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A COURSE OF

DIFFERENTIAL GEOMETRY

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A COURSE OF

DIFFERENTIAL GEOMETRY

BY THE LATE

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PREFACE

My father had spent most of his spare time since the War in writing this book. Only two months before his death, while on our summer holiday in 1924, he had brought some of the chapters with him, and sent off the final draft of them to the Clarendon Press. Even on these holidays, which he greatly enjoyed, we were all accustomed to a good deal of work, and it was an unexpected pleasure to find that with these once dispatched to the press he took an unusually complete holiday.

While rejoicing that he was so far able to complete the book, we are sorry that a last chapter or appendix in which he was greatly interested was hardly begun. Apparently this was to deal with the connexion between the rest of the book and Einstein's theory. To the mathematical world his interest in this was shown by his Presidential address to the London Mathematical Society in 1920—to his friends by the delight he took on his frequent walks in trying to explain in lucid language something of what Einstein's theory meant.

We cannot be too grateful to Professor Elliott, F.R.S., an old friend of many years standing, for preparing the book for the press and reading and correcting the proofs. No labour has been too great for him to make the book as nearly as possible what it would have been. And the task has been no light one.

We should like to thank the Clarendon Press for their unfailing courtesy and for the manner in which the book has been produced.

J. M. H. C.

Christmas 1925.

EDITOR'S NOTE

My dear friend the author of this book has devoted to preparation for it years of patient study and independent thought. Now that he has passed away, it has been a labour of love to me to do my best for him in seeing it through the press. As I had made no special study of Differential Geometry beforehand, and was entirely without expertness in the methods of which Mr. Campbell had been leading us to realize the importance, there was no danger of my converting the treatise into one partly my own. It stands the work of a writer of marked individuality, with rather unusual instincts as to naturalness in presentation. A master's hand is shown in the analysis.

Before his death he had written out, and submitted to the Delegates of the University Press, nearly all that he meant to say. An appendix, bearing on the Physics of Einstein, was to have been added; but only introductory statements on the subject have been found among his papers. Unfortunately finishing touches, to put the book itself in readiness for printing, had still to be given to it. The chapters were numbered in an order which, rightly or wrongly, is in one place here departed from, but they stood almost as separate monographs, with only a very few references in general terms from one to another. To connect them as the author would have done in due course is beyond the power of another. The articles, however, have now been numbered, and headings have been given to them. Also some references have been introduced. The text has not been tampered with, except in details of expression; but a few foot-notes in square brackets have been appended.

E. B. E.

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CHAPTER I

TENSOR THEORY

§ 1. **The n-way differential quadratic form.** Let us consider the expression

$$a_{ik}dx_i dx_k, \quad {}^i_k = 1 \dots n \tag{1.1}$$

which is briefly written for the sum of n^2 such terms, obtained by giving to i, k independently the values 1, 2, ... n. If, for instance, $n = 2$, the expression is a short way of writing

$$a_{11}dx_1^2 + 2a_{12}dx_1dx_2 + a_{22}dx_2^2;$$

for we are assuming that

$$a_{ik} = a_{ki}. \tag{1.2}$$

Let us also denote by a^{ik} the result of dividing by a itself $(-1)^{i+k}$ times the determinant obtained by erasing the row and the column which contain a_{ik} in the determinant

$$a \equiv \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{vmatrix}. \tag{1.3}$$

The coefficients a_{ik} ... are at present arbitrarily assigned functions of the variables $x_1 \dots x_n$, limited only by the condition that a is not zero.

When we are given the coefficients a_{ik} as functions of their arguments, there must exist r functions

$$X_1 \dots X_r, \quad r = \tfrac{1}{2}n(n+1),$$

of the variables $x_1 \dots x_n$, such that

$$dX_1^2 + \dots + dX_r^2 \equiv a_{ik}dx_i dx_k. \tag{1.4}$$

The differential equations which will determine these functions are

$$\frac{\partial X_p}{\partial x_i}\frac{\partial X_p}{\partial x_k} = a_{ik}. \tag{1.5}$$

Just as in the expression $a_{ik}dx_idx_k$ the law of the notation is that, whenever a suffix, which occurs in one factor of a product, is repeated in another factor, the sum of all such products is to be taken, so here the above differential equation is the short way of writing

$$\frac{\partial X_1}{\partial x_i}\frac{\partial X_1}{\partial x_k}+\ldots+\frac{\partial X_r}{\partial x_i}\frac{\partial X_r}{\partial x_k}=a_{ik}. \qquad (1.6)$$

As there are just as many unknown functions as there are differential equations to be satisfied, we know that the functions $X_1 \ldots X_r$ must exist. The actual solution of this system of differential equations is, however, quite another matter, and questions connected with the solution form a chief part in the study of Differential Geometry.

§ 2. The distance element. Euclidean and curved spaces. If we regard $x_1 \ldots x_n$ as the coordinates of a point in an n-way space, then, $X_1 \ldots X_r$ being functions of $x_1 \ldots x_n$, we may regard this space as a locus in r-way Euclidean space; and we may regard ds as the distance between two neighbouring points $x_1 \ldots x_n$ and $x_1+dx_1 \ldots x_n+dx_n$, where ds is defined by

$$ds^2 = a_{ik}dx_i dx_k. \qquad (2.1)$$

Thus, if $n = 2$, the two-way space given by

$$ds^2 = a_{ik}dx_i dx_k$$

lies within our ordinary Euclidean space, and it is with this space that Differential Geometry has hitherto been chiefly concerned.

If $n = 3$, the 'curved' three-way space lies, in general, within a Euclidean six-way space. If, however, the coefficients a_{ik}, instead of being arbitrarily assigned functions of their arguments x_1, x_2, x_3, satisfy certain conditions, the Euclidean space may be only a five-way space, or even only a four-way space. In yet more special cases the three-way space may not be 'curved' at all, but only ordinary Euclidean space with a different coordinate system of reference.

If $n = 4$, the curved four-way space lies, in general, within a Euclidean ten-way space, and so on.

We know what a curved two-way space within a Euclidean three-way space means, being a surface: but what does a 'curved' three-way space mean? We have not, and we cannot have, a conception of a four-way space, Euclidean or otherwise, within which the three-way space is to be curved. But by thinking of the geometry associated with the form

$$ds^2 = a_{11}dx_1^2 + 2a_{12}dx_1dx_2 + a_{22}dx_2^2 \qquad (2.2)$$

we say that it is that of a curved two-way space; and we know that it is, in general, different from the flat Euclidean plane geometry associated with the form

$$ds^2 = dx_1^2 + dx_2^2. \qquad (2.3)$$

We can distinguish these two geometries without any reference to the Euclidean three-way space, or any other three-way space. This distinction we, with our knowledge of a three-way Euclidean space, characterize by saying that the first space is curved and the second flat, or Euclidean.

This is what we mean when we say that the space given by

$$ds^2 = a_{ik}dx_idx_k \qquad (2.4)$$

is, in general, a curved space, whilst that given by

$$ds^2 = dX_1^2 + \ldots + dX_r^2 \qquad (2.5)$$

is a flat space. We shall find that a geometrical property will be associated with a curved space, which will distinguish it from a flat space.

If we have no real knowledge of a space of more than three dimensions, we have at least no knowledge that it does not exist: and, by analogy from our knowledge both of a two-way space and a three-way space, we are able to make use of the ideas of higher space to express analytical results in an interesting form.

The space in which we live may, or may not, be flat or Euclidean. Up till quite recently it has been assumed to be flat, and the geometry which has been built up has been that associated with the form

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

The geometry which we wish to know about to-day would be that associated with the form

$$ds^2 = a_{ik}dx_idx_k, \quad {}^i_k = 1 \ldots 4,$$

where $x_1 \ldots x_4$ are functions of the three variables which locate an event in space, and a fourth variable which locates it in time.

The geometry of Euclidean space is much simpler than the geometry associated with the more general form, and its properties have been more studied. It may therefore be of advantage, at least in some ways, to regard the form

$$ds^2 = a_{ik} dx_i dx_k \tag{2.1}$$

as that of an n-way locus in a flat r-way space, although r is generally a much larger number than n.

§ **3. Vectors in a Euclidean space which trace out the space of a form.** Let $\iota', \iota'', \iota''' \ldots$ be r unit vectors in the Euclidean space and let y and z be vectors given by

$$\left.\begin{aligned} y &\equiv y'\iota' + y''\iota'' + \ldots \\ z &\equiv z'\iota' + z''\iota'' + \ldots \end{aligned}\right\}. \tag{3.1}$$

What we call the scalar product of the two vectors y and z is denoted by $\underbrace{yz}$ and defined by

$$\underbrace{yz} + y'z' + y''z'' + \ldots = 0. \tag{3.2}$$

The cosine of the angle between the vectors is defined as

$$\frac{y'z' + y''z'' + \ldots}{\sqrt{y'^2 + y''^2 + \ldots}\sqrt{z'^2 + z''^2 + \ldots}},$$

and may be written

$$\frac{-\underbrace{yz}}{\sqrt{\underbrace{yy}\,\underbrace{zz}}}. \tag{3.3}$$

We shall generally write $\underbrace{yy}$ as y^2, but we must remember then that the root of y^2 is not y.

The numbers $y', y'' \ldots$ are called the components of the vector y: they are ordinary scalar numbers.

Now let z be a vector whose components are functions of the n parameters $x_1 \ldots x_n$. Denoting the derivative of z with respect to x_r by z_r, we have $dz = z_p dx_p$ in the notation we have explained, which is the foundation of the Tensor Calculus. We therefore have

$$\underbrace{dz\,dz} = \underbrace{z_i z_k} dx_i dx_k. \tag{3.4}$$

The vector z traces out an n-way space within the Euclidean r-way space, and in this n-way space the element of 'length' is given by

$$ds^2 = -\underbrace{dz dz}; \qquad (3.5)$$

and therefore, if we take

$$a_{ik} = -\underbrace{z_i z_k}, \qquad (3.6)$$

we have

$$ds^2 = a_{ik} dx_i dx_k.$$

We say that dz is an element in this space, and we notice that an element has direction as well as length. The element is localized at the extremity of the vector z; the element lies in the n-way space, but the vector lies in the r-way Euclidean space.

The direction cosines of the element in the r-way space are

$$z'_p \frac{dx_p}{ds}, \quad z''_p \frac{dx_p}{ds}, \; \ldots . \qquad (3.7)$$

We write

$$\xi^p \equiv \frac{dx_p}{ds},$$

and we speak of $\xi^1, \xi^2, \ldots \xi^n$ as the direction cosines of the element in the n-way space given by, or associated with,

$$ds^2 = a_{ik} dx_i dx_k.$$

The upper affixes in $\xi^1 \ldots \xi^n$ have, of course, no implication of powers as in ordinary algebra. The notation introduced is in accordance with that of the tensor calculus which we are leading up to. In accordance with that calculus we ought to write the variables $x_1 \ldots x_n$ as $x^1 \ldots x^n$, but we do not do so, as the notation $x_1 \ldots x_n$ is at present too firmly fixed perhaps.

If ω is the angle between two elements, drawn through the extremity of z, whose direction cosines with respect to the n-way space are

$$\xi^1, \xi^2, \ldots \xi^n,$$
$$\eta^1, \eta^2, \ldots \eta^n,$$

respectively,

$$\begin{aligned}\cos\omega &= \xi^p \eta^q (z'_p z'_q + z''_p z''_q + \ldots) \\ &= -\xi^p \eta^q \underbrace{z_p z_q} = a_{pq} \xi^p \eta^q. \qquad (3.8)\end{aligned}$$

It should be noticed that $a_{pq}\xi^p\eta^q$ means precisely the same thing as $a_{ik}\xi^i\eta^k$. Repeated suffixes are called dummy suffixes and can be replaced by any other dummy suffixes. The chief rule that we need to follow is not to use the same dummy more than twice in an expression containing a number of factors.

It should be noticed that the angle, for which we have found an expression, is that between two elements drawn through the same point, viz. the same extremity of the vector z. We have no expression for the angle between two elements at different points in our n-way space. This is something that distinguishes the geometry connected with the form $ds^2 = a_{ik}dx_i dx_k$ from the geometry of Euclidean space.

§ **4. Christoffel's two symbols of three indices.** Let

$$(ikt) \equiv \frac{1}{2}\left(\frac{\partial a_{it}}{\partial x_k} + \frac{\partial a_{kt}}{\partial x_i} - \frac{\partial a_{ik}}{\partial x_t}\right). \qquad (4.1)$$

This is the definition of Christoffel's three-index symbol of the first kind. It is exceedingly important in the theory of differential geometry. The first two suffixes are interchangeable. We may write it sometimes in the form T_t when we regard i and k as fixed suffixes.

Since $$a_{ik} = -\underbrace{z_i z_k},$$

we see that $$(ikt) = -\underbrace{z_{ik} z_t}, \qquad (4.2)$$

where $$z_{ik} \equiv \frac{\partial^2 z}{\partial x_i \partial x_k}.$$

We introduce the symbol ϵ^i_k to denote zero if i and k are unequal and unity if i and k are equal. We do not write ϵ^i_i as equal to unity, for by our convention

$$\epsilon^i_i = \epsilon^1_1 + \epsilon^2_2 + \ldots + \epsilon^n_n = n.$$

In employing dummy suffixes it is best to employ a letter to which we have not attached a definite connotation.

From the property of determinants and their first minors we see that $$a^{it}a_{kt} = \epsilon^i_k. \qquad (4.3)$$

Let $$\{ikj\} \equiv a^{jt}(ikt); \tag{4.4}$$
then $\{ikj\}$ is Christoffel's three-index symbol of the second kind. The first two suffixes are interchangeable and it may be written T^j when we regard i and k as fixed suffixes.

We have at once $$(ikj) = a_{jt}\{ikt\}. \tag{4.5}$$

§ 5. **Some important operators.** Even already we have come across a number of functions of the variables which we denote by integers attached to a certain letter. Thus we have the fundamental functions denoted by $a_{ik}\ldots$; we have the direction cosines denoted by $\xi^1 \ldots \xi^n$, and the functions

$$a^{ik}\ldots.$$

More generally we may have a number of functions of the variables, say θ, ϕ, ψ, ... and we may form a function of θ, ϕ, ψ, ... and their derivatives with respect to the variables. It may be that the function thus arrived at may be denoted by

$$T^{\alpha, \beta, \ldots}_{a, b, \ldots} \tag{5.1}$$

where α, β, ... are integers of the upper row, upper integers we call them, and a, b, ... are lower integers. These integers may take independently any of the values 1, 2, ... n and thus indicate how the function $T^{\alpha, \beta, \ldots}_{a, b, \ldots}$ is formed. The number of the upper integers is not necessarily equal to the number of the lower integers. It may be that there are no integers in the upper row, or none in the lower, or even none in either.

We shall come across many functions which may be expressed in this manner, and we have come across some.

In connexion with functions which are expressed in the above form there are n operators which are of fundamental importance in tensor theory. These operators may be written

$$\bar{1}, \bar{2}, \bar{3}, \ldots \bar{n},$$

where $\bar{p}$ denotes the operator

$$\frac{\partial}{\partial x_p} + \{tp\lambda\}\binom{t}{\lambda} - \{\mu pt\}\binom{t}{\mu}; \tag{5.2}$$

and where $\binom{t}{\lambda}$ denotes the operation of substituting t for λ,

λ being any upper integer, and where $\binom{t}{\mu}$ has a similar meaning with respect to a lower integer.

Thus $\bar{p}\, T^{\alpha,\beta}_{a}$

$$= \frac{\partial}{\partial x_p} T^{\alpha,\beta}_{a} + \{tp\alpha\}\, T^{t,\beta}_{a} + \{tp\beta\}\, T^{\alpha,t}_{a} - \{apt\}\, T^{\alpha,\beta}_{t}; \tag{5.3}$$

the t which occurs on the right is a dummy suffix, and thus, for instance,

$\{tp\alpha\}\, T^{t\beta}_{a}$

$$= \{1p\alpha\}\, T^{1\beta}_{a} + \{2p\alpha\}\, T^{2\beta}_{a} + \ldots + \{np\alpha\}\, T^{n\beta}_{a}. \tag{5.4}$$

We notice that the definite integers $1, 2, \ldots n$ are not dummies, and we should avoid the use of n as a dummy.

We write $$T^{\alpha\beta}_{a\cdot p} \equiv \bar{p}\, T^{\alpha\beta}_{a}. \tag{5.5}$$

By aid of the symbolism thus introduced we can avoid a prolixity which would otherwise almost bar progress. A very little practice will enable one to use this symbolism freely, and when necessary to express the results explicitly.

§ **6. Conclusions as to derivatives of a_{ik}, a^{ik}, and $\frac{1}{2}\log a$.** We see from the definition that

$$\frac{\partial a_{ik}}{\partial x_p} = (ipk) + (kpi)$$
$$= a_{kt}\{ipt\} + a_{it}\{kpt\}; \tag{6.1}$$

and therefore the operator $\bar{p}$ annihilates each of the functions a_{ik}, which, of course, could have been written T_{ik}.

We have $$a^{it} a_{tk} = \epsilon^{i}_{k};$$

and therefore

$$a_{tk}\frac{\partial}{\partial x_p} a^{it} + a^{it}(tpk) + a^{it}(kpt) = 0.$$

It follows that

$$a^{kq} a_{tk}\frac{\partial}{\partial x_p} a^{it} + a^{it}\{tpq\} + a^{kq}\{kpi\} = 0;$$

that is, $$\frac{\partial}{\partial x_p} a^{iq} + a^{it}\{tpq\} + a^{qt}\{tpi\} = 0. \tag{6.2}$$

It follows that the operator $\bar{p}$ also annihilates each of the functions a^{ik}.

By the rule for the differentiation of a determinant

$$\frac{\partial a}{\partial x_t} = a\, a^{pq} \frac{\partial}{\partial x_t} a_{pq}$$

$$= a\, a^{pq}(ptq) + a\, a^{pq}(qtp)$$

$$= a\,\{ptp\} + a\,\{qtq\}$$

$$= 2a\,\{ptp\},$$

or
$$\frac{\partial}{\partial x_t} a^{\frac{1}{2}} = a^{\frac{1}{2}}\,\{ptp\}. \qquad (6\,.\,3)$$

This formula will be required later. [It should be remembered that the symbol on the right stands for the sum of n symbols, with $p = 1, 2, \dots n$.]

§ 7. **Tensors and tensor components defined.** We must now explain what is meant by a tensor. We have seen how functions denoted by

$$T^{\alpha,\,\beta,\,\dots}_{a,\,b,\,\dots}\dots$$

may be derived from functions θ, ϕ, ψ ... and their derivatives with respect to $x_1 \dots x_n$. The different functions obtained by allowing the integers to take all values from 1 up to n are called components of the set.

Suppose that we transform to new variables $x'_1 \dots x'_n$, and that θ' denotes the expression of θ in terms of the new variables, and that ϕ', ψ' ... have similar meanings. Suppose further that

$$T'^{\alpha',\,\beta',\,\dots}_{a',\,b',\,\dots}\dots$$

are functions formed from θ', ϕ', ψ', ... and their derivatives with respect to the new variables $x'_1 \dots x'_n$ by exactly the same rules as the functions

$$T^{\alpha,\,\beta,\,\dots}_{a,\,b,\,\dots}\dots$$

were formed from θ, ϕ, ψ, and their derivatives with respect to $x_1 \dots x_n$.

We say that $$T^{\alpha, \beta, \dots}_{a, b, \dots} \dots$$

are components of a tensor if

$$T'^{\alpha', \beta', \dots}_{a', b', \dots} = \frac{\partial x_a}{\partial x'_{a'}} \frac{\partial x_b}{\partial x'_{b'}} \dots \frac{\partial x'_{\alpha'}}{\partial x_\alpha} \frac{\partial x'_{\beta'}}{\partial x_\beta} \dots T^{\alpha, \beta, \dots}_{a, b, \dots}.$$

Notice that the integers on the left are not dummies but that the integers $\alpha, \beta, \dots a, b, \dots$ on the right are. Notice also that the above equation must hold for all values of the integers on the left if the expressions

$$T^{\alpha, \beta, \dots}_{a, b, \dots} \dots$$

are to be tensor components.

This is the formal definition: we shall immediately come across examples of tensors which will illustrate the definition.

§ 8. **The functions a_{ik} and a^{ik} are tensor components.** If we transform to new variables $x'_1 \dots x'_n$, the expression for the square of the element of length must remain unaltered in magnitude though its form may change We therefore have

$$a_{pq} dx_p dx_q = a'_{\lambda\mu} dx'_\lambda dx'_\mu$$

and so $$a'_{\lambda\mu} = \frac{\partial x_p}{\partial x'_\lambda} \frac{\partial x_q}{\partial x'_\mu} a_{pq}. \tag{8.1}$$

Thus the functions $a_{ik} \dots$ satisfy the condition for being tensor components.

Again from the fundamental equality

$$a_{pq} dx_p dx_q = a'_{\lambda\mu} dx'_\lambda dx'_\mu$$

we have $$a_{pq} \frac{\partial x_p}{\partial x'_\lambda} = a'_{\lambda\mu} \frac{\partial x'_\mu}{\partial x_q}.$$

Notice that q and λ are no longer dummy suffixes in this equality. Multiply across by $a'^{\lambda s} \dfrac{\partial x_r}{\partial x'_s}$, then we have

$$a_{pq} a'^{\lambda s} \frac{\partial x_r}{\partial x'_s} \frac{\partial x_p}{\partial x'_\lambda} = a'_{\lambda\mu} a'^{\lambda s} \frac{\partial x'_\mu}{\partial x_q} \frac{\partial x_r}{\partial x'_s}.$$

The expression on the right hand of this equality is

$$\epsilon^s_\mu \frac{\partial x'_\mu}{\partial x_q} \frac{\partial x_r}{\partial x'_s} = \frac{\partial x'_s}{\partial x_q} \frac{\partial x_r}{\partial x'_s} = \frac{\partial x_r}{\partial x_q} = \epsilon^r_q = a_{pq} a^{rp},$$

and therefore $\qquad a_{pq}\left(a^{rp} - a'^{\lambda s} \frac{\partial x_r}{\partial x'_s} \frac{\partial x_p}{\partial x'_\lambda}\right) = 0.$

This equation holds for all values of p, q, and r, and therefore, as the determinant a is not zero, we must have

$$a^{rp} = a'^{\lambda s} \frac{\partial x_r}{\partial x'_s} \frac{\partial x_p}{\partial x'_\lambda}. \tag{8.2}$$

It follows that the functions a^{ik} ... also satisfy the condition of being tensor components.

§ 9. Expressions for second derivatives when $a_{ik} dx_i dx_k = a'_{ik} dx'_i dx'_k$.

We have $\qquad z'_r = z_t \frac{\partial x_t}{\partial x'_r};$

$$z'_{pq} = z_{\lambda\mu} \frac{\partial x_\lambda}{\partial x'_p} \frac{\partial x_\mu}{\partial x'_q} + z_\lambda \frac{\partial^2 x_\lambda}{\partial x'_p \partial x'_q},$$

where z' is the expression of z in terms of the new variables, and z'_r denotes $\frac{\partial z'}{\partial x'_r}$ and z'_{pq} denotes $\frac{\partial^2 z'}{\partial x'_p \partial x'_q}$. It follows, since by (4.2) $(pqr)' = -\underbrace{z'_{pq} z'_r}$, that

$$(pqr)' = (\lambda\mu t) \frac{\partial x_\lambda}{\partial x'_p} \frac{\partial x_\mu}{\partial x'_q} \frac{\partial x_t}{\partial x'_r} + a_{\lambda t} \frac{\partial x_t}{\partial x'_r} \frac{\partial^2 x_\lambda}{\partial x'_p \partial x'_q}. \tag{9.1}$$

Notice, that we see, from this equation, that Christoffel's three-index symbols of the first kind do not satisfy the condition of being tensor components.

Multiply across by $a'^{rs} \frac{\partial x_k}{\partial x'_s}$, and we have

$$\{pqs\}' \frac{\partial x_k}{\partial x'_s}$$
$$= a'^{rs} (\lambda\mu t) \frac{\partial x_\lambda}{\partial x'_p} \frac{\partial x_\mu}{\partial x'_q} \frac{\partial x_t}{\partial x'_r} \frac{\partial x_k}{\partial x'_s} + a'^{rs} \frac{\partial x_t}{\partial x'_r} \frac{\partial x_k}{\partial x'_s} a_{\lambda t} \frac{\partial^2 x_\lambda}{\partial x'_p \partial x'_q}.$$

But, by (8.2), $\qquad a'^{rs} \frac{\partial x_t}{\partial x'_r} \frac{\partial x_k}{\partial x'_s} = a^{tk},$

and therefore the right-hand member of this equation becomes

$$a^{tk}(\lambda\mu t)\frac{\partial x_\lambda}{\partial x'_p}\frac{\partial x_\mu}{\partial x'_q}+a^{tk}a_{\lambda t}\frac{\partial^2 x_\lambda}{\partial x'_p\partial x'_q},$$

that is,
$$\{\lambda\mu k\}\frac{\partial x_\lambda}{\partial x'_p}\frac{\partial x_\mu}{\partial x'_q}+\epsilon^k_\lambda\frac{\partial^2 x_\lambda}{\partial x'_p\partial x'_q}. \qquad (9.2)$$

We therefore have the fundamental formula in the transformation theory

$$\frac{\partial^2 x_k}{\partial x'_p\partial x'_q}-\{pqs\}'\frac{\partial x_k}{\partial x'_s}=-\{\lambda\mu k\}\frac{\partial x_\lambda}{\partial x'_p}\frac{\partial x_\mu}{\partial x'_q}. \qquad (9.3)$$

Similarly we have

$$\frac{\partial^2 x'_k}{\partial x_p\partial x_q}+\{\lambda\mu k\}'\frac{\partial x'_\lambda}{\partial x_p}\frac{\partial x'_\mu}{\partial x_q}=\{pqs\}\frac{\partial x'_k}{\partial x_s}. \qquad (9.4)$$

§ **10. Tensor derivatives of tensor components are tensor components.** We must now show that the operators

$$\bar{1},\ \bar{2},\ \ldots\ \bar{n},$$

when applied to any tensor components, generate other tensor components.

Let
$$M\equiv\frac{\partial x_a}{\partial x'_{a'}}\frac{\partial x_b}{\partial x'_{b'}}\cdots,\qquad N\equiv\frac{\partial x'_{\alpha'}}{\partial x_\alpha}\frac{\partial x'_{\beta'}}{\partial x_\beta}\cdots,$$

and assume that $T^{\alpha,\beta,\ldots}_{a,b,\ldots}$ ... are tensor components.

We have
$$T'^{\alpha',\beta',\ldots}_{a',b',\ldots}=T^{\alpha,\beta,\ldots}_{a,b,\ldots}MN$$

which we briefly write $T'=TMN$.

Expanding $\frac{\partial}{\partial x'_{p'}}M$, using the formulae of the transformation theory which have been obtained,

$$\frac{\partial}{\partial x'_{p'}}M=\left(\{\mu p't\}'\binom{t}{\mu}-\frac{\partial x_q}{\partial x'_{p'}}\{tq\lambda\}\binom{t}{\lambda}\right)M,$$

and therefore

$$\bar{p}'M=\left(\{tp'\lambda\}'\binom{t}{\lambda}-\frac{\partial x_q}{\partial x'_{p'}}\{tq\lambda\}\binom{t}{\lambda}\right)M. \qquad (10.1)$$

Similarly we have

$$\frac{\partial}{\partial x'_{p'}} N = \left(\frac{\partial x_q}{\partial x'_{p'}} \{\mu q t\} \tbinom{t}{\mu} - \{t p' \lambda\}' \tbinom{t}{\lambda}\right) N,$$

and therefore

$$\bar{p}' N = \left(\frac{\partial x_q}{\partial x'_{p'}} \{\mu q t\} \tbinom{t}{\mu} - \{\mu p' t\}' \tbinom{t}{\mu}\right) N. \qquad (10.2)$$

Now

$$\bar{p}' T = \frac{\partial x_q}{\partial x'_{p'}} \frac{\partial}{\partial x_q} T + \left(\{t p' \lambda\}' \tbinom{t}{\lambda} - \{\mu p' t\}' \tbinom{t}{\mu}\right) T,$$

$$= \frac{\partial x_q}{\partial x'_{p'}} \bar{q} T - \frac{\partial x_q}{\partial x'_{p'}} \left(\{t q \lambda\} \tbinom{t}{\lambda} - \{\mu q t\} \tbinom{t}{\mu}\right) T$$

$$+ \left(\{t p' \lambda\}' \tbinom{t}{\lambda} - \{\mu p' t\} \tbinom{t}{\mu}\right) T. \qquad (10.3)$$

We have written $\frac{\partial x_a}{\partial x'_{a'}} \frac{\partial x_b}{\partial x'_{b'}} \dots$ simply as M, but we must note that M has the upper integers $a, b, \dots$ (as well as the lower integers $a', b', \dots$) and that the upper integers in M are the same as the lower integers in T.

Similarly we note that the lower integers in N are the upper integers in T.

It follows that

$$N \{t q \lambda\} \tbinom{t}{\lambda} T = T \{\mu q t\} \tbinom{t}{\mu} N, \qquad (10.4)$$

if we remember that these lower integers in N and upper integers in T are just dummies.

We have similarly

$$M \{\mu q t\} \tbinom{t}{\mu} T = T \{t q \lambda\} \tbinom{t}{\lambda} M, \qquad (10.5)$$

$$N \{t p' \lambda\}' \tbinom{t}{\lambda} T = T \{\mu p' t\}' \tbinom{t}{\mu} N, \qquad (10.6)$$

$$M \{\mu p' t\}' \tbinom{t}{\mu} T = T \{t p' \lambda\}' \tbinom{t}{\lambda} M, \qquad (10.7)$$

and therefore

$$\bar{p}' (TMN) = \frac{\partial x_q}{\partial x'_{p'}} M N \bar{q} T.$$

That is,

$$\bar{p}' T' = \bar{q} T \frac{\partial x_a}{\partial x'_a} \frac{\partial x_b}{\partial x'_{b'}} \dots \frac{\partial x_q}{\partial x'_{p'}} \frac{\partial x'_{\alpha'}}{\partial x_\alpha} \frac{\partial x'_{\beta'}}{\partial x_\beta} \dots, \qquad (10.8)$$

and therefore $\bar{p}\, T^{\alpha, \beta, \dots}_{a, b, \dots} \dots$ are tensor components.

· This is a very important theorem in the tensor calculus. It is the rule of taking what we call the tensor derivative and we see that the tensor derivative of a tensor component is a tensor component. We denote the p derivative by

$$T^{\alpha, \beta, \ldots}_{a, b, \ldots . p}. \tag{10.9}$$

§ **11. Rules and definitions of the tensor calculus.** We have now proved the most important theorem in the tensor calculus: its proof depended on the transformation theorems. These theorems, having served their purpose, disappear, as it were, from the calculus.

There are some simple rules of the calculus which we now consider.

The product of two tensors is a tensor whose components are the products of each component of the first and each component of the second tensor. The upper integers of the product are the upper integers of the two factors, and the lower integers of the product are the lower integers of the two factors.

Two tensors of the same character—that is, with the same number of each kind of integers, upper and lower—can be added, if we take together the components which have the same integers. They can also be combined in other ways, as we shall see.

We form the tensor derivative of the product of two tensors by the same rule as in ordinary differentiation.

The tensors a_{ik} and a^{ik} are called fundamental tensors. We have seen that they have the property of being annihilated by any operator $\bar{p}$. As regards tensor derivation they therefore play the part of constants.

The symbol ϵ_k satisfies the definition of a tensor. It also is called a fundamental tensor.

Any tensor, formed by taking the product of a tensor and a fundamental tensor, is said to be an associate tensor of the tensor from which it is derived.

Suppose that $T^{\alpha, \beta, \gamma, \ldots}_{a, b, c, \ldots}$ is any tensor. The tensor itself is the entity made up of all its components, formed by allowing $\alpha, \beta, \gamma, \ldots, a, b, c, \ldots$ to take all integral values from 1 up to n. Suppose now, that instead of taking all the components, we take those in which one of the upper integers, say α, is equal to one of the lower integers, say b. The entity we thus arrive at will be a tensor. For

$$\frac{\partial x_b}{\partial x'_{b'}} \frac{\partial x'_{b'}}{\partial x_b} = 1.$$

The tensor thus arrived at is denoted by

$$T^{p,\, \beta,\, \gamma,\, \ldots}_{a,\, p,\, c,\, \ldots}$$

and is said to be $\quad T^{\alpha,\, \beta,\, \gamma,\, \ldots}_{a,\, b,\, c,\, \ldots}$

contracted with respect to α, b.

We can contract a tensor with respect to any number of upper integers and an equal number of lower integers.

If we take the tensor $\quad T^{\alpha,\, \beta}_{a,\, b}$,

an associate tensor would be

$$\epsilon^{a}_{\alpha}\, T^{\alpha,\, \beta}_{a,\, b},$$

and we might write this $\quad T^{a,\, \alpha.\, \beta}_{\alpha,\, a,\, b}$;

and as it is contracted with respect to two upper integers and two lower we might write this simply as T^{β}_{b}.

So we may write $a^{pq}\, T^{\alpha\beta}_{pq}$ as $T^{\alpha\beta}$.

We shall often use this contraction when we are considering associate tensors

The rank of a tensor is the number of integers, upper and lower, in any component. When the rank is zero the tensor is an invariant. When the rank is even we can form an associate tensor which will be an invariant. When the rank is odd we can form an associate tensor of rank unity. When the rank is unity the tensor may be said to be a vector in the n-way space: a contravariant vector if the integer is an upper one, a covariant vector if the integer is a lower one. But it must be carefully noticed that when we think of a vector in the flat r-way space, we are thinking of the word vector in a different sense. Thus the vector z which traces out the n-way space is not an invariant, but rather the entity of r invariants, and so as regards the derivatives of z. In the r-way space they are all vectors, but the coefficients of the vectors ι', ι'' ... come under the classification of tensors. If we bear this distinction in mind we shall not be misled, and we may gain an advantage by combining the two notions. It is a useless exaggeration of the great advantages of the tensor calculus to ignore the calculus of Quaternions. We certainly cannot afford to give up the aid of the directed vector notation in the differential geometry of flat space within which lies our n-way curved space.

§ **12. Beltrami's three differential parameters.** If we take any function U of the variables, then

$$U_1, U_2, \dots U_n$$

will be tensor components. The tensor derivative of an invariant is just the ordinary derivative; and therefore the above functions are just the same as

$$U_{\cdot 1}, U_{\cdot 2}, \dots U_{\cdot n}. \tag{12.1}$$

[For the notation see (5.5) and (10.9).]

But if we take the second tensor derivatives we come across different functions from the ordinary second derivatives. These second tensor derivatives we denote by $U_{\cdot ik} \dots$ where

$$U_{\cdot ik} \equiv U_{ik} - \{ikt\}\, U_t. \tag{12.2}$$

These we have proved are tensor components (§ 10), whereas the ordinary second derivatives U_{ik} are not. It would be a useful exercise to prove that the functions $U_{\cdot ik} \dots$ are tensor components: it might make the general theorem, whose proof is rather complicated, more easily understood.

The square of the tensor whose components are $U_1 \dots U_n$ is a tensor whose components are $U_i U_k$. If we form the associate tensor $a^{ik} U_i U_k$ we have an invariant which is denoted by $\Delta(U)$, so that

$$\Delta(U) \equiv a^{ik} U_i U_k. \tag{12.3}$$

This is Beltrami's first differential parameter.

Similarly by forming the tensor which is the product of the two tensors whose components are $U_1 \dots U_n$ and $V_1 \dots V_n$, and taking the associate tensor $a^{ik} U_i V_k$, we have Beltrami's 'mixed' differential parameter

$$\Delta(U, V) \equiv a^{ik} U_i V_k. \tag{12.4}$$

We also have Beltrami's second differential parameter

$$\Delta_2(U) \equiv a^{ik} U_{.ik}. \tag{12.5}$$

Clearly all these 'differential parameters' as they are called are invariants. They are of great utility, as we shall find, in differential geometry.

§ 13. **Two associated vector spaces. Normals to surfaces.** Returning now to the vector z, whose extremity traces out the n-way space within the flat r-way space, we have, see (5.2) and (12.2),

$$z_{.ik} = z_{ik} - \{ikt\} z_t. \tag{13.1}$$

Clearly the components of this vector $z_{.ik}$ are tensor components.

We have $\frac{1}{2}n(n+1)$ vectors $z_{.ik}$ and we have n vectors z_i; as these $(\frac{1}{2}n(n+1)+n)$ vectors all lie in a $\frac{1}{2}n(n+1)$ flat space there must be n linear equations connecting them. These vectors all depend on the parameters $x_1 \dots x_n$, and we may regard them as all localized at the extremity of the vector z.

Now, see § 4,

$$\begin{aligned} \underbrace{z_{.ik} z_p} &= \underbrace{z_{ik} z_p} - \{ikt\} \underbrace{z_t z_p}, \\ &= -(ikp) + a_{tp}\{ikt\} = 0. \end{aligned} \tag{13.2}$$

We thus see that the vector $z_{.ik}$ is perpendicular to every element in the n-way space drawn through the extremity of z.

Let one of the n equations which connect the vectors $z_{.ik}$, z_i be

$$b_{ik} z_{.ik} + b_t z_t = 0,$$

where $b_{ik} \dots b_t \dots$ are scalars. Multiply the equation by z_p and take the scalar product: then, since

$$\underbrace{z_{.ik} z_p} = 0,$$

we have $$b_t \underbrace{z_t z_p} = 0;$$

that is, $$b_t a_{tp} = 0,$$

and therefore, since the determinant a cannot be zero, we have

$$b_1 = b_2 = .. = 0. \qquad (13.3)$$

It follows that the n linear equations connect the vectors $z._{ik}$ ... only.

At any point of the n-way space, therefore, there are n vectors $z_1 \dots z_n$ generating a flat n-way space; and there are $\frac{1}{2}n(n+1)$ vectors $z._{ik}$, only $\frac{1}{2}n(n-1)$ of which are linearly independent, and these generate a $\frac{1}{2}n(n-1)$ flat space. These two flat spaces, associated with the point $x_1 \dots x_n$, are such that every element in the one space, drawn through the extremity of z, is perpendicular to every element in the other space, drawn through the extremity of z.

Thus when n is equal to 2, as it is in ordinary differential geometry, the vectors $z._{11}, z._{12}, z._{22}$ (13.3)

are parallel to the normal at the extremity of z which traces out the surface we are concerned with.

§ **14. Euclidean coordinates at a point.** Associated with every point $x_1 \dots x_n$ we have a special system of coordinates which we call the Euclidean coordinates of the point. They are very helpful in proving tensor identities, which without their aid would prove very laborious.

At the point under consideration a_{ik} ... (ikj) ... are constants. Let another set of constants be defined by

$$a_{ik} = b_{it}b_{kt}, \quad b_{ik} = b_{ki}, \qquad (14.1)$$

and then another set by

$$(ikj) = b_{jt}c_{tik}, \quad c_{tik} = c_{tki}, \qquad (14.2)$$

and consider the transformation scheme

$$x'_i = b_{it}x_t + c_{ipq}x_p x_q. \qquad (14.3)$$

We have

$$z_j = z'_p(b_{pj} + c_{pjq}x_q),$$

$$z_{ik} = z'_{\lambda\mu}(b_{\lambda i} + c_{\lambda it}x_t)(b_{\mu k} + c_{\mu kr}x_r) + z'_\lambda c_{\lambda ik};$$

and therefore at the point

$$a_{ik} = a'_{\lambda\mu}b_{\lambda i}b_{\mu k}, \qquad (14.4)$$

$$(ikj) = (\lambda\mu p)' b_{\lambda i}b_{\mu k}b_{pj} + a'_{\lambda p}b_{pj}c^\lambda{}_{ik}. \qquad (14.5)$$

Now $$a_{ik} = b_{\mu i} b_{\mu k} = \epsilon_{\mu}^{\lambda} b_{\lambda i} b_{\mu k},$$
and the determinant a is equal to the square of the determinant b, so that the determinant b cannot be zero.

It follows that $$a'_{\lambda\mu} = \epsilon_{\mu}^{\lambda}; \tag{14.6}$$
and therefore $$(ikj) = (\lambda\mu p)' b_{\lambda i} b_{\mu k} b_{pj} + b_{j\lambda} c_{\lambda ik},$$
that is, $$(\lambda\mu p)' b_{\lambda i} b_{\mu k} b_{pj} = 0;$$
and therefore $$(\lambda\mu p)' = 0. \tag{14.7}$$

In this coordinate system the ground form *at the point* is
$$dx_1^2 + \ldots + dx^{\ 2}, \tag{14.8}$$
and the first derivatives of $a_{ik} \ldots$ vanish at the point. Of course it is only *at the point* that these results hold.

§ 15. Two symbols of four indices which are tensor components. Let us now consider the expression
$$\underbrace{z_{\cdot ri} z_{\cdot kh}} - \underbrace{z_{\cdot rh} z_{\cdot ik}}.$$

We see that since
$$\underbrace{z'_{\cdot ri} z'_{\cdot kh}} = \frac{\partial x_\lambda}{\partial x'_r} \frac{\partial x_\mu}{\partial x'_i} \frac{\partial x_\nu}{\partial x'_k} \frac{\partial x_\rho}{\partial x'_h} \underbrace{z_{\cdot\lambda\mu} z_{\cdot\nu\rho}},$$
and
$$\underbrace{z'_{\cdot rh} z'_{\cdot ki}} = \frac{\partial x_\lambda}{\partial x'_r} \frac{\partial x_\rho}{\partial x'_h} \frac{\partial x_\nu}{\partial x'_k} \frac{\partial x_\mu}{\partial x'_i} \underbrace{z_{\cdot\lambda\rho} z_{\cdot\nu\mu}},$$
we have
$$\underbrace{z'_{\cdot ri} z'_{\cdot kh}} - \underbrace{z'_{\cdot rh} z'_{\cdot ki}} = \frac{\partial x_\lambda}{\partial x'_r} \frac{\partial x_\nu}{\partial x'_k} \frac{\partial x_\rho}{\partial x'_h} \frac{\partial x_\mu}{\partial x'_i} (\underbrace{z_{\cdot\lambda\mu} z_{\cdot\nu\rho}} - \underbrace{z_{\cdot\lambda\rho} z_{\cdot\nu\mu}}), \tag{15.1}$$
that is, the expression is a tensor component which should be denoted by T_{rkhi}, but as is customary we denote it by
$$(rkhi). \tag{15.2}$$

This is Christoffel's four-index symbol of the first kind.

We see that if the two first integers are interchanged the sign is reversed, if the last two integers are interchanged the sign is reversed, and if the two extreme integers are interchanged and also the two middle integers there is no change.

The expression $$a^{kt} z_{\cdot it} \tag{15.3}$$
is a vector whose components are tensor components: it is an associate vector to $z_{\cdot it}$ and may be denoted by z^k.

We then have

$$z_{\cdot ri} z^t_h - z_{\cdot rh} z^t_i$$
$$= a^{kt}(rkhi). \qquad (15.4)$$

This is Christoffel's four-index symbol of the second kind, which should be denoted by T^t_{rhi}, but is denoted by

$$\{rthi\}. \qquad (15.5)$$

Like the four-index symbol of the first kind it is a tensor component. If the last two integers are reversed the sign is changed, so that

$$\{rthi\} = -\{rtih\}. \qquad (15.6)$$

The three-index symbols, it will be remembered, unlike the four-index symbols, are not tensor components.

We can express the four-index symbols in terms of the fundamental tensor components $a_{ik}\ldots$ and their derivatives.

We have

$$(rkhi) = \underbrace{z_{\cdot ri} z_{\cdot kh}} - \underbrace{z_{\cdot rh} z_{\cdot ik}} = \underbrace{z_{ri} z_{\cdot kh}} - \underbrace{z_{rh} z_{\cdot ik}},$$
$$= \underbrace{z_{ri} z_{kh}} - \underbrace{z_{rh} z_{ik}} - \underbrace{z_{ri} z_t}\{kht\} + \underbrace{z_{rh} z_t}\{ikt\};$$

and, as

$$\underbrace{z_{ri} z_t} = -(rit), \quad \underbrace{z_{rh} z_t} = -(rht),$$

$$\frac{\partial}{\partial x_h}\underbrace{z_{ri} z_k} - \frac{\partial}{\partial x_i}\underbrace{z_{rh} z_k} = \underbrace{z_{ri} z_{kh}} - \underbrace{z_{rh} z_{ki}},$$

we therefore have

$$(rkhi) = \frac{\partial}{\partial x_i}(rhk) - \frac{\partial}{\partial x_h}(rik) + (rit)\{kht\} - (rht)\{ikt\}. \qquad (15.7)$$

This formula may be written

$$(rkhi) = \bar{i}\,(rhk) - \bar{h}\,(rik),$$

if we make the convention that the operators are only to act on the last integer, the first two being regarded as fixed, and the last as a lower integer.

We also have

$$\{rkhi\} = a^{kt}(rthi)$$
$$= a^{kt}\left(\bar{i}\,(rht) - \bar{h}\,(rit)\right)$$
$$= \bar{i}a^{kt}(rht) - \bar{h}a^{kt}(rit)$$
$$= \bar{i}\,\{rhk\} - \bar{h}\,\{rik\};$$

and therefore

$$\{rkhi\} = \frac{\partial}{\partial x_i}\{rhk\} - \frac{\partial}{\partial x_h}\{rik\} + \{tik\}\{rht\} - \{thk\}\{rit\}, \tag{15.8}$$

since the last integer in $\{ikt\}$ is to be regarded as an upper integer.

It may be noticed that

$$\{ikt\}\,(rht) = a^{tp}\,(ikp)\,(rht) = (ikp)\,\{rhp\}, \tag{15.9}$$

so that in the product $\{ikt\}\,(rht)$ the two symbols $\{\ \}$ and $(\ )$ may be interchanged.

§ **16. A four-index generator of tensor components from tensor components.** If we consider the expression

$$(\bar{p}\bar{q} - \bar{q}\bar{p})\,T^{\alpha,\,\beta,\,\dots}_{a,\,b,\,\dots}, \tag{16.1}$$

we see at once that it is a tensor component. To find out what it is we employ Euclidean coordinates at a specified point.

At this point we see that it is

$$\frac{\partial}{\partial x_p}\left(\frac{\partial}{\partial x_q}T + \{tq\lambda\}\binom{t}{\lambda}T - \{\mu qt\}\binom{t}{\mu}T\right)$$
$$- \frac{\partial}{\partial x_q}\left(\frac{\partial}{\partial x_p}T + \{tp\lambda\}\binom{t}{\lambda}T - \{\mu pt\}\binom{t}{\mu}T\right),$$

that is,

$$\left(\frac{\partial}{\partial x_p}\{tq\lambda\} - \frac{\partial}{\partial x_q}\{tp\lambda\}\right)\binom{t}{\lambda}T - \left(\frac{\partial}{\partial x_p}\{\mu qt\} - \frac{\partial}{\partial x_q}\{\mu pt\}\right)\binom{t}{\mu}T,$$

that is, $\qquad \left(\{t\lambda qp\}\binom{t}{\lambda} - \{\mu tqp\}\binom{t}{\mu}\right)T.$

At the specified point we therefore have

$$\bar{p}\bar{q} - \bar{q}\bar{p} = \{t\lambda qp\}\binom{t}{\lambda} - \{\mu tqp\}\binom{t}{\mu}; \tag{16.2}$$

and, as this is a tensor identity, it must therefore hold at every point.

The proof of this important theorem is a good example of the utility of Euclidean coordinates, *at a point*. The three-index symbols of Christoffel vanish at any point when referred to the Euclidean coordinates of that point. If they had been tensor components they would therefore have vanished in any system of coordinates. The four-index symbols do not vanish when referred to Euclidean coordinates. The four-index symbols and the tensor components which are associate to them are the indispensable tools of the calculus when we apply it to differential geometry and to the Modern Einstein Physics.

§ 17. **Systems of invariants.** We have (§ 12)

$$\Delta (u) \equiv a^{ik} u_i u_k,$$

and, in accordance with the notion of associate tensors, we may write

$$u^k \equiv a^{ik} u_i,$$

and therefore

$$\Delta (u) = u^t u_t. \tag{17.1}$$

Similarly we have

$$\Delta (u, v) = u^t v_t = u_t v^t. \tag{17.2}$$

In accordance with the same notion of associate tensors we might say that

$$u = a^{ik} u_{\cdot ik}; \tag{17.3}$$

but this is a rather dangerous use of the notation, as it suggests that the u on the left is the same as the u from which we formed $u_{\cdot ik}$, which is absurd. However, a very moderate degree of caution will enable us to use the Calculus of Tensors without making absurd mistakes on the one hand, or, on the other hand, introducing a number of extra symbols, and thus destroying the simplicity of the calculus, for the sake of avoiding mistakes which no one is likely to make.

We have proved, in § 6, the formulae

$$\frac{\partial}{\partial x_p} a^{iq} + a^{it} \{tpq\} + a^{qt} \{tpi\} = 0,$$

$$\frac{\partial}{\partial x_t} a^{\frac{1}{2}} = a^{\frac{1}{2}} \{ptp\},$$

and therefore we have

$$\frac{\partial}{\partial x_t} a^{\frac{1}{2}} a^{it} = -a^{\frac{1}{2}} a^{pq} \{pqi\}.$$

It follows that

$$\frac{\partial}{\partial x_t} a^{\frac{1}{2}} a^{it} u_i = a^{\frac{1}{2}} a^{ik} u_{\cdot ik}, \tag{17.3}$$

and therefore

$$\Delta_2 (u) = a^{-\frac{1}{2}} \frac{\partial}{\partial x_t} a^{\frac{1}{2}} u^t. \tag{17.4}$$

If we have any invariant of the quadratic form $a_{ik}dx_i dx_k$, say ϕ, we can obtain other invariants $\Delta(\phi)$, $\Delta_2(\phi)$ by means of the differential parameters; and when we have two invariants, ϕ and ψ, we also have the invariant $\Delta(\phi, \psi)$.

Clearly there cannot be more than n independent invariants.

Suppose that we have obtained, in any way, n independent invariants $u_1, \ldots u_n$. Here the suffixes have no meaning of differentiation or of being tensor components.

If we take these n invariants as the variables, then we have

$$a^{ik} = \Delta(u_i u_k), \tag{17.5}$$

and we can express the ground form in terms of the invariants.

In this case we can say that the necessary and sufficient conditions that two ground forms may be equivalent—that is, transformable the one into the other—are that for each form the equations

$$\Delta(u_i \; u_k) = \phi_{ik}(u_1 \ldots u_n) \tag{17.6}$$

may be the same.

For special forms of the ground form we may not be able to find the required n invariants to apply this method. Thus if the form is that of Euclidean space there are no invariants which are functions of the variables.

§ 18. An Einstein space, and its vanishing invariants.

Let us write $$A_{rkih} \equiv (rkih), \tag{18.1}$$

then $$\{rkih\} = a^{kp} A_{rpih}, \tag{18.2}$$

and therefore $$(rkih) = a_{kp}\{rpih\}.$$

We form associate tensor components (§ 11) of $(rkih) \ldots$, and we know that they will be tensor components. Thus we know that

$$a^{ki}(rkih) \tag{18.3}$$

will be a ten or component. We write

$$a^{ki} A_{rkih} \equiv A_{rh} \equiv A_{hr}.$$

A space for which all the tensor components $A_{ik} \ldots$ vanish is what is called an Einstein space. A space for which

$$a^{ik} A_{rkih} \equiv A_{rh} = m a_{rh},$$

where m is independent of the integers r, h, is called an extended Einstein space.

We can form invariants from the associate tensor components. Thus

$$a^{ik}A_{ik} \tag{18.4}$$

is an invariant which we may denote by A.

Again,

$$a^{ik}A_{ip} \tag{18.5}$$

is a tensor component which we may write A_p^k. We thus have the series of invariants

$$A_p^i A_i^p,\ A_p^i A_q^p A_i^q,\ A_p^i A_q^p A_r^q A_i^r \dots \tag{18.6}$$

All of these invariants vanish for an Einstein space.

We can form another series of invariants which do not vanish for an Einstein space. Thus we have

$$\begin{gathered} a^{\lambda\alpha}a^{\mu\beta}a^{\nu\gamma}a^{\rho\delta}A_{\lambda\mu\nu\rho}A_{\alpha\beta\gamma\delta}, \\ a^{\lambda\alpha}a^{\mu q}a^{\nu p}a^{\rho\delta}a^{\beta r}a^{\gamma s}A_{\lambda\mu\nu\rho}A_{\alpha\beta\gamma\delta}A_{pqrs}, \end{gathered} \tag{18.7}$$

and so on.

CHAPTER II*

THE GROUND FORM WHEN $n = 2$

§ **19. Alternative notations.** We now consider the ground form $a_{ik} dx_i dx_k$ for the particular case when $n = 2$. That is, we are to consider the geometry on a surface which lies in ordinary three-dimensional Euclidean space.

The square of an element of length on any surface is given by

$$ds^2 = a_{11} dx_1^2 + 2a_{12} dx_1 dx_2 + a_{22} dx_2^2, \qquad (19.1)$$

where a_{11}, a_{12}, a_{22} are functions of the coordinates x_1, x_2 which define the position of a point on the surface.

We often avoid the use of the double suffix notation, and take u and v to be the coordinates of a point on the surface, when we write

$$ds^2 = edu^2 + 2fdudv + gdv^2; \qquad (19.2)$$

or in yet another form

$$ds^2 = A^2 du^2 + 2AB\cos\alpha\, du dv + B^2 dv^2, \qquad (19.3)$$

where α is the angle at any point between the parametric curves, that is, the u curve along which only u varies and the v curve along which only v varies, and Adu and Bdv are the elementary arcs on these curves.

There is no difficulty in passing from one notation to the other. The double suffix is the one in which general theorems are best stated: it alone falls in with the use of the tensor calculus which so much lessens the labour of calculation.

* [The packets of MS. containing Chapters II and III, as submitted to the Delegates of the University Press, were numbered by the author in the reverse order, and that order would probably have been made suitable, by some rearrangement of matter, had he lived to put the work in readiness for printing. It has seemed best, however, to revert to the order of a list of headings found among the author's papers, an order in which the chapters, as they stand, were almost certainly written.]

In connexion with the form

$$ds^2 = edu^2 + 2fdu\,dv + gdv^2$$

we use h to denote the positive square root of $eg - f^2$: that is

$$h = a^{\frac{1}{2}} = AB\sin\alpha, \text{ where } a = a_{11}a_{22} - a_{12}^2. \quad (19.4)$$

The element of area on the surface is

$$hdudv = AB\sin\alpha\, dudv = a^{\frac{1}{2}}dudv. \quad (19.5)$$

§ **20. An example of applicable surfaces.** If we are given the equation of any surface in the Euclidean space, we can express the Cartesian coordinates of any point on the surface in terms of two parameters and thus obtain e, f, g in terms of these parameters,

$$e = x_1^2 + y_1^2 + z_1^2, \quad f = x_1x_2 + y_1y_2 + z_1z_2, \quad g = x_2^2 + y_2^2 + z_2^2, \quad (20.1)$$

where the suffixes indicate differentiation with regard to the two parameters.

Thus, if u is the length of any arc of a plane curve, we may write the equation of the curve $y = \phi(u)$, and the surface of revolution obtained by rotating the curve about the axis of x will have the ground form

$$ds^2 = du^2 + \phi(u))^2 dv^2,$$

where v is the angle turned through.

Can we infer that, if a surface has this ground form, it is a surface of revolution? We shall see that we cannot make this inference.

Thus consider the catenoid, that is, the surface obtained by the revolution of the catenary about its directrix. The ground form is $ds^2 = du^2 + (u^2 + c^2)\,dv^2$.

Take the right helicoid, given by the equation

$$z = c\tan^{-1}\frac{y}{x};$$

this is clearly a ruled surface, and we can express the coordinates of any point on it by

$$x = u\cos v, \quad y = u\sin v, \quad z = cv.$$

Its ground form is then

$$ds^2 = du^2 + (u^2 + c^2)\, dv^2,$$

and it is not a surface of revolution.

It is, however, applicable on the catenoid ; two surfaces which have the same ground form being said to be applicable, the one on the other.

§ 21. **Spherical and pseudospherical surfaces. The tractrix revolution surface.** There are two distinct classes of theorems about surfaces: there are the theorems which are concerned with the surface regarded as a locus in space; and there are the theorems about the surface regarded as a two-way space, and not as regards its position in a higher space. It is the latter type of theorems about which the ground form gives us all the information we require.

Thus all the formulae of spherical trigonometry can be, as we shall see [in the next chapter], deduced from the ground form

$$ds^2 = du^2 + \sin^2 u\, dv^2, \qquad (21.1)$$

where u is the colatitude and v the longitude.

We shall prove the fundamental formula

$$\cos c = \cos a \cos b + \sin a \sin b \cos C, \qquad (21.2)$$

and the formula for the area

$$A + B + C - \pi, \qquad (21.3)$$

and from these all the other formulae may be deduced.

So from the ground form

$$ds^2 = du^2 + \sinh^2 u\, dv^2 \qquad (21.4)$$

we can obtain the formulae of pseudospherical trigonometry —the trigonometry on a sphere of imaginary radius.

The fundamental formula is here

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cosh C, \qquad (21.5)$$

and the area of a triangle is

$$\pi - A - B - C. \qquad (21.6)$$

If in (21.4) we make the substitution (c being a constant)

$$u = u' - c, \quad v = 2e^{-c}v',$$

this ground form becomes

$$ds^2 = du^2 + (e^{-u} - e^{u-2c})^2 dv^2;$$

and if we take c to be a large constant it approximates to the ground form

$$ds^2 = du^2 + e^{-2u} dv^2, \tag{21.7}$$

and to this form pseudospherical trigonometry will also apply.

The formulae of spherical trigonometry or of pseudospherical trigonometry will apply to any surfaces which have the same ground form as the sphere or the pseudosphere. A real surface may have as its ground form

$$ds^2 = du^2 + e^{-2u} dv^2.$$

Thus if we take a tractrix, the involute, that is, of a catenary which passes through its vertex, the equation of the catenary is

$$y = \cosh x, \tag{21.8}$$

taking the directrix of the catenary as the axis of x; and if we take u as the arc of the tractrix, measured from its cusp, the vertex of the catenary, the equation of the tractrix is

$$y = e^{-u}. \tag{21.9}$$

If we now revolve the tractrix about the axis of x we get a surface of revolution with the ground form

$$ds^2 = du^2 + e^{-2u} dv^2. \tag{21.7}$$

The figure of the tractrix is something like Fig. 1; and its surface of revolution like Fig. 2.

§ **22. Ruled and developable surfaces. The latter applicable on a plane.** Let us now consider the most general ruled surface, formed by taking any curve in space as base, or as we shall say as directrix, and drawing, through each point of the directrix, a straight line in any direction determined by the position of the point on the directrix.

If x, y, z are the coordinates of any point on the directrix, and l, m, n the direction cosines of the line, then these coordinates of the point and these direction cosines of the line will be functions of a parameter v. We take u to be the distance of any point on the line from the point where the line intersects the directrix. Then the current coordinates of any point on the line may be written

$$x' = x + ul, \quad y' = y + um, \quad z' = z + un;$$

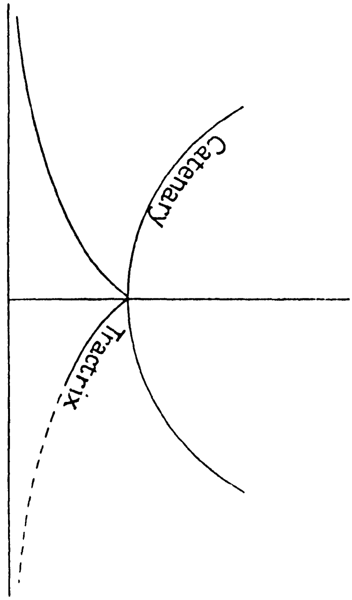


Fig. 1.

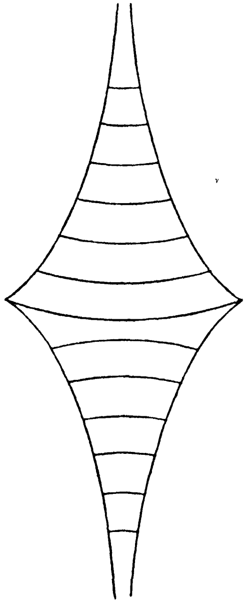


Fig. 2.

and for the ruled surface we have the ground form

$$ds^2 = du^2 + 2f du\, dv + g dv^2,$$

where

$$f = lx_2 + my_2 + nz_2, \tag{22.1}$$

$$g = x_2^2 + y_2^2 + z_2^2 + 2u\,(l_2x_2 + m_2y_2 + n_2z_2) + u^2\,(l_2^2 + m_2^2 + n_2^2). \tag{22.2}$$

That is, f is a function of v only and g is of the form

$$\alpha u^2 + 2\beta u + \gamma,$$

where α, β, γ are functions of v only.

We have $$ds^2 = (du + f dv)^2 + (g - f^2)\, dv^2; \tag{22.3}$$

the coordinates of any line of the ruled surface are functions of v only, and therefore the shortest distance between the point u, v and a neighbouring point on the line whose coordinates are functions of $v + dv$ is $(g - f^2)\, dv^2$.

The value of u for which this shortest distance will be least is then given by $\frac{\partial g}{\partial u} = 0$; that is, the equation of the line of striction is

$$\frac{\partial g}{\partial u} = 0. \tag{22.4}$$

If we take, as we may, the directrix to be a curve crossing the generators at right angles, and dv to be the angle between two neighbouring generators, we have

$$ds^2 = du^2 + \big((u-a)^2 + b^2\big)\, dv^2,$$

where a and b are functions of v. The line of striction is now $u = a$, and the shortest distance between two neighbouring generators is bdv.

For a developable surface therefore we have

$$ds^2 = du^2 + (u-a)^2\, dv^2. \tag{22.5}$$

If we take

$$u' = u \sin v - \int a \cos v\, dv, \quad v' = u \cos v - \int a \sin v\, dv, \tag{22.6}$$

we see that referred to the new coordinate system

$$ds^2 = du'^2 + dv'^2; \tag{22.7}$$

so that the above transformation formulae establish a correspondence, between the points on any developable and points on a Euclidean plane, such that the distance between neighbouring points on the developable and the distance between the corresponding neighbouring points on the plane are the same. The developable is therefore said to be applicable on the plane.

§ **23. Elliptic coordinates.** Consider now the system of confocal quadrics

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1.$$

We know that the relations

$$x^2=\frac{(a^2+u)(a^2+v)(a^2+w)}{(a^2-b^2)(a^2-c^2)},\quad y^2=\frac{(b^2+u)(b^2+v)(b^2+w)}{(b^2-a^2)(b^2-c^2)},$$

$$z^2=\frac{(c^2+u)(c^2+v)(c^2+w)}{(c^2-a^2)(c^2-b^2)} \tag{23.1}$$

give the coordinates of any point in space in terms of the focal coordinates u, v, w; and that the perpendiculars from the centre on the three confocals through any point are given by

$$p^2=\frac{(a^2+u)(b^2+u)(c^2+u)}{(u-v)(u-w)},\quad q^2=\frac{(a^2+v)(b^2+v)(c^2+v)}{(v-u)(v-w)},$$

$$r^2=\frac{(a^2+w)(b^2+w)(c^2+w)}{(w-u)(w-v)}.$$

From the formula

$$p^2=(a^2+u)\cos^2\alpha+(b^2+u)\cos^2\beta+(c^2+u)\cos^2\gamma, \tag{23.2}$$

where $$x\cos\alpha+y\cos\beta+z\cos\gamma=p$$

is the tangent plane to the surface $u=$ constant, we see that

$$2p\,dp=du,\quad 2q\,dq=dv,\quad 2r\,dr=dw,$$

and therefore $$4\,ds^2=\frac{du^2}{p^2}+\frac{dv^2}{q^2}+\frac{dw^2}{r^2}. \tag{23.3}$$

If we now take $w=0$ and write

$$a^2+u=U^2,\quad a^2+v=V^2,\quad K_1^2=a^2-b^2,\quad K_2^2=a^2-c^2,$$

so that the new coordinates are the semi-major axes of the two confocals through any point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

we have

$$ds^2 = (U^2-V^2)\left(\frac{(U^2-a^2)\,dU^2}{(U^2-K_1^2)(U^2-K_2^2)}+\frac{(a^2-V^2)\,dV^2}{(V^2-K_1^2)(V^2-K_2^2)}\right). \quad (23.4$$

It follows, as a particular case, that the ground form for a plane may be taken to be

$$ds^2 = (u^2-v^2)\left(\frac{du^2}{u^2-c^2}+\frac{dv^2}{c^2-v^2}\right). \qquad (23.5)$$

We thus have as ground forms of a plane

$$ds^2 = dx^2+dy^2,$$

$$ds^2 = dr^2+r^2 d\theta^2,$$

$$ds^2 = (u^2-v^2)\left(\frac{du^2}{u^2-c^2}+\frac{dv^2}{c^2-v^2}\right),$$

and we could find an infinite number of other forms for the plane, or for any other surface.

We are thus led to inquire as to the tests by which we can decide whether two given ground forms are equivalent; that is whether by a change of the variables the one form can be transformed into the other.

§ 24. The invariant K. $\Delta\phi$ and $\Delta_2\phi$ when K is constant. Consider the form $a_{ik}dx_i dx_k$, $\;{}^{i}_{k} = 1, 2$,

and let us use the methods of the tensor calculus.

In terms of the four-index symbols of Christoffel we have one and only one invariant

$$(1212) \div a,^* \qquad (24.1)$$

where $\quad a = a_{11}a_{22}-(a_{12})^2.$

* [The invariants A of (18.4) reduce to one. Also, as the equalities

$$(1212) = -(2112) = -(1221) = (2121)$$

hold, and the other symbols (1112), &c., vanish, the sum equal to $(1212)'$ in (15.1), with the notation of (15.2), is

$$\left(\frac{\partial x_1}{\partial x'_1}\frac{\partial x_2}{\partial x'_2}-\frac{\partial x_1}{\partial x'_2}\frac{\partial x_2}{\partial x'_1}\right)^2 (1212), \text{ i.e. } \frac{a'}{a}(1212).$$

For the explicit expression of K see Chap. III, § 43.]

We denote this invariant by K.

Let us first take the case when K is a constant and consider the differential equations

$$\phi_{\cdot 11} + K a_{11}\phi = 0, \quad \phi_{\cdot 12} + K a_{12}\phi = 0, \quad \phi_{\cdot 22} + K a_{22}\phi = 0. \tag{24.2}$$

We shall prove that they form a complete system: that is, a system such that no equation of the first order can be deduced from them by differentiation only.

We have $\phi_{\cdot 112} + K a_{11}\phi_2 = 0, \quad \phi_{\cdot 121} + K a_{12}\phi_1 = 0,$

and therefore [§ 16]

$$(\overline{21} - \overline{12})\,\phi + K\,(a_{11}\,\phi_2 - a_{12}\phi_1) = 0,$$

that is,
$$-\{1t\,12\}\,\phi_t + K\,(a_{11}\,\phi_2 - a_{12}\phi_1) = 0. \tag{24.3}$$

We have then to show that this is a mere identity.

Now $\{1t\,12\} = a^{tp}\,(1p\,12) = a^{t2}\,(1212)$

and therefore

$$\{1t\,12\}\,\phi_t = Ka\,(a^{12}\phi_1 + a^{22}\phi_2) = K\,(a_{11}\phi_2 - a_{12}\phi_1),$$

so that the equation of the first order turns out to be a mere identity. Similarly we see that the other equation of the first order is a mere identity.

If ϕ and ψ are any two integrals of the complete system we have

$$\begin{aligned}
&\frac{\partial}{\partial x_p}\left(\Delta\,(\phi, \psi) + K\phi\psi\right) \\
&= \bar{p}\,(a^{ik}\phi_i\psi_k) + K\frac{\partial}{\partial x_p}\phi\psi \\
&= a^{ik}\,(\phi_{\cdot ip}\psi_k + \phi_i\psi_{\cdot kp}) + K\,(\phi_p\psi + \phi\psi_p) \\
&= -Ka^{ik}\,(a_{ip}\psi_k + a_{kp}\phi_i) + K\,(\phi_p\psi + \phi\psi_p) \\
&= -K\,(\epsilon^k_p\psi_k + \epsilon^i_p\phi_i) + K\,(\phi_p\psi + \phi\psi_p) \\
&= 0.
\end{aligned} \tag{24.4}$$

We therefore have

$$\Delta\,(\phi) + K\phi^2 = \text{constant}. \tag{24.5}$$

We also have at once from (12.5) and the equations (24.2)

$$\Delta_2\,(\phi) + 2K\phi = 0. \tag{24.6}$$

§ **25. Determination of a ψ such that $\Delta(\phi, \psi) = 0$.** We shall now prove that if we are given any function u, such that $\Delta(u)$ and $\Delta_2(u)$ are both functions of u, then, in all cases (not merely when K is a constant), we can obtain by quadrature a function v such that $\Delta(u, v) = 0$.

Let
$$\mu = e^{-\int \frac{\Delta_2(u)}{\Delta(u)} du}, \tag{25.1}$$

then
$$\mu_1 = -\mu \frac{\Delta_2(u)}{\Delta(u)} u_1, \quad \mu_2 = -\mu \frac{\Delta_2(u)}{\Delta(u)} u_2.$$

The condition that $\mu a^{\frac{1}{2}} (u^2 dx_1 - u^1 dx_2)$,
where $u^1 \equiv a^{11} u_1 + a^{12} u_2, \quad u^2 \equiv a^{12} u_1 + a^{22} u_2,$
may be a perfect differential is
$$\frac{\partial}{\partial x_1}(\mu a^{\frac{1}{2}} u^1) + \frac{\partial}{\partial x_2}(\mu a^{\frac{1}{2}} u^2) = 0;$$
that is, $\mu a^{\frac{1}{2}} \Delta_2(u) + a^{\frac{1}{2}} (\mu_1 u^1 + \mu_2 u^2) = 0;$
and this condition is fulfilled.

We can therefore by quadrature find a function v such that
$$v_1 = \mu a^{\frac{1}{2}} u^2, \quad v_2 = -\mu a^{\frac{1}{2}} u^1, \tag{25.2}$$
and therefore
$$v_1 u^1 + v_2 u^2 = 0,$$
that is,
$$\Delta(u, v) = 0. \tag{25.3}$$

§ **26. Reduction of a ground form when K is constant.** Returning now to the case when K is a constant, we have seen that, if ϕ is an integral of the complete system,
$$\Delta_2(\phi) + 2K\phi = 0, \quad \Delta(\phi) + K\phi^2 = \text{constant},$$
and we can therefore by quadrature obtain ψ, where
$$\Delta(\phi, \psi) = 0.$$

First let us take the case when K is zero.

Without loss of generality we may suppose that
$$\Delta(\phi) = 1, \quad \Delta(\phi, \psi) = 0, \tag{26.1}$$
and we may take as new variables
$$x_1 = \phi, \quad x_2 = \psi,$$
and the ground form becomes
$$ds^2 = dx_1^2 + a_{22} dx_2^2.$$

Since x_1 is an integral of the complete system, we have

$$\{111\} = 0, \ \{121\} = 0, \ \{221\} = 0,$$

and therefore $(111) = 0, \quad (121) = 0, \quad (221) = 0.$

From the fact that a_{12} is zero, we have

$$(122)+(221) = 0,$$

and therefore $$(212) = 0;$$

so that a_{22} is a function of x_2 only.

We can therefore take the ground form to be

$$ds^2 = dx_1^2 + dx_2^2. \tag{26.2}$$

We next take the case when K is a positive constant, say R^{-2}. We then have

$$\Delta(\phi) + R^{-2}\phi^2 = \text{constant},$$

and, without loss of generality, we may suppose

$$\Delta(\phi) = R^{-2}(1-\phi^2), \tag{26.3}$$

and, by quadrature, we can find ψ so that

$$\Delta(\phi, \psi) = 0. \tag{26.4}$$

Take as new variables

$$x_1 = R\cos^{-1}\phi, \quad x_2 = \psi, \tag{26.5}$$

and the ground form becomes

$$ds^2 = dx_1^2 + a_{22}dx_2^2.$$

We have, since a_{12} is zero,

$$(122)+(221) = 0,$$

and, since $\cos\frac{x_1}{R}$ satisfies

$$\phi_{\cdot 22} + Ka_{22}\phi = 0,$$

we have $$(221) + \frac{a_{22}}{R}\cot\left(\frac{x_1}{R}\right) = 0;$$

and therefore $$(212) = \frac{a_{22}}{R}\cot\left(\frac{x_1}{R}\right),$$

that is, $$\frac{\partial}{\partial x_1}a_{22} = 2a_{22}\frac{\partial}{\partial x_1}\log\sin\left(\frac{x_1}{R}\right),$$

so that
$$\frac{\sin^2\left(\frac{x_1}{R}\right)}{a_{22}}$$
is a function of x_2 only.

We may therefore take the ground form as

$$ds^2 = dx_1^2 + \sin^2 \frac{x_1}{R} dx_2^2,$$

or if we take $\quad x_1 = Rx'_1, \quad x_2 = Rx'_2,$

we may take the ground form as

$$ds^2 = R^2 (dx_1^2 + \sin^2 x_1 dx_2^2). \qquad (26.6)$$

When K is a negative constant $-R^{-2}$, we see that the ground form is $ds^2 = -R^2 (dx_1^2 + \sin^2 x_1 dx_2^2)$; (26.7)

or if we take $\quad x'_1 = \iota x_1, \quad x'_2 = x_2,$

the ground form becomes

$$ds^2 = R^2 (dx_1^2 + \sinh^2 x_1 dx_2^2). \qquad (26.8)$$

We have seen in § 21 how the ground form

$$ds^2 = R^2 (dx_1^2 + e^{-2x_1} dx_2^2) \qquad (26.9)$$

may be deduced from this.

§ **27. The case of $\Delta(K) = 0$.** We have now seen the forms to which the ground forms are reducible when the invariant K is a constant; and we see that the necessary and sufficient condition that two ground forms may be equivalent, when for one of them K is a constant, is that for the other K may be the same constant.

We must now consider how we are to proceed when K is not a constant.

If $\Delta(K)$ is zero, we choose as our variables $x_1 = K$, $x_2 = v$, where v is any other function of the coordinates of the assigned ground form.

Since $\Delta(x_1)$ is zero, a^{11} is zero and the ground form may be written

$$ds^2 = edu^2 + 2\phi_2 du dv, \qquad (27.1)$$

where e and $\phi_2 \left(\equiv \frac{\partial \phi}{\partial v}\right)$ are some functions of u and v.

The equation which determines the invariant K which we have taken to be u is therefore

$$2u\phi_2 = \frac{\partial}{\partial v}\left(\frac{e_2 - 2\phi_{12}}{\phi_2}\right),$$

or

$$\frac{\partial}{\partial v}\left(\frac{e_2 - 2\phi_{12}}{\phi_2} - 2u\phi\right) = 0. \qquad (27.2)$$

We may therefore take

$$e = 2\phi_1 + u\phi^2 + \alpha\phi + \beta,$$
$$f = \phi_2,$$

where α and β are functions of u only.

The ground form now becomes, if we take

$$x_1 = K, \; x_2 = \phi,$$
$$ds^2 = (x_1 x_2^2 + \alpha x_2 + \beta)\, dx_1^2 + 2\, dx_1 dx_2, \qquad (27.3)$$

where α and β are functions of x_1 only.

We can then decide at once whether two ground forms for each of which $\Delta(K)$ is zero are equivalent.

§ 28. **The case when $\Delta_2 K$ and ΔK are functions of K.** We may now dismiss this special case when $\Delta(K)$ is zero: it is not of much interest, as it cannot arise in the case of a real surface.

We now consider the case when K is not a constant and $\Delta(K)$ is not zero, but $\Delta_2(K)$ and $\Delta(K)$ are both functions of K. This arises when the surface is applicable on a surface of revolution.

Let us take
$$u = K, \qquad (28.1)$$
and let v be the function which we have seen can be obtained by quadrature to satisfy the equation
$$\Delta(u, v) = 0, \qquad (28.2)$$
when $\Delta_2(u)$ and $\Delta(u)$ are both functions of u, though the reasoning would have held equally had $\Delta_2(u) \div \Delta(u)$ been only assumed to be a function of u.

We saw that if
$$\mu = e^{-\int \frac{\Delta_2(u)}{\Delta(u)} du},$$
$$v_1 = \mu a^{\frac{1}{2}} u^2, \quad v_2 = -\mu a^{\frac{1}{2}} u^1.$$

If then we take $$\theta = \int \mu du,$$

we have $$\theta_1 = \mu u_1, \quad \theta_2 = \mu u_2.$$

and therefore $$v_1 = a^{\frac{1}{2}}(a^{12}\theta_1 + a^{22}\theta_2) = a^{\frac{1}{2}}\theta^2.$$

Similarly we have $$v_2 = -a^{\frac{1}{2}}\theta^1;$$

and therefore $$\theta_1 = -a^{\frac{1}{2}}v^2, \quad \theta_2 = a^{\frac{1}{2}}v^1.$$

It follows that $$\theta_1\theta^1 = v_2 v^2, \quad \theta_2\theta^2 = v_1 v^1,$$

and therefore $$\Delta(\theta) = \Delta(v), \quad \Delta(\theta, v) = 0. \tag{28.3}$$

We also have $$(d\theta)^2 = (\mu du)^2, \quad \Delta(\theta) = (\mu)^2 \Delta(u),$$

and therefore $$\frac{d\theta^2}{\Delta(\theta)} = \frac{du^2}{\Delta(u)}. \tag{28.4}$$

If we now take as the new variables θ and v, the ground form becomes

$$\begin{aligned} ds^2 &= \frac{d\theta^2}{\Delta(\theta)} + \frac{dv^2}{\Delta(v)}, \\ &= \frac{du^2}{\Delta(u)} + \frac{dv^2}{\Delta(\theta)}, \\ &= \frac{du^2}{\Delta(u)} + e^{2\int \frac{\Delta_2(u)}{\Delta(u)} du} \frac{dv^2}{\Delta(u)}. \end{aligned}$$

We therefore see that the ground form may be written

$$ds^2 = (\Delta(K))^{-1} \left((dK)^2 + e^{2\int \frac{\Delta_2(K)}{\Delta(K)} dK} dv^2\right), \tag{28.5}$$

where v may be expressed by quadrature in terms of K and integrals of functions of it.

We thus see that given two ground forms, for each of which $\Delta(K)$ is a function of K and also $\Delta_2(K)$ is a function of K, the two forms are equivalent if, and only if, the functional forms are the same.

§ 29. Conditions for equivalence in the general case. Finally we have the general and the simplest case when K is not a constant, and $\Delta(K)$ and $\Delta_2(K)$ are not both functions of K.

In this case we have two invariants, say u and v. We take these invariants as the coordinates, when the ground form becomes

$$ds^2 = \frac{\Delta(v)\,du^2 - 2\Delta(u,v)\,du\,dv + \Delta(u)\,dv^2}{\Delta(u)\,\Delta(v) - (\Delta(u,v))^2}. \qquad (29.1)$$

The necessary and sufficient conditions, that two such ground forms may be equivalent, are that, for each of the forms,

$$\Delta(u), \quad \Delta(u,v), \quad \Delta(v) \qquad (29.2)$$

may be respectively the same functions of u and v.

We now know in all cases the tests which will determine whether two assigned ground forms are, or are not, equivalent.

§ **30. The functions called rotation functions.** When the measure of curvature * is constant we saw [§ 24] that the reduction of the ground form to its canonical form depends on finding an integral of the complete system of differential equations

$$\phi_{\cdot 11} + Ka_{11}\phi = 0, \quad \phi_{\cdot 12} + Ka_{12}\phi = 0, \quad \phi_{\cdot 22} + Ka_{22}\phi = 0. \qquad (30.1)$$

We shall now show how this integral may be found by aid of Riccati's equation.

Take any four functions, which we denote by q_1, q_2, r_1, r_2 and which will satisfy the three algebraic equations

$$Ka_{11} = q_1^2 + r_1^2, \quad Ka_{12} = q_1 q_2 + r_1 r_2, \quad Ka_{22} = q_2^2 + r_2^2. \qquad (30.2)$$

The functions thus chosen are not tensor components, but we shall operate on them in accordance with our notation with $\bar{1}$ and $\bar{2}$.

These two operators annihilate Ka_{11}, Ka_{12}, Ka_{22}, and therefore we have

$$q_1 q_{1\cdot 1} + r_1 r_{1\cdot 1} = 0, \quad q_1 q_{1\cdot 2} + r_1 r_{1\cdot 2} = 0,$$

$$q_1 q_{2\cdot 1} + q_2 q_{1\cdot 1} + r_1 r_{2\cdot 1} + r_2 r_{1\cdot 1} = 0,$$

$$q_1 q_{2\cdot 2} + q_2 q_{1\cdot 2} + r_1 r_{2\cdot 2} + r_2 r_{1\cdot 2} = 0,$$

$$q_2 q_{2\cdot 1} + r_2 r_{2\cdot 1} = 0, \quad q_2 q_{2\cdot 2} + r_2 r_{2\cdot 2} = 0.$$

We define two other functions p_1 and p_2 by

$$q_{1\cdot 1} = r_1 p_1, \quad q_{1\cdot 2} = r_1 p_2. \qquad (30.3)$$

* [This name for the invariant K will be shown later to have geometrical fitness. See § 37.]

It at once follows by simple algebra that

$$r_{1\cdot 1}+p_1 q_1 = 0, \quad q_{1\cdot 1}-p_1 r_1 = 0,$$
$$r_{1\cdot 2}+p_2 q_1 = 0, \quad q_{1\cdot 2}-p_2 r_1 = 0,$$
$$r_{2\cdot 1}+p_1 q_2 = 0, \quad q_{2\cdot 1}-p_1 r_2 = 0,$$
$$r_{2\cdot 2}+p_2 q_2 = 0, \quad q_{2\cdot 2}-p_2 r_2 = 0.$$

We then have $r_{1\cdot 12}+p_{1\cdot 2}q_1+p_1 q_{1\cdot 2} = 0,$

$$r_{1\cdot 21}+p_{2\cdot 1}q_1+p_2 q_{1\cdot 1} = 0,$$

and therefore $(\bar{2}\bar{1}-\bar{1}\bar{2})\,r_1+q_1(p_{1\cdot 2}-p_{2\cdot 1}) = 0,$

that is, $$\{1\,t\,12\}\,r_t = q_1(p_{1\cdot 2}-p_{2\cdot 1}), \tag{30.4}$$

or $$a^{tk}(1\,k\,12)\,r_t = q_1(p_{1\cdot 2}-p_{2\cdot 1}). \tag{30.5}$$

Now

$$a^{tk}(1\,k\,12)\,r_t = a^{t2}(12\,12)\,r_t = aK(a^{12}r_1+a^{22}r_2),$$
$$= K(a_{11}r_2-a_{12}r_1) = r_2(q_1^2+r_1^2)-r_1(q_1q_2+r_1r_2),$$
$$= q_1(r_2q_1-r_1q_2),$$

and therefore $p_{1\cdot 2}-p_{2\cdot 1} = q_1r_2-q_2r_1,$

that is, $$\frac{\partial p_1}{\partial x_2}-\frac{\partial p_2}{\partial x_1} = q_1r_2-q_2r_1. \tag{30.6}$$

Similarly we have

$$\frac{\partial q_1}{\partial x_2}-\frac{\partial q_2}{\partial x_1} = r_1p_2-r_2p_1, \tag{30.7}$$

$$\frac{\partial r_1}{\partial x_2}-\frac{\partial r_2}{\partial x_1} = p_1q_2-p_2q_1. \tag{30.8}$$

Any six functions $p_1, p_2, q_1, q_2, r_1, r_2$ which satisfy these three equations are called rotation functions. They have important geometrical properties and are much used by Darboux, but here we simply regard them as algebraically defined functions.

§ 31. Integration of the complete system of § 24. Now consider the equations

$$\frac{\partial l}{\partial x_1}-mr_1+nq_1 = 0, \quad \frac{\partial l}{\partial x_2}-mr_2+nq_2 = 0;$$

$$\frac{\partial m}{\partial x_1}-np_1+lr_1 = 0, \quad \frac{\partial m}{\partial x_2}-np_2+lr_2 = 0;$$

$$\frac{\partial n}{\partial x_1}-lq_1+mp_1 = 0, \quad \frac{\partial n}{\partial x_2}-lq_2+mp_2 = 0. \tag{31.1}$$

These equations are consistent because the functions

$$p_1, p_2, q_1, q_2, r_1, r_2$$

are rotation functions; and we see that $l^2+m^2+n^2$ is a constant.

Let $$\sigma = \frac{l+\iota m}{\sqrt{l^2+m^2+n^2}-n}, \qquad (31.2)$$

where ι stands for $\sqrt{-1}$: we have

$$\frac{\partial\sigma}{\partial x_1} = \frac{\iota p_1+q_1}{2}\sigma^2 - \iota r_1\sigma + \frac{q_1-\iota p_1}{2},$$

$$\frac{\partial\sigma}{\partial x_2} = \frac{\iota p_2+q_2}{2}\sigma^2 - \iota r_2\sigma + \frac{q_2-\iota p_2}{2}. \qquad (31.3)$$

We can therefore find σ by the solution of Riccati's equation.

To determine l, m, n we have

$$\sigma = \frac{l+\iota m}{\sqrt{l^2+m^2+n^2}-n}, \quad \sigma^{-1} = \frac{l-\iota m}{\sqrt{l^2+m^2+n^2}+n},$$

and $$l^2+m^2+n^2 = \text{constant}:$$

and we thus see how, when we are given the rotation functions, we can determine l, m, n.

We can now at once verify that

$$l._{11}+Ka_{11}l = 0, \quad l._{12}+Ka_{12}l = 0, \quad l._{22}+Ka_{22}l = 0, \qquad (31.4)$$

and that $$\Delta(l) = K(m^2+n^2) = K(\text{constant}-l^2). \qquad (32.5)$$

We have thus shown how a common integral of the complete system can be obtained by aid of Riccati's equation. The l, m, n which we thus obtain will be the direction cosines of the normal to the surface, but we do not make any use of our knowledge of a third dimension in obtaining l, m, n.

CHAPTER III*

GEODESICS IN TWO-WAY SPACE

§ 32. **Differential equation of a geodesic.** We have now considered the ground form of a surface, and we know the method by which we are to determine when two given ground forms are equivalent; that is when they are transformable the one into the other by a change of the variables.

We now wish to consider the geometry on the surface regarded as a two-way space; and we are thus led to the theory of geodesics. We have

$$ds^2 = a_{ik} dx_i dx_k, \tag{32.1}$$

and therefore

$$2\frac{d\delta s}{ds} = a_{ik}\frac{dx_i}{ds}\frac{d\delta x_k}{ds} + a_{ik}\frac{dx_k}{ds}\frac{d\delta x_i}{ds} + \frac{dx_i}{ds}\frac{dx_k}{ds}\delta a_{ik},$$

$$= \frac{d}{ds}\left(a_{ik}\frac{dx_i}{ds}\delta x_k + a_{ik}\frac{dx_k}{ds}\delta x_i\right)$$

$$-\delta x_k\frac{d}{ds}\left(a_{ik}\frac{dx_i}{ds}\right) - \delta x_i\frac{d}{ds}\left(a_{ik}\frac{dx_k}{ds}\right)$$

$$+\frac{dx_i}{ds}\frac{dx_k}{ds}\frac{\partial a_{ik}}{\partial x_t}\delta x_t.$$

For a path of critical length we therefore have

$$\frac{d}{ds}\left(a_{it}\frac{dx_i}{ds}\right) + \frac{d}{ds}\left(a_{tk}\frac{dx_k}{ds}\right) = \frac{\partial a_{ik}}{\partial x_t}\frac{dx_i}{ds}\frac{dx_k}{ds}. \tag{32.2}$$

Now
$$\frac{\partial a_{it}}{\partial x_t} = (itk) + (kti);$$

and we notice that, though $(itk) \neq (kti)$, yet

$$(itk)\frac{dx_i}{ds}\frac{dx_k}{ds} = (kti)\frac{dx_i}{ds}\frac{dx_k}{ds}. \tag{32.3}$$

* [See foot-note on p. 25.]

The path of critical length therefore satisfies the equation

$$\frac{d}{ds}\left(a_{it}\frac{dx_i}{ds}\right) = (itk)\frac{dx_i}{ds}\frac{dx_k}{ds}. \qquad (32 : 4)$$

§ 33. **Another form of the equation.** The expressions

$$\frac{dx_1}{ds} \text{ and } \frac{dx_2}{ds}$$

are called the direction cosines of the element. They determine the position of the element at the point x_1, x_2, but they are not, in general, the cosines of the angles the element makes with the parametric lines. We denote them by ξ^1, ξ^2 in tensor notation.

For a geodesic we therefore have

$$\frac{d}{ds}(a_{it}\xi^i) = (itk)\,\xi^i\xi^k. \qquad (33.1)$$

We can put this equation in another form. We have

$$\frac{d}{ds}a_{it} = \frac{dx_p}{ds}\frac{\partial a_{it}}{\partial x_p} = \xi^p(ipt) + \xi^p(tpi),$$

and therefore

$$a_{it}\frac{d}{ds}\xi^i + \xi^i\xi^p(ipt) + \xi^i\xi^p(tpi) = (itk)\,\xi^i\xi^k. \qquad (33.2)$$

Now $\qquad \xi^i\xi^p(tpi) \equiv \xi^i\xi^k(tki) \equiv \xi^i\xi^k(kti)$

and $\qquad \xi^i\xi^k(itk) \equiv \xi^i\xi^k(kti).$

We then have for a geodesic

$$a_{it}\frac{d}{ds}\xi^i + (ipt)\,\xi^i\xi^p = 0, \qquad (33.3)$$

and therefore, multiplying by a^{tq} and summing,

$$\frac{d}{ds}\xi^q + \{ipq\}\,\xi^i\xi^p = 0. \qquad (33.4)$$

For a geodesic we thus have either of the two equivalent equations

$$\frac{d}{ds}(a_{it}\xi^t) = (ipq)\,\xi^p\xi^q, \qquad (33.1)'$$

$$\frac{d}{ds}\xi^i + \{pqi\}\,\xi^p\xi^q \doteq 0. \qquad (33.4)'$$

§ **34. Condition that orthogonal trajectories be geodesics.** If two elements at x_1, x_2 are perpendicular to one another we have

$$a_{11} dx_1 \delta x_1 + a_{12}(dx_1 \delta x_2 + dx_2 \delta x_1) + a_{22} dx_2 \delta x_2 = 0. \quad (34.1)$$

The elements perpendicular to the curve ϕ = constant will then satisfy the equation

$$\phi_2 (a_{11} dx_1 + a_{12} dx_2) = \phi_1 (a_{12} dx_1 + a_{22} dx_2), \quad (34.2)$$

or

$$a_{11}\xi^1 + a_{12}\xi^2 = \mu\phi_1,$$
$$a_{12}\xi^1 + a_{22}\xi^2 = \mu\phi_2, \quad (34.3)$$

where μ is some multiplier.

We have

$$\xi^1 = \mu(a^{11}\phi_1 + a^{12}\phi_2), \quad \xi^2 = \mu(a^{21}\phi_1 + a^{22}\phi_2), \quad (34.4)$$

and, as $$a_{ik}\xi^i\xi^k = 1,$$

we see that $$\mu(\xi^1\phi_1 + \xi^2\phi_2) = 1;$$

and therefore $$\mu^2 \Delta(\phi) = 1. \quad (34.5)$$

We thus have

$$a_{11}\xi^1 + a_{12}\xi^2 = \frac{\phi_1}{\sqrt{\Delta(\phi)}}, \quad a_{12}\xi^1 + a_{22}\xi^2 = \frac{\phi_2}{\sqrt{\Delta(\phi)}}, \quad (34.6)$$

$$\xi^1 = \frac{a^{11}\phi_1 + a^{12}\phi_2}{\sqrt{\Delta(\phi)}}, \quad \xi^2 = \frac{a^{12}\phi_1 + a^{22}\phi_2}{\sqrt{\Delta(\phi)}}. \quad (34.7)$$

Now $\frac{d}{ds}(a_{it}\xi^t) - (ipq)\xi^p\xi^q$

$$= \xi^p\left(\frac{\partial}{\partial x_p}\frac{\phi_i}{\sqrt{\Delta(\phi)}} - (ipq)\,a^{qt}\frac{\phi_t}{\sqrt{\Delta(\phi)}}\right),$$

$$= \xi^p\left(\frac{\partial}{\partial x_p}\frac{\phi_i}{\sqrt{\Delta(\phi)}} - \{ipt\}\frac{\phi_t}{\sqrt{\Delta\phi}}\right),$$

$$= \xi^p\left(\frac{\phi_{\cdot ip}}{\sqrt{\Delta(\phi)}} - \frac{\phi_i(\Delta(\phi))_p}{2(\Delta(\phi))^{\frac{3}{2}}}\right),$$

$$= \frac{a^{pq}\phi_q}{\Delta(\phi)}\left(\phi_{\cdot ip} - \frac{\phi_i(\Delta\phi)_p}{2\Delta(\phi)}\right),$$

$$= \frac{a^{pq}\phi_q\phi_{\cdot ip}}{\Delta(\phi)} - \frac{\phi_i\Delta(\phi, \Delta(\phi))}{2(\Delta(\phi))^2}.$$

And, as $$\Delta(\phi) \equiv a^{pq}\phi_p\phi_q,$$
we see that $$(\Delta(\phi))_i = a^{pq}\phi_{\cdot pi}\phi_q + a^{pq}\phi_p\phi_{\cdot qi},$$
$$= 2a^{pq}\phi_q\phi_{\cdot pi}.$$

We therefore have

$$\frac{d}{ds}(a_{it}\xi^t) - (ipq)\,\xi^p\xi^q = \frac{(\Delta(\phi))_i}{2\,\Delta(\phi)} - \frac{\phi_i\Delta(\phi,\Delta\phi)}{2(\Delta(\phi))^2}. \quad (34.8)$$

Suppose now that ϕ is a function of the parameters such that
$$\Delta(\phi) = F(\phi). \quad (34.9)$$

We see at once that the right-hand member of the above equation vanishes; and therefore the orthogonal trajectories of the curves $\phi =$ constant are geodesics, if $\Delta(\phi)$ is a function of ϕ.

Conversely we see that, the orthogonal trajectories of any system of geodesics being $\phi =$ constant, $\Delta(\phi)$ must be a function of ϕ.

§ **35. Geodesic curvature.** If ξ^1, ξ^2 are the direction cosines of an element of the curve $\phi =$ constant, we have
$$\xi^p\phi_p = 0, \quad a_{ik}\xi^i\xi^k = 1,$$
and therefore
$$\phi_1 = a^{\frac{1}{2}}(\Delta\phi)^{\frac{1}{2}}\xi^2, \quad \phi_2 = -a^{\frac{1}{2}}(\Delta\phi)^{\frac{1}{2}}\xi^1. \quad (35.1)$$

Differentiating $$\xi^p\phi_p = 0$$
with respect to the arc, we have
$$\phi_p\frac{d}{ds}\xi^p + \xi^p\xi^q(\phi_{\cdot pq} + \{pqt\}\,\phi_t) = 0;$$

that is, $$\phi_p\left(\frac{d}{ds}\xi^p + \{ikp\}\,\xi^i\xi^k\right) + \phi_{\cdot pq}\xi^p\xi^q = 0.$$

We therefore have, summing along the curve,
$$\int(\xi^2(d\xi^1 + \{ik1\}\,\xi^i\xi^k ds) - \xi^1(d\xi^2 + \{ik2\}\,\xi^i\xi^k ds))$$
$$+\int(\phi_{\cdot 11}(\phi_2)^2 - 2\phi_{\cdot 12}\phi_1\phi_2 + \phi_{\cdot 22}(\phi_1)^2)\,a^{-\frac{3}{2}}(\Delta(\phi))^{-\frac{3}{2}}ds = 0. \quad (35.2)$$

The first integral if summed along a small length of the curve only differs by a small quantity of the second order from the same integral if summed along the curve formed by the geodesic tangents at its extremities.

Now, summed along a geodesic, we know that the first integral vanishes, but summed along the curve formed by two geodesics the integral is

$$\xi^1 \eta^2 - \xi^2 \eta^1, \tag{35.3}$$

where ξ^1, ξ^2 and η^1, η^2 are the direction cosines of the two geodesics at their common point.

The angle α between the two elements whose direction cosines are ξ^1, ξ^2 and η^1, η^2 is given by

$$a^{\frac{1}{2}} (\xi^1 \eta^2 - \xi^2 \eta^1) = \sin \alpha. \tag{35.4}$$

We therefore have the formula

$$a^{-\frac{1}{2}} \frac{d\theta}{ds} = \left(\phi_{\cdot 11} (\phi_2)^2 - 2\phi_{\cdot 12} \phi_1 \phi_2 + \phi_{\cdot 22} (\phi_1)^2\right) a^{-\frac{3}{2}} (\Delta \phi)^{-\frac{3}{2}}, \tag{35.5}$$

where $d\theta$ is the small angle at which the geodesic tangents at the extremities of ds intersect. The formula for the geodesic curvature of the curve $\phi =$ constant is therefore

$$\frac{1}{\rho_g} = \left(\phi_{\cdot 11} (\phi_2)^2 - 2\phi_{\cdot 12} \phi_1 \phi_2 + \phi_{\cdot 22} (\phi_1)^2\right) a^{-1} (\Delta \phi)^{-\frac{3}{2}}. \tag{35.6}$$

§ **36.** We can express the above formula in a better form: to prove this we employ the coordinates which are Euclidean at a specified point.

We have at the specified point

$$(\Delta\phi)_1 = 2\phi_1\phi_{11} + 2\phi_2\phi_{12},$$
$$(\Delta\phi)_2 = 2\phi_1\phi_{12} + 2\phi_2\phi_{22},$$

and therefore

$$\Delta(\phi, \Delta\phi) = \phi_1 (\Delta\phi)_1 + \phi_2 (\Delta\phi)_2,$$
$$= 2\left(\phi_{11}(\phi_1)^2 + 2\phi_{12}\phi_1\phi_2 + \phi_{22}(\phi_2)^2\right). \tag{36.1}$$

Now $\quad \Delta_2(\phi) = \phi_{11} + \phi_{22}, \quad \Delta(\phi) = (\phi_1)^2 + (\phi_2)^2,$

and therefore at the specified point

$$\phi_{11}(\phi_2)^2 - 2\phi_{12}\phi_1\phi_2 + \phi_{22}\phi_1^2$$
$$= \Delta_2(\phi)\,\Delta(\phi) - \tfrac{1}{2}\Delta(\phi, \Delta\phi), \tag{36.2}$$

and at the specified point a is unity.

We thus have at the specified point of the curve, and therefore at every point of the curve,

$$\frac{1}{\rho_g} = \frac{\Delta_2(\phi)}{(\Delta\phi)^{\frac{1}{2}}} + \Delta\left(\phi, (\Delta\phi)^{-}\right). \tag{36.3}$$

The method of thus employing Euclidean coordinates is very helpful in proving formulae in the tensor calculus. The direct proof of the equality

$$\frac{\Delta_2(\phi)}{(\Delta\phi)^{\frac{1}{2}}} + \Delta\left(\phi. (\Delta\phi)^{-\frac{1}{2}}\right)$$
$$= \left(\phi._{11}(\phi_2)^2 - 2\,\phi._{12}\phi_1\phi_2 + \phi._{22}(\phi_1)^2\right) a^{-1} (\Delta\phi)^{-\frac{3}{2}} \tag{36.4}$$

would be much longer.

§ **37. Polar geodesic coordinates. The measure of curvature *K*.** The geodesic curvature of a curve is given by the formula

$$\frac{1}{\rho_g} = a^{\frac{1}{2}}\xi^1\left(\frac{d\xi^2}{ds} + \{ik\,2\}\,\xi^i\xi^k\right) - a^{\frac{1}{2}}\xi^2\left(\frac{d\xi^1}{ds} + \{ik\,1\}\,\xi^i\xi^k\right), \tag{37.1}$$

where ξ^1, ξ^2 are the direction cosines of an element of the curve.

If we take, and we shall see that we can take, the ground form of the surface to be

$$du^2 + B^2 dv^2, \tag{37.2}$$

where u is the geodesic distance of any point on the surface from a fixed point on the surface, and the curves $v =$ constant are geodesics passing through the fixed points, dv being the angle at the point between two neighbouring geodesics, B being a function of u and v which on expansion in the neighbourhood of the fixed point is of the form $u + \ldots$, where the terms denoted by $+\ldots$ are of degree above the first, we employ what we may call polar geodesic coordinates with respect to the fixed point.

Let us now employ polar geodesic coordinates to interpret the formula for geodesic curvature. We have

$$\{111\} = 0, \quad \{112\} = 0, \quad \{121\} = 0, \quad \{122\} = \frac{B_1}{B},$$
$$\{221\} = -BB_1, \quad \text{and} \quad \{222\} = \frac{B_2}{B}. \tag{37.3}$$

If θ is the angle at which the curve crosses the geodesics through the fixed point,

$$\xi^1 = \cos\theta, \quad B\xi^2 = \sin\theta,$$

and [see § 43] the measure of curvature K of the surface is given by

$$KB + B_{11} = 0. \tag{37.4}$$

We have

$$a^{\frac{1}{2}}\left(\frac{d\xi^1}{ds} + \{ik\,1\}\,\xi^i\xi^k\right) = B\left(-\sin\theta\frac{d\theta}{ds} - \frac{B_1}{B}\sin^2\theta\right),$$

$$a^{\frac{1}{2}}\left(\frac{d\xi^2}{ds} + \{ik\,2\}\,\xi^i\xi^k\right)$$

$$= B\left(\frac{\cos\theta}{B}\frac{d\theta}{ds} - \frac{\sin\theta}{B^2}\left(B_1\cos\theta + \frac{B_2}{B}\sin\theta\right)\right.$$

$$\left. + \frac{2B_1}{B^2}\sin\theta\cos\theta + \frac{B_2}{B^3}\sin^2\theta\right),$$

and therefore

$$\frac{1}{\rho_g} = \frac{d\theta}{ds} + \frac{B_1}{B}\sin\theta. \tag{37.5}$$

Now consider the expression

$$\iint K\,dS, \tag{37.6}$$

where dS is an element of area of the surface, and take the summation over the small strip bounded by two neighbouring geodesics through the origin of the polar geodesic coordinates and an element of the curve.

The expression is $\iint KB\,du\,dv,$

and this is equal to $-\iint B_{11}du\,dv$

$$= -\int (B_1 - 1)\,dv,$$

$$= dv - \int \frac{B_1}{B}\sin\theta\,ds. \tag{37.7}$$

It follows that $\int \frac{ds}{\rho_g}$

taken over the boundary of any closed curve surrounding the point is equal to

$$2\pi - \iint K\,dS, \tag{37.8}$$

where the integral is to be taken over the area of the curve.

We thus have a geometrical interpretation of the measure of curvature: it is the excess of 2π over the angle turned through by the geodesic tangent, as we describe a small closed curve, divided by the area of the curve. It will be noticed that in this definition we do not make use of any knowledge of a space other than the two-way space of the surface itself. This is what the curvature of a two-way space must mean to a mathematician to whom the knowledge of a three-way space can only be apprehended in the same vague way as we speak of a four-dimensional space.

§ 38. **Recapitulation. Parallel curves.** It may be convenient to bring together the various formulae which so far we have proved in connexion with direction cosines and geodesics before we proceed further.

$$\xi^1 = \frac{dx_1}{ds}, \quad \xi^2 = \frac{dx_2}{ds};$$

$$\xi^1 = -a^{-\frac{1}{2}}(\Delta\phi)^{-\frac{1}{2}}\phi_2, \quad \xi^2 = a^{-\frac{1}{2}}(\Delta\phi)^{-\frac{1}{2}}\phi_1, \qquad (35.1)'$$

where the direction cosines are those of an element of the curve $\phi = \text{constant}$;

$$\xi^1 = (a^{11}\phi_1 + a^{12}\phi_2)(\Delta\phi)^{-\frac{1}{2}}, \quad \xi^2 = (a^{12}\phi_1 + a^{22}\phi_2)(\Delta\phi)^{-\frac{1}{2}}, \qquad (34.7)'$$

where the direction cosines are those of an element perpendicular to the curve; for a geodesic we have

$$\frac{d}{ds}(a_{it}\xi^i) = (tik)\,\xi^i\xi^k, \qquad (33.1)''$$

$$\frac{d}{ds}\xi^i + \{pqi\}\,\xi^p\xi^q = 0. \qquad (33.4)'$$

The orthogonal trajectories of the curves $\phi = \text{constant}$, where $\Delta(\phi)$ is a function of ϕ, are geodesics, and the orthogonal trajectories of any system of geodesics are curves $\phi = \text{constant}$, where $\Delta(\phi)$ is a function of ϕ. (34.9)'

Leaving aside the case when $\Delta(\phi)$ is zero, we can choose the function ϕ so that $\Delta(\phi) = 1$.

If we know any integral of this partial differential equation involving an arbitrary constant

$$\phi(x_1, x_2, \alpha) = 0,$$

then the system of curves $\dfrac{\partial\phi}{\partial\alpha} = \beta$

will be geodesics: for $\Delta\left(\phi, \dfrac{\partial\phi}{\partial\alpha}\right) = 0,$

and, as the condition that two families of curves

$$\phi = \text{constant}, \quad \psi = \text{constant},$$

may cut orthogonally is $\Delta(\phi, \psi) = 0$, we conclude that the curves

$$\frac{\partial\phi}{\partial\alpha} = \beta \tag{38.1}$$

will be geodesics, since they cut the curves $\phi = \text{constant}$ orthogonally.

If we choose the arbitrary constant β so that the geodesics given by

$$\frac{\partial\phi}{\partial\alpha} = \beta$$

may all pass through a fixed point, and if we take the equation of the geodesics to be $v = \text{constant}$ and take v as one of our parametric coordinates and ϕ to be the other parametric coordinate u, we have

$$\Delta(u) = 1, \quad \Delta(u, v) = 0.$$

The ground form of the surface then takes the form

$$du^2 + B^2 dv^2, \tag{37.2}$$

and in the neighbourhood of the fixed point, through which the geodesics pass, we may clearly take from elementary geometry that $B = u + \dots.$

We thus have what we called the polar geodesic coordinates.

We have $d\phi = \phi_1 dx_1 + \phi_2 dx_2$, and therefore, ξ^1, ξ^2 being the direction cosines of an element perpendicular to the curve $\phi = \text{constant}$, the length dn of the normal element is given by

$$d\phi = dn(\phi_1\xi^1 + \phi_2\xi^2) = dn\sqrt{\Delta(\phi)},$$

or

$$\frac{\partial\phi}{\partial n} = \sqrt{\Delta(\phi)}. \tag{38.2}$$

The curves which satisfy the equation

$$\Delta(\phi) = 1 \qquad (38.3)$$

are called parallel curves. We see thus that two parallel curves cut off equal intercepts on the geodesics which cut them orthogonally.

If particles, constrained to lie on a smooth surface and acted on by no forces but the normal reaction of the surface, are projected at the same instant, with the same velocities normal to any curve, they will at any other instant lie on a parallel curve.

From the theory of partial differential equations we know that *any* curve on the surface will have a series of curves parallel to it, though the finding of them involves the solution of the equation $\Delta(\phi) = 1.$

The explicit forms of the differential equations

$$\frac{d\xi^i}{ds} + \xi^p \xi^q \{pqi\} = 0$$

of a geodesic are

$$\ddot{x}_1 + \{111\}\dot{x}_1^2 + 2\{121\}\dot{x}_1\dot{x}_2 + \{221\}\dot{x}_2^2 = 0,$$
$$\ddot{x}_2 + \{112\}\dot{x}_1^2 + 2\{122\}\dot{x}_1\dot{x}_2 + \{222\}\dot{x}_2^2 = 0, \qquad (38.4)$$

where the dot denotes differentiation with respect to the arc.

If we write the variables as x and y and let

$$p = \frac{dy}{dx}, \quad q = \frac{d^2y}{dx^2},$$

we have $\dot{y} = \dot{x}p, \quad \ddot{y} = \ddot{x}p + \dot{x}^2 q,$

and the equation of the geodesic becomes

$$q - \{221\}p^3 + (\{222\} - 2\{121\})p^2$$
$$+ (2\{122\} - \{111\})p + \{112\} = 0. \qquad (38.5)$$

§ **39. Notes regarding geodesic curvature.** Now considering geodesic curvature, in the figure on p. 52 P and Q are two neighbouring points on any curve, PT and TQ are the geodesic tangents at P and Q, and QM is an element of arc perpendicular to the geodesic tangent PTM.

By definition the geodesic curvature of the given curve at

P is the limiting ratio of the angle QTM to the arc PQ as Q approaches P. We therefore have

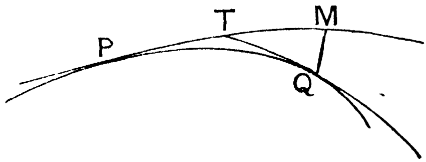
$$\frac{1}{\rho_g} = \frac{2QM}{PQ^2}, \tag{39.1}$$

and thus have the analogue of Newton's measure of curvature of a plane curve for a curve on the surface. It is the geodesic curvature only that has a meaning when we confine our attention to the two-way space on a surface.

We have the formula

$$\frac{1}{\rho_g} = \frac{\Delta_2(\phi)}{\sqrt{\Delta(\phi)}} + \Delta\left(\phi, (\Delta\phi)^{-\frac{1}{2}}\right), \tag{36.3}$$

and we may apply it to find the geodesic curvature of the curve all the points of which are at a constant geodesic



distance from the origin, in the polar geodesic coordinate system. We have $ds^2 = du^2 + B^2 dv^2$,

$$\phi \equiv u,$$

and therefore $$\frac{1}{\rho_g} = \frac{\partial}{\partial u} \log B.$$

The curvature will be constant if, and only if,

$$B = f(u)\, F(v),$$

that is, if the surface is applicable on one of revolution.

The curvature will then only be $\frac{1}{u}$, as it would be in a plane, if

$$ds^2 = du^2 + u^2 dv^2,$$

that is, if the surface is applicable on a plane.

If we take the case where K is positive unity and

$$ds^2 = du^2 + \sin^2 u\, dv^2,$$

we see that the geodesic curvature of a small circle is $\cot u$.

If we take the form

$$ds^2 = du^2 + e^{-2u} dv^2,$$

which is applicable to the tractrix or any surface applicable on it, we see that the geodesic curvature of the curves $u =$ constant is minus unity.

§ **40.** The formula for the geodesic curvature may be written

$$\frac{1}{\rho_g} = \frac{\Delta_2(\phi) + \Delta\left(\phi, \log(\Delta\phi)^{-\frac{1}{2}}\right)}{\sqrt{\Delta\phi}}. \tag{40.1}$$

Let μ be an integrating factor of

$$a^{\frac{1}{2}}(\phi^2\, dx_1 - \phi^1\, dx_2),$$

where $\quad \phi^1 \equiv a^{11}\phi_1 + a^{12}\phi_2, \quad \phi^2 \equiv a^{12}\phi_1 + a^{22}\phi_2,$

so that $\quad \mu a^{\frac{1}{2}}\phi^2 = \psi_1, \quad \mu a^{\frac{1}{2}}\phi^1 = -\psi_2,$

and therefore $\quad \dfrac{\partial}{\partial x_1}(\mu a^{\frac{1}{2}}\phi^1) + \dfrac{\partial}{\partial x_2}(\mu a^{\frac{1}{2}}\phi^2) = 0.$

Now $\quad a^{\frac{1}{2}}\Delta_2(\phi) = \dfrac{\partial}{\partial x_t} a^{\frac{1}{2}}\phi^t, \quad \Delta(\phi, \mu) = \phi^t \mu_t,$

and therefore $\quad \mu\Delta_2(\phi) + \Delta(\phi, \mu) = 0,$

that is, $\quad \Delta_2(\phi) + \Delta(\phi, \log\mu) = 0.$

The formula for curvature may therefore be written

$$-\frac{1}{\rho_g} = \frac{\Delta\left(\phi, \log\mu(\Delta\phi)^{\frac{1}{2}}\right)}{(\Delta\phi)^{\frac{1}{2}}}. \tag{40.2}$$

This is an equation to give the integrating factor. When the integrating factor is known we can find the function ψ by quadrature; and, as

$$\Delta(\phi, \psi) = \phi^t\psi_t = 0, \tag{40.3}$$

we have then the equation of the orthogonal trajectories of the curves $\phi =$ constant.

In particular when the curves $\phi =$ constant are geodesics, we may take

$$\mu = (\Delta\phi)^{-\frac{1}{2}}, \tag{40.4}$$

and we thus see that the orthogonal trajectories of any system of geodesics may be found by quadrature.

In general we have $\quad \mu\sqrt{\Delta\phi} = \sqrt{\Delta\psi},$

and thus the formula for the geodesic curvature of the curves $\phi =$ constant may be written

$$-\frac{1}{\rho_g} = \frac{\Delta(\phi, \log\sqrt{\Delta\psi})}{\sqrt{\Delta\phi}}, \tag{40.5}$$

where the curves $\psi =$ constant are the orthogonal trajectories.

§ 41. **Integration of geodesic equations when K is constant.** We have obtained the differential equation of a geodesic on any surface, but, in general, we cannot solve the equation we have arrived at. Sometimes we can. Thus when the measure of curvature is positive unity we may take the ground form to be $ds^2 = du^2 + \sin^2 u\, dv^2$. (41 . 1)

We then have as the equation of the geodesic

$$\frac{d\theta}{ds} + \frac{B_1}{B} \sin\theta = 0,$$

that is, $$\cos\theta \frac{d\theta}{du} + \cot u \sin\theta = 0,$$

or $$\sin\theta \sin u = \text{constant}.$$

Now $$\sin\theta = \sin u \frac{dv}{ds},$$

and therefore $$\sin^2 u \frac{dv}{ds} = \sin\alpha, \qquad (41 . 2)$$

where α is some constant.

We could have obtained this equation directly, as we easily see, by the rules of the Calculus of Variations.

We deduce that

$$\sin^2 u \left(1 - \left(\frac{du}{ds}\right)^2\right) = \sin^2 \alpha,$$

and therefore $$\cos u = \cos\alpha \cos s,$$

and we thus obtain the equations

$$\sin v = \frac{\sin s}{\sin u}, \quad \cos v = \frac{\sin\alpha \cos s}{\sin u}, \quad \tan v = \frac{\tan s}{\sin\alpha}. \qquad (41 . 3)$$

We now see that

$$\cos u_1 \cos u_2 + \sin u_1 \sin u_2 \cos(v_1 - v_2)$$
$$= \cos^2\alpha \cos s_1 \cos s_2 + \sin^2\alpha \cos s_1 \cos s_2 + \sin s_1 \sin s_2,$$
$$= \cos(s_1 - s_2).$$

This is just the well-known formula of spherical trigonometry

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \qquad (41 . 4)$$

Similarly we could obtain the formula

$$\cosh(s_1 - s_2) = \cosh u_1 \cosh u_2 - \sinh u_1 \sinh u_2 \cos(v_1 - v_2); \qquad (41 . 5)$$

which would be applicable to a surface of constant negative curvature.

The formula $$\int \frac{ds}{\rho_g} + \iint K dS = 2\pi,$$

when applied to a geodesic triangle on a surface of curvature positive unity, gives us the well-known formula for the area of a spherical triangle $$A+B+C-\pi; \tag{41.6}$$

and more generally for any surface of constant curvature
$$K^{-1}(A+B+C-\pi). \tag{41.7}$$

§ 42. Focal coordinates. If we take, as the coordinates of a point on a surface, the geodesic distances of the point from two fixed points on the surface, the ground form will take the form $$(\sin\alpha)^{-2}(du^2+dv^2-2\cos\alpha\, du\, dv),$$

where α is the angle between the two geodesic distances.

We easily see this geometrically, using the property that the locus of a point at a constant geodesic distance from a fixed point is a curve cutting the geodesic radii vectores orthogonally. Analytically we prove the formula from the fact that $\Delta(u)$ and $\Delta(v)$ are both unity, and applying this to the general ground form

$$A^2 du^2 + B^2 dv^2 - 2AB\cos\alpha\, du\, dv,$$

when we have $$A^2 = B^2 = \operatorname{cosec}^2\alpha.$$

If we take $$2x = u+v, \quad 2y = u-v,$$

we have $$ds^2 = \sec^2\frac{\alpha}{2}dx^2 + \operatorname{cosec}^2\frac{\alpha}{2}dy^2. \tag{42.1}$$

This system of coordinates may be called focal coordinates: the curves $x =$ constant will represent confocal ellipses; that is, curves the sum of whose geodesic distances from two fixed points, which we call the foci, is constant.

Similarly the curves $y =$ constant will represent confocal hyperbolas, and we see that the ellipses and hyperbolas intersect orthogonally.

§ 43. Explicit expressions for symbols $\{ikj\}$ and for K. It will be convenient here to give in explicit form Christoffel's three-index symbols of the second kind,* as we so often need them, and expressions for the measure of curvature.

We take the ground form

$$ds^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2, \tag{43.1}$$

and we then have $\quad a = h^2 = eg - f^2, \quad (43.2)$

$$2h^2\{111\} = ge_1 + f(e_2 - 2f_1), \quad 2h^2\{112\} = e(2f_1 - e_2) - fe_1,$$
$$2h^2\{121\} = ge_2 - fg_1, \quad 2h^2\{122\} = eg_1 - fe_2,$$
$$2h^2\{221\} = g(2f_2 - g_1) - fg_2, \quad 2h^2\{222\} = eg_2 + f(g_1 - 2f_2),$$
$$(43.3)\ \ldots\ (43.8)$$

$$\begin{aligned}4h^4K = {} & e\left(g_2(e_2 - 2f_1) + g_1^2\right) + g\left(e_1(g_1 - 2f_2) + e_2^2\right) \\ & - 2h^2(e_{22} + g_{11} - 2f_{12}) \\ & + f\left(2f_1(f_2 - g_1) + 2f_2(f_1 - e_2) + e_1g_2 - e_2g_1\right), \end{aligned} \tag{43.9}$$

$$\text{or} \quad 4hK + \frac{\partial}{\partial u}\left(\frac{f}{h}\left(\frac{g_2}{g} - \frac{e_2}{e}\right) + 2\frac{(g_1 - f_2)}{h}\right)$$
$$+ \frac{\partial}{\partial v}\left(\frac{f}{h}\left(\frac{e_1}{e} - \frac{g_1}{g}\right) + 2\frac{(e_2 - f_1)}{h}\right) = 0. \tag{43.9$'$}$$

If we take as the ground form

$$A^2du^2 + 2AB\cos\alpha\,du\,dv + B^2dv^2,$$

the last formula becomes

$$AB\sin\alpha K + \alpha_{12} + \frac{\partial}{\partial u}\left(\frac{B_1 - A_2\cos\alpha}{A\sin\alpha}\right) + \frac{\partial}{\partial v}\left(\frac{A_2 - B_1\cos\alpha}{B\sin\alpha}\right) = 0, \tag{43.10}$$

which is Darboux's form.

* [Those of the first kind are at once

$$(111) = \tfrac{1}{2}e_1, \quad (112) = f_1 - \tfrac{1}{2}e_2,$$
$$(121) = (211) = \tfrac{1}{2}e_2, \quad (122) = (212) = \tfrac{1}{2}g_1,$$
$$(221) = f_2 - \tfrac{1}{2}g_1, \quad (222) = \tfrac{1}{2}g_2.$$

We also have

$$\Delta\phi = \frac{1}{h^2}\left\{e\left(\frac{\partial\phi}{\partial v}\right)^2 - 2f\frac{\partial\phi}{\partial v}\frac{\partial\phi}{\partial u} + g\left(\frac{\partial\phi}{\partial u}\right)^2\right\},$$

$$\Delta(\phi,\psi) = \frac{1}{h^2}\left\{e\frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial v} - f\left(\frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial u} + \frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial v}\right) + g\frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial u}\right\},$$

$$\Delta_2\phi = \frac{1}{h}\frac{\partial}{\partial u}\left\{\frac{1}{h}\left(g\frac{\partial\phi}{\partial u} - f\frac{\partial\phi}{\partial v}\right)\right\} + \frac{1}{h}\frac{\partial}{\partial v}\left\{\frac{1}{h}\left(e\frac{\partial\phi}{\partial v} - f\frac{\partial\phi}{\partial u}\right)\right\}.]$$

In particular if the parametric coordinates are geodesics we have
$$\{221\} = 0, \quad \{112\} = 0, \tag{43.11}$$
and therefore
$$\alpha_2 + \frac{B_1 - A_2 \cos\alpha}{A \sin\alpha} = 0, \quad \alpha_1 + \frac{A_2 - B_1 \cos\alpha}{B \sin\alpha} = 0,$$
and the formula for the curvature takes the simple form
$$AB \sin\alpha K = \alpha_{12}. \tag{43.12}$$
From this formula we could easily deduce again the formula
$$\int \frac{ds}{\rho_g} + \iint K\, dS = 2\pi.$$
When we take the ground form to be
$$A^2 du^2 + B^2 dv^2,$$
we have
$$ABK + \frac{\partial}{\partial u}\left(A^{-1}\frac{\partial B}{\partial u}\right) + \frac{\partial}{\partial v}\left(B^{-1}\frac{\partial A}{\partial u}\right) = 0. \tag{43.13}$$
When we take the ground form to be
$$e(du^2 + dv^2),$$
we have
$$\{111\} = \frac{e_1}{2e}, \quad \{112\} = -\frac{e_2}{2e}, \quad \{121\} = \frac{e_2}{2e}, \quad \{122\} = \frac{e_1}{2e},$$
$$\{221\} = -\frac{e_1}{2e}, \quad \{222\} = \frac{e_2}{2e},$$
(43.14) ... (43.19)
$$2eK + \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\log e = 0, \tag{43.20}$$
$$\Delta_2(\phi) = e^{-1}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\phi, \tag{43.21}$$
$$\Delta(\phi) = e^{-1}\left(\left(\frac{\partial\phi}{\partial u}\right)^2 + \left(\frac{\partial\phi}{\partial v}\right)^2\right). \tag{43.22}$$
Finally, when we take the ground form to be
$$2f\,du\,dv,$$
$$fK + \frac{\partial^2}{\partial u\,\partial v}(\log f) = 0, \tag{43.23}$$

$$\Delta_2(\phi) = \frac{2}{f}\frac{\partial^2 \phi}{\partial u \partial v}, \tag{43.24}$$

$$\Delta(\phi) = \frac{2}{f}\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v}. \tag{43.25}$$

§ 44. **Liouville's special form.** When the ground form of a surface takes the special form—Liouville's form—

$$ds^2 = (U+V)(du^2+dv^2), \tag{44.1}$$

U and V denoting functions of u and v respectively, we can find a first integral of the equation of geodesic lines.

For the form $$e\,(du^2+dv^2),$$

$$e^{\frac{1}{2}}\xi^1 = \cos\theta, \quad e^{\frac{1}{2}}\xi^2 = \sin\theta,$$

and the equation of a geodesic becomes

$$2e^{\frac{3}{2}}\frac{d\theta}{ds} + e_1\sin\theta - e_2\cos\theta = 0,$$

that is,
$$\frac{\partial}{\partial u}(e^{\frac{1}{2}}\sin\theta) + \frac{\partial}{\partial v}(e^{\frac{1}{2}}\cos\theta) = 0. \tag{44.2}$$

We therefore have

$$e^{\frac{1}{2}}\sin\theta = \phi_2, \quad e^{\frac{1}{2}}\cos\theta = -\phi_1,$$

and
$$e = \phi_1^2 + \phi_2^2,$$

that is,
$$\Delta(\phi) = 1. \tag{44.3}$$

In the particular case of Liouville's surface

$$\phi_1^2 - U = V - \phi_2^2,$$

and we obtain a complete integral of this partial differential equation by equating the above expressions to a constant.

We thus have $\phi_1 = \sqrt{U+a}, \quad \phi_2 = \sqrt{V-a},$

giving the first integral

$$e^{\frac{1}{2}}\cos\theta = -\sqrt{U+a}, \quad e^{\frac{1}{2}}\sin\theta = \sqrt{V-a}, \tag{44.4}$$

or
$$\frac{du^2}{U+a} = \frac{dv^2}{V-a}. \tag{44.5}$$

§ 45. **Null lines. Complex functions of position.** We shall now consider a further application of Beltrami's differential parameters to the geometry of surfaces.

The null lines of a surface are the lines which satisfy the equation

$$a_{ik}dx_i dx_k = 0. \tag{45.1}$$

These lines play the part in the geometry of a surface which the circular lines play in plane Euclidean geometry.

If the equation of the null lines is $\phi =$ constant, then

$$\Delta(\phi) = 0. \tag{45.2}$$

To obtain the null lines we must therefore be able to solve this equation.

The equation will have two independent integrals. If we take these integrals as the parameters we employ what we call null coordinates. The ground form takes the form

$$2f\,du\,dv, \tag{45.3}$$

and, as we have seen, $\Delta(\phi) = \frac{2}{f}\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v}.$

If then $\Delta(\phi) = 0,$

ϕ must be a function of u only, or a function of v only. A function satisfying the equation may be called a complex function of position. There are therefore only two types of complex functions of position, viz. the two functions whose differentials are multiples of the factors of ds^2. The first we shall take as that which corresponds to the factor

$$\left(a_{11}dx_1 + (a_{12} + \iota\sqrt{a})\,dx_2\right) \div a_{11}^{\frac{1}{2}}, \tag{45.4}$$

and the second that which corresponds to

$$\left(a_{11}dx_1 + (a_{12} - \iota\sqrt{a})\,dx_2\right) \div a_{11}^{\frac{1}{2}}. \tag{45.5}$$

We need only consider those which correspond to the first factor, and, if we do this, we can say that every function of position is a function of every other such complex function.

Thus in the case of the plane, where we have

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2,$$

$x + \iota y$ is a complex function of position since its differential is a multiple (unity) of $dx + \iota dy$ of the first factor of $dx^2 + dy^2$, and $\log r + \iota\theta$ is a complex function of position since its differential $\frac{dr}{r} + \iota d\theta$ is a multiple $\left(\frac{1}{r}\right)$ of $dr + \iota r d\theta$ of the first factor of $dr^2 + r^2 d\theta^2$; and $\log r + \iota\theta$ is a function of

$$x + \iota y.$$

Just as the position of any point in the plane is given by means of the complex variable $x+\iota y$, so the position of any point on a surface is given by means of the complex variable u where u is an integral of

$$\Delta(\phi) = 0. \tag{45.2}$$

§ **46. Conjugate Harmonic Functions. Mapping on a plane.** Let us now consider the equation

$$\Delta_2(\phi) = 0, \tag{46.1}$$

that is,
$$\frac{\partial}{\partial x_1} a^{\frac{1}{2}}\phi^1 + \frac{\partial}{\partial x_2} a^{\frac{1}{2}}\phi^2 = 0, \tag{46.2}$$

where
$$\phi^1 \equiv a^{11}\phi_1 + a^{12}\phi_2, \quad \phi^2 \equiv a^{12}\phi_1 + a^{22}\phi_2. \tag{46.3}$$

The expression $\quad a^{\frac{1}{2}}\phi^2 dx_1 - a^{\frac{1}{2}}\phi^1 dx_2$

is thus a perfect differential if $\Delta_2(\phi)$ is zero; and we have

$$a^{\frac{1}{2}}\phi^1 = \psi_2, \quad a^{\frac{1}{2}}\phi^2 = -\psi_1, \tag{46.4}$$

and therefore
$$a^{\frac{1}{2}}\psi^1 = -\phi_2, \quad a^{\frac{1}{2}}\psi^2 = \phi_1, \tag{46.5}$$

It follows that
$$\Delta_2(\psi) = 0, \quad \Delta(\phi, \psi) = 0. \tag{46.6}$$

Thus if ϕ is any integral of $\Delta_2(\phi) = 0$ we can by quadrature find ψ, another integral of the equation, and the two curves $\phi =$ constant, $\psi =$ constant will cut orthogonally.

A real function, annihilated by the linear operator Δ_2 of the second order, is said to be a harmonic function. The function ψ, obtained as explained by quadrature from ϕ, is called the conjugate harmonic function to ϕ. It will be noticed that the function conjugate to ψ is not ϕ but $-\phi$.

We also have
$$\Delta(\phi) = \Delta(\psi), \tag{46.7}$$

and therefore, since
$$\Delta(\phi, \psi) = 0,$$

we see that
$$\Delta(\phi + \iota\psi) = 0. \tag{46.8}$$

The function $\phi + \iota\psi$ is thus a complex function of position on the surface.

If we take
$$u = \phi, \; v = \psi$$

we have
$$ds^2 = (\Delta(\phi))^{-1}(du^2 + dv^2). \tag{46.9}$$

Thus the problem of mapping any surface on a plane, so that the map may be a true representation of the surface as regards similarity of small figures in each, just depends on the solution of the equation

$$\Delta_2(\phi) = 0. \tag{46.1}$$

The magnifying factor from the surface to the plane is $\Delta(\phi)$.

Thus to map any surface, applicable on a sphere of unit radius, and whose ground form may therefore be taken as

$$du^2 + \sin^2 u\, dv^2, \tag{46.10}$$

upon a plane we have $\Delta_2(\phi) = 0$;
and this tells us that ϕ must be a function of

$$\log\left(\tan\frac{u}{2}\right) + \iota v.$$

We thus obtain Mercator's Projection

$$x = \log\left(\tan\frac{u}{2}\right),\quad y = v. \tag{46.11}$$

The theory of conjugate functions of position on a surface can be applied to problems in Hydrodynamics and Electricity as has been done in the case of the plane. Thus if ϕ is a harmonic function on the surface, we may take it to be the velocity potential in the irrotational motion of a liquid over the surface, and ψ, the conjugate harmonic function, will then be the stream function.

Conversely, if the ground form is taken to be

$$ds^2 = e\,(du^2 + dv^2), \tag{46.12}$$

u and v will be conjugate harmonic functions.

CHAPTER IV

TWO-WAY SPACE AS A LOCUS IN EUCLIDEAN SPACE

§ **47. A quaternion notation.** So far we have been thinking of the two-way space associated with the ground form

$$ds^2 = a_{ik} dx_i dx_k;$$

we must now think of that space as a surface locus in Euclidean space.

Let ι', ι'', ι''' be three symbols which are to obey the associative law and the following self-consistent laws:

$$\iota''\iota''' = \iota', \qquad \iota'''\iota' = \iota'', \qquad \iota'\iota'' = \iota''',$$
$$\iota'''\iota'' = -\iota', \quad \iota'\iota''' = -\iota'', \quad \iota''\iota' = -\iota''',$$
$$\iota'\iota' = -1, \quad \iota''\iota'' = -1, \quad \iota'''\iota''' = -1. \qquad (47.1)$$

Let x', x'', x''' be three ordinary numbers called scalar quantities, then, if

$$x = x'\iota' + x''\iota'' + x'''\iota''', \qquad (47.2)$$

x may be said to be a complex number.

If we take

$$y = y'\iota' + y''\iota'' + y'''\iota''',$$

we see that

$$xy = -(x'y' + x''y'' + x'''y''') + (x''y''' - x'''y'')\iota' + (x'''y' - x'y''')\iota'' + (x'y'' - x''y')\iota''', \quad (47.3)$$

so that xy consists of two parts, a scalar part and a complex number. We write the scalar part

$$Sxy \text{ or } \underset{\smile}{xy}, \qquad (47.4)$$

and the complex part

$$Vxy \text{ or } \overset{\frown}{xy}. \qquad (47.5)$$

It follows that x^2 is a pure scalar.

We may easily verify the following results:

$$\underset{\smile}{xy} = \underset{\smile}{yx}, \quad \overset{\frown}{xy} = -\overset{\frown}{yx}, \quad xy + yx = 2\underset{\smile}{xy},$$
$$xy - yx = 2\overset{\frown}{xy}, \quad (x+y)^2 = x^2 + y^2 + 2\underset{\smile}{xy},$$

and, by multiplying the two matrices

$$\begin{Vmatrix} x' & x'' & x''' \\ y' & y'' & y''' \end{Vmatrix} \quad \begin{Vmatrix} z' & z'' & z''' \\ w' & w'' & w''' \end{Vmatrix}, \quad . \quad .$$

we verify that

$$S\overset{\frown}{xy}\overset{\frown}{zw} = \underset{\smile}{xw}\underset{\smile}{yz} - \underset{\smile}{xz}\underset{\smile}{yw},$$

$$Vx\overset{\frown}{yz} = z\underset{\smile}{xy} - y\underset{\smile}{zx},$$

$$Vx\overset{\frown}{yz} + Vy\overset{\frown}{zx} + Vz\overset{\frown}{xy} = 0.$$

If we take ι', ι'', ι''' to be unit vectors in the positive directions along the axes of rectangular Cartesian coordinates, then x will be the vector from the origin to the point whose coordinates are x', x'', x'''. The length of the vector x will be denoted by $|x|$. The symbol $\overset{\frown}{xy}$ will denote a vector at right angles to x and y, and in the sense that, if the left hand is along x and the right hand along y, then the direction $\overset{\frown}{xy}$ will be from foot to head; the magnitude of the vector will be $|x|\ |y| \sin\theta$, where θ is the angle between x and y from left to right.

The scalar $\underset{\smile}{xy}$ will be equal to $-|x|\ |y| \cos\theta$.

§ **48. Introduction of new fundamental magnitudes and equations.** Now let z be a vector whose components z', z'', z''' are functions of the parameters x_1 and x_2, that is, of the coordinates of the two-way space. We have

$$dz = z_p dx_p$$

and

$$a_{ik} = -\underset{\smile}{z_i z_k}.$$

The vector z traces out a surface. Let the unit vector drawn at the extremity of z normal to this surface be denoted by λ. We have proved [in § 13] that $z_{\cdot ik}$ is parallel to λ. We therefore have

$$z_{\cdot ik} = \Omega_{ik}\lambda, \tag{48.1}$$

where Ω_{ik} is a scalar quantity.

We know that

$$z_{\cdot ikh} - z_{\cdot ihk} = (\overline{hk} - \overline{kh})\, z_i = \{ithk\}\, z_t,$$

and therefore

$$(\Omega_{ik\cdot h} - \Omega_{ih\cdot k})\,\lambda + \Omega_{ik}\lambda_h - \Omega_{ih}\lambda_k = \{ithk\}\, z_t. \tag{48.2}$$

Multiplying across by λ and taking the scalar product we have, since $\underset{\smile}{\lambda\lambda_h} = 0$, $\underset{\smile}{\lambda\lambda_k} = 0$, $\underset{\smile}{\lambda z_t} = 0$,

the equation $$\Omega_{ik\cdot h} = \Omega_{ih\cdot k}. \tag{48.3}$$

This is true for all values of i, h, k from 1 to 2 inclusive, and $\Omega_{ik} = \Omega_{ki}$, so that

$$\Omega_{11\cdot 2} = \Omega_{12\cdot 1}, \quad \Omega_{22\cdot 1} = \Omega_{12\cdot 2}. \tag{48.4}$$

These equations are known as Codazzi's equations.

§ **49. Connexion of the magnitudes with curvature.** The length of the perpendicular from a point at the extremity of the vector $z+\delta z$ (where δz is not necessarily small) on the tangent plane at the extremity of z is

$$-\underset{\smile}{\lambda\delta z}. \tag{49.1}$$

If we now take δz so small that cubes of δx_1, δx_2 may be neglected, the length becomes

$$-\tfrac{1}{2}(\underset{\smile}{\lambda z}_{\cdot 11}\delta x_1^2 + 2\underset{\smile}{\lambda z}_{\cdot 12}\delta x_1\delta x_2 + \underset{\smile}{\lambda z}_{\cdot 22}\delta x_2^2),$$

that is, $$\tfrac{1}{2}(\Omega_{11}\delta x_1^2 + 2\Omega_{12}\delta x_1\delta x_2 + \Omega_{22}\delta x_2^2). \tag{49.2}$$

The radius of curvature of any normal section of the surface is therefore given by

$$\frac{1}{R} = \frac{\Omega_{ik}dx_i dx_k}{a_{ik}dx_i dx_k} \tag{49.3}$$

in the tensor notation, and the principal radii of curvature are consequently given by

$$\begin{vmatrix} R\Omega_{11}-a_{11}, & R\Omega_{12}-a_{12} \\ R\Omega_{12}-a_{12}, & R\Omega_{22}-a_{22} \end{vmatrix} = 0. \tag{49.4}$$

The product of the reciprocals of the principal radii of curvature is therefore $$\frac{\Omega_{11}\Omega_{22}-\Omega_{12}^2}{a_{11}a_{22}-a_{12}^2}. \tag{49.5}$$

Now we saw that

$$\begin{aligned}(1212) &= \underset{\smile}{z_{\cdot 12}z_{\cdot 12}} - \underset{\smile}{z_{\cdot 11}z_{\cdot 22}}, \\ &= \lambda^2(\Omega_{12}^2 - \Omega_{11}\Omega_{22}), \\ &= \Omega_{11}\Omega_{22} - \Omega_{12}^2,\end{aligned} \tag{49.6}$$

and therefore the invariant K is just the measure of curvature.

We thus have the equations

$$Ka = \Omega_{11}\Omega_{22} - \Omega_{12}^2, \qquad (49.7)$$

$$\Omega_{11\cdot 2} = \Omega_{12\cdot 1}, \quad \Omega_{22\cdot 1} = \Omega_{12\cdot 2}, \qquad (48.4)$$

wherewith we are to determine the functions

$$\Omega_{11}, \ \Omega_{12}, \ \Omega_{22}.* \qquad (49.8)$$

When we have found these functions we can find the principal radii of curvature by aid of the equation

$$\frac{a}{R^2} - \frac{1}{R}(a_{11}\Omega_{22} + a_{22}\Omega_{11} - 2a_{12}\Omega_{12}) + \Omega_{11}\Omega_{22} - \Omega_{12}^2 = 0, \qquad (49.9)$$

which may be written

$$\frac{1}{R^2} - \frac{1}{R}a^{ik}\Omega_{ik} + a^{-1}(\Omega_{11}\Omega_{22} - \Omega_{12}^2) = 0, \qquad (49.10)$$

applying the tensor notation to the coefficient of $\frac{1}{R}$.

If we were to keep strictly to the tensor notation we should write $\Omega_{11}\Omega_{22} - \Omega_{12}^2$ as Ω. We must distinguish between the integer which denotes merely a power, as in Ω_{12}^2 denoting the square of Ω_{12}, and the integer which we called the upper integer in a tensor component. The two meanings are not likely to cause any practical difficulty in reality.

§ 50. The normal vector determinate when the functions Ω_{ik} are known. We must now show how we may determine the unit vector λ when the functions Ω_{ik} are known.

* [It is usual to speak of the functions Ω_{11}, Ω_{12}, Ω_{22}, i.e. (by § 50) $\underbrace{z_1\lambda_1}$, $\underbrace{z_1\lambda_2} = \underbrace{z_2\lambda_1}$, $\underbrace{z_2\lambda_2}$, as the fundamental magnitudes of the second order, those of the first order being the a_{11}, a_{12}, a_{22} or e, f, g of the ground form ds^2, and to say that the six are connected by Gauss's equation (49.7), in which K (§ 43) is a known function of the magnitudes of the first order and their derivatives, and by the two Codazzi equations (48.4). Written at greater length these two equations are

$$\frac{\partial}{\partial x_2}\Omega_{11} - \{121\}\Omega_{11} - \{122\}\Omega_{12} = \frac{\partial}{\partial x_1}\Omega_{12} - \{111\}\Omega_{12} - \{112\}\Omega_{22},$$

$$\frac{\partial}{\partial x_1}\Omega_{22} - \{122\}\Omega_{22} - \{121\}\Omega_{12} = \frac{\partial}{\partial x_1}\Omega_{12} - \{222\}\Omega_{12} - \{221\}\Omega_{11},$$

and their explicit forms are obtained by substituting in these for {111}, &c., from § 43.]

We denote the ground form of the spherical image, that is, of the sphere traced out by a unit vector drawn through the origin, parallel to the normal at the extremity of z, by

$$a'_{ik}dx_i dx_k, \tag{50.1}$$

so that

$$a'_{ik} = -\underbrace{\lambda_i\lambda_k}. \tag{50.2}$$

If

$$\lambda_{\cdot ik} \equiv \lambda_{ik} - \{ikt\}'\lambda_t,$$

where $\{ikt\}'$ refers to the ground form of the spherical image, we see as before that $\lambda_{\cdot ik}$ is parallel to the normal to the sphere at the extremity of λ: that is $\lambda_{\cdot ik}$ is parallel to λ.

Now $\underbrace{\lambda\lambda_i}$ is zero, and differentiating we have

$$\underbrace{\lambda\lambda_{ik}} + \underbrace{\lambda_i\lambda_k} = 0,$$

so that

$$\underbrace{\lambda\lambda_{\cdot ik}} + \underbrace{\lambda_i\lambda_k} = 0.$$

It follows that

$$\lambda_{\cdot ik} = \underbrace{\lambda_i\lambda_k}\lambda = -a'_{ik}\lambda\,; \tag{50.3}$$

and as we have shown [in § 30] how, when $a'_{ik}\ldots$ are given, λ can be obtained by aid of Riccati's equation, we have only to show how $a'_{ik}\ldots$ can be expressed in terms of $a_{ik}\ldots$ and $\Omega_{ik}\ldots$.

Along a line of curvature we have

$$dz + R d\lambda = 0\,; \tag{50.4}$$

let R' and R'' be the principal radii of curvature, and let us choose the lines of curvature so that they may be the parametric lines, that corresponding to R' being

$$dx_2 = 0,$$

and that corresponding to R'' being

$$dx_1 = 0.$$

We therefore have

$$z_1 + R'\lambda_1 = 0, \quad z_2 + R''\lambda_2 = 0. \tag{50.5}$$

And it follows that

$$\begin{gathered} a_{11} = R'\Omega_{11}, \quad a_{12} = R'\Omega_{12}, \quad a_{12} = R''\Omega_{12}, \quad a_{22} = R''\Omega_{22}, \\ \Omega_{11} = R'a'_{11}, \quad \Omega_{12} = R'a'_{12}, \quad \Omega_{12} = R''a'_{12}, \quad \Omega_{22} = R''a'_{22}, \end{gathered} \tag{50.6}$$

so that
$$a_{11}-(R'+R'')\,\Omega_{11}+a'_{11}R'R''=0,$$
$$a_{12}-(R'+R'')\,\Omega_{12}+a'_{12}R'R''=0,$$
$$a_{22}-(R'+R'')\,\Omega_{22}+a'_{22}R'R''=0. \qquad (50.7)$$

Now the expressions on the left in these equations are tensor components, and therefore, as they vanish for one particular coordinate system, they vanish for all systems. That is, the equations are identities.

We may express the identity in the form
$$dz^2+(R'+R'')\,\underbrace{dz\,d\lambda}+R'R''d\lambda^2=0. \qquad (50.8)$$

We thus see how $a'_{ik}\ldots$ are obtained.

We see that
$$\Omega_{ik}=\underbrace{z_i\lambda_k}=\underbrace{z_k\lambda_i}; \qquad (50.9)$$
for
$$\underbrace{\lambda z_i}=0,$$
and therefore
$$\underbrace{\lambda z_{ik}}+\underbrace{\lambda_k z_i}=0,$$
which gives
$$\underbrace{\lambda z_{\cdot ik}}+\underbrace{\lambda_k z_i}=0.$$

From the equations
$$\underbrace{\lambda_1 z_1}=\Omega_{11},\ \underbrace{\lambda_1 z_2}=\underbrace{\lambda_2 z_1}=\Omega_{12},\ \underbrace{\lambda_2 z_2}=\Omega_{22},\ \underbrace{\lambda z_1}=0,\ \underbrace{\lambda z_2}=0 \qquad (50.10)$$
we can find z_1 and z_2 when λ is known, and thus determine z by quadrature.

We have now shown how the determination of the surfaces applicable to the ground form
$$a_{ik}dx_i dx_k$$
depends on the determination of the functions Ω_{ik}.

But here comes the difficulty: the equations to determine these functions
$$\Omega_{11\cdot 2}=\Omega_{12\cdot 1},\ \Omega_{22\cdot 1}=\Omega_{12\cdot 2},\ Ka=\Omega_{11}\Omega_{22}-\Omega_{12}^2$$
are differential equations of the second order which, in general, we cannot solve.

In one very special case we can solve them, viz. when the invariant K is zero. In this case we have shown that the ground form may be taken to be
$$dx_1^2+dx_2^2. \qquad (50.11)$$

The equations now become

$$\frac{\partial}{\partial x_1}\Omega_{12} = \frac{\partial}{\partial x_2}\Omega_{11}, \quad \frac{\partial}{\partial x_2}\Omega_{12} = \frac{\partial}{\partial x_1}\Omega_{22},$$

and therefore

$$\Omega_{11} = \frac{\partial^2 \phi}{\partial x_1^2}, \quad \Omega_{12} = \frac{\partial^2 \phi}{\partial x_1 \partial x_2}, \quad \Omega_{22} = \frac{\partial^2 \phi}{\partial x_2^2}, \qquad (50.12)$$

where

$$\frac{\partial^2 \phi}{\partial x_1^2}\frac{\partial^2 \phi}{\partial x_2^2} = \left(\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right)^2.$$

We can easily prove that we are now led to developable surfaces.

§ 51. Reference to lines of curvature. The measure of curvature. When we refer to lines of curvature as parametric lines we have, in (50.6),

$$a_{12} = R'\Omega_{12}, \quad a_{12} = R''\Omega_{12},$$

and therefore, unless R' and R'' are equal, we must have

$$a_{12} = \Omega_{12} = 0. \qquad (51.1)$$

If the radii of curvature are equal, operating with $\bar{1}$ and $\bar{2}$ which annihilate a_{ik}, we have

$$R_1\Omega_{12} + R\Omega_{12\cdot 1} = 0,$$
$$R_2\Omega_{12} + R\Omega_{12\cdot 2} = 0. \qquad (51.2)$$

Similarly by operating on

$$a_{11} = R\Omega_{11}, \quad a_{22} = R\Omega_{22},$$

we have

$$R_2\Omega_{11} + R\Omega_{11\cdot 2} = 0,$$
$$R_1\Omega_{22} + R\Omega_{22\cdot 1} = 0. \qquad (51.3)$$

From Codazzi's equations we deduce that

$$R_1\Omega_{12} = R_2\Omega_{11},$$
$$R_1\Omega_{22} = R_2\Omega_{12}.$$

As we cannot have $\Omega_{11}\Omega_{22} = \Omega_{12}^2$, unless R is infinite, we must have

$$R_1 = 0, \; R_2 = 0, \qquad (51.4)$$

that is, R is constant and the surface must be a sphere.

Leaving aside the special case of a sphere, we have when the parametric lines are the lines of curvature

$$a_{12} = \Omega_{12} = a'_{12} = 0, \qquad (51.5)$$

and we can often simplify proofs of theorems by referring to lines of curvature as parametric lines.

The vector $\widehat{z_1 z_2}$ is clearly normal to the surface at the extremity of z: its magnitude is $a^{\frac{1}{2}}$ (or h as it is generally written) and therefore $\widehat{z_1 z_2} = a^{\frac{1}{2}}\lambda$.

Similarly we have $\widehat{\lambda_1 \lambda_2} = a'^{\frac{1}{2}}\lambda$.

The expression $K\widehat{z_i z_k} - \widehat{\lambda_i \lambda_k}$
is a tensor component. It obviously vanishes when we refer to lines of curvature: it therefore vanishes identically and we have

$$K\widehat{z_1 z_2} = \widehat{\lambda_1 \lambda_2}. \tag{51.6}$$

We then have

$$Ka^{\frac{1}{2}} = a'^{\frac{1}{2}}, \tag{51.7}$$

that is, the measure of curvature is the ratio of a small element of area on the spherical image to the corresponding area on the surface.

§ **52. Tangential equations. Minimal surfaces.** We shall now develop some further formulae. We have

$$\Omega_{11}\Omega_{22} - \Omega_{12}^2 = aK = a'K^{-1} = (aa')^{\frac{1}{2}};$$

$$a^{ik}\Omega_{ik} = \frac{1}{R'} + \frac{1}{R''}, \tag{52.1}$$

and, from the formulae connecting

$$a_{ik},\ a'_{ik},\ \Omega_{ik},$$

we easily deduce $a'^{ik}\Omega_{ik} = R' + R''$,

$$a^{ik}a'_{ik} = \left(\frac{1}{R'}\right)^2 + \left(\frac{1}{R''}\right)^2, \quad a'^{ik}a_{ik} = (R')^2 + (R'')^2. \tag{52.2}$$

We can also obtain formulae applicable to a surface given by its tangential equation. This means that instead of beginning with a vector z, given in terms of parameters x_1 and x_2, we begin with assuming that λ is known in terms of these parameters, and also p, the perpendicular from the origin on the tangent plane to the surface.

The lines of curvature are given by

$$(R\Omega_{11} - a_{11})\,dx_1 + (R\Omega_{12} - a_{12})\,dx_2 = 0,$$
$$(R\Omega_{12} - a_{12})\,dx_1 + (R\Omega_{22} - a_{22})\,dx_2 = 0.$$

They are therefore also given by

$$(\Omega_{11}-Ra'_{11})\,dx_1+(\Omega_{12}-Ra'_{12})\,dx_2=0,$$
$$(\Omega_{12}-Ra'_{12})\,dx_1+(\Omega_{22}-Ra'_{22})\,dx_2=0, \qquad (52.3)$$

as we see at once from the connecting equations.

The tangential equation of a surface is

$$p+\lambda z=0. \qquad (52.4)$$

By differentiation we deduce that

$$p_i+\underset{\smile}{\lambda_i}z=0,$$
$$p_{ik}+\underset{\smile}{\lambda_{ik}}z+\underset{\smile}{\lambda_i}z_k=0.$$

With reference to the ground form of the spherical image we therefore have $\quad p_{\cdot ik}+\underset{\smile}{\lambda_{\cdot ik}}z+\Omega_{ik}=0.$

Now $\qquad \lambda_{\cdot ik}=-a'_{ik}\lambda,$

and therefore $\qquad p_{\cdot ik}+a'_{ik}p+\Omega_{ik}=0. \qquad (52.5)$

When therefore we are given the tangential equation of a surface, the lines of curvature and the radii of curvature are given by the formulae

$$\left(p_{\cdot 11}+a'_{11}(p+R)\right)dx_1+\left(p_{\cdot 12}+a'_{12}(p+R)\right)dx_2=0,$$
$$\left(p_{\cdot 12}+a'_{12}(p+R)\right)dx_1+\left(p_{\cdot 22}+a'_{22}(p+R)\right)dx_2=0. \qquad (52.6)$$

In particular if we want the parametric lines to be lines of curvature on the surface we must have

$$a'_{12}=\Omega_{12}=0,$$

and therefore p must satisfy the equation

$$p_{\cdot 12}=0. \qquad (52.7)$$

There is a particular type of surface with which we shall have to do: the minimal surface characterized by the property that the principal radii of curvature are equal and opposite.

The expression $\qquad S\lambda\lambda_i z_k - S\lambda\lambda_k z_i \qquad (52.8)$

is a tensor component. If the surface is a minimal one it vanishes when we refer to lines of curvature, and therefore it vanishes always if, and, we see, only if, the surface is a minimal one.

We always have the formula, as we easily see,

$$\Delta'_2 p + 2p + R' + R'' = 0.$$

The tangential equation of a minimal surface is therefore given by

$$\Delta'_2 p + 2p = 0. \tag{52.9}$$

If we refer to the null lines of the spherical image as parametric lines, the ground form of the sphere becomes

$$4\,(1 + x_1 x_2)^{-2} dx_1\, dx_2;$$

and the equation which p has to satisfy becomes

$$(1 + x_1 x_2)^2 \frac{\partial^2 p}{\partial x_1 \partial x_2} + 2p = 0.$$

It may be shown by Laplace's method that the most general solution of this equation is

$$(1 + x_1 x_2)\, p = 2 x_1 f(x_1) + 2 x_2 \phi(x_2) + (1 + x_1 x_2)\big(x_1^2 f'(x_1) + x_2^2 \phi'(x_2)\big), \tag{52.10}$$

and we have thus obtained the tangential equation of the minimal surface.

§ **53. Weingarten or *W* surfaces.** We now proceed to consider more generally surfaces which, like the minimal surface, are characterized by the property that their radii of curvature are functionally connected. These surfaces are called *W* surfaces, after Weingarten, who studied their properties.

When we refer to the lines of curvature as parametric lines we have (50.5) $z_1 + R'\lambda_1 = 0, \quad z_2 + R''\lambda_2 = 0,$

and therefore $(R' - R'')\,\lambda_{12} = R''_1 \lambda_2 - R'_2 \lambda_1.$

Let

$$R'' = f(R')$$

and

$$\phi(x) \equiv e^{\int \frac{dx}{x - f(x)}}.$$

We easily verify that

$$\frac{f'(x)\,\phi'(x)}{\phi(x)} = \frac{\phi''(x)}{\phi'(x)}, \quad \frac{1}{x - f(x)} = \frac{\phi'(x)}{\phi(x)}. \tag{53.1}$$

Now

$$\frac{R''_1}{R'-R''} = \frac{f'(R')R'_1}{R'-f(R')} = \frac{\phi''(R')R'_1}{\phi'(R')} = \frac{\partial}{\partial x_1}\log(\phi'(R')),$$

$$\frac{R'_2}{R'-R''} = \frac{R'_2}{R'-f(R')} = \frac{\phi'(R')R'_2}{\phi(R')} = \frac{\partial}{\partial x_2}\log(\phi(R')).$$

The equation satisfied by λ thus becomes

$$\lambda_{12} = \lambda_2 \frac{\partial}{\partial x_1}\log(\phi'(R')) - \lambda_1 \frac{\partial}{\partial x_2}\log(\phi(R')), \qquad (53.2)$$

and therefore, since $\lambda_1\lambda_2$ is zero,

$$\frac{\partial}{\partial x_1}\left(\frac{\lambda_2^2}{(\phi'(R'))^2}\right) = 0; \quad \frac{\partial}{\partial x_2}(\lambda_1^2(\phi(R'))^2) = 0.$$

We may therefore, the lines of curvature still remaining the parametric lines, take

$$\lambda_1^2(\phi(R'))^2 + 1 = 0; \quad \lambda_2^2(\phi'(R'))^{-2} + 1 = 0. \qquad (53.3)$$

The spherical image of the W surface (that is, it will be remembered, the surface traced out by a unit vector parallel to the normal at the extremity of the vector z, and expressed in the coordinates which give z), when the W surface is referred to the lines of curvature as parametric lines, will be therefore

$$\frac{dx_1^2}{(\phi(R'))^2} + (\phi'(R'))^2 dx_2^2. \qquad (53.4)$$

It will be sometimes more convenient to express the parametric coordinates by u and v.

Conversely, if we are given the ground form of a sphere in the form $pdu^2 + qdv^2$, where p and q are functionally connected, it will be the spherical image of a W surface referred to its lines of curvature.

§ **54. An example of W surfaces.** We may now consider some examples. We saw (§ 42) that, referred to what we called focal coordinates, the ground form of any surface may be taken as

$$ds^2 = \sec^2\theta\, du^2 + \operatorname{cosec}^2\theta\, dv^2, \qquad (54.1)$$

where

$$2u \equiv PA + PB, \; 2v \equiv PA - PB,$$

and A and B are any two points on the surface which we call the foci; PA and PB are geodesic distances and 2θ is the angle APB.

If the surface is applicable on a sphere we see that

$$\tan^2\theta = \frac{\sin(c-v)\sin(c+v)}{\sin(u+c)\sin(u-c)},$$

where $2c$ is the geodesic distance between A and B.

Thus $\phi(R') = \cos\theta,\ \phi'(R') = \operatorname{cosec}\theta,$

and therefore $\operatorname{cosec}\theta \dfrac{dR'}{d\theta} = -\sin\theta.$

If we now integrate this equation we have

$$4R' = \sin 2\theta - 2\theta + \epsilon,$$

where ϵ is some constant.

But $R' - R'' = \dfrac{\phi(R')}{\phi'(R')} = \sin\theta\cos\theta,$

so that $4R'' = -2\theta - \sin 2\theta + \epsilon,$

and therefore $2(R'-R'') = \sin(\epsilon - 2R' - 2R'').$ (54 . 2)

This is the relation between the principal radii of curvature of the W surface which corresponds to the spherical image

$$\sec^2\theta du^2 + \operatorname{cosec}^2\theta dv^2. \qquad (54\,.\,1)$$

In this case we know the radii of curvature in terms of the parameters since θ is so known. We thus know the ground form both of the surface and of the spherical image, and therefore can find the surface as a locus in space.

§ **55. The spherical and pseudo-spherical examples.** In the above example we began with a known ground form for the spherical image and deduced the relation between the curvatures.

If we take any known ground form for the spherical image

$$pdu^2 + qdv^2,$$

where p and q are functionally related, and known in terms of the parameters, we could proceed similarly. We could find the relation between the curvatures and we should obtain in known terms of the parameters the ground form of the surface. We could then obtain the surface as a locus in space. In my exposition of the method I have followed

Darboux and taken the example he gives, as I also do in what follows.

When on the other hand we begin with a known relation between the curvatures, we cannot in general find the surface as a locus in space. Thus, let us apply the method to the problem of finding the surfaces applicable on a sphere of unit radius.

Here we have $$R'R'' = 1, \tag{55.1}$$
and we may take $R' = \coth\theta,\ R'' = \tanh\theta.$

The function which expresses R'' in terms of R' is

$$f(x) = \frac{1}{x},$$

and

$$\phi(R') = e^{\int \frac{dR'}{R' - \frac{1}{R'}}} = \operatorname{cosech}\theta,$$

$$\phi'(R') = \cosh\theta.$$

The ground form of the spherical image is thus

$$\sinh^2\theta\, du^2 + \cosh^2\theta\, dv^2. \tag{55.2}$$

On the sphere the measure of curvature is unity, and therefore our formula for K gives

$$\theta_{11} + \theta_{22} + \sinh\theta\cosh\theta = 0. \tag{55.3}$$

Now if we knew how to solve this equation we should have an expression for θ in terms of the parameters u and v, and we should thus be able to write down the ground forms of the surface and of the spherical image in terms of the parameters; and thus have the means of determining as loci in space all the surfaces which are applicable on the sphere.

Unfortunately we cannot solve the equation generally. This example shows how ultimately nearly all questions in Differential Geometry come to getting a differential equation; and that the complete answer depends on the solution of the equation. But even when we cannot solve the equation we gain in knowledge by having the differential equation in explicit form. Thus it happens sometimes that two apparently quite different geometrical problems may depend on the same insoluble differential equation. The surfaces connected with the problems are thus brought into relationship with one another; and the relationship is sometimes very simple and very beautiful. Illustrations of this will occur later. All we can say now is that the differential equation

$$\theta_{11}+\theta_{22}+\sinh\theta\cosh\theta = 0$$

is that on which depends the obtaining of all surfaces which are applicable on the sphere: that is, the surfaces whose geodesic geometry may be considered as absolutely known, being just spherical trigonometry.

Similarly we might consider the problem of finding the surfaces applicable on a pseudosphere. Here we have

$$R'R'' = -1, \tag{55.4}$$

and we take $\quad R' = \cot\theta,\ R'' = -\tan\theta.$

We find that

$$\phi(\cot\theta) = \operatorname{cosec}\theta,\ \phi'(\cot\theta) = \cos\theta,$$

so that the ground form of the spherical image is

$$\sin^2\theta\, du^2+\cos^2\theta\, dv^2, \tag{55.5}$$

and the equation to determine θ is

$$\theta_{22}-\theta_{11}+\sin\theta\cos\theta = 0. \tag{55.6}$$

If we apply the substitution

$$2u' \equiv u+v,\ 2v' \equiv u-v,\ 2\theta \equiv \theta',$$

the equation takes the simpler form

$$\theta_{12} = \sin\theta\,; \tag{55.7}$$

and on this equation depends the obtaining of the surfaces with the known pseudospherical trigonometry, obtainable from spherical trigonometry by writing ιa, ιb, ιc, for the arcs of a spherical triangle.

§ 56. **Reference to asymptotic lines.** We have now considered the surface when referred to lines of curvature as parametric coordinates, and the equations resulting,

$$z_1 = R'\lambda_1,\ z_2 = R''\lambda_2,$$

where R' and R'' are the principal radii of curvature and λ is the unit vector parallel to the normal at the extremity of z.

We now proceed to consider another special system of coordinates.

The elements dz and δz on the surface which are drawn through the extremity of the vector z are perpendicular if

$$\underset{\smile}{dz\,\delta z} = 0;$$

that is, if

$$\underset{\smile}{z_1^2}\, dx_1\,\delta x_1 + \underset{\smile}{z_1 z_2}\,(dx_1\,\delta x_2 + dx_2\,\delta x_1) + \underset{\smile}{z_2^2}\, dx_2\,\delta x_2 = 0,$$

or
$$a_{11}\, dx_1\,\delta x_1 + a_{12}\,(dx_1\,\delta x_2 + dx_2\,\delta x_1) + a_{22}\, dx_2\,\delta x_2 = 0. \tag{56.1}$$

The elements dz and δz at the extremity of z are said to be conjugate, if the tangent planes at the extremity of z and at the extremity of $z+dz$ both contain the element δz; that is, if δz is perpendicular to the normals at the extremities of z and of $z+dz$. We therefore have for conjugate elements

$$\underset{\smile}{\delta z\, d\lambda} = 0,$$

that is,

$$\underset{\smile}{z_1\lambda_1}\, dx_1\,\delta x_1 + \underset{\smile}{z_1\lambda_2}\,(dx_1\,\delta x_2 + dx_2\,\delta x_1) + \underset{\smile}{z_2\lambda_2}\, dx_2\,\delta x_2 = 0,$$

or
$$\Omega_{11}\, dx_1\,\delta x_1 + \Omega_{12}\,(dx_1\,\delta x_2 + dx_2\,\delta x_1) + \Omega_{22}\, dx_2\,\delta x_2 = 0. \tag{56.2}$$

Thus we see that the lines of curvature at any point of a surface are both orthogonal and conjugate, and conversely we see that lines which at any point are both orthogonal and conjugate are lines of curvature.

An element which is conjugate to itself satisfies the equation

$$\Omega_{11}\, dx_1^2 + 2\,\Omega_{12}\, dx_1\, dx_2 + \Omega_{22}\, dx_2^2 = 0.$$

The self-conjugate elements at a point form the asymptotic lines

$$\Omega_{11}\, dx_1^2 + 2\,\Omega_{12}\, dx_1\, dx_2 + \Omega_{22}\, dx_2^2 = 0; \tag{56.3}$$

and we see that the radius of curvature of a normal section in the direction of an asymptotic line is infinite.

We call $\dfrac{dx_1}{ds}$ and $\dfrac{dx_2}{ds}$ the 'direction cosines' of an element on the surface. They tell us the direction but they are not the cosines of the angles the element makes with the parametric lines. We often write them in the tensor notation ξ^1, ξ^2; but we must remember ξ^2 is not the square of ξ, nor is $(\xi^1)^2$ a tensor component ξ^{12}, but the square of ξ^1. We have identically

$$a_{ik}\xi^i\xi^k = 1 \; ;$$

and, if R is the radius of curvature of any normal section of the surface,

$$R\Omega_{ik}\xi^i\xi^k = 1. \qquad (56.4)$$

Take now the asymptotic lines as parametric lines. We have

$$\Omega_{11} = 0, \; \Omega_{22} = 0,$$

and therefore by Codazzi's equations

$$\Omega_{11\cdot 2} = \Omega_{12\cdot 1}, \; \Omega_{22\cdot 1} = \Omega_{12\cdot 2},$$

$$\Omega_{11}\Omega_{22} - \Omega_{12}\Omega_{12} - Ka = 0$$

we have

$$\frac{\partial}{\partial u}\log \Omega_{12} = \{111\} - \{212\},$$

$$\frac{\partial}{\partial v}\log \Omega_{12} = \{222\} - \{121\}.$$

Now we saw (§ 6) that the determinant a satisfied the equations

$$\frac{\partial}{\partial u} a^{\frac{1}{2}} = a^{\frac{1}{2}}(\{111\} + \{212\}),$$

$$\frac{\partial}{\partial v} a^{\frac{1}{2}} = a^{\frac{1}{2}}(\{121\} + \{222\}),$$

and therefore

$$\frac{\partial}{\partial u}(\log K^{\frac{1}{2}}) + 2\{212\} = 0,$$

$$\frac{\partial}{\partial v}(\log K^{\frac{1}{2}}) + 2\{121\} = 0. \qquad (56.5)$$

These are the equations which the coefficients a_{ik} must satisfy if the parametric lines are to be asymptotic.

If we are given any ground form, and if we could transform it so that the new coefficients would satisfy the above equations, then we could, since in this case we would know the functions Ω_{11}, Ω_{12}, Ω_{22} and the ground form, find the surfaces to which the form would be applicable. But the transformation would itself involve the solution of differential equations of as great difficulty as Codazzi's equations.

Taking the asymptotic lines as coordinate axes we have

$$z_{\underset{\smile}{1}}\lambda_1 = 0, \; z_{\underset{\smile}{2}}\lambda_2 = 0,$$

and therefore $$z_1 = p\widehat{\lambda\lambda_1}, \tag{56.6}$$
where p is some scalar.

Similarly we have $$z_2 = q\widehat{\lambda\lambda_2}, \tag{56.7}$$
where q is a scalar.

As $$\underset{\smile}{z_1\lambda_2} = \underset{\smile}{z_2\lambda_1}$$
we have $$pS\lambda\lambda_1\lambda_2 = qS\lambda\lambda_2\lambda_1 = -qS\lambda\lambda_1\lambda_2,$$
and therefore $$p = -q. \tag{56.8}$$

Since $$\widehat{\lambda_1\lambda_2} = K\widehat{z_1z_2},$$
we have $$-Kp^2 V\widehat{\lambda\lambda_1}\widehat{\lambda\lambda_2} = \widehat{\lambda_1\lambda_2},$$
that is, $$-Kp^2(\lambda_2 S\lambda\lambda_1\lambda - \lambda S\lambda\lambda_1\lambda_2) = \widehat{\lambda_1\lambda_2},$$
or $$Kp^2\lambda S\lambda^2 = \lambda, \tag{56.9}$$
since $\widehat{\lambda_1\lambda_2}$ is parallel to λ.

We therefore have $$p = (-K)^{-\frac{1}{2}}, \tag{56.10}$$
and $$z_1 = (-K)^{-\frac{1}{2}}\widehat{\lambda\lambda_1},\ z_2 = -(-K)^{-\frac{1}{2}}\widehat{\lambda\lambda_2}. \tag{56.11}$$

These are the exceedingly important equations which we have when we choose the asymptotic lines to be the parametric lines.

§ 57. **Equations determining a surface.** If we now take
$$Z \equiv (-K)^{-\frac{1}{4}}\lambda, \tag{57.1}$$
so that Z is a vector, parallel to the normal at the extremity of z, and of length $(-K)^{-\frac{1}{4}}$, we can write the equations which determine the surface in the simple form
$$z_1 = \widehat{ZZ_1},\ z_2 = -\widehat{ZZ_2}.$$

From these equations we have
$$\widehat{ZZ_{12}} = 0,$$
and therefore $$Z_{12} = pZ, \tag{57.2}$$
where p is some scalar [not the p of (56.10)].

In order to find the asymptotic lines of a given surface we have to solve the ordinary differential equation
$$\Omega_{11}du^2 + 2\Omega_{12}du\,dv + \Omega_{22}dv^2 = 0,$$

and when we have done this we can bring the equation of the surface to the form stated.

We have
$$Z = c\lambda, \tag{57.1)'}$$
and we notice that c is an absolute invariant.

Differentiating we see that
$$c\lambda_{12}+c_1\lambda_2+c_2\lambda_1+c_{12}\lambda = pc\lambda,$$
and therefore
$$-c\lambda_1\lambda_2 = c_{12}-pc,$$
that is,
$$ca'_{12} = c_{12}-pc. \tag{57.3}$$

From the formulae
$$a'^{ik}\Omega_{ik} = R'+R'',$$
$$\Omega_{11}\Omega_{22}-(\Omega_{12})^2 = a'K,$$
we see that
$$a'_{12} = \Omega_{12}\left(\frac{1}{R'}+\frac{1}{R''}\right),$$
and therefore
$$p = \frac{c_{12}}{c}-\Omega_{12}\left(\frac{1}{R'}+\frac{1}{R''}\right). \tag{57.4}$$

The equation of the surface referred to the asymptotic lines is therefore
$$z_1 = \widehat{ZZ}_1, \quad z_2 = -\widehat{ZZ}_2, \tag{57.5}$$
where
$$Z_{12} = \left(\frac{c_{12}}{c}-\Omega_{12}\left(\frac{1}{R'}+\frac{1}{R''}\right)\right)Z. \tag{57.6}$$

§ 58. **The equation for the normal vector in tensor form.** We can express the equation which the vector Z must satisfy in tensor form so as to be independent of any particular coordinate system.

The null lines on the surface applicable on the ground form
$$a_{ik}dx_i dx_k$$
are the lines which satisfy the equation
$$a_{ik}dx_i dx_k = 0.$$
On a real surface they are of course imaginary and are characterized by the property that the distance, measured along the curve, between any two points on a null curve is zero.

Let us now consider the ground form
$$\Omega_{ik}dx_i dx_k, \tag{58.1}$$

remembering that any quadratic differential expression is the ground form of some set of surfaces. The surface, to which this form applies, will have as its null lines the correspondents of the asymptotic lines on the surface we are considering.

Let Beltrami's differential operator with reference to the ground form

$$\Omega_{ik} dx_i dx_k$$

be denoted by $${}_{\omega}\Delta_2. \qquad (58.2)$$

Now we saw (43 . 24) that, with reference to the null lines as parametric lines, that is with reference to the asymptotic lines on the surface we are considering,

$${}_{\omega}\Delta_2 = \frac{2}{\Omega_{12}} \frac{\partial}{\partial u} \frac{\partial}{\partial v}.$$

The equation $Z_{12} = \left(\frac{c_{12}}{c} - \Omega_{12}\left(\frac{1}{R'} + \frac{1}{R''}\right)\right) Z$

may be written

$$\frac{2}{\Omega_{12}} \frac{\partial}{\partial u} \frac{\partial}{\partial v} Z = \left(\frac{2}{\Omega_{12} c} \frac{\partial}{\partial u} \frac{\partial}{\partial v} c - 2\left(\frac{1}{R'} + \frac{1}{R''}\right)\right) Z,$$

that is, $${}_{\omega}\Delta_2 Z = \left(\frac{{}_{\omega}\Delta_2 c}{c} - 2\left(\frac{1}{R'} + \frac{1}{R''}\right)\right) Z; \qquad (58.3)$$

and this is a tensor equation independent of any coordinate system.

§ **59. Introduction of a new vector ζ.** We may write this tensor equation briefly in the form

$$\Delta_2 Z = pZ. \qquad (59.1)$$

Let θ be any scalar quantity which satisfies the equation

$$\Delta_2 \theta = p\theta.$$

We then have $$\theta \Delta_2 Z = Z \Delta_2 \theta.$$

Now we saw (17 . 4) in the chapter on tensors that

$$\Delta_2 u \equiv a^{-\frac{1}{2}} \frac{\partial}{\partial x_t} a^{\frac{1}{2}} u^t,$$

where $$u^t \equiv a^{tq} u_q,$$

and therefore $$u_t v^t \equiv v_t u^t.$$

We may then write the equation

$$\theta\,\Delta_2 Z - Z\,\Delta_2\,\theta = 0$$

in the form
$$\theta\,\frac{\partial}{\partial x_t}\sqrt{\Omega}Z^t = Z\,\frac{\partial}{\partial x_t}\sqrt{\Omega}\,\theta^t,$$

or
$$\frac{\partial}{\partial x_t}\sqrt{\Omega}\,(\theta Z^t - Z\theta^t) = 0 \qquad (59.2)$$

[where Ω denotes $\Omega_{11}\Omega_{22}-\Omega_{12}^2$].

If the asymptotic lines are real Ω will be negative: we therefore write this equation

$$\frac{\partial}{\partial x_1}\sqrt{-\Omega}\,(\theta Z^1 - Z\theta^1) + \frac{\partial}{\partial x_2}\sqrt{-\Omega}\,(\theta Z^2 - Z\theta^2) = 0. \quad (59.3)$$

We can then by quadrature find a vector ζ such that

$$\zeta_1 = \sqrt{-\Omega}\,(\theta Z^2 - Z\theta^2), \quad -\zeta_2 = \sqrt{-\Omega}\,(\theta Z^1 - Z\theta^1),$$

that is,

$$\zeta_1\sqrt{-\Omega} = Z(\Omega_{12}\theta_1 - \Omega_{11}\theta_2) - \theta\,(\Omega_{12}Z_1 - \Omega_{11}Z_2),$$
$$\zeta_2\sqrt{-\Omega} = Z(\Omega_{12}\theta_2 - \Omega_{22}\theta_1) - \theta\,(\Omega_{12}Z_2 - \Omega_{22}Z_1). \quad (59.4)$$

It should be noticed that to find ζ required a solution of the equation
$$\Delta_2\theta = p\theta. \qquad (59.5)$$

§ **60. Orthogonally corresponding surfaces.** We have

$$V(\Delta_2 Z)\,Z = 0,$$

and therefore

$$VZ\frac{\partial}{\partial x_1}(\sqrt{-\Omega}Z^1) + VZ\frac{\partial}{\partial x_2}(\sqrt{-\Omega}Z^2) = 0,$$

or, since
$$VZ_t Z^t = 0,$$

$$\frac{\partial}{\partial x_1}V\sqrt{-\Omega}ZZ^1 + \frac{\partial}{\partial x_2}V\sqrt{-\Omega}ZZ^2 = 0. \qquad (60.1)$$

We can therefore by quadrature find a vector z such that

$$z_1 = \sqrt{-\Omega}\,\widehat{ZZ^2}, \quad -z_2 = \sqrt{-\Omega}\,\widehat{ZZ^1},$$

that is,
$$z_1\sqrt{-\Omega} = VZ(\Omega_{12}Z_1 - \Omega_{11}Z_2),$$
$$z_2\sqrt{-\Omega} = VZ(\Omega_{22}Z_1 - \Omega_{12}Z_2). \qquad (60.2)$$

If the parametric lines are asymptotic these are just the equations we began with.

We see at once that

$$\underbrace{z_1\zeta_1} = 0, \quad \underbrace{z_1\zeta_2}+\underbrace{z_2\zeta_1} = 0, \quad \underbrace{z_2\zeta_2} = 0,$$

and therefore corresponding elements of the surfaces traced out by z and by ζ are connected by the equation

$$\underbrace{dz\,d\zeta} = 0, \tag{60.3}$$

that is, corresponding elements are perpendicular to one another. The surfaces are then said to correspond orthogonally to one another.

§ **61. Recapitulation.** We may now restate the results we have arrived at.

Consider the ground form

$$\Omega_{ik}\,dx_i\,dx_k$$

and let Δ_2 have reference to this form. Let Z be a vector which satisfies the equation

$$VZ(\Delta_2 Z) = 0.$$

Then $$z_1 = \sqrt{-\Omega}\,\widehat{ZZ^2}, \quad z_2 = -\sqrt{-\Omega}\,\widehat{ZZ^1},$$

define a surface traced out by a vector z.

On this surface the unit vector parallel to the normal at the extremity of z is given by

$$Z = c\lambda,$$

where $$c = (-K)^{-\frac{1}{4}},$$

and K is the measure of curvature of the surface z.

We have $$\Delta_2 Z = pZ,$$

where $$p = \frac{\Delta_2 c}{c} - 2\left(\frac{1}{R'} + \frac{1}{R''}\right).$$

The asymptotic lines on z are

$$\Omega_{ik}\,dx_i\,dx_k = 0.$$

The surfaces given by

$$\zeta_1 = \sqrt{-\Omega}\,(\theta Z^2 - Z\theta^2), \quad \zeta_2 = -\sqrt{-\Omega}\,(\theta Z^1 - Z\theta^1),$$

where θ is any scalar satisfying the equation

$$\Delta_2\theta = p\theta,$$

correspond orthogonally to the surface z.

§ **62. Relationship of surfaces z and ζ.** When the parametric lines are asymptotic on z, that is, when

$$\Omega_{11} = 0, \quad \Omega_{22} = 0,$$

$$\zeta_1 = \theta_1 Z - \theta Z_1, \quad \zeta_2 = \theta Z_2 - \theta_2 Z,$$

and therefore
$$\zeta_{12} = \frac{\theta_1}{\theta}\zeta_2 + \frac{\theta_2}{\theta}\zeta_1. \tag{62.1}$$

The parametric lines on ζ are now conjugate lines: for if we have an equation of the form

$$z_{12} = p z_1 + q z_2,$$

where p and q are any scalars,

$$\underset{\smile}{\lambda z}_{12} = 0.$$

If
$$p_2 = q_1$$
the conjugate lines have equal invariants in Laplace's sense. The parametric lines on ζ are therefore said to be conjugate lines with equal invariants. To the asymptotic lines on z there correspond therefore conjugate lines with equal invariants on ζ.

If on any surface ζ we are given the conjugate lines with equal invariants, we can find by mere quadrature a surface which will correspond orthogonally to ζ. For if

$$\zeta_{12} = \frac{\theta_1}{\theta}\zeta_2 + \frac{\theta_2}{\theta}\zeta_1,$$

then
$$\zeta_{12} + \frac{\phi_1}{\phi}\zeta_2 + \frac{\phi_2}{\phi}\zeta_1 = 0,$$

where
$$\theta\phi = 1;$$

and therefore
$$\frac{\partial}{\partial u}(\phi^2\zeta_2) + \frac{\partial}{\partial v}(\phi^2\zeta_1) = 0, \tag{62.2}$$

where ϕ^2 means the square of ϕ and is not a tensor notation.

We can therefore find by quadrature a vector Z such that

$$-\phi^2\zeta_1 = (\phi Z)_1, \quad \phi^2\zeta_2 = (\phi Z)_2;$$

that is,
$$\zeta_1 = \theta_1 Z - \theta Z_1, \quad \zeta_2 = \theta Z_2 - \theta_2 Z.$$

The surface given by

$$z_1 = \widehat{ZZ}_1, \quad z_2 = -\dot{\widehat{ZZ}}_2$$

will correspond orthogonally to ζ and will have the asymptotic lines as parametric lines.

We have now seen the relationship to one another of the surfaces z and ζ, and the method by which, given either, we are to obtain the other.

§ 63. Association of two other surfaces with a z-surface. Let a vector m be defined by the equation

$$\zeta = m\theta.$$

We have, taking as parametric lines the conjugate lines with equal invariants on ζ,

$$\theta_1 Z - \theta Z_1 = m\theta_1 + m_1\theta, \quad \theta Z_2 - \theta_2 Z = m\theta_2 + m_2\theta,$$

and therefore

$$\theta_1 (Z - m) = \theta (Z_1 + m_1), \quad \theta_2 (Z + m) = \theta (Z_2 - m_2). \tag{63.1}$$

From these equations we see that

$$\widehat{Z_{12}Z} = 0, \quad \widehat{m_{12}m} = 0,$$

and, as

$$V(Z-m)(Z_1+m_1) = 0, \quad V(Z+m)(Z_2-m_2) = 0,$$

$$\widehat{ZZ_1} - \widehat{mm_1} + \widehat{Zm_1} + \widehat{Z_1m} = 0,$$

$$\widehat{ZZ_2} - \widehat{mm_2} - \widehat{Zm_2} - \widehat{Z_2m} = 0. \tag{63.2}$$

We can take $\quad z_1 = \widehat{ZZ_1}, \quad z_2 = -\widehat{ZZ_2},$

$$y_1 = \widehat{mm_1}, \quad y_2 = -\widehat{mm_2},$$

and we have $\quad y_1 - z_1 = \dfrac{\partial}{\partial u}\widehat{Zm},$

$$y_2 - z_2 = \frac{\partial}{\partial v}\widehat{Zm}.$$

It follows that y only differs by a constant vector from

$$z + \widehat{Zm}.$$

We have thus obtained the surface y, where

$$y \equiv z + \widehat{Zm}, \tag{63.3}$$

directly from z and ζ, and the asymptotic lines on this surface correspond to the asymptotic lines on z.

§ **64.** We obtain yet another surface directly from the definition

$$\eta = Z\phi, \tag{64.1}$$

where $\theta\phi = 1$;

and we see that

$$\eta_1 = Z_1\phi + Z\phi_1 = \phi^2(Z_1\theta - Z\theta_1) = -\phi^2(m\theta_1 + m_1\theta),$$

so that

$$\eta_1 = m\phi_1 - m_1\phi. \tag{64.2}$$

Similarly we see that

$$\eta_2 = m_2\phi - m\phi_2. \tag{64.3}$$

The surface η will therefore correspond orthogonally to the surface y; and to the asymptotic lines on y will correspond on η conjugate lines with equal invariants.

We have thus four mutually related surfaces,

$$z, \quad \zeta, \quad y, \quad \eta,$$

which are intimately connected with two problems in the Theory of Surfaces, viz. the theory of the deformation of a surface, and a particular class of congruences of straight lines. The relations between the four surfaces will be more completely stated when eight other surfaces are introduced, as they will be when we consider the Deformation Theory.

CHAPTER V

DEFORMATION OF A SURFACE, AND CONGRUENCES

§ 65. Continuous deformation of a surface. We have seen that the problem of determining the surfaces in Euclidean space, to which a given ground form

$$a_{ik}dx_i dx_k$$

appertains, depends on the solution of the equations

$$\Omega_{11\cdot 2} = \Omega_{12\cdot 1}, \quad \Omega_{22\cdot 1} = \Omega_{12\cdot 2},$$
$$\Omega_{11}\Omega_{22} - \Omega_{12}^2 = Ka,$$

and we have pointed out the difficulty of solving these differential equations.

There is a related problem the solution of which is simpler. This problem is the determination of a surface differing infinitesimally from a given surface and applicable upon the given surface. Let z be the vector of the given surface, and $z+t\zeta$ the vector which describes the neighbouring surface which we are seeking, t being a small constant.

We may regard t as a small interval of time and ζ as a linear velocity vector, descriptive of the rate of increase of z, as we pass to the neighbouring surface which is applicable upon the given surface; or as the growth of the vector z under the condition of preserving unaltered the element of length.

If we can obtain ζ we have the vector which defines the continuous deformation of a surface.

We have at once

$$\underbrace{z_1\zeta_1} = 0\,; \quad \underbrace{z_1\zeta_2} + \underbrace{z_2\zeta_1} = 0\,; \quad \underbrace{z_2\zeta_2} = 0, \qquad (65.1)$$

that is, the vector ζ describes a surface corresponding orthogonally with the given surface described by z.

An interesting and immediately verifiable theorem on surfaces which correspond orthogonally is the following:

'If z and ζ correspond orthogonally, then the surfaces traced out by $z+\zeta$ and $z-\zeta$ are applicable on one another; and conversely, if z and ζ are the vectors of two surfaces applicable on one another, $z+\zeta$ and $z-\zeta$ will be the vectors of two surfaces which correspond orthogonally.'

§ **66. A vector of rotation.** From the kinematical relation of the vectors z and ζ, we see that $d\zeta$ is the relative velocity of the extremities of dz in the deformation of the surface z.

In the deformed surface the element which corresponds to dz will have the same length but will have turned through an angle. Let the rotation necessary to produce this be represented by the vector tr.

Now if a vector α, drawn from a point, is made to rotate with an angular velocity whose magnitude and direction is represented by a vector r, drawn through the same point, the linear velocity of the extremity of α will be given by $\overset{\frown}{r\alpha}$.

It therefore follows that $d\zeta = \overset{\frown}{rdz}$,

or
$$\zeta_1 = \overset{\frown}{rz_1}\,; \quad \zeta_2 = \overset{\frown}{rz_2}. \tag{66.1}$$

The vector r is parallel to the normal to the surface ζ, at the extremity of the vector ζ. We therefore have

$$r = a\overset{\frown}{\zeta_1\zeta_2},$$

where a is some scalar; and therefore

$$\zeta_1 = aVz_1\overset{\frown}{\zeta_2\zeta_1} = a\zeta_1\underset{\smile}{z_1\zeta_2},$$

since
$$\underset{\smile}{z_1\zeta_1} = 0\,; \quad \underset{\smile}{z_1\zeta_2}+\underset{\smile}{z_2\zeta_1} = 0\,; \quad \underset{\smile}{z_2\zeta_2} = 0,$$

so that
$$r = \frac{\overset{\frown}{\zeta_1\zeta_2}}{\underset{\smile}{z_1\zeta_2}}; \tag{66.2}$$

and thus r is uniquely obtained, when z and ζ are known.

§ **67. Geometrical relationship of surfaces traced out by certain vectors.** In exactly the same way we see that

$$z_1 = \overset{\frown}{\rho\zeta_1}; \quad z_2 = \overset{\frown}{\rho\zeta_2}, \tag{67.1}$$

where
$$\rho = \frac{\overset{\frown}{z_1 z_2}}{\underset{\smile}{\zeta_1 z_2}}. \tag{67.2}$$

By differentiation of the equations

$$\zeta_1 = \overset{\frown}{rz_1}, \quad \zeta_2 = \overset{\frown}{rz_2},$$

we see that
$$\overset{\frown}{r_2 z_1} = \overset{\frown}{r_1 z_2}; \tag{67.3}$$

and therefore the vectors r_1, r_2, z_1, z_2 are all parallel to the same plane. It follows that the normals to the surfaces traced out by z and r are parallel at corresponding points.

Similarly we see that the normals to the surfaces traced out by the vectors ρ and ζ are parallel at corresponding points.

But the vector r is parallel to the normal at the corresponding point of ζ: it is therefore parallel to the normal at the corresponding point of ρ.

From the equations
$$r = \frac{\overset{\frown}{\zeta_1\zeta_2}}{\underset{\smile}{z_1\zeta_2}}; \quad \rho = \frac{\overset{\frown}{z_1 z_2}}{\underset{\smile}{\zeta_1 z_2}},$$

we see that
$$\underset{\smile}{r\rho} = 1. \tag{67.4}$$

It follows that the r and ρ surfaces are polar reciprocals with respect to a sphere whose centre is at the origin and radius the square root of minus unity.

§ **68.** The angular velocity r is applied at the extremity of the vector z. Now an angular velocity r, at the extremity of the vector z, and an angular velocity $-r$ at the origin, are equivalent to a linear velocity $\overset{\frown}{zr}$.

It follows that a linear velocity ζ and an angular velocity r, at the extremity of z, are equivalent in effect to a linear velocity $\zeta + \overset{\frown}{zr}$, and an angular velocity r at the origin. We are thus led to consider two other vectors,

$$\zeta + \overset{\frown}{zr} \text{ and } z + \overset{\frown}{\zeta\rho}.$$

§ **69. A group of operators, and a system of twelve associated surfaces traced out by vectors.** The fundamental relations between the vectors z, ζ, r, ρ, are expressed by the equations

$$d\zeta = \overset{\frown}{r dz};\quad dz = \overset{\frown}{\rho d\zeta}. \tag{69.1}$$

These relations are unaltered by the transformation scheme in z, ζ, r, ρ,

$$z' = \zeta + \overset{\frown}{zr};\quad \zeta' = r;\quad r' = \frac{\rho}{\underset{\smile}{z\rho}};\quad \rho' = z, \tag{69.2}$$

which we shall denote by the operator A.

They are also unaltered by the transformation scheme

$$z' = \zeta;\quad \zeta' = z;\quad r' = \rho;\quad \rho' = r, \tag{69.3}$$

which we shall denote by the operator B.

We see that the operators A^2, A^3, A^4, A^5 are respectively the transformation schemes

$$z' = \frac{z+\overset{\frown}{\zeta\rho}}{\underset{\smile}{z\rho}};\quad \zeta' = \frac{\rho}{\underset{\smile}{z\rho}};\quad r' = \frac{z}{\underset{\smile}{z\zeta}};\quad \rho' = \zeta + \overset{\frown}{zr};$$

$$z' = \frac{\zeta}{\underset{\smile}{z\zeta}};\quad \zeta' = \frac{z}{\underset{\smile}{z\zeta}};\quad r' = \frac{\zeta+\overset{\frown}{zr}}{\underset{\smile}{\zeta r}};\quad \rho' = \frac{z+\overset{\frown}{\zeta\rho}}{\underset{\smile}{z\rho}};$$

$$z' = \frac{r}{\underset{\smile}{r\zeta}};\quad \zeta' = \frac{\zeta+\overset{\frown}{zr}}{\underset{\smile}{r\zeta}};\quad r' = z+\overset{\frown}{\zeta\rho};\quad \rho' = \frac{\zeta}{\underset{\smile}{z\zeta}};$$

$$z' = \rho;\quad \zeta' = z+\overset{\frown}{\zeta\rho};\quad r' = \zeta;\quad \rho' = \frac{r}{\underset{\smile}{r\zeta}}.$$

We see that

$$A^6 = 1;\quad B^2 = 1, \tag{69.4}$$

and $A^5B = BA$; $A^4B = BA^2$; $A^3B = BA^3$; $A^2B = BA^4$;

$$AB = BA^5,$$

and so the operators A and B form a group of order twelve.

The operators A form a sub-group of order six; the operators B form a sub-group of order two.

If we take $P \equiv A^3$; $Q \equiv BA$; $R \equiv A^2$

we have

$$P^2 = 1;\quad Q^2 = 1;\quad R^3 = 1, \tag{69.5}$$

$$PQ = QP;\quad PR = RP;\quad QR = R^2Q;\quad QR^2 = RQ,$$

and the operators P, Q, R will generate the same group. Of this group the operators P form one sub-group, the operators

Q another sub-group, and the operators P and Q together a sub-group of order four. The operators R form a sub-group of order three.

We thus obtain directly from the four vectors z, ζ, r, ρ a system of twelve vectors which trace out twelve surfaces connected in various ways at corresponding points.

§ 70. We may arrange the twelve surfaces in tabular form thus

$$\begin{array}{cccc}
z, & \zeta, & r, & \rho, \\[1ex]
\dfrac{z+\overset{\frown}{\zeta\rho}}{\underset{\smile}{z\rho}}, & \dfrac{\rho}{\underset{\smile}{z\rho}}, & \dfrac{z}{\underset{\smile}{z\zeta}}, & \zeta+\overset{\frown}{zr}, \\[2ex]
\dfrac{r}{\underset{\smile}{z\zeta}}, & \dfrac{\zeta+\overset{\frown}{zr}}{\underset{\smile}{r\zeta}}, & z+\overset{\frown}{\zeta\rho}, & \dfrac{\zeta}{\underset{\smile}{z\zeta}}, \\[2ex]
\dfrac{\zeta}{\underset{\smile}{z\zeta}}, & \dfrac{z}{\underset{\smile}{z\zeta}}, & \dfrac{\zeta+\overset{\frown}{zr}}{\underset{\smile}{\zeta r}}, & \dfrac{z+\overset{\frown}{\zeta\rho}}{\underset{\smile}{z\rho}}, \\[2ex]
\rho, & z+\overset{\frown}{\zeta\rho}, & \zeta, & \dfrac{r}{\underset{\smile}{r\zeta}}, \\[2ex]
\zeta+\overset{\frown}{zr}, & r, & \dfrac{\rho}{\underset{\smile}{z\rho}}, & z.
\end{array}$$

The first column will denote a vector of a surface; the second the vector of the surface which corresponds orthogonally to the surface in the first column and in the same row; the third column will denote the vector which gives the angular velocity corresponding to the surface in the same row but in the first column; the fourth will denote the angular velocity which corresponds to the surface in the same row but in the second column.

The vectors in the third column are parallel to the normals to the surfaces in the second column and in the same row; the vectors in the fourth column are parallel to the normals to the surfaces in the first column and in the same row. Finally the surfaces in the same rows and in the third and fourth columns respectively are reciprocal to one another.

§ 71. **The twelve surfaces form three classes of four.** Let us now recall what we proved about the four surfaces which we denoted in §§ 62–4 by z, ζ, y, η, and the equations of connexion when z is referred to its asymptotic lines.

We had

$$z_1 = \widehat{ZZ_1},\ z_2 = -\widehat{ZZ_2},\ \zeta_1 = \theta_1 Z - \theta Z_1,\ \zeta_2 = \theta Z_2 - \theta_2 Z,$$

$$\zeta = m\theta,\ Z = \eta\theta,\ y = z + \widehat{Zm}.$$

We see that Z is parallel to the normal at the extremity of z, and ρ is parallel to the same normal. Therefore

$$Z = p\rho, \tag{71.1}$$

where p is some scalar.

Now $$z_1 = \widehat{\rho\zeta_1} = \theta_1\widehat{\rho Z} - \theta\widehat{\rho Z_1} = -\theta\widehat{\rho Z_1},$$

but $$z_1 = \widehat{ZZ_1},$$

and therefore $$Z = -\theta\rho,$$

that is, $$\eta = -\rho. \tag{71.2}$$

Now $$y = z + \widehat{Zm} = z + \widehat{\eta\zeta},$$

and therefore $$y = z + \widehat{\zeta\rho}. \tag{71.3}$$

The four surfaces are therefore in the present notation (merely changing the sign of the vector η)

$$z,\quad \zeta,\quad z + \widehat{\zeta\rho},\quad \rho,$$

that is, $$z,\quad Bz,\quad BAz,\quad A^5 z,$$

or $$z,\quad PQRz,\quad Qz,\quad PRz.$$

Now the asymptotic lines correspond on two surfaces which are polar reciprocal to one another, since conjugate lines reciprocate into conjugate lines; and we know that the asymptotic lines correspond on

$$z \text{ and } z + \widehat{\zeta\rho}.$$

The asymptotic lines therefore correspond on

$$z,\quad \frac{\zeta}{\underset{\smile}{z\zeta}},\quad z + \widehat{\zeta\rho},\quad \frac{\rho}{\underset{\smile}{z\rho}};$$

that is, on $$z,\quad Pz,\quad Qz,\quad PQz.$$

The surfaces which correspond to these orthogonally are respectively

$$\zeta, \quad \frac{z}{\undergroup{z\zeta}}, \quad \rho, \quad \frac{z+\overgroup{\zeta\rho}}{\undergroup{z\rho}},$$

that is, $\quad PQRz, \quad QRz, \quad PRz, \quad Rz.$

On these surfaces there correspond to the asymptotic lines conjugate lines with equal invariants. We will say conjugate lines with equal point invariants.

The surfaces which are respectively reciprocal to these four are

$$\frac{r}{\undergroup{r\zeta}}, \quad \zeta+\overgroup{zr}, \quad r, \quad \frac{\zeta+\overgroup{rz}}{\undergroup{\zeta r}},$$

that is, $\quad R^2z, \quad PR^2z, \quad QR^2z, \quad PQR^2z.$

We say that on these surfaces there correspond, to the conjugate lines with equal invariants on their reciprocals, conjugate lines with equal tangential invariants.

The twelve surfaces thus fall into three classes: viz. those on which the asymptotic lines correspond; those on which conjugate lines with equal invariants correspond; those on which conjugate lines with equal tangential invariants correspond. The surfaces of any class are permuted amongst themselves by the operations of the sub-group

$$1; \quad P; \quad Q; \quad PQ.$$

§ **72. A case in which one surface is minimal.** If the vector z is of constant length we can prove that the surface

$$\zeta+\overgroup{zr} \tag{72.1}$$

is a minimal surface.

We saw that the normals at corresponding points of z and of r were parallel. If then z is of constant length, the vector z is parallel to its own normal and therefore equal to $k\lambda$, where k is a constant, and λ is the unit vector parallel to the normal at the extremity of r. But

$$\undergroup{\lambda_1 r_2} = \undergroup{\lambda_2 r_1}$$

and therefore

$$\undergroup{z_1 r_2} = \undergroup{z_2 r_1}.$$

We saw (52.8) that the condition that z might be a minimal surface was $Sz_1\lambda\lambda_2 = Sz_2\lambda\lambda_1$,

and clearly this condition will remain the same if we replace λ by any vector parallel to it.

Let
$$y \equiv \zeta + \widehat{zr}.$$

We see by the table that z is parallel to the normal at the extremity of y. The condition that y may be a minimal surface is then
$$Sy_1zz_2 = Sy_2zz_1. \qquad (72.2)$$

But by the fundamental formula of connexion and by the table we see that
$$y_1 = \widehat{zr_1}, \quad y_2 = \widehat{zr_2}.$$

The surface y will therefore be a minimal surface if
$$S\widehat{zr_1}\widehat{zz_2} = S\widehat{zr_2}\widehat{zz_1},$$

that is, if
$$\undergroup{zz_2}\,\undergroup{zr_1} - z^2\undergroup{r_1z_2} = \undergroup{zz_1}\,\undergroup{zr_2} - z^2\undergroup{r_2z_1}.$$

Now z being of constant length this condition becomes
$$\undergroup{r_1z_2} = \undergroup{r_2z_1}:$$

and this we have seen is true.

This theorem will be used in proving an interesting theorem of Ribaucour's in connexion with a particular class of congruences.

We now proceed to consider the theory of congruences of straight lines in connexion with which the twelve surfaces will be of interest.

§ 73. **Congruences of straight lines.** If we wish to consider not merely the geometry on one particular surface but the relation of points on that surface to corresponding points on another surface, we are led naturally to consider the congruence of straight lines which join the corresponding points.

Let z be a vector depending on two parameters u and v, and μ a unit vector depending on the same two parameters, and drawn through the extremity of z. Let w be a length taken along the vector μ; the congruence will then be defined by
$$z' = z + w\mu. \qquad (73.1)$$

We regard z and μ as functions of the parameters u and v, and therefore the current vector z' will be a function of the three parameters u, v, and w.

The unit vector μ will trace out a sphere which we call the spherical image of the congruence.

Let
$$\underbrace{\mu_i\mu_k} = -a_{ik},$$
so that
$$d\sigma^2 = a_{11}du^2 + 2a_{12}dudv + a_{22}dv^2 \qquad (73.2)$$
is the ground form of the spherical image.

Let
$$\underbrace{\mu_i z_k} \equiv \omega_{ik},$$
$$h^2 \equiv a_{11}a_{22} - a_{12}^2;$$
and notice that in general
$$\omega_{ik} \neq \omega_{ki}.$$

If we take two neighbouring rays of the congruence we have
$$dz' = dz + wd\mu + \mu dw.$$

If λ is a unit vector perpendicular to μ and $d\mu$,
$$d\sigma\lambda = \overset{\frown}{\mu d\mu}, \quad \overset{\frown}{\mu_1\mu_2} = h\mu;$$
and therefore
$$\begin{aligned} hd\sigma\lambda &= V\overset{\frown}{\mu_1\mu_2}d\mu, \\ &= \mu_1\underbrace{\mu_2 d\mu} - \mu_2\underbrace{\mu_1 d\mu}, \\ &= \mu_2(a_{11}du + a_{12}dv) - \mu_1(a_{12}du + a_{22}dv). \end{aligned}$$

It follows that
$$\begin{aligned} hd\sigma\underbrace{\lambda dz} = (\omega_{21}du + \omega_{22}dv)(a_{11}du + a_{12}dv) \\ -(\omega_{11}du + \omega_{12}dv)(a_{12}du + a_{22}dv). \end{aligned}$$

But, if δ is the shortest distance between the two neighbouring lines,
$$\delta = -\underbrace{\lambda dz},$$
and therefore
$$hd\sigma\delta = \begin{vmatrix} \omega_{11}du + \omega_{12}dv, & \omega_{21}du + \omega_{22}dv \\ a_{11}du + a_{12}dv, & a_{12}du + a_{22}dv \end{vmatrix}. \qquad (73.3)$$

§ 74. **Focal planes and focal points of a ray.** The value of w which corresponds to the shortest distance between two neighbouring rays is given by the fact that dz' is perpendicular to μ and $\mu+d\mu$; and therefore

$$\underset{\smile}{dz'd\mu} = 0.$$

We thus have $$\underset{\smile}{dzd\mu} + wd\mu^2 = 0,$$

so that $$w = \frac{\omega_{11}du^2 + (\omega_{12}+\omega_{21})\,du\,dv + \omega_{22}dv^2}{a_{11}du^2 + 2a_{12}du\,dv + a_{22}dv^2}. \qquad (74.1)$$

The critical values of w, say w' and w'', as we vary the ratio $du:dv$, are therefore given by

$$\begin{vmatrix} \omega_{11} - wa_{11}, & \tfrac{1}{2}(\omega_{12}+\omega_{21}) - wa_{12} \\ \tfrac{1}{2}(\omega_{12}+\omega_{21}) - wa_{12}, & \omega_{22} - wa_{22} \end{vmatrix} = 0, \quad (74.2)$$

and the corresponding values of the ratio $du:dv$ are given by

$$\begin{vmatrix} \omega_{11}du + \tfrac{1}{2}(\omega_{12}+\omega_{21})\,dv, & \tfrac{1}{2}(\omega_{12}+\omega_{21})\,du + \omega_{22}dv \\ a_{11}du + a_{12}dv, & a_{12}du + a_{22}dv \end{vmatrix} = 0. \quad (74.3)$$

There are, by (73.3), two values of the ratio $du:dv$ which make $\delta = 0$. Through each ray of the congruence there thus pass two developable surfaces defined by

$$\begin{vmatrix} \omega_{11}du + \omega_{12}dv, & \omega_{21}du + \omega_{22}dv \\ a_{11}du + a_{12}dv, & a_{12}du + a_{22}dv \end{vmatrix} = 0. \qquad (74.4)$$

The planes which pass through this ray and touch the developables are called the *focal planes* of the ray. The points where the ray is intersected by these neighbouring rays are called the *focal points* of the ray.

The developables are defined by

$$\omega_{11}du + \omega_{12}dv = p\,(a_{11}du + a_{12}dv),$$
$$\omega_{21}du + \omega_{22}dv = p\,(a_{12}du + a_{22}dv),$$

where p is some multiplier; and we see that this multiplier is w. The focal distances, f' and f'', are therefore the values of w which satisfy the equation

$$\begin{vmatrix} \omega_{11} - wa_{11}, & \omega_{12} - wa_{12} \\ \omega_{21} - wa_{12}, & \omega_{22} - wa_{22} \end{vmatrix} = 0. \qquad (74.5)$$

§ 75. Limiting points. The Hamiltonian equation. Principal planes. If we have any two real quadratic forms

$$a_{11}du^2 + 2a_{12}du\,dv + a_{22}dv^2,$$
$$b_{11}du^2 + 2b_{12}du\,dv + b_{22}dv^2,$$

we can, by a real transformation, bring them to such a form that in the new variables

$$a_{12} = b_{12} = 0.$$

It is therefore possible by a real transformation to make

$$\omega_{12} + \omega_{21} = 0, \quad a_{12} = 0. \tag{75.1}$$

The points on the ray given by w', w'' are called the limiting points of the ray. These points are therefore real.

If we suppose the transformation applied which makes

$$\omega_{12} + \omega_{21} = 0, \quad a_{12} = 0,$$

we have $\qquad \omega_{11} = w'a_{11}, \quad \omega_{22} = w''a_{22};$

and the value of w which corresponds to the shortest distance between two neighbouring rays is given by

$$w = \frac{w'a_{11}du^2 + w''a_{22}dv^2}{a_{11}du^2 + a_{22}dv^2}.$$

We may take

$$\cos^2\theta = \frac{a_{11}du^2}{a_{11}du^2 + a_{22}dv^2}, \quad \sin^2\theta = \frac{a_{22}dv^2}{a_{11}du^2 + a_{22}dv^2},$$

and we have the Hamiltonian equation

$$w = w'\cos^2\theta + w''\sin^2\theta, \tag{75.2}$$

showing that the shortest distance between any two neighbouring rays lies between the two limiting points.

The values of the ratio $du:dv$ which correspond to the limiting points are given by

$$(w' - w'')\,du\,dv = 0.$$

Leaving aside the special congruence when the limiting points may coincide, we see that corresponding to the limiting point w', du is zero, and the shortest distance is parallel to $\widehat{\mu\mu_2}$. Similarly the shortest distance corresponding to w'' is parallel to $\mu\mu_1$; and these shortest distances are perpendicular to one another since

$$S\widehat{\mu\mu_1}\widehat{\mu\mu_2} = \underbrace{\mu\mu_2}\underbrace{\mu\mu_1} - \mu^2 \underbrace{\mu_1\mu_2} = 0. \qquad (75\,.\,3)$$

The planes through the ray μ which are perpendicular to these shortest distances are called the *principal planes* of the ray: and they are perpendicular to one another.

§ 76. **Principal surfaces, and the central surface.** Returning now to general coordinates we see that

$$f' + f'' = w' + w'',$$

$$4h^2(f'f'' - w'w'') = (\omega_{12} - \omega_{21})^2,$$

and therefore, in the important class of congruences for which $\omega_{12} = \omega_{21}$, the limiting points and the focal points coincide. We see also that the focal planes will then coincide with the principal planes.

When we take any equation connecting the parameters u and v of the congruence we obtain a ruled surface of the congruence. The directrices of the ruled surface will be curves lying on the surface z. If u and v are functions of a variable p, then p and w will be the coordinates of the ruled surface. The lines of striction on the ruled surface will be given by

$$w = \frac{\omega_{11}du^2 + (\omega_{12} + \omega_{21})\,du\,dv + \omega_{22}dv^2}{a_{11}du^2 + 2a_{12}du\,dv + a_{22}dv^2}, \qquad (76\,.\,1)$$

where u and v are connected by the equation which defines the ruled surface.

The ruled surfaces given by

$$\begin{vmatrix} \omega_{11}du + \frac{1}{2}(\omega_{12} + \omega_{21})\,dv, & \frac{1}{2}(\omega_{12} + \omega_{21})\,du + \omega_{22}dv \\ a_{11}du + a_{12}dv, & a_{12}du + a_{22}dv \end{vmatrix} = 0 \qquad (76\,.\,2)$$

are called the principal surfaces of the congruence.

The locus of the points on rays midway between the foci, and therefore midway between the limiting points, is called the *central surface* of the congruence.

§ 77. **The focal surface.** Any ray of the congruence will be intersected by a neighbouring ray if

$$dz + w d\mu + \mu dw = 0.$$

The developables which pass through the ray are therefore given by

$$S dz d\mu \mu = 0;$$

that is, by $S(z_1 du + z_2 dv)(\mu_1 du + \mu_2 dv)\mu = 0.$

The focal points are given by

$$(z_1 + w\mu_1) du + (z_2 + w\mu_2) dv + \mu dw = 0;$$

that is, by $S(z_1 + w\mu_1)(z_2 + w\mu_2)\mu = 0.$

The focal surface of the congruence is defined as the locus of the focal points on the rays of the congruence. If we so choose the parameters that the equation defining the developables is

$$du dv = 0,$$

then $S z_1 \mu_1 \mu = 0, \quad S z_2 \mu_2 \mu = 0;$

so that $z_1 = a\mu_1 + b\mu, \quad z_2 = c\mu_2 + d\mu,$

where a, b, c, d are scalars.

Substituting in the equation

$$S(z_1 + w\mu_1)(z_2 + w\mu_2)\mu = 0,$$

which defines the focal points, we see that the focal surface has two sheets given by

$$z' = z - a\mu, \quad z' = z - c\mu.$$

§ 78. **Rays touch both sheets of the focal surface. The congruence of rays of light.** For the first sheet

$$z'_1 = (b - a_1)\mu, \quad z'_2 = (c - a)\mu_2 + (d - a_2)\mu,$$

$$z'_{12} = (b - a_1)\mu_2 + (b_2 - a_{12})\mu,$$

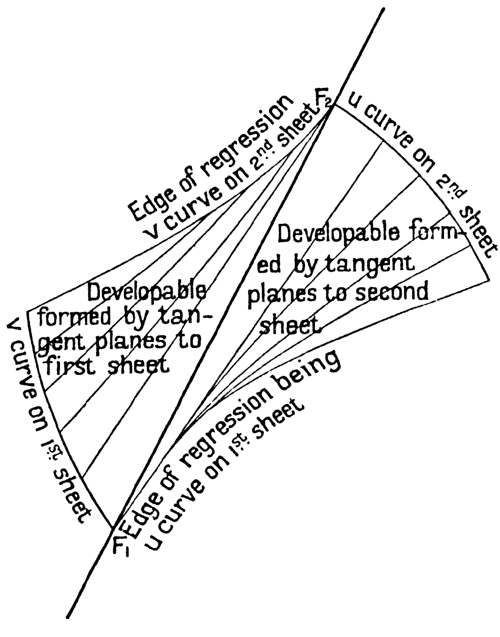
so that the normal to the first sheet is parallel to $\widehat{\mu\mu_2}$; and the ray touches the first sheet along the u curve on it—that is, the curve along which only u varies; and the v curve is conjugate to the u curve.

Similarly we see that the ray touches the second sheet along the v curve on it, and the u curve on it is conjugate to this.

Thus any ray of the congruence touches both sheets of the focal surface; and the tangent planes to the focal surface at the two points of contact are the tangent planes to the developables through the ray.

The edges of regression of the developables are the u curves on the first sheet, and the v curves on the second sheet.

If the congruence is formed by rays of light, the focal points on the ray are the foci as defined in the theory of thin



pencils. F_1 and F_2 are the foci on what is called the principal ray of the thin pencil. The tangent plane at F_2 to the second sheet, which is the tangent plane at F_1 to the developable, is called the first focal plane: so the tangent plane at F_1 to the first sheet, which is the tangent plane at F_2 to the other developable, is called the second focal plane.

The developables through any ray are somewhat like the above figure.

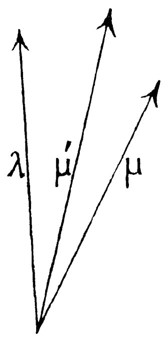
The focal lines as defined in some text-books on Geometrical

Optics have no meaning at all; but it has been pointed out that the lines conjugate to the principal ray on each sheet have a physical meaning which might entitle them to the name of focal lines.*

§ 79. Refraction of a congruence. Malus's theorem. A congruence is given in terms of the coefficients a_{ik} of its spherical image and of the coefficients

$$\omega_{11},\ \omega_{12},\ \omega_{21},\ \omega_{22}.$$

We can see how the congruence, when we regard it as formed by rays of light, is altered by refraction at any surface z, whose normal is parallel to the unit vector λ.



Let μ' be the unit vector into which μ is refracted: that is, let μ' trace out the new spherical image.

We have $\mu' = a\mu + b\lambda$, where a and b are scalars. In the ordinary notation of optics, if ϕ is the angle of incidence, ϕ' the angle of refraction, and k the index of refraction,

$$k \sin \phi' = \sin \phi.$$

Now $\widehat{\lambda\mu'} = a\widehat{\lambda\mu}, \quad \widehat{\mu'\mu} = b\widehat{\lambda\mu};$

and therefore

$$a \sin \phi = \sin \phi', \quad b \sin \phi = \sin (\phi - \phi'). \tag{79.1}$$

We thus see that a is a constant independent of the angle ϕ, but b depends on ϕ. We have

$$a^2 + b^2 - 2ab\underset{\smile}{\lambda\mu} = 1,$$

$$\underset{\smile}{\lambda\mu} + \cos \phi = 0.$$

Since $\qquad \mu_i' = a\mu_i + b\lambda_i + b_i\lambda,$

we have $\qquad \omega_{ik}' = a\omega_{ik} + b\Omega_{ik},$

where Ω_{ik} refers with its usual meaning to the surface of refraction.

* [Probably the allusion is to a note 'On focal lines of congruences of rays': Elliott, *Messenger of Mathematics*, xxxix, p. 1.]

We see that $$\cos\phi' = a\cos\phi + b,$$
and therefore if we multiply
$$\mu'_i = a\mu_i + b\lambda_i + b_i\lambda$$
by μ', that is, by $$a\mu + b\lambda,$$
and take the scalar product, we get, since $\underset{\smile}{\mu'\mu'_i}$ is zero,
$$ab\,(\underset{\smile}{\lambda\mu_i} + \underset{\smile}{\lambda_i\mu}) = b_i\cos\phi'. \tag{79.2}$$

We notice that if $$\omega_{12} = \omega_{21}$$
then $$\omega'_{12} = \omega'_{21}.$$

We shall see (83.2) that the condition
$$\omega_{12} = \omega_{21} \tag{79.3}$$
means that the rays of the congruence are normal to a system of surfaces and we now see that this property is unaltered by refraction. This is Malus's theorem.

We have now given the equations which would determine any refracted congruence, when we are given the refracting surface. Unfortunately the equations are complicated.

§ **80. The Ribaucourian congruence.** We shall now consider some special classes of congruences.

Consider the congruence formed by rays drawn from every point of a surface, parallel to the normal at the corresponding point of a surface which corresponds orthogonally to the given surface. This is the Ribaucourian congruence, so called as Ribaucour was the first to consider it.

We take ζ to be the surface from which the rays are drawn parallel to the normals to the surface z.

Taking the asymptotic lines on z as the parametric lines we had $$\zeta_1 = \theta_1 Z - \theta Z_1, \quad \zeta_2 = \theta Z_2 - \theta_2 Z,$$
and $$Z = c\lambda,$$
where $$c = (-K)^{-\frac{1}{4}},$$
K being the measure of curvature on z.

To bring this into accordance with our notation for congruences we write μ for λ, and we have
$$\zeta_1 = (\theta_1 c - \theta c_1)\,\mu - \theta c\mu_1, \quad \zeta_2 = (\theta c_2 - \theta_2 c)\,\mu + \theta c\mu_2.$$

Since $$S\zeta_1\mu_1\mu = 0 \text{ and } S\zeta_2\mu_2\mu = 0,$$
the equation which defines the developables is
$$du\,dv = 0\,;$$
and the focal points are given by
$$w = c\theta, \quad w = -c\theta.$$

The surface ζ is then the central surface of the congruence, and the developables intersect it in conjugate lines with equal invariants. These lines correspond to the asymptotic lines on z, the surface which corresponds orthogonally to the central surface.

§ 81. **The Isotropic congruence. Ribaucour's theorem.** We have a particular, and most interesting, case of this congruence, when the surface which corresponds orthogonally with ζ is a sphere with the origin as centre.

In this case c is a constant and ζ corresponds orthogonally with μ itself.

The congruence is $$z' = \zeta + w\mu$$
and is called the isotropic congruence.

For the isotropic congruence,
$$\omega_{11} = 0, \quad \omega_{12} + \omega_{21} = 0, \quad \omega_{22} = 0, \tag{81.1}$$
and therefore the limiting points of any ray coincide and are on the central surface. Any plane through a ray is a principal plane and any surface may be regarded as a principal surface. The lines of striction of all the ruled surfaces of the congruence lie on the central surface.

In the chapter on the ruled surface [see § 108] we prove that any two ruled surfaces of the congruence intersect at the same angle all along their common generator.

The developables and the focal points we see are imaginary.

We have proved that $$y \equiv \zeta + \widehat{\mu r}$$
is a minimal surface and that μ is the unit vector parallel to the normal at the extremity of y. The perpendicular p on the tangent plane to this surface is given by
$$p + \underset{\smile}{y\mu} = 0,$$
that is, by $$p + \underset{\smile}{\mu\zeta} = 0.$$

The tangent plane is therefore the plane drawn through the extremity of ζ perpendicular to the ray of the congruence. We thus have Ribaucour's theorem that 'The envelope of the plane, drawn through the extremity of the vector which traces out the central surface, perpendicular to the corresponding ray of an isotropic congruence, is a minimal surface'.

The surface corresponding orthogonally to the sphere is therefore the pedal of a minimal surface.

If two surfaces are applicable on one another, and if the distance between corresponding points is constant, we see that the line joining these points traces out an isotropic congruence. For if μ is the unit vector parallel to the join of the points, and z is the vector to the middle point of the join, and $2c$ is the length of the joining line,

$$(z_1+c\mu_1)^2 = (z_1-c\mu_1)^2\,; \quad (z_2+c\mu_2)^2 = (z_2-c\mu_2)^2,$$

$$S\,(z_1+c\mu_1)\,(z_2+c\mu_2) = S\,(z_1-c\mu_1)\,(z_2-c\mu_2),$$

from which equations we at once deduce the result stated.

§ 82. **W congruences.** Let us now consider again the two surfaces which we denoted by z and $z+\widehat{\zeta\rho}$, and consider the congruence formed by the line joining corresponding points on these surfaces. Looking at the tabular arrangement of the twelve surfaces we see that ρ is parallel to the normal to z at the corresponding point, and that ζ is parallel to the normal to $z+\widehat{\zeta\rho}$ at the corresponding point. The line joining corresponding points on the two surfaces z and $z+\widehat{\zeta\rho}$, being perpendicular to both ρ and ζ, is perpendicular to the normals to z and to $z+\widehat{\zeta\rho}$, and therefore touches each of these surfaces.

Now if a ray of a congruence touches a surface, that surface must be a focal surface of the congruence. For, taking z to be the vector to the surface, and μ the unit vector parallel to the ray,

$$S\mu z_1 z_2 = 0\,;$$

and therefore, the focal points being given by

$$S\,(z_1+w\mu_1)\,(z_2+w\mu_2)\,\mu = 0,$$

we see that one of the focal surfaces is given by $w=0$.

It follows that z and $z+\widehat{\zeta\rho}$ are the focal surfaces of the congruence we are considering.

Now on these surfaces the asymptotic lines correspond. Conversely it may be shown that if the asymptotic lines correspond on the two sheets of the focal surface the focal surfaces are z and $z+\widehat{\zeta\rho}$.

Congruences of this type may be called W congruences.

§ **83. Congruence of normals to a surface.** We now come to the case of congruences where the rays are normal to a surface. The theory of such congruences is of special interest in geometrical optics as well as in geometry.

Instead of μ we shall write λ, where λ is the unit vector normal to the surface from which the rays emanate.

We now have
$$\underbrace{z_1\lambda_2} = \underbrace{z_2\lambda_1} \tag{83.1}$$
as a necessary condition that the congruence may be a normal one.

This necessary condition is also sufficient: for if
$$\underbrace{\mu_1 z_2} = \underbrace{\mu_2 z_1}$$
then
$$\frac{\partial}{\partial u}\underbrace{\mu z_2} = \frac{\partial}{\partial v}\underbrace{\mu z_1},$$
and we can therefore determine a function w such that
$$w_1 = \underbrace{\mu z_1}; \quad w_2 = \underbrace{\mu z_2}.$$

Let
$$z' = z + w\mu,$$
then
$$\underbrace{z'_1\mu} = \underbrace{z_1\mu} + w_1\mu^2 = 0,$$
$$\underbrace{z'_2\mu} = \underbrace{z_2\mu} + w_2\mu^2 = 0,$$
so that the rays are normal to the surface z'.

The normal congruence is therefore defined by
$$\omega_{21} = \omega_{12}, \tag{83.2}$$
and the limiting points coincide with the focal points, and the focal planes with the principal planes. The focal planes are therefore perpendicular to one another.

Conversely if the focal planes are perpendicular to one another the congruence is a normal one: for we see that the condition that the focal planes may be perpendicular is

$$(\omega_{12}-\omega_{21})(a_{11}a_{22}-a_{12}^2) = 0,$$

and therefore, since $a_{11}a_{22}-a_{12}^2$ is not zero,

$$\omega_{12} = \omega_{21}.$$

§ **84. Reference to lines of curvature.** We now take the parametric lines on the surface z to be the lines of curvature, and we have $\quad z_1 = -R'\lambda_1, \quad z_2 = -R''\lambda_2,$

where R' and R'' are the principal radii of curvature.

We have

$$\omega_{11} = -R'\lambda_1^2, \quad \omega_{12} = \omega_{21} = 0, \quad \omega_{22} = -R''\lambda_2^2,$$

that is, $\omega_{11} = R'a_{11}, \quad \omega_{22} = R''a_{22}, \quad \omega_{12} = \omega_{21} = a_{12} = 0.$

The focal points are given by

$$f' = R', \quad f'' = R'',$$

and the two focal surfaces are now given by

$$z' = z+R'\lambda, \quad z' = z+R''\lambda.$$

The equation of the developables is

$$(R'-R'')\,du\,dv = 0.$$

As we need not consider the case where $R' = R''$ any further than we have already done we see that the equations of the developables are

$$du = 0, \quad dv = 0. \tag{84.1}$$

For the focal surfaces we have

$$dz' = -(R''-R')\lambda_2 dv+\lambda dR'. \tag{84.2}$$

Calling this the first sheet of the focal surface, its ground form is

$$(dR')^2+(R''-R')^2 a_{22}dv^2, \tag{84.3}$$

and therefore the u curve is a geodesic on the first sheet. Similarly we see that the v curve is a geodesic on the second sheet.

§ **85. Tangents to a system of geodesics.** Conversely if we take any surface, and draw any singly infinite system of geodesics on it, the tangents to these geodesics will generate a normal congruence.

For take a surface with the ground form

$$du^2 + B^2 dv^2,$$

and consider the congruence formed by the tangents to the curves $v =$ constant, that is, by the tangents to this family of geodesics. We have $\mu = z_1$ and as

$$z_1^2 = -1, \quad \underbrace{z_1 z_2} = 0,$$

we must have $\underbrace{z_1 z_{12}} = 0, \quad \underbrace{z_{11} z_2} + \underbrace{z_1 z_{12}} = 0,$

so that $$\underbrace{z_2 z_{11}} = 0.$$

Now $$\underbrace{z_2 \mu_1} = \underbrace{z_2 z_{11}} = 0.$$

and $$\underbrace{z_1 \mu_2} = \underbrace{z_1 z_{12}} = 0,$$

so that $$\underbrace{z_2 \mu_1} = \underbrace{z_1 \mu_2},$$

and the congruence is a normal one.

§ 86. **Connexion of W congruences which are normal with W surfaces.** Now let us consider the asymptotic lines on the two sheets of the focal surface.

The vector to the first sheet is

$$z' = z + R'\lambda;$$

and we have

$$z'_1 = R'_1 \lambda, \quad z'_2 = -(R'' - R')\lambda_2 + R'_2 \lambda,$$

and therefore $(R' - R'')\lambda_{12} = R''_1 \lambda_2 - R'_2 \lambda_1.$

The equation of the asymptotic lines is

$$\underbrace{dz' dV} = 0,$$

if V is the unit vector parallel to the normal at the extremity of z'.

Now λ_1 is parallel to V, and therefore the equation of the asymptotic lines is $\underbrace{dz' d\lambda_1} = 0;$

that is, $S\left((R'' - R')\lambda_2 dv - \lambda\, dR'\right)(\lambda_{11} du + \lambda_{12} dv) = 0.$

We have, since $\underbrace{\lambda_1 \lambda_2}$ is zero,

$$\underbrace{\lambda_{11} \lambda_2} = -\underbrace{\lambda_1 \lambda_{12}} = R_2' \lambda_1^2 \div (R' - R''),$$

$$\underbrace{\lambda \lambda_{11}} = -\lambda_1^2, \quad \underbrace{\lambda \lambda_{12}} = 0,$$

$$\underbrace{\lambda_2 \lambda_{12}} = R_1'' \lambda_2^2 \div (R' - R''),$$

and therefore the equation of the asymptotic lines on the first sheet is

$$\lambda_1^2 R'_1 du^2 - \lambda_2^2 R''_1 dv^2 = 0. \qquad (86.1)$$

Similarly we see that the asymptotic lines on the second sheet of the focal surface are given by

$$\lambda_1^2 R'_2 du^2 - \lambda_2^2 R''_2 dv^2 = 0. \qquad (86.2)$$

The necessary and sufficient condition that the asymptotic lines on the two sheets may correspond is therefore that R' and R'' may be functionally connected.

We thus have the theorem that in a *W* congruence, if it is also a normal one, the surfaces which intersect the rays orthogonally have their radii of curvature functionally connected: that is, they are *W* surfaces.

§ **87. Surfaces applicable to surfaces of revolution, and *W* normal congruences.** We saw (§ 84) that the ground form of the first sheet of the focal surface of a normal congruence was

$$(dR')^2 + (R''-R')^2 a_{22} dv^2, \qquad (87.1)$$

and similarly we see that the ground form for the second sheet is

$$(dR'')^2 + (R''-R')^2 a_{11} du^2. \qquad (87.2)$$

If the congruence is also a *W* congruence we know that

$$(R''-R')^2 a_{22} = (\phi(R'))^2,$$
$$(R''-R')^2 a_{11} = (\phi'(R'))^{-2};$$

the ground forms of the first and second sheet are then respectively

$$(dR')^2 + (\phi(R'))^2 dv^2, \qquad (87.3)$$
$$(dR'')^2 + (\phi'(R'))^{-2} du^2. \qquad (87.4)$$

The two sheets are therefore applicable on surfaces of revolution, the u curves on the first sheet corresponding to the meridians, and the v curves on the second sheet.

Conversely, if we have any surface applicable on a surface of revolution, the curves which correspond to the meridians will be geodesics, and the tangents to these curves will therefore trace out a normal congruence which will be a *W* congruence; and the surfaces which cut the rays orthogonally will be *W* surfaces.

If the surface is one of constant curvature we need to solve an equation of Riccati's form to obtain the curves which correspond to the meridians, but in other cases we can find the curves by quadrature.

An interesting property of any given W surface, which is not of constant curvature, is that we can find the lines of curvature on it by quadrature.

For we can find the two sheets of the focal surface, and on these sheets we can find by quadrature the curves which correspond to the meridians. These curves will have as their correspondents on the given W surface the lines of curvature. This theorem was discovered by Lie.

§ 88. **Surfaces of constant negative curvature.** Returning to the ground forms of the two sheets of the focal surface

$$(dR')^2 + (\phi(R'))^2 dv^2,$$
$$(dR'')^2 + (\phi'(R'))^{-2} du^2,$$

we see by aid of the formula

$$K + \frac{B_{11}}{B} = 0,$$

when the ground form is $du^2 + B^2 dv^2$, that, K' denoting the measure of curvature on the first sheet,

$$K' + \frac{\phi''(R')}{\phi(R')} = 0. \qquad (88.1)$$

Similarly we find for the measure of curvature K'' of the second sheet $K'' + \{\phi'(R')\}^4 \div [(\phi(R'))^3 \phi''(R')] = 0$ (88.2)

since $\qquad R'' = f(R'), \quad R' - f(R') = \dfrac{\phi(R')}{\phi'(R')}.$

If the two sheets are applicable on one another at corresponding points we must have $K' = K''$ and therefore we must have

$$\phi''(R')\,\phi(R') = \pm(\phi'(R'))^2. \qquad (88.3)$$

Taking the upper sign we see that

$$\phi(R') = be^{-\frac{R'}{a}},$$

where a and b are constants.

We now see that $$R'' - R' = a, \tag{88.4}$$

from the equations $$R' - f(R') = \frac{\phi(R')}{\phi'(R')},$$
$$R'' = f(R').$$

The measure of curvature is found to be $-a^{-2}$ from the formula
$$K' + \frac{\phi''(R')}{\phi(R')} = 0. \tag{88.5}$$

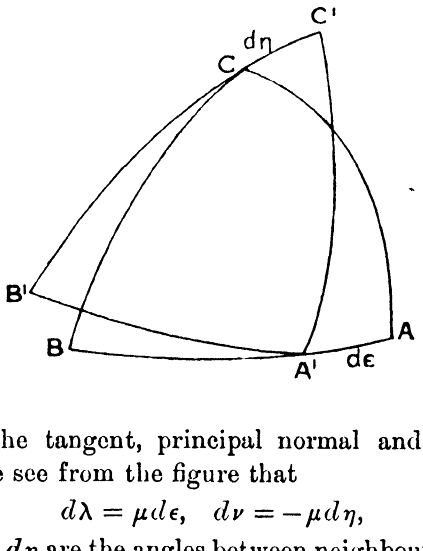
The two sheets have then the same constant negative measure of curvature $-a^{-2}$, and the distance between the corresponding points is equal to the constant a.

We therefore see how, when we are given a surface of constant negative curvature, we can construct another surface of the same constant curvature. We find a system of geodesics on the given surface—this involves the solution of an equation of Riccati's form—and draw the tangents and take a constant distance a along the tangent: the locus of the point so obtained will be the surface required.

CHAPTER VI

CURVES IN EUCLIDEAN SPACE AND ON A SURFACE. MOVING AXES

§ 89. **Serret's formulae. Rotation functions.** Let λ, μ, ν be three unit vectors drawn through the origin, respectively



parallel to the tangent, principal normal and binormal of a curve. We see from the figure that

$$d\lambda = \mu\, d\epsilon, \quad d\nu = -\mu\, d\eta,$$

where $d\epsilon$ and $d\eta$ are the angles between neighbouring positions of the tangents and osculating planes respectively—in the sense of the figure.

We thus have

$$\dot{\lambda} = \frac{\mu}{\rho}, \quad \dot{\nu} = -\frac{\mu}{\sigma},$$

where the dot denotes differentiation with respect to the arc of the curve, and ρ and σ are the radii of curvature and torsion respectively. We thus have

$$\underset{\smile}{\mu\dot{\lambda}} = -\frac{1}{\rho}, \quad \underset{\smile}{\mu\dot{\nu}} = \frac{1}{\sigma},$$

and therefore, since $\underset{\smile}{\lambda\mu} = 0, \quad \underset{\smile}{\mu\nu} = 0,$

we have $\underset{\smile}{\dot{\mu}\lambda} = \frac{1}{\rho}, \quad \underset{\smile}{\dot{\mu}\nu} = -\frac{1}{\sigma}, \quad \underset{\smile}{\dot{\mu}\mu} = 0.$

It follows that

$$\dot{\lambda} = \frac{\mu}{\rho}, \quad \dot{\mu} = -\frac{\lambda}{\rho} + \frac{\nu}{\sigma}, \quad \dot{\nu} = -\frac{\mu}{\sigma}. \qquad (89.1)$$

These are the formulae of Serret.

If we were to take unit vectors through the origin mutually at right angles, the first, λ, parallel to the tangent to the curve, and the second, μ, making an angle ϕ with the principal normal, we could easily deduce that

$$\dot{\lambda} = \mu r - \nu q, \quad \dot{\mu} = \nu p - \lambda r, \quad \dot{\nu} = \lambda q - \mu p,$$

where $p = \dot{\phi} + \frac{1}{\sigma}, \quad q = \frac{\sin\phi}{\rho}, \quad r = \frac{\cos\phi}{\rho}.$

More generally, if λ, μ, ν are three unit vectors mutually at right angles which are given angular displacements

$$p\,ds, \quad q\,ds, \quad r\,ds,$$

we have

$$\dot{\lambda} = \mu r - \nu q, \quad \dot{\mu} = \nu p - \lambda r, \quad \dot{\nu} = \lambda q - \mu p, \qquad (89.2)$$

as we see from the figure.

The functions p, q, r may be called rotation functions.

If ωds denotes the angular displacement which the vectors regarded as a rigid system receive, where

$$\omega = p\lambda + q\mu + r\nu,$$

we can write our equations in the more elegant form

$$\dot{\lambda} = \overset{\frown}{\omega\lambda}, \quad \dot{\mu} = \overset{\frown}{\omega\mu}, \quad \dot{\nu} = \overset{\frown}{\omega\nu}. \qquad (89.3)$$

§ **90. Codazzi's equations.** It will be useful to consider a more general displacement.

Let the vectors λ, μ, ν regarded as a rigid system receive three angular displacements

$$\omega' du, \quad \omega'' dv, \quad \omega''' dw.$$

We then have

$$\lambda_1 = \overset{\frown}{\omega'\lambda}, \quad \lambda_2 = \overset{\frown}{\omega''\lambda}, \quad \lambda_3 = \overset{\frown}{\omega'''\lambda};$$

and therefore $\overset{\frown}{\omega'\lambda_2} + \overset{\frown}{\omega'_2\lambda} = \overset{\frown}{\omega''\lambda_1} + \overset{\frown}{\omega''_1\lambda};$

or $$V\omega'\widehat{\omega''\lambda} - V\omega''\widehat{\omega'\lambda} = V(\omega''_1 - \omega'_2)\lambda,$$

that is, $$V\widehat{\omega'\omega''}\lambda = V(\omega''_1 - \omega'_2)\lambda. \tag{90.1}$$

We have exactly the same equation for μ and therefore we have identically $$\widehat{\omega'\omega''} = \omega''_1 - \omega'_2.$$

Similarly we obtain two other vectorial equations, and we have

$$\omega'''_2 - \omega''_3 = \widehat{\omega''\omega'''}, \quad \omega'_3 - \omega'''_1 = \widehat{\omega'''\omega'}, \quad \omega''_1 - \omega'_2 = \widehat{\omega'\omega''}. \tag{90.2}$$

Suppose now that the vectors λ, μ, ν instead of being drawn through the origin are drawn at the extremity of the vector z, which depends on the three parameters u, v, w. If we regard the extremity of the vector z as the new origin then we may say that the linear displacements of the origin are

$$z_1 du, \quad z_2 dv, \quad z_3 dw.$$

Let
$$z_1 = \xi'\lambda + \eta'\mu + \zeta'\nu,$$
$$z_2 = \xi''\lambda + \eta''\mu + \zeta''\nu,$$
$$z_3 = \xi'''\lambda + \eta'''\mu + \zeta'''\nu.$$

We therefore have

$$\xi'_2\lambda + \eta'_2\mu + \zeta'_2\nu + \xi'(\mu r'' - \nu q'') + \eta'(\nu p'' - \lambda r'') + \zeta'(\lambda q'' - \mu p'')$$
$$= \xi''_1\lambda + \eta''_1\mu + \zeta''_1\nu + \xi''(\mu r' - \nu q') + \eta''(\nu p' - \lambda r') + \zeta''(\lambda q' - \mu p'),$$

so that
$$\xi''_1 - \xi'_2 = \zeta' q'' - \zeta'' q' + \eta'' r' - \eta' r'',$$
$$\eta''_1 - \eta'_2 = \xi' r'' - \xi'' r' + \zeta'' p' - \zeta' p'',$$
$$\zeta''_1 - \zeta'_2 = \eta' p'' - \eta'' p' + \xi'' q' - \xi' q''.$$

Similarly we obtain two other sets of equations:

$$\xi'''_2 - \xi''_3 = \zeta'' q''' - \zeta''' q'' + \eta''' r'' - \eta'' r''',$$
$$\eta'''_2 - \eta''_3 = \xi'' r''' - \xi''' r'' + \zeta''' p'' - \zeta'' p''',$$
$$\zeta'''_2 - \zeta''_3 = \eta'' p''' - \eta''' p'' + \xi''' q'' - \xi'' q''';$$
$$\xi'_3 - \xi'''_1 = \zeta''' q' - \zeta' q''' + \eta' r''' - \eta''' r',$$
$$\eta'_3 - \eta'''_1 = \xi''' r' - \xi' r''' + \zeta' p''' - \zeta''' p',$$
$$\zeta'_3 - \zeta'''_1 = \eta''' p' - \eta' p''' + \xi' q''' - \xi''' q'.$$

If we ignore the parameter w, we have the six equations:

$$p'_2 - p''_1 = q'r'' - q''r'; \quad q_2' - q''_1 = r'p'' - r''p';$$
$$r'_2 - r''_1 = p'q'' - p''q';$$
$$\xi'_2 - \xi''_1 = q'\zeta'' - q''\zeta' + r''\eta' - r'\eta'';$$
$$\eta'_2 - \eta''_1 = r'\xi'' - r''\xi' + p''\zeta' - p'\zeta'';$$
$$\zeta'_2 - \zeta''_1 = p'\eta'' - p''\eta' + q''\xi' - q'\xi''. \qquad (90.3)$$

These are the equations of Codazzi of which Darboux makes so much use in his Theory of Surfaces.

§ **91. Expressions for curvature and torsion.** Returning now to the case of a curve, Serret's equations may be written

$$\dot{\lambda} = \widehat{\omega\lambda}, \quad \dot{\mu} = \widehat{\omega\mu}, \quad \dot{\nu} = \widehat{\omega\nu};$$

where
$$\omega = \frac{\lambda}{\sigma} + \frac{\nu}{\rho}. \qquad (91.1)$$

If z is the vector which describes the curve to which we are applying Serret's equations we may write

$$z = z'\iota' + z''\iota'' + z'''\iota''',$$

where $\iota', \iota'', \iota'''$ are three fixed orthogonal vectors through the origin, so that z', z'', z''' are the Cartesian coordinates of any point on the curve.

We have $\dot{z} = \lambda$ and therefore

$$\mu = \rho\ddot{z}, \quad \nu = \sigma\rho\dddot{z} + \sigma\dot{\rho}\ddot{z} + \frac{\sigma}{\rho}\dot{z}.$$

Denoting the components of the vectors λ, μ, ν with respect to $\iota', \iota'', \iota'''$ in the usual way, we know that

$$\begin{vmatrix} \lambda', & \lambda'', & \lambda''' \\ \mu', & \mu'', & \mu''' \\ \nu', & \nu'', & \nu''' \end{vmatrix} = 1;$$

and therefore

$$\rho^2\sigma \begin{vmatrix} \dot{z}', & \dot{z}'', & \dot{z}''' \\ \ddot{z}', & \ddot{z}'', & \ddot{z}''' \\ \dddot{z}', & \dddot{z}'', & \dddot{z}''' \end{vmatrix} = 1, \qquad (91.2)$$

and
$$\frac{1}{\rho^2} = (\ddot{z}')^2 + (\ddot{z}'')^2 + (\ddot{z}''')^2. \qquad (91.3)$$

These are the usual formulae in the theory of curves.

If we take, as is more usual, x, y, z to be the Cartesian coordinates of any point on the curve and regard them as functions, not of the arc, but of any variable, we see that

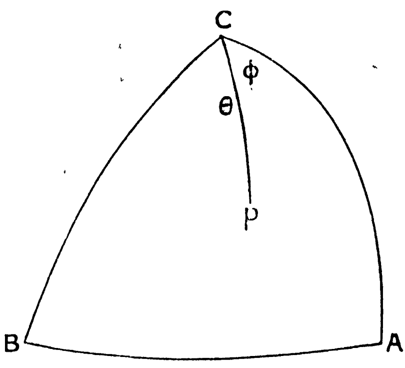
$$\rho^{-2} = \left\| \begin{matrix} \dot{x}, & \dot{y}, & \dot{z} \\ \ddot{x}, & \ddot{y}, & \ddot{z} \end{matrix} \right\|^2 \div (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^3, \tag{91.4}$$

$$\sigma^{-1} = \begin{vmatrix} \dot{x}, & \dot{y}, & \dot{z} \\ \ddot{x}, & \ddot{y}, & \ddot{z} \\ \dddot{x}, & \dddot{y}, & \dddot{z} \end{vmatrix} \div \left\| \begin{matrix} \dot{x}, & \dot{y}, & \dot{z} \\ \ddot{x}, & \ddot{y}, & \ddot{z} \end{matrix} \right\|^2. \tag{91.5}$$

§ 92. **Determination of a curve from Serret's equations.** We must now show how the equations

$$\dot{\lambda} = \frac{\mu}{\rho}, \quad \dot{\mu} = -\frac{\lambda}{\rho} + \frac{\nu}{\sigma}, \quad \dot{\nu} = -\frac{\mu}{\sigma},$$

determine the curve when we are given the natural equations



of the curve; that is, when we are given ρ and σ in terms of the arc.

Any unit vector may be written

$$\sin\theta\cos\phi \,.\, \lambda + \sin\theta\sin\phi \,.\, \mu + \cos\theta \,.\, \nu.$$

Expressing a fixed vector in this way, and noticing that there can be no relation between the vectors λ, μ, ν of the form

$$p\lambda + q\mu + r\nu = 0,$$

where p, q, r are scalars, we find, by aid of Serret's equations, that

$$\dot{\theta} = \frac{\sin\phi}{\sigma}, \quad \dot{\phi} + \frac{1}{\rho} = \frac{\cot\theta\cos\phi}{\sigma}. \tag{92.1}$$

Let $$\psi = \cot\frac{\theta}{2}e^{\iota\phi},$$

then we find that $$\dot{\psi} = \frac{\iota}{2\sigma}(\psi^2-1)-\frac{\iota\psi}{\rho}. \qquad (92.2)$$

This is an equation of Riccati's form. When we have solved it, we know θ and ϕ, and thus the position of a fixed vector with reference to λ, μ, ν. When we have thus found three fixed vectors, with reference to λ, μ, ν, we know λ, μ, ν in terms of the arc.

When we have obtained λ in terms of the arc we can find z by aid of the equation $\dot{z} = \lambda$. (92.3)

It must now be shown how, when we are given any curve in space, any other curve, with the same natural equations, can by a mere movement in space be brought into coincidence with the given curve.

If λ_0, μ_0, ν_0 denote the positions of the vectors λ, μ, ν when the arc s is equal to s_0 or, say, to zero, then we see, by repeated applications of Serret's equations, that

$$\lambda = a'\lambda_0 + a''\mu_0 + a'''\nu_0,$$
$$\mu = b'\lambda_0 + b''\mu_0 + b'''\nu_0,$$
$$\nu = c'\lambda_0 + c''\mu_0 + c'''\nu_0, \qquad (92.4)$$

where the coefficients of λ_0, μ_0, ν_0 are known series in powers of s.

By a mere rotation we can bring λ_0, μ_0, ν_0 into coincidence with the tangent, principal normal, and binormal at the point from which we measure the arc on the given curve.

It follows that λ, μ, ν will be unit vectors coinciding with the directions of the tangent, principal normal and binormal at the point s on the given curve.

A mere translation will therefore bring the curve into coincidence with the given curve when the required rotation has been carried out, since we have

$$\dot{z} = \lambda, \quad \dot{z}' = \lambda,$$

and thus $$z' = z + \alpha$$

where α is a fixed vector, that is, a vector not depending on the arc.

§ 93. **Associated Bertrand curves. The right helicoid.** Let us now consider the curve defined by

$$z' = z + k\mu \tag{93.1}$$

where k is some function of the arc, and let us find the conditions that the two curves defined by z and z' may have the same principal normal.

We have $\dot{z}' = \left(\lambda + \mu\dot{k} + k\left(\frac{\nu}{\sigma} - \frac{\lambda}{\rho}\right)\right)\frac{ds}{ds'}$,

and therefore $\lambda' = \left(\lambda + \mu\dot{k} + k\left(\frac{\nu}{\sigma} - \frac{\lambda}{\rho}\right)\right)\frac{ds}{ds'}$.

Since $$\lambda' \smile \mu = 0,$$

we must have $\dot{k}$ equal to zero [i.e. k a constant].

Again, differentiating with respect to the arc s',

$$\frac{\mu}{\rho'} = \left(\frac{\mu}{\rho} + k\left(\frac{\lambda\dot{\rho}}{\rho^2} - \frac{\nu\dot{\sigma}}{\sigma^2} - \frac{\mu}{\rho^2} - \frac{\mu}{\sigma^2}\right)\right)\left(\frac{ds}{ds'}\right)^2 + \left(\lambda + k\left(\frac{\nu}{\sigma} - \frac{\lambda}{\rho}\right)\right)\frac{d^2s}{ds'^2};$$

and therefore $$k\frac{\dot{\rho}}{\rho^2}\left(\frac{ds}{ds'}\right)^2 + \left(1 - \frac{k}{\rho}\right)\frac{d^2s}{ds'^2} = 0,$$

$$k\frac{\dot{\sigma}}{\sigma^2}\left(\frac{ds}{ds'}\right)^2 - \frac{k}{\sigma}\frac{d^2s}{ds'^2} = 0.$$

Eliminating $\frac{ds}{ds'}$ and $\frac{d^2s}{ds'^2}$ we obtain

$$\frac{\dot{\rho}}{\rho - k} - \frac{\dot{\rho}}{\rho} + \frac{\dot{\sigma}}{\sigma} = 0,$$

and integrating we have $$\frac{k}{\rho} + \frac{k'}{\sigma} = 1, \tag{93.2}$$

where k' is a constant introduced on integration.

A curve satisfying the above equation is called a Bertrand curve. We see that the property of a Bertrand curve is to be associated with another Bertrand curve having the same principal normal, the distance between corresponding points being the constant k.

If a Bertrand curve has more than one corresponding curve it will have an infinite number of such curves and will clearly be a circular helix, for ρ and σ will each be constant.

We can immediately deduce that the only ruled minimal surface is the right helicoid. For consider the curved asymptotic line on a ruled surface. We know that the osculating plane of any asymptotic line on any surface is a tangent plane to the surface. The generator of the ruled surface therefore lies in the osculating plane of the other asymptotic line through any point on it. If the surface is a minimal one it must therefore be a principal normal, and since an infinite number of asymptotic lines cut any generator orthogonally the asymptotic lines must be circular helices. The surface is therefore a right helicoid.

§ **94. A curve on a surface in relation to that surface.** We now pass on to consider the curves which lie on a given surface. Since such curves are defined by a relation between the parameters u and v, and since z, the vector of the given surface, is a function of these parameters, we are really given z in terms of one parameter along the curve defined by an equation

$$F(u, v) = 0.$$

But since we want to consider the curves in relation to the surface we proceed by a different method.

We have the formulae

$$\dot{\lambda} = \mu r - \nu q, \quad \dot{\mu} = \nu p - \lambda r, \quad \dot{\nu} = \lambda q - \mu p,$$

where λ is a unit vector parallel to the tangent to the curve, μ a unit vector parallel to the normal to the surface and making an angle ϕ with the principal normal to the curve; and we have seen (§ 89) that

$$p = \dot{\phi} + \frac{1}{\sigma}, \quad q = \frac{\sin\phi}{\rho}, \quad r = \frac{\cos\phi}{\rho}, \quad . \qquad (94.1)$$

where ρ and σ are the radii of curvature and torsion of the curve.

We know that

$$\underset{\smile}{\mu z_1} = 0, \quad \underset{\smile}{\mu z_2} = 0, \quad \underset{\smile}{\mu_1 z_1} = \Omega_{11}, \quad \underset{\smile}{\mu_1 z_2} = \underset{\smile}{\mu_2 z_1} = \Omega_{12},$$

$$\underset{\smile}{\mu_2 z_2} = \Omega_{22};$$

we can therefore easily verify the formulae

$$h^2\mu_1 = (f\Omega_{12} - g\Omega_{11}) z_1 + (f\Omega_{11} - e\Omega_{12}) z_2,$$
$$h^2\mu_2 = (f\Omega_{22} - g\Omega_{12}) z_1 + (f\Omega_{12} - e\Omega_{22}) z_2;$$

and from these formulae we deduce

$$h\widehat{\mu_1 z_1} = (e\Omega_{12} - f\Omega_{11})\mu, \quad h\widehat{\mu_1 z_2} = (f\Omega_{12} - g\Omega_{11})\mu,$$
$$h\widehat{\mu_1 z_1} = (e\Omega_{22} - f\Omega_{12})\mu, \quad h\widehat{\mu_2 z_2} = (f\Omega_{22} - g\Omega_{12})\mu.$$

It follows that

$$\underset{\smile}{\mu\dot{z}} = S(\mu_1\dot{u} + \mu_2\dot{v})(z_1\dot{u} + z_2\dot{v})$$
$$= \Omega_{11}\dot{u}^2 + 2\Omega_{12}\dot{u}\dot{v} + \Omega_{22}\dot{v}^2, \tag{94.2}$$

and that

$$h\widehat{\mu\dot{z}} = hV(\mu_1\dot{u} + \mu_2\dot{v})(z_1\dot{u} + z_2\dot{v})$$
$$= \mu\left((e\Omega_{12} - f\Omega_{11})\dot{u}^2 + (e\Omega_{22} - g\Omega_{11})\dot{u}\dot{v} + (f\Omega_{22} - g\Omega_{12})\dot{v}^2\right)$$
$$= \mu\begin{vmatrix} e\dot{u} + f\dot{v}, & f\dot{u} + g\dot{v} \\ \Omega_{11}\dot{u} + \Omega_{12}\dot{v}, & \Omega_{12}\dot{u} + \Omega_{22}\dot{v} \end{vmatrix}. \tag{94 3}$$

But $\underset{\smile}{\mu\dot{z}} = \underset{\smile}{\mu\lambda} = r = \dfrac{\cos\phi}{\rho}$

and $h\widehat{\mu\dot{z}} = h\widehat{\mu\lambda} = hp\mu,$

and therefore
$$\frac{\cos\phi}{\rho} = \Omega_{11}\dot{u}^2 + 2\Omega_{12}\dot{u}\dot{v} + \Omega_{22}\dot{v}^2, \tag{94.4}$$

$$h\left(\dot{\phi} + \frac{1}{\sigma}\right) = \begin{vmatrix} e\dot{u} + f\dot{v}, & f\dot{u} + g\dot{v} \\ \Omega_{11}\dot{u} + \Omega_{12}\dot{v}, & \Omega_{12}\dot{u} + \Omega_{22}\dot{v} \end{vmatrix}. \tag{94.5}$$

We have thus expressed the two angular velocity components p and r of the curve under consideration in terms of the derivatives of the parameters u and v with respect to the arc and the functions e, f, g and $\Omega_{11}, \Omega_{12}, \Omega_{22}$.

We must consider the remaining component q.

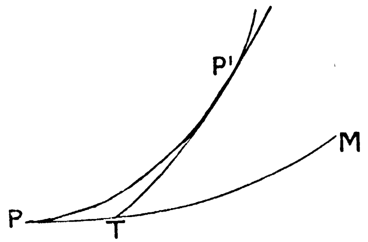
As the vectors λ, μ, ν are displaced from their positions at P to their positions at P', a neighbouring point of the curve under consideration, we may consider that they are displaced along the geodesic PT and then along the geodesic TP'.

As we pass along PT the displacement $q ds$ is zero and as we pass along TP' the displacement $q ds$ is also zero. The total displacement qds is therefore just the angle $P'TM$: that is [§ 39]

$$q = \frac{1}{\rho_g} \tag{94.6}$$

since the geodesic curvature of the curve is defined by the formula

$$\frac{1}{\rho_g} = \mathrm{Lt}_{P' \to P} \frac{P'TM}{P'P}.$$



We should notice that unlike p and r the angular velocity q depends on the first ground form only and the derivatives of u and v and not on Ω_{11}, Ω_{12}, Ω_{22}.

We have proved earlier (36 . 3) that

$$\frac{1}{\rho_g} = \frac{\Delta_2(F)}{\sqrt{\Delta(F)}} + \Delta\left(F, (\Delta(F))^{-\frac{1}{2}}\right). \tag{94.7}$$

We express this formula in a more convenient form for some purposes without the aid of the differential parameters by

$$\frac{1}{\rho_g} = \frac{a^{\frac{1}{2}}(F_{.11}F_2^2 - 2F_{.12}F_1F_2 + F_{.22}F_1^2}{\{a_{11}F_2^2 - 2a_{12}F_1F_2 + a_{22}F_2^2\}^{\frac{3}{2}}}, \tag{94.8}$$

where

$$F(u, v) = 0$$

is the equation of the curve, or, since

$$F_1\dot{u} + F_2\dot{v} = 0,$$

and

$$F_1(\ddot{u} + \{111\}\dot{u}^2 + 2\{121\}\dot{u}\dot{v} + \{221\}\dot{v}^2)$$
$$+ F_2(\ddot{v} + \{112\}\dot{u}^2 + 2\{212\}\dot{u}\dot{v} + \{222\}\dot{v}^2)$$
$$+ F_{.11}\dot{u}^2 + 2F_{.12}\dot{u}\dot{v} + F_{.22}\dot{v}^2 = 0,$$

and

$$a_{11}\dot{u}^2 + 2a_{12}\dot{u}\dot{v} + a_{22}\dot{v}^2 = 1,$$

in the form

$$\frac{1}{\rho_g} = h\begin{vmatrix} \dot{u}, & \ddot{u} + \{111\}\dot{u}^2 + 2\{121\}\dot{u}\dot{v} + \{221\}\dot{v}^2 \\ \dot{v}, & \ddot{v} + \{112\}\dot{u}^2 + 2\{212\}\dot{u}\dot{v} + \{222\}\dot{v}^2 \end{vmatrix}. \tag{94.9}$$

We have thus found expressions for the angular velocities

$$p = \dot{\phi} + \frac{1}{\sigma}, \quad q = \frac{\sin\phi}{\rho}, \quad r = \frac{\cos\phi}{\rho}, \tag{94.1}$$

along the curve in terms of the derivatives of u and v and the functions which define the ground forms. We notice that p and r depend only on the first derivatives, but q depends on the second derivatives and is the geodesic curvature of the curve.

We have seen [§ 49] that the curvature of the normal section of the surface in the direction of the tangent to the curve is given by

$$\frac{1}{R} = \Omega_{11}\dot{u}^2 + 2\Omega_{12}\dot{u}\dot{v} + \Omega_{22}\dot{v}^2.$$

We thus have Meunier's theorem that

$$\frac{\cos\phi}{\rho} = \frac{1}{R}. \tag{94.10}$$

The expression

$$\dot{\phi} + \frac{1}{\sigma} \tag{94.11}$$

is the same for all curves having the same tangent at the point under consideration. It is therefore the torsion of the geodesic curve which touches the curve at that point.

§ 95. **Formulae for geodesic torsion and curvature.** We can find another formula to express the torsion of the geodesic by aid of the formula already proved

$$\dot{z}^2 + (R' + R'')\,\dot{z}\dot{\mu} + R'R''\dot{\mu}^2 = 0.$$

Since $\dot{z} = \lambda$ and $\dot{\mu} = \nu p - \lambda r$,

we have $1 - (R' + R'')\,r + R'R''\,(p^2 + r^2) = 0,$

that is,

$$p^2 + \left(\frac{1}{R'} - r\right)\left(\frac{1}{R''} - r\right) = 0. \tag{95.1}$$

If we take the parametric lines as the lines of curvature, so that

$$r = \frac{\cos^2\theta}{R'} + \frac{\sin^2\theta}{R''},$$

this becomes

$$p = \cos\theta\sin\theta\left(\frac{1}{R'} - \frac{1}{R''}\right)$$

or

$$\dot{\phi} + \frac{1}{\sigma} = \cos\theta\sin\theta\left(\frac{1}{R'} - \frac{1}{R''}\right). \tag{95.2}$$

Since q is the angular velocity about the normal to the surface, as we pass along the curve we are considering, we see that

$$q = -\dot{\theta} + q'\dot{u} + q''\dot{v},$$

where

$$q'\dot{u} + q''\dot{v}$$

is the angular velocity about the normal of the rigid system made up of the normal and the tangents to the two lines of curvature.

We thus have the formula for the geodesic curvature

$$\frac{1}{\rho_g} = -\dot{\theta} + q'\dot{u} + q''\dot{v}. \qquad (95.3)$$

We have

$$\dot{r} = 2\sin\theta\cos\theta\left(\frac{1}{R''} - \frac{1}{R'}\right)\dot{\theta} + \cos^2\theta\left(\dot{u}\frac{\partial}{\partial u}\left(\frac{1}{R'}\right) + \dot{v}\frac{\partial}{\partial v}\left(\frac{1}{R'}\right)\right)$$

$$+\sin^2\theta\left(\dot{u}\frac{\partial}{\partial u}\left(\frac{1}{R''}\right) + \dot{v}\frac{\partial}{\partial v}\left(\frac{1}{R''}\right)\right),$$

and therefore

$$\dot{r} - 2pq \qquad (95.4)$$

depends only on the first derivatives of the parameters u and v, and so is the same for all curves on the surface having the same tangent at the point under consideration. This theorem is due to Laguerre.

In connexion with the formulae

$$q = \frac{1}{\rho_g} = \frac{\sin\phi}{\rho} = \frac{\tan\phi}{R}, \qquad (95.5)$$

where R is the radius of curvature of the normal section of the surface in the direction of the tangent to the curve, it is useful to remember that if a particle describes a curve on any surface with velocity V, the acceleration normal to the path and tangential to the surface is $\frac{V^2}{\rho_g}$.

§ **96. Surfaces whose lines of curvature are plane curves.** So far the curve we have been considering has been any curve on the surface: suppose now that it is a line of curvature.

We have $p = 0$, as we see from the formula

$$p = \cos\theta \sin\theta\left(\frac{1}{R'} - \frac{1}{R''}\right),$$

and therefore
$$\dot{\phi} + \frac{1}{\sigma} = 0. \qquad (96.1)$$

If therefore the line of curvature is a plane curve its plane makes a constant angle with the surface all along it; and conversely if the osculating plane at each point of a line of curvature makes the same angle with the surface the line of curvature is a plane curve.

We now propose to find the form of a surface if all its lines of curvature are plane curves.

Let α be a vector perpendicular to the plane line of curvature along which only v varies so that α depends on u only.

Similarly let β be a vector perpendicular to the plane line of curvature along which only u varies.

In accordance with our general notation in the theory of surfaces, let λ be a unit vector normal to the surface at the extremity of the vector z.

We have, since the parametric lines are lines of curvature,

$$z_1 = R'\lambda_1, \quad z_2 = R''\lambda_2;$$

and as
$$\underset{\smile}{\alpha z_2} = 0, \quad \underset{\smile}{\beta z_1} = 0,$$

we also have
$$\underset{\smile}{\alpha\lambda_2} = 0, \quad \underset{\smile}{\beta\lambda_1} = 0.$$

It follows that $\alpha = p\lambda_1 + q\lambda, \quad \beta = r\lambda_2 + s\lambda,$
where p, q, r, s are scalars.

We thus obtain the two equations

$$p\lambda_{12} + p_2\lambda_1 + q\lambda_2 + q_2\lambda = 0,$$
$$r\lambda_{12} + r_1\lambda_2 + s_1\lambda + s\lambda_1 = 0. \qquad (96.2)$$

Now since
$$S\lambda_{12}\lambda_1\lambda_2 = 0,$$
as the lines of curvature are conjugate lines,

$$q_2 = s_1 = 0;$$

and as there can be no relation between λ_1 and λ_2 of the form

$$a\lambda_1 + b\lambda_2 = 0,$$

where a and b are scalars, we must have

$$\frac{p_2}{p} = \frac{s}{r}, \quad \frac{r_1}{r} = \frac{q}{p}. \tag{96.3}$$

It follows that $(\log p)_{12} = (\log r)_{12} = -\frac{sq}{rp}$,

and therefore we may take

$$p = F(u)\,e^{\theta}, \qquad r = f(v)\,e^{\theta},$$

$$q = F(u)\,e^{\theta}\,\theta_1, \quad s = f(v)\,e^{\theta}\,\theta_2. \tag{96.4}$$

We also have $$\theta_{12} + \theta_1\,\theta_2 = 0,$$

so that $$\lambda_{12} + \theta_2\lambda_1 + \theta_1\lambda_2 = 0, \tag{96.5}$$

and $$\theta_{12} + \theta_1\,\theta_2 = 0. \tag{96.6}$$

Let us now start again with these two equations.

We see that $$\theta_{12} + \theta_1\,\theta_2 = 0$$

tells us that $$e^{\theta} = f(u) + \phi(v);$$

and, since $$\lambda_1\lambda_2 = 0,$$

the lines of curvature being at right angles, the equation

$$\lambda_{12} + \theta_2\lambda_1 + \theta_1\lambda_2 = 0$$

tells us that $$\frac{\partial}{\partial v}(\lambda_1^2) + 2\,\theta_2\lambda_1^2 = 0,$$

$$\frac{\partial}{\partial u}(\lambda_2^2) + 2\,\theta_1\lambda_2^2 = 0,$$

so that $$\lambda_1^2 + e^{-2\theta}F(u) = 0,$$

$$\lambda_2^2 + e^{-2\theta}\phi(v) = 0. \tag{96.7}$$

We can now so choose the parameters that

$$\lambda_1^2 + e^{-2\theta} = 0, \quad \lambda_2^2 + e^{-2\theta} = 0.$$

The spherical image of the surface is therefore given by

$$ds^2 = A^2\,(du^2 + dv^2)$$

where $$A^{-1} = U + V,$$

U being a function of u only and V a function of v only.

But, from the expression for the measure of curvature of the surface
$$ds^2 = A^2 du^2 + B^2 dv^2,$$
$$KAB + \frac{\partial}{\partial u}\left(\frac{1}{A}\frac{\partial B}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{B}\frac{\partial A}{\partial v}\right) = 0.$$

We must therefore have
$$(U+V)^{-2} = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\log(U+V);$$
and therefore
$$1 = \ddot{U}U - \dot{U}^2 + \ddot{V}V - \dot{V}^2 + \ddot{U}V + \ddot{V}U. \qquad (96.8)$$

It now easily follows that without loss of generality we may take
$$U = \operatorname{cosec}\alpha \cosh u,$$
$$V = -\cot\alpha \cos v,$$
so that
$$e^{\theta} = \frac{\cosh u - \cos\alpha\cos v}{\sin\alpha}. \qquad (96.9)$$

If p is the perpendicular on the tangent plane to the surface of which we have found the spherical image we have
$$p + \underset{\smile}{\lambda z} = 0.$$

It follows that
$$p_1 + \underset{\smile}{\lambda_1 z} = 0, \quad p_2 + \underset{\smile}{\lambda_2 z} = 0, \quad p_{12} + \underset{\smile}{\lambda_{12} z} = 0,$$
since
$$\underset{\smile}{\lambda z_1} = 0, \quad \underset{\smile}{\lambda z_2} = 0, \quad \underset{\smile}{\lambda_1 z_2} = 0, \quad \underset{\smile}{\lambda_2 z_1} = 0;$$
and therefore
$$p_{12} + \theta_2 p_1 + \theta_1 p_2 = 0,$$
that is, since
$$\theta_{12} + \theta_1\theta_2 = 0,$$
$$pe^{\theta} = U + V, \qquad (96.10)$$
where U is a function of u only and V a function of v only.

We know that
$$\lambda_1^2 + e^{-2\theta} = 0, \quad \underset{\smile}{\lambda_1\lambda_2} = 0, \quad \lambda_2^2 + e^{-2\theta} = 0,$$
where
$$e^{\theta} = \frac{\cosh u - \cos\alpha\cos v}{\sin\alpha},$$
and therefore we can find λ by the solution of equations of Riccati's form.

We see that $(\cosh u - \cos\alpha\cos v)\lambda$

$$= \sin\alpha\sinh u i + \sin\alpha\sin v j + (\cos\alpha\cosh u - \cos v)\, k,$$

where i, j, k are fixed unit vectors at right angles, will satisfy the conditions; and we know that any other possible value of the vector λ can be obtained from this vector by a mere fixed rotation.

The surface may therefore be regarded as the envelope of the plane

$$x\sin\alpha\sinh u + y\sin\alpha\sin v + z(\cos\alpha\cosh u - \cos v) = U + V. \tag{96.11}$$

§ 97. **Enneper's theorem** Let us now consider a curve which is an asymptotic line on the surface.

We have $\frac{1}{R} = 0$ for an asymptotic line and therefore

$$\frac{\cos\phi}{\rho} = 0.$$

If ρ is infinite the asymptotic line is straight and therefore the surface is ruled.

Leaving aside the case of ruled surfaces, $\cos\phi$ is zero and therefore $\phi = \frac{\pi}{2}$, that is, the osculating plane of an asymptotic line is a tangent plane to the surface.

For an asymptotic line the angular velocities are

$$p = \frac{1}{\sigma}, \quad q = \frac{1}{\rho}, \quad r = 0,$$

and the formula $p^2 + \left(\frac{1}{R'} - r\right)\left(\frac{1}{R''} - r\right) = 0$

gives
$$\frac{1}{\sigma^2} + \frac{1}{R'R''} = 0, \tag{97.1}$$

that is, the torsion is $\sqrt{-K}$. This is Enneper's theorem.

We also see that the geodesic curvature of an asymptotic line is just the ordinary curvature.

§ 98. **The method of moving axes.** If we now return to the equations of Codazzi (90.3), which are the foundation of a considerable portion of Darboux's method of treating problems of differential geometry, a method which is in effect the method of moving axes, we may take ζ', ζ'' to be zero.

The rotations are p', q', r'; p'', q'', r'', and the translations ξ', η', 0; ξ'', η'', 0; and the connexions are

$$p'_2 - p''_1 = q'r'' - q''r', \quad q'_2 - q''_1 = r'p'' - r''p',$$
$$r'_2 - r''_1 = p'q'' - p''q', \quad \xi'_2 - \xi''_1 = r''\eta' - r'\eta'',$$
$$\eta'_2 - \eta''_1 = r'\xi'' - r''\xi', \quad p'\eta'' - p''\eta' = q'\xi'' - q''\xi'.$$

The displacements of a point whose coordinates with reference to the moving axes are x, y, z are, with reference to fixed axes with which the moving axes instantaneously coincide,

$$dx + \xi' du + \xi'' dv - y(r'du + r''dv) + z(q'du + q''dv),$$
$$dy + \eta' du + \eta'' dv - z(p'du + p''dv) + x(r'du + r''dv),$$
$$dz - x(q'du + q''dv) + y(p'du + p''dv).$$

Thus for a curve on the surface making an angle ω with the axis of x

$$ds\cos\omega = \xi' du + \xi'' dv, \quad ds\sin\omega = \eta' du + \eta'' dv. \quad (98.1)$$

A point on the normal to the surface and at unit distance from the surface traces out what we call the spherical image of the surface.

Thus the spherical image of the curve is given by

$$d\sigma\cos\theta = q'du + q''dv, \quad d\sigma\sin\theta = -p'du - p''dv. \quad (98.2)$$

The direction of the line element conjugate to the line element whose direction is ω is $\theta + \frac{\pi}{2}$, and therefore the two elements du, dv and δu, δv will be conjugate if

$$\frac{\xi'\delta u + \xi''\delta v}{p'du + p''dv} = \frac{\eta'\delta u + \eta''\delta v}{q'du + q''dv}. \quad (98.3)$$

The asymptotic lines, being the lines traced out by self-conjugate elements, will therefore be given by

$$\frac{\xi' du + \xi'' dv}{p'du + p''dv} = \frac{\eta' du + \eta'' dv}{q'du + q''dv}. \quad (98.4)$$

The spherical image of the surface will be given by

$$d\sigma^2 = (p'du + p''dv)^2 + (q'du + q''dv)^2. \qquad (98.5)$$

The principal radii of curvature and the lines of curvature will be deduced from the fact that the point whose coordinates are

$$0 \quad 0 \quad R$$

will have no displacement in space and therefore

$$(\xi' + Rq')\,du + (\xi'' + Rq'')\,dv = 0,$$
$$(\eta' - Rp')\,du + (\eta'' - Rp'')\,dv = 0.$$

It follows that the measure of curvature will be given by

$$K = \frac{p'q'' - p''q'}{\xi'\eta'' - \xi''\eta'} = \frac{r_2' - r_1''}{\xi'\eta'' - \xi''\eta'}. \qquad (98.6)$$

Here we should notice that the translation functions depend only on the ground form, as

$$e = (\xi')^2 + (\xi'')^2, \quad f = \xi'\eta' + \xi''\eta'', \quad g = (\eta')^2 + (\eta'')^2,$$

and that r' and r'' can be expressed in terms of the translation functions, so that we see again and very simply that the measure of curvature is an invariant.

If the surface is referred to parametric lines at right angles we may take

$$ds^2 = A^2du^2 + B^2dv^2,$$

and

$$\xi' = A, \quad \xi'' = 0, \quad \eta' = 0, \quad \eta'' = B.$$

We then have

$$r' = -\frac{A_2}{B}, \quad r'' = \frac{B_1}{A},$$

and at once deduce the formula

$$KAB + \frac{\partial}{\partial u}\left(\frac{B_1}{A}\right) + \frac{\partial}{\partial v}\left(\frac{A_2}{B}\right) = 0. \qquad (98.7)$$

If we refer the surface to the lines of curvature as parametric lines we have $p' = q'' = 0$, and the principal radii of curvature are

$$R' = -\frac{A}{q'}, \quad R'' = \frac{B}{p''}. \qquad (98.8)$$

§ **99. Orthogonal surfaces.** To illustrate the employment of moving axes depending on three parameters we might consider the case of orthogonal surfaces

$$u = \text{constant}, \quad v = \text{constant}, \quad w = \text{constant},$$

and take as axes the normals to these three surfaces at a point of intersection.

We have $\xi'' = \xi''' = 0, \quad \eta''' = \eta' = 0, \quad \zeta' = \zeta'' = 0$;
and we may write $\xi' = \xi, \quad \eta'' = \eta, \quad \zeta''' = \zeta.$

The equations satisfied by the translation functions now become $\xi_2 = -r'\eta, \quad \xi_3 = \zeta q', \quad \eta_3 = -\zeta p'', \quad \eta_1 = \xi r'',$

$$\xi_1 = -\xi q''', \quad \zeta_2 = \eta p''',$$

$$\zeta q'' + \eta r''' = 0, \quad \xi r''' + \zeta p' = 0, \quad \eta p' + \xi q'' = 0.$$

We therefore have

$$p' = 0, \quad q'' = 0, \quad r''' = 0,$$

and we have the well-known theorem that the lines of curvature on the surfaces are the lines where the orthogonal surfaces intersect them.

We shall return to the theory of orthogonal surfaces later and so shall not pursue the study further here.

CHAPTER VII

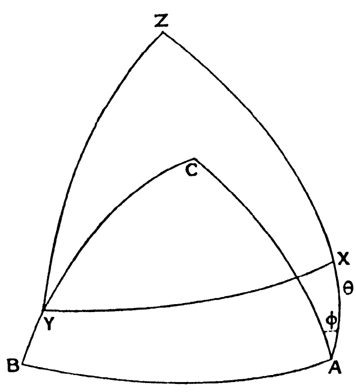
THE RULED SURFACE

§ **100.** Let a vector ζ trace out any curve in space, and let λ', μ', ν' be unit vectors drawn through its extremity, parallel respectively to the tangent, principal normal and binormal of the curve. Let

$$\lambda \equiv \cos\theta \,.\, \lambda' - \sin\theta \sin\phi \,.\, \mu' + \sin\theta \cos\phi \,.\, \nu',$$
$$\mu \equiv \cos\phi \,.\, \mu' + \sin\phi \,.\, \nu',$$
$$\nu \equiv -\sin\theta \,.\, \lambda' - \cos\theta \sin\phi \,.\, \mu' + \cos\theta \cos\phi \,.\, \nu', \qquad (100.1)$$

then λ, μ, ν will also be unit vectors mutually at right angles.

On the unit sphere, whose centre is the origin, vectors



parallel to these two sets will cut out the vertices A, B, C and X, Y, Z of two spherical triangles as in the figure.

Let $\frac{1}{\rho}$ and $\frac{1}{\sigma}$ denote respectively the curvature and torsion of the curve and let

$$p' = \dot{\phi} + \frac{1}{\sigma}, \quad q' = \frac{\sin\phi}{\rho}, \quad r' = \frac{\cos\phi}{\rho}, \qquad (100.2)$$

and $p = p'\cos\theta + r'\sin\theta, \quad q = q' - \dot{\theta}, \quad r = r'\cos\theta - p'\sin\theta.$ (100.3)

By aid of Serret's equations we see that

$$\dot{\lambda} = r\mu - q\nu, \quad \dot{\mu} = p\nu - r\lambda, \quad \dot{\nu} = q\lambda - p\mu, \qquad (100.4)$$

so that p, q, r are the rotation functions for the moving triangle XYZ. The dot above any symbol denotes that it is the symbol differentiated with respect to the arc of the curve traced out by ζ: we denote the arc by v.

§ 101. **The ground form and fundamental magnitudes.*** Let $z = \zeta + u\lambda$, so that u is the distance of the extremity of the vector z from the extremity of the vector ζ. As u and v vary, the vector z will trace out a ruled surface of the most general kind if θ and ϕ are functions of v.

The curve traced out by ζ will lie on the ruled surface: it is called the directrix of the ruled surface. Any curve on the surface may be taken as directrix.

We have

$$z_1 = \lambda, \quad z_2 = \lambda' + u\dot{\lambda} = \cos\theta\,\lambda + ur\mu - (\sin\theta + uq)\,\nu.$$

The ground form of the ruled surface will be

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + \left(u^2(q^2+r^2) + 2uq\sin\theta + 1\right) dv^2. \qquad (101.1)$$

We may write $q = M\cos\psi, \quad r = M\sin\psi,$

when the ground form becomes

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + (u^2M^2 + 2uM\sin\theta\cos\psi + 1)\, dv^2, \qquad (101.2)$$

so that q and r are given when the ground form is given.

The function h is given by

$$h^2 = u^2M^2 + 2uM\sin\theta\cos\psi + \sin^2\theta,$$
$$= (uM + \sin\theta\cos\psi)^2 + \sin^2\theta\sin^2\psi. \qquad (101.3)$$

The angle between two neighbouring generators is

$$M dv, \qquad (101.4)$$

and the shortest distance between them is

$$\sin\theta\sin\psi\, dv. \qquad (101.5)$$

The unit vector whose direction is the shortest distance is

$$\cos\psi\,\mu + \sin\psi\,\nu. \qquad (101.6)$$

* [See also § 22.]

Since $z_1 = \lambda$, $z_2 = \cos\theta\lambda + uM\sin\psi\mu - (uM\cos\psi + \sin\theta)\nu$,

we have $\quad \widehat{z_1 z_2} = (uM\cos\psi + \sin\theta)\mu + uM\sin\psi\nu$.

The unit vector normal to the surface at the extremity of z is therefore Z, where

$$Z \equiv h^{-1}\left[(uM\cos\psi + \sin\theta)\mu + uM\sin\psi\nu\right]. \qquad (101.7)$$

If we calculate z_{11}, z_{12}, z_{22} we deduce, by aid of the formulae

$$\Omega_{11} = -\underset{\smile}{z_{11}Z}, \quad \Omega_{12} = -\underset{\smile}{z_{12}Z}, \quad \Omega_{22} = -\underset{\smile}{z_{22}Z},$$

that $\qquad \Omega_{11} = 0, \quad \Omega_{12} = h^{-1}M\sin\psi\sin\theta,$

$$\Omega_{22} = ph + h^{-1}\big((u^2M^2 + uM\cos\psi\sin\theta)\dot\psi + u\dot M\sin\psi\sin\theta$$
$$+ M\cos\theta\sin\psi(\sin\theta - u\dot\theta)\big). \qquad (101.8)$$

We may write N for $M\cos\psi\sin\theta$ when we are only considering the ground form.

§ **102. Bonnet's theorem on applicable ruled surfaces.** We saw that one of the most difficult problems in the Theory of Surfaces was, given the ground form, to determine the surfaces in space to which the form was applicable; and we saw that the solution of the problem depended on a partial differential equation of the second order. In general we cannot solve this equation, but there is a striking exception in the case of the ruled surface.

Let us first consider a theorem on ruled surfaces.

If on the surface with the ground form $a_{ik}dx_i dx_k$ the curves $x_2 =$ constant are geodesics, we must have $\{112\} = 0$. If the curves $x_2 =$ constant are asymptotic lines we must have $\Omega_{11} = 0$. If both these conditions are fulfilled the surface is ruled; that is, if

$$\Omega_{11} = 0 \text{ and } \{112\} = 0, \qquad (102.1)$$

the surface is ruled and the generators are

$$x_2 = \text{constant}. \qquad (102.2)$$

Now suppose that we have a second ruled surface with the same ground form and therefore applicable on the first surface, and suppose if possible that its generators are not the lines

$$x_2 = \text{constant}.$$

We can therefore choose our coordinates so that the two surfaces will have the same ground form and that in the first surface $x_2 =$ constant will be the equation of the generators and on the second surface $x_1 =$ constant will be the equation of the generators.

We have $\Omega_{11} = 0$ and $\Omega'_{22} = 0$,

and as for the two surfaces $\Omega_{11}\Omega_{22} - \Omega^2_{12}$

and $\Omega'_{11}\Omega'_{22} - \Omega'^2_{12}$,

are the same, we must have

$$\Omega^2_{12} = \Omega'^2_{12}. \tag{102.3}$$

From Codazzi's equation (48.4) for the two surfaces we have

$$\frac{\partial}{\partial x_1}\Omega_{12} + (\{122\} - \{111\})\Omega_{12} = 0,$$

$$\frac{\partial}{\partial x_2}\Omega_{12} + (\{211\} - \{222\})\Omega_{12} = 0.$$

It is therefore possible to satisfy Codazzi's equation for the given ground form with

$$\{112\} = 0, \quad \{221\} = 0, \tag{102.4}$$

by taking Ω_{11} and Ω_{22} both zero: that is, it is possible to find a surface with both systems of asymptotic lines straight lines; that is, to find a quadric applicable to the given ground form.

Unless then the form

$$du^2 + 2\cos\theta\, du\, dv + (M^2u^2 + 2Nu + 1)\, dv^2 \tag{102.5}$$

is applicable to a quadric, the generators of any ruled surface which is applicable to it must be

$$v = \text{constant}. \tag{102.6}$$

This is Bonnet's Theorem and Bianchi's proof of it.

When therefore the ground form is given in the form

$$ds^2 = du^2 + 2\cos\theta\, du dv + (M^2u^2 + 2Nu + 1)\, dv^2, \tag{102.5}$$

we know that, leaving aside the case of quadrics, the surfaces which are ruled and applicable on it must be generated in the method we have described [so that their rectilinear generators are applied to its rectilinear generators].

When the ground form is given we are given q and r. We may take p as any arbitrary function of v. We then know ρ and σ of the directrix, and so can find it by the solution of Riccati's equation. Similarly we obtain λ and thus find the ruled surface.

§ **103. Ground forms applicable on a ruled surface.** If we are given the ground form of a surface, how are we to decide whether it is applicable on a ruled surface? It will be applicable on such a surface if the ground form can be brought to the form

$$du^2 + 2\cos\theta\, du\, dv + (M^2u^2 + 2Nu + 1)\, dv^2, \qquad (102.5)$$

where θ, M, and N are functions of v only, but unless these are given functions of the parameter the general method will not immediately apply. This is the question we now wish to consider.

The expressions
$$\frac{du}{ds} \text{ and } \frac{dv}{ds},$$
where u and v are the parameters of a point on the surface, are tensor components. We may denote them by T^1 and T^2.

The difficulty of the tensor notation comes in when we want to express the power of a tensor component with an upper integer. Thus the square of T^2 would have to be written T^2T^2, and in calculations this is inconvenient.

We therefore generally write the above two components as ξ and η and try just to remember that they are tensor components when we apply the methods of the tensor calculus.

The equations of a geodesic are (§ 38)

$$\frac{d\xi}{ds} + \{111\}\,\xi^2 + 2\{121\}\,\xi\eta + \{221\}\,\eta^2 = 0,$$

$$\frac{d\eta}{ds} + \{112\}\,\xi^2 + 2\{122\}\,\xi\eta + \{222\}\,\eta^2 = 0,$$

and

$$\frac{d\xi}{ds} = \xi\frac{\partial \xi}{\partial x_1} + \eta\frac{\partial \xi}{\partial x_2},$$

$$\frac{d\eta}{ds} = \xi\frac{\partial \eta}{\partial x_1} + \eta\frac{\partial \eta}{\partial x_2}.$$

The equations of a geodesic may therefore be written

$$\xi\left(\frac{\partial \xi}{\partial x_1} + \{111\}\,\xi + \{121\}\,\eta\right) + \eta\left(\frac{\partial \xi}{\partial x_2} + \{121\}\,\xi + \{221\}\,\eta\right) = 0,$$

$$\xi\left(\frac{\partial \eta}{\partial x_1} + \{112\}\,\xi + \{122\}\,\eta\right) + \eta\left(\frac{\partial \eta}{\partial x_2} + \{122\}\,\xi + \{222\}\,\eta\right) = 0. \qquad (103.1)$$

Now these equations are very simply expressed in the tensor notation by

$$T^1 T^1_{\cdot 1} + T^2 T^1_{\cdot 2} = 0, \quad T^1 T^2_{\cdot 1} + T^2 T^2_{\cdot 2} = 0. \qquad (103.2)$$

The equation of the asymptotic lines is given by

$$\Omega_{11} T^1 T^1 + 2\,\Omega_{12} T^1 T^2 + \Omega_{22} T^2 T^2 = 0. \qquad (103.3)$$

Now remembering that on a ruled surface one of the asymptotic lines is a geodesic, and taking the tensor derivatives of this equation, we have

$$\Omega_{11\cdot 1} T^1 T^1 + 2\,\Omega_{12\cdot 1} T^1 T^2 + \Omega_{22\cdot 1} T^2 T^2 + 2\,T^1 (\Omega_{11} T^1_{\cdot 1} + \Omega_{12} T^2_{\cdot 1}) + 2\,T^2 (\Omega_{12} T^1_{\cdot 1} + \Omega_{22} T^2_{\cdot 1}) = 0, \qquad (103.4)$$

and $$\Omega_{11\cdot 2} T^1 T^1 + 2\,\Omega_{12\cdot 2} T^1 T^2 + \Omega_{22\cdot 2} T^2 T^2 + 2\,T^1 (\Omega_{11} T^1_{\cdot 2} + \Omega_{12} T^2_{\cdot 2} (+ 2\,T^2 (\Omega_{12} T^1_{\cdot 2} + \Omega_{22} T^2_{\cdot 2}) = 0. \qquad (103.5)$$

Multiplying the first equation by T^1 and the second by T^2, and adding, and making use of the equations for a geodesic, we see that if the surface is ruled we have for the equations of that asymptote which is a generator

$$\Omega_{11} \xi^2 + 2\,\Omega_{12} \xi\eta + \Omega_{22} \eta^2 = 0, \qquad (103.6)$$

$$\Omega_{11\cdot 1} \xi^3 + 3\,\Omega_{11\cdot 2} \xi^2 \eta + 3\,\Omega_{22\cdot 1} \xi \eta^2 + \Omega_{22\cdot 2} \eta^3 = 0. \qquad (103.7)$$

If we write these two equations

$$(A, B, C)(\xi, \eta)^2 = 0,$$

$$(a, b, c, d)(\xi, \eta)^3 = 0,$$

the eliminant is (Salmon, *Higher Algebra*, § 198)

$$a^2 C^3 - 6\,abBC^2 + 6\,acC\,(2\,B^2 - AC) + ad\,(6\,ABC - 8\,B^3) + 9\,b^2 AC^2 - 18\,bcABC + 6\,bdA\,(2\,B^2 - AC) + 9\,c^2 A^2 C - 6\,cdBA^2 + d^2 A^3 = 0. \qquad (103.8)$$

This vanishes for a ruled surface.

Now we know that

$$\Omega_{11}\Omega_{22}-\Omega_{12}^2 = K(a_{11}a_{22}-a_{12}^2),$$

and, since an arbitrary function is needed to express Ω_{11}, Ω_{22}, Ω_{12} in terms of the parameters, there can only be one other equation connecting these functions.

Applying tensor derivation to this equation we have, using Codazzi's equations,

$$\Omega_{11}\Omega_{22\cdot 1}+\Omega_{22}\Omega_{11\cdot 1}-2\,\Omega_{12}\Omega_{11\cdot 2} = K_1\,(a_{11}a_{22}-a_{12}^2),$$
$$\Omega_{11}\Omega_{22\cdot 2}+\Omega_{22}\Omega_{11\cdot 2}-2\,\Omega_{12}\Omega_{22\cdot 1} = K_2\,(a_{11}a_{22}-a_{12}^2). \qquad (103.9)$$

We thus have three equations, viz. these two and the eliminant we have found. We conclude that this system must be complete if the surface is ruled. For if another equation of the first order in the derivatives of Ω_{11} and Ω_{22} could be obtained—the function Ω_{12} is known in terms of Ω_{11} and Ω_{22}—we could obtain Ω_{11} and Ω_{22} by quadratures, and no arbitrary function would appear.

This method, though tedious actually to carry out, will enable us to determine whether any given ground form is applicable to a ruled surface.

§ **104. Case of applicability to a quadric.** We must now consider the ground form

$$du^2+2\cos\theta\, du\,dv+(M^2u^2+2Nu+1)\,dv^2,$$

as regards its special form when it is applicable to a quadric.

The Cartesian coordinates of any point on a fixed generator of a quadric may be taken to be

$$x=\frac{a_1v+b_1}{av+b},\quad y=\frac{a_2v+b_2}{av+b},\quad z=\frac{a_3v+b_3}{av+b}, \qquad (104.1)$$

where the variable v denotes distance on that generator.

We have similar expressions for the coordinates on any other generator; and the variables v and v' of the points which lie on the same generator of the opposite system will be connected by a bilinear equation.

It follows that if P_1 is a point on the first generator and P_2

its correspondent on the second generator the direction cosines of their join may be taken as

$$\frac{a_1 d_2 - a_2 d_1}{D}, \quad \frac{b_1 d_2 - b_2 d_1}{D}, \quad \frac{c_1 d_2 - c_2 d_1}{D}, \tag{104.2}$$

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$ are linear functions of v and

$$D^2 = d_1^2(a_2^2+b_2^2+c_2^2)+d_2^2(a_1^2+b_1^2+c_1^2) - 2d_1 d_2(a_1 a_2 + b_1 b_2 + c_1 c_2). \tag{104.3}$$

The coordinates of any point on the quadric may then be expressed in terms of u and v in the form

$$x = \frac{a_1}{d_1} + u\frac{a_1 d_2 - a_2 d_1}{D}, \quad y = \frac{b_1}{d_1} + u\frac{b_1 d_2 - b_2 d_1}{D},$$

$$z = \frac{c_1}{d_1} + u\frac{c_1 d_2 - c_2 d_1}{D}. \tag{104.4}$$

It follows that M^2, ND, $\cos\theta D$ are rational functions of v which can be calculated, and that D^2 is a quartic in v.

§ 105. **Special ground forms. Binormals to a curve. Line of striction.** We have found in §§ 100, 101 the chief formulae required in the study of the general ruled surface. When the ground form is given we are given q and r, and we find the different ruled surfaces which are applicable to the form by varying p. This generally means that we vary ϕ, the angle of inclination of the osculating plane of the directrix to the corresponding normal section of the surface. We cannot however take ϕ to be zero unless the ground form is special: for, if ϕ is zero, $q+\dot\theta$ is zero: that is,

$$M\cos\psi + \dot\theta = 0;$$

which would give the special ground form

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + (M^2u^2 - 2\dot\theta\sin\theta\, u + 1)\, dv^2. \tag{105.1}$$

Thus the binormals to a curve in space trace out a ruled surface with the special ground form

$$ds^2 = du^2 + \left(\frac{u^2}{\sigma^2} + 1\right) dv^2, \tag{105.2}$$

where σ is the radius of torsion.

If we take as directrix an orthogonal trajectory of the generators, θ is $\frac{\pi}{2}$, and the ground form is

$$ds^2 = du^2 + (M^2u^2 + 2\,Mu\cos\psi + 1)\,dv^2. \qquad (105\,.\,3)$$

In seeking the surfaces which are ruled and applicable to this form we may take for one of them $\phi = \frac{\pi}{2}$. The directrix of this surface will be an asymptotic line and the surface will be generated by the principal normals of this directrix.

We obtain the equation of the line of striction from the ground form itself. We have to find for given values of v and dv the values of u and du which will make

$$du^2 + 2\cos\theta\,du\,dv + (M^2u^2 + 2\,Mu\cos\psi\sin\theta + 1)\,dv^2$$

least.

Clearly we must have

$$du + \cos\theta\,dv = 0,$$

$$Mu + \cos\psi\sin\theta = 0.$$

The equation of the line of striction is therefore

$$Mu + \cos\psi\sin\theta = 0. \qquad (105\,.\,4)$$

Let us take the line of striction as the directrix. We must then have

$$\cos\psi\sin\theta = 0. \qquad (105\,.\,5)$$

We cannot have $\sin\theta$ equal to zero unless the shortest distance between neighbouring generators vanishes: that is, unless the ruled surface is a developable. We must therefore have in general, when the line of striction is taken as the directrix, $\psi = \frac{\pi}{2}$, and therefore $q = 0$.

It follows that $\dot{\theta} = \frac{\sin\phi}{\rho}$, i.e. that $\dot{\theta}$, the rate of increase of the angle at which the line of striction crosses the generators, is equal to the geodesic curvature of the line of striction. It follows that the line of striction will cross the generators at a constant angle if, and only if, it is a geodesic. In this case the ground form will be

$$ds^2 = du^2 + 2\cos\alpha\,du\,dv + (M^2u^2 + 1)\,dv^2, \qquad (105\,.\,6)$$

where α is the constant angle of crossing.

If $\alpha = \frac{\pi}{2}$ the form will be applicable on the surface generated by the binormals of a curve in space.

§ **106. Constancy of anharmonic ratios. Applicable ruled surfaces and surfaces of Revolution.** We shall now consider the equation of the asymptotic line which is not a generator.

The equation is $2\Omega_{12}du + \Omega_{22}dv = 0.$ (106.1)

Referring to the values given for Ω_{12} and Ω_{22} we see that this is an equation of Riccati's form. It follows that the equation of an asymptotic line is

$$u = \frac{\alpha k + \beta}{\gamma k + \delta} \tag{106.2}$$

where α, β, γ, δ are some functions of v only, and k is an arbitrary constant.

We thus see that every generator is cut in a constant anharmonic ratio by any four fixed asymptotic lines.

We also notice from the property of Riccati's equation that if we are given any one asymptotic line we can find the others by quadrature.

We have also seen in § 101 that the normal to the ruled surface is parallel to

$$uM(\cos\psi\mu + \sin\psi\nu) + \sin\theta\mu.$$

It follows that the anharmonic ratio of four tangent planes through any generator is

$$\frac{(u_1 - u_3)(u_2 - u_4)}{(u_1 - u_4)(u_2 - u_3)}; \tag{106.3}$$

that is, the anharmonic ratio of the planes is the same as that of the points of contact.

Suppose now that P is any point on a generator, and that the tangent plane at P intersects a neighbouring generator in P'. Then in the limit PP' is the element of the asymptotic line at P. It follows that the asymptotic lines through four points on a generator intersect a neighbouring generator in the cross ratio of the tangent planes: that is, in the cross ratio of the points of contact.

We thus have a second and more geometrical proof of the theorem that every generator is cut in a constant cross ratio by four fixed asymptotic lines. This theorem also is due to Bonnet.

The condition that the normals to a ruled surface, at two points u_1, u_2 on the same generator, may be perpendicular is

$$u_1 u_2 M^2 + (u_1 + u_2) M \sin\theta \cos\psi + \sin^2\theta = 0. \qquad (106.4)$$

The points are therefore corresponding points in an involution range whose centre is on the line of striction.

No ruled surface exists which is also a surface of revolution except the quadric of revolution. We see this at once by considering a surface of revolution in relation to any meridian line. The asymptotic lines, through any point on this line, must be symmetrically placed with respect to the line. If then one of these is a straight line so will the other be. The surface will therefore, if it is a ruled one, be a quadric.

But the ruled surface may be applicable on a surface of revolution without being a surface of revolution. We now inquire what property the ground form must have if it is to be applicable to a surface of revolution with generators corresponding to the meridian lines.

Taking as directrix an orthogonal trajectory of the generators we have

$$ds^2 = du^2 + (M^2 u^2 + 2Nu + 1)\, dv^2. \qquad (106.5)$$

If then this form is to be applicable to a surface of revolution M and N must be constants, and we see that the ground form may be written

$$ds^2 = du^2 + (u^2 + a^2)\, dv^2 \qquad (106.6)$$

where a is a constant.

Thus the catenoid and the helicoid will both have this form applicable to them.

§ **107. Surfaces cutting at one angle all along a generator.** We now wish to investigate the condition that two ruled surfaces with a common generator may intersect at the same angle all along that generator. The condition will be found to have an interesting connexion with a particular class of congruences.

We have seen that

$$uM(\cos\psi\mu+\sin\psi\nu)+\sin\theta\mu$$

is a vector parallel to the normal to the ruled surface at the extremity of the vector z.

As we move along the generator this vector turns through an angle, remaining of course perpendicular to the generator.

The vector product of the above vector and the neighbouring vector

$$(u+du)M(\cos\psi\mu+\sin\psi\nu)+\sin\theta\mu$$

is

$$Mdu\sin\theta\sin\psi\lambda.$$

But the vector product is also

$$(M^2u^2+2Mu\cos\psi\sin\theta+\sin^2\theta)\,d\epsilon\lambda,$$

where $d\epsilon$ is the angle turned through; and therefore

$$\frac{d\epsilon}{du}=\frac{M\sin\theta\sin\psi}{M^2u^2+2Mu\sin\theta\cos\psi+\sin^2\theta}. \qquad (107.1)$$

Let

$$M=k\sin\theta\sin\psi$$

where k is the ratio of the angle between two neighbouring generators to the shortest distance between them. Then

$$\frac{d\epsilon}{du}=\frac{k}{(ku+\cot\psi)^2+1}.$$

The equation of the line of striction is

$$ku+\cot\psi=0.$$

If therefore we measure u not from the directrix but from the line of striction we have the formula

$$\frac{d\epsilon}{du}=\frac{k}{k^2u^2+1}. \qquad (107.2)$$

It follows that if we have two ruled surfaces, for which k is the same, and one of the surfaces is given a movement in space, bringing one of its generators into coincidence with the corresponding generator of the other, and the corresponding points of the line of striction into coincidence, then the two ruled surfaces will intersect at the same angle all along that generator.

§ **108. The ruled surfaces of an isotropic congruence.** Let us now consider a ruled surface referred to its line of striction as directrix.

Now (§ 100) $\quad \dot{\zeta} = \lambda' = \cos\theta\lambda - \sin\theta\nu.$

$$\dot{\lambda} = r\mu - q\nu,$$

and, since the line of striction is the directrix, q is zero. We therefore have
$$\underset{\smile}{\dot{\zeta}\dot{\lambda}} = 0. \qquad (108.1)$$

Suppose now that ζ is a vector depending on two parameters u and v, and that λ is a unit vector depending on the same two parameters.

Consider the congruence $z = \zeta + w\lambda$.

The congruence is said to be isotropic if ζ and λ correspond orthogonally. [See § 81.]

We have as the conditions for orthogonal correspondence

$$\underset{\smile}{\zeta_1\lambda_1} = 0, \quad \underset{\smile}{\zeta_1\lambda_2} + \underset{\smile}{\zeta_2\lambda_1} = 0, \quad \underset{\smile}{\zeta_2\lambda_2} = 0,$$

and therefore $\qquad \lambda_1 = a\overset{\frown}{\lambda\zeta_1}, \quad \lambda_2 = a\overset{\frown}{\lambda\zeta_2},$

where a is some scalar function of u and v.

We thus have
$$d\lambda = a\overset{\frown}{\lambda d\zeta}, \qquad (108.2)$$

whatever be the values of du and dv.

The ruled surfaces of the congruence are obtained by connecting u and v by some equation. For any ruled surface of the congruence we therefore have

$$\underset{\smile}{\dot{\zeta}\dot{\lambda}} = 0, \quad \dot{\lambda} = a\overset{\frown}{\lambda\dot{\zeta}},$$

where the dot denotes differentiation along the arc of the curve chosen. This arc will be the line of striction since

$$\underset{\smile}{\dot{\zeta}\dot{\lambda}} = 0;$$

and, since q is therefore zero,

$$\dot{\lambda} = r \operatorname{cosec} \theta \widehat{\lambda \dot{\zeta}}.$$

It follows that $$a = r \operatorname{cosec} \theta,$$

that is, from our definition of k (§ 107),

$$a = k. \tag{108.3}$$

The ruled surfaces of the isotropic congruence therefore intersect at the same angle all along their common generator. They have all the same k at the point where the common generator intersects the surface $w = 0$, and their lines of striction all lie on this surface. This surface is the central surface of the congruence.

CHAPTER VIII

THE MINIMAL SURFACE

§ **109. Formulae and a characteristic property.** If we give to z, the vector which traces out any surface, a small arbitrary displacement normal to the surface at the extremity of z, we have $z' = z + \lambda t$, where t is a small arbitrary parameter.

Since $\quad z'_1 = z_1 + \lambda_1 t + \lambda t_1, \quad z'_2 = z_2 + \lambda_2 t + \lambda t_2,$
we have

$$a'_{11} = a_{11} - 2\underbrace{z_1\lambda_1}t, \quad a'_{12} = a_{12} - 2\underbrace{z_1\lambda_2}t, \quad a'_{22} = a_{22} - 2\underbrace{z_2\lambda_2}t;$$

that is, by (50 . 9),

$$a'_{11} = a_{11} - 2t\Omega_{11}, \quad a'_{12} = a_{12} - 2t\Omega_{12}, \quad a'_{22} = a_{22} - 2t\Omega_{22}. \tag{109 . 1}$$

If the area of the surface is to be stationary, under this variation, then a must be stationary, where

$$a = a_{11}a_{22} - a_{12}^2,$$

since the area is $\quad \int a^{\frac{1}{2}} du dv.$

We therefore have

$$a_{11}\Omega_{22} + a_{22}\Omega_{11} - 2a_{12}\Omega_{12} = 0; \tag{109 . 2}$$

that is, the sum of the principal radii of curvature must be zero.

The surface of minimum area, the minimal surface as it is called, is therefore characterized by the property

$$R' + R'' = 0 \tag{109 . 3}$$

where R' and R'' are the principal radii of curvature.

If we refer to lines of curvature as parametric lines on any surface $\quad z_1 = R'\lambda_1, \quad z_2 = R''\lambda_2,$

and therefore, if $\quad a_{11}du^2 + a_{22}dv^2$

is the ground form of the surface, the ground form of the spherical image will be

$$\frac{a_{11}du^2}{(R')^2}+\frac{a_{22}dv^2}{(R'')^2}. \tag{109.4}$$

The surface and its spherical image will therefore be similar at corresponding points if, and only if,

$$(R')^2=(R'')^2, \tag{109.5}$$

that is, if the surface is a sphere or a minimal surface.

On a minimal surface

$$z_1=R'\lambda_1,\quad z_2=-R'\lambda_2,$$

and therefore $\quad 2R'\lambda_{12}+R'_2\lambda_1+R'_1\lambda_2=0.$

It follows that

$$\frac{\partial}{\partial u}(R'\lambda_2^2)=0,\quad \frac{\partial}{\partial v}(R'\lambda_1^2)=0,$$

and therefore without loss of generality we may say

$$R'\lambda_1^2=-1,\quad R'\lambda_2^2=-1. \tag{109.6}$$

The ground form may then be taken as

$$R'(du^2+dv^2), \tag{109.7}$$

the asymptotic lines as $du^2-dv^2=0$, (109.8)

and the ground form of the spherical image as

$$(R')^{-1}(du^2+dv^2). \tag{109.9}$$

We may now write R instead of R', and since the ground form of a sphere of unit radius is

$$d\theta^2+\sin^2\theta\, d\phi^2,$$

we must have $R(d\theta^2+\sin^2\theta\, d\phi^2)=du^2+dv^2.$ (109.10)

If we take $\quad w=\cot\frac{\theta}{2}e^{i\phi},$

we see that the complex variable w is the complex variable on the plane on to which the sphere of unit radius can be projected stereographically from the pole, if we take the pole as the origin from which θ is measured and take the plane as the corresponding equator.

If $\overline{w}$ denotes the conjugate complex

$$\cot\frac{\theta}{2}e^{-\iota\phi},$$

we see that $$4\sin^4\frac{\theta}{2}dw\,d\overline{w} = d\theta^2+\sin^2\theta\,d\phi^2,$$

and therefore $$4R\sin^4\frac{\theta}{2}dw\,d\overline{w} = du^2+dv^2. \qquad (109.11)$$

If we regard $u+\iota v$ as the complex variable of another plane and denote it by x, we have

$$R = \frac{1}{4}\operatorname{cosec}^4\frac{\theta}{2}\left|\frac{dx}{dw}\right|^2.$$

Now the curvature of the form

$$4R\sin^4\frac{\theta}{2}dw\,d\overline{w}$$

is zero; and, from the formula for the curvature of the ground form $$ds^2 = 2f\,du\,dv,$$

we have $$f^3K = f_1f_2-ff_{12},$$

and therefore $$R\sin^4\frac{\theta}{2} = f(w)\,F(\overline{w}), \qquad (109.12)$$

where f and F are functional forms. If the surface is to be a real surface these forms must be conjugate forms.

Since $$\operatorname{cosec}^2\frac{\theta}{2} = 1+w\overline{w}$$

the formula for R may be written

$$R = (1+w\overline{w})^2f(w)\,F(\overline{w}). \qquad (109.13)$$

We notice that in a minimal surface the asymptotic lines are perpendicular to one another in general though not necessarily so at a singular point. This property is characteristic of the minimal surface.

§ **110. Reference to null lines. Stereographic projection.** We now choose as the parametric lines on the minimal surface its null lines (§ 45) and, instead of writing w and $\overline{w}$, we take u and v to represent these complex quantities.

The spherical image will therefore also be referred to its null, lines and their parameters will be the same w and $\bar{w}$ or u and v.

The normal to the surface is therefore given by

$$(1+uv)\,\lambda = (u+v)\,\iota' - \iota\,(u-v)\,\iota'' + (uv-1)\,\iota''', \qquad (110.1)$$

where ι', ι'', ι''' are three fixed unit vectors mutually at right angles and ι denotes $\sqrt{-1}$.

It is now convenient to introduce two vectors defined by

$$2\rho \equiv (1-u^2)\,\iota' + \iota\,(1+u^2)\,\iota'' + 2u\,\iota''',$$
$$2\sigma \equiv (1-v^2)\,\iota' - \iota\,(1+v^2)\,\iota'' + 2v\,\iota'''. \qquad (110.2)$$

These vectors are conjugate vectors and of course not real. They are, in fact, generators of the point sphere whose centre is the origin.

Such point spheres must play in solid geometry the same part that the circular lines through a point play in plane geometry. We may easily verify the following relations between ρ, σ, and λ;

$$2\,\overset{\frown}{\rho\sigma} = \iota\,(1+uv)^2\,\lambda, \qquad 2\,\underset{\smile}{\rho\sigma} = -(1+uv)^2,$$
$$2\rho = (1+uv)^2\,\lambda_2, \qquad 2\sigma = (1+uv)^2\,\lambda_1,$$
$$\overset{\frown}{\rho\lambda} = -\iota\rho, \qquad \overset{\frown}{\sigma\lambda} = \iota\sigma,$$
$$\overset{\frown}{\rho\lambda_1} = \iota\lambda, \qquad \overset{\frown}{\sigma\lambda_2} = -\iota\lambda,$$
$$(1+uv)^2\,\overset{\frown}{\lambda_1\lambda_2} = -2\iota\lambda, \qquad \rho^2 = 0, \quad \sigma^2 = 0. \qquad (110.3)$$

We have seen (§ 109) that the complex variable u on the sphere

$$u = \cot\frac{\theta}{2}\,e^{\iota\phi} \qquad (110.4)$$

is the complex variable on the equator when we take the ground form of the sphere to be

$$d\theta^2 + \sin^2\theta\,d\phi^2,$$

and project the sphere, from the pole from which we measure θ, stereographically on to the equator.

The conjugate complex v is the image of u in the real axis of the plane.

The complex u fixes a real point on the sphere, since when u is given its two parts are given and so its conjugate v is given. If u_1 is the complex which fixes a point P_1 on the sphere and u_2 is the complex which fixes the diametrically opposite point on the sphere, we have

$$1 + u_1 v_2 = 0, \tag{110.5}$$

and consequently we also have

$$1 + u_2 v_1 = 0.$$

We should notice that we cannot have

$$uv + 1 = 0.$$

The complexes which correspond to the two opposite ends of a diameter may be called inverse complexes.

§ **111. The vector of a null curve.** A null curve is defined as a curve whose tangent at every point intersects the circle at infinity. Another way of stating the same definition is to say that the tangent at every point is a generator of the point sphere at the point. If z is the vector which traces out a null curve we therefore have

$$dz^2 = 0. \tag{111.1}$$

Now the components of a vector which satisfies the equation $x^2 = 0$ may be taken as proportional to

$$1 - u^2, \quad \iota\,(1 + u^2), \quad 2u,$$

and therefore we must have

$$dz = f'''(u)\,\rho\,du, \tag{111.2}$$

where $f'''(u)$ is some scalar function of the parameter u.

It follows that the vector z which traces out a null curve may be defined by

$$z = \rho f''(u) - \rho_1 f'(u) + \rho_{11} f(u)$$

since the third derivative of ρ vanishes.

We now denote the vector of the null curve by α, where

$$\alpha = \rho f''(u) - \rho_1 f'(u) + \rho_{11} f(u). \tag{111.3}$$

§ **112. Self-conjugate null curves. They may be (1) unicursal, (2) algebraic.** The conjugate null curve to α is clearly

$$\bar{\alpha} = \sigma \bar{f}''(v) - \sigma_2 \bar{f}'(v) + \sigma_{22} \bar{f}(v), \tag{112.1}$$

where $\bar{f}$ is the conjugate function to f, and σ the vector we have defined in terms of its parameter v.

A null curve is said to be self-conjugate, when for each value of u a value v' can be found, where v' is the conjugate complex to a complex u', such that

$$\alpha_u = \bar{\alpha}_{v'}. \qquad (112.2)$$

We generally write ρ without specifying its parameter u, but sometimes we may need to bring the parameter into evidence and then we write it ρ_u.

Differentiating the equation

$$\alpha_u = \bar{\alpha}_{v'},$$

we have
$$\rho_u f'''(u) = \sigma_{v'} \bar{f}'''(v') \frac{dv'}{du},$$

so that
$$V \rho_u \sigma_{v'} = 0,$$

and therefore
$$1 + uv' = 0. \qquad (112.3)$$

If we now write ρ for ρ_u, and σ for $\sigma_{v'}$, we have

$$\rho + \sigma u^2 = 0,\ \rho_1 + \sigma_2 + 2\sigma u = 0,\ u^2 \rho_{11} + \sigma_{22} + 2u\sigma_2 + 2u^2\sigma = 0,$$

and we can write

$$\alpha_u = -\sigma u^2 f''(u) + (\sigma_2 + 2\sigma u) f'(u)$$
$$- (\sigma_{22} + 2u\sigma_2 + 2u^2\sigma) \frac{f(u)}{u^2}, \qquad (112.4)$$

$$\bar{\alpha}_{v'} = \sigma \bar{f}''(v') - \sigma_2 \bar{f}'(v') + \sigma_{22} \bar{f}(v'). \qquad (112.5)$$

If we now equate the coefficients of the vector σ_{22} on the two sides of the equation $\alpha_u = \bar{\alpha}_{v'}$

we see that
$$\bar{f}(u) \equiv -u^2 f\left(-\frac{1}{u}\right) \qquad (112.6)$$

and we see further that this single condition is sufficient to satisfy the equation
$$\alpha_u = \bar{\alpha}_{-\frac{1}{u}}. \qquad (112.7)$$

In order then that a null curve may be self-conjugate it is necessary and sufficient that the function f which defines it should have the property

$$\bar{f}(u) \equiv -u^2 f\left(-\frac{1}{u}\right). \qquad (112.8)$$

If we take

$$f(u) \equiv a_0(1-u^2) + a_1 u + \sum_{p=1}^{p=r} a_{2p+1}(u^{2p+1} + u^{1-2p})$$
$$+ \sum_{p=2}^{p=s} a_{2p}(u^{2p} - u^{2-2p}) + \iota b_0(1+u^2)$$
$$+ \iota \sum_{p=1}^{p=r'} b_{2p+1}(u^{2p+1} - u^{1-2p}) + \iota \sum_{p=2}^{p=s'} b_{2p}(u^{2p} + u^{2-2p}), \tag{112.9}$$

where the coefficients are any real constants and the summations may pass to any limits, we see that the function will satisfy the condition necessary to determine a self-conjugate null curve; and we see that this is the most general function which will do so.

If we only take a finite number of constants the self-conjugate null curve which results will be unicursal.

More generally, if we take $f(u)$ to be an algebraic function of u, then $f'(u)$ and $f''(u)$ will also be algebraic functions of u. We can then express the Cartesian coordinates of any point on the self-conjugate null curve rationally in terms of $f(u)$, $f'(u)$, $f''(u)$, and u. We shall then have six algebraic equations, connecting the three Cartesian coordinates and the four quantities $f(u)$, $f'(u)$, $f''(u)$, and u. We can eliminate these four quantities and there will result two algebraic equations connecting the Cartesian coordinates.

We have now seen how to construct null curves and self-conjugate null curves; and also how we can construct self-conjugate null curves which will be unicursal; and yet more generally how to construct self conjugate null curves which will be algebraic.

§ **113. Generation of minimal surfaces from null curves. Double minimal surfaces.** When the minimal surface is referred to the null lines on it as parametric lines we have

$$a_{11} = 0, \quad a_{22} = 0,$$

and therefore, since $a_{11}\Omega_{22} + a_{22}\Omega_{11} = 2a_{12}\Omega_{12}$, (113.1)
we must have Ω_{12} equal to zero.

That is, we have

$$z_1^2 = 0, \quad z_2^2 = 0, \quad \underbrace{z_1\lambda_2} = \underbrace{z_2\lambda_1} = 0:$$

and therefore, since $\underbrace{\lambda z_1} = 0$, $\underbrace{\lambda z_2} = 0$,

we have $$\underbrace{\lambda z_{12}} = 0.$$

We also have, from $z_1^2 = 0 \quad z_2^2 = 0$,

that $$\underbrace{z_1 z_{12}} = 0, \quad \underbrace{z_2 z_{12}} = 0,$$

and therefore $$z_{12} = p\lambda,$$

where p is a scalar. But $\underbrace{\lambda z_{12}} = 0$,

and therefore $$z_{12} = 0. \qquad (113.2)$$

The minimal surface is therefore a particular case of a translation surface.

A translation surface is defined by

$$z = \alpha + \beta, \qquad (113.3)$$

where α is a vector describing a curve whose parameter is u and β a vector describing a curve whose parameter is v. We see why it is called a translation surface as we can generate it by translating the u curve along the v curve or translating the v curve along the u curve.

We might also define a translation surface by

$$2z = \alpha + \beta, \qquad (113.4)$$

when we see that it is the locus of the middle points of chords one extremity of which lies on one curve and one on the other.

In the case of the minimal surface we also have

$$(d\alpha)^2 = 0 \text{ and } (d\beta)^2 = 0,$$

since $$z_1^2 = 0 \text{ and } z_2^2 = 0.$$

The minimal surface is therefore given by

$$2z = \alpha + \beta$$

where α and β are vectors tracing out null curves.

If we confine ourselves to real minimal surfaces the null curves must be conjugate and the parameters of the two points must be conjugate complexes. It is obvious that such conjugate null curves will, if the corresponding parameters are conjugate complexes, give a real surface, and the converse may be proved.

If the null curve is a self-conjugate curve, however, we must take as the corresponding complex, not the conjugate complex, but the inverse complex.

Thus the general real minimal surface is given by

$$2z = \alpha_u + \bar{\alpha}_v; \tag{113.5}$$

and the real minimal surface generated from a self-conjugate null curve is given by

$$2z = \alpha_u + \alpha_{\left(-\frac{1}{v}\right)}, \tag{113.6}$$

where the suffix is the parameter of the null curve which is to be taken.

We notice that in the minimal surface

$$2z = \alpha_u + \alpha_{\left(-\frac{1}{v}\right)},$$

as we pass from the point whose parameters are u, v by a continuous path to a point whose parameters are $-\frac{1}{v}$, $-\frac{1}{u}$, we return to the point from which we started; the z of the point will be the same but the λ will be changed into $-\lambda$. That is, we are on the other side of the surface. For this reason the surface is called a double minimal surface.

§ **114. Henneberg's surface.** We have now seen how minimal surfaces are generated from null curves, and how real minimal surfaces are to be obtained, and how real double minimal surfaces may be generated.

From what we said about the construction of null curves we see how to obtain minimal surfaces which will be rational functions of their parameters and how to obtain more generally algebraic minimal surfaces; and from what we said about the construction of self-conjugate null curves we can construct these surfaces to be double minimal surfaces.

Thus

$$2z = \left(\left(\frac{1-u^2}{u}\right)^3 + \left(\frac{1-v^2}{v}\right)^3\right)\iota' + \iota\left(\left(\frac{1+u^2}{u}\right)^3 - \left(\frac{1+v^2}{v}\right)^3\right)\iota''$$
$$+ 3\,(u^2 + u^{-2} + v^2 + v^{-2})\,\iota''' \qquad (114.1)$$

will be an example of a real double minimal surface as may easily be verified. It is known as Henneberg's surface.

It may easily be shown that a minimal surface will then only be algebraic when the null curves which generate it are algebraic.

§ 115. **Lines of curvature and asymptotic lines on minimal surfaces.** We have for a minimal surface

$$2z = \alpha + \beta$$

and, if the surface is to be real,

$$2z = \alpha_u + \bar{\alpha}_v.$$

It follows that

$$2z_1 = \rho f'''(u), \quad 2z_2 = \sigma \bar{f}'''(v),$$

and therefore $\qquad 4\,\widehat{z_1 z_2} = f'''(u)\,\bar{f}'''(v)\,\widehat{\rho\sigma}.$

But $\qquad 4\,\widehat{\rho\sigma} = (1+uv)^4\,\widehat{\lambda_2\lambda_1},$

and, if R, $-R$ are the principal radii of curvature of the surface, $\qquad \widehat{z_1 z_2} = -R^2\widehat{\lambda_1\lambda_2}.$

We therefore have

$$16\,R^2 = f'''(u)\,\bar{f}'''(v)\,(1+uv)^4. \qquad (115.1)$$

If we write $f'''(u) \equiv \left(\frac{\partial\psi}{\partial u}\right)^2, \quad \bar{f}'''(v) \equiv \left(\frac{\partial\bar{\psi}}{\partial v}\right)^2,$

then $\qquad 4\,R = \frac{\partial\psi}{\partial u}\frac{\partial\bar{\psi}}{\partial v}(1+uv)^2,$

and $\qquad 2z_1 = \rho\left(\frac{\partial\psi}{\partial u}\right)^2, \quad 2z_2 = \sigma\left(\frac{\partial\bar{\psi}}{\partial v}\right)^2.$

We then have

$$dz^2 = \tfrac{1}{2}\rho\sigma\left(\frac{\partial\psi}{\partial u}\right)^2\left(\frac{\partial\psi}{\partial v}\right)^2 du\,dv$$

$$= -\tfrac{1}{4}(1+uv)^2\left(\frac{\partial\psi}{\partial u}\right)^2\left(\frac{\partial\bar\psi}{\partial v}\right)^2 du\,dv$$

$$= -R\,d\psi\,d\bar\psi,$$

so that if $\psi \equiv \xi + \iota\eta, \quad \bar\psi \equiv \xi - \iota\eta$

we come back to the ground form

$$ds^2 = R\,(d\xi^2 + d\eta^2)$$

for the surface.

The lines of curvature are

$$\xi = \text{constant}, \quad \eta = \text{constant},$$

and the asymptotic lines are

$$\xi + \eta = \text{constant}, \quad \xi - \eta = \text{constant}.$$

We may therefore say, if $R\phi(u)$ denotes the real part of $\phi(u)$, that one family of the lines of curvature is

$$R\int \sqrt{f'''(u)}\,du = \text{constant}, \qquad (115.2)$$

and the other $\quad R\iota\int \sqrt{f'''(u)}\,du = \text{constant}; \qquad (115.3)$

whilst the asymptotic lines are given by

$$R\int \sqrt{\iota f'''(u)}\,du = \text{constant}, \qquad (115.4)$$

$$R\int \sqrt{-\iota f'''(u)}\,du = \text{constant}. \qquad (115.5)$$

§ **116. Associate and adjoint minimal surfaces.** The surface obtained by substituting for f the function $e^{\iota\alpha}f$ where α is a real constant is said to be an *associate* minimal surface to f; and when we take for α the number $\frac{\pi}{2}$ it is said to be the adjoint minimal surface.

An associate minimal surface is applicable on the surface to which it is associate and the normals are parallel at corresponding points.

If ζ is the vector which traces out the adjoint surface to z

$$2\,dz = f'''(u)\,\rho\,du + \bar{f}'''(v)\,\sigma\,dv,$$

$$2\,d\zeta = \iota\left(f'''(u)\,\rho\,du - \bar{f}'''(v)\,\sigma\,dv\right), \qquad (116.1)$$

so that these two surfaces will also correspond orthogonally.

We see that $z - \iota\zeta$ traces out not a surface but a null curve, and $z + \iota\zeta$ traces out the conjugate null curve.

Since $$\widehat{\rho\lambda} = -\iota\rho, \quad \widehat{\sigma\lambda} = \iota\sigma,$$

we also see that $$d\zeta = \widehat{\lambda\,dz}. \qquad (116.2)$$

If then we are given a curve on the surface we shall know the ζ which will correspond to z along this curve, if we know the normal to the surface along the curve. We shall therefore know $z + \iota\zeta$ and $z - \iota\zeta$ along the given curve, and thus have the null curves which generate the minimal surface.

The formula $$\alpha_u = z - \iota\int \widehat{\lambda\,dz} \qquad (116.3)$$

is due to Schwartz.

CHAPTER IX

THE PROBLEM OF PLATEAU AND CONFORMAL REPRESENTATION

§ 117. **The minimal surface with a given closed boundary.** Any account of minimal surfaces would be incomplete without some reference to the problem proposed by Lagrange: 'To determine the minimal surface with a given closed boundary, and with no singularity on the surface within the boundary.' This problem is known as the Problem of Plateau, who solved it experimentally. The problem has not yet been solved mathematically in its general form; but has been solved in some particular cases, where the bounding curve consists of straight lines and plane arcs of curves.

Consider a part of the bounding curve, which is a straight line, on a minimal surface. This line must be an asymptotic line on the surface. Now we saw (§ 109) that, when the surface is referred to the lines of curvature, as parametric lines, the equation of the asymptotic lines is

$$du^2 - dv^2 = 0; \tag{117.1}$$

and the ground form of the surface is

$$R\,(du^2 + dv^2), \tag{117.2}$$

and the ground form of the spherical image is

$$R^{-1}\,(du^2 + dv^2). \tag{117.3}$$

We conclude that when the surface is conformally represented on the plane, on which u and v are the rectangular coordinates, the asymptotic lines are conformally represented by lines parallel to the bisectors of the angle between the axes, and the lines of curvature, and also their spherical images, are conformally represented by lines parallel to the axes.

If a part of the bounding curve is a plane curve, whose plane cuts the minimal surface orthogonally, and is therefore a geodesic, it must be a line of curvature. It will therefore be conformally represented on the plane by a line parallel to one of the axes.

If then the whole of the bounding curve is composed of straight lines and such curves, the bounding curve will be conformally represented on the plane by a figure, bounded by straight lines, parallel either to the axes or to the bisectors; and the part of the minimal surface, within the boundary, will be represented by the area of the plane within the polygon.

Next let us consider the spherical image of the surface within and on the boundary. At each point of the boundary, the normal to the surface will be perpendicular to a direction, which will not change as we pass along a continuous part of the boundary, but will change at each angle of the boundary.

The boundary will therefore consist of arcs of great circles.

If therefore we can find a function of w, the complex variable which defines the position of any point on the sphere, which will transform the spherical boundary into the plane boundary, and points within the spherical boundary to points within the plane boundary, we shall have $u+\iota v$ known in terms of w, and can proceed to find the required surface as follows.

We have (109.10) for an element of the sphere

$$d\sigma^2 = d\theta^2 + \sin^2\theta\, d\phi^2, \tag{117.4}$$

so that

$$w = \cot\frac{\theta}{2} e^{\iota\phi} \tag{117.5}$$

is the complex variable which defines the position of points on the sphere.

The normal, to the sphere, which is given by w, is by (110.1)

$$(1+w\bar{w})\lambda = (w+\bar{w})\iota' - \iota(w-\bar{w})\iota'' + (w\bar{w}-1)\iota''', \tag{117.6}$$

where ι', ι'', ι''' are fixed unit vectors, mutually at right angles, ι is $\sqrt{-1}$, and $\bar{w}$ the conjugate complex to w.

Now we know that in terms of u and v

$$d\sigma^2 = R^{-1}(du^2 + dv^2)$$

and therefore $R^{-1}(du^2 + dv^2) = d\theta^2 + \sin^2\theta\, d\phi^2$. (117.7)

As we have seen in § 109, R is therefore known, being given by

$$R = \tfrac{1}{4}(1 + w\bar{w})^2 \left| \frac{dx}{dw} \right|^2. \qquad (117.8)$$

We can therefore construct the surface since R and λ are known in terms of w and w.

We can retrace our steps and see that the surface we have obtained satisfies the conditions required.

We are thus led to the problem of conformal representation, and this we proceed to discuss, so far as it bears on the question before us.

§ 118. **The notation of a linear differential equation of the second order with three singularities.** Let a, c, b be three real quantities in ascending order of magnitude, and let x be a complex variable.

When x lies on the real axis between $-\infty$ and a, or between b and $+\infty$, we see that

$$\frac{b-c}{b-a}\,\frac{x-a}{x-c}$$

lies between zero and positive unity. When x lies between a and c

$$\frac{c-b}{c-a}\,\frac{x-a}{x-b}$$

lies between zero and positive unity. We also see that the reciprocals of these two expressions lie between zero and positive unity, when x lies between c and b.

When x is complex we see that the modulus of one of the first two expressions is less than unity, or the modulus of each of the reciprocals is less than unity.

Let α_1, α_2, β_1, β_2, γ_1, γ_2 be six quantities real or complex, but such that $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 1$; (118.1)

and such that the real parts of

$$\alpha_2 - \alpha_1, \quad \beta_2 - \beta_1, \quad \gamma_2 - \gamma_1$$

are each positive.

Let $$P \equiv \frac{1-\alpha_1-\alpha_2}{x-a} + \frac{1-\beta_1-\beta_2}{x-b} + \frac{1-\gamma_1-\gamma_2}{x-c}, \quad (118.2)$$

$$(x-a)(x-b)(x-c)Q$$
$$\equiv \frac{\alpha_1\alpha_2(a-b)(a-c)}{x-a} + \frac{\beta_1\beta_2(b-a)(b-c)}{x-b} + \frac{\gamma_1\gamma_2(c-a)(c-b)}{x-c}, \quad (118.3)$$

and let $(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, a, b, c, x)$

denote the hypergeometric series

$$F(p, q, r, \xi) = 1 + \frac{p \cdot q}{1 \cdot r}\xi + \frac{p(p+1)}{1 \cdot 2}\frac{q(q+1)}{r(r+1)}\xi^2 + \ldots \quad (118.4)$$

where $p = \alpha_1+\beta_1+\gamma_1$, $q = \alpha_1+\beta_2+\gamma_1$, $r = 1+\alpha_1-\alpha_2$,

$$\xi = \frac{c-b}{c-a}\frac{x-a}{x-b}.$$

We notice that P and Q are unaltered by the following substitutions:

$$(\alpha_1\alpha_2), \quad (\beta_1\beta_2), \quad (\gamma_1\gamma_2), \quad (\beta_1\gamma_1)(\beta_2\gamma_2)(bc),$$
$$(\gamma_1\alpha_1)(\gamma_2\alpha_2)(ca), \quad (\alpha_1\beta_1)(\alpha_2\beta_2)(ab). \quad (118.5)$$

§ **119. Conformal representation on a triangular area.** Consider now the differential equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0. \quad (119.1)$$

It is known, and may easily be proved, that

$$y \equiv \left(\frac{x-a}{x-b}\right)^{\alpha_1}\left(\frac{x-c}{x-b}\right)^{\gamma_1}\frac{(a-b)^{\alpha_1+\gamma_1}}{(a-c)^{\gamma_1}}$$
$$(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, a, b, c, x) \quad (119.2)$$

is a power series, beginning with $(x-a)^{\alpha_1}$ for its first term and expansible in powers of $x-a$ in the neighbourhood of $x = a$, which will satisfy the differential equation. This series when x lies on the real axis is valid when x lies between a and c. It is therefore valid at any point in the plane, the circle through which, having a and b as limiting points, intersects the real axis between a and c.

Another power series also beginning with $(x-a)^{\alpha_1}$ can be obtained from the first by applying the substitution

$$(\beta_1\gamma_1)(\beta_2\gamma_2)(bc)$$

to it. The two series will therefore be identical at any point where they are both valid. The second is valid for real values of x between $-\infty$ and a and between b and $+\infty$. The region for which it is valid, when x is complex, can be obtained by a similar rule to that which was used as regards the first series.

When one series is valid, but not the other, the valid series is a continuation of the other. We denote these series by $Y\alpha_1$.

By applying the substitution $(\alpha_1\alpha_2)$ we obtain two other series beginning with $(x-a)^{\alpha_2}$, valid over the same part of the plane. We denote these series by $Y\alpha_2$.

By applying the substitution $(\gamma_1\alpha_1)(\gamma_2\alpha_2)(ca)$ we get two series, $Y\gamma_1$ beginning with $(x-c)^{\gamma_1}$, and $Y\gamma_2$ beginning with $(x-c)^{\gamma_2}$, valid over the part of the plane which corresponds to real values of x between c and b.

By applying the substitution $(\beta_1\gamma_1)(\beta_2\gamma_2)(bc)$ to these last two series we get two other series, $Y\beta_1$ beginning with $(x-b)^{\beta_1}$, and $Y\beta_2$ beginning with $(x-b)^{\beta_2}$, valid over the same part of the plane. All these series, when valid, satisfy the equation.

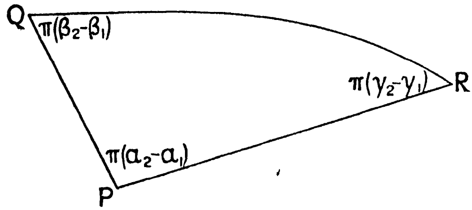
Let
$$w = \frac{Y\alpha_2}{Y\alpha_1}.$$

Then we see that, as x describes the real axis from b to $+\infty$, and then from $-\infty$ to a, w varies continuously and its argument is $\pi(\alpha_2-\alpha_1)$, if we agree that the argument of a positive

quantity is to be taken as zero, and the argument of a negative quantity as π, as x describes the real axis in this definite way.

Let Q be the point in the w plane which corresponds to

$x = b$, and let P be the origin in the w plane corresponding to $x = a$. As x describes the path defined, w describes the



straight line QP.

When x describes the semicircle about a the argument of w diminishes by $\pi(\alpha_2 - \alpha_1)$, and as x describes the real part of the axis the argument of w remains zero, till we come to R, which corresponds to $x = c$.

We must now consider what happens as x describes the semicircle round c, and then, passing along the real axis, comes to b and passes round the semicircle there.

Over any part of the plane which corresponds to real values of x between c and b we can express w in either of the forms

$$\frac{A + B\dfrac{Y\gamma_2}{Y\gamma_1}}{C + D\dfrac{Y\gamma_2}{Y\gamma_1}},$$

or

$$\frac{A' + B'\dfrac{Y\beta_2}{Y\beta_1}}{C' + D'\dfrac{Y\beta_2}{Y\beta_1}},$$

where A, B, C, D and A', B', C', D' are certain constants.

We see this from the known properties of a linear differential equation of the second order.

Now the argument of $\dfrac{Y\gamma_2}{Y\gamma_1}$ is the same as that of

$$\left(\frac{x-c}{x-a}(c-a)\right)^{\gamma_2-\gamma_1},$$

and therefore zero, as x passes along the real axis from c to b.

It follows that w describes a circle which passes through Q and R.

The increment of the argument of $w - \frac{A}{C}$ as we pass along the semicircle c is the same as the increment of the argument of $\frac{Y\gamma_2}{Y\gamma_1}$; that is, it diminishes by $\pi(\gamma_2 - \gamma_1)$. The circular arc through R therefore makes an angle $\pi(\gamma_2 - \gamma_1)$ with RP.

In the same way we see that the angle at Q is $\pi(\beta_2 - \beta_1)$.

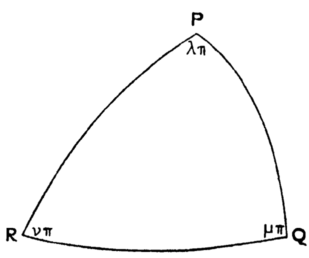
Since, when x moves from its real axis to the positive side of its plane, w must move to the inner part of the triangle PQR, we see that the positive part of the plane of x is conformally represented by the inner part of the triangle.

§ **120. The w-plane or part of it covered with curvilinear triangles.** Consider now the transformation

$$x' = \frac{px + q}{rx + s}$$

where p, q, r, s are any constants, real or complex.

If x describes a circle (or as a particular case a straight line) in its plane, so will x'. If x_1 and x_2 are any two points



inverse to the circle x, then x'_1 and x'_2 will be inverse to the circle x'.

We thus see that if P, Q, R are the three points which, in the above transformation, with $Y\alpha_2 \div Y\alpha_1$ substituted for x, correspond to the singularities at a, b, c, the curvilinear triangle PQR, formed by three circular arcs intersecting at angles $\lambda\pi$, $\mu\pi$, $\nu\pi$, where

$$\lambda = \alpha_2 - \alpha_1, \quad \mu = \beta_2 - \beta_1, \quad \nu = \gamma_2 - \gamma_1,$$

will enclose the part of the w plane, which conformally represents the upper part of the x plane.

Let w be the complex variable which defines any point S within the triangle PQR, and let w_1 be the complex variable which defines the point S_1, which is inverse to S with respect to the arc RQ.

Let $$w' = \frac{pw+q}{rw+s} \tag{120.1}$$

be the substitution which transforms the arc RQ to a part of the real axis of w in its plane.

Then $$\frac{pw+q}{rw+s} \text{ and } \frac{pw_1+q}{rw_1+s} \tag{120.2}$$

are inverse to one another with respect to the real axis of w.

Let $f(x)$ be the function of x which we found would in this case transform the upper part of the x plane to within the curvilinear triangle in the w plane. We now assume the quantities $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ to be all real. Along the real axis of the plane x the coefficients in $f(x)$ will be real, and therefore $f(\bar{x})$ will be the function which will transform the lower part of the plane x to points without the curvilinear triangle, where $\bar{x}$ denotes the conjugate variable to x.

We therefore have

$$\frac{pw+q}{rw+s} = f(x) \text{ and } \frac{pw_1+q}{rw_1+s} = f(\bar{x}).$$

It follows that

$$w = \frac{sf(x)-q}{p-rf(x)} \text{ and } w_1 = \frac{sf(\bar{x})-q}{p-rf(\bar{x})}; \tag{120.3}$$

and consequently we have

$$\bar{w} = \frac{\bar{s}f(\bar{x})-\bar{q}}{\bar{p}-\bar{r}f(\bar{x})}. \tag{120.4}$$

Eliminating $f(\bar{x})$ we have

$$w_1 = \frac{(\bar{p}s-q\bar{r})\,\bar{w}+\bar{q}s-q\bar{s}}{(p\bar{r}-r\bar{p})\,\bar{w}+p\bar{s}-\bar{q}r}.$$

If then $$w = F(x),$$

and $$w_1 = \phi(\bar{x}),$$

then $$\phi(\bar{x}) = \frac{(\bar{p}s - q\bar{r})\bar{F}(\bar{x}) + \bar{q}s - q\bar{s}}{(p\bar{r} - r\bar{p})\bar{F}(\bar{x}) + p\bar{s} - \bar{q}r}. \qquad (120.5)$$

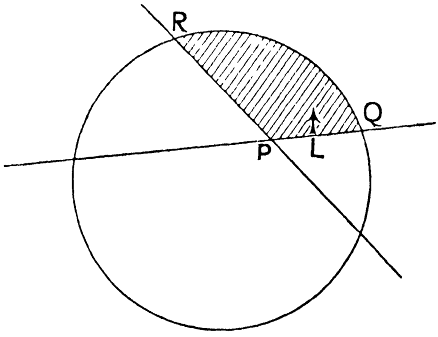
If P_1 is the inverse of P in the arc QR, we thus see how the lower part of the x plane is conformally represented on the triangle P_1QR in the w plane.

Similarly if Q_1 is the inverse of Q in RP, and R_1 the inverse of R in PQ, we can conformally represent the lower part of the x plane on the triangle Q_1RP, and on the triangle R_1PQ.

Just in the same way from the triangle P_1QR we can by inversion obtain three other triangles, one of which will be the triangle PQR. These triangles will give conformal representations of the upper part of the plane x on the plane of w.

Proceeding thus we cover the whole, or a part, of the w plane with curvilinear triangles.

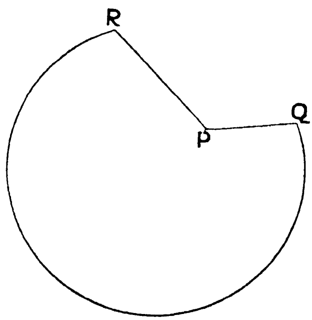
§ 121. Consideration of the case when triangles do not overlap. In general these triangles will overlap, so that a point in the w plane may be counted many times over: in



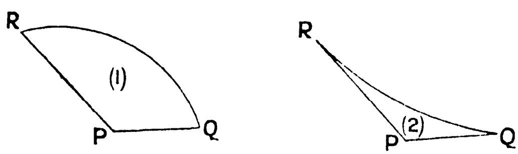
fact, unless λ, μ, ν are commensurable, a point in the w plane which lies within any triangle will lie within an infinite number of triangles. If, however, λ, μ, ν are each the reciprocal of a whole number there will be no overlapping at all. We now confine ourselves to this case.

One and only one circle can be drawn to cut orthogonally the arcs of the fundamental curvilinear triangle in the w plane. By inversion we may take PQ and PR to be straight lines.

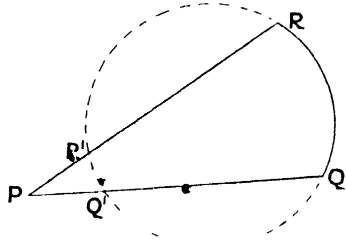
We see that the two straight lines and the circle divide the w plane into eight parts. We see, however, by considering the original figure with which we began this discussion, that the triangle with which we are concerned is the shaded one. For at the point L the variable w will move in the direction of the arrow, for a corresponding movement of z to the upper part of the x plane; and, as w will not move off to infinity, the triangle could not be the outward part of



The triangle PQR is therefore of one of the two forms



In case (1) P must lie within the circle of which RQ is the arc.



For, otherwise, the sum of the angles at Q and R being—for we are now assuming $\lambda = \frac{1}{p}$, $\mu = \frac{1}{q}$, $\nu = \frac{1}{r}$ —

$$\pi\left(\frac{1}{q} + \frac{1}{r}\right),$$

the sum of the angles at Q' and R' would be

$$\pi\left(2 - \frac{1}{q} - \frac{1}{r}\right);$$

and therefore $$2 - \frac{1}{q} - \frac{1}{r} < 1.$$

But this is not possible if q and r are integers. No real circle can therefore be drawn with P as centre to cut the arc QR orthogonally in case (1).

The two cases are therefore thus distinguished: in case (1)

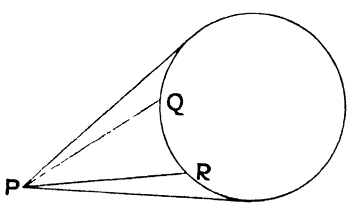
$$\lambda + \mu + \nu > 1, \qquad (121.1)$$

and the orthogonal circle is imaginary: in case (2)

$$\lambda + \mu + \nu < 1, \qquad (121.2)$$

and the circle which is orthogonal to the three arcs is real.

§ **122. Case of a real orthogonal circle as natural boundary.** Taking case (2), the circle, whose centre is at P and which cuts the arc QR orthogonally, must intersect the circle QR at the points of contact of tangents to the circle from P. Clearly these points are without the arc QR, since the arc QR is



convex with respect to P. The points P, Q, R therefore lie within the orthogonal circle. When we invert with respect to a point outside the orthogonal circle we have three circular arcs within the new orthogonal circle. By considering the point P_1 which is the inverse of P with respect to QR, we see that P_1 also lies within the orthogonal circle. Proceeding thus we see that all the curvilinear triangles are within the real orthogonal circle which corresponds to the case

$$\lambda+\mu+\nu < 1.$$

In this case, therefore, only the part of the w plane which lies within the orthogonal circle is covered with the curvilinear triangles, which conformally represent the x plane on the w plane. This circle is therefore the natural boundary of the function which, with its various continuations across the real axis of x, conformally transforms the x plane to the w plane.

Since there are an infinite number of solutions of the inequality

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}< 1,$$

where p, q, and r are integers, we get an infinite number of triangles which grow smaller and smaller as we continue to invert and invert: and as we approach the boundary—the orthogonal circle—the triangles tend to become mere point triangles.

§ **123. Fundamental spherical triangles when there is no natural boundary.** We now consider the first case when

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}> 1$$

and the orthogonal circle is imaginary.

If we stereographically project the w plane on to a sphere which touches the w plane at the real centre of the orthogonal circle, the fundamental curvilinear triangle becomes a spherical triangle which we shall now denote by ABC.

The only possible solutions of the inequality are

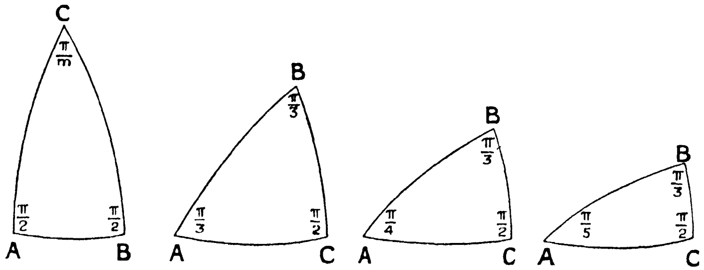
(1) $p=2,\ q=2,\ r=m$; (2) $p=2,\ q=3,\ r=3$;

(3) $p=2,\ q=3,\ r=4$; (4) $p=2,\ q=3,\ r=5$;

or equivalent results obtained by permutation of the integers. We lose nothing by taking A, B, C to be the correspondents to the singular points a, c, b in the x plane.

We may thus have for the fundamental spherical triangle any of the four figures which follow.

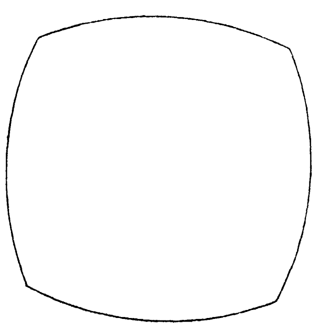
The operation of inversion is now replaced by the simple operation of taking the reflexion of each vertex with respect to the opposite side. We see at once that the whole surface of the w sphere is covered by the triangles and their images.



In the first case we have $2m$ triangles in the upper part of the hemisphere and $2m$ triangles in the lower part.

In case two we have a triangle whose area is $\frac{1}{24}$ that of the sphere, and by taking the six triangles with a common vertex at A we have an equilateral triangle whose area is $\frac{1}{4}$ that of the sphere: that is, we have the face of a regular tetrahedron.

In case three, which is just that of the triangle formed by bisecting the angle C in case two, we have a triangle whose area is $\frac{1}{48}$ that of the sphere. By taking the eight triangles with a common vertex at A we have the equilateral quadrilateral



whose area is $\frac{1}{6}$ that of the sphere, that is, the face of a regular cube. Its angles are each $\frac{2\pi}{3}$; and it is also the figure formed by planes, through the centre of the sphere circumscribing a regular tetrahedron, perpendicular to two pairs of opposite edges.

In case four we have a triangle whose area is $\frac{1}{120}$ that of the sphere. By taking the six triangles, with a common vertex at B, we obtain an equilateral triangle, whose area is $\frac{1}{20}$ that of the sphere: that is, a face of the regular icosahedron.

§ 124. **Summary of conclusions.** When λ, μ, and ν are then the reciprocals of integers, we have found functions w of the complex variable, which will conformally transform the upper and lower halves of the x plane into the area within the curvilinear triangles in the w plane. To each point in the x plane there will correspond, in the w plane, one point in each triangle or in the triangle adjacent which is its inverse. The real axis will be transformed into the circular boundaries of these triangles.

Two different points in the x plane cannot have the same w to correspond to them. For by taking λ, μ, and ν to be the reciprocals of integers we have provided against any overlapping in the w plane.

It follows that x is a uniform function of w.

In the case where $\lambda+\mu+\nu>1$ there are only a finite number of values of w which will make x zero or infinite; and therefore x will be a rational function of w. We could express each value of w which makes x zero in terms of any one, and thus obtain the numerator of the rational function. Similarly we could find the denominator. As we only wish to give a general explanation we do not enter into any details.

We have now shown how to represent the w plane, or its equivalent sphere, on the x plane.

§ 125. **Representation of the x-plane on a given polygon.** To complete the problem of conformal representation in so far as it bears on the problem of Plateau, we have now only to show how the x plane can be conformally represented on a given polygon. The procedure is much the same as in the problem we have just discussed, but much simpler.

Let a, c, b be defined as earlier and let α, β, γ be three real constants which are positive, and such that

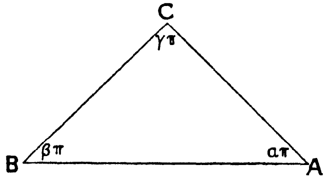
$$\alpha+\beta+\gamma = 1. \tag{125.1}$$

Let $$X \equiv \int_{-\infty}^{x} (x-a)^{\alpha-1}(x-b)^{\beta-1}(x-c)^{\gamma-1}\,dx, \tag{125.2}$$

and let A be the position which X attains as x moving along its real axis approaches a.

As x moves along the real axis in its plane from $-\infty$ to a, the argument of X is zero, so that it too moves along the real axis of its plane. As x moves along the small semicircle with centre at a, the argument of X diminishes by $\alpha\pi$. As x then moves along the real axis to c, X moves along a straight line AC to C, the point which corresponds to c. When x describes the semicircle at c, the argument of X again diminishes by $\gamma\pi$. Then as x moves along the real axis from c to b, X describes a straight line CB to B the point which corresponds to b. X is now again on its real axis; and as x describes the semicircle at b the argument of X diminishes by $\beta\pi$. Finally as x moves along the real axis to $+\infty$ and then from $-\infty$ to a, X describes the straight line BA.

We thus have the figure



in the plane of X, and the upper half of the x plane is conformally represented by the area within this triangle.

By a transformation of the form $X' = pX+q$ where p and q are constants the triangle may be transformed into any similar and similarly placed triangle in the plane of X; and thus the upper half of the x plane may be conformally represented by the area within the triangle ABC which lies in the plane of X anywhere.

We thus see, as before, that the plane of x can be represented by a series of triangles in the plane of X, which will cover it completely. But if there is to be no overlapping we must have α, β, and γ to be the reciprocals of integers.

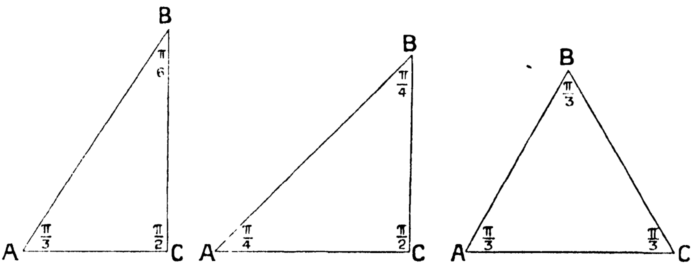
These integers must satisfy the equation

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \qquad (125.3)$$

and we see that the only solutions of this equation are

$$p = 6, \quad q = 3, \quad r = 2;$$
$$p = 4, \quad q = 4, \quad r = 2;$$
$$p = 3, \quad q = 3, \quad r = 3. \qquad (125.4)$$

We thus have three cases



and we see into what kind of triangles the given polygon must be decomposable in order that x may be a uniform function of X.

We see that X is a doubly periodic function of x; and from the above triangles, and their images in the sides, with respect to the opposite vertex, we can construct the period parallelograms.

§ **126.** We have found corresponding to each value of w, the complex variable of the sphere, a definite value of x. This value of x will under certain circumstances which we have considered be a rational function of w. To this value of x we must choose, as its correspondent X, that value, or those values, which lie within the given polygon. Since the values of w which lie on the boundary of the spherical polygon are to correspond to values of X lying on the boundary of the plane polygon, and since these values of w correspond to points on the real axis of x, we see that the polygon must have its boundary made up of sides of the elementary triangles in the X plane.

The principal results in the theory I have tried to explain in outline are due to Riemann, to Weierstrass, and to Schwartz; and my presentation is based on the treatises of Darboux and Bianchi. The connexion of this branch of Geometry with the Theory of Functions is interesting.

CHAPTER X

ORTHOGONAL SURFACES

§ 127. A certain partial differential equation of the third order. We now want to consider the theory of a triply infinite system of mutually orthogonal surfaces; and we begin by considering the partial differential equation of the third order

$$p\frac{\partial \theta}{\partial x}+q\frac{\partial \theta}{\partial y}+\operatorname{sech}^2 x\frac{\partial \theta}{\partial w}=q\tanh x, \qquad (127.1)$$

where
$$2\theta \equiv \tan^{-1}\frac{2s+2q\tanh x}{r-t+2p\tanh x} \qquad (127.2)$$

and z is the dependent variable and x, y, and w the independent variables. [Here p, q, r, s, t denote respectively

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x\,\partial y}, \frac{\partial^2 z}{\partial y^2}.]$$

We shall see that it is on this equation that we depend when we wish to obtain the general system of orthogonal surfaces.

Let z be any function which satisfies this equation, and let

$$U\equiv\frac{\partial}{\partial x}-\cot\theta\frac{\partial}{\partial y}, \quad V\equiv\frac{\partial}{\partial x}+\tan\theta\frac{\partial}{\partial y},$$

$$W\equiv p\cosh^2 x\frac{\partial}{\partial x}+q\cosh^2 x\frac{\partial}{\partial y}+\frac{\partial}{\partial w};$$

then it is not difficult to verify that

$$UW-WU=U(p\cosh^2 x).U,$$

$$VW-WV=V(p\cosh^2 x).V. \qquad (127.3)$$

It follows that a function u exists which is annihilated by the operators V and W, and also a function v annihilated by U and W.

We may therefore regard x and y as functions of u, v, and w, and we have

$$\frac{\partial u}{\partial x}x_2+\frac{\partial u}{\partial y}y_2=0,\quad \frac{\partial u}{\partial x}x_3+\frac{\partial u}{\partial y}y_3+\frac{\partial u}{\partial w}=0,$$

$$\frac{\partial v}{\partial x}x_1+\frac{\partial v}{\partial y}y_1=0,\quad \frac{\partial v}{\partial x}x_3+\frac{\partial v}{\partial y}y_3+\frac{\partial v}{\partial w}=0;$$

where the suffixes 1, 2, 3 respectively denote differentiation with respect to u, v, w.

But from the definition of u and v we have

$$\frac{\partial u}{\partial x}+\tan\theta\frac{\partial u}{\partial y}=0,\quad p\cosh^2 x\frac{\partial u}{\partial x}+q\cosh^2 x\frac{\partial u}{\partial y}+\frac{\partial u}{\partial w}=0,$$

$$\frac{\partial v}{\partial x}-\cot\theta\frac{\partial v}{\partial y}=0,\quad p\cosh^2 x\frac{\partial v}{\partial x}+q\cosh^2 x\frac{\partial v}{\partial y}+\frac{\partial v}{\partial w}=0;$$

and therefore it follows that

$$x_1+y_1\tan\theta=0,\qquad x_2-y_2\cot\theta=0,$$
$$x_3-p\cosh^2 x=0,\quad y_3-q\cosh^2 x=0.\qquad (127.4)$$

We now see that

$$\frac{\partial}{\partial u}=x_1 U,\quad \frac{\partial}{\partial v}=x_2 V,\quad \frac{\partial}{\partial w}=W;$$

so that the equation with which we began becomes

$$\theta_3=q\sinh x\cosh x,\qquad (127.5)$$

that is, $\theta_3=y_3\tanh x.$

We thus have the three equations

$$x_1+y_1\tan\theta=0,\quad x_2-y_2\cot\theta=0,\quad \theta_3=y_3\tanh x.\qquad (127.6)$$

Now let

$$\xi\equiv e^{-x}\frac{\cos\frac{y+\theta}{2}}{\cos\frac{y-\theta}{2}},\quad \eta\equiv e^{-x}\frac{\sin\frac{y+\theta}{2}}{\cos\frac{y-\theta}{2}},\quad \zeta\equiv\tan\frac{y-\theta}{2},$$

so that $$\tan\theta=\frac{\eta-\xi\zeta}{\xi+\eta\zeta},$$

then we can verify that

$$\xi_1-\eta\zeta_1+\zeta\eta_1=0,\quad \eta_2-\zeta\xi_2+\xi\zeta_2=0,\quad \zeta_3-\xi\eta_3+\eta\xi_3=0.\qquad (127.7)$$

§ **128. A solution led to when functions satisfying a set of three equations are known.** By retracing our steps we may verify that if we have any three functions which satisfy these equations we shall be led back to the solution of the equation of the third order. For we have clearly

$$x_1 + y_1 \tan\theta = 0, \quad x_2 - y_2 \cot\theta = 0, \quad \theta_3 = y_3 \tanh x;$$

and therefore $$\frac{\partial}{\partial u} = x_1 U, \quad \frac{\partial}{\partial v} = x_2 V.$$

Now $$\frac{\partial}{\partial x} = \sin^2\theta\, U + \cos^2\theta\, V, \quad \frac{\partial}{\partial y} = \sin\theta\cos\theta\,(V-U);$$

and therefore we can verify that

$$\frac{\partial}{\partial y}\frac{x_3}{\cosh^2 x} = \frac{\partial}{\partial x}\frac{y_3}{\cosh^2 x},$$

so that $$x_3 = p\cosh^2 x, \quad y_3 = q\cosh^2 x. \qquad (128.1)$$

From $$x_{13} + y_{13}\tan\theta + y_1\sec^2\theta\,\theta_3 = 0$$
we verify that

$$U(p\cosh^2 x) + \tan\theta\, U(q\cosh^2 x)$$
$$= (\cot\theta + \tan\theta)\, q\sinh x\cosh x,$$

and therefore $$\tan 2\theta = \frac{2s + 2q\tanh x}{r - t + 2p\tanh x}. \qquad (128.2)$$

Finally we see that $$\frac{\partial}{\partial w} = W,$$

and thus the equation $\theta_3 = y_3 \tanh x$
or $\theta_3 = q\sinh x\cosh x$
becomes $$W\theta = q\sinh x\cosh x. \qquad (128.3)$$

The equations

$$\xi_1 - \eta\zeta_1 + \zeta\eta_1 = 0, \quad \eta_2 - \zeta\xi_2 + \xi\zeta_2 = 0, \quad \zeta_3 - \xi\eta_3 + \eta\xi_3 = 0$$

are thus connected in the way we have described with the partial differential equation of the third order.

§ **129. The vector $q\alpha q^{-1}$, where α is a vector and q a quaternion.** We now pass on to the geometry which we associate with these three equations.

If we are given any vector and any scalar quantity we can take the vector to be $r \sin\theta \,.\, \epsilon$ and the scalar quantity to be $r \cos\theta$, where ϵ is a unit vector.

Let $$q = r(\cos\theta + \epsilon \sin\theta), \tag{129.1}$$

then q is called a quaternion, ϵ is called the axis of the quaternion, θ is called its argument, and r is called its modulus.

A quaternion is thus just the ordinary complex variable of the plane perpendicular to the axis of the quaternion.

We have $$q^{-1} = r^{-1}(\cos\theta - \epsilon \sin\theta). \tag{129.2}$$

Any other vector may be written

$$x\epsilon' + y\epsilon, \tag{129.3}$$

where x and y are scalars and ϵ' is some unit vector at right angles to ϵ.

We see that $$q(x\epsilon' + y\epsilon)q^{-1}$$

is equal to $$x(\epsilon' \cos 2\theta + \epsilon'' \sin 2\theta) + y\epsilon, \tag{129.4}$$

where ϵ'' is the unit vector perpendicular to ϵ and ϵ'. That is, if α is any vector, $$q\alpha q^{-1} \tag{129.5}$$

is just the vector α rotated about the axis of the quaternion through an angle double the argument of the quaternion.

§ **130. Passage from set to set of three orthogonal vectors.** Let us now consider the quaternion

$$q = 1 + \xi i + \eta j + \zeta k, \tag{130.1}$$

where i, j, k are fixed unit vectors at right angles to one another and ξ, η, ζ are any three scalar functions of the parameters u, v, and w.

$$Dq^{-1} = 1 - \xi i - \eta j - \zeta k,$$

where $$D = 1 + \xi^2 + \eta^2 + \zeta^2. \tag{130.2}$$

Let $$\lambda = qiq^{-1}, \quad \mu = qjq^{-1}, \quad \nu = qkq^{-1}, \tag{130.3}$$

then λ, μ, ν will also be three unit vectors mutually at right angles to one another, no longer fixed vectors but depending on the parameters u, v, w.

Any system of mutually orthogonal unit vectors can be so defined.

We easily see that

$$D\widehat{q^{-1}q_1} = (\xi_1 - \eta\zeta_1 + \zeta\eta_1)\,i + (\eta_1 - \zeta\xi_1 + \xi\zeta_1)\,j + (\zeta_1 - \xi\eta_1 + \eta\xi_1)\,k. \tag{130.4}$$

Now $$qq^{-1}q_1q^{-1} = q_1q^{-1},$$

and therefore $$q\widehat{q^{-1}q_1}q^{-1} = \widehat{q_1q^{-1}}.$$

It follows that

$$D\widehat{q_1q^{-1}} = (\xi_1 - \eta\zeta_1 + \zeta\eta_1)\,\lambda + (\eta_1 - \zeta\xi_1 + \xi\zeta_1)\,\mu + (\zeta_1 - \xi\eta_1 + \eta\xi_1)\,\nu. \tag{130.5}$$

From $$qi = \lambda q$$

we have $$q_1q^{-1}\lambda - \lambda q_1q^{-1} = \lambda_1,$$

and therefore $$\widehat{q_1q^{-1}}\lambda - \lambda\widehat{q_1q^{-1}} = \lambda_1.$$

It follows that, since

$$\mu\nu - \nu\mu = 2\lambda, \quad \nu\lambda - \lambda\nu = 2\mu, \quad \lambda\mu - \mu\lambda = 2\nu,$$

$$D\lambda_1 = -2\,(\eta_1 - \zeta\xi_1 + \xi\zeta_1)\,\nu + 2\,(\zeta_1 - \xi\eta_1 + \eta\xi_1)\,\mu. \tag{130.6}$$

Let $\tfrac{1}{2}Dp' \equiv \xi_1 - \eta\zeta_1 + \zeta\eta_1$, $\tfrac{1}{2}Dq' \equiv \eta_1 - \zeta\xi_1 + \xi\zeta_1$,

$$\tfrac{1}{2}Dr' \equiv \zeta_1 - \xi\eta_1 + \eta\xi_1, \quad \tfrac{1}{2}Dp'' \equiv \xi_2 - \eta\zeta_2 + \zeta\eta_2,$$
$$\tfrac{1}{2}Dq'' \equiv \eta_2 - \zeta\xi_2 + \xi\zeta_2, \quad \tfrac{1}{2}Dr'' \equiv \zeta_2 - \xi\eta_2 + \eta\xi_2,$$
$$\tfrac{1}{2}Dp''' \equiv \xi_3 - \eta\zeta_3 + \zeta\eta_3, \quad \tfrac{1}{2}Dq''' \equiv \eta_3 - \zeta\xi_3 + \xi\zeta_3,$$
$$\tfrac{1}{2}Dr''' \equiv \zeta_3 - \xi\eta_3 + \eta\xi_3;$$

then we have proved (130.6) that $\lambda_1 = \mu r' - \nu q'$; and similarly we prove the other equations of the system

$$\begin{aligned} \lambda_1 &= \mu r' - \nu q', & \mu_1 &= \nu p' - \lambda r', & \nu_1 &= \lambda q' - \mu p', \\ \lambda_2 &= \mu r'' - \nu q'', & \mu_2 &= \nu p'' - \lambda r'', & \nu_2 &= \lambda q'' - \mu p'', \\ \lambda_3 &= \mu r''' - \nu q''', & \mu_3 &= \nu p''' - \lambda r''', & \nu_3 &= \lambda q''' - \mu p'''. \end{aligned} \tag{130.7}$$

It may be noticed that the q', q'', q''' as here defined have no direct relationship to the quaternion q where

$$q = 1 + \xi i + \eta j + \zeta k.$$

If $\omega' \equiv p'\lambda + q'\mu + r'\nu$, $\omega'' \equiv p''\lambda + q''\mu + r''\nu$,

$$\omega''' \equiv p'''\lambda + q'''\mu + r'''\nu,$$

we see that $\lambda_1 = \widehat{\omega'\lambda}, \quad \mu_1 = \widehat{\omega'\mu}, \quad \nu_1 = \widehat{\omega'\nu},$

$$\lambda_2 = \widehat{\omega''\lambda}, \quad \mu_2 = \widehat{\omega''\mu}, \quad \nu_2 = \widehat{\omega''\nu},$$

$$\lambda_3 = \widehat{\omega'''\lambda}, \quad \mu_3 = \widehat{\omega'''\mu}, \quad \nu_3 = \widehat{\omega'''\nu}, \qquad (130.8)$$

and we can easily verify from the formulae given that

$$\omega' = 2\widehat{q_1 q}^{-1}, \quad \omega'' = 2\widehat{q_2 q}^{-1}, \quad \omega''' = 2\widehat{q_3 q}^{-1} \quad (130.9)$$

where q is the quaternion.

We also have—as we proved earlier [see § 90]—the equations

$$\omega'''_2 - \omega''_3 = \widehat{\omega''\omega'''}, \quad \omega'_3 - \omega'''_1 = \widehat{\omega'''\omega'}, \quad \omega''_1 - \omega'_2 = \widehat{\omega'\omega''}. \qquad (130.10)$$

The angular displacements of the vectors λ, μ, ν regarded as a rigid body are $\omega' du, \quad \omega'' dv, \quad \omega''' dw.$ (130.11)

§ **131. Rotation functions.** So far we have been considering a system of three unit orthogonal vectors of the most general kind depending on three parameters, and we have seen how they depend on the quaternion

$$1 + \xi i + \eta j + \zeta k.$$

We now want to consider the particular system characterized by the property that $p' = q'' = r''' = 0,$ (131.1)

that is, by the property that ξ, η, ζ satisfy the three equations (127.7) which in § 128 we connected with the partial differential equation of the third order with which we began our discussion.

We now have from (130.7)

$$\lambda_1 = \mu r' - \nu q', \quad \lambda_2 = \mu r'', \quad \lambda_3 = -\nu q''',$$

$$\mu_1 = -\lambda r', \quad \mu_2 = \nu p'' - \lambda r'', \quad \mu_3 = \nu p''',$$

$$\nu_1 = \lambda q', \quad \nu_2 = -\mu p'', \quad \nu_3 = \lambda q''' - \mu p'''.$$

It will be convenient to write

$$q' = (31), \quad r'' = (12), \quad p''' = (23),$$

$$r' = -(21), \quad p'' = -(32), \quad q''' = -(13),$$

when the above equations become

$$\lambda_1 + \mu(21) + \nu(31) = 0, \quad \mu_2 + \nu(32) + \lambda(12) = 0,$$

$$\nu_3 + \lambda(13) + \mu(23) = 0,$$

$$\lambda_2 = \mu(12), \quad \mu_1 = \lambda(21), \quad \nu_1 = \lambda(31),$$

$$\lambda_3 = \nu(13), \quad \mu_3 = \nu(23), \quad \nu_2 = \mu(32). \qquad (131.2)$$

The six functions

$$(23),\quad (32),\quad (31),\quad (13),\quad (12),\quad (21)$$

we shall call rotation functions. They are connected by the laws

$$(23)_2+(32)_3+(12)(13)=0,\quad (31)_3+(13)_1+(23)(21)=0,$$
$$(12)_1+(21)_2+(31)(32)=0,$$
$$(23)_1=(21)(13),\quad (31)_2=(32)(21),\quad (12)_3=(13)(32),$$
$$(32)_1=(31)(12),\quad (13)_2=(12)(23),\quad (21)_3=(23)(31),\qquad (131.3)$$

as we can at once verify from the equations satisfied by λ, μ, ν. We can express these rotation functions, as we have done, in terms of ξ, η, ζ and their derivatives.

§ **132. A vector which traces out a triply orthogonal system.** Now consider the system of equations

$$\alpha_2=\beta(12),\quad \beta_3=\gamma(23),\quad \gamma_1=\alpha(31),$$
$$\alpha_3=\gamma(13),\quad \beta_1=\alpha(21),\quad \gamma_2=\beta(32),\qquad (132.1)$$

where α, β, γ are scalars to be determined by these equations. We see at once from the set of conditions

$$(23)_1=(21)(13),\quad (31)_2=(32)(21),\quad (12)_3=(13)(32),$$
$$(32)_1=(31)(12),\quad (13)_2=(12)(23),\quad (21)_3=(23)(31)$$

that they are consistent.

Let α, β, γ be any three functions which satisfy them, and let

$$z=\alpha\lambda+\beta\mu+\gamma\nu.\qquad (132.2)$$

We have

$$z_1=(\alpha_1+\beta(21)+\gamma(31))\,\lambda,$$
$$z_2=(\beta_2+\gamma(32)+\alpha(12))\,\mu,$$
$$z_3=(\gamma_3+\alpha(13)+\beta(23))\,\nu,\qquad (132.3)$$

and therefore the vector z traces out a triply orthogonal system of surfaces.

Conversely we see that there is no triply orthogonal system of surfaces which cannot be obtained by this method.

§ 133. **Lines and measures of curvature.** If we take

$$a \equiv \alpha_1 + \beta(21) + \gamma(31), \quad b \equiv \beta_2 + \gamma(32) + \alpha(12),$$
$$c \equiv \gamma_3 + \alpha(13) + \beta(23),$$

we have $$z_1 = a\lambda, \quad z_2 = b\mu, \quad z_3 = c\nu,$$

and we see that

$$a_2 = b(21), \quad b_3 = c(32), \quad c_1 = a(13),$$
$$a_3 = c(31), \quad b_1 = a(12), \quad c_2 = b(23),$$
$$c_{12} = c_1 \frac{\partial}{\partial v} \log(13) + c_2 \frac{\partial}{\partial u} \log(23),$$
$$c_{13} = c_1 \frac{\partial}{\partial w} \log(13) + c(31)(13),$$
$$c_{23} = c_2 \frac{\partial}{\partial w} \log(23) + c(32)(23), \qquad (133.1)$$

and that these three last equations together with

$$a = \frac{c_1}{(13)}, \quad b = \frac{c_2}{(23)} \qquad (133.2)$$

are equivalent with the first six.

The three orthogonal surfaces are

$$u = \text{constant}, \quad v = \text{constant}, \quad w = \text{constant};$$

the unit vectors parallel to the normals at the extremity of the vector z are respectively λ, μ, ν.

We have $$z_2 = \frac{b}{(12)}\lambda_2, \quad z_3 = \frac{c\lambda_3}{(13)}, \qquad (133.3)$$

and therefore the curves along which only v and w respectively vary are the lines of curvature on the surface $u =$ constant, and its principal radii of curvature are

$$\frac{b}{(12)} \text{ and } \frac{c}{(13)}. \qquad (133.4)$$

We thus have the fundamental theorem about lines of curvature of orthogonal surfaces, viz. that they are the lines in which the two other surfaces intersect one of the surfaces.

If we consider the curve in space along which only u varies, and if we suppose its principal normal to make an angle θ' with the vector μ, and ρ' and σ' to be its two curvatures, we have in the notation we used in considering curves in space (see § 94)

$$\dot{\lambda} = \frac{\cos\theta'}{\rho'}\mu - \frac{\sin\theta'}{\rho'}\nu; \quad \dot{\mu} = \left(\dot{\theta}' + \frac{1}{\sigma'}\right)\nu - \frac{\cos\theta'}{\rho'}\lambda;$$

$$\dot{\nu} = \frac{\sin\theta'}{\rho'}\lambda - \left(\dot{\theta}' + \frac{1}{\sigma'}\right)\mu. \qquad (133.5)$$

Now $\quad a\dot{\lambda} = \lambda_1, \quad a\dot{\mu} = \mu_1, \quad a\dot{\nu} = \nu_1,$

since adu is the element of arc of the curve, and as we have

$$\lambda_1 + \mu(21) + \nu(31) = 0, \quad \mu_1 = \lambda(21), \quad \nu_1 = \lambda(31),$$

we must have

$$\dot{\theta}' + \frac{1}{\sigma'} = 0, \quad (21) = \frac{-a\cos\theta'}{\rho'}, \quad (31) = \frac{a\sin\theta'}{\rho'}.$$

Thus considering the three curves we get

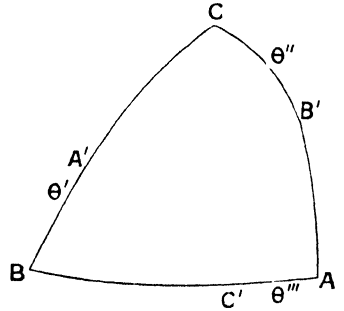
$$\dot{\theta}' + \frac{1}{\sigma'} = 0, \qquad \dot{\theta}'' + \frac{1}{\sigma''} = 0, \qquad \dot{\theta}''' + \frac{1}{\sigma'''} = 0,$$

$$(23) = \frac{c\sin\theta'''}{\rho'''}, \qquad (31) = \frac{a\sin\theta'}{\rho'}, \qquad (12) = \frac{b\sin\theta''}{\rho''},$$

$$(32) = \frac{-b\cos\theta''}{\rho''}, \qquad (13) = \frac{-c\cos\theta'''}{\rho'''}, \qquad (21) = \frac{-a\cos\theta'}{\rho'}; \qquad (133.6)$$

and we thus see another interpretation of the rotation functions.

In the figure here given A, B, C represent the points where



the vectors λ, μ, ν intersect the unit sphere whose centre is the origin; that is, the points where parallels to the three tangents to the curves intersect the sphere; and A', B', C' the points where the parallels to the corresponding principal normals intersect the sphere.

The principal radii of curvature of the surface $u =$ constant were, we saw,

$$\frac{b}{(12)} \text{ and } \frac{c}{(13)},$$

that is,

$$\frac{\rho''}{\sin\theta''}, \quad \frac{-\rho'''}{\cos\theta'''},$$

and therefore the measure of curvature is

$$K' = \frac{-2\theta''\cos\theta'''}{\rho''\rho'''} = \frac{-\cos B'C'}{\rho''\rho'''}. \qquad (133.7)$$

Similarly we have

$$K'' = \frac{-\cos C'A'}{\rho'''\rho'}, \quad K''' = \frac{-\cos A'B'}{\rho'\rho''}. \qquad (133.8)$$

Again, from the formulae

$$\dot{\theta}' + \frac{1}{\sigma'} = 0, \quad \dot{\theta}'' + \frac{1}{\sigma''} = 0, \quad \dot{\theta}''' + \frac{1}{\sigma'''} = 0 \qquad (133.9)$$

we at once see that, if a line of curvature is a plane curve, its plane cuts the surface at the same angle all along it.

§ 134. **Linear equations on whose solution depends that of the equation of the third order.** We now return to the equation of the third order (127.1),

$$p\cosh^2 x\frac{\partial\theta}{\partial x} + q\cosh^2 x\frac{\partial\theta}{\partial y} + \frac{\partial\theta}{\partial w} = q\sinh x\cosh x,$$

where

$$2\theta = \tan^{-1}\frac{2s+2q\tanh}{r-t+2p\tanh x}.$$

Suppose that z is any integral of this equation: we may suppose it expressed in the series

$$z = f(x, y) + w\phi(x, y) + w^2\psi(x, y) + \ldots, \qquad (134.1)$$

and if the integral is a general one we may take f to be any arbitrary function of x and y.

We shall show how when f is a known function the function ϕ depends for its determination on a differential equation of the second order.

Let

$$P \equiv 2\frac{\partial^2 f}{\partial x\,\partial y} + 2\tanh x\frac{\partial f}{\partial y}, \quad Q \equiv \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} + 2\tanh x\frac{\partial f}{\partial x};$$

and let $$\Omega \equiv \cosh^2 x\frac{\partial f}{\partial x}\frac{\partial}{\partial x} + \cosh^2 x\frac{\partial f}{\partial y}\frac{\partial}{\partial y}.$$

The equation which determines ϕ is then

$$Q\Omega P - P\Omega Q - 2\frac{\partial f}{\partial y}\sinh x\cosh x\,(P^2+Q^2)$$
$$= P\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} + 2\tanh x\frac{\partial\phi}{\partial x}\right) - 2Q\left(\frac{\partial^2\phi}{\partial x\,\partial y} + \tanh x\frac{\partial\phi}{\partial y}\right). \tag{134.2}$$

Now let $w = w' + w_0$, where w_0 is a small constant whose square may be neglected, then

$$z = f + w_0\phi + w'(\phi + 2w_0\psi) + \ldots, \tag{134.3}$$

and by solving a similar equation to the above with $f + w_0\phi$ substituted for f we should find

$$\phi + 2w_0\psi, \tag{134.4}$$

and thus obtain ψ.

Proceeding thus we see the system of linear partial differential equations on whose solution we depend for obtaining the coefficients of the different powers of w in the series for z.

A particular solution of the equation of the third order would be obtained by taking f to satisfy the equation

$$Q\Omega P - P\Omega Q = 2\frac{\partial f}{\partial y}\sinh x\cosh x\,(P^2+Q^2), \tag{134.5}$$

when we could take ϕ to be equal to f.

§ **135. Synopsis of the general argument.** It may be useful at this stage to give a *résumé* of the general argument.

z is a function of x, y, and w which satisfies the equation

$$p\frac{\partial\theta}{\partial x} + q\frac{\partial\theta}{\partial y} + \operatorname{sech}^2 x\frac{\partial\theta}{\partial w} = q\tanh x,$$

where $$2\theta \equiv \tan^{-1}\frac{2s+2q\tanh x}{r-t+2p\tanh x}.$$

$$U \equiv \frac{\partial}{\partial x} - \cot\theta\frac{\partial}{\partial y}; \quad V \equiv \frac{\partial}{\partial x} + \tan\theta\frac{\partial}{\partial y};$$

$$W \equiv p\cosh^2 x\frac{\partial}{\partial x} + q\cosh^2 x\frac{\partial}{\partial y} + \frac{\partial}{\partial w};$$

and u and v are defined by

$$Vu = 0, \quad Wu = 0; \quad Uv = 0, \quad Wv = 0.$$

We can now express x, y, and θ in terms of u, v, and w; and having done so we define ξ, η, ζ by

$$\xi \equiv e^{-x}\frac{\cos\frac{y+\theta}{2}}{\cos\frac{y-\theta}{2}}, \quad \eta \equiv e^{-x}\frac{\sin\frac{y+\theta}{2}}{\cos\frac{y-\theta}{2}}, \quad \zeta \equiv \tan\frac{y-\theta}{2},$$

and we have

$$\xi_1 - \eta\zeta_1 + \zeta\eta_1 = 0, \quad \eta_2 - \zeta\xi_2 + \xi\zeta_2 = 0, \quad \zeta_3 - \xi\eta_3 + \eta\xi_3 = 0.$$

The functions ξ, η, ζ now define a quaternion

$$q = 1 + \xi i + \eta j + \zeta k,$$

where i, j, k are any fixed unit vectors at right angles to one another.

Three unit vectors mutually at right angles are now defined by $\lambda = qiq^{-1}, \quad \mu = qjq^{-1}, \quad \nu = qkq^{-1},$

where $$Dq^{-1} = 1 - \xi i - \eta j - \zeta k$$

and $$D = 1 + \xi^2 + \eta^2 + \zeta^2.$$

These vectors are not fixed.

We have

$$\lambda_1 + \mu(21) + \nu(31) = 0, \quad \mu_2 + \nu(32) + \lambda(12) = 0,$$

$$\nu_3 + \lambda(13) + \mu(23) = 0,$$

$$\lambda_2 = \mu(12), \quad \mu_1 = \lambda(21), \quad \nu_1 = \lambda(31),$$

$$\lambda_3 = \nu(13), \quad \mu_3 = \nu(23), \quad \nu_2 = \mu(32),$$

and thus the six rotation functions

$$(23), \quad (32), \quad (31), \quad (13), \quad (12), \quad (21)$$

are defined. These functions satisfy the conditions

$$(23)_2+(32)_3+(12)(13)=0, \quad (31)_3+(13)_1+(23)(21)=0,$$
$$(12)_1+(21)_2+(31)(32)=0,$$
$$(23)_1=(21)(13), \quad (31)_2=(32)(21), \quad (12)_3=(13)(32),$$
$$(32)_1=(31)(12), \quad (13)_2=(12)(23), \quad (21)_3=(23)(31).$$

The vectors λ, μ, ν are parallel to the normals at the extremity of some vector z depending on three parameters which traces out the three orthogonal surfaces

$$u=\text{constant}, \quad v=\text{constant}, \quad w=\text{constant}.$$

This vector z is defined by

$$z=\alpha\lambda+\beta\mu+\gamma\nu,$$

where α, β, γ are scalars to be determined by the six equations

$$\alpha_2=\beta(12), \quad \beta_3=\gamma(23), \quad \gamma_1=\alpha(31),$$
$$\alpha_3=\gamma(13), \quad \beta_1=\alpha(21), \quad \gamma_2=\beta(32).$$

Corresponding to each solution of this equation system we obtain a system of orthogonal surfaces, and the different systems thus obtained have the property of having their normals parallel at corresponding points.

If $\quad a\equiv\alpha_1+\beta(21)+\gamma(31), \quad b\equiv\beta_2+\gamma(32)+\alpha(12),$

$$c\equiv\gamma_3+\alpha(13)+\beta(23), \tag{135.1}$$

then $$z_1=a\lambda, \quad z_2=b\mu, \quad z_3=c\nu,$$

and $$a_2=b(21), \quad b_3=c(32), \quad c_1=a(13),$$
$$a_3=c(31), \quad b_1=a(12), \quad c_2=b(23),$$

so that the ground form for the Euclidean space is

$$ds^2=a^2du^2+b^2dv^2+c^2dw^2. \tag{135.2}$$

§ 136. **An alternative method indicated.** The functions a, b, c of u, v, w must satisfy certain conditions which can at once be obtained by expressing the rotation functions in terms of a, b, c and their derivatives and using the conditions which the rotation functions must satisfy. But we can more rapidly obtain these conditions by just saying that the space defined by $ds^2 = a^2du^2 + b^2dv^2 + c^2dw^2$ (136.1)

is flat, and therefore $(rkih) = 0.$ (136.2)

The conditions then are seen to be

$$a_{23} = a_2\frac{b_3}{b} + a_3\frac{c_2}{c}, \quad b_{31} = b_3\frac{c_1}{c} + b_1\frac{a_3}{a},$$

$$c_{12} = c_1\frac{a_2}{a} + c_2\frac{b_1}{b}, \tag{136.3}$$

and $$\left(\frac{b_3}{c}\right)_3 + \left(\frac{c_2}{b}\right)_2 + \frac{b_1c_1}{a^2} = 0, \quad \left(\frac{c_1}{a}\right)_1 + \left(\frac{a_3}{c}\right)_3 + \frac{c_2a_2}{b^2} = 0,$$

$$\left(\frac{a_2}{b}\right)_2 + \left(\frac{b_1}{a}\right)_1 + \frac{a_3b_3}{c^2} = 0. \tag{136.4}$$

These six equations if we could solve them would equally lead to orthogonal surfaces, and this is the usual method by which the problem of orthogonal surfaces is approached. There seems, however, to be an advantage in making the whole theory depend on one equation of the third order as we have done.

§ 137. **Three additional conditions which may be satisfied.** We now wish to consider a special class of orthogonal surfaces, and we begin by inquiring whether there are any rotation functions which, in addition to satisfying the nine necessary conditions which all rotation functions must satisfy, also satisfy the three additional conditions

$$(23)_3 + (32)_2 + (21)(31) = 0, \quad (31)_1 + (13)_3 + (32)(12) = 0,$$

$$(12)_2 + (21)_1 + (13)(23) = 0. \tag{137.1}$$

If we take

$$(23) \equiv x + \xi, \quad (31) \equiv y + \eta, \quad (12) \equiv z + \zeta,$$

$$(32) \equiv x - \xi, \quad (13) \equiv y - \eta, \quad (21) \equiv z - \zeta,$$

and $$2u' \equiv v + w, \quad 2v' \equiv w + u, \quad 2w' \equiv u + v,$$

and, f being a function of the parameters u', v', w', denote

$$\frac{\partial f}{\partial u'}, \quad \frac{\partial f}{\partial v'}, \quad \frac{\partial f}{\partial w'}, \quad -\frac{\partial f}{\partial u'} - \frac{\partial f}{\partial v'} - \frac{\partial f}{\partial w'}$$

respectively by $f_1, \; f_2, \; f_3, \; f_4,$

we see that the twelve conditions which the rotation functions now have to satisfy are expressed by

$$
\begin{aligned}
x_1 &= -2yz, & y_2 &= -2zx, & z_3 &= -2xy,\\
x_4 &= -2\eta\zeta, & y_4 &= -2\zeta\xi, & z_4 &= -2\xi\eta,\\
\xi_2 &= -2\eta z, & \eta_3 &= -2\zeta x, & \zeta_1 &= -2\xi y,\\
\xi_3 &= -2\zeta y, & \eta_1 &= -2\xi z, & \zeta_2 &= -2\eta x. \qquad (137.2)
\end{aligned}
$$

Now it is easily seen that these equations are satisfied by taking

$$
\begin{aligned}
(23) &= \tfrac{1}{2}\sqrt{V_{23}}+\tfrac{1}{2}\sqrt{V_{14}}, & (32) &= \tfrac{1}{2}\sqrt{V_{23}}-\tfrac{1}{2}\sqrt{V_{14}},\\
(31) &= \tfrac{1}{2}\sqrt{V_{31}}+\tfrac{1}{2}\sqrt{V_{24}}, & (13) &= \tfrac{1}{2}\sqrt{V_{31}}-\tfrac{1}{2}\sqrt{V_{24}},\\
(12) &= \tfrac{1}{2}\sqrt{V_{12}}+\tfrac{1}{2}\sqrt{V_{31}}, & (21) &= \tfrac{1}{2}\sqrt{V_{12}}-\tfrac{1}{2}\sqrt{V_{34}},
\end{aligned}
\qquad (137.3)
$$

where V is a function satisfying the 'complete' system of equations

$$
\begin{aligned}
V_{234}+2\sqrt{V_{23}V_{34}V_{42}} &= 0, & V_{341}+2\sqrt{V_{34}V_{41}V_{13}} &= 0,\\
V_{412}+2\sqrt{V_{41}V_{12}V_{24}} &= 0, & V_{123}+2\sqrt{V_{12}V_{23}V_{31}} &= 0.
\end{aligned}
\qquad (137.4)
$$

We thus see that such rotation functions exist. A particular solution of such a system of equations would be obtained by taking

$$V_4 = 0, \quad V_{123}+2\sqrt{V_{12}V_{23}V_{31}} = 0; \qquad (137.5)$$

and in this case

$$(23) = (32), \quad (31) = (13), \quad (12) = (21). \qquad (137.6)$$

It may be shown that this solution corresponds to the particular solution of the original equation of the third order when we take z to be independent of w.

§ 138. Orthogonal systems from which others follow by direct operations. We must now consider the special property which the orthogonal surfaces will have which correspond to rotation functions satisfying the twelve conditions. We return to the original variables u, v, w in what follows.

Let α, β, γ be any scalars which satisfy the equations

$$
\begin{aligned}
\alpha_2 &= \beta(12), & \beta_3 &= \gamma(23), & \gamma_1 &= \alpha(31),\\
\alpha_3 &= \gamma(13), & \beta_1 &= \alpha(21), & \gamma_2 &= \beta(32), \qquad (138.1)
\end{aligned}
$$

and let a, b, c be defined by

$$a \equiv \alpha_1 + \beta(21) + \gamma(31), \quad b \equiv \beta_2 + \gamma(32) + \alpha(12),$$
$$c \equiv \gamma_3 + \alpha(13) + \beta(23).$$

We then have

$$a_2 = b(21), \quad b_3 = c(32), \quad c_1 = a(13),$$
$$a_3 = c(31), \quad b_1 = a(12), \quad c_2 = b(23). \qquad (138.2)$$

Let $\alpha' \equiv a_1 + b(12) + c(13), \quad \beta' \equiv b_2 + c(23) + a(21),$

$$\gamma' \equiv c_3 + a(31) + b(32).$$

We can at once verify that

$$\alpha'_2 = \beta'(12), \quad \beta'_3 = \gamma'(23), \quad \gamma'_1 = \alpha'(31),$$
$$\alpha'_3 = \gamma'(13), \quad \beta'_1 = \alpha'(21), \quad \gamma'_2 = \beta'(32); \qquad (138.3)$$

and therefore $\qquad z' = \alpha'\lambda + \beta'\mu + \gamma'\nu$

will trace out another system of orthogonal surfaces. This second system is thus obtained from the first by direct operations not involving integration. We thus see that when we are given any one system of orthogonal surfaces of this particular class we can deduce by direct operations an infinite system of such surfaces.

CHAPTER XI

DIFFERENTIAL GEOMETRY IN n-WAY SPACE

§ **139. Geodesics in n-way space.** In order to see what kind of geometry we may associate with the ground form

$$ds^2 = a_{ik} dx_i dx_k$$

of an n-way space, we naturally think of the simple case when n was 2, and the space a Euclidean plane. The most elementary part of that geometry was that associated with straight lines; that is, the shortest distances between two points. We are thus led to consider the theory of geodesics in our n-way space.

We have

$$2\frac{d\delta s}{ds} = a_{ik}\frac{dx_i}{ds}\frac{d\delta x_k}{ds} + a_{ik}\frac{dx_k}{ds}\frac{d\delta x_i}{ds} + \frac{dx_i}{ds}\frac{dx_k}{ds}\delta a_{ik},$$

$$= \frac{d}{ds}\left(a_{ik}\frac{dx_i}{ds}\delta x_k + a_{ik}\frac{dx_k}{ds}\delta x_i\right) - \delta x_k \frac{d}{ds}\left(a_{ik}\frac{dx_i}{ds}\right)$$

$$- \delta x_i \frac{d}{ds}\left(a_{ik}\frac{dx_k}{ds}\right) + \frac{dx_i}{ds}\frac{dx_k}{ds}\frac{\partial a_{ik}}{\partial x_t}\delta x_t.$$

For a path of critical length therefore we must have

$$\frac{d}{ds}\left(a_{it}\frac{dx_i}{ds}\right) + \frac{d}{ds}\left(a_{tk}\frac{dx_k}{ds}\right) = \frac{\partial a_{ik}}{\partial x_t}\frac{dx_i}{ds}\frac{dx_k}{ds}. \quad (139.1)$$

Now (§ 6)

$$\frac{d}{ds}a_{ik} = \frac{\partial a_{ik}}{\partial x_p}\frac{dx_p}{ds} = \frac{dx_p}{ds}((ipk) + (kpi)),$$

and therefore

$$\frac{dx_i}{ds}\frac{da_{ik}}{ds} = \frac{dx_i}{ds}\frac{dx_p}{ds}((ipk) + (kpi)).$$

It follows that

$$a_{it}\frac{d^2x_i}{ds^2}+a_{tk}\frac{d^2x_k}{ds^2}+\frac{dx_i}{ds}\frac{dx_p}{ds}((ipt)+(tpi))$$
$$+\frac{dx_k}{ds}\frac{dx_p}{ds}((kpt)+(tpk))=\frac{dx_i}{ds}\frac{dx_k}{ds}((itk)+(kti));$$

and therefore $$a_{it}\frac{d^2x_i}{ds^2}+(ikt)\frac{dx_i}{ds}\frac{dx_k}{ds}=0. \qquad (139.2)$$

Multiplying by a^{tp} and summing we have

$$\frac{d^2x_p}{ds^2}+\{ikp\}\frac{dx_i}{ds}\frac{dx_k}{ds}=0. \qquad (139.3)$$

We thus have n equations wherewith to obtain the coordinates of any point on a geodesic in terms of the length s.

But the equations are differential equations of the second order; and in general we can only solve them so as to obtain the coordinates in the form of infinite series. This is a practical difficulty and one of the reasons why we cannot have the same kind of knowledge of the theory of geodesics in n-way space that we have in Euclidean geometry of straight lines.

The direction cosines of an element of length in n-way space are defined by

$$\xi^p=\frac{dx_p}{ds},\quad p=1\ldots n. \qquad (139.4)$$

Going along a geodesic, therefore, we have

$$\frac{d\xi^p}{ds}+\{ikp\}\xi^i\xi^k=0, \qquad (139.5)$$

and we see that, unlike the direction cosines of a straight line in a plane, associated with the form

$$ds^2=dx_1^2+dx_2^2,$$

these direction cosines vary as we pass along the geodesic.

Thus we are familiar with the difficulty of keeping to the shortest course between two given points at sea, viz. a great circle. In this case the differential equations are soluble in finite terms; but even with this advantage we should need a continuously calculating machine to find the direction cosines at each point of the course. If the ocean instead of being spherical were ellipsoidal, we should not even have the advantage of being given the equations of the geodesic in finite form, and the difficulty of keeping to the shortest course would be even greater.

Now if we had built up our plane geometry by using the form
$$ds^2 = dx_1^2 + x_1^2 dx_2^2,$$
the direction cosines of a straight line would also have varied from point to point of the straight line and yet we would not say that the direction of the straight line varied from point to point. The navigator on the ellipsoidal ocean might hope—till he had learnt a little more geometry—to mend the want of constancy in his direction cosines as the plane geometer could mend his by a proper choice of coordinates.

He could not mend this want of constancy by any choice of coordinates, but though the direction cosines change in passing along a geodesic there is no need to think of the 'direction' as changing.

We will then say that the direction in an n-way space *is* the same all along a geodesic.

§ **140. Geodesic polar coordinates and Euclidean coordinates at a point.** We recall the fact (§ 2) that any n-way space may be regarded as lying in a Euclidean r-fold where $r = \frac{1}{2}n(n+1)$, and that the vector z which lies in this r-fold, depending on the n parameters $x_1 \dots x_n$, has the property that its extremity traces out our n-way space.

In the n-way space, unless it happens to be merely a Euclidean space, we cannot think of a vector as lying in it: it is only the extremity of the vector with which we are concerned.

But at any particular point of the n-way space there is a Euclidean n-fold which we may usefully associate with the point.

Let z be the vector to the point under consideration, and let $z_1 \dots z_n$ be its derivatives at the point with respect to $x_1 \dots x_n$, the parameters of the point.

Let $\xi^1 \dots \xi^n$ be the direction cosines of any element of the n-way space at the point so that

$$a_{ik}\xi^i\xi^k = 1, \tag{140.1}$$

then the vector ξ defined by

$$\xi \equiv \xi^1 z_1 + \dots + \xi^n z_n \tag{140.2}$$

will lie in the Euclidean n-fold at the point. It will clearly be a unit vector since

$$\underbrace{\xi\xi} = \xi^i\xi^k \underbrace{z_i z_k},$$

$$= -a_{ik}\xi^i\xi^k = -1. \tag{140.3}$$

We shall call this Euclidean n-fold in which ξ lies the tangential n-fold at the point.

The coordinates of any point in the tangential n-fold may be taken as $\xi_1 \dots \xi_n$, where

$$\xi_i = s\xi^i, \tag{140.4}$$

s being a scalar.

We establish a correspondence between the points of our n-way space and the points of the tangential n-fold by taking the coordinates of the n-way space to be $\xi_1 \dots \xi_n$.

Consider the geodesic which starting at the point under consideration has the direction cosines $\xi^1 \dots \xi^n$.

From the equation of a geodesic

$$\frac{d^2 x_i}{ds^2} + \{\lambda\mu i\} \frac{dx_\lambda}{ds}\frac{dx_\mu}{ds} = 0$$

we see that the current coordinates are given by

$$x'_i = x_i + \xi^i s - \{\lambda\mu i\}\,\xi^\lambda\xi^\mu \frac{s^2}{2} + \dots, \tag{140.5}$$

where s is the arc from the initial point.

Let z' be the vector which traces out the n-way space at the point $x'_1 \dots x'_n$, and let ζ' denote the same vector expressed in terms of the coordinates $\xi_1 \dots \xi_n$.

We have [see § 4 for the notation]

$$\zeta'_t = z'_i(\epsilon^i_t - \{t\mu i\}\,\xi^\mu s + \dots).$$

It follows that

$$\underbrace{\zeta'_i\zeta'_k} = \underbrace{z'_p z'_q}(\epsilon^p_i - \{i\lambda p\}\,\xi^\lambda s + \dots)(\epsilon^q_k - \{k\mu q\}\,\xi^\mu s + \dots),$$

and therefore the transformation formula is

$$a'_{ik} = a_{ik} - (k\mu i)\,\xi^{\mu} s - (i\lambda k)\,\xi^{\lambda} s + \ldots, \qquad (140.6)$$

where $+\ldots$ refers to higher powers of s.

We thus have at the origin

$$\frac{\partial a'_{ik}}{\partial \xi_t} = \epsilon_t^p \frac{\partial a_{ik}}{\partial x_p} - (kti) - (itk), \qquad (140.7)$$

that is, in the coordinates we have chosen the first derivative of each of the coefficients in the ground form vanishes.

It follows that in this system of coordinates, which establishes a correspondence between the points of the tangential n-fold and the points of the n-way space, every three-index symbol of Christoffel vanishes at the point under consideration.

As regards the four-index symbol $(rkih)'$ we have

$$\begin{aligned}(rkih)' &= (pqst)\frac{\partial x'_p}{\partial \xi_r}\frac{\partial x'_q}{\partial \xi_k}\frac{\partial x'_s}{\partial \xi_i}\frac{\partial x'_t}{\partial \xi_h} \\ &= (pqst)\,\epsilon_r^p\,\epsilon_k^q\,\epsilon_i^s\,\epsilon_h^t \\ &= (rkih). \qquad (140.8)\end{aligned}$$

We may call this transformation a transformation to geodesic polar coordinates at a specified point.

We can combine the transformation with any linear transformation in the tangential n-fold. To do this suppose $x_1 \ldots x_n$ to be the original coordinates, taken to be zero at the point to be considered.

Let
$$x_i = c_{ik}\,x'_k, \qquad (140.9)$$

where $c_{ik}\ldots$ denote constants.

We can now so choose these coordinates as to make the coefficients take any assigned values at the point We can then apply the geodesic transformation, and can thus arrange that the coefficients a_{ik} may have any values we like (provided the determinant is not zero), and at the same time have all the three-index symbols vanishing at the point.

In particular we can so choose the constants that

$$a_{ik} = \epsilon_k^i \qquad (140.10)$$

at the point and that the three-index symbols may vanish. Such a system of coordinates may be said to be the Euclidean coordinates of the n-way space at the point.*

§ 141. Riemann's measure of curvature of n-way space. If we take the transformation

$$x_t = \xi^t x'_1 + \eta^t x'_2 + \dots,$$

where ξ, η, ... are fixed vectors at the point, we find that

$$(12\,12)' = (pqrs)\frac{\partial x_p}{\partial x'_1}\frac{\partial x_q}{\partial x'_2}\frac{\partial x_r}{\partial x'_1}\frac{\partial x_s}{\partial x'_2}$$

$$= (pqrs)\,\xi^p\eta^q\xi^r\eta^s, \qquad (141.1)$$

$$a'_{11} = a_{ik}\xi^i\xi^k,$$

$$a'_{12} = a_{ik}\xi^i\eta^k,$$

$$a'_{22} = a_{ik}\eta^i\eta^k, \qquad (141.2)$$

and

$$a'_{11}a'_{22} - a'^2_{12} = \tfrac{1}{4}(\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(a_{ik}a_{pq} - a_{iq}a_{pk}), \qquad (141.3)$$

and therefore

$$\frac{(12\,12)'}{a'_{11}a'_{22} - a'^2_{12}} = \frac{(\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(ipkq)}{(\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(a_{ik}a_{pq} - a_{iq}a_{pk})}. \qquad (141.4)$$

Now let us consider the expression on the left-hand side of this equation.

In general the four-index symbol as applied to n-way space is not the same thing as when applied to the lower space in which the coordinates whose integers do not occur in the symbol are put equal to constants. But in geodesic polar coordinates at the point the equality holds, since the three-index symbols vanish.

It follows (see §§ 24, 37) that the expression on the left is the measure of curvature at the point of the two-way surface formed by keeping all the geodesic coordinates constant except two. The expression on the right is therefore the measure of

* This system of coordinates has been called the system of Galilean coordinates at the point.

curvature of the geodesic surface—that is, the surface formed by the geodesics—through the assigned point, which touches at the point the Euclidean plane generated by the two vectors ξ and η.

This is Riemann's measure of curvature of the n-way space. We see how it is connected with Gauss's measure of curvature, and we should notice how in this respect the tangential n-fold takes the place of the mere tangent plane when $n = 2$. In the flat n-fold we consider all the Euclidean planes by taking any two vectors in the n-fold. We see that these two-way surfaces have different curvatures and so different geometries.

§ **142. Further study of curvature. The Gaussian measures for geodesic surfaces. Orientation.** We have now obtained Riemann's measure of curvature and have seen how it is connected with Gauss's measure of curvature of a surface.

We must now consider this curvature from another point of view.

We saw that we were to consider the direction to be the same at all points of a geodesic in n-way space. This leads us to define two 'parallel' displacements at neighbouring points $x_1 \ldots x_n$ and $x_1 + dx_1 \ldots x_n + dx_n$ as displacements whose direction cosines $\xi^1 \ldots \xi^n$ and $\xi^1 + d\xi^1 \ldots \xi^n + d\xi^n$ are connected by the equations

$$d\xi^p + \{ikp\}\, \xi^i dx_k = 0. \tag{142.1}$$

Thus in this sense of 'parallel' the tangents are parallel at all points on the same geodesic.

It may be noted that the equation defining parallel displacements does not entitle us to say that

$$\frac{\partial \xi^p}{\partial x_k} + \{ikp\}\, \xi^i = 0.$$

If this equation system held, the tensor component ξ^p would be annihilated by every operator $\bar{1}, \bar{2}, \ldots \bar{n}$, and therefore

$$\xi^p_{\cdot ik} - \xi^p_{\cdot ki} = \{tpki\}\, \xi^t = 0,$$

which could only be true in flat space.

Let ζ be any vector in the tangential n-fold at $x_1 \dots x_n$ and $\zeta + d\zeta$ be the 'parallel' vector in the tangential n-fold at

$$x_1 + dx_1 \dots x_n + dx_n.$$

We have $$d\zeta = \zeta^i (z._{ik} + \{ikt\} z_t) dx_k + z_t d\zeta_t$$
$$= \zeta^i z._{ik} dx_k.$$

As we pass from the point of departure with an assigned value for ζ, and the vector is carried parallel to itself, its value at any other point is defined by the integral

$$\zeta = \int \zeta^i z._{ik} dx_k, \qquad (142.2)$$

and this value depends on the path of integration.

Consider the small parallelogram in the n-way space whose edges are parallel to the vectors ξ and η, the lengths of the edges being respectively a and b. We want to find the change in ζ by integrating round the parallelogram.

We have $\zeta^i = (\zeta^i)_0 + \delta\zeta^i = (\zeta^i)_0 - \{\lambda\mu i\}_0 (\zeta^\lambda)_0 \delta x_\mu$,

where δx_μ is the increment in the coordinate, neglecting powers of small quantities of the second order, and

$$z._{ik} = (z._{ik})_0 + [z._{ik\mu} + \{i\mu t\} z._{tk} + \{k\mu t\} z._{it}]_0 \delta x_\mu,$$

and therefore

$$\zeta^i z._{ik} = (\zeta^i z._{ik})_0 + [\zeta^i z._{ik\mu} + \zeta^i \{i\mu t\} z._{tk} + \zeta^i \{k\mu t\} z._{it}]_0 \delta x_\mu$$
$$- [\zeta^\lambda \{\lambda\mu i\} z._{ik}]_0 \delta x_\mu.$$

We thus have

$$\zeta^i z._{ik} = (\zeta^i z._{ik})_0 + [\zeta^i (z._{ik\mu} + \{k\mu t\} z._{it})]_0 \delta x_\mu.$$

On the first edge at a point distant s this is

$$(\zeta^i z._{ik})_0 + [\zeta^i (z._{ik\mu} + \{k\mu t\} z._{it}) \xi^\mu]_0 s;$$

on the second edge it becomes

$$(\zeta^i z._{ik})_0 + [\zeta^i (z._{ik\mu} + \{k\mu t\} z._{it}) \xi^\mu]_0 a$$
$$+ [\zeta^i (z._{ik\mu} + \{k\mu t\} z._{it}) \eta^\mu]_0 s.$$

The change in ζ by integrating

$$\zeta^i z._{ik} dx_k$$

along the two edges is

$$[\zeta^i z_{\cdot ik}\xi^k]_0 a + [\zeta^i z_{\cdot ik}\eta^k]_0 b + [\zeta^i (z_{\cdot ik\mu} + \{k\mu t\} z_{\cdot it})\xi^\mu\xi^k]_0 \frac{a^2}{2}$$
$$+ [\zeta^i (z_{\cdot ik\mu} + \{k\mu t\} z_{\cdot it})\xi^\mu\eta^k]_0 ab$$
$$+ [\zeta^i (z_{\cdot ik\mu} + \{k\mu t\} z_{\cdot it})\eta^\mu\eta^k]_0 \frac{b^2}{2}. \qquad (142.3)$$

If we had integrated in the opposite sense along the other two edges we should have interchanged ξ and η, a and b, and we thus see that the change in ζ by going round the parallelogram in the same sense is

$$[\zeta^i (z_{\cdot ik\mu} - z_{\cdot i\mu k})\xi^\mu\eta^k]_0 ab,$$

that is,
$$[\zeta^i \{itk\mu\} z_t \xi^\mu\eta^k]_0 ab.$$

The change in ζ^t is therefore

$$\tfrac{1}{2}[\zeta^i \{itk\mu\} (\xi^\mu\eta^k - \xi^k\eta^\mu)]_0 ab. \qquad (142.4)$$

If θ is the angle between ξ and η,

$$\cos\theta = a_{ik}\xi^i\eta^k,$$

$$\sin^2\theta = \tfrac{1}{4}(\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(a_{ik}a_{pq} - a_{iq}a_{kp}). \qquad (142.5)$$

Let us now consider how the angle θ is changed, if, keeping η fixed, we carry ξ parallel to itself round the parallelogram.

$$\begin{aligned} -\sin\theta\,\delta\theta &= a_{ik}\eta^k\delta\xi^i \\ &= -\tfrac{1}{2} a_{ik}\eta^k\xi^t \{tipq\}(\xi^p\eta^q - \xi^q\eta^p)\, ab \\ &= -\tfrac{1}{4}(\xi^t\eta^k - \xi^k\eta^t)(\xi^p\eta^q - \xi^q\eta^p)(tkpq)\, ab. \end{aligned}$$

It follows that $\delta\theta$ divided by the area of the small parallelogram is equal to

$$\frac{(\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(ipkq)}{(\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(a_{ik}a_{pq} - a_{iq}a_{kp})}. \qquad (142.6)$$

That is, $\delta\theta$ divided by the area of the small parallelogram is equal to the curvature of the geodesic surface which touches the Euclidean plane generated by the two vectors formed by the sides of the parallelogram.

From the equation

$$d\zeta = \zeta^i z_{\cdot ik} dx_k$$

we see that the rate of change of ζ in parallel displacement in the direction of the vector ξ is $\xi^k \zeta^i z_{\cdot ik}$. We may then express this result in the notation

$$\zeta_\xi = \xi^k \zeta^i z_{\cdot ik}.$$

Thus we have

$$\xi_\xi = \xi^k \xi^i z_{\cdot ik},$$

$$\xi_\eta = \eta_\xi = \xi^k \eta^i z_{\cdot ik},$$

$$\eta_\eta = \eta^k \eta^i z_{\cdot ik}.$$

Also

$$\underbrace{\xi\xi} = \xi^k \xi^i a_{ik},$$

$$\underbrace{\xi\eta} = \underbrace{\eta\xi} = \xi^k \eta^i a_{ik},$$

$$\underbrace{\eta\eta} = \eta^k \eta^i a_{ik}.$$

It follows that

$$\underbrace{\xi_\xi \eta_\eta} - \underbrace{\xi_\eta \eta_\xi} = \xi^i \xi^k \eta^p \eta^q (z_{\cdot ik} z_{\cdot pq} - z_{\cdot iq} z_{\cdot kp}),$$

$$\underbrace{\xi\xi}\,\underbrace{\eta\eta} - \underbrace{\xi\eta}\,\underbrace{\eta\xi} = \xi^i \xi^k \eta^p \eta^q (a_{ik} a_{pq} - a_{iq} a_{kp}),$$

and therefore

$$\frac{\underbrace{\xi_\xi \eta_\eta} - \underbrace{\xi_\eta \eta_\xi}}{\underbrace{\xi\xi}\,\underbrace{\eta\eta} - \underbrace{\xi\eta}\,\underbrace{\eta\xi}} = \frac{(\xi^i \eta^p - \xi^p \eta^i)(\xi^k \eta^q - \xi^q \eta^k)(ipqk)}{(\xi^i \eta^p - \xi^p \eta^i)(\xi^k \eta^q - \xi^q \eta^k)(a_{ik} a_{pq} - a_{iq} a_{kp})}. \tag{142.7}$$

Riemann's measure of curvature may therefore be written

$$\frac{\underbrace{\xi_\xi \eta_\eta} - \underbrace{\xi_\eta \eta_\xi}}{\underbrace{\xi\xi}\,\underbrace{\eta\eta} - \underbrace{\xi\eta}\,\underbrace{\eta\xi}}. \tag{142.8}$$

Again the rate of change of z in displacement along the vector ξ is just $\xi^1 z_1 + \ldots + \xi^n z_n$, and therefore the vector ξ itself may be written z_ξ. Here we may notice that the vector z unlike the displacement vector ξ is not a vector in the n-way space but only in the containing r-way flat space; the vector ξ on the other hand marks a direction, or displacement, in the n-way space, although it has only an elemental length in this space.

There is then as regards the vector z in its parallel displacement just the ordinary Euclidean idea of translation.

Riemann's measure of curvature may therefore be written

$$\frac{\underbrace{z_{\xi\xi}z_{\eta\eta}} - \underbrace{z_{\xi\eta}z_{\eta\xi}}}{\underbrace{z_{\xi}z_{\xi}}\,\underbrace{z_{\eta}z_{\eta}} - \underbrace{z_{\xi}z_{\eta}}\,\underbrace{z_{\xi}z_{\eta}}}. \qquad (142.9)$$

This is Gauss's measure of curvature of the geodesic surface, made up of the singly infinite system of geodesic curves drawn through an assigned point at which we require the measure of curvature: the curves at the point all touch the Euclidean plane generated by the vectors ξ and η.

Riemann's measure of curvature has an 'orientation' given by the vectors ξ and η; and at the assigned point, by varying this plane, we get the different Gaussian measures.

§ **143. A notation for oriented area.** So far we have in using vectors only considered their products as scalar products. There is another product which we ought now to consider. When $n = 3$ and the ground form is that appertaining to Euclidean space, we know what the vector product means and how useful it proved in Geometry, but it does not seem to be capable of useful extension. We shall now think of the product of two vectors ξ and η as defining an area in the Euclidean plane formed by ξ and η. This area has then an orientation, and we shall understand by $\xi\eta$ the area of the parallelogram whose edges are the ξ and η drawn through the point.

The angle the vector ξ makes with the vector η being θ, by parallel displacement of the vector ξ round the parallelogram whose edges are in the directions ξ and η, and whose lengths are ds and δs, this angle is increased by

$$\frac{\underbrace{z_{\xi\xi}z_{\eta\eta}} - \underbrace{z_{\xi\eta}z_{\eta\xi}}}{\underbrace{z_{\xi}z_{\xi}}\,\underbrace{z_{\eta}z_{\eta}} - \underbrace{z_{\xi}z_{\eta}}\,\underbrace{z_{\xi}z_{\eta}}}\,\xi\eta\, ds\,\delta s,$$

which may be written

$$\frac{\underbrace{z_{\xi\xi}z_{\eta\eta}} - \underbrace{z_{\xi\eta}z_{\eta\xi}}}{\underbrace{z_{\xi}z_{\xi}}\,\underbrace{z_{\eta}z_{\eta}} - \underbrace{z_{\xi}z_{\eta}}\,\underbrace{z_{\xi}z_{\eta}}}\, dz\,\delta z, \qquad (143.1)$$

where dz, δz represent the sides of the small parallelogram in magnitude and direction.

We should notice that area has a sign as well as a magnitude: we express this by the equation

$$\xi\eta + \eta\xi = 0. \tag{143.2}$$

§ **144. A system of geodesics normal to one surface are normal to a system of surfaces.** If the direction cosines of a geodesic are $\xi^1, \dots \xi^n$, we have seen (139.5) that the equations of a geodesic are

$$\frac{d}{ds}\xi^p + \{ikp\}\,\xi^i\xi^k = 0.$$

The geometrical interpretation of these equations is that the tangent remains 'parallel' to itself as we move along the geodesic.

We can put the equations in another form,

$$a_{\lambda p}\frac{d}{ds}\xi^p + (ik\lambda)\,\xi^i\xi^k = 0,$$

and therefore

$$\frac{d}{ds}(a_{\lambda p}\xi^p) - \xi^p\xi^t\frac{\partial}{\partial x_t}a_{\lambda p} + (pt\lambda)\,\xi^p\xi^t = 0,$$

since

$$\frac{d}{ds} \equiv \xi^t\frac{\partial}{\partial x_t}\cdot$$

Now

$$\frac{\partial}{\partial x_t}a_{\lambda p} = (\lambda tp) + (pt\lambda),$$

and therefore the equations of a geodesic may be written

$$\frac{d}{ds}(a_{\lambda p}\xi^p) = (\lambda tp)\,\xi^p\xi^t. \tag{144.1}$$

We now wish to consider the expression

$$T_\lambda \equiv a_{\lambda p}\xi^p.$$

We know, from the theory of differential equations, that the necessary and sufficient condition that $T_\lambda\,dx_\lambda$ may be rendered a perfect differential by multiplying by a factor is that

$$T_\lambda\left(\frac{\partial}{\partial x_\mu}T_\nu - \frac{\partial}{\partial x_\nu}T_\mu\right) + T_\mu\left(\frac{\partial}{\partial x_\nu}T_\lambda - \frac{\partial}{\partial x_\lambda}T_\nu\right) + T_\nu\left(\frac{\partial}{\partial x_\lambda}T_\mu - \frac{\partial}{\partial x_\mu}T_\lambda\right) \tag{144.2}$$

should vanish identically for all values of λ, μ, ν; and that the necessary and sufficient condition that $T_\lambda dx_\lambda$ may vanish, wherever $\phi(x_1 \ldots x_n) = 0$, is that

$$T_\lambda\left(\frac{\partial}{\partial x_\mu} T_\nu - \frac{\partial}{\partial x_\nu} T_\mu\right) + T_\mu\left(\frac{\partial}{\partial x_\nu} T_\lambda - \frac{\partial}{\partial x_\lambda} T_\nu\right) + T_\nu\left(\frac{\partial}{\partial x_\lambda} T_\mu - \frac{\partial}{\partial x_\mu} T_\lambda\right)$$

should vanish for all values of λ, μ, ν wherever

$$\phi(x_1 \ldots x_n) = 0.$$

Geometrically interpreted these are the conditions that the curves

$$\frac{dx_1}{\xi^1} = \frac{dx_2}{\xi^2} = \ldots = \frac{dx_n}{\xi^n} \tag{144.3}$$

should (1) be cut orthogonally by a system of surfaces, (2) be cut orthogonally by the definite surface

$$\phi(x_1 \ldots x_n) = 0. \tag{144.4}$$

We are now going to prove that if the curves are geodesics, and if the condition (2) is satisfied, (1) is satisfied also.

Let

$$\frac{\partial T_p}{\partial x_q} - \frac{\partial T_q}{\partial x_p} \equiv (pq) \tag{144.5}$$

and

$$T_p(qr) + T_q(rp) + T_r(pq) \equiv [p, q, r]. \tag{144.6}$$

Since

$$\frac{d}{ds}\frac{\partial}{\partial x_q} \equiv \frac{\partial}{\partial x_q}\frac{d}{ds} - \frac{\partial \xi^t}{\partial x_q}\frac{\partial}{\partial x_t},$$

$$\frac{d}{ds}(pq) = \frac{\partial}{\partial x_q}\frac{d}{ds}T_p - \frac{\partial}{\partial x_p}\frac{d}{ds}T_q - \frac{\partial \xi^t}{\partial x_q}\frac{\partial}{\partial x_t}T_p + \frac{\partial \xi^t}{\partial x_p}\frac{\partial}{\partial x_t}T_q.$$

Now

$$\frac{\partial}{\partial x_q}\frac{d}{ds}T_p = \frac{\partial}{\partial x_q}(p\lambda\mu)\xi^\lambda\xi^\mu$$

$$= (p\lambda\mu)\left(\xi^\lambda\frac{\partial \xi^\mu}{\partial x_q} + \xi^\mu\frac{\partial \xi^\lambda}{\partial x_q}\right) + \xi^\lambda\xi^\mu\frac{\partial}{\partial x_q}(p\lambda\mu)$$

$$= \xi^\lambda\frac{\partial \xi^\mu}{\partial x_q}((p\lambda\mu) + (p\mu\lambda)) + \xi^\lambda\xi^\mu\frac{\partial}{\partial x_q}(p\lambda\mu)$$

$$= \xi^\lambda\frac{\partial \xi^\mu}{\partial x_q}\frac{\partial \dot{a}_{\lambda\mu}}{\partial x_p} + \xi^\lambda\xi^\mu\frac{\partial}{\partial x_q}(p\lambda\mu).$$

It follows that

$$\frac{d}{ds}(pq) = \xi^\lambda \xi^\mu \left(\frac{\partial}{\partial x_q}(p\lambda\mu) - \frac{\partial}{\partial x_p}(q\lambda\mu)\right)$$
$$+ \frac{\partial \xi^\lambda}{\partial x_q}\left(\xi^\mu \frac{\partial a_{\lambda\mu}}{\partial x_p} - \frac{\partial}{\partial x_\lambda}(a_{p\mu}\xi^\mu)\right)$$
$$- \frac{\partial \xi^\lambda}{\partial x_p}\left(\xi^\mu \frac{\partial a_{\lambda\mu}}{\partial x_q} - \frac{\partial}{\partial x_\lambda}(a_{q\mu}\xi^\mu)\right).$$

Now $\frac{\partial}{\partial x_q}(p\lambda\mu) - \frac{\partial}{\partial x_p}(q\lambda\mu)$

$$= (\lambda\mu pq) + (\mu qt)\{\lambda pt\} - \{\mu pt\}(\lambda qt),$$

and therefore we see that the first term in this expression vanishes, since λ and μ are interchangeable.

We therefore have

$$\frac{d}{ds}(pq)$$
$$= \frac{\partial \xi^\lambda}{\partial x_q}\left(\frac{\partial}{\partial x_p}T_\lambda - \frac{\partial}{\partial x_\lambda}T_p\right) + \frac{\partial \xi^\lambda}{\partial x_p}\left(\frac{\partial}{\partial x_\lambda}T_q - \frac{\partial}{\partial x_q}T_\lambda\right)$$
$$= \frac{\partial \xi^\lambda}{\partial x_q}(\lambda p) - \frac{\partial \xi^\lambda}{\partial x_p}(\lambda q). \tag{144.7}$$

It follows that

$$\frac{d}{ds}[p, q, r]$$
$$= (qr)\frac{d}{ds}T_p + (rp)\frac{d}{ds}T_q + (pq)\frac{d}{ds}T_r$$
$$+ T_p\left(\frac{\partial \xi^\lambda}{\partial x_r}(\lambda q) - \frac{\partial \xi^\lambda}{\partial x_q}(\lambda r)\right) + T_q\left(\frac{\partial \xi^\lambda}{\partial x_p}(\lambda r) - \frac{\partial \xi^\lambda}{\partial x_r}(\lambda p)\right)$$
$$+ T_r\left(\frac{\partial \xi^\lambda}{\partial x_q}(\lambda p) - \frac{\partial \xi^\lambda}{\partial x_p}(\lambda q)\right)$$
$$= (qr)\xi^\lambda \frac{\partial}{\partial x_\lambda}T_p + (rp)\xi^\lambda \frac{\partial}{\partial x_\lambda}T_q + (pq)\xi^\lambda \frac{\partial}{\partial x_\lambda}T_r$$
$$+ \frac{\partial \xi^\lambda}{\partial x_p}((qr)T_\lambda + [q, \lambda, r]) + \frac{\partial \xi^\lambda}{\partial x_q}((rp)T_\lambda + [r, \lambda, p]),$$
$$+ \frac{\partial \xi^\lambda}{\partial x_r}((pq)T_\lambda + [p, \lambda, q]). \tag{144.8}$$

Since $$a_{\lambda\mu}\xi^\lambda\xi^\mu = 1,$$
we have $$\xi^\lambda T_\lambda = 1,$$
and therefore $$T_\lambda \frac{\partial \xi^\lambda}{\partial x_p} + \xi^\lambda \frac{\partial T_\lambda}{\partial x_p} = 0.$$

It follows that
$$\frac{d}{ds}[p, q, r]$$
$$= (qr)(p\lambda) + (rp)(q\lambda) + (pq)(r\lambda)$$
$$+ \frac{\partial \xi^\lambda}{\partial x_p}[q, \lambda, r] + \frac{\partial \xi^\lambda}{\partial x_q}[r, \lambda, p] + \frac{\partial \xi^\lambda}{\partial x_r}[p, \lambda, q]. \qquad (144.9)$$

Suppose now that over a given surface
$$[p, q, r] = 0.$$
We then have
$$\frac{d}{ds}[p, q, r]$$
$$= (qr)(p\lambda) + (rp)(q\lambda) + (pq)(r\lambda).$$

Since $[p, q, r]$ is zero for all values of the integers over the given surface, we have
$$T_q(rs) + T_r(sq) + T_s(qr) = 0,$$
$$T_p(sr) + T_r(ps) + T_s(rp) = 0,$$
$$T_p(qs) + T_q(sp) + T_s(pq) = 0,$$
$$T_p(rq) + T_q(pr) + T_r(qp) = 0,$$
and therefore
$$\begin{vmatrix} 0, & (rs), & (sq), & (qr) \\ (sr), & 0, & (ps), & (rp) \\ (qs), & (sp), & 0, & (pq) \\ (rq), & (pr), & (qp), & 0 \end{vmatrix} \equiv 0,$$
that is, $$(pq)(rs) + (qr)(ps) + (rp)(qs) \equiv 0. \qquad (144.10)$$

It follows that if $[p, q, r]$ is zero over a surface it is zero everywhere, and therefore if a system of geodesics are normal to any one surface they are normal to a system of surfaces.

The direction cosines of the system of geodesics therefore satisfy the equation system

$$a_{pt}\xi^t = K\phi_p.$$

It follows that $K^2 a^{pq}\phi_p\phi_q = a^{pq}a_{pt}\xi^t a_{qr}\xi^r$

$$= \epsilon^q_t \xi^t a_{qr}\xi^r$$

$$= a_{qr}\xi^q\xi^r = 1, \qquad (144.11)$$

and therefore

$$a_{pt}\xi^t = \frac{\phi_p}{\sqrt{\Delta(\phi)}}. \qquad (144.12)$$

§ **145. The determination of surfaces orthogonal to geodesics and of geodesics orthogonal to surfaces.** We can now find the equation which ϕ must satisfy when the surfaces

$$\phi = \text{constant}$$

are those which cut the geodesics orthogonally.

We have

$$\xi^p = a^{pt}\frac{\phi_t}{\sqrt{\Delta(\phi)}}. \qquad (145.1)$$

The equations of a geodesic being (139.5)

$$\frac{d}{ds}\xi^p + \{ikp\}\xi^i\xi^k = 0,$$

we must have

$$a^{pt}\frac{\phi_t}{\sqrt{\Delta(\phi)}}\frac{\partial}{\partial x_p}\left(a^{k\lambda}\frac{\phi_\lambda}{\sqrt{\Delta(\phi)}}\right) + \{\lambda\mu k\}\frac{a^{\lambda t}\phi_t a^{\mu s}\phi_s}{\Delta(\phi)} = 0,$$

that is,

$$a^{pt}\frac{\phi_t}{\sqrt{\Delta(\phi)}}\left(a^{k\lambda}\frac{\phi_\lambda}{\sqrt{\Delta(\phi)}}\right)_{\cdot p} = 0,$$

and therefore

$$a^{pt}\frac{\phi_t}{\sqrt{\Delta(\phi)}}\left(a_{kq}a^{k\lambda}\frac{\phi_\lambda}{\sqrt{\Delta(\phi)}}\right)_{\cdot p} = 0,$$

or

$$a^{pt}\frac{\phi_t}{\sqrt{\Delta(\phi)}}\left(\frac{\phi_q}{\sqrt{\Delta(\phi)}}\right)_{\cdot p} = 0. \qquad (145.2)$$

Expanding,

$$\Delta(\phi, \phi_q) = \frac{\phi_q}{2\Delta(\phi)}\Delta(\phi, \Delta(\phi)),$$

that is,

$$\frac{\partial}{\partial x_q}(\Delta(\phi)) = \phi_q\frac{\Delta(\phi, \Delta(\phi))}{\Delta(\phi)}.$$

It follows that

$$\phi_p \frac{\partial}{\partial x_q}(\Delta(\phi)) = \phi_q \frac{\partial}{\partial x_p}(\Delta(\phi)), \qquad (145.3)$$

and therefore $\Delta(\phi)$ must be a function of ϕ.

Without loss of generality we may therefore say that

$$\Delta(\phi) = 1. \qquad (145.4)$$

We thus see that if we take any surface and consider the geodesics drawn from every point on it perpendicular to the surface, they are cut orthogonally by a system of surfaces $\phi =$ constant, where $\Delta(\phi) = 1$.

In ordinary Euclidean space this is the theorem that the lines normal to any surface are cut orthogonally by the surfaces $\phi =$ constant, where

$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 = 1.$$

Conversely, let ϕ be any integral of the equation $\Delta(\phi) = 1$; then we shall show that the orthogonal trajectories of the surfaces $\phi =$ constant will be a system of geodesics.

It will be convenient now to think of an $(n+1)$-way space and to take, instead of the variables $x_1 \ldots x_{n+1}$, a new system of variables $y_1 \ldots y_{n+1}$, where

$$y_{n+1} \equiv \phi, \qquad (145.5)$$

so that $\Delta(y_{n+1}) = 1,$

and to choose $y_1 \ldots y_n$ as n independent integrals of the equation

$$\Delta(y_{n+1}, y) = 0. \qquad (145.6)$$

Let the ground form of the $(n+1)$-way space be

$$b_{ik} dy_i dy_k, \quad {}^i_k = 1, 2, \ldots n+1. \qquad (145.7)$$

Since $\Delta(y_{n+1}) = 1$ and $\Delta(y_{n+1}, y_r) = 0$ if $r \neq (n+1)$, we see that $b_{n+1,\, n+1} = 1, \quad b_{n+1,\, r} = 0,$

and therefore the ground form is

$$dy^2_{n+1} + b_{ik} dy_i dy_k, \quad {}^i_k = 1, 2, \ldots n. \qquad (145.8)$$

It will therefore be convenient to take as the ground form in the $(n+1)$-way space

$$du^2 + b_{ik} dx_i dx_k, \quad {}^i_k = 1 \ldots n, \qquad (145.9)$$

where b_{ik} is a function of $x_1 \ldots x_n$ and u.

The surface $u = 0$ is any arbitrary surface in the $(n+1)$-way space, and when $u = 0$ we may write $b_{ik} = a_{ik}$. We may consider

$$a_{ik}dx_i dx_k \tag{145.10}$$

to be the ground form of the n-way space deduced from the $(n+1)$-way space by putting $u = 0$.

The lower ground form may be said to be the ground form of a surface in the higher space.

By varying u we obtain a series of surfaces cut orthogonally by the curves whose direction cosines are given by

$$\xi^1 = 0, \quad \xi^2 = 0, \dots \xi^n = 0.$$

These curves are geodesics, since

$$\{n+1,\, n+1,\, p\} = 0, \quad p = 1 \dots n.$$

It will be noticed that the first of the surfaces cut orthogonally may be any whatever, but the other surfaces are given by

$$\Delta(u) = 1. \tag{145.11}$$

When we know the geodesics normal to $u = 0$, we know the whole series of surfaces which are cut orthogonally, or at least can find them by quadrature, since

$$a_{ik}\xi^k = u_i. \tag{145.12}$$

We obtain the geodesics, on the other hand, by the solution of the linear equation $\Delta(u, v) = 0$, when we know u.*

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§ **146. A useful reference in $(n+1)$-way space.** We have shown (145.9) that the ground form of any $(n+1)$-way space may be taken to be

$$du^2 + b_{ik}dx_i dx_k, \quad {}^{i}_{k} = 1 \dots n,$$

where b_{ik} is a function of $x_1 \dots x_n$ and u.

The surface $u = 0$ is any surface whatever in the $(n+1)$-way space. By drawing the geodesics perpendicular to this surface we obtain a series of curves which are cut orthogonally by the surfaces u = constant, u being the geodesic distance of any point from the surface $u = 0$.

* [At this point in the author's MS. there is a memorandum 'New Chapter'.]

The surfaces $u =$ constant are said to be parallel surfaces, and we have $\Delta(u) = 1$. Travelling along any of the geodesics from the surface $u = 0$, only u varies.

It will, however, be found useful to consider a more general system of surfaces in the $(n+1)$-way space.

We therefore consider any system of surfaces whatever in this space, $u =$ constant, where we no longer have $\Delta(u) = 1$, and by taking the orthogonal trajectories of these surfaces, as the parametric lines

$$x_1 = \text{constant}, \ldots x_n = \text{constant}$$

we may take the ground form of the space to be

$$\phi^2 du^2 + b_{ik} dx_i dx_k, \quad {}_k^i = 1 \ldots n. \tag{146.1}$$

The orthogonal trajectories are now no longer geodesics.

The function b_{ik} depends on the coordinates $x_1 \ldots x_n$ and u, and, when $u = 0$, $b_{ik} = a_{ik}$.

We now wish to consider the two ground forms

$$\phi^2 du^2 + b_{ik} dx_i dx_k \tag{146.1}$$

and

$$a_{ik} dx_i dx_k \tag{146.2}$$

in connexion with Christoffel's symbols, where after calculating their values for the higher space we put $u = 0$. We can obtain the special case of parallel surfaces by putting $\phi = 1$.

We shall thus be shown how to generate the $(n+1)$-way space which as it were surrounds any given n-way space.

When we place the suffix b outside a symbol this will indicate that the symbol belongs to the higher space. The suffixes will always be $1 \ldots n$. When we have to consider the suffix which should correspond to the variable u it will be denoted by a dot.

Let

$$\frac{\partial b_{ik}}{\partial u} \equiv -2\Omega_{ik}\phi. \tag{146.3}$$

We see that

$$(ikh)_b = (ikh)_a; \qquad (ik\cdot)_b = \Omega_{ik}\phi; \qquad (i\cdot k)_b = -\Omega_{ik}\phi;$$

$$\{ikh\}_b = \{ikh\}_a; \quad \{ik\cdot\}_b = \Omega_{ik}\phi^{-1}; \quad \{i\cdot k\}_b = -a^{kt}\Omega_{it}\phi;$$

$$(rkhi)_b = (rkhi)_a + \Omega_{ri}\Omega_{hk} - \Omega_{rh}\Omega_{ik};$$

$$(rkh\cdot)_b = \frac{\partial}{\partial x_k}(\Omega_{rh}\phi) - \frac{\partial}{\partial x_r}(\Omega_{hk}\phi) + \Omega_{hk}\phi_r - \Omega_{rh}\phi_k$$
$$- \{hkt\}_a \Omega_{rt}\phi + \{rht\}_a \Omega_{kt}\phi$$
$$= \phi(\Omega_{rh\cdot k} - \Omega_{hk\cdot r}),$$

in tensor notation,

$$(r\cdot h\cdot)_b = \phi\left(\frac{\partial}{\partial u}\Omega_{rh} - \phi_{\cdot rh}\right) + a^{\lambda\mu}\phi^2\Omega_{r\lambda}\Omega_{h\mu}. \quad (146.4)$$

§ 147. **Geometry of the functions Ω_{ik}.** If we are given any n-way space we shall see that it may be surrounded by what is called an $(n+1)$-way Einstein space. We shall define this space shortly. A particular n-way space may be surrounded by a Euclidean space of $n+1$ dimensions; but before we consider particular kinds of this $(n+1)$-way space we had better consider the geometrical meaning of the functions Ω_{ik} which together with ϕ are to generate the space.

With this end in view let us consider two geodesics going out from the same point $x_1 \dots x_n$, $u = 0$, and having the same direction cosines $\xi^1 \dots \xi^n$, 0 at this point, the first geodesic being in the n-way space denoted by a, and the second in the $(n+1)$-way space denoted by b.

We have (140.5) for the current coordinates on these geodesics $x'_1 \dots x'_n$ and $x''_1 \dots x''_n$

$$x'_i = x_i + s\xi^i - \frac{s^2}{2}\{\lambda\mu i\}_a \xi^\lambda \xi^\mu + \dots,$$

$$x''_i = x_i + s\xi^i - \frac{s^2}{2}\{\lambda\mu i\}_b \xi^\lambda \xi^\mu + \dots,$$

and therefore, neglecting terms of the third order in the arc s, we see that the coordinates are the same for the two geodesics. But for the first geodesic the coordinate u is zero, and for the second geodesic

$$u = -\frac{s^2}{2}\{\lambda\mu\cdot\}_b \xi^\lambda \xi^\mu + \dots.$$

The distance between two points, one on each of the geodesics, is therefore

$$\phi\frac{s^2}{2}\{\lambda\mu\cdot\}_b \xi^\lambda \xi^\mu + \dots, \quad (147.1)$$

s being the distance of each point measured along its own geodesic from the initial point.

This mutual distance we may consider to be normal to the geodesics if we neglect terms of the third order in the arc.

The curvature of the first geodesic is defined by Voss as the ratio of twice this distance to s^2. This is obviously a proper definition, agreeing with the ordinary definition when we are dealing with Euclidean space.

We therefore have

$$\frac{1}{\rho} = \phi\{\lambda\mu\cdot\}_b \xi^\lambda \xi^\mu$$
$$= \frac{\phi\Omega_{\lambda\mu}\phi^{-1}dx_\lambda dx_\mu}{a_{\lambda\mu}dx_\lambda dx_\mu}$$
$$= \frac{\Omega_{\lambda\mu}dx_\lambda dx_\mu}{a_{\lambda\mu}dx_\lambda dx_\mu}. \tag{147.2}$$

The curvature of the first geodesics may be called the normal curvature in the $(n+1)$-way space of the surface $u = 0$ in the direction $dx_1, dx_2, \ldots dx_n$.

Looked at in this way we may write our formula

$$\frac{1}{R} = \frac{\Omega_{\lambda\mu}dx_\lambda dx_\mu}{a_{\lambda\mu}dx_\lambda dx_\mu}. \tag{147.3}$$

To get what we may call the directions of principal curvature we require the directions dx_i which make $\frac{1}{R}$ critical.

The directions of principal curvature are therefore given by

$$(a_{\lambda\mu} - R\Omega_{\lambda\mu})\,dx_\mu = 0, \tag{147.4}$$

where the values of R, the principal radii of curvature, are given by the determinant

$$|\,a_{\lambda\mu} - R\Omega_{\lambda\mu}\,| = 0. \tag{147.5}$$

We shall now show that the directions of principal curvature are in general mutually orthogonal.

At any *given* point we can choose the coordinates so that corresponding to the principal radius R_i only the coordinate x_i varies at the *given* point. We therefore have at the point

$$a_{ik} = R_k\Omega_{ik},$$
$$a_{ki} = R_i\Omega_{ki}.$$

If, then, all the radii of curvature are distinct,

$$a_{ik} = \Omega_{ik} = 0, \text{ if } i \neq k, \qquad (147.6)$$

and therefore the coordinates are mutually orthogonal at the given point: that is, the lines of curvature are in general mutually orthogonal.

§ **148. The sum of the products of two principal curvatures at a point.** We now wish to obtain an extension of the well-known formula of Gauss

$$\frac{1}{R_1 R_2} = \frac{(12\,12)}{a_{11}a_{22}-a_{12}^2}$$

for a surface lying in ordinary Euclidean space.

Consider any determinant

$$\begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{n1} & \ldots & a_{nn} \end{vmatrix}$$

and the corresponding determinant

$$\begin{vmatrix} a^{11} & \ldots & a^{1n} \\ \vdots & & \\ a^{n1} & \ldots & a^{nn} \end{vmatrix}.$$

If $|a|$ denotes the first determinant we see that any of its minors is equal to $|a|$ multiplied by the complementary minor of the second determinant.

Expanding the determinant (147.5), or say

$$|\, a_{ik} - \lambda a'_{ik} \,|,$$

we see that the determinant divided by $|a|$ is equal to

$$1 - \frac{\lambda}{(1!)^1} a^{ik}a'_{ik} + \frac{\lambda^2}{(2!)^2}(a^{ri}a^{hk} - a^{rh}a^{ik})(a'_{ri}a'_{hk} - a'_{rh}a'_{ik}) - \ldots, \qquad (148.1)$$

the numerical factors $\frac{1}{(1!)^1}, \frac{1}{(2!)^2}, \frac{1}{(3!)^3}, \ldots$ being introduced in accordance with the convention about repeated factors.

We therefore have for the principal curvatures

$$\sum \frac{1}{R_i} = a^{pq}\Omega_{pq}. \qquad (148.2)$$

$$4 \sum \frac{1}{R_i R_k} = (a^{r\lambda}a^{h\mu} - a^{rh}a^{\lambda\mu})(\Omega_{r\lambda}\Omega_{h\mu} - \Omega_{rh}\Omega_{\lambda\mu}). \qquad (148.3)$$

Now consider the expression, a tensor component clearly,

$$A_{ri} \equiv a^{kh}\,(rkhi)\,;$$

and similarly $\qquad B_{ri} \equiv b^{kh}\,(rkhi).$

We have seen in § 146 that

$$(rkhi)_b = (rkhi)_a + \Omega_{ri}\Omega_{hk} - \Omega_{rh}\Omega_{ik},$$
$$(rkh\cdot)_b = \phi\,(\Omega_{rh\cdot k} - \Omega_{hk\cdot r}),$$
$$(r\cdot\cdot i)_b = (\cdot ri\cdot)_b = \phi\left(\phi_{\cdot ri} - \frac{\partial}{\partial u}\Omega_{ri}\right) - a^{\lambda\mu}\phi^2\Omega_{r\lambda}\Omega_{i\mu}.$$

From the last equation, we see that

$$(B..)_b = a^{ri}\phi\left(\phi_{\cdot ri} - \frac{\partial}{\partial u}\Omega_{ri}\right) - a^{ri}a^{\lambda\mu}\phi^2\,\Omega_{r\lambda}\Omega_{i\mu}. \qquad (148.4)$$

From the second equation, we see that

$$(B_{r}\cdot)_b = a^{kh}\phi\,(\Omega_{rh\cdot k} - \Omega_{hk\cdot r}). \qquad (148.5)$$

From the first and last equation, that

$$(B_{ri})_b = (A_{ri})_a + a^{kh}\,(\Omega_{ri}\Omega_{hk} - \Omega_{rh}\Omega_{ik})$$
$$+\,\phi^{-1}\left(\phi_{\cdot ri} - \frac{\partial}{\partial u}\Omega_{ri}\right) - a^{\lambda\mu}\Omega_{r\lambda}\Omega_{i\mu}. \qquad (148.6)$$

The expression $a^{ri}A_{ri}$ is an invariant which we denote by A. We thus obtain

$$B = A + a^{ri}a^{kh}\,(\Omega_{ri}\Omega_{hk} - \Omega_{rh}\Omega_{ik}) + 2\phi^{-2}B..\,;$$

or, interchanging i and h,

$$B = A + a^{rh}a^{ki}\,(\Omega_{rh}\Omega_{ik} - \Omega_{ri}\Omega_{hk}) + 2\phi^{-2}B..\,;$$

and therefore

$$2B = 2A + (a^{ri}a^{kh} - a^{rh}a^{ki})\,(\Omega_{rh}\Omega_{ik} - \Omega_{ri}\Omega_{hk}) + 4\phi^{-2}B\ldots \qquad (148.7)$$

It follows that

$$2B = 2A + 4\Sigma\frac{1}{R_1 R_2} + 4\phi^{-2}B\ldots \qquad (148.8)$$

Remembering that B is an invariant in the $(n+1)$-way space, and $B..$ a tensor component in this space, and that A is an invariant in the n-way space obtained as a section of the $(n+1)$-way space by the surface $u = 0$, we may express the result at which we have arrived in the following way.

Consider any $(n+1)$-way space, and a section of it by any surface.

Let $\xi^1 \dots \xi^{n+1}$ be the direction cosines of this surface, regarded as a locus in the $(n+1)$-way space.

The direction cosines are connected by the identical equation

$$1 = b_{ri} \xi^r \xi^i.$$

The expression $b^{kh}(rkhi)_b \xi^r \xi^i$ is an invariant of the $(n+1)$-way and the surface we have chosen. When we take the ground form $\phi^2 du^2 + b_{ik} dx_i dx_k$ and the surface $u = 0$ the expression becomes $\phi^{-2} B_{..}$, since $\xi^1 \dots \xi^n$ are zero, and

$$\phi^2 \xi^{n+1} \xi^{n+1} = 1.$$

If B is the invariant of the $(n+1)$-way space,

$$b^{ri} b^{kh} (rkhi)_b,$$

and if A is the corresponding invariant of the n-way space which is the section of the $(n+1)$-way space by the given surface, then what we have proved is the following.

The sum of the products, two at a time, of the reciprocals of the principal radii of curvature of the surface, regarded as a locus in the $(n+1)$-way space, is equal to

$$\tfrac{1}{2}(B-A) - b^{kh}(rkhi)_b \xi^r \xi^i. \qquad (148.9)$$

§ **149. Einstein space.** Suppose, now, that instead of taking any surface we choose a surface whose direction cosines satisfy the equation

$$\tfrac{1}{2} B = b^{kh}(rkhi) \xi^r \xi^i; \qquad (149.1)$$

we shall have

$$\tfrac{1}{2} A + \Sigma \frac{1}{R_i R_k} = 0. \qquad (149.2)$$

For the case $n = 2$, this becomes Gauss's well-known formula

$$\frac{1}{R_1 R_2} = \frac{(12\,12)}{a_{11} a_{22} - a_{12}^2}.$$

If, then, the $(n+1)$-way space is to be such that for all surfaces lying in it the formula of Gauss will hold, the equation

$$\tfrac{1}{2} B = B_{ri} \xi^r \xi^i$$

must be identical with $\quad 1 = b_{ri} \xi^r \xi^i.$

We must therefore have

$$B_{ri} = \tfrac{1}{2} B b_{ri}, \tag{149.3}$$

and in consequence of this

$$b^{ri} B_{ri} = \tfrac{1}{2} B b^{ri} b_{ri},$$

that is,

$$B = \tfrac{1}{2}(n+1) B. \tag{149.4}$$

Leaving aside the case when $n = 1$, we must have

$$B = 0, \tag{149.5}$$

and therefore

$$B_{ri} = 0. \tag{149.6}$$

A space with this property is what is called an Einstein space.

It is interesting to see how from mere considerations of purely geometrical ideas we should be led to Einstein space.

§ **150. An $(n+1)$-way Einstein space surrounds any given n-way space** (§§ 150–4). We shall now show how, being given the ground form of any n-way space, we may obtain the ground form of a surrounding $(n+1)$-way Einstein space.

We look on $a_{ik}\ldots$ as functions of $x_1 \ldots x_n$ and u whose values are known when $u = 0$. We have if possible to determine functions $\Omega_{ik}\ldots$ and ϕ which will satisfy the equations

$$a^{kh}(\Omega_{rh\cdot k} - \Omega_{hk\cdot r}) = 0, \tag{150.1}$$

$$\phi A_{ri} + a^{\lambda\mu}(\Omega_{ri}\Omega_{\lambda\mu} - 2\Omega_{r\lambda}\Omega_{i\mu})\phi + \phi_{\cdot ri} - \frac{\partial}{\partial u}\Omega_{ri} = 0, \tag{150.2}$$

$$a^{ri}\left(\phi_{\cdot ri} - \frac{\partial}{\partial u}\Omega_{ri} - \phi a^{\lambda\mu}\Omega_{r\lambda}\Omega_{i\mu}\right) = 0, \tag{150.3}$$

$$\frac{\partial a_{ik}}{\partial u} = -2\Omega_{ik}\phi, \tag{150.4}$$

when $u = 0$.

If we can find such functions, we take

$$b_{ik} = (a_{ik})_{u=0} - 2\phi\Omega_{ik} u + \ldots,$$

and thus find the Einstein form

$$\phi^2 du^2 + b_{ik} dx_i dx_k \tag{150.5}$$

in the immediate neighbourhood of the surface $u = 0$, that is, in the immediate neighbourhood of the given n-way space, and proceeding thus by the method of infinitesimal stages we ultimately obtain the Einstein space which we require.

Let $$A^{\lambda}_{\mu} \equiv a^{\lambda t} \{tpp\mu\}. \tag{150.6}$$

We shall first prove the fundamental identity

$$A^{\lambda}_{\mu \cdot \lambda} = \tfrac{1}{2} A_{\mu}. \tag{150.7}$$

Employing the geodesic coordinates at a given point,

$$\begin{aligned} A^{\lambda}_{\mu} &= (\lambda pp\mu) \\ &= \frac{\partial}{\partial x_{\mu}}(\lambda pp) - \frac{\partial}{\partial x_{p}}(\lambda \mu p), \end{aligned}$$

and therefore in this system of coordinates we have at the given point

$$A^{\lambda}_{\mu \cdot \lambda} = \frac{\partial^2}{\partial x_{\lambda} \partial x_{u}}(\lambda pp) - \frac{\partial^2}{\partial x_{p} \partial x_{\lambda}}(\lambda \mu p). \tag{150.8}$$

Similarly we have

$$\tfrac{1}{2} A_{\mu} = \tfrac{1}{2} A^{\lambda}_{\lambda \cdot \mu} = \frac{1}{2} \frac{\partial}{\partial x_{\mu}}\left(\frac{\partial}{\partial x_{\lambda}}(\lambda pp) - \frac{\partial}{\partial x_{p}}(\lambda\lambda p)\right), \tag{150.9}$$

and therefore $A^{\lambda}_{\mu \cdot \lambda} - \tfrac{1}{2} A_{\mu}$

$$\begin{aligned} &= \frac{1}{2} \frac{\partial^2}{\partial x_{\lambda} \partial x_{\mu}}(\lambda pp) + \frac{1}{2} \frac{\partial^2}{\partial x_{\mu} \partial x_{p}}(\lambda\lambda p) \\ &\quad - \frac{1}{2} \frac{\partial^2}{\partial x_{p} \partial x_{\lambda}}(\lambda \mu p) - \frac{1}{2} \frac{\partial^2}{\partial x_{p} \partial x_{\lambda}}(p\mu\lambda) \\ &= \frac{1}{2} \frac{\partial^2}{\partial x_{\lambda} \partial x_{\mu}}[(\lambda pp) + (pp\lambda)] \\ &\quad - \frac{1}{2} \frac{\partial^2}{\partial x_{p} \partial x_{\lambda}}[(\lambda \mu p) + (p\mu\lambda)] \\ &= \frac{1}{2} \frac{\partial^3 a_{\lambda p}}{\partial x_{\lambda} \partial x_{\mu} \partial x_{p}} - \frac{1}{2} \frac{\partial^3 a_{p\lambda}}{\partial x_{p} \partial x_{\lambda} \partial x_{\mu}} \\ &= 0. \end{aligned} \tag{150.7}$$

The required identity holds therefore universally, since it is a tensor equation which vanishes for geodesic coordinates.

§ **151.** We now transform the functions $\Omega_{ik} \ldots$, which are to be found thus.

Let $$V_i^k \equiv a^{kt}\Omega_{it},$$

and therefore $$\Omega_{ik} = \Omega_{ki} = a_{k\lambda} V_i^\lambda = a_{i\lambda} V_k^\lambda,$$

$$\frac{\partial}{\partial u}\Omega_{,i} = a_{i\lambda}\frac{\partial}{\partial u} V_r^\lambda + V_r^\lambda \frac{\partial}{\partial u} a_{i\lambda}$$

$$= a_{i\lambda}\left(\frac{\partial}{\partial u} V_r^\lambda - 2\phi V_r^t V_t^\lambda\right). \qquad (151.1)$$

Writing V for V_λ^λ we now have

$$V_{\mu\cdot\lambda}^\lambda = V_\mu, \qquad (151.2)$$

$$\phi\,(A_{\lambda\mu} + a_{\lambda t} V V_\mu^t) + \phi_{\cdot\lambda\mu} - a_{\lambda t}\frac{\partial}{\partial u} V_\mu^t = 0,$$

that is, multiplying by $a^{\lambda p}$ and summing,

$$\phi\,(A_\mu^\lambda + V V_\mu^\lambda) + a^{\lambda t}\phi_{\cdot t\mu} = \frac{\partial}{\partial u} V_\mu^\lambda. \qquad (151.3)$$

The third equation is

$$a^{\lambda\mu}\phi_{\cdot\lambda\mu} + \phi V_\lambda^\mu V_u^\lambda = \frac{\partial}{\partial u} V,$$

and by aid of the second equation this may be replaced by

$$A = V_\lambda^\mu V_\mu^\lambda - V^2. \qquad (151.4)$$

We thus have the equations

$$V_{\mu\cdot\lambda}^\lambda = V_\mu; \quad A = V_\lambda^\mu V_\mu^\lambda - V^2,$$

and, writing ϕ_μ^λ for $a^{\lambda t}\phi_{\,t\mu}$ in the other equation (151.3),

$$\frac{\partial}{\partial u} V^\lambda = \phi\,(A_\mu^\lambda + V V_\mu^\lambda) + \phi_\mu^\lambda. \qquad (151.5)$$

These are only assumed to hold when $u = 0$.

§ **152.** Noticing that $\phi_\lambda^\lambda = \Delta_2(\phi)$,

we have $$\phi_{\mu\cdot\lambda}^\lambda = \bar\lambda a^{\lambda t}\phi_{\cdot\mu t} = a^{\lambda t}\bar\lambda\bar\mu\phi_t$$

$$= a^{\lambda t}(\bar\mu\bar\lambda\phi_t - \{tq\,\mu\lambda\}\phi_q)$$

$$= \bar\mu a^{\lambda t}\phi_{\cdot\lambda t} - a^{\lambda t}a^{qp}(pt\lambda\mu)\,\phi_q$$

$$= \bar\mu\,\Delta_2(\phi) - a^{qp}\{p\lambda\lambda\mu\}\,\phi_q,$$

and therefore $$\phi_{\mu\cdot\lambda}^\lambda + A_\mu^\lambda\phi_\lambda = \bar\mu\,\Delta_2(\phi). \qquad (152.1)$$

We now operate with $\bar{\lambda}$ on the equation

$$\frac{\partial}{\partial u} V_\mu^\lambda = \phi (A_\mu^\lambda + V V_\mu^{r\lambda}) + \phi_\mu^\lambda,$$

where $$\bar{\lambda} \equiv \frac{\partial}{\partial x_\lambda} + \{t\lambda s\}\binom{t}{s} - \{\mu\lambda t\}\binom{t}{\mu},$$

s being the upper integer—here λ itself so that we employ another integer s—and μ the lower integer in V_μ^λ.

We easily verify that

$$\bar{\lambda} \frac{\partial}{\partial u} = \frac{\partial}{\partial u} \bar{\lambda} - \binom{t}{s} \frac{\partial}{\partial u} \{t\lambda s\} + \binom{t}{\mu} \frac{\partial}{\partial u} \{\mu\lambda t\},$$

and therefore

$$\frac{\partial}{\partial u} V_{\mu \cdot \lambda}^\lambda = V_\mu^t \frac{\partial}{\partial u} \{t\lambda\lambda\} - V_t^\lambda \frac{\partial}{\partial u} \{\mu\lambda t\} + \phi_\lambda (A_\mu^\lambda + V V_\mu^\lambda)$$
$$+ \phi (A_{\mu \cdot \lambda}^\lambda + V_\lambda V_\mu^\lambda + V V_{\mu \cdot \lambda}^\lambda) + \phi_{\mu \cdot \lambda}^\lambda. \quad (152.2)$$

Now, since (4.3) $\quad a^{\lambda\mu} a_{\mu s} = \epsilon_s^\lambda,$

we see that $$\frac{\partial}{\partial u} a^{\lambda\mu} = 2\phi a^{\lambda t} V_t^\mu, \quad (152.3)$$

and $$\frac{\partial}{\partial u} \{rkh\} = \frac{\partial}{\partial u} a^{ht} (rkt)$$
$$= \tfrac{1}{2} a^{ht} \frac{\partial}{\partial u} \left(\frac{\partial a_{kt}}{\partial x_r} + \frac{\partial a_{rt}}{\partial x_k} - \frac{\partial a_{rk}}{\partial x_t} \right)$$
$$+ (rkt)(2\phi a^{tp} V_p^h)$$
$$= \tfrac{1}{2} a^{ht} \left(\frac{\partial}{\partial x_t} 2\phi\Omega_{rk} - \frac{\partial}{\partial x_r} 2\phi\Omega_{kt} - \frac{\partial}{\partial x_k} 2\phi\Omega_{rt} \right)$$
$$+ 2\phi a^{tp} (rkt) V_p^h. \quad (152.4)$$

Since

$$\frac{\partial}{\partial x_t} \phi\Omega_{rk} = \phi_t \Omega_{rk} + \phi (\Omega_{rk \cdot t} + \{rtp\} \Omega_{pk} + \{ktp\} \Omega_{pr}),$$

we see that

$$\frac{\partial}{\partial u} \{rkh\} = a^{ht} \begin{pmatrix} \phi_t \Omega_{rk} + \phi\Omega_{rk \cdot t} - \phi_r \Omega_{kt} - \phi\Omega_{kt \cdot r} \\ \quad - \phi_k \Omega_{rt} - \phi\Omega_{rt \cdot k} - 2\{krp\} \phi\Omega_{pt} \end{pmatrix}$$
$$+ 2\phi a^{tp} (rkt) V_p^h$$
$$= a^{ht} a_{kp} (\phi_t V_r^p + \phi V_{r \cdot t}^p) - \phi_r V_k^h - \phi_k V_r^h$$
$$- \phi (V_{k \cdot r}^h + V_{r \cdot k}^h) - 2\phi \{krp\} V_p^h + 2\phi \{rkp\} V_p^h. \quad (152.5)$$

Now if we multiply each side of this equation by V^k_h and sum, remembering that

$$V^k_h a_{kp} = V^k_p a_{kh},$$

we get

$$V^k_h \frac{\partial}{\partial u}\{rkh\} = V^k_p a^{ht} a_{kh}(\phi_t V^p_r + \phi V^p_{r\cdot t}) - \phi_r V^h_k V^k_h$$
$$- \phi_k V^h_r V^k_h - \phi V^k_h (V^h_{k\cdot r} + V^h_{r\cdot k})$$
$$= V^k_p(\phi_k V^p_r + \phi V^p_{r\cdot k}) - \phi_r V^h_k V^k_h - \phi_k V^h_r V^k_h$$
$$- \phi V^k_h (V^h_{k\cdot r} + V^h_{r\cdot k})$$
$$= - V^k_h (\phi V^h_{k\cdot r} + \phi_r V^h_k). \qquad (152.6)$$

We deduce from the equation for $\frac{\partial}{\partial u}\{rkh\}$ that

$$\frac{\partial}{\partial u}\{rkk\} = \phi_t V^t_r + \phi V^t_{r\cdot t} - \phi_r V - \phi_t V^t_r - \phi(V_r + V^t_{r\cdot t})$$
$$= -\phi_r V - \phi V_r. \qquad (152.7)$$

Combining the formulae we have proved, we now see that

$$\frac{\partial}{\partial u} V^\lambda_{\mu\cdot\lambda} = V^k_h(\phi V^h_{k\cdot\mu} + \phi_\mu V^h_k) - V^r_\mu(\phi_r V + \phi V_r)$$
$$+ \mu\Delta_2(\phi) + \phi_\lambda V V^\lambda_\mu + \phi(A^\lambda_{\mu\cdot\lambda} + V_\lambda V^\lambda_\mu + V V^\lambda_{\mu\cdot\lambda}). \quad (152.8)$$

We also have $\frac{\partial}{\partial u} V = \phi(A + V^2) + \Delta_2(\phi),$

and therefore

$$\frac{\partial}{\partial u} V_u = \phi_\mu(A + V^2) + \phi(A_\mu + 2VV_\mu) + \mu\Delta_2\phi,$$

so that $\frac{\partial}{\partial u}(V^\lambda_{\mu\cdot\lambda} - V_\mu)$

$$= \phi_\mu(V^k_h V^h_k - A - V^2)$$
$$+ \phi(V^k_h V^h_{k\cdot\mu} + A^\lambda_{\mu\cdot\lambda} + V V^\lambda_{\mu\cdot\lambda} - A_\mu - 2VV_\mu). \quad (152.9)$$

Now $\qquad A + V^2 = V^h_k V^k_h,$

and therefore $\qquad A_\mu + 2VV_\mu = 2V^k_h V^h_{k\cdot\mu}.$

We have proved (150.?) that $A^\lambda_{\mu\cdot\lambda} \equiv \frac{1}{2}A_\mu,$

and we have $\qquad V^\lambda_{\mu\cdot\lambda} = V_\mu;$

it follows that we have

$$\frac{\partial}{\partial u}(V^{\lambda}_{\mu\cdot\lambda} - V_{\mu}) = 0. \qquad (152.10)$$

§ **153.** Now A is an invariant for any transformation of the coordinates $x_1 \dots x_n$ in the n-way space, and so therefore is $\frac{\partial A}{\partial u}$. We shall therefore, in finding an expression for $\frac{\partial A}{\partial u}$, employ geodesic coordinates at a given point and thus materially simplify the necessary algebra.

$$\frac{\partial A}{\partial u}$$

$$= \frac{\partial}{\partial u} a^{\lambda\mu}\left(\frac{\partial}{\partial x_{\mu}}\{\lambda tt\} - \frac{\partial}{\partial x_t}\{\lambda\mu t\} + \{p\mu t\}\{\lambda tp\} - \{ptt\}\{\lambda\mu p\}\right).$$

Now $$\frac{\partial}{\partial u}\{\lambda\mu t\} = a^{tp}a_{\mu q}(\phi_p V^q_{\lambda} + \phi V^q_{\lambda\cdot p})$$
$$- \phi_{\lambda} V^t_{\mu} - \phi_{\mu} V^t_{\lambda} - \phi(V^t_{\mu\cdot\lambda} + V^t_{\lambda\cdot\mu})$$

and therefore

$$a^{\lambda\mu}\frac{\partial}{\partial u}\{\lambda\mu t\} = a^{tp}(\phi_p V + \phi V_p) - a^{\lambda\mu}(\phi_{\lambda} V^t_{\mu} + \phi_{\mu} V^t_{\lambda})$$
$$- a^{\lambda\mu}\phi(V^t_{\mu\cdot\lambda} + V^t_{\lambda\cdot\mu}),$$

and we proved in the last article (152.7) that

$$\frac{\partial}{\partial u}\{\lambda tt\} = -\phi_{\lambda} V - \phi V_{\lambda}.$$

Hence, in geodesic coordinates, $\frac{\partial A}{\partial u}$ is equal to

$$\{\lambda tt\mu\}\frac{\partial}{\partial u}a^{\lambda\mu} + a^{\lambda\mu}\frac{\partial}{\partial u}\{\lambda tt\mu\}$$

$$= 2\phi a^{\lambda p} V^{\mu}_{p}\{\lambda tt\mu\} - a^{\lambda\mu}(\phi_{\cdot\lambda\mu} V + \phi V_{\cdot\lambda\mu} + \phi_{\lambda} V_{\mu} + \phi_{\mu} V_{\lambda})$$
$$- a^{tp}(\phi_{\cdot pt} V + \phi V_{\cdot pt} + \phi_p V_t + \phi_t V_p)$$
$$+ a^{\lambda\mu}(\phi_{\lambda} V^t_{\mu\cdot t} + \phi_{\mu} V^t_{\lambda\cdot t} + \phi_{\cdot\lambda t} V^t_{\mu} + \phi_{\cdot\mu t} V^t_{\lambda})$$
$$+ a^{\lambda\mu}\phi_t(V^t_{\mu\cdot\lambda} + V^t_{\lambda\cdot\mu}) + a^{\lambda\mu}\phi(V^t_{\mu\cdot\lambda t} + V^t_{\lambda\cdot\mu t}). \qquad (153.1)$$

Now $$V^t_{\mu\cdot\lambda t} = \overline{t\lambda}\, V^t_{\mu} = \overline{\lambda t}\, V^t_{\mu} + \{qt\lambda t\} V^q_{\mu} - \{\mu q\lambda t\} V^t_q$$
$$= V^t_{\mu\cdot t\lambda} + \{qt\lambda t\} V^q_{\mu} - \{\mu q\lambda t\} V^t_q,$$
$$V^t_{\lambda\cdot\mu t} = V^t_{\lambda\cdot t\mu} + \{qt\mu t\} V^q_{\lambda} - \{\lambda q\mu t\} V^t_q,$$

and therefore

$$a^{\lambda\mu}\phi\,(V^t_{\mu\cdot\lambda t}+V^t_{\lambda\cdot\mu t})$$

$$= a^{\lambda\mu}\phi\,(V^t_{\mu\cdot t\lambda}+V^t_{\lambda\cdot t\mu})+2a^{\lambda\mu}\phi\,\{qt\lambda t\}\,V^q_\mu$$

$$-a^{\lambda\mu}\phi a^{qp}(\mu p\lambda t)\,V^t_q-a^{\lambda\mu}\phi a^{qp}(\mu p\lambda t)\,V^t_q$$

$$= 2a^{\lambda\mu}\phi V_{\cdot\lambda\mu}-2a^{\lambda\mu}\phi\,\{qtt\lambda\}\,V^q_\mu+2a^{qp}\,\{p\lambda\lambda t\}\,V^t_q\phi$$

$$= 2\phi\Delta_2(V)-2a^{\lambda p}\phi\,\{\lambda tt\mu\}\,V^\mu_p+2A^q_t V^t_q\phi. \qquad (153.2)$$

It follows that $\dfrac{\partial A}{\partial u}$ is equal to

$$-V\Delta_2(\phi)-\phi\Delta_2(V)-2\Delta(\phi,V)$$
$$-V\Delta_2(\phi)-\phi\Delta_2(V)-2\Delta(\phi,V)$$
$$+a^{\lambda\mu}(\phi_\lambda V_\mu+\phi_\mu V_\lambda)+\phi^\mu_t V^t_\mu+\phi^\lambda_t V^t_\lambda$$
$$+2a^{\lambda\mu}\phi_t V^t_{\mu\cdot\lambda}+2\phi\Delta_2(V)+2A^q_t V^t_q\phi. \qquad (153.3)$$

Again, $\qquad a^{\lambda\mu}V^t_\mu=a^{t\mu}V^\lambda_\mu,$

and therefore

$$a^{\lambda\mu}\phi_t V^t_{\mu\cdot\lambda}=\phi_t a^{t\mu}V^\lambda_{\mu\cdot\lambda}=a^{t\mu}\phi_t V_\mu=\Delta(\phi,V),$$

so that $\dfrac{\partial A}{\partial u}$ is equal to

$$-2V\Delta_2(\phi)+2\phi^\lambda_\mu V^\mu_\lambda+2A^\lambda_\mu V^\mu_\lambda\phi. \qquad (153.4)$$

Since, then, the equation

$$\frac{\partial A}{\partial u}=-2V\Delta_2(\phi)+2\phi^\lambda_\mu V^\mu_\lambda+2A^\lambda_\mu V^\mu_\lambda\phi \qquad (153.5)$$

is expressed in invariant form, it is true not merely at the point, whose geodesic coordinates we employed, but universally.

Now $\dfrac{\partial}{\partial u}(V^\mu_\lambda V^\lambda_\mu-V^2)$

$$=2V^\mu_\lambda(\phi(A^\lambda_\mu+VV^\lambda_\mu)+\phi^\lambda_\mu)-2V(\phi(A+V^2)+\Delta_2(\phi))$$

$$=-2V\Delta_2\phi+2\phi^\lambda_\mu V^\mu_\lambda+2A^\lambda_\mu V^\mu_\lambda\phi,$$

so that $\qquad \dfrac{\partial}{\partial u}(V^\mu_\lambda V^\lambda_\mu-V^2-A)=0. \qquad (153.6)$

§ **154.** If, then, we are given any n-way space, and we find functions V_i^k such that

$$a_{k\lambda} V_i^\lambda = a_{i\lambda} V_k^\lambda \qquad (154.1)$$

which will satisfy the equations

$$V^\lambda_{\mu\cdot\lambda} = V_\mu, \quad A = V^\lambda_\mu V^\mu_\lambda - V^2, \qquad (154.2)$$

and if, taking arbitrarily any function ϕ of $x_1, x_2, \ldots x_n$ and a new variable u, we allow a_{ik} and V_i^k to grow in accordance with the laws

$$\frac{\partial a_{ik}}{\partial u} = -2\phi a_{i\lambda} V_k^\lambda \qquad (154.3)$$

$$\frac{\partial V_i^k}{\partial u} = \phi (A_i^k + V V_i^k) + a^{kt}\phi_{\cdot ti}, \qquad (154.4)$$

taking as their initial values when $u = 0$ the given values of a_{ik} in terms of $x_1 \ldots x_n$, and the values initially found for V_i^k, the equations

$$V^\lambda_{\mu\cdot\lambda} = V_\mu, \quad A = V^\lambda_\mu V^\mu_\lambda - V^2 \qquad (154.2)$$

will remain true when u takes any value whatever, and the form

$$\phi^2 du^2 + a_{ik} dx_i dx_k \qquad (154.5)$$

will be the ground form of an $(n+1)$-way Einstein space.

The surfaces $u =$ constant may be any whatever in the Einstein space; and we see (149.2) that the property of this space is that the sum of the products two at a time of the reciprocals of the principal radii of curvature of any surface in this space is equal to $-\frac{1}{2}A$, where A refers to the n-way space given as the section by the surface of the $(n+1)$-way Einstein space.

CHAPTER XII

THE GENERATION OF AN $(n+1)$-WAY STATIONARY EINSTEIN SPACE FROM AN n-WAY SPACE

§ **155. Conditions that the $(n+1)$-way Einstein space surrounding a given n-way space be stationary.** We have shown that *any* n-way space is surrounded by an $(n+1)$-way Einstein space, and that the equations which lead to the Einstein space are

$$V^{\lambda}_{\mu\cdot\lambda} = V_{\mu}; \quad A = V^{\lambda}_{\mu} V^{\mu}_{\lambda} - V^2,$$

$$\frac{\partial a_{ik}}{\partial u} = -2\phi a_{i\lambda} V^{\lambda}_{k},$$

$$\frac{\partial}{\partial u} V^{\lambda}_{\mu} = \phi (A^{\lambda}_{\mu} + V V^{\lambda}_{\mu}) + \phi^{\lambda}_{\mu},$$

where $\quad A^{\lambda}_{\mu} \equiv a^{\lambda t}\{tpp\mu\}, \quad \phi^{\lambda}_{\mu} \equiv a^{\lambda t}\phi_{\cdot t\mu}.$

The Einstein space has the ground form

$$\phi^2 du^2 + b_{ik} dx_i dx_k,$$

where b_{ik} is equal to a_{ik} when $u = 0$.

We now inquire what properties the n-way space ground form must have if ϕ and b_{ik} are to be independent of u.

Clearly the necessary and sufficient conditions are that

$$V^{k}_{i} = 0. \tag{155.1}$$

We therefore have $\quad A = 0, \quad \phi A^{\lambda}_{\mu} + \phi^{\lambda}_{\mu} = 0;$

that is, $\quad \phi\{\lambda tt\mu\} + \phi_{\cdot\lambda\mu} = 0; \quad \Delta_2(\phi) = 0. \qquad (155.2)$

We now want to transform the ground form

$$a_{ik} dx_i dx_k$$

and the function ϕ by the transformation

$$a_{ik} = b_{ik} e^{2V}, \quad \phi = e^{(2-n)V}. \tag{155.3}$$

We have $\quad a^{ik} = e^{-2V} b^{ik},$

$$\{ikj\}_a = \{ikj\}_b + a^{jt}(a_{it} V_k + a_{kt} V_i - a_{ik} V_t).$$

Let* $\theta^j_{ik} \equiv \epsilon^j_i \theta_k + \epsilon^j_k \theta_i - a_{ik} \theta^j,$

where $\theta^j \equiv a^{jt} \theta_t$

and therefore $\theta_j = a_{jt} \theta^t.$

We also have $\Delta(\theta) = \theta^t \theta_t, \quad \Delta_2(\theta) = \theta^t_{\cdot t}.$

It is easy to verify the following relations :

$$\theta^t_{rh} \theta^i_{ti} = 2n\, \theta_r \theta_h - n a_{rh} \Delta(\theta),$$

$$\theta^i_{th} \theta^t_{ri} = (n+2)\, \theta_r \theta_h - 2 a_{rh} \Delta(\theta),$$

$$\theta^i_{ri \cdot h} - \theta^i_{rh \cdot i} = (n-2)\, \theta_{\cdot rh} + a_{rh} \Delta_2(\theta). \qquad (155.4)$$

Now $\{ikj\}_a = \{ikj\}_b + V^j_{ik},$

and

$$\{rkih\} = \frac{\partial}{\partial x_h} \{rik\} - \frac{\partial}{\partial x_i} \{rhk\} + \{rit\}\{thk\} - \{rht\}\{tik\},$$

and therefore

$$\{rkih\}_a = \{rkih\}_b + \frac{\partial}{\partial x_h} V^k_{ri} - \frac{\partial}{\partial x_i} V^k_{rh}$$
$$+ \{rit\}_a V^k_{th} + V^t_{ri}(\{thk\}_a - V^k_{th})$$
$$- \{rht\}_a V^k_{ti} - V^t_{rh}(\{tik\}_a - V^k_{ti})$$
$$= \{rkih\}_b + V^k_{ri \cdot h} - V^k_{rh \cdot i} + V^k_{ti} V^t_{rh} - V^k_{th} V^t_{ri}. \qquad (155.5)$$

We also see that

$$(\Delta_2(V))_a - (n-2)(\Delta(V))_a = e^{-2V}(\Delta_2(V))_b,$$

$$\Delta_2 e^{(2-n)V} = (2-n)\, e^{(2-n)V}(\Delta_2(V) - (n-2)\Delta(V)),$$

and therefore

$$(\Delta_2(V) - (n-2)\Delta(V))_a = (\Delta_2(V))_b = 0. \qquad (155.6)$$

Now $V^t_{rt \cdot h} - V^t_{rh \cdot t} + V^t_{rh} V^s_{ts} - V^s_{th} V^t_{rs}$

$$= (n-2)(V_{\cdot rh} + V_r V_h) + a_{rh}(\Delta_2(V) - (n-2)\Delta V),$$

and therefore

$$\{rtth\}_a = \{rtth\}_b + (n-2)(V_{\cdot rh} + V_r V_h). \qquad (155.7)$$

It follows that, since

$$\{rtth\}_a + \frac{\phi_{\cdot rh}}{\phi} = 0,$$

$$\{rtth\}_b + (n-2)(n-1)\, V_r V_h = 0. \qquad (155.8)$$

* [This is not the introduction of some new function θ, but an assignment of meanings to θ^j_{ik} and θ^i in connexion with any known θ_i. The meaning of ϵ^k_i is that assigned in § 4.]

For the form $b_{ik}dx_i dx_k$ we therefore have

$$\Delta_2(V) = 0, \quad \{rtth\}_b + (n-1)(n-2)\, V_r V_h = 0. \qquad (155.9)$$

If we can find a ground form to satisfy these conditions, then

$$a_{ik} = b_{ik}e^{2V}, \quad \phi = e^{(2-n)V} \qquad (155.10)$$

will give a ground form which will lead to an $(n+1)$-way Einstein form, the coefficients of which will be independent of u.

§ 156. Infinitesimal generation of the $(n+1)$-way from the n-way form. In order to simplify the problem of finding the ground form $b_{ik}dx_i dx_k$ we shall regard it as an $(n+1)$-way form and bring it to the form

$$*\ \phi^2 du^2 + b_{ik}dx_i dx_k, \quad {}^i_k = 1 \ldots n, \qquad (156.1)$$

as we have done before.

The equations

$$\Delta_2(V) = 0; \quad \{rtth\} + n(n-1)\, V_r V_h = 0$$

now become, if we take $V\sqrt{n(1-n)} = u$,

$$\Delta_2(u) = 0; \quad \{\cdot tt \cdot\} = 1; \quad \{rtth\} = 0;$$
$$\{rtt\cdot\} = 0. \qquad (156.2)$$

If we regard the form as generated from $a_{ik}dx_i dx_k$ we have (§ 146) the equations

$$\frac{\partial b_{ik}}{\partial u} = -2\Omega_{ik}\phi,$$

$$(rkhi)_b = (rkhi)_a + \Omega_{ri}\Omega_{hk} - \Omega_{rh}\Omega_{ik},$$

$$(rkh\cdot)_b = \phi\,(\Omega_{rh\cdot k} - \Omega_{hk\cdot r}),$$

$$(r\cdot h\cdot)_b = \phi\left(\frac{\partial}{\partial u}\Omega_{rh} - \phi_{\cdot rh}\right) + a^{\lambda\mu}\phi^2\Omega_{r\lambda}\Omega_{h\mu}.$$

We therefore have

$$1 = a^{ri}\phi\left(\phi_{\cdot ri} - \frac{\partial}{\partial u}\Omega_{ri}\right) - a^{ri}a^{\lambda\mu}\phi^2\Omega_{r\lambda}\Omega_{i\mu}, \qquad (156.3)$$

$$0 = a^{kh}(\Omega_{rh\cdot k} - \Omega_{hk\cdot r}), \qquad (156.4)$$

$$0 = \phi\,(A_{ri} + a^{kh}(\Omega_{ri}\Omega_{hk} - \Omega_{rh}\Omega_{ik}) - a^{\lambda\mu}\Omega_{r\lambda}\Omega_{i\mu})$$
$$+ \phi_{\cdot ri} - \frac{\partial}{\partial u}\Omega_{ri}. \qquad (156.5)$$

* The ϕ used here is not the ψ of the Einstein form.

The first equation may be replaced by

$$-\phi^{-2} = A + a^{\lambda\mu}(\Omega_{ri}\Omega_{\lambda\mu} - \Omega_{r\lambda}\Omega_{i\mu})\, a^{ri}.$$

The geometrical meaning of this equation, since it may be written

$$-\phi^{-2} = A + \tfrac{1}{2}(a^{ri}a^{\lambda\mu} - a^{r\lambda}a^{i\mu})(\Omega_{ri}\Omega_{\lambda\mu} - \Omega_{r\lambda}\Omega_{i\mu}),$$

is

$$\tfrac{1}{2}(A + \phi^{-2}) = -\sum \frac{1}{R_1 R_2}, \tag{156.6}$$

that is, the sum of the products two at a time of the principal curvatures of the equipotential surface $V = 0$ regarded as a locus in the $(n+1)$-way space is equal to $-\tfrac{1}{2}(A + \phi^{-2})$.

Making the transformation to the functions V^{ι}_{λ} we have

$$\begin{aligned} \frac{\partial}{\partial u} V^{\lambda}_{\mu} &= \phi\,(A^{\lambda}_{\mu} + V V^{\lambda}_{\mu}) + \phi^{\lambda}_{\mu}, \\ V^{\lambda}_{\mu\cdot\lambda} &= V_{\mu}, \\ \phi^{-2} + A + V^2 - V^{\mu}_{\lambda} V^{\lambda}_{\mu} &= 0, \\ \frac{\partial}{\partial u} a_{ik} &= -2\phi a_{i\lambda} V^{\lambda}_{k}, \end{aligned} \tag{156.7}$$

and, from $[\Delta_2(u)]_b = 0$, we have

$$-a^{\lambda\mu}\{\lambda\mu\cdot\} - \phi^{-2}\{\cdots\} = 0;$$

that is,

$$-a^{\lambda\mu}\Omega_{\lambda\mu}\phi^{-1} - \phi^{-4}\phi\frac{\partial\phi}{\partial u} = 0,$$

or

$$-V\phi^{-1} - \phi^{-3}\frac{\partial\phi}{\partial u} = 0;$$

so that

$$\frac{\partial\phi}{\partial u} + V\phi^2 = 0. \tag{156.8}$$

We have, a denoting the determinant of the form $a_{ik}dx_i dx_k$,

$$\begin{aligned} \frac{\partial a}{\partial u} &= a a^{ik}\frac{\partial a_{ik}}{\partial u} \\ &= -2a a^{ik}\Omega_{ik}\phi = -2aV\phi; \end{aligned}$$

and therefore

$$\frac{\partial}{\partial u}(a\phi^{-2}) = -2aV\phi^{-1} - 2a\phi^{-3}\frac{\partial\phi}{\partial u};$$

so that the equation

$$\frac{\partial\phi}{\partial u} + V\phi^2 = 0$$

may be replaced by

$$\frac{\partial}{\partial u}(a\phi^{-2}) = 0. \tag{156.9}$$

We have the equations

$$V^{\lambda}_{\mu\cdot\lambda} = V_{\mu}; \quad \phi^{-2} + A + V^2 - V^{\mu}_{\lambda} V^{\lambda}_{\mu} = 0,$$

$$\frac{\partial}{\partial u} V^{\lambda}_{\mu} = \phi (A^{\lambda}_{\mu} + V V^{\lambda}_{\mu}) + \phi^{\lambda}_{\mu},$$

$$\frac{\partial}{\partial u} a_{ik} = -2\phi a_{i\lambda} V^{\lambda}_{k},$$

$$\frac{\partial}{\partial u} \phi = -V\phi^2.$$

As earlier, we therefore see that

$$\frac{\partial}{\partial u}(V^{\lambda}_{\mu\cdot\lambda} - V_{\mu})$$
$$= \phi_{\mu}(V^{\lambda}_{\rho} V^{\rho}_{\lambda} - A - V^2)$$
$$+ \phi (V_{\rho} V^{\rho}_{\lambda\cdot\mu} + A^{\lambda}_{\mu\cdot\lambda} + V V^{\lambda}_{\mu\cdot\lambda} - A_{\mu} - 2 V V_{\mu})$$
$$= \phi_{\mu}\phi^{-2} + \phi (V^{\lambda}_{\rho} V^{\rho}_{\lambda\cdot\mu} + \tfrac{1}{2} A_{\mu} + V V_{\mu} - A_{\mu} - 2 V V_{\mu});$$
$$(156.10)$$

and this is equal to zero, since

$$\phi^{-3}\phi_{\mu} = \tfrac{1}{2} A_{\mu} - V^{\lambda}_{\rho} V^{\rho}_{\lambda\cdot\mu} + V V_{\mu}.$$

We also have, from what we proved earlier (153.5),

$$\frac{\partial A}{\partial u} = -2 V \Delta_2(\phi) + 2\phi^{\lambda}_{\mu} V^{\mu}_{\lambda} + 2 A^{\lambda}_{\mu} V^{\mu}_{\lambda} \phi,$$

and we have

$$\frac{\partial}{\partial u}(\phi^{-2} + V^2 - V^{\mu}_{\lambda} V^{\lambda}_{\mu}) = -2\phi^{-3}\frac{\partial \phi}{\partial u} + 2V(\phi(A + V^2) + \Delta_2(\phi))$$
$$-2 V^{\mu}_{\lambda}(\phi(A^{\lambda}_{\mu} + V V^{\lambda}_{\mu}) + \phi^{\lambda}_{\mu});$$

and therefore $\quad \dfrac{\partial}{\partial u}(A + \phi^{-2} + V^2 - V^{\mu}_{\lambda} V^{\lambda}_{\mu})$

$$= -2\phi^{-3}\frac{\partial \phi}{\partial u} + 2V\phi(A + V^2 - V^{\mu}_{\lambda} V^{\lambda}_{\mu})$$
$$= 2V\phi(A + V^2 - V^{\mu}_{\lambda} V^{\lambda}_{\mu} + \phi^{-2}) = 0. \qquad (156.11)$$

We thus see that the required $(n+1)$-way form can be generated from any n-way form infinitesimally by choosing V^i_k and ϕ to satisfy the equations

$$V^{\lambda}_{\mu\cdot\lambda} = V_{\mu}; \quad A + \phi^{-2} + V^2 - V^{\mu}_{\lambda} V^{\lambda}_{\mu} = 0. \qquad (156.12)$$

§ 157. **Restatement and interpretation of results.** We may now restate the result at which we have arrived.

Let $a_{ik}dx_i dx_k$ be the ground form of any n-way space whatever.

Find functions V_i^k such that $a_{k\lambda}V_i^\lambda = a_{i\lambda}V_k^\lambda$, and which satisfy the equations $V_{\mu\cdot\lambda}^\lambda = V_\mu$.

Define a function ϕ by the equation

$$A + \phi^{-2} + V^2 - V_\mu^\lambda V_\lambda^\mu = 0,$$

where $$A \equiv a^{\lambda\mu}\{\lambda tt\mu\}.$$

Let the coefficients a_{ik} and the functions ϕ and V_i^k grow, with respect to a new variable u, according to the laws

$$\frac{\partial a_{ik}}{\partial u} = -2\phi a_{i\lambda}V_k^\lambda,$$

$$\frac{\partial}{\partial u}V_i^k = \phi(A_i^k + VV_i^k) + a^{kt}\phi_{\cdot ti},$$

$$\frac{\partial}{\partial u}\phi = -V\phi^2,$$

having as initial values, when $u = 0$, the given values of a_{ik} in terms of $x_1 \ldots x_n$ and the values initially found for V_i^k and ϕ.

The equations $$V_{\mu\cdot\lambda}^\lambda = V_\mu,$$

$$A + \phi^{-2} + V^2 - V_\mu^\lambda V_\lambda^\mu = 0$$

will remain true when u takes any value whatever; and, a denoting the determinant of $a_{ik}\ldots$, $a\phi^{-2}$ will remain a function of $x_1 \ldots x_n$ only.

The $(n+1)$-way form

$$\phi^2 du^2 + a_{ik}dx_i dx_k, \quad {}_k^i = 1 \ldots n,$$

will now have the properties

$$\Delta_2(u) = 0; \quad \{\cdot tt\cdot\} = 1; \quad \{rtth\} = 0; \quad \{rtt\cdot\} = 0.$$

Transform now to any new variables which we may still denote by $x_1 \ldots x_n$, x_{n+1}, and let

$$V\sqrt{-n(n-1)} = u,$$

and thus let the $(n+1)$-way form be

$$a_{ik}dx_i dx_k, \quad {}_k^i = 1 \ldots n+1.$$

It will now have the properties

$$\{rtth\} + n(n-1) V_r V_h = 0,$$

where
$$\Delta_2(V) = 0.$$

From this $(n+1)$-way form let us pass to the form

$$b_{ik} dx_i dx_k,$$

where
$$b_{ik} = a_{ik} e^{2V},$$

and let
$$\theta = e^{(1-n)V}.$$

We now have, for the $(n+1)$-way form

$$b_{ik} dx_i dx_k, \quad {}^{i}_{k} = 1 \ldots n+1,$$

$$\theta\{rtth\} + \theta_{\cdot rh} = 0, \quad \Delta_2(\theta) = 0,$$

or
$$B = 0, \quad \theta B^\lambda_\mu + \theta^\lambda_\mu = 0,$$

where
$$B^\lambda_\mu \equiv b^{\lambda r}\{rtt\mu\}, \quad \theta^\lambda_\mu \equiv b^{\lambda r}\theta_{\cdot r\mu}.$$

The $(n+2)$-way form $\theta^2 du^2 + b_{ik} dx_i dx_k$ will now be an Einstein form and the coefficients b_{ik} and θ will be independent of u.

§ **158. A particular case examined when** $n = 2$. As a particular case we might consider what properties the n-way space must have if in the $(n+1)$-way form which it generates, namely
$$\phi^2 du^2 + a_{ik} dx_i dx_k, \tag{158.1}$$
the coefficients a_{ik} and the function ϕ are to be independent of u.

We must have, as the necessary and sufficient conditions,

$$A + \phi^{-2} = 0,$$

$$\phi A^\lambda_\mu + a^{\lambda t}\phi_{\cdot t\mu} = 0,$$

that is,
$$A + \phi^{-2} = 0,$$

$$\phi\{\lambda tt\mu\} + \phi_{\cdot \lambda\mu} = 0. \tag{158.2}$$

Now the chief interest of an Einstein space is when its dimension is 4. We shall therefore only consider this special case when $n = 2$. We thus have

$$(12\,12) = \frac{a_{11}a_{22} - a_{12}^2}{2\phi^2},$$

that is,
$$2\phi^2 K = 1. \tag{158.3}$$

The other equations become

$$\phi a_{11}K-\phi_{\cdot 11}=0;\quad \phi a_{12}K-\phi_{\cdot 12}=0;\quad \phi a_{22}K-\phi_{\cdot 22}=0. \tag{158.4}$$

We wish to find the properties of the two-way form which will satisfy these tensor equations.

The element of length on the corresponding surface we take to be given by $ds^2 = 2e^{-\theta}du\,dv.$

We then have $\{111\}+\theta_1 = 0,\quad \{222\}+\theta_2 = 0,$

$$\{112\} = \{121\} = \{122\} = \{221\} = 0,$$

and $$K = e^{\theta}\,\theta_{12}.$$

The equations which have to be satisfied will now be

$$\phi_{11}+\theta_1\phi_1 = 0;\quad 2\phi\phi_{12} = e^{-\theta};\quad \phi_{22}+\theta_2\phi_2 = 0;$$
$$2\phi^2 e^{\theta}\,\theta_{12} = 1. \tag{158.5}$$

We should notice that the suffixes in these differential equations denote ordinary differentiation, and not tensor derivation which would be indicated by the dot before the suffixes.

By means of the equation $2\phi\phi_{12} = e^{-\theta}$ we can eliminate θ from the other three equations, and we see that they reduce to the two equations

$$\phi\phi_1\phi_{112} = \phi_{12}(\phi\phi_{11}-\phi_1^2),$$
$$\phi\phi_2\phi_{221} = \phi_{12}(\phi\phi_{22}-\phi_2^2). \tag{158.6}$$

These two equations may be written

$$\frac{\partial}{\partial u}(\log\phi_{12}+\log\phi-\log\phi_1) = 0,$$

$$\frac{\partial}{\partial v}(\log\phi_{12}+\log\phi-\log\phi_2) = 0.$$

Consequently $$\frac{\phi_{12}\phi}{\phi_1}\ \text{and}\ \frac{\phi_{12}\phi}{\phi_2} \tag{158.7}$$

are respectively functions of v only and of u only.

We do not lose in generality by assuming that

$$\phi_{12}\phi = \phi_1 = \phi_2$$

and therefore $$\phi = F(u+v),$$

where $$F''F = F'.$$

The ground form of the surface is therefore

$$ds^2 = 4F' du\,dv, \tag{158.8}$$

where F is a function of $u+v$ given by $F''F = F'$.

If we take the parameters on the surface

$$x = u+v, \quad \iota y = u-v,$$

then
$$ds^2 = F'(x)\,(dx^2+dy^2), \tag{158.9}$$

and
$$F''(x)\,F(x) = F'(x). \tag{158.10}$$

The surface is thus a particular case of a Liouville surface.

§ **159. General procedure in looking for a four-way stationary Einstein space.** In general, when we want a four-way Einstein space of the form

$$\theta^2 du^2 + b_{ik}\,dx_i\,dx_k, \quad {}^i_k = 1, 2, 3,$$

in which the coefficients b_{ik} and θ are to be independent of u, that is, what is called the 'stationary' form, we begin with the ground form
$$a_{11}\,dx_1^2 + 2\,a_{12}\,dx_1\,dx_2 + a_{22}\,dx_2^2. \tag{159.1}$$

We then find in any way three functions

$$\Omega_{11}, \quad \Omega_{12} = \Omega_{21}, \quad \Omega_{22}$$

of the parameters x_1 and x_2 which satisfy the tensor equations

$$\Omega_{11\cdot 2} = \Omega_{12\cdot 1}, \quad \Omega_{22\cdot 1} = \Omega_{12\cdot 2}. \tag{159.2}$$

We define a function ϕ by the equation

$$\frac{a}{2\phi^2} \equiv (12\,12) - \Omega_{11}\Omega_{22} + \Omega_{12}^2. \tag{159.3}$$

We now let the coefficients a_{ik} and the functions Ω_{ik} grow with respect to a new variable u in accordance with the laws

$$\frac{\partial a_{ik}}{\partial u} = -2\phi\,\Omega_{ik}, \tag{159.4}$$

$$\frac{\partial \Omega_{ik}}{\partial u} = \phi_{\cdot ik} - \frac{a_{ik}}{2\phi} - a^{\lambda\mu}\,\Omega_{i\lambda}\Omega_{k\mu}\phi. \tag{159.5}$$

The equation which defines ϕ will be unaltered, and the equations
$$\Omega_{11\cdot 2} = \Omega_{12\cdot 1}, \quad \Omega_{22\cdot 1} = \Omega_{12\cdot 2} \tag{159.2}$$
will remain true.

We thus attain the three-way form

$$\phi^2 du^2 + a_{11} dx_1^2 + 2 a_{12} dx_1 dx_2 + a_{22} dx_2^2, \qquad (159.6)$$

in which in general ϕ and the coefficients a_{ik} will be functions of x_1, x_2, and u.

In the particular case we considered in the last article, when the two-way form appertained to a particular class of Liouville surface, the function ϕ and the coefficients a_{ik} will not involve u.

But in all cases the three-way space with the form

$$\phi^2 du^2 + a_{11} dx_1^2 + 2 a_{12} dx_1 dx_2 + a_{22} dx_2^2$$

will have the property $\quad \Delta_2(u) = 0, \qquad (159.7)$

$$\{ittk\} = 0, \quad \{itt\cdot\} = 0, \quad \{\cdot tt\cdot\} = 1, \qquad (159.8)$$

and therefore

$$(2323) = -\tfrac{1}{2}a_{22}; \quad (3131) = -\tfrac{1}{2}a_{11}; \quad (3123) = \tfrac{1}{2}a_{12};$$

$$(2312) = 0; \quad (3112) = 0; \quad (1212) = \frac{a}{2\phi^2}. \qquad (159.9)$$

From this three-way form we can deduce the ground form of a stationary four-way space by the rules we have given in the general case.

We should notice that if we begin with the proper Liouville surface, the Einstein stationary form at which we arrive can be, by a proper choice of the parameters, thrown into a form in which all the coefficients will be functions of two parameters only.

§ **160. Conclusions as to curvature.** The three-way space with the form

$$\phi^2 du^2 + a_{11} dx_1^2 + 2 a_{12} dx_1 dx_2 + a_{22} dx_2^2$$

is such that, if we regard the surface $u =$ constant as lying in it, the product of the reciprocals of what we have called its principal radii of curvature is

$$\frac{\Omega_{11}\Omega_{22} - \Omega_{12}^2}{a_{11}a_{22} - a_{12}^2}. \qquad (160.1)$$

We must not confuse these radii of curvature with the radii of curvature of the surface $u =$ constant regarded (as it may be) as lying in Euclidean space.

We have in fact the theorem: 'the product of the reciprocals of the principal radii of curvature of the surface, $u =$ constant, is equal to
$$K - \tfrac{1}{2}\phi^{-2}, \qquad (160.2)$$
where K is Gauss's measure of curvature.'

Riemann's measure of curvature corresponding to the vectors ξ and η which lie in the tangential Euclidean space at any point is (§ 141)
$$\frac{(\xi^i \eta^p - \xi^p \eta^i)(\xi^k \eta^q - \xi^q \eta^k)(ipkq)}{(\xi^i \eta^p - \xi^p \eta^i)(\xi^k \eta^q - \xi^q \eta^k)(a_{ik}a_{pq} - a_{iq}a_{pk})}, \qquad (160.3)$$
where the direction cosines of the vectors ξ and η are respectively ξ^1, ξ^2, ξ^3, and η^1, η^2, η^3.

If the vectors ξ and η touch the surface $u =$ constant, this becomes $\tfrac{1}{2}\phi^{-2}$. If their plane contains the normal to the surface it becomes $-\tfrac{1}{2}\phi^{-2}$.

In the particular case when we start with the proper Liouville surface these are respectively K and $-K$, where K is Gauss's measure of curvature.

CHAPTER XIII

n-WAY SPACE OF CONSTANT CURVATURE

§ **161. Ground form for a space of zero Riemann curvature.** We shall now consider the simplest form in which the ground form of a space may be expressed in which Riemann's measure of curvature is zero everywhere and for all orientations.

For such a space
$$\{trpq\} = 0 \qquad (161\,.\,1)$$
for all values of the integers.

Consider the system of differential equations
$$\phi_{\cdot pq} = 0. \qquad (161\,.\,2)$$

We have
$$\phi_{\cdot pqr} = 0, \quad \phi_{\cdot prq} = 0,$$
and therefore
$$(\bar{r}\bar{q}-\bar{q}\bar{r})\,\phi_p = 0,$$
that is,
$$\{ptrq\}\,\phi_t = 0. \qquad (161\,.\,3)$$

A system of equations with the property that no equation of lower order can be deduced from them by the processes of algebra and of the differential calculus is said to be 'complete'.

The necessary and sufficient conditions that the system of differential equations $\phi_{\cdot pq} = 0$ may be complete is then
$$\{ptrq\} = 0, \qquad (161\,.\,1)$$
that is, that Riemann's measure of curvature is everywhere zero.

If u and v are any two integrals of the complete system
$$\phi_{\cdot pq} = 0,$$
$$\begin{aligned}\bar{r}\Delta(u, v) &= \bar{r}a^{\lambda\mu}u_\lambda v_\mu \\ &= a^{\lambda\mu}u_{\cdot\lambda r}v_\mu + a^{\lambda\mu}u_\lambda v_{\cdot\mu r} \\ &= 0, \end{aligned} \qquad (161\,.\,4)$$
and therefore, since $\Delta(u, v)$ is an invariant, it is a mere constant.

If, then, we take any n independent integrals of the equation system

$$\phi_{\cdot pq} = 0 \tag{161.2}$$

as our new variables, the ground form will take such a form that each a_{ik} is a mere constant.

The ground form can therefore be so chosen as to have the Euclidean form

$$dx_1^2 + \ldots + dx_n^2. \tag{161.5}$$

§ **162. Ground form for a space of constant curvature for all orientations.** We next consider the ground form which corresponds to a space for which Riemann's measure of curvature is the same constant for all orientations.

We have
$$(\lambda\nu\mu p) = K(a_{\lambda\mu}a_{\nu p} - a_{\lambda p}a_{\mu\nu}),$$

and therefore
$$\{\lambda t\mu p\} = K(a_{\lambda\mu}\epsilon^t_p - a_{\lambda p}\epsilon^t_\mu). \tag{162.1}$$

If, then,
$$t \neq p \text{ and } t \neq \mu,$$

we have
$$\{\lambda t\mu p\} = 0, \tag{162.2}$$

and
$$\{\lambda p\mu p\} = Ka_{\lambda\mu} \tag{162.3}$$

if p is not equal to μ. Here *the repeated integer p is not to have the usual implication of summation.*

Consider now the system of equations

$$u_{\cdot pq} + Ka_{pq}u = 0. \tag{162.4}$$

We see that the system is complete: for

$$u_{\cdot pqr} + Ka_{pq}u_r = 0,$$

$$u_{\cdot prq} + Ka_{pr}u_q = 0,$$

and therefore
$$\{ptrq\}\, u_t + K(a_{pq}u_r - a_{pr}u_q) = 0,$$

and for a space such as we are considering this is a mere identity and not a differential equation of the first order. The system is therefore complete.

Now let u be any integral of the complete system (162.4).

We have
$$\begin{aligned} \bar{r}\Delta(u) &= \bar{r}a^{\lambda\mu}u_\lambda u_\mu \\ &= a^{\lambda\mu}u_{\cdot\lambda r}u_\mu + a^{\lambda\mu}u_\lambda u_{\cdot\mu r} \\ &= -a^{\lambda\mu}Ka_{\lambda r}uu_\mu - a^{\lambda\mu}Ka_{\mu r}uu_\lambda \\ &= -K(uu_r + uu_r), \end{aligned}$$

and therefore $$\Delta u + K u^2 \tag{162.5}$$
is a mere constant.

As u does not satisfy any equation of the first order, being defined as any solution of the complete system, we can choose u so that $\Delta(u) + Ku^2$ is equal to zero at the origin, and therefore zero everywhere.

Let $$K = -\frac{1}{R^2}. \tag{162.6}$$

We now have $$\Delta(\log u) = \frac{1}{R^2}.$$

Take now as new variables

$$y_1 = R \log u, \quad y_2, \ldots y_n,$$

where $$\Delta(y_1, y_k) = 0, \quad k = 2, \ldots n.$$

The ground form will take the form

$$dx_1^2 + a_{ik} dx_i dx_k, \quad {}^i_k = 2, \ldots n,$$

and, since $$u_{.pq} = \frac{a_{pq} u}{R^2},$$

and $$u = e^{\frac{x_1}{R}},$$

we have $$\{ik1\} + \frac{a_{ik}}{R} = 0;$$

that is, $$(ik1) + \frac{a_{ik}}{R} = 0,$$

or $$\frac{\partial}{\partial x_1}(\log a_{ik}) = \frac{2}{R}$$

It follows that $$a_{ik} = e^{\frac{2x_1}{R}} b_{ik}, \tag{162.7}$$
where b_{ik} is a function of $x_2 \ldots x_n$ only.

As regards the form $b_{ik} dx_i dx_k, \quad {}^i_k = 2, \ldots n,$

we see that, since

$$(rkih)_a = e^{\frac{2x_1}{R}} (rkih)_b + \{ik1\}(rh1) - \{hk1\}(ri1)$$

$$= e^{\frac{2x_1}{R}} (rkih)_b + \frac{a_{rh} a_{ik} - a_{ri} a_{hk}}{R^2},$$

and since (162.1) $(rkih)_a = \dfrac{a_{rh}a_{ik} - a_{ri}a_{hk}}{R^2}$,

we must have $$(rkih)_b = 0. \qquad (162.8)$$

It follows that the ground form of a space of constant negative curvature may be taken as

$$dx_1^2 + e^{\frac{2x_1}{R}}(dx_2^2 + \ldots + dx_n^2). \qquad (162.9)$$

By the substitution

$$y_1 = e^{-\frac{x_1}{R}}, \quad Ry_2 = x_2, \ldots Ry_n = x_n,$$

the form may be written

$$\frac{R^2}{x_1^2}(dx_1^2 + dx_2^2 + \ldots + dx_n^2). \qquad (162.10)$$

The corresponding form for a space of constant positive curvature may be taken as

$$\frac{R^2}{x_1^2}(-dx_1^2 + dx_2^2 + \ldots + dx_n^2). \qquad (162.11)$$

§ 163. **Different forms for these spaces.** We may find other forms for these spaces.

Taking the case of positive curvature, instead of choosing u so that

$$\Delta(u) + Ku^2 = 0,$$

we may choose u so that

$$\Delta(u) = \frac{1-u^2}{R^2}. \qquad (163.1)$$

Let $$u = \cos\frac{x_1}{R},$$

then $$\Delta(x_1) = 1,$$

and the ground form may be taken

$$dx_1^2 + a_{ik}dx_i dx_k, \quad {}^i_k = 2, \ldots n. \qquad (163.2)$$

Since $$u_{\cdot pq} + a_{pq}\frac{u}{R^2} = 0,$$

we now have $$(ik1) + \frac{a_{ik}}{R}\cot\frac{x_1}{R} = 0,$$

and therefore $$a_{ik} = \sin^2\frac{x_1}{R}\, b_{ik}, \tag{163 3}$$

where b_{ik} depends on $x_2 \dots x_n$ only.

We have

$$(rkih)_a = \sin^2\frac{x_1}{R}(rkih)_b + \{ik1\}\,(rh1) - \{hk1\}\,(ri1),$$

and therefore

$$\frac{a_{ri}a_{hk} - a_{rh}a_{ki}}{R^2} = \sin^2\frac{x_1}{R}(rkih)_b + \cot^2\frac{x_1}{R}\,\frac{a_{rh}a_{ki} - a_{hk}a_{ri}}{R^2};$$

that is, $$(rkih)_b = \frac{b_{ri}b_{hk} - b_{rh}b_{ik}}{R^2}.$$

The ground form may therefore be written

$$dx_1^2 + \sin^2\frac{x_1}{R}\, b_{ik}\,dx_i\,dx_k, \tag{163 . 4}$$

where $b_{ik}dx_idx_k$ is a ground form in $x_2 \dots x_n$ only, with the same constant positive measure of curvature.

It at once follows that the ground form for a space of constant positive curvature may be written

$$dx_1^2 + \sin^2\frac{x_1}{R}\,dx_2^2 + \sin^2\frac{x_1}{R}\sin^2\frac{x_2}{R}\,dx_3^2 + \dots;$$

or perhaps better as

$$R^2\,(dx_1^2 + \sin^2 x_1\,dx_2^2 + \sin^2 x_1 \sin^2 x_2\,dx_3^2 + \dots). \tag{163 . 5}$$

A form obviously equivalent would be

$$R^2\,(dx_1^2 + \cos^2 x_1\,dx_2^2 + \cos^2 x_1 \cos^2 x_2\,dx_3^2 + \dots). \tag{163 . 6}$$

The latter form when applied to a space of constant negative curvature would become

$$-R^2\,(dx_1^2 + \cos^2 x_1\,dx_2^2 + \cos^2 x_1 \cos^2 x_2\,dx_3^2 + \dots), \tag{163 . 7}$$

and this may be written

$$R^2\,(dx_1^2 + \cosh^2 x_1\,dx_2^2 + \cosh^2 x_1 \cosh^2 x_2\,dx_3^2 + \dots). \tag{163 . 8}$$

The surface $x_1 =$ constant, that is, the $(n-1)$-way space $x_2 \dots x_n$, regarded as a locus in the n-way space of constant curvature given by the form

$$dx_1^2 + \sin^2\frac{x_1}{R}\,dx_2^2 + \dots, \tag{163 . 9}$$

has all its principal radii of curvature equal to

$$-R\tan\frac{x_1}{R},$$

and any line on the surface is a line of principal curvature.

§ **164. Geodesic geometry for a space of curvature +1.** We shall now consider the geodesic geometry of a space whose curvature is positive unity: that is, the space corresponding to the form

$$ds^2 = dx_1^2 + \sin^2 x_1\, dx_2^2 + \sin^2 x_1 \sin^2 x_2\, dx_3^2 + \dots \quad (164\,.\,1)$$

We shall first find the equation which a path must satisfy if it is to be stationary with respect to variation of the coordinate x_1.

If we write $\dot{x}_i$ for $\frac{dx_i}{ds}$ we must have

$$\frac{d}{ds}\dot{x}_1 = \sin x_1 \cos x_1 (\dot{x}_2^2 + \sin^2 x_2 \dot{x}_3^2 + \dots);$$

and therefore, since

$$1 = \dot{x}_1^2 + \sin^2 x_1 \dot{x}_2^2 + \sin^2 x_1 \sin^2 x_2 \dot{x}_3^2 + \dots,$$

we have $$\ddot{x}_1 = \cot x_1 (1 - \dot{x}_1^2). \quad (164\,.\,2)$$

It follows that $$\frac{d}{ds}\sin^2 x_1 (\dot{x}_1^2 - 1) = 0,$$

and therefore $$\cos x_1 = \cos\alpha_1 \cos(s + \epsilon_1), \quad (164\,.\,3)$$

where α_1 and ϵ_1 are constants, and s is the arc measured from some point on the path.

It follows that

$$\frac{\sin^2\alpha_1}{(1 - \cos^2\alpha_1 \cos^2(s + \epsilon_1))^2} = \dot{x}_2^2 + \sin^2 x_2 \dot{x}_3^2 + \dots \quad (164\,.\,4)$$

Let $$s_1 = \tan^{-1}\frac{\tan(s + \epsilon_1)}{\sin\alpha_1},$$

then we have $$ds_1^2 = dx_2^2 + \sin^2 x_2 dx_3^2 + \dots \quad (164\,.\,5)$$

Here s_1 is the arc in an $(n-1)$-way space of curvature positive unity, and if s is to be stationary for variation of x_2, then s_1 must also be stationary.

Proceeding thus we see that the equations which define a geodesic are

$$\cos x_1 = \cos \alpha_1 \cos (s+\epsilon_1),$$

$$\cos x_2 = \cos \alpha_2 \cos (s_1+\epsilon_2), \ldots, \cos x_{n-1} = \cos \alpha_{n-1} \cos (s_{n-2}+\epsilon_{n-1}),$$

$$x_n = s_{n-1}+\epsilon_n,$$

$$\sin \alpha_1 \tan s_1 = \tan (s+\epsilon_1), \quad \sin \alpha_2 \tan s_2 = \tan (s_1+\epsilon_2), \ldots,$$

$$\sin \alpha_{n-1} \tan s_{n-1} = \tan (s_{n-2}+\epsilon_{n-1}). \qquad (164.6)$$

If we take

$$\xi_1 = r\cos x_1, \quad \xi_2 = r\sin x_1 \cos x_2, \quad \xi_3 = r\sin x_1 \sin x_2 \cos x_3,$$

$$\xi_n = r\sin x_1 \ldots \sin x_{n-1} \cos x_n, \quad \xi_{n+1} = r\sin x_1 \ldots \sin x_{n-1} \sin x_n,$$

we see that $\xi_1^2+\xi_2^2+\ldots+\xi_{n+1}^2 = r^2$,

and we easily verify that

$$d\xi_1^2+d\xi_2^2+\ldots+d\xi_{n+1}^2 = dr^2+r^2(dx_1^2+\sin^2 x_1 dx_2^2+\ldots). \qquad (164.7)$$

The n-way space of curvature positive unity is then the section of an $(n+1)$-way Euclidean space by a sphere of radius unity.

§ **165. Geodesics as circles.** We shall now prove that every geodesic is a circle of unit radius in ordinary Euclidean space of three dimensions, but generally two geodesics will not lie in the same Euclidean three-fold.

We have for a geodesic

$$\sin x_r \cos s_r = \sin \alpha_r \cos (s_{r-1}+\epsilon_r), \quad \sin x_r \sin s_r = \sin (s_{r-1}+\epsilon_r),$$

and therefore

$$\sin x_r \cos x_{r+1} = A_r \cos s_{r-1} + B_r \sin s_{r-1}$$

$$= a_r \cos (s_{r-1}+\epsilon_r) + b_r \sin (s_{r-1}+\epsilon_r), \qquad (165.1)$$

where A_r, B_r, a_r, b_r are some constants.

It follows that

$$\sin x_r \sin x_{r+1} \cos x_{r+2}$$

$$= \sin x_r (A_{r+1} \cos s_r + B_{r+1} \sin s_r)$$

$$= A_{r+1} \sin \alpha_r \cos (s_{r-1}+\epsilon_r) + B_{r+1} \sin (s_{r-1}+\epsilon_r),$$

and therefore

$$b_r \cos \alpha_r \sin x_r \sin x_{r+1} \cos x_{r+2} - B_{r+1} \cos \alpha_r \sin x_r \cos x_{r+1}$$

$$= (B_{r+1} a_r - \sin \alpha_r b_r A_{r+1}) \cos x_r. \qquad (165.2)$$

We thus have a linear relation between the three coordinates

$$\xi_r, \quad \xi_{r+1}, \quad \xi_{r+2}.$$

By a linear transformation in the $(n+1)$-way Euclidean space $\xi_1 \dots \xi_{n+1}$ we can take it that the first such relation is $\xi_1 = 0$, and that $\xi_1^2 + \xi_2^2 + \dots + \xi_{n+1}^2 = 1$.

Proceeding thus with respect to any one geodesic we can take it that the equations which define it are

$$\xi_1 = 0, \quad \xi_2 = 0, \quad \dots \quad \xi_{n-1} = 0,$$

that is, $x_1 = \frac{\pi}{2}, \quad x_2 = \frac{\pi}{2}, \quad \dots \quad x_{n-1} = \frac{\pi}{2}.$

It is therefore just a circle in the space given by

$$ds^2 = d\xi_n^2 + d\xi_{n+1}^2, \tag{165.3}$$

and its equation is $\xi_n^2 + \xi_{n+1}^2 = 1,$ (165.4)

with $\xi_1 = 0, \quad \xi_2 = 0, \quad \dots \quad \xi_{n-1} = 0.$

§ **166. Geodesic distance between two points.** We shall now find an expression for the geodesic distance between any two points in the n-way space whose measure of curvature is positive unity.

Let the two points whose coordinates are

$$x_1 \dots x_n \quad \text{and} \quad y_1 \dots y_n$$

be denoted by x and y, and consider the geodesic which joins the two points. Let $s, s_1, \dots s_{n-1}$ be the arcs which correspond to x, and $s', s'_1, \dots s'_{n-1}$ the arcs which correspond to y.

We have

$$\cos x_1 \cos y_1 + \sin x_1 \sin y_1 \cos s_1 \cos s'_1 + \sin x_1 \sin y_1 \sin s_1 \sin s'_1$$
$$= \cos^2\alpha_1 \cos(s+\epsilon_1)\cos(s'+\epsilon_1) + \sin^2\alpha_1 \cos(s+\epsilon_1)\cos(s'+\epsilon_1)$$
$$+ \sin(s+\epsilon_1)\sin(s'+\epsilon_1),$$

and therefore

$$\cos(s'-s) = \cos x_1 \cos y_1 + \sin x_1 \sin y_1 \cos(s'_1 - s_1). \tag{166.1}$$

Similarly we see that

$$\cos(s'_1 - s_1) = \cos x_2 \cos y_2 + \sin x_2 \sin y_2 \cos(s'_2 - s_2) \tag{166.2}$$

$\vdots$

$$\cos(s'_{n-2} - s_{n-2})$$
$$= \cos x_{n-1} \cos y_{n-1} + \sin x_{n-1} \sin y_{n-1} \cos(s'_{n-1} - s_{n-1}),$$
$$s'_{n-1} - s_{n-1} = y_n - x_n. \tag{166.3}$$

It follows that, denoting the geodesic distance between the points x and y by (xy),

$$\cos(xy) = \cos x_1 \cos y_1 + \sin x_1 \sin y_1 \cos x_2 \cos y_2 + \sin x_1 \sin y_1 \sin x_2 \sin y_2 \cos x_3 \cos y_3 + \ldots + \sin x_1 \sin y_1 \sin x_2 \sin y_2 \ldots \sin x_{n-1} \sin y_{n-1} \cos(x_n - y_n). \quad (166.4)$$

This is the formula which is fundamental in the metrical geometry of n-way space of curvature positive unity.

§ 167. **Coordinates analogous to polar coordinates.** We can now employ a system of coordinates, to express geometrically the position of any point in our space, which will be analogous to the use of polar coordinates in ordinary Euclidean space.

We take any point in the space as origin, that is, the point from which we are to measure x_1, the geodesic distance from the origin.

It will be convenient to denote this distance by $\tan^{-1} r$, so that

$$r = \tan x_1. \quad (167.1)$$

Let us now consider the system of geodesics which pass through this origin. For any one of these geodesics $x_2, \ldots x_n$ are fixed, and we may therefore regard $x_2, \ldots x_n$ as the coordinates which define the geodesic, and thus regard $r, x_2, \ldots x_n$ as the polar coordinates of a point in our space.

The geodesics through the origin cut the surface $r =$ constant in an $(n-1)$-way space of positive curvature $1 + r^{-2}$.

In particular the surface

$$r = \text{infinity} \quad (167.2)$$

is an $(n-1)$-way space of curvature positive unity, and the coordinates of any point in this space define a geodesic through the origin.

The geodesic distance between two points at small distances x_1 and y_1 from the origin is given by

$$\cos(x_1 y_1) = \cos x_1 \cos y_1 + \sin x_1 \sin y_1 (\cos x_2 \cos y_2 + \sin x_2 \sin y_2 \cos x_3 \cos y_3 + \ldots),$$

and therefore

$$1 - \tfrac{1}{2}(x_1 y_1)^2 = 1 - \frac{x_1^2}{2} - \frac{y_1^2}{2} + x_1 y_1 (\cos x_2 \cos y_2 + \ldots).$$

But, if θ is the angle between the geodesics through these points and the origin,

$$(x_1 y_1)^2 = x_1^2 + y_1^2 - 2x_1 y_1 \cos\theta.$$

It follows that

$$\cos\theta = \cos x_2 \cos y_2 + \sin x_2 \sin y_2 \cos x_3 \cos y_3 + \ldots, \qquad (167.3)$$

that is, the angle between the two geodesics is the geodesic distance between the points where the geodesics intersect the surface r = infinity.

The geodesic distance between any two points is therefore the geodesic distance between two points, on a sphere of unit radius, whose polar distances from a point on that sphere are x_1 and y_1, and the difference of whose longitudes is the angle which the geodesics through the points cut out on the surface r = infinity.

§ **168. The three-way space of curvature** $+1$. We now limit ourselves to the case where $n = 3$, that is, the three-way space of curvature positive unity. For this space x_1 is the geodesic distance from the origin; and x_2 and x_3 may be taken as the polar coordinates of the point—on the two-way surface of positive curvature unity, $x_1 = \frac{\pi}{2}$—where the geodesic, through the point x_1, x_2, x_3 and the origin, intersects the surface.

We may without loss of generality suppose that x_1 lies between 0 and $\frac{\pi}{2}$, x_2 between 0 and π, and x_3 between 0 and 2π. In the surrounding four-way Euclidean space ξ_1 will then always be positive.

Through two points in our space one, and only one, geodesic can be drawn, unless the two points lie on the same geodesic through the origin, and are the two points where that geodesic intersects the surface $x_1 = \frac{\pi}{2}$.

Through three points in the space we can in general draw one, and only one, two-way locus of positive curvature unity.

We see this by noticing that three points (x_1, x_2, x_3), (y_1, y_2, y_3), and (z_1, z_2, z_3) determine the plane

$$a_1\xi_1 + a_2\xi_2 + a_3\xi_3 + a_4\xi_4 = 0$$

in the surrounding Euclidean space. The exceptional case would be when the three points lie on the same geodesic.

By a linear transformation, in the Euclidean four-way space, we may take the plane to be $\xi_1 = 0$ and the locus of the points of intersection with the sphere to be given by

$$\xi_2^2 + \xi_3^2 + \xi_4^2 = 1.$$

There will then be a corresponding set of coordinates x_1, x_2, x_3 such that the locus is given by $x_1 = \frac{\pi}{2}$ in the new coordinate system.

It will be convenient to call any two-way locus of curvature positive unity a plane, though we should remember that it is only properly a plane in the Euclidean four-fold. Similarly we shall call any geodesic a line.

Plane geometry in our space is therefore just spherical trigonometry.

§ **169. The geometry of the space.** We may now introduce a different system of coordinates in order to bring out the relationship between the geometry of space of curvature positive unity and that of ordinary Euclidean space.

Let $$x = \tan x_1 \sin x_2 \cos x_3,$$

$$y = \tan x_1 \sin x_2 \sin x_3, \quad z = \tan x_1 \cos x_2. \qquad (169.1)$$

In this system of coordinates the geodesic distance between two points (x, y, z) and (x', y', z') will be

$$\cos^{-1}\frac{1 + xx' + yy' + zz'}{(1 + r^2)^{\frac{1}{2}}(1 + r'^2)^{\frac{1}{2}}}, \qquad (169.2)$$

where $$r^2 = x^2 + y^2 + z^2.$$

The square of the element of length will be given by

$$ds^2 = (1 + r^2)^{-1}\left(dx^2 + dy^2 + dz^2 - (1 + r^2)^{-1} r^2 dr^2\right); \quad (169.3)$$

but in this geometry, as in Euclidean geometry, having the expression for the actual distance between any two points, we do not need to make so much use of the expression for the element of length.

The equation of any plane is

$$\lambda x + \mu y + \nu z + \delta = 0.$$

Now a plane, we know, is a two-way surface of curvature positive unity. Let x_1, y_1, z_1 be the coordinates of its centre, that is, the point at a geodesic distance $\frac{\pi}{2}$ from every point of it.

We then have $$x x_1 + y y_1 + z z_1 + 1 = 0, \qquad (169.4)$$

and therefore $\delta x_1 = \lambda, \quad \delta y_1 = \mu, \quad \delta z_1 = \nu.$

The angle between two planes is, as in spherical trigonometry, the supplement of the angle—that is, the geodesic distance—between their centres.

The cosine of the angle between the two planes

$$\lambda_1 x + \mu_1 y + \nu_1 z + \delta_1 = 0,$$

$$\lambda_2 x + \mu_2 y + \nu_2 z + \delta_2 = 0$$

is therefore $$\frac{\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 + \delta_1\delta_2}{(\lambda_1^2 + \mu_1^2 + \nu_1^2 + \delta_1^2)^{\frac{1}{2}}(\lambda_2^2 + \mu_2^2 + \nu_2^2 + \delta_2^2)^{\frac{1}{2}}}. \qquad (169\ 5)$$

The equation of a plane, given in terms of the coordinates of its centre, is $$x x_1 + y y_1 + z z_1 + 1 = 0. \qquad (169.4)$$

The condition that the plane passes through the origin, that is, the point where x, y, and z are each zero, is that its centre should lie on the plane whose centre is the origin.

The equation of a line is given as the intersection of two planes $$x x_1 + y y_1 + z z_1 + 1 = 0,$$

$$x x_2 + y y_2 + z z_2 + 1 = 0.$$

In connexion with this line we consider the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

The plane whose centre is A may be called the polar plane of A. We see that if B lies on the polar plane of A, then A lies on the polar plane of B.

We now see that if (x_1, y_1, z_1) and (x_2, y_2, z_2) are any two points on a line, then every other point on the line is given by

$$x = \frac{px_1 + qx_2}{p+q}, \quad y = \frac{py_1 + qy_2}{p+q}, \quad z = \frac{pz_1 + qz_2}{p+q}, \qquad (169.6)$$

where $p : q$ is an arbitrary parameter.

The line given as the intersection of the planes

$$xx_1 + yy_1 + zz_1 + 1 = 0,$$
$$xx_2 + yy_2 + zz_2 + 1 = 0$$

stands therefore to the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) in the relationship, that the distance between any point on the one line and any point on the other line is $\frac{\pi}{2}$. The lines which are in this relationship will be called polar lines.

We now wish to consider two lines, viz. the line given by

$$xx_1 + yy_1 + zz_1 + 1 = 0,$$
$$xx_2 + yy_2 + zz_2 + 1 = 0,$$

and the line given by

$$xx_3 + yy_3 + zz_3 + 1 = 0,$$
$$xx_4 + yy_4 + zz_4 + 1 = 0.$$

If these lines intersect, the four points

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3), \quad (x_4, y_4, z_4)$$

lie on a plane, and we thus see that if two lines intersect their polar lines also intersect, and the plane on which they lie is the polar plane of the point of intersection.

§ **170. Formulae for lines in the space, and an invariant.** Just as in Euclidean geometry, a line has six coordinates. We define these coordinates

$$l \equiv x_2 - x_1, \quad m \equiv y_2 - y_1, \quad n \equiv z_2 - z_1, \quad \lambda \equiv y_1 z_2 - y_2 z_1,$$
$$\mu \equiv z_1 x_2 - z_2 x_1, \quad \nu \equiv x_1 y_2 - x_2 y_1. \qquad (170.1)$$

The six coordinates are those of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2), and they are connected by the relation

$$l\lambda + m\mu + n\nu \equiv 0. \qquad (170.2)$$

We easily see that if $l, m, n, \lambda, \mu, \nu$ are the coordinates of a line, the coordinates of its polar line are $\lambda, \mu, \nu, l, m, n$.

Let $\quad s_{12} \equiv x_1 x_2 + y_1 y_2 + z_1 z_2, \quad s_1 \equiv x_1^2 + y_1^2 + z_1^2.$

If (12) denotes the geodesic distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2),

$$\cos(12) = \frac{1 + s_{12}}{(1 + s_1)^{\frac{1}{2}}(1 + s_2)^{\frac{1}{2}}},$$

$$\sin(12) = \left\{\frac{(1 + s_1)(1 + s_2) - (1 + s_{12})^2}{(1 + s_1)(1 + s_2)}\right\}^{\frac{1}{2}},$$

and therefore $$\sin^2(12) = \frac{l^2 + m^2 + n^2 + \lambda^2 + \mu^2 + \nu^2}{(1 + s_1)(1 + s_2)}. \tag{170.3}$$

Consider the expression

$$\frac{ll' + mm' + nn' + \lambda\lambda' + \mu\mu' + \nu\nu'}{(l^2 + m^2 + n^2 + \lambda^2 + \mu^2 + \nu^2)^{\frac{1}{2}}(l'^2 + m'^2 + n'^2 + \lambda'^2 + \mu'^2 + \nu'^2)^{\frac{1}{2}}},$$

where $(l, m, n, \lambda, \mu, \nu)$ and $(l', m', n', \lambda', \mu', \nu')$ are the coordinates of the lines which respectively join the points 1 and 2, and the points 3 and 4.

It is easily verified that the numerator of the expression is

$$(1 + s_{13})(1 + s_{24}) - (1 + s_{14})(1 + s_{23})$$

and the denominator is

$$\sqrt{(1 + s_1)(1 + s_2)(1 + s_3)(1 + s_4)}\sin(12)\sin(34).$$

The expression is therefore equal to

$$\frac{\cos(13)\cos(24) - \cos(14)\cos(23)}{\sin(12)\sin(34)}, \tag{170.4}$$

and this is clearly an invariant.

Suppose now that the points 1 and 3 coincide. The expression becomes $$\frac{\cos(24) - \cos(14)\cos(12)}{\sin(12)\sin(14)}, \tag{170.5}$$

and we see that this is the cosine of the angle between the lines 12 and 14.

Suppose next that the line 13 is perpendicular to the lines 12 and 34.

Clearly, from the formula

$$\cos(24) = \cos(14)\cos(12) \qquad (170.6)$$

when the lines 12 and 14 are perpendicular, the line 13 will be the shortest distance between the lines 12 and 34.

The planes 132 and 134 will be the planes through the shortest distance and the lines 12 and 34.

We may, to interpret the expression

$$\frac{ll' + mm' + nn' + \lambda\lambda' + \mu\mu' + \nu\nu'}{(l^2 + m^2 + n^2 + \lambda^2 + \mu^2 + \nu^2)^{\frac{1}{2}} (l'^2 + m'^2 + n'^2 + \lambda'^2 + \mu'^2 + \nu'^2)^{\frac{1}{2}}},$$

since we have seen that it is an invariant, take the points 1, 2, 3, 4 to be

$$0, 0, 0; \quad x_2, 0, 0; \quad 0, 0, z_3; \quad x_4, y_4, z_3;$$

it now becomes $$\frac{x_4}{\sqrt{x_4^2 + y_4^2}\sqrt{1 + z_3^2}}.$$

The equations of the planes 132 and 134 become respectively

$$y = 0, \quad yx_4 - xy_4 = 0.$$

The angle between these planes is

$$\cos^{-1}\frac{x_4}{\sqrt{x_4^2 + y_4^2}}.$$

The shortest distance between the lines is

$$\cos^{-1}\frac{1}{\sqrt{1 + z_3^2}},$$

and therefore the invariant expression is equal to the product of the cosine of the shortest distance between the lines into the cosine of the angle between the two planes drawn through the shortest distance and the two given lines.

The invariant vanishes if the lines are polar lines. It also vanishes if the planes through the shortest distance and the two lines are perpendicular.

If the lines are not polar lines and if the invariant vanishes, we see that the polar line of 12 intersects 34 and the polar line of 34 intersects 12.

§ **171. Volume in the space.** The expression for the element of volume in a space of three dimensions and with the measure of curvature positive and equal to unity is

$$\sin^2 x_1 \sin x_2 \, dx_1 \, dx_2 \, dx_3, \qquad (171.1)$$

returning to the original notation of § 168.

The volume enclosed by an area of any plane—that is, a two-way surface of curvature positive unity—and the lines joining the origin to the perimeter of the area is

$$\tfrac{1}{2}\iint (x_1 - \sin x_1 \cos x_1) \sin x_2 \, dx_2 \, dx_3, \qquad (171.2)$$

where x_1 is the geodesic distance from the origin to a point within the perimeter.

If the plane is at a geodesic distance p from the origin we can use the equation $\tan p = \tan x_1 \cos x_2$,

and express the above integral in the form

$$\tfrac{1}{2}\int (p - \tan p \,.\, x_1 \cot x_1)\, dx_3, \qquad (171.3)$$

where x_1 is now the geodesic distance to a point on the perimeter from the origin.

If we take r to be the geodesic distance of a point on the perimeter from the foot of the perpendicular, and take x_3 to be the corresponding longitude θ in the plane, the above formula becomes

$$\tfrac{1}{2}\int \left(p - \frac{\tan p \cos p \cos r \cos^{-1}(\cos p \cos r)}{\sqrt{1 - \cos^2 p \cos^2 r}}\right) d\theta. \qquad (171.4)$$

If the foot of the perpendicular lies within the area, this formula gives us for the volume the expression

$$p\pi - \int \frac{\sin p \cos r \cos^{-1}(\cos p \cos r)}{\sqrt{1 - \cos^2 p \cos^2 r}}\, d\theta, \qquad (171.5)$$

where the integral is to be taken round the perimeter.

We notice that in space of curvature positive unity when s, the variable in the equation of a geodesic, increases by 2π, then $s_1, s_2, \ldots$ also increase by 2π, and therefore the coordinates $x_1, \ldots x_n$ all increase by 2π. We thus, in proceeding along a geodesic, come back to the point we started from. We cannot have any two points at a greater distance from one another than π.

§ 172. An n-way space of constant curvature as a section of an extended Einstein space. We now wish to consider the n-way space of constant curvature as a section of an $(n+1)$-way surrounding space.

We take the $(n+1)$-way space ground form to be

$$\phi^2 du^2 + b_{ik} dx_i dx_k, \quad {}^{i}_{k} = 1 \ldots n, \tag{172.1}$$

where $b_{ik} = a_{ik}$ when $u = 0$.

We have (§ 146)

$$\begin{aligned}
(rkhi)_b &= (rkhi)_a + \Omega_{,i}\Omega_{hk} - \Omega_{,h}\Omega_{ik},\\
(rkh\cdot)_b &= \phi\,(\Omega_{,h\cdot k} - \Omega_{hk\cdot r}),\\
(r\cdot h\cdot)_b &= \phi\left(\frac{\partial}{\partial u}\Omega_{rh} - \phi\cdot_{rh}\right) + a^{\lambda\mu}\phi^2\Omega_{r\lambda}\Omega_{h\mu},\\
\frac{\partial b_{ik}}{\partial u} &= -2\Omega_{ik}\phi.
\end{aligned} \tag{172.2}$$

Extending the definition of an Einstein space, we shall now say that a space is an Einstein space if

$$b^{ki}(rkih) = cb_{,h}, \tag{172.3}$$

where c is a constant.

We have

$$\begin{aligned}
(B\cdot\cdot)_b &= a^{ri}\phi\left(\phi\cdot_{,i} - \frac{\partial}{\partial u}\Omega_{,i}\right) - a^{ri}a^{\lambda\mu}\phi^2\Omega_{,\lambda}\Omega_{i\mu},\\
(B_{r}\cdot)_b &= a^{kh}\phi\,(\Omega_{rh\cdot k} - \Omega_{hk\cdot r}),\\
(B_{ri})_b &= (A_{,i})_a + a^{kh}(\Omega_{ri}\Omega_{hk} - \Omega_{,h}\Omega_{ik})\\
&\quad + \phi^{-1}\left(\phi\cdot_{,i} - \frac{\partial}{\partial u}\Omega_{,i}\right) - a^{\lambda\mu}\Omega_{,\lambda}\Omega_{i\mu}.
\end{aligned} \tag{172.4}$$

If the surrounding $(n+1)$-way space is to be Einstein space according to the new definition,* we must have

$$\begin{aligned}
c &= a^{ri}\phi^{-1}\left(\phi\cdot_{,i} - \frac{\partial}{\partial u}\Omega_{ri}\right) - a^{ri}a^{\lambda\mu}\Omega_{,\lambda}\Omega_{i\mu},\\
0 &= a^{kh}(\Omega_{,h\cdot k} - \Omega_{hk\cdot ,}),
\end{aligned}$$

* [Called in § 18 an *extended Einstein space.*]

$$ca_{ri} = A_{ri} + a^{kh}(\Omega_{ri}\Omega_{hk} - \Omega_{rh}\Omega_{ik})$$
$$+ \phi^{-1}\left(\phi_{\cdot ri} - \frac{\partial}{\partial u}\Omega_{ri}\right) - a^{\lambda\mu}\Omega_{r\lambda}\Omega_{i\mu}.$$

That is, if
$$V_i^k \equiv a^{kt}\Omega_{it} \tag{172.5}$$
where
$$\Omega_{ik} = \Omega_{ki} = a_{k\lambda}V_i^\lambda = a_{i\lambda}V_k^\lambda,$$
we must have
$$c = \phi^{-1}\left(\Delta_2(\phi) - a^{ri}\frac{\partial}{\partial u}\Omega_{ri}\right) - V_\mu^\lambda V_\lambda^\mu, \tag{172.6}$$
$$V_{\mu\cdot\lambda}^\lambda = V_\mu. \tag{172.7}$$

Now
$$\frac{\partial}{\partial u}\Omega_{ri} = a_{i\lambda}\left(\frac{\partial}{\partial u}V_r^\lambda - 2\phi V_r^t V_t^\lambda\right),$$
and therefore
$$ca_{ri} = A_{ri} + a^{kh}\Omega_{ri}\Omega_{hk} - 2a^{\lambda\mu}\Omega_{r\lambda}\Omega_{i\mu}$$
$$+ \phi^{-1}\left(\phi_{\cdot ri} - a_{i\lambda}\frac{\partial}{\partial u}V_r^\lambda + 2\phi a_{i\lambda}V_r^t V_t^\lambda\right)$$
$$= A_{ri} + \Omega_{ri}V + \phi^{-1}\left(\phi_{\cdot ri} - a_{i\lambda}\frac{\partial}{\partial u}V_r^\lambda\right).$$

Multiplying by a^{ip} and summing,
$$c\epsilon_r^p = A_r^p + V_r^p V + \phi^{-1}\left(\phi_r^p - \frac{\partial}{\partial u}V_r^p\right). \tag{172.8}$$

We also have
$$c = \phi^{-1}\left(\Delta_2(\phi) - \frac{\partial}{\partial u}V + 2\phi V_\mu^\lambda V_\lambda^\mu\right) - V_\mu^\lambda V_\lambda^\mu.$$

That is, we have
$$\frac{\partial}{\partial u}V_\mu^\lambda = \phi_\mu^\lambda + \phi(A_\mu^\lambda + VV_\mu^\lambda - c\epsilon_\mu^\lambda),$$
$$\frac{\partial}{\partial u}V = \Delta_2(\phi) + \phi V_\mu^\lambda V_\lambda^\mu - c\phi,$$
$$V_{\mu\cdot\lambda}^\lambda = V_\mu.$$

We may replace the equation
$$\frac{\partial}{\partial u}V = \Delta_2(\phi) + \phi V_\mu^\lambda V_\lambda^\mu - c\phi$$
by
$$A + V^2 - V_\mu^\lambda V_\lambda^\mu = (n-1)c.$$

The equations therefore become

$$A + V^2 - V^\lambda_\mu V^\mu_\lambda = (n-1)\,c, \qquad (172.9)$$

$$V^\lambda_{\mu\cdot\lambda} = V_\mu, \qquad (172.10)$$

$$\frac{\partial}{\partial u} V^\lambda_\mu = \phi^\lambda_\mu + \phi\,(A^\lambda_\mu + V V^\lambda_\mu - c\epsilon^\lambda_\mu), \qquad (172.11)$$

$$\frac{\partial a_{ik}}{\partial u} = -2\,\phi a_{it} V^t_k. \qquad (172.12)$$

We may easily verify that the results which we have proved for the case $c = 0$ still hold in this more general Einstein space.

The special conditions that the coefficients b_{ik} and the function ϕ may be independent of u become

$$A = (n-1)\,c, \quad \phi A^\lambda_\mu + \phi^\lambda_\mu = c\phi\epsilon^\lambda_\mu,$$

that is, $\phi\,\{\lambda tt\mu\} + \phi_{\cdot\lambda\mu} = c\phi a_{\lambda\mu}, \quad \Delta_2\phi = c\phi.$ (172.13)

Now let us assume that the n-way space is of constant curvature K. We have

$$\{\lambda tt\mu\} = (1-n)\,K a_{\lambda\mu}.$$

If we choose K so that $Kn = -c$, the conditions that the surrounding Einstein space may satisfy the required conditions become

$$\phi_{\lambda\mu} + K a_{\lambda\mu}\phi = 0,$$

$$\Delta_2(\phi) + Kn\phi = 0.$$

The second condition is a consequence of the first set, and we see that all that we need is that the system

$$\phi_{\cdot\lambda\mu} + K a_{\lambda\mu}\phi = 0 \qquad (172.14)$$

may be complete.

We know it is, and thus we may take $\phi = \cos x_1$, and the space given by

$$ds^2 = \cos^2 x_1\,du^2 + R^2\,(dx_1^2 + \sin^2 x_1\,dx_2^2 + \ldots) \qquad (172.15)$$

will be an Einstein space of the kind required.

If the space is of constant negative curvature we should see that regarded as a locus in $(n+1)$-way Euclidean space it would be an imaginary section.

The expression for the geodesic distance in space of negative curvature unity is given by

$$\cosh s + \sinh x_1 \sinh y_1 + \cosh x_1 \cosh y_1 \sinh x_2 \sinh y_2 + \ldots$$
$$+ \sinh x_1 \ldots \sinh x_{n-2} \sinh y_1 \ldots \sinh y_{n-2} \cosh x_{n-1} \cosh y_{n-1}$$
$$= \cosh x_1 \ldots \cosh x_{n-1} \cosh y_1 \ldots \cosh y_{n-1} \cosh(x_n - y_n).$$

CHAPTER XIV

n-WAY SPACE AS A LOCUS IN $(n+1)$-WAY SPACE

§ 173. **A space by which any n-way space may be surrounded.** We now consider again the ground form of an $(n+1)$-way space $du^2 + b_{ik}dx_i dx_k$ which, when we put $u = 0$, becomes $a_{ik}dx_i dx_k$.

We have, by the formulae of § 146,

$$\{ikh\}_b = \{ikh\}_a; \quad \{ik\cdot\}_b = \Omega_{ik}; \quad \{i\cdot k\}_b = -a^{kt}\Omega_{it},$$

$$(rkih)_b = (rkih)_a + \Omega_{rh}\Omega_{ik} - \Omega_{ri}\Omega_{hk},$$

$$(rki\cdot)_b = \Omega_{ri\cdot k} - \Omega_{ik\cdot r},$$

$$(r\cdot i\cdot)_b = \frac{\partial \Omega_{ri}}{\partial u} + a^{\lambda\mu}\Omega_{r\lambda}\Omega_{i\mu},$$

$$0 = \frac{\partial b_{ik}}{\partial u} + 2\Omega_{ik}.$$

We shall prove that we may surround any n-way space with a space for which

$$(rki\cdot)_b = 0; \quad (r\cdot i\cdot)_b = 0. \tag{173.1}$$

Let

$$\dot{\Omega}_{ik} = \frac{\partial \Omega_{ik}}{\partial u},$$

then

$$\frac{\partial}{\partial u}\Omega_{ri\cdot k} = \frac{\partial}{\partial u}\left(\frac{\partial}{\partial x_k}\Omega_{ri} - \{rkt\}\,\Omega_{ti} - \{ikt\}\,\Omega_{tr}\right)$$

$$= \frac{\partial}{\partial x_k}\dot{\Omega}_{ri} - \{rkt\}\,\dot{\Omega}_{ti} - \{ikt\}\,\dot{\Omega}_{tr}$$

$$-\Omega_{ti}\frac{\partial}{\partial u}\{rkt\} - \Omega_{tr}\frac{\partial}{\partial u}\{ikt\}$$

$$= \dot{\Omega}_{ri\cdot k} - \Omega_{ti}\frac{\partial}{\partial u}\{rkt\} - \Omega_{tr}\frac{\partial}{\partial u}\{ikt\}.$$

Now we have seen that

$$\frac{\partial}{\partial u}\{rkt\} = a^{t\lambda}a_{k\mu}V^{\mu}_{r\cdot\lambda} - V^{t}_{k\cdot r} - V^{t}_{r\cdot k}$$
$$= a^{t\lambda}\Omega_{kr\cdot\lambda} - a^{t\lambda}\Omega_{\lambda k\cdot r} - a^{t\lambda}\Omega_{r\lambda\cdot k}$$
$$= -a^{t\lambda}\Omega_{r\lambda\cdot k},$$

and therefore

$$\frac{\partial}{\partial u}\Omega_{ri\cdot k} = -a^{\lambda\mu}\Omega_{r\lambda}\Omega_{i\mu\cdot k} - a^{\lambda\mu}\Omega_{r\lambda\cdot k}\Omega_{i\mu}$$
$$+ a^{t\lambda}\Omega_{ti}\Omega_{r\lambda\cdot k} + a^{t\lambda}\Omega_{rt}\Omega_{i\lambda\cdot k};$$

that is, $$\frac{\partial}{\partial u}\Omega_{ri\cdot k} = a^{\lambda\mu}(\Omega_{r\mu}\Omega_{i\lambda\cdot k} - \Omega_{r\lambda}\Omega_{i\mu\cdot k})$$
$$= 0. \qquad (173.2)$$

Thus we see that the relations $\Omega_{ri\cdot k} = \Omega_{ik\cdot r}$ persist when a_{ik} and Ω_{ik} grow in accordance with the laws

$$\frac{\partial\Omega_{ik}}{\partial u} + a^{\lambda\mu}\Omega_{i\lambda}\Omega_{k\mu} = 0, \qquad (173.3)$$

$$\frac{\partial a_{ik}}{\partial u} + 2\Omega_{ik} = 0. \qquad (173.4)$$

We can therefore surround the n-way space with a space for which $$(rki\cdot)_b = 0;\ (r\cdot i\cdot)_b = 0.$$

§ **174. Curvature properties of this surrounding space.** We will now consider what properties such a surrounding space would have as regards curvature.

Consider the ground form of the surrounding space, which we denote by the suffix b, $du^2 + b_{ik}dx_i dx_k$.

Let $$\acute{\xi} \equiv \xi^1 z_1 + \ldots + \xi^n z_n + \dot{\xi}\dot{z},$$
$$\acute{\eta} \equiv \eta^1 z_1 + \ldots + \eta^n z_n + \dot{\eta}\dot{z}$$

be two vectors of lengths $|\acute{\xi}|$ and $|\acute{\eta}|$ inclined at an angle θ' which lie in the tangential $(n+1)$-fold and therefore in Euclidean space.

Let $$\xi \equiv \xi^1 z_1 + \ldots + \xi^n z_n,$$
$$\eta \equiv \eta^1 z_1 + \ldots + \eta^n z_n$$

be two corresponding vectors of lengths $|\xi|$ and $|\eta|$ inclined at an angle θ and lying in the tangential n-fold.

We have
$$|\xi|^2 = a_{ik}\xi^i\xi^k,$$
$$|\eta|^2 = a_{ik}\eta^i\eta^k,$$
$$|\xi||\eta|\cos\theta = a_{ik}\xi^i\eta^k,$$
$$4|\xi|^2|\eta|^2\sin^2\theta = (\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(a_{ik}a_{pq} - a_{iq}a_{kp}).$$

The measure of curvature K_a, according to Riemann, which corresponds to the orientation given by the vectors ξ and η, satisfies the equation
$$4\sin^2\theta\,|\xi|^2|\eta|^2 K_a = (\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(ipkq)_a.$$

Now consider the vectors $\acute{\xi}$, $\acute{\eta}$ when $u = 0$.
$$|\acute{\xi}|^2 = |\xi|^2 - \dot{\xi}^2, \quad |\acute{\eta}|^2 = |\eta|^2 - \dot{\eta}^2,$$
$$4|\acute{\xi}|^2|\acute{\eta}|^2\sin^2\theta'$$
$$= 4|\xi|^2|\eta|^2\sin^2\theta + 4(\xi^i\dot{\eta} - \dot{\xi}\eta^i)(\xi^k\dot{\eta} - \dot{\xi}\eta^k)a_{ik}$$
$$= 4(|\xi|^2|\eta|^2\sin^2\theta + \dot{\xi}^2|\eta|^2 + \dot{\eta}^2|\xi|^2 - 2\dot{\xi}\dot{\eta}|\xi||\eta|\cos\theta).$$

We therefore have
$$4K_b(|\xi|^2|\eta|^2\sin^2\theta + \dot{\xi}^2|\eta|^2 + \dot{\eta}^2|\xi|^2 - 2\dot{\xi}\dot{\eta}|\xi||\eta|\cos\theta)$$
$$= (\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(ipkq)_b,$$
since
$$(ipk\cdot)_b = 0, \quad (i\cdot k\cdot)_b = 0.$$

But
$$(ipkq)_b = (ipkq)_a + \Omega_{iq}\Omega_{pk} - \Omega_{ik}\Omega_{pq},$$
and therefore
$$4K_b(|\xi|^2|\eta|^2\sin^2\theta + \dot{\xi}^2|\eta|^2 + \dot{\eta}^2|\xi|^2 - 2\dot{\xi}\dot{\eta}|\xi||\eta|\cos\theta)$$
$$= 4K_a|\xi|^2|\eta|^2\sin^2\theta$$
$$+ (\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(\Omega_{iq}\Omega_{pk} - \Omega_{ik}\Omega_{pq}). \quad (174.1)$$

Here K_b is the Riemann curvature in the $(n+1)$-way space corresponding to the orientation of the vectors $\xi + \dot{\xi}z$ and $\eta + \dot{\eta}z$; and K_a is the Riemann curvature in the n-way space corresponding to the vectors ξ and η.

§ **175.** We may express the result in yet another form.

Consider the ground form $c\Omega_{ik}dx_i dx_k$, where c is a constant introduced to keep the dimensions right, and let a vector ζ be defined by the equations $\zeta_i\zeta_k + c\Omega_{ik} = 0$.

The vector ζ will then trace out in some Euclidean r-fold an n-way space $x_1 \ldots x_n$.

In this space let us consider two vectors $\acute{\xi}$ and $\acute{\eta}$ of lengths $|\acute{\xi}|$ and $|\acute{\eta}|$ inclined at an angle θ', where

$$\acute{\xi} \equiv \xi^1\zeta_1 + \ldots + \xi^n\zeta_n,$$
$$\acute{\eta} \equiv \eta^1\zeta_1 + \ldots + \eta^n\zeta_n.$$

We have

$$4\sin^2\theta' \, |\acute{\xi}|^2 \, |\acute{\eta}|^2$$
$$= c^2 (\xi^i\eta^p - \xi^p\eta^i)(\xi^k\eta^q - \xi^q\eta^k)(\Omega_{ik}\Omega_{pq} - \Omega_{iq}\Omega_{kp}),$$

and therefore

$$|\xi|^2 \, |\eta|^2 \sin^2\theta\,(K_b - K_a) + |\acute{\xi}|^2 \, |\acute{\eta}|^2 \sin^2\theta' c^{-2}$$
$$+ K_b (\dot{\xi}^2 |\eta|^2 + \dot{\eta}^2 |\xi|^2 - 2\,\dot{\xi}\dot{\eta}\,|\xi|\,|\eta| \cos\theta) = 0. \tag{175.1}$$

We see that the curvature of this $(n+1)$-way space which surrounds the given n-way space depends, then, on the knowledge of the ground form $\Omega_{ik}dx_i dx_k$ with the property that

$$\Omega_{ri\cdot k} = \Omega_{rk\cdot i}.$$

§ **176. A condition that the surrounding space may be Euclidean.** We now ask whether the surrounding space can be Euclidean?

If it is Euclidean we must have

$$(rkih) + \Omega_{rh}\Omega_{ik} - \Omega_{ri}\Omega_{hk} = 0,$$
$$\Omega_{ri\cdot k} = \Omega_{rk\cdot i},$$
$$\frac{\partial \Omega_{ri}}{\partial u} + a^{\lambda\mu}\Omega_{r\lambda}\Omega_{i\mu} = 0,$$
$$\frac{\partial b_{ik}}{\partial u} + 2\Omega_{ik} = 0.$$

We have seen in § 173 that, if when $u = 0$ the equations

$$\Omega_{ri\cdot k} = \Omega_{rk\cdot i} \tag{176.1}$$

hold, they will persist for any value of u whilst Ω_{ik} and a_{ik} grow in accordance with the laws

$$\frac{\partial a_{ik}}{\partial u} + 2\Omega_{ik} = 0. \tag{176.2}$$

$$\frac{\partial \Omega_{ik}}{\partial u} + a^{\lambda\mu}\Omega_{i\lambda}\Omega_{k\mu} = 0. \tag{176.3}$$

We shall now prove that if these equations hold, then, if

$$(rkih) + \Omega_{rh}\Omega_{ik} - \Omega_{ri}\Omega_{hk} = 0 \qquad (176.4)$$

holds when $u = 0$, it also will persist when u has any value.

The expression

$$\frac{\partial}{\partial u}(rkih) + \{rtih\}\,\Omega_{kt} - \{ktih\}\,\Omega_{rt}$$

is a tensor component: when we refer to the geodesic coordinates of any given point we shall find that it vanishes at that given point and therefore vanishes identically.

So referred,

$$\frac{\partial}{\partial u}(rkih) = \frac{\partial}{\partial u}\left(\frac{\partial}{\partial x_h}(rik) - \frac{\partial}{\partial x_i}(rhk)\right)$$

$$= \frac{\partial^2 \Omega_{ri}}{\partial x_h \partial x_k} - \frac{\partial^2 \Omega_{ik}}{\partial x_h \partial x_r} - \frac{\partial^2 \Omega_{rh}}{\partial x_i \partial x_k} + \frac{\partial^2 \Omega_{hk}}{\partial x_i \partial x_r}.$$

Now
$$\frac{\partial \Omega_{ri}}{\partial x_k} = \Omega_{ri\cdot k} + \{rkt\}\,\Omega_{it} + \{ikt\}\,\Omega_{rt},$$

$$\frac{\partial \Omega_{ik}}{\partial x_r} = \Omega_{ik\cdot r} + \{irt\}\,\Omega_{kt} + \{krt\}\,\Omega_{it};$$

and therefore

$$\frac{\partial}{\partial u}(rkih) = \frac{\partial}{\partial x_h}(\{ikt\}\,\Omega_{rt} - \{irt\}\,\Omega_{kt})$$

$$- \frac{\partial}{\partial x_i}(\{hkt\}\,\Omega_{rt} - \{hrt\}\,\Omega_{kt})$$

$$= \Omega_{rt}\,\{ktih\} - \Omega_{kt}\,\{rtih\}, \qquad (176.5)$$

which proves the required formula.

Again

$$\frac{\partial}{\partial u}(\Omega_{rh}\Omega_{ik} - \Omega_{ri}\Omega_{hk}) = \Omega_{ri}a^{\lambda\mu}\Omega_{h\lambda}\Omega_{k\mu} + \Omega_{hk}a^{\lambda\mu}\Omega_{r\lambda}\Omega_{i\mu}$$

$$- a^{\lambda\mu}\Omega_{rh}\Omega_{\lambda i}\Omega_{\mu k} - a^{\lambda\mu}\Omega_{ik}\Omega_{r\lambda}\Omega_{h\mu}$$

$$= \Omega_{k\mu}a^{\lambda\mu}(\Omega_{ri}\Omega_{h\lambda} - \Omega_{rh}\Omega_{\lambda i})$$

$$+ \Omega_{r\lambda}a^{\lambda\mu}(\Omega_{hk}\Omega_{i\mu} - \Omega_{ik}\Omega_{h\mu})$$

$$= \Omega_{k\mu}a^{\lambda\mu}(r\lambda ih) + \Omega_{r\lambda}a^{\lambda\mu}(k\mu hi)$$

$$= \Omega_{k\mu}\,\{r\mu ih\} - \Omega_{r\lambda}\,\{k\lambda ih\}.$$

It follows that

$$\frac{\partial}{\partial u}\left((rkih) + \Omega_{,rh}\Omega_{ik} - \Omega_{,i}\Omega_{hk}\right) = 0\,;$$

that is, the equations

$$(rkih) + \Omega_{,rh}\Omega_{ik} - \Omega_{,i}\Omega_{hk} = 0, \qquad (176.6)$$

if true when $u = 0$, will always be true.

The condition that an n-way space may be contained in a Euclidean $(n+1)$-way space is that the equations

$$\Omega_{ri\cdot k} = \Omega_{,k\cdot i}, \qquad (176.7)$$

$$(rkih) = \Omega_{,i}\Omega_{hk} - \Omega_{,h}\Omega_{ik} \qquad (176.8)$$

may be consistent.

§ **177. Procedure for applying the condition when $n > 2$.** There is now an essential distinction between the case $n = 2$ and the case $n > 2$.

A two-way space is always contained in a Euclidean space of three dimensions, and we have considered the problems associated with this case.

If $n > 2$ we can uniquely determine the functions Ω_{ik} in terms of the four-index symbols of Christoffel, by aid of the equations $$(rkih) = \Omega_{,i}\Omega_{hk} - \Omega_{,rh}\Omega_{ik}$$ alone. If $n > 3$ we even have relations between the four-index symbols from the consistency of these equations. It is a problem of algebra merely to determine the functions Ω_{ik}, and the functions so determined are tensor components.

If the surrounding space is to be Euclidean, the functions so determined must satisfy the equations $\Omega_{,i\cdot k} = \Omega_{,rk\cdot i}$. We can therefore, when we are given the ground form $a_{ik}dx_i dx_k$, determine, by algebraic work merely, whether the space to which the ground form refers is or is not contained within a Euclidean $(n+1)$-way space. The actual work would, however, be laborious.

§ **178. The n-way space as a surface in the Euclidean space when this exists.** Suppose, now, that we are given the ground form $a_{ik}dx_i dx_k$, and that we have found that the space to which it refers is contained in a Euclidean $(n+1)$-way space and have calculated the functions Ω_{ik}: we may ask, what is the surface in Euclidean space which is the given n-way space?

Let z be the vector in the Euclidean $(n+1)$-way space which traces out the given n-way space. We know from our earlier work that $z_{\cdot ik}$ is normal to each element of the space drawn through the extremity of z. Now there is only one such vector in the Euclidean $(n+1)$-fold. Let λ be the unit vector which is normal to the surface. Then

$$z_{\cdot ik} = w_{ik}\lambda, \qquad (178.1)$$

where w_{ik} is some scalar.

We have
$$z_{\cdot rik} = w_{ri}\lambda_k + w_{ri\cdot k}\lambda,$$
$$z_{\cdot rki} = w_{rk}\lambda_i + w_{rk\cdot i}\lambda,$$

and therefore, since

$$\overline{ki} - \overline{ik} = \{tpik\}\binom{t}{p} - \{qtik\}\binom{t}{q},$$

where p is an upper integer and q a lower integer, we have

$$-\{rtik\}\, z_t = w_{ri}\lambda_k - w_{rk}\lambda_i + (w_{ri\cdot k} - w_{rk\cdot i})\,\lambda. \qquad (178.2)$$

Multiplying by λ, and taking the scalar product, and noting that $\underbrace{\lambda\lambda_p} = 0$, we have $\quad w_{ri\cdot k} = w_{rk\cdot i}.$

We also have $\quad \underbrace{z_{\cdot ri} z_{\cdot kh}} - \underbrace{z_{\cdot rh} z_{\cdot ki}} = (rkhi),$

and therefore $\quad w_{rh}w_{ki} - w_{ri}w_{kh} = (rkhi).$

It follows that $\quad w_{ik} = \Omega_{ik}, \qquad (178.3)$

and we have $\quad z_{\cdot ik} = \Omega_{ik}\lambda. \qquad (178.4)$

We also have $\quad \{rtik\}\, z_t = \Omega_{rk}\lambda_i - \Omega_{ri}\lambda_k, \qquad (178.5)$

so that when we know λ we can find z by quadrature and thus determine the surface save for a translation.

§ **179.** We have now to show how to determine λ.

$$\underbrace{\lambda z_p} = 0,$$

and therefore $\quad \underbrace{\lambda_q z_p} + \underbrace{\lambda z_{\cdot pq}} = 0.$

It follows that $\quad \underbrace{\lambda_i z_k} = \underbrace{\lambda_k z_i} = \Omega_{ik}.$

From the equation

$$\{rtik\}\, z_t = \Omega_{rk}\lambda_i - \Omega_{ri}\lambda_k, \qquad (179.1)$$

$$\{rtik\}\, \Omega_{tp} = \Omega_{rk}\underbrace{\lambda_i\lambda_p} - \Omega_{ri}\underbrace{\lambda_k\lambda_p};$$

and therefore

$$a^{tq}\,(rqik)\,\Omega_{tp} = \Omega_{rk}\underbrace{\lambda_i\lambda_p} - \Omega_{ri}\underbrace{\lambda_k\lambda_p};$$

that is, $a^{tq}\Omega_{tp}\,(\Omega_{ri}\Omega_{qk} - \Omega_{rk}\Omega_{qi}) = \Omega_{rk}\underbrace{\lambda_i\lambda_p} - \Omega_{ri}\underbrace{\lambda_k\lambda_p},$

or $\Omega_{rk}\,(\underbrace{\lambda_i\lambda_p} + a^{tq}\Omega_{tp}\Omega_{qi}) = \Omega_{ri}\,(\underbrace{\lambda_k\lambda_p} + a^{tq}\Omega_{tp}\Omega_{qk}).$ (179.2)

Unless, then, the coefficients of Ω_{rk} and Ω_{ri} are zero, we must have

$$\Omega_{rk}\Omega_{si} = \Omega_{ri}\Omega_{sk},$$

which would mean that $(rski) = 0$ and that the n-way space was Euclidean, a case we need not consider. We conclude that

$$\underbrace{\lambda_i\lambda_k} + a^{pq}\Omega_{pi}\Omega_{qk} = 0. \qquad (179.3)$$

We thus know the ground form of the surface traced out by the unit vector λ.

Let

$$\lambda_{\cdot ik} \equiv \frac{\partial^2\lambda}{\partial x_i \partial x_k} - \{ikt\}'\,\frac{\partial\lambda}{\partial x_t}, \qquad (179.4)$$

where $\{ikt\}'$ is formed with reference to this ground form.

We have

$$\underbrace{\lambda\lambda_i} = 0,$$

and therefore

$$\underbrace{\lambda\lambda_{\cdot ik}} + \underbrace{\lambda_i\lambda_k} = 0.$$

Now $\lambda_{\cdot ik}$ is parallel to the normal to the surface traced out by λ, and therefore, as λ is a unit vector, is parallel to λ.

Since

$$\underbrace{\lambda\lambda_{\cdot ik}} = a^{pq}\Omega_{pi}\Omega_{qi},$$

it follows that

$$\lambda_{\cdot ik} + a^{pq}\Omega_{pi}\Omega_{qk}\lambda = 0. \qquad (179.5)$$

We thus have the equations which determine λ.

These equations may be written

$$\lambda_{\cdot ik} + a'_{ik}\lambda = 0, \qquad (179.6)$$

where a'_{ik} denotes a coefficient in the ground form of λ. As this ground form is that of a space of constant positive curvature we see that the system is 'complete'.

It follows that we can allow $\lambda_1 \ldots \lambda_n$ to take any initial values and thus we can determine λ save as to a 'movement' in Euclidean space.

We have considered three ground forms; these may be written

$$-\underbrace{dz\,dz} = a_{ik}dx_i dx_k,$$
$$-\underbrace{dz\,d\lambda} = \Omega_{ik}dx_i dx_k,$$
$$-\underbrace{d\lambda\,d\lambda} = a^{pq}\Omega_{pi}\Omega_{qk}dx_i dx_k.$$

We saw (147.4) that the lines of principal curvature were given by the equations $(a_{pq}-R\Omega_{pq})\,dx_q = 0$,

that is now, by $\underbrace{z_p\,(dz-Rd\lambda)} = 0$,

and as we also have $\underbrace{\lambda\,(dz-Rd\lambda)} = 0$,

we conclude that $$dz = Rd\lambda \tag{179.7}$$

is the equation of the line of curvature corresponding to the principal radius of curvature R.

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