four right angles, the two remaining angles will be together equal to two right
‡I. 33. angles. And because they are opposite angles, they are‡ equal to one another; therefore each of them will be a right angle.

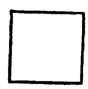
Cor. 2. If one of the angles be not a right angle, none of the angles shall be right angles; but there shall be two angles greater than right angles, and two less.

For if one of the angles be *greater* than a right angle, the angle opposite to it will also be greater; and the two will be together greater than two right angles; and the two remaining angles will be together less than two right angles; and because they are equal to one another, each of them will be less than a right angle. And in the same way, if one angle and its opposite be *less* than right angles, the others will be greater.

Cor. 1 above. Nomenclature 1. A parallelogram of which the angles are right angles, is called a *rectangle.*

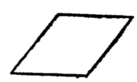
A rectangle is said to be *made* by any two of the straight lines between which is one of its right angles. Or for brevity, the rectangle made by two straight lines AB, BC, is called the rectangle AB, BC; or more briefly still, the rectangle ABC.

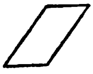
Nom. 2. A rectangle of which all the sides are equal, is called a *square.*



Nom. 3. A rectangle of which only the opposite sides are equal, is called an *oblong.*

Nom. 4. A parallelogram of which all the sides
†Cor. 2 above. are equal, but the angles are† not right angles, is called a *rhombus.*





Nom. 5. A parallelogram of which only the opposite sides are equal, and the angles are not right angles, is called a *rhomboid.*

Nom. 6. All other quadrilateral rectilinear figures beside these, are called *trapeziums.*

Nom. 7. The perpendicular drawn from one of the angular points of a triangle to the side opposite or to its prolongation, is called the *altitude* of the triangle in that direction; and the side to which or to its prolongation such perpendicular is drawn, is (with relation to that altitude) called the *base.*

Nom. 8. The perpendicular drawn from any point in one side of a parallelogram to the side opposite or to its prolongation,
*I.33.Cor.2. [*which perpendicular has been shown to be* of the same length from whatever point in the side it may be drawn*], is called the *altitude* of the parallelogram in that direction; and the side to which or to its prolongation such perpendicular is drawn, is (with relation to that altitude) called the *base.*

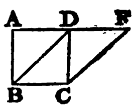
Nom. 9. When two or more triangles or parallelograms are placed between the same parallels, [*which is only another way*
†I.33.Cor.2. *of declaring that*† *their altitudes are equal*]; the sides which are placed in the same straight line in one of the parallels for this purpose, are in like manner called their *bases.*

Scholium.—Though the terms *altitude* and *base* are borrowed from the relation between a building and its foundation, they are not confined to objects in any particular position with respect to the horizon; any more than the term *perpendicular,* which originally meant 'the line determined by a plummet.'

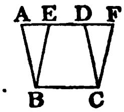
PROPOSITION XXXV.

Theorem.—*Parallelograms upon the same base, and between the same parallels, are equal to one another.*

Let the parallelograms AC, BF, be upon the same base BC, and between the same parallels AF, BC. The parallelograms shall be equal to one another.

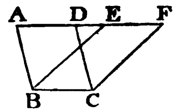


First Case; if the parallelograms have two of the sides opposite to the base BC, terminated in
‡ I. 33. the same point D; each of the parallelograms is‡
* Interc. 1. Cor. 10. double of the triangle BDC; and they are* therefore equal to one another.



Second Case; if the side EF of the one parallelogram be terminated in some point *within* the side AD of the other. Because ABCD is a parallelogram, AD is* equal to BC; for the same reason EF is equal to BC; wherefore AD is† equal to EF. From each of the equals AD, EF, take away ED; and the remainder AE is‡ equal to the remainder DF. AB also is* equal to DC, and BE to CF. Wherefore in the triangles BAE, CDF, the sides BA, AE are equal to the sides CD, DF respectively; and the third side BE is equal to the third side CF; therefore the triangles BAE, CDF are† equal to one another. Add to each the trapezium EBCD, and the sums will be‡ equal; that is, the parallelogram ABCD will be equal to the parallelogram EBCF.

* I. 33.
† Interc. 1.
‡ Interc. 1. Cor. 7.
* I. 33.
† I. 8.
‡ Interc. 1. Cor. 4.



Third Case; if the side EF of the one parallelogram be terminated in some point *without* the side AD of the other. AD is equal to EF, as before. To each of the equals AD and EF, add DE; and the sum AE is* equal to the sum DF. Wherefore the triangles BAE, CDF have their sides respectively equal as before, and the triangles are† equal. From the trapezium ABCF take the triangle CDF, and from the same trapezium take the triangle BAE, and the remainders will be‡ equal; that is, the parallelogram ABCD will be equal to the parallelogram EBCF.

* Interc. 1. Cor. 4.
† I. 8.
‡ Interc. 1. Cor. 7.

And by parity of reasoning, the like may be proved of all other parallelograms under the same conditions. Wherefore, universally, parallelograms upon the same base, &c. Which was to be demonstrated.

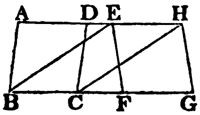
See Note.

Cor. Every parallelogram is equal to the rectangle made by any one of its sides, and its altitude as measured to such side or to its prolongation.

For such parallelogram and rectangle are parallelograms upon the same base and between the same parallels.

PROPOSITION XXXVI.

THEOREM.—*Parallelograms upon equal bases, and between the same parallels, are equal to one another.*



Let ABCD, EFGH be parallelograms upon equal bases BC, FG, and between the same parallels AH, BG. The parallelogram ABCD is equal to EFGH.

Join BE, CH.

* Hyp. † I. 33. ‡ Interc. 1. Cor. 1. * Hyp. † I. 34. ‡ I. 32 *bis*. Nom. * I. 35.

Because BC is* equal to FG, and FG to† EH, BC is‡ equal to EH. And they are* parallel, and joined towards the same parts by the straight lines BE, CH; therefore BE, CH are† equal and parallel, and EBCH is a‡ parallelogram. And because EBCH, ABCD are parallelograms upon the same base BC, and between the same parallels AH, BG, they are* equal to one another. And because EBCH, EFGH are parallelograms upon the same base EH and between the same parallels, they are equal to one another. Therefore the parallelograms ABCD,

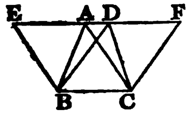
† Interc. 1. EFGH (being each equal to EBCH) are† equal to one another.

[In the same way might the parallelograms have been shown to be equal, if the points E, F, in which are terminated the sides of the parallelogram EFGH, had been situate *within* the points D and C, in which are terminated the sides of the other parallelogram; or if one had been situate within, and one without; or if one, or both, had coincided with the point in which is terminated the side of the other parallelogram.]

And by parity of reasoning, the like may be proved of all other parallelograms under the same conditions. Wherefore, universally, parallelograms upon equal bases, &c. Which was to be demonstrated.

PROPOSITION XXXVII.

THEOREM.—*Triangles upon the same base, and between the same parallels, are equal to one another.*



Let the triangles ABC, DBC be upon the same base BC, and between the same parallels AD, BC. The triangle ABC is equal to the triangle DBC.

‡ Interc. 12, Cor. 6. * I. 3.

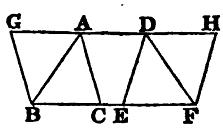
Prolong‡ AD both ways, and take* AE,

DF, each equal to BC; and join BE, CF. Because AE, CB are
* Constr. † Hyp. ‡ I. 34. equal* and parallel†, BE and CA are‡ equal and parallel; and
* I. 32 *bis.* Nom. BCAE is* a parallelogram. For the same reason BCFD is a parallelogram. And because BCAE, BCFD are parallelograms upon the same base BC and between the same parallels BC, EF,
† I. 35. they are† equal to one another. But the triangle ABC is the
‡ I. 33. half of the parallelogram BCAE, because the diagonal AB‡ bisects it; and for the same reason the triangle DBC is the half of the parallelogram BCFD; and the halves of equal magnitudes
* Interc. 1. Cor. 12. are* equal; therefore the triangle ABC is equal to the triangle DBC.

And by parity of reasoning, the like may be proved of all other triangles under the same conditions. Wherefore, universally, triangles upon the same base, &c. Which was to be demonstrated.

PROPOSITION XXXVIII.

Theorem.—*Triangles upon equal bases, and between the same parallels, are equal to one another.*



Let the triangles ABC, DEF be upon equal bases BC, EF, and between the same parallels AD, BF. The triangle ABC is equal to the triangle DEF.

† Interc. 12. Cor. 6. ‡ I. 3. Prolong† AD both ways, and take‡ AG equal to CB, and DH to EF; and join BG, FH.

* Constr. † Hyp. ‡ I. 34. Because AG and CB are equal* and parallel†, BG and CA‡ are equal and parallel, and BCAG is* a parallelogram. For the
* I. 32 *bis.* Nom. same reason EFHD is a parallelogram. And because the parallelograms BCAG, EFHD are upon equal† bases BC, EF, and
† Hyp. ‡ I. 36. between the same parallels BF, GH, they are‡ equal to one another. But the triangle ABC is the half of the parallelogram
* I. 33. BCAG, because the diagonal AB* bisects it; and for the same reason the triangle DEF is the half of the parallelogram EFHD;
† Interc. 1. Cor. 12. and the halves of equal magnitudes are† equal; therefore the triangle ABC is equal to the triangle DEF.

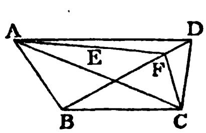
[In the same way might the triangles have been shown to be equal, if the point E in which is terminated a side of the triangle DEF, had been situate *within* the base BC of the other triangle, or had coincided with the point C; or if, in addition to this, the point D had been on the other side of the point A, or had coincided with it.]

And by parity of reasoning, the like may be proved of all other triangles under the same conditions. Wherefore, universally, triangles upon equal bases, &c. Which was to be demonstrated.

PROPOSITION XXXIX.

See Note. THEOREM.—*Equal triangles upon the same base, and on the same side of it, are between the same parallels.*

Let the triangles ABC, BCD, which are upon the same base BC and on the same side of it, be equal to one another. They are between the same parallels.



Join AD; the straight lines AD and BC shall be parallel.

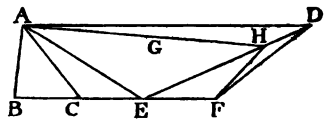
For if the angles DAB, ABC are not together equal to two right angles, let it be assumed that they are greater; and
I. 28. from the point A draw AE parallel to BC. Because
†I.29.Cor.1. AE, BC are parallel, the angles EAB, ABC must be† together equal to two right angles; and because the angles DAB, ABC are supposed to be together greater than two right angles,
‡INTERC. 1. Cor. 2. they must be‡ together greater than the angles EAB, ABC; and taking away the common angle ABC, the remaining angle
INTERC. 1. Cor. 8. DAB must be greater than the remaining angle EAB. Wherefore the straight line AE must lie *within* the straight
†I.13.Cor.5. line AD, and being prolonged must† meet BD. Let it meet it in F; and join FC. Because the triangles ABC, BCF are upon the same base BC and between the same parallels BC
‡I. 37. and AF, the triangle BCF must be‡ equal to the triangle
Hyp. ABC. But the triangle BCD is equal to the triangle ABC;
†INTERC. 1. therefore the triangle BCD must be† equal to the triangle BCF, the
‡I.Nom.26. greater to the less, which is impossible. The assumption‡, therefore, which involves this impossible consequence, cannot be true; or DAB, ABC are not together greater than two right angles. And in the same way may be shown that ADC, DCB cannot be together greater than two right angles. Also they cannot be *less*; for because the interior angles of the quadri-
I.32.Cor.1. lateral figure ABCD are together equal to four right angles, is ADC, DCB are together less than two right angles, the remaining angles DAB, ABC must be together greater than two right

angles; and it has been shown that they cannot be greater; wherefore ADC and DCB cannot be less. And because the angles ADC and DCB together are neither greater nor less than two right angles, they are equal to two right angles. But because the angles ADC, DCB are together equal to two right angles,
I.27.Cor.2. AD and BC are parallel.

And by parity of reasoning, the like may be proved of all other triangles under the same conditions. Wherefore, universally, equal triangles upon the same base, &c. Which was to be demonstrated.

PROPOSITION XL.

See Note. THEOREM.—*Equal triangles, upon equal bases in the same straight line, and on the same side of it, are between the same parallels.*



Let the triangles ABC, EFD, which are upon equal bases BC and EF in the same straight line BF, and on the same side of it, be equal to one another. They are between the same parallels.

Join AD, and AE; the straight lines AD and BF shall be parallel.

For if the angles DAB, ABF are not together equal to two right angles, let it be assumed that they are greater; and
† I. 28. from the point A draw† AG parallel to BF. Because AG,
‡ I.29.Cor.1. BF are parallel, the angles GAB, ABF must be ‡ together equal to two right angles; and because the angles DAB, ABF are supposed to be together greater than two right angles, they must
Interc. 1. Cor. 2. be together greater than the angles GAB, ABF; and taking
† Interc. 1. Cor. 8. away the common angle ABF, the remaining angle DAB must be† greater than the remaining angle GAB. Wherefore the straight line AG must lie *within* the straight line AD; and moreover it must lie on the side of AE which is towards AD, for if not, it would meet BE, which cannot be, for they are parallel; there-
‡ I.13.Cor.5. fore being prolonged it must‡ meet DE, in some point between D and E. Let it meet it in H; and join HF. Because the triangles ABC, EFH are upon equal bases BC and EF, and between the same parallels BF and AH, the
* I. 38. triangle EFH must be* equal to the triangle ABC. But the
† Hyp. triangle EFD is† equal to the triangle ABC; therefore the

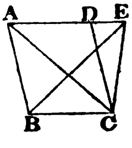
Interc. 1. triangle EFD must be equal to the triangle EFH, the greater
†I.Nom.26. to the less, which is impossible. The assumption†, therefore, which involves this impossible consequence, cannot be true; or DAB, ABF are not together greater than two right angles. And in the same way may be shown that ADF, DFB cannot be together greater than two right angles. Also they cannot be *less*; for
‡I.32.Cor.1. because the interior angles of the quadrilateral figure ABFD are‡ together equal to four right angles, if ADF, DFB are together less than two right angles, the remaining angles DAB, ABF must be together greater than two right angles; and it has been shown that they cannot be greater; wherefore ADF and DFB cannot be less. And because the angles ADF and DFB together are neither greater nor less than two right angles, they are equal to two right angles. But because the angles ADF, DFB
I.27.Cor.2. are together equal to two right angles, AD and BF are parallel.

[In the same way might the triangles have been shown to be between the same parallels, if the point E in which is terminated a side of the triangle EFD, had been situate *within* the base BC of the other triangle, or had coincided with the point C; or if the point D had coincided with the point A, or been on the other side of it.]

And by parity of reasoning, the like may be proved of all other triangles under the same conditions. Wherefore, universally, equal triangles, upon equal bases &c. Which was to be demonstrated.

PROPOSITION XLI.

Theorem.—*If a parallelogram and triangle be upon the same base, and between the same parallels; the parallelogram shall be double of the triangle.*



Let the parallelogram ABCD and the triangle EBC be upon the same base BC, and between the same parallels BC, AE; the parallelogram ABCD is double of the triangle EBC.

Join AC.

The triangle ABC is equal to half of the parallelogram
†I. 33. ABCD, because the diagonal AC† bisects it; and the triangle
‡I. 37. ABC is‡ equal to the triangle EBC, because they are on the same base BC and between the same parallels; wherefore half the
Interc. 1. parallelogram ABCD is equal to the triangle ECB, and the

INTERC. 1. Cor. 10. parallelogram ABCD is equal to the double of the triangle EBC.

And by parity of reasoning, the like may be proved of every other parallelogram and triangle under the same conditions. Wherefore, universally, if a parallelogram and triangle be upon the same base, &c. Which was to be demonstrated.

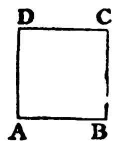
SCHOLIUM. In a similar manner might be demonstrated, that if a parallelogram and triangle be upon *equal* bases and between the same parallels, the parallelogram shall be double of the triangle.

See Note. COR. Every triangle is equal to half the rectangle made by any one of its sides, and its altitude as measured to such side or to its prolongation.

For such a rectangle is a parallelogram upon the same base, and between the same parallels. Therefore (by the Prop. above) it is the double of the triangle.

PROPOSITION XLII.

PROBLEM.—*To describe a square upon a given straight line.*



Let AB be the given straight line; it is required to describe a square upon it.

†I. 10. From the points A and B, draw† AD and BC at right angles to AB and on the same side of it;

‡I. 3. and make‡ AD and BC, each equal to AB; and join CD. ABCD is the square required.

Because the angles BAD and ABC are two right angles, AD

I.27.Cor.2. †Constr. ‡I.34. and BC are parallel. But they are† also equal; wherefore DC and AB which join their extremities, are‡ equal and parallel, and

* I. 32 *bis*. Nom. ABCD is* a parallelogram. And because the opposite sides are equal to one another, and also AB equal to AD; all the four sides are equal to one another. Also because one of the angles, as BAD,

† I. 34 *bis*. Cor. 1. is a right angle, all the angles are† right angles. Wherefore ABCD

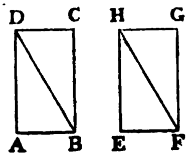
‡ I. 34 *bis*. Nom. 2. is‡ a square; and it is described upon AB. Which was to be done.

And by parity of reasoning, the like may be done in every other instance.

PROPOSITION XLIII.

See Note. THEOREM.—*Rectangles which are made by equal straight lines, are equal to one another.*

Let ABCD, EFGH be two rectangles, of which the side AB



is equal to the side EF, and the side AD to the side EH. The rectangles are equal to one another.

Join BD, FH.

* Hyp. Because AB, AD are* equal to EF, EH
† I. 11. respectively, and the angle BAD is† equal to
‡ I. 4. the angle FEH, the triangle BAD is‡ equal to the triangle FEH.
* I. 33. And because a diagonal of any parallelogram divides* it into two equal parts, the rectangles ABCD, EFGH are respectively double of the triangles BAD, FEH; and because the triangles are equal
† Interc. 1. Cor. 10. to one another, the rectangles which are the double of them are† equal to one another.

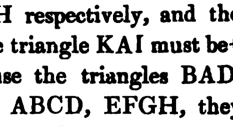
And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, rectangles which are made by equal straight lines, &c. Which was to be demonstrated.

Cor. The squares described upon equal straight lines, are equal to one another.

For such squares are rectangles made by equal straight lines.

PROPOSITION XLIV.

Theorem.—*The sides of equal squares, are equal to one another.*



Let ABCD, EFGH be two equal squares. The side AB of the one square, is equal to the side EF of the other square.

For if AB be not equal to EF, one of
‡ I. 3. them must be the greater. Let AB be the greater; and make‡ AK and AI each equal to EF; and join KI. Join also BD, FH.

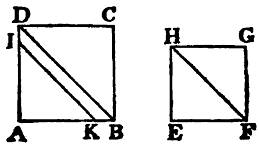
* I. 11. Because AK, AI are equal to EF, EH respectively, and the
† I. 4. angle KAI is* equal to the angle FEH, the triangle KAI must be† equal to the triangle FEH. But because the triangles BAD,
‡ I. 33. FEH are‡ the halves of the equal squares ABCD, EFGH, they
* Interc. 1. Cor. 12. are* equal to one another; therefore the triangle KAI (which is
† Interc. 1. equal to the triangle FEH) must be† equal to the triangle BAD, the less to the greater, which is impossible. Therefore AB is not greater than EF; and in like manner may be shown that EF is not greater than AB. And because neither is greater than the other, they are equal.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, the sides of equal squares, &c. Which was to be demonstrated.

PROPOSITION XLV.

THEOREM.—*The square described upon a greater straight line, is greater than the square described upon a less.*

Let the straight line AB be greater than the straight line EF. The square ABCD which is described upon AB, is greater than the square EFGH which is described upon EF.



* I. 3. Make* AK, AI equal to EF, EH; and join KI. Join also BD, FH.

Because AK, AI are equal to EF, EH respectively, and the
† I. 11. angle KAI is† equal to FEH, the triangle KAI is‡ equal to the
‡ I. 4. triangle FEH. But the triangle BAD is greater than the triangle
Interc. 1. Cor. 2. KAI; therefore it is also greater than the triangle FEH,
† I. 33. Wherefore the square ABCD which is† double of the triangle
‡Interc. 1. Cor. 11. BAD, is‡ greater than the square EFGH which is double of the triangle FEH.

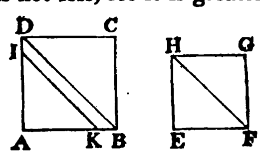
And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, the square described upon a greater straight line, &c. Which was to be demonstrated.

PROPOSITION XLVI.

See Note. THEOREM.—*The greater square has the greater side.*

Let the square ABCD be greater than the square EFGH. The side AB is greater than the side EF.

For if it be not greater, it must either be equal or less. It is not
I. 43. Cor. equal; for then the square ABCD must be equal to the square
† Hyp. EFGH; but ABCD is not equal to EFGH, for it is† greater.
‡ I. 45. Neither is it less; for then the square ABCD must be‡ less than the square EFGH; but ABCD is not less, for it is greater. Wherefore, because the side AB is neither equal to the side EF, nor less, it is greater.

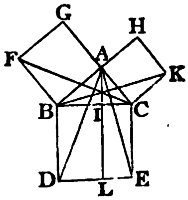


And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, the greater square &c. Which was to be demonstrated.

PROPOSITION XLVII.

THEOREM.—*In any right-angled triangle, the square described on the side which is opposite to the right angle, is equal to the sum of the squares described on the sides between which is the right angle.*

Let ABC be a triangle having the right angle BAC. The square described on the side BC which is opposite to the right angle, is equal to the sum of the squares described on the sides BA, AC, between which is the right angle.



I. 42. On BC describe the square BCED ; and on BA and AC, the squares BAGF, ACKH.
†I. 23. From A draw† AI perpendicular to BC,
‡INTERC. 12. Cor. 6. and prolong‡ it to an unlimited length ; and join AD, AE, CF, BK.

I. 17. Cor. 1. Because ABC and ACB are acute angles, AI will fall† between
†I. 23. Cor. 3. the extremities of BC. And because AIB, IBD are right angles,
‡I. 27. Cor. 1. AL and BD are‡ parallel; and in the same manner AL and CE are parallel. Wherefore AL lies between BD and CE, and is parallel to them; and the quadrilateral figures BL, CL
*I. 32 *bis.* Nom. have their opposite sides parallel, and are* parallelograms. Be-
† Hyp. cause the angle BAC is† a right angle, and the angle BAG
‡ Constr. is‡ also a right angle, AC and AG are* in the same straight line.
* I. 13. For the like reason, AH and AB are in the same straight line.
† I. 11. And because the angle DBC is† equal to the angle FBA (each of them being a right angle), add to each the angle ABC; there-
‡ INTERC. 1. Cor. 4. fore the whole angle DBA is‡ equal to the whole angle CBF; and because the two sides AB, BD are equal to the two FB, BC respectively, and the angle DBA is equal to the angle CBF, the
* I. 4. side DA is* equal to the side CF, and the triangle DBA is equal
† I. 41. to the triangle CBF. Now the parallelogram BL is† double of the triangle DBA, because they are on the same base BD, and between the same parallels BD, AL; and the square BAGF is double of the triangle CBF, because they are on the same base FB, and between the same parallels FB, GC; but the doubles of
‡ INTERC. 1. Cor. 10. equal things are‡ equal to one another; therefore the parallelogram BL is equal to the square BAGF. In the same way may be shown, that the parallelogram CL is equal to the square ACKH. Therefore, by adding equals to equals, the sum of the

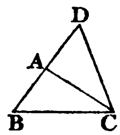
INTERC. 1. Cor. 5. parallelogram BL and the parallelogram CL, is equal to the sum of the square BAGF and the square ACKH. But the sum of the parallelogram BL and the parallelogram CL, is the square †INTERC. 1. BCED; therefore the square BCED is† equal to the sum of the square BAGF and the square ACKH.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, in any right-angled triangle, &c. Which was to be demonstrated.

PROPOSITION XLVIII.

THEOREM.—*If the square described on one of the sides of a triangle, be equal to the sum of the squares described on the other two sides of it; the angle made by those two sides is a right angle.*

Let ABC be a triangle, which is such that the square described on one of its sides BC, is equal to the sum of the squares described on BA and AC. The angle BAC is a right angle.



‡I. 10. *I. 3. From the point A draw‡ AD at right angles to AC; and make* AD equal to AB; and join DC.

†Constr. ‡I. 43. Cor. *INTERC. 1. Cor. 4. †I. 47. ‡Constr. *Hyp. †INTERC. 1. Cor. 3. ‡I. 44. Because AD is† equal to AB, the square on AD is‡ equal to the square on AB. To each add the square on AC; and the sum of the square on AD and the square on AC, is* equal to the sum of the square on AB and the square on AC. But the sum of the square on AD and the square on AC, is† equal to the square on DC; because the angle DAC is‡ a right angle. And the sum of the square on AB and the square on AC is* equal to the square on BC. Therefore the square on DC is† equal to the square on BC. And because the square on DC is equal to the square on BC, DC is‡ equal to BC. But because in the triangles BAC, DAC, the sides BA, AC are equal to the sides DA, AC respectively, and also the third side BC is equal to the third side *I. 8. †Constr. ‡INTERC. 1. Cor. 1. DC; the angle BAC is* equal to the angle DAC. But the angle DAC is† a right angle; therefore the angle BAC, which is equal to it, is‡ also a right angle.

And by parity of reasoning, the like may be proved in every other instance. Wherefore, universally, if the square described on one of the sides of a triangle, &c. Which was to be demonstrated.

END OF THE FIRST BOOK.

NOTES,

FAMILIAR AND GEOMETRICAL.

NOMENCLATURE I.—BOOK I.

'The Way to improve our Knowledge, is not, I am sure, blindly, and with an implicit Faith, to receive and swallow Principles; but is, I think, to get and fix in our minds clear, distinct, and compleat Ideas, as far as they are to be had, and annex to them proper and constant Names.'—*Locke on the Human Understanding*. iv. 12. 6.

NOM. VI.—BOOK I.

It may occur here, that for some purposes requiring extreme accuracy of division, as for instance on the nonius and scales of astronomical instruments, the divisions might be constituted by alternate layers of two metals of different colours, so as to leave a visible line absolutely without breadth. The same result might in a certain degree be obtained, by painting or staining the alternate intervals of different colours. It is not known whether this has ever been put in practice.

NOM. VII.—BOOK I.

The descriptions of a point and a line, as given by the ancients, might be supposed intended to perplex beginners. The description of the first, in particular, appears to approach as nearly as possible to a description of '*nothing.*' The difficulty is referable to a love for enigmatical forms; and vanishes on explanation.

NOM. VIII.—BOOK I.

It is not essential to a figure, that it should inclose a space. Parabolas, hyperbolas, and spirals, are figures; but they do not inclose a space.

Figures of this last kind, may be advanced as examples of *linear* figures, as distinguished from *superficial.*

NOM. XV AND XVI.—BOOK I.

From Euclid's Book of Data, with slight alterations.

NOM. XIX.—BOOK I.

The term Corollary by no means implies the non-necessity of demonstration, as has been sometimes represented. On the contrary, there must be an assignable reason for Corollaries, as for every thing else. It is true the reason may sometimes be one that can hardly fail to present itself though not given; and sometimes Corollaries are little more than recapitulations, of truths which have been evolved in the course of demonstrating some ulterior truth. But with the express view of opposing the notion that Corollaries are of the nature of the imaginary things styled 'self-evident truths,' it has been determined to insert no Corollary without its being followed by a statement of the reason; which is a demonstration in little.

Those who will pursue this plan, will be surprised to find how often they have considerable difficulty in making out a clear statement of the reason of a Corollary, which at first sight seemed perfectly obvious and inevitable. The cause of which appears to be, that in all people the habit of judging by a sort of guess-work, useful enough for many of the purposes of common life but incompetent to the severe distinguishing of truth from falsehood, has greatly outrun the habit and power of rigid demonstration.

The practical use of inserting subsidiary propositions under the form of Corollaries, is, first, to save room, and secondly, to render the connexion more visible between what may be called the leading Propositions. If every Corollary in the First Book of Euclid were presented as a distinct Proposition, it is easy to see how the connexion would be obscured.

NOM. XX.—BOOK I.

In an elementary science like geometry, it would appear to be unnecessary to call anything a Lemma. The term, however, is used by Euclid; and is frequently employed in the progress of the sciences. For example, in optics or astronomy, if the chain of argument is interrupted to introduce some preliminary proposition from geometry or algebra; such interloping proposition is usefully distinguished as a 'Lemma.'

NOM. XXVI.—BOOK I.

As a help to clearness, the use of the word 'must' will be confined to the tracing of false or impossible consequences in the manner here described.

PROPOSITION I.—INTERCALARY BOOK.

COR. 14 and 15. The substance of these two Corollaries is demonstrated by Euclid in Propositions I and V of the Fifth Book; but is previously assumed by him without proof, in the Proposition called by Simson the Twentieth of the Third Book, the object of which is to show that the angle at the centre of a circle is double of the angle at the circumference.

COR. 16. If evidence is demanded of the existence of magnitudes which, being added together to any number however great, cannot surpass a given magnitude, the simplest instance that can be given is in the magnitudes in the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$ &c. Suppose there is first given to a person the *half* of some particular thing; then the *fourth* (which is the half of what was left); then the *eighth* (which is the half of what was left, again); then the *sixteenth*; and so on. It is plain that what this person has received, is increased every time. But it is also plain that it can never amount to the whole thing, or 1; for there will always be a piece left. Still less can it ever amount to more than one, as for instance to two. And if each successive magnitude, instead of being *one half* of the preceding, was in any other constantly diminished proportion, (as for instance $\frac{999}{1000}$ths), it would be equally true (though not so easily proved) that the sum would never surpass some given magnitude.

Hence, to have shown that a magnitude receives perpetual additions, is never evidence that it will arrive at a certain specified amount. Both mathematicians and political economists have fallen into snares for want of attention to this.

COR. 18. Euclid's proof as given in his Tenth Book, and by Simson at the beginning of the Twelfth, establishes the fact on a magnitude vastly smaller than there is any necessity for, and thereby destroys the simplicity of analogy. For example, if C should require to be taken sixteen times in order to be greater than AB, Euclid instead of establishing that the sixteenth part of AB is less than C, establishes it of the 32768th part, being 2048 times less than necessary.

PROP. II.—INTERC. BOOK.

The First Case of this Proposition may to some appear frivolous and vexatious; though it could not well be omitted if the Second was to be demonstrated.

But in the Second Case, the constant equality of distance between the two points, which is the basis of the possibility of their always occupying the same places during the turning, is a fair specimen of a geometrical reason, and one that might easily go unobserved long after an experimental acquaintance with the truth of the proposition enounced.

PROP. V.—INTERC. BOOK. First Case.

If the surface in which the spheres are assumed to coincide, was an annular one or had a void in the middle, the proof would continue to be applicable; and would be so as long as there was any breadth at all between the inner and outer boundaries of the surface. If there is no breadth, the case merges into that of a self-rejoining line which next follows.

PROP. IX.—INTERC. BOOK. Nomenclature.

What is given as the description of a straight line in the translations of Euclid, is nothing but an identical proposition. For to say that a straight line lies 'evenly' between its extreme points, is only to say that it lies 'straightly,' or in other words a straight line is a straight line. At the same time this is not exactly what Euclid has said. For the phrase translated 'evenly,' is in the original ἐξ ἴσου, 'under circumstances of equality.' By which may be understood to be meant, that the line is similarly situated with respect to what may be on each of the two sides of it; a very darkling description, but not amounting to an identical proposition.

It is at the same time remarkable, that Euclid never afterwards makes any reference to this; but when he has occasion for a distinguishing property of straight lines, always has recourse to the property conveyed in what in the translations is called the 10th Axiom, viz. that two straight lines cannot inclose a space. Euclid's description of straight lines therefore practically is, that they are those of which two cannot inclose a space.

Archimedes took as an Axiom, that a straight line is 'the shortest between two points*;' and thence derived the conclusion, that there can be only one straight line between them. This is virtually only a begging of the question, that two sides of a triangle are greater than the third. A parallel case would have been, if a circle had been stated to be 'the plane figure which incloses a given area under the shortest boundary line.' It might be ultimately true; but it would be an odd place to start from, for ascertaining the properties of the circle.

Plato declared a straight line to be 'one in which what is between, is in front of both the ends†;' meaning, apparently, that when the two ends are brought into one by the eye, all the intermediate parts are at the same time brought into one with the two ends and with one another. This appears to be a notification of the fact that rays of light move in straight lines; or at most, a declaration that straight lines are such as examples may be seen of in rays of light. And in this view it seems to have been taken by Proclus, who in quoting it, illustrates it by the occurrence of an eclipse when the moon is in a straight line between the sun and the eye‡. A parallel case would have been, if a circle had been stated to be 'the figure of which examples may be seen in the disks of the sun and full moon.'

Among the new descriptions given by the moderns, that of Bonnycastle is, that a straight line 'is that which has all its parts lying in the same direction§.' This is little more than the identical proposition formerly pointed out; for what is a line whose parts *do not* lie all in the same direction, but a line of which the parts do not all lie in the straight line leading to a particular point or object? Professor Leslie in his 'Rudiments of Plane Geometry' has given nearly the same description of a straight line; and has built on it as consequences, that no more than one straight line can join two points, and that if a straight line be conceived to turn like an axis about both extremities, its intermediate points

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* λαμβάνω δὲ ταῦτα, τῶν τὰ αὐτὰ πέρατα ἐχουσῶν γραμμῶν ἐλαχίστην εἶναι τὴν εὐθεῖαν·—*Archim. De Sphær. et Cyl.*

† καὶ μὴν εὐθύ γε, οὗ ἂν τὸ μέσον ἀμφοῖν τοῖν ἐσχάτοιν ἐπίπροσθεν ᾖ.—*Platonis Parmenid.*

‡ ὅθεν δὴ καὶ οἱ ἀστρολογικοὶ φασὶν τότε τοῦ ἥλιου ἐκλείπειν, ὅταν ἐπὶ μιᾶς εὐθείας γένηται αὐτός τε καὶ ἡ σελήνη, καὶ τὸ ὄμμα τὸ ἡμέτερον. τότε γὰρ ὑπὸ τῆς σελήνης ἐπιπροσθεῖσθαι, μέσης αὐτοῦ τε καὶ ἡμῶν γενομένης.—*Comment. Procli in Primum Euclidis Librum. Lib.* 2.

§ Bonnycastle's Elements of Geometry. p. 2.

will not change their position*. Professor Playfair's description is, that 'if two lines are such that they cannot coincide in any two points, without coinciding altogether, each of them is called a straight line;' after which he inserts under the title of a Corollary, that 'hence two straight lines cannot inclose a space. Neither can two straight lines have a common segment; that is, they cannot coincide in part, without coinciding altogether†.' The Corollary is a very reasonable inference from the premises; but the fault is in the premises, which present the fallacy of attempting to wrap up the debateable matter under a name. 'Coinciding altogether if in any two points,' can no more be called being a straight line, than having the square on the opposite side equal to the sum of the squares of the containing sides can be called being a right angle. No reason is ever given, why the naked fact should exist at all; the inquirer has a right to demand how anybody knows that a line endowed all over with such properties is within the range of possibility,—how it is to be constructed,—where it is to be seen,—what evidence there is that the thing pointed out for it, has the properties asserted. If experience (which is experiment) be referred to, experience will equally prove the ratio of the sphere to the circumscribing cylinder; for whatever is ultimately true, cannot fail to be accordant with experience if tried. But the object of the geometer is to discern *reasons*.

In the subsequent nomenclature, *prolonged* appears to be a better term than 'produced;' if it was only for the advantage of employing the term *prolongation*, which is so often of use in the practical applications of science. The same terms are afterwards applied to planes; to which it would be possible to object, that a plane is not extended in *length* only, but in length and breadth. The French geometricians, however, have not thought the objection of weight; and it is apprehended that the 'prolongation of a plane' is a phrase in general use in English.

NOTE TO THE SCHOLIUM.—For an analogy in the case of angles, see Note to Nom. 36, Book I.

* 'The uniform tracing of a line, which is stretched through its whole extent in the same direction, gives the idea of a *straight* line. No more than one straight line can hence join two points; and if a straight line be conceived to turn like an axis about both extremities, its intermediate points will not change their position.'—*Rudiments of Plane Geometry*. p. 19.

† Playfair's Elements of Geometry. p. 1.

PROP. XII.—INTERC. BOOK.

What is given as the demonstration of this Theorem in the Corollary to the 11th Proposition of Simson's Euclid, is null. For if the operation of drawing a perpendicular to the second straight line or to what is assumed to be such, is actually performed according to the directions laid down for it, the result is a second perpendicular distinct from the one previously drawn, and the intended proof falls to the ground.

PROP. XIII.—INTERC. BOOK.

NOTE 1. It may occur as an objection here, that it has not been proved that CL may not be met by the surface of the sphere whose radius is AR, in more points than S. But even if this was the case, it would only follow that every such other point as well as S, must be in the self-rejoining line which is the intersection of the spheres described with the radii AR and BR, and must by the revolution of CL be made to pass through R. By all which, no prejudice would be done to the fact that R is in the surface described by the revolution of CL.

NOTE 2. It may appear as if the demonstration might be shortened in this place, by making MT turn about M till MT or its prolongation passes through any second point that shall have been assigned, whether *in* MQOP, or *within* it, or *without*. But the proof would fail, if the second point should happen to lie in a straight line which met MQOP in M but not in any other point. Which difficulty, though in a case of necessity it might not be held fatal by a mathematician familiar with the treatment of ultimate values, would be embarrassing to a beginner, and is therefore best obviated by some such process as is given.

NOM. XXXVI.—BOOK I.

The thing treated of by geometers under the name of an angle, appears to bear to 'divergence' precisely the same relation that a straight line bears to 'distance.' It is easy, previously to all mention of straight lines, to assign the meaning of *equal* distances, by appealing to the capability of coincidence as is done in Nom. XI; but it would not be so easy, clearly to lay down a description of *greater* or *less* distances. The nearest perhaps that could be come to it, would be by propounding some such criterion, as that a *greater* distance was that where the sphere described with it about a common centre, was exterior to the sphere described with the other; and still there would remain the question how distances, without possessing the aid of anything on which they are to be represented or marked off, are to be added together or subtracted. There wants a *substratum* of some kind (as the logicians would possibly have termed it), in which the quality of distance shall be embodied, so as to furnish a ready method of comparison by means of addition and subtraction, which are the tests of being what is meant by *greater* or *less*; and this want is supplied by the straight line. From the moment therefore that the straight line can be brought upon the scene, all mention of *distances* should be abolished as belonging to an earlier and inferior state of knowledge; or at all events confined to cases where it is distinctly specified what straight line is intended by the term.

The same analogy may be traced with precision, in the case of divergence which may be termed 'rotatory distance.' It is easy to assign the meaning of *equal* divergences, by appealing to the fact that the two pairs of straight lines in question can be made to coincide. But for laying down a description of *greater* or *less* divergences, there would appear to be no resource unless it were the very clumsy one of saying a greater divergence was that where the cone described by one of the pairs of straight lines turning round one of its legs as a common axis with a common vertex, was exterior to the cone described by the other; and still there would remain the question how divergences, without possessing the aid of anything on which they are to be represented or marked off, are to be added together or subtracted. What is wanted therefore, is the introduction of the plane, to assist, as the straight line did before, by its capability of addition and subtraction. Allow that the thing spoken of shall be a plane surface and not an abstract divergence;—as it is admitted that the thing spoken of in the other case is a straight line and not an abstract distance;—and all difficulty vanishes in one case as in the other. Refuse it, and there seems to be induced the same state of obscurity and perplexity, that would be the result of insisting on rejecting the introduction of the straight line into the comparison of distances.

Though the term 'angle' originally meant a *corner*, the geometrical idea of an angle equally includes cases where the straight lines diverge in such manner as to lie in one and the same straight line, or even to form a *corner* on the other side. If one hand of a watch stood still at twelve o'clock, while the other moved round, first to three, then to six, then to nine, and so on till it coincided with the position from which it set out, all the portions of plane thus passed over would be in geometry called angles, and the moving hand is the *radius vectus*. The principal use of introducing this last term, is to give the power of designating the larger and last-mentioned kind of angles; which would be difficult without it.

PROPOSITION I.—BOOK I.

In this and other Propositions, the word 'finite' has been omitted, as being unnecessary where magnitudes are stated to be either 'given' or 'equal to one another.' The observation that equality (meaning of magnitude, not of ratio) implies being finite, is as old as the *Philebus* of Plato.

PROP. II.—BOOK I.

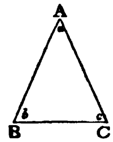
This Proposition and the next have been sometimes objected to as unnecessary; because, it is urged, a straight line might be transported from its place and applied to another straight line, and its two extremities marked off. There is no doubt that this might be done if it were a case of necessity. But Euclid desired to do without it, as being one demand the less. And the same reason appears valid for following his example.

PROP. IV.—BOOK I.

In this and several other Propositions, the term 'base' has been avoided as unnecessary; and its use confined to places where its meaning is distinctly laid down.

PROP. V.—BOOK I.

Professor Playfair (Elem. of Geom. Notes, p. 395) has noticed the ease with which this Proposition might be demonstrated, if it was allowed to suppose the isoskeles triangle transported to a new situation, and the new one applied to the old so as to apply different equal sides to one another. This might be mechanically exhibited, by cutting out the isoskeles triangle, as *acb*, with scissors, and applying it to the old situation ABC, so that the point *a* should coincide with A, and the side *ac* with AB; whereupon it is manifest that by reason of the equality of the angles



a and A, and of the sides *ab* and AC, the point *b* will also coincide with C, and the third side *cb* with BC; whence the angles *c* and B, as coinciding with one another, will be equal.

This species of superimposition might be practically objectionable, as difficult of comprehension for a beginner. It is at the same time worth noticing, how very little substantial difference there is in Euclid's method. For all he does is to append an additional triangle to ABC in two different quarters or directions; and with these appendages, he unscrupulously applies the triangle ABC to the triangle ACB as proposed above. The readiest way of seeing the truth of this, is to imagine BF (*See the figure in the text of Prop.* V.) to be diminished till it becomes evanescent.

PROP. VI.—BOOK I. Cor.

This Corollary may be used with advantage in the demonstration of the Proposition IV. 15. of Simson, which is to inscribe an equilateral and equiangular hexagon in a given circle.

PROP. VII.—BOOK I.

The sides terminated in the first-mentioned extremity of AB, have been made visibly equal in the figures; which it was imagined would be advantageous to a beginner.

PROP. IX.—BOOK I.

The Proposition to bisect a given angle, has been moved to be Proposition XIV, for reasons stated in that place. Other alterations in numbers have been made in consequence; but to the smallest extent that could be contrived.

PROP. XI.—BOOK I.

The insertion of the matter of this Proposition as an Axiom, is an instance of the carelessness of the ancients on what may be called the philosophy of Geometry. It is true that Euclid does not seem to have troubled himself with the fact that two straight lines cannot have a common segment. But by taking EF equal to BC, and EH to BG, and applying C and G to F and H, the necessity of the two straight lines coinciding in order not to inclose a space, and of the point B coinciding with E in consequence of the equality of CB and FE, would have put the proof in the same train as here given.

PROP. XIV.—BOOK I.

This Proposition, as it stood as Prop. IX, was only a proposition to bisect an angle less than two right angles. It therefore became necessary to remove it to the other side of the Propositions numbered XII and XIII.

PROP. XXI.—BOOK I.

Referred to in the Proposition III. 8. of Simson, which shows that of straight lines from a point without the circle to the circumference, the least is that between the point without the circle and the diameter; &c.

PROP. XXII.—BOOK I.

Considerable changes have been made in the details of this Proposition, for the purpose of showing that the circles used in the construction will necessarily cut one another. And the Proposition which usually stood as Prop. XXIII has been inserted as Cor. 2, to make room for the insertion of another Proposition as XXIII, for reasons next hereafter stated.

PROP. XXIII.—BOOK I.

The Proposition here numbered XXIII was usually given as Prop. XII, and is removed to avoid the necessity of proving that a straight line which cuts the circumference of a circle but without passing through the centre, cuts it only in a point; a thing necessary to the establishment of the old demonstration, but which, to be proved, would require to be presented as a Corollary to some of the Propositions in the Third Book.

Cor. 1. Is inserted as being of frequent practical application; and as capable of being referred to on the subject of straight lines 'equally distant from the centre,' in the Third Book.

PROP. XXIV.—BOOK I.

The construction and proof of this Proposition are altered, for the purpose of showing that the point G will necessarily fall on the other side of BC from the point A, or AG be greater than AH; which is essential to the demonstration, and is not so clearly established by Simson in his Note to this Proposition.

PROP. XXVI.—BOOK I.

The truth of the First Case is apparent almost upon inspection, if the two triangles are supposed to be applied to one another, beginning with the sides that are equal in each. But to establish this application with accuracy, would involve the reintroduction of the subject of the coincidence of planes, in the same manner as in Prop. IV; and therefore it may be a question on the whole, whether the shortest way is not to follow Euclid in employing the agency of that Proposition.

This Proposition is the last in which Euclid establishes the equality of triangles and their sides and angles respectively, from the equality of certain of their parts. He never notices the case of two triangles having two sides of the one equal to two sides of the other respectively, and an angle *opposite to* equal sides equal in both. This is probably, first, because he never found any necessity for making use of the proposition; and secondly, because it is burthened with an additional condition, which is, that the other angle opposite to equal sides *shall not be acute in one of the triangles and obtuse in the other*. Without this proviso, the equality does not necessarily exist, for there may often be two different triangles answering the conditions; but with it, the demonstration is easy by the help of the Propositions in the Third Book.

PROP. XXVII.—BOOK I.

The demonstration of this Proposition is altered, and the figure changed; in order to avoid the representation of two straight lines with an angle in the middle of each, which was peculiarly abhorrent to beginners, and may reasonably be so to all other persons. Euclid also has omitted to say anything of the lines being in the same plane; which is necessary to their being parallel.

PROP. XXVIII.—BOOK I.

This Proposition has been removed, in order to bring it on the hither side of that part of the question of parallel lines which has given so much trouble to geometers. Which clearly ought to be done where practicable.

To preserve the numbers of the Propositions as nearly as possible, the Proposition usually numbered XXVIII has been added as a Corollary to Prop. XXVII, which it really is; and the two theorems which have usually been joined under the title of Prop. XXXII, have been separated.

PROP. XXX.—BOOK I.

This Proposition has hitherto always been demonstrated in the case which needed no demonstration. For if two straight lines are each parallel to a straight line that is *between* them; because they can never meet the intermediate straight line, they can never meet one another.

This Proposition is referred to, in the Proposition numbered XLV of the First Book by Simson, but omitted in this work for the purpose of removing to the Sixth Book as hereafter noted.

PROP. XXXII.—BOOK I. Cor. 2.

Referred to in the Appendix, on the question of Parallel Lines.

If any of the angles of the figure are *re-entering* or greater than two right angles, the assertion in the Corollary becomes inapplicable altogether. *(From Lardner's Euclid.)*

PROP. XXXIII.—BOOK I.

This Proposition and the next following it have been transposed, for a supposed improvement in the order.

Cor. 1. From Lardner's Euclid. The division of a given straight line into any assigned number of equal parts, has generally been effected by means of the principle of proportionals in the Sixth Book (VI. 9. of Simson). But as it can be performed upon the earlier principle, it ought to be.

PROP. XXXIV *bis.*—BOOK I.

This Proposition, in addition to its use in throwing light on the properties of quadrilateral figures, is noticeable as containing the principle of the *parallel ruler.*

PROP. XXXV.—BOOK I. Cor.

Inserted as being the foundation of the mensuration of the area of parallelograms in general.

PROP. XXXIX.—BOOK I.

What has usually been given as the demonstration, does not provide for the case where the straight line drawn through the point A parallel to the base, should not fall *within* the straight line that joins the vertices of the triangles.

PROP. XL.—BOOK I.

What has usually been given as the demonstration, has the same defect as is pointed out in the preceding Proposition; and further, it was not shown that AG being prolonged will of necessity meet DE, and not fall upon EB at some point between E and B.

This Proposition does not appear to be ever afterwards referred to by Euclid. But it is given as having been given by him; and as being moreover likely enough to be practically useful.

PROP. XLI.—BOOK I. Cor.

Inserted as being the foundation of the mensuration of the area of rectilinear triangles in general.

PROP. XLIII.—BOOK I.

This and the three Propositions next following it, are in substance from Bonnycastle's Geometry.

PROP. XLVI.—BOOK I.

Wanted in the Proposition III. 15. of Simson, which teaches that of straight lines in a circle, that which is nearer to the centre is greater than that which is more remote.

Observations on Propositions omitted for insertion in other parts of the Elements.

The Proposition which in Simson's Euclid is the 43rd of the First Book, would appear to be more properly placed in the Second Book, in some part antecedent to the Proposition which is the 4th of Simson.

The Propositions which in Simson's Euclid are the 42nd, 44th, and 45th of the First Book, should be placed in the Sixth Book, in some part antecedent to the 25th of the Sixth Book of Simson; and they would appear to be most conveniently introduced, next after the Proposition which in Simson's Euclid is the 18th of the Sixth Book.

These Propositions in their former place have always been felt to be a great impediment to learners in their progress towards the demonstration of the 47th Proposition, which may be considered as the crowning Proposition of the First Book; and as such they have usually been passed over by intelligent teachers.

END OF THE NOTES.

APPENDIX

Containing Notices of Methods at different times proposed for getting over the difficulty in the Twelfth Axiom of Euclid.

THE uses of such a Collection, are to throw light on the particulars that must not be left unguarded in any attempt at solution, and to prevent future explorers from consuming their time unnecessarily in exhausted tracks.

1. The objection to Euclid's Axiom (independently of the objections common to all Axioms), is that there is no more reason why it should be taken for granted without proof, than numerous other propositions which are the subjects of formal demonstration, and the taking any one of which for granted would equally lead to the establishment of the matter in dispute.

2. Ptolemy the astronomer, who wrote a treatise on Parallel Lines of which extracts are preserved by Proclus, proposed to prove that if a straight line cuts two parallel straight lines the two interior angles on each side are together equal to two right angles, by saying that if the interior angles on the one side are greater than two right angles, then because the lines on one side are* *no more parallel* than those on the other, the two interior angles on the other side must likewise be together greater than two right angles, and the whole greater than four, which is impossible; and in the same way if they were supposed less. In which the palpable weakness is, that there is no proof, evidence, or cause of probability assigned, why parallelism should be connected with the angles on one side being together equal to those on the other; the very question in debate being, whether they may not be a little more than two right angles on one side and a little less on the other, and still the straight lines not meet.

3. Clavius announces that 'a line every point in which is equally distant from a straight line in the same plane, is a straight line;' upon taking which for granted, he finds himself able to infer the properties of Parallel Lines. And he supports it on the ground that because the acknowledged straight line is one which lies evenly [*ex æquo*] between its extreme points, the other line must do the same, or it would be impossible that it should be everywhere equidistant from

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* *οὐδὲν γαρ μᾶλλον αἱ αζ γη παράλληλοι ἢ αἱ ζδ ηβ.—Procli Comment. in Primum Euclidis Librum. Lib.* 4.

It is but right to notice, that Proclus calls this παραλογισμὸς and δείξεως ἀσθένεια; and Barocius the Venetian translator in 1560, notes it in the margin as *Flagitiosa Ptolemæi ratiocinatio.* Professor Playfair says it is curious to observe in Proclus's account an argument founded on the principle known to the moderns by the name of the *sufficient reason* (El. of Geo. p.405). If the allusion is to this part, the *sufficient reason* of the moderns must be something very feeble.

the first*. Which is only settling one unknown by a reference to another unknown.

4 & 5. In a tract printed in 1604 by Dr. Thomas Oliver of Bury, entitled *De rectarum linearum parallelismo et concursu doctrina Geometrica* (Mus. Brit.), two demonstrations are proposed; both of them depending on taking for granted, that if a perpendicular of fixed length moves along a straight line, its extremity describes a straight line.

6. Wolfius, Boscovich, Thomas Simpson in the first edition of his 'Elements,' and Bonnycastle, alter the definition of parallels, and substitute in substance, 'that straight lines are parallel which preserve always the same distance from one another;' by distance being understood the length of the perpendicular drawn from a point in one of the straight lines to the other. Attempts to get rid of a difficulty by throwing it into the definition, are always to be suspected of introducing a theorem in disguise; and in the present instance, it is only the introduction of the proposition of Clavius. No evidence is adduced that straight lines in any assignable position, *will* always preserve the same distance from one another; or that if a perpendicular of fixed length travels along a straight line keeping always at right angles to it, what the mathematicians call the *locus* of the distant extremity is necessarily a straight line at all.

7. D'Alembert proposed to define parallels as being 'straight lines one of which has *two* points equally distant from the other;' but acknowledged the absence of proof, that any other points besides these two would be equidistant in consequence.

8. Thomas Simpson in the second edition of his 'Elements' proposed that the Axiom should be, that 'If two points in a straight line are posited at unequal distances from another straight line in the same plane, those two lines being indefinitely produced on the side of the least distance will meet one another†.'

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* 'Nam si omnia puncta lineæ AB, æqualiter distant à rectâ DC, ex æquo sua interjacebit puncta, hoc est, nullum in eâ punctum intermedium ab extremis sursùm, aut deorsùm, vel huc, atque illuc deflectendo subsultabit, nihilque in eâ flexuosum reperietur, sed æquabiliter semper inter sua puncta extendetur, quemadmodum recta DC. Alioquin non omnia ejus puncta æqualem à rectâ DC, distantiam haberent, quod est contra hypothesin. Neque verò cogitatione apprehendi potest aliam lineam præter rectam, posse habere omnia sua puncta à rectâ lineâ, quæ in eodem cum illâ plano existat, æqualiter distantia.'—*Clavii Opera. In Euclid. Lib. I.* p. 50.

† It is impossible to avoid mentioning, as indicative of the spirit in which books on geometry have been written, that this author absolutely proposes that the Proposition which states the equality of triangles having two sides and the included angle equal respectively, should be *introduced as an Axiom*; but 'if this Axiom should not appear sufficiently evident,' he directs one triangle to be applied to the other in the manner of Euclid (*Elements of Geometry, by Thomas Simpson. First Edition, p.* 8). Now what is this but saying, 'Persuade your scholar to believe without proof if you can; but if you cannot, you may give him the proof.' The world may be challenged to show, in what way it is possible to arrive at any perception of the truth of the Proposition mentioned, except by directly or indirectly, formally or informally, going through something equivalent to Euclid's proof. On what conceivable principle, therefore, was the proof to be sunk, and the scholar invited to believe without? This is not teaching geometry, but teaching to do without geometry.

9. Robert Simson proposes that the Axiom should be, 'that a straight line cannot first come nearer to another straight line, and then go further from it, before it cuts it*.' By coming nearer or going from it, being understood the diminution or increase of the perpendicular from one to the other.

The objection to all these is, that no information has been given on the subject of the things termed straight lines, which points

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* This and most of what has preceded, is in the Arabic. In a manuscript copy of Euclid in Arabic but in a Persian hand, bought at Ahmedabad in 1817, the editor on the introduction of Euclid's Axiom breaks out as follows.

'I maintain that the last proposition is not among the universally-acknowledged truths, nor anything that is demonstrated in any other part of the science of geometry. The best way therefore would be that it should be put among the questions instead of the principles; and I shall demonstrate it in a suitable place. And I lay down for this purpose another proposition, which is, that straight lines in the same plane, if they are subject to an increase of distance on one side, will not be subject to a diminution of distance on that same side, and the contrary; but will cut one another. And in the demonstration of this I shall employ another proposition, which Euclid has employed in the tenth book and elsewhere, which is, that of any two finite magnitudes of the same kind, the smallest by being doubled over and over will become greater than the greatest. And it will further require to be laid down, that one straight line cannot be in the same straight line with straight lines more than one that do not coincide with one another; and that the angle which is equal to a right angle, is a right angle.'

اقول القضية الاخرة ليست من العلوم المتعارفة ولا مما يتضح في غير علم الهندسة فاذن الاولي ان يترتب في المسايل دون المصادرات وانا ساوضحها في موضع يليق بها ووضعت بذلك قضية اخري هي ان الخطوط المستقيمة الكاينة في سطح مستو ان كانت موضوعة علي التباعد في جهة فهي لا تكون موضوعة علي التقارب في تلك الجهة بعينها وبالعكس الا ان يتقاطعا واَستَعمِلُ في بيانها قضية اخري قد استعملها اقليدس في المقالة العاشرة وغيرها وهي ان كل مقدارين محدودين من جنس واحد فان الاصغر منهما يصير بالتضعيف مرة بعد اخري اعظم من الاعظم ومما يجب ايضا ان يوضع ان الخط المستقيم الواحد لا يتصل علي الاستقامة باكثر من خط واحد مستقيم غير مسامت بعضها ببعض وان الزاوية المساوية القايمة قايمـــة

to any reason why the perpendicular's growing smaller should be necessarily followed by the meeting of the lines. It may be true; but the reason why, is not upon the record. If the answer is that 'straight line' means a certain figure familiar to the eye, viz. the figure taken by a plumb-line, and that it is accordant with experience that lines of this kind do not first approach and then not meet; —it would only be an extension of the same process to say it is accordant with experience that the rectangles under the segments in a circle are equal, for there is no doubt of everybody who has tried having found it true.

10. Varignon, Bezout, and others, propose to define parallels to be 'straight lines which are equally inclined to a third straight line,' or in other words, make the exterior angle equal to the interior and opposite on the same side of the line. By which they either intend to take for granted the principal fact at issue, which is whether no straight lines but those that make such angles can fail to meet; or if their project is to admit none to be parallel lines of which it shall not be predicated that they make equal angles as above *with some one straight line* either expressed or understood, then they intend to take for granted that because they make equal angles with *one* straight line, they shall also do it with any other that shall in any way be drawn across them,—a thing utterly unestablished by any previous proof.

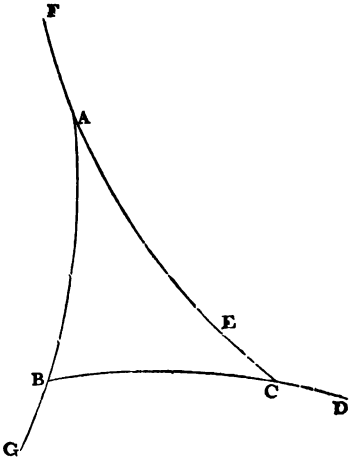
11. Professor Playfair proposes to employ as an Axiom, that 'two straight lines, which cut one another, cannot be both parallel to the same straight line;' in which he had been preceded by Ludlam and others, and which he says 'is a proposition readily enough admitted as self-evident.' The misfortune of which is, that instead of being self-evident, a man cannot see it if he tries. What he sees is, that he does not see it. He sees that a straight line's making certain angles with one of the parallels, causes it to meet the other; and he sees that by increasing the distance of the point of meeting, he can cause the angle with the first parallel to grow less and less. But if he feels a curiosity to know whether he can go on thus reducing the angle till he makes it less than any assignable magnitude, (or in other words whether there may not possibly be some angle so small as to fail of causing the straight line to reach the other parallel), he discovers that this is the very thing nature has denied to his sight; an odd thing certainly, to call self-evident.

12. The same objections appear to lie against Professor Leslie's proposed demonstration in p. 34 of his 'Rudiments of Plane Geometry.'

13. Professor Playfair in the Notes to his 'Elements of Geometry,' p. 409, has proposed another demonstration, founded on a remarkable *non causa pro causâ*. It purports to collect the fact* that (on the sides being successively prolonged to the same hand) the exterior angles of a rectilinear triangle are together equal to four right angles, from the circumstance that a straight line carried round the perimeter of a triangle by being applied to all the sides in succession, is brought into its old situation again; the argument being, that because this line has made the sort of somerset it would do by being turned through four right angles about a fixed point, the exterior angles of the triangle have necessarily been equal to four right angles. The answer to which is, that there is no connexion between

*I.32.Cor.2.

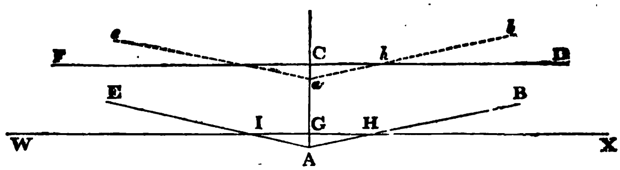
the things at all, and that the result will just as much take place where the exterior angles are avowedly not equal to four right angles. Take, for example, the plane triangle formed by three small arcs of the same or equal circles, as in the margin; and it is manifest that an arc of this circle may be carried round precisely in the way described and return to its old situation, and yet there be no pretence for inferring that the exterior angles were equal to four right angles. And if it is urged that these are *curved* lines and the statement made was of *straight*; then the answer is by demanding to know, what property of straight lines has been laid down or established, which determines that what is not true in the case of other lines is true in theirs. It has been shown that, as a general proposition, the connexion between a line returning to its place and the exterior angles having been equal to four right angles, is a *non sequitur*; that it is a thing that may be or may not be; that the notion that it returns to its place *because* the exterior angles have been equal to four right angles, is a mistake. From which it is a legitimate conclusion, that if it had pleased nature to make the exterior angles of a triangle greater or less than four right angles, this would not have created the smallest impediment to the line's returning to its old situation after being carried round the sides; and consequently the line's returning is no evidence of the angles not being greater or less than four right angles.



14. Franceschini, Professor of Mathematics in the University of Bologna, in an Essay entitled *La Teoria delle parallele rigorosamente dimostrata*, printed in his *Opuscoli Matematichi* at Bassano in 1787, offers* a proof which may be reduced to the statement, that if two straight lines make with a third the interior angles on the same side one a right angle and the other an acute, perpendiculars drawn to the third line from points in the line which makes the acute angle, will cut off successively greater and greater portions of the line they fall on. From which it is inferred, that because the portions so cut off go on increasing, they must increase till they reach the other of the two first straight lines, which implies that these two straight lines will meet. Being a conclusion founded on neglect of the very early mathematical truth, that continually increasing is no evidence of ever arriving at a magnitude assigned.

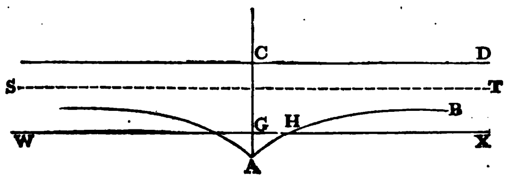
15. A fallacy somewhat more subtle than Franceschini's, though akin to it, may be framed on the consideration of the angle of inter-

* See the Notes to Playfair's Elements of Geometry, p. 406; where there is a figure.



section. Let AB, CD be two straight lines in the same plane, making with a third straight line AC the angles CAB, ACD, of which ACD is a right angle and CAB less than a right angle. And to improve the appearance (though this is not indispensable) draw a straight line AE on the other side of AC and in the same plane, making an angle CAE equal to CAB. And from A let a straight line of unlimited length as WX travel along the straight lines AB and AE, cutting AC always at right angles in some point G between A and C. This line will represent Franceschini's succession of perpendiculars. But instead of arguing from its continually cutting off greater and greater portions AG, let it be argued that because it at any time makes with AB an angle AHG or BHX, it may always be removed to a position farther from A without ceasing to cut AB and AE. From which it at first sight might appear to be a reasonable conclusion, that the straight line WX may be carried forward without the possibility of failing to cut AB and AE, till it arrives at C. And the fallacy will perhaps be still more taking, if AB and AE are made to begin by being placed at C, and so are moved from C towards A, as represented by *ab* and *ae*; under which circumstances the allegation that there must always be an angle of some kind at *h*, has a very inviting appearance as a reason why *ab* and CD, being continually prolonged, cannot quit one another or fail to meet and make an angle of some magnitude or other, the consequence of which would be that *ab* and *ae* might be moved till *a* coincides with A and *ab* with AB, without the possibility of parting company with CD by the way.

The answer to this is by inquiring, whether there are no lines in which there may be the same kind of evidence on the subject of the angle, but where it is certain that a straight line as WX cannot be carried on to an unlimited extent as proposed. And here it is easy to show that there may. Take, for example, any hyperbola, and from the vertex draw a perpendicular to each of the asymptotes; and let the two halves of the linear hyperbola, together with the perpendiculars and the portions of the asymptotes cut off by them on the side remote from the intersection of the asymptotes, be placed so that the perpendiculars shall coincide and the asymptotes in consequence be in one straight line, as ST in the figure below. Upon which it is clear, that however it may be pleaded that there may always be an angle smaller than AHG or BHX



between WX and AB, WX cannot be carried beyond the line of the asymptotes ST without ceasing to meet AB; and consequently cannot be carried till it meets CD, if CD lies on the other side of ST as represented in the figure.

It follows therefore, that to say there will always be the possibility of a further diminution of the angle, is not enough. It is the sophism of Achilles and the tortoise; which argued that because after running a mile, half a mile, a quarter of a mile, &c. Achilles would always be behind by the last-mentioned fraction of a mile, he would never overtake or pass the tortoise. The solution resolving itself into the fact, that these quantities though infinite in number are finite and surpassable in amount.

To establish the union of the lines to any particular extent that may be desired, it is consequently necessary to prove, not only that the angle at the intersection is capable of diminution, but that the angle each way (that is to say, both the angle AHG and the angle GHB) shall never be *reduced to less than some given angle*. Which is what is done accordingly, in the Proposition numbered XXVIII *D* of the present work.

16. In a tract entitled 'The Theory of Parallel Lines perfected; or the Twelfth Axiom of Euclid's Elements demonstrated. By Thomas Exley, A. M.—London. Hatchard. 1818.'—the proof rests on taking for granted (in the Second Proposition) that if *four* equal straight lines in the same plane, making right angles with one another successively towards the same hand, do not meet and inclose a space, a *fifth* if prolonged both ways must inevitably accomplish it. A conclusion which may be resolved into taking for granted that the three angles of a rectilinear triangle are greater than a right angle and a half; for if they were equal to this, the angles of an equilateral and equiangular oktagon would be right angles, and the fifth straight line in the series proposed would never meet the first; still more if they were less. And in the same manner if it was urged that a *sixth, seventh*, &c. perpendicular must meet the first straight line, it would only resolve itself into a demand for admitting without proof, that the three angles of a triangle are greater than some other amount capable of being specified. There is no obscurity about the fact that four such straight lines, and still more five, are found on experiment to meet; but the object was to discover *why* they necessarily meet. And between the observed fact and the explained fact, there is a difference of the same kind as between Kepler's observation of the proportion between the periodic times and distances of the planets, and Newton's explanation of the cause.

17. The demonstration presented by M. Legendre in the earlier editions of his '*Eléments de Géométrie*,' consisted in first establishing that the three angles of a rectilinear triangle cannot be *greater* than two right angles (which may be passed over as irrefragable and liable to no remark), and afterwards proceeding to show cause why they should not be *less*. But the evidence offered on this latter point, depended on taking for granted that two straight lines (DE and BE in fig. 35 a in the Fourth Edition, and probably in the subsequent editions as far as the Eighth inclusive) meet when they make with a third straight line (DB) angles *of which one (as EDB) is, or may be made to be, less than a right angle, and the other* looks *less than a right angle, but without further evidence*. In the Seventh

Edition an attempt was made to show that the lines must meet; but the proof advanced involves the same fallacy as that of the Bolognese Professor*.

18. This demonstration was withdrawn in the Ninth Edition, and a new one inserted in the Twelfth. The new one depended upon taking in any triangle an angle that is *not less*† than any other in the triangle, and a second that is *not greater* (See *Douzième édition p.* 20, and Plate); bisecting the side opposite to the second angle, and drawing a straight line from the angular point to the point of bisection; cutting off in this straight line and its prolongation a part from the angular point equal to the side opposite to the first-mentioned angle (viz. that angle which is *not less* than any other in the triangle), and in this side and its prolongation towards the same hand a part equal to double the straight line between the angular point and the point of bisection formerly mentioned, and joining the extremities of the two parts thus cut off. It is not difficult to show, that in the new triangle thus last constructed, the sum of the three angles is the same as in the original triangle; and moreover that of the angles of these two triangles which are at a common point, that belonging to the new triangle is not greater than half that of the old, while another of the angles of the new triangle is equal to their difference. And if these operations be applied in like manner to the last constructed triangle, a third triangle will be constructed having the same relations to the second; and so on. Whence it follows, that the described process may be continued, till two of the angles of the last-resulting triangle are together less than any magnitude that shall have been assigned; and consequently the third or remaining angle may be made to approach, within any magnitude however small it may be chosen to assign, to the sum of the three angles of the original or any of the intervening triangles.

All this is irrefragable; but not so the proposition next taken for granted, which is that the third angle last-mentioned approaches within any magnitude however small it may be chosen to assign, to *the sum of two right angles*. That it approaches it (that is that the angle continually grows larger) is certain; but that it approaches to it within any magnitude however small, is the point which, as in so many parallel instances, is taken for granted without sufficing proof. The weakness in the actual case, is in the fact that the base or side opposite to the continually increasing angle, becomes itself of unlimited length. If the resulting triangles had been all on the same base, the inference might perhaps have been conceded to be good. But it is precisely because by the extension of the base to an unlimited magnitude the progress of the operation is removed from human eyes, that the force of the inference is diluted and done away. Just as fast as the diminution of the two acute angles appears to induce a necessity for the obtuse angle's approximating to the sum of two right angles, does the increase of the length of the sides hold forth an augmented probability that the angle may after all evade increasing by the quantity required to make it attain to two right angles in the

* Elém. de Géom. Par A.M. Legendre, 7ème édit. p. 280. Note II.

† *Not less* and *not greater* are substituted for the *greatest* and *least* of the original, from a persuasion that these last are an oversight. The demonstration is plainly intended to be applicable to any triangle; but the terms used would not apply to an equilateral triangle, or any kind of isoskeles.

end. To argue that when the acute angles are nothing, or the lines coincide, the third angle will make a straight line,—is substituting for what really happens, what by the hypothesis is never to happen. The demonstration is therefore finally of the same strength as Franceschini's and others that have been mentioned. There is evidence of a perpetual approach towards a given magnitude; but there is not evidence of the degree and rapidity of approach which are necessary to ensure arriving at it.

19. Another demonstration, or step towards a demonstration, presented by the same author (See *Note II. p.* 279, 12*ème édition*), consists in representing, that if any angle less than two right angles is bisected, all perpendiculars to the bisecting straight line must meet the sides, because otherwise there would be a straight line shut up between the lines that make an angle, '*which is repugnant to the nature of the straight line.*' On which it is sufficient to observe, that the existence, cause, and origin of this repugnance, are precisely what it was in question to establish.

20. The next paragraph in the same page is directed to establishing the sort of postulate assumed in the last, viz. that a straight line cannot be shut up within an angle. The argument appears to be, that either of the straight lines which make an angle, being prolonged will divide the infinite plane in which it exists into equal parts, and any other straight line must do the same; but a straight line that should be shut up within the angle, would cut off more on one side and less on the other; therefore a straight line cannot be shut up within an angle. Whoever examines this closely, will see that it would equally prove that two straight lines cannot be parallel to one another; for in that case it might equally be urged, that if the one cuts the plane in halves, the other must cut off more on one side and less on the other. The whole is manifestly a mistake arising from overlooking Plato's observation, that equality of magnitude can only be predicated of things finite.

21. The next in order is the so-called *analytical* proof, which professes to demonstrate that if two angles in one rectilinear triangle are respectively equal to two in another, the remaining angles are necessarily equal. If two angles of a triangle and the side between them are given, the rest of the sides and angles of that one triangle are determined; that is to say, they can severally be only of one certain magnitude and no other. Hence, said the advancers of this demonstration, the angle opposite to the given side is a *function* of the two angles and the given side;—their meaning by this term being, that a quantity is a *function* of other quantities, when on those other quantities being fixed and determined in magnitude, the first quantity is necessarily fixed and determined in magnitude, or is what Euclid in his Book of Data would call *given*. 'Let the right angle be equal to unity or 1, and then the angles will all be numbers somewhere between 0 and 2; and since the third angle is *a function* of the two other angles and the side between them, it will follow that the side cannot enter as an element into the determination of the magnitude of the angle.' And this, they said, because the side is heterogeneous with the other quantities which are numbers, and no equality can be compounded or made to exist between them*.

* 'Il faut donc que l'angle C soit entièrement déterminé, lorsqu'on connâit

In this the reference is to what the analysts had denominated the principle of *homogeneity;* a principle in itself irrefragable, but like all others, capable of being ill applied. Wherever quantities are to be equal, they must be homogeneous or of the same kind; for equality is nothing but the capability of coincidence, and things heterogeneous cannot coincide. A mile of length, or two, or three, or four miles, can by no possibility be equal to an hour of time; the assertion would be *ipso facto* foolish and unmeaning. But there is no objection to saying that $\frac{\text{four miles}}{\text{two miles}} = \frac{\text{ten hours}}{\text{five hours}}$; because the first of these expressions means only the number of times that the quantity two miles can be taken in the quantity four miles, which is the number *two;* and five hours may be taken the same number of times in ten. And by the same rule, there is no objection to saying that four miles = two miles $\times \frac{\text{ten hours}}{\text{five hours}}$; for this means nothing but that four miles = two miles × the number which results from seeing how often five hours can be taken in ten. It follows therefore that heterogeneous quantities *enter equations by pairs;* or at all events are reducible to pairs by running some two or more of them into one by the operation of addition or subtraction. There cannot be the slightest idea of questioning this, or any of the legitimate results of what has been called the principle of homogeneity.

But the application in this instance was not legitimate, or at all events not legitimately conducted. There was on the face of it an attempt at fallacy, consisting in substituting for the angles the numbers which expressed their ratios. Professor Leslie brought this into full light, by pointing out that if the same reasoning was applied to the case where two sides (a and b) were given and the angle between them P, it would produce the statement that the remaining side $c = \phi : (a, b, \text{P})$, in which, on substituting for a, b, c, the numbers which express their ratios, there would be the same argument for inferring that c would be the same whatever was the angle, which is notoriously untrue. And this brought out the avowal, that his opponents in the case of the angles intended to substitute the ratios, and in the case of the sides, not; a mode of

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les angles A et B, avec le côté p; car, si plusieurs angles C pouvaient correspondre aux trois données A, B, p, il y aurait autant de triangles différents qui auraient un côté égal adjacent à deux angles égaux, ce qui est impossible : donc l'angle C doit être une fonction déterminée des trois quantités A, B, p; ce que j'exprime ainsi, $\text{C} = \phi : (\text{A}, \text{B}, p)$.'

'Soit l'angle droit égal à l'unité, alors les angles A, B, C, seront des nombres compris entre 0 et 2; et puisque $\text{C} = \phi : (\text{A}, \text{B}, p)$, je dis que la ligne p ne doit point entrer dans la fonction ϕ. En effet, on a vu que C doit être entièrement déterminé par les seules données A, B, p, sans autre angle ni ligne quelconque, mais la ligne p est hétérogene avec les nombres A, B, C; et si on avait une équation quelconque entre A, B, C, p, on en pourrait tirer la valeur de p en A, B, C; d'où il résulterait que p est égal à un nombre, ce qui est absurde : donc p ne peut entrer dans la fonction ϕ, et on a simplement $\text{C} = \phi : (\text{A}, \text{B})$.'—*Legendre. Elém. de Géom.* 12ème *édit. Notes. p.* 281.

The entire passage is inserted together, to show that no alteration has been made either in order or connexion.

arguing comparable only to the ingenuity of the artist, who in playing at 'odd or even,' holds a ball which he has the power of projecting or not as is required to make him win.

When pushed on this point, they replied, that their reason for substituting the ratios in the case of the angles and not of the sides, was 'because the right angle was the natural unit of angles*.' But the fact of a right angle (or more properly four right angles or a turning from the place started from till arriving at it again) being a convenient object of reference for the comparison of angles in general, is devoid of any proved connexion with the propriety of substituting the ratios in one case, and not substituting them in the other.

When pressed, however, they produced a reason. They said it was because 'the angle is a portion of a finite whole, the straight line a portion of an infinite whole; so that every given angle is a finite quantity, while every given straight line is a quantity infinitely small, and only the ratios of given straight lines can enter into our calculations with given angles†.' And this was repeated as 'a very subtle and very just metaphysical idea; and at the same time strictly analytical‡.' On which all that can be done, is to remark on the complete absence of any reasonable or demonstrated connexion (even supposing the terms correct, which might be disputed), between the facts alleged and the consequences assigned to them.

But a circumstance that appears to have escaped Professor Leslie, is that his opponents, till his counter case appeared, had been at the expense of a useless wrong. Whether this arose from mistake, or from foresight of the argument that might be brought against them, might be a curious speculation; but certain it is, that there was in the first instance no necessity for the substitution, to produce their argument. They would seem to have been beset by the idea, that when the angles A and B appeared on the same side of their equation, one must of necessity be *divided by* the other; else why did they insist on the substitution of numbers at all? Whereas the fact they were themselves aiming to prove, was that $C = 2R - (A + B)$; where R stands for a right angle. They gained nothing by the substitution of numbers, that they might not have had without; for p would have been just as intractable a companion for the angles as for the numbers. The suspicion consequently may be, that they were lying in wait for Leslie's argument. And when that came, they should have said that they would eject p the side, as incapable of homogeneity, but for P the angle they would substitute $\frac{P}{R}$, and then it would be a number, which need not be ejected. This would at least have held together; but it would have sunk under the unreasonableness

* 'L'angle est une quantité que je mesure toujours par son rapport avec l'angle droit, car l'angle droit est l'unité naturelle des angles. Dans cette notion très simple, un angle est toujours un nombre. Il n'en est pas de même des lignes: une ligne ne peut entrer dans le calcul, dans une équation quelconque, qu'avec une autre ligne qui sera prise pour unité, ou qui aura un rapport connu avec la ligne unité.'—*Letter of M. Legendre*. Leslie's Rudiments of Plane Geometry. Fourth Edition. Notes and Illustrations, p. 296.

† Paper of M. le Baron Maurice; as given in Dr. Brewster's Edition of Legendre's Geometry, Notes, p.235.

‡ Note by M. Legendre, *Ibid.*

of the substitution demanded in one case with intention to refuse it in the other.

And this leads to the substantial inference from the whole of the somewhat perplexed controversy which took place; which is, that the original mistake consisted in confounding two sets of things essentially distinct,—the quantities the fixation of which causes another quantity to be necessarily fixed or what Euclid in his Book of Data would call *given*, and the quantities which must be employed as elements in its actual calculation. These two sets are not necessarily alike, either in number or in kind. Take for example Professor Leslie's case, where $c = \varphi : (a, b, P)$. It is quite true that when a, b, P are fixed, c is fixed. But proceed to the actual calculation of c, and very different things appear upon the scene. For the value of c has to be collected from the well-known trigonometrical formula, that $a + b : a - b :: \text{tang. of} \frac{2R - P}{2} : \text{tang. of}$ semi-difference of the angles at the base c. Here then instead of the solitary and heterogeneous angle P, start up among the practical elements of the calculation two straight lines in the shape of the tangents of two arcs; which of course do not afterwards fail to conduct themselves with perfect submission to the law of homogeneity. And with all this the proposers of the *analytical* proof are bound to make their argument square; for the concession of their own demands ends in establishing the results of vulgar trigonometry, and not in altering them. On the whole therefore, the pretence of knowing what quantities must be ejected to preserve the law of homogeneity, is visionary till it is known what quantities may or may not subsequently appear among the practical elements of the calculation; which is impossible in the preliminary stage.

The point, then, which the supporters of the *analytical* proof must be called on to establish, is why the possibility of the apparition of new elements which is visible in other cases (and which in Professor Leslie's case they actually claim by demanding the admission of R), is non-existent in their own. Take for example the case of what may be called the hyperbolic triangle AHG in p. 140. In this it is plain that if the line AB and the straight lines AG and GH are fixed and determined, the angle AHG must be one fixed angle and no other. But proceed to calculate the comparative magnitude of the angle to different values of AG, and there immediately start into action new elements in no stinted number, viz. two constant straight lines under the denominations of a major and a minor axis, and a varying straight line under the title of *abscissa*, to say nothing of the radius of a circle and such sines or tangents of different arcs thereof as may be found necessary in the process. *How then do the opponents know that there are no more elements in the other case?* If it had pleased nature that the three angles of a triangle should not be always equal to two right angles, the proportionality of the sides of similar triangles would not have held good, and in making Tables, for example, of the tangents to different arcs of a circle, the magnitude of the radius of the circle must in some guise or other have been an element. The tangent of 45° to a radius of one foot would have borne some given ratio to a foot, and the tangent of the same angle to a radius of two feet, instead of bearing the same ratio to two feet, would have borne some different one. There must have been a column of numbers to be applied according to the length of the radius, to obtain the true tangent of the angle to a given length of radius; in the same manner as would be necessary if it was desired to frame a Table for finding the perpendiculars in the hyperbolic triangle for different lengths of base. That this is not so, may be a happy event; but by what evidence included in their proposed demonstration, do they know that it is not? All they can say is, that they have no evidence that it *is* so. Their fallacy therefore, is that of putting what they do not know to be, for what they know *not to be*. Or if they trust to the difficulty of finding anything in the case of straight lines by which the variation of the angle could have been regulated,—how do they know, for example, that nature instead of making the angle $C = 2R - (A + B)$, has not made it $= \overline{2R - (A + B)} \times \frac{m - p}{m}$, where m the modulus is some given straight line; m again being equal in different triangles to $z \times \frac{2R}{A + B}$, where z shall be some grand modulus existing in nature, which (for the sake of removing the argument from vulgar experience) may be supposed to be of very great dimension, as for instance equal to the radius of the earth's orbit? If an astronomer should arise and declare he had found astronomical evidence that this was true, how would the supporters of the *analytical* proof proceed to put him down?—and would they not find themselves in the situation of those prophets, who find it easier to prophesy after the fact, than while the result is in abeyance?*

22. A demonstration is offered in the *Elémens de Géométrie, par Lacroix (13ème édition, p. 23)* and attributed to M. Bertrand, which is perhaps the hardest of all to convince of weakness, and takes the strongest hold of the difficulty which exists in distinguishing between observation and mathematical proof. M. Bertrand begins by stating, that an angle however small may be multiplied till it equals or exceeds a right angle; whence it may be considered as intercepting a surface equal to or greater than some given fraction of the infinite space intercepted between the straight lines that form a right angle, as for argument's sake a thousandth. If then there be taken another right angle, and at any distance from the angular point be drawn a perpendicular from one of the straight lines that make the right angle and of course a parallel to the other, and other perpendiculars at the same distance in succession from one another; a thousand of the intercepted surfaces or bands, it is argued, will not fill up the infinite space aforesaid; wherefore, it is concluded, the space between two of the diverging lines must finally be greater than the space between two of the parallel ones, and consequently a straight line making an angle of a thousandth part of a right angle with the straight line to which the first perpendicular was drawn parallel, must finally overtake and pass and cut that perpendicular.

All references to the equality of magnitude of infinite areas, are intrinsically paralogisms. When it is affirmed that the surface of the thousand small angles is equal to the surface of the right angle and the surface of the thousand bands is not, to reduce this to

* For reference to a number of places where this subject is agitated in various senses, see Legendre's *Elém. de Géom.* 12ème *édition*, *Notes*, *p.* 287.

anything reasonable and precise it is necessary that it be understood to mean, that if a circle be drawn about the angular point, the portion of its area intercepted between M. Bertrand's two perpendiculars to the radius, will, on increasing the radius of the circle while the distance between the perpendiculars remains unaltered, diminish in ratio to the area of the whole quadrant, and may be reduced below the thousandth or other assigned part; from which it would follow that the line making the angle of a thousandth part of a right angle with the straight line to which the first perpendicular was drawn parallel, will have cut that perpendicular. And there is no doubt that this is accordant with experience in ordinary cases; but the question is, whether there is evidence that it will of necessity take place in cases where experience cannot be said to act at all. The real basis on which the mind inclines to receive it, is neither more nor less than the confidence felt in the indications derived from experiment in other cases, that the perpendiculars between parallel lines are equal, and consequently the areas of parallelograms vary as their length; but if this is not to be taken for granted, the question is, what evidence, and of what kind, there is in an extreme case such as may be said to remove itself from experimental examination. Suppose, for example, two perpendiculars to the earth's diameter, and a straight line making with one of them an angle of a second, or 324,000th part of a right angle. In a case like this, where even on taking for granted the theory in dispute the meeting could not take place under a distance equal to above 206,000 diameters of the earth or 17 times the distance of the sun, and in which the difference of the lines from parallelism at any possible portion of their course is totally imperceptible to human sense,—has the mind any convincing evidence, either from experience or otherwise, that it would be contrary to the nature and constitution of straight lines that the two perpendiculars should fail at some time to intercept an area less than the 324,000th part of the quadrant, and by a gradual deflexion totally beyond the perception of the senses should evade the approaches of the would-be secant, and take a form resembling that of a parallel to it, or of a line to which it is an asymptote? Man has experience in *some* cases that the like would not happen; and it is not denied that in *some* cases the section will take place. But has he experience in this? and if he has not, has he geometrical evidence to put in its room? It is imagined, not. This then may be concluded to be another, though a very complicated and ingenious case, in which man's empirical knowledge of what he *has* tried, is substituted in what he has not tried and cannot try. Which, whatever may be thought of the chances of its final validity, is not geometrical proof.

The number of demonstrations proposed on the subject of Parallel Lines is evidence of the anxiety felt by geometrical writers upon the subject. If an erroneous account has been given of any cited above, provision has been made for furnishing the reader with the references required to put him in possession of the truth.

THE END.

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