Trees of Rationals

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Abstract

The positive rationals can be arranged in three types of tree with fixed matrix descent and a single root 1/1. Their descent can be taken by bits either high to low or low to high for total six trees which have been described separately by various authors.

Some results are obtained on turn sequences, convex hull, minimum area rectangle, and inertia.

Contents

Notation

The Fibonacci numbers F_n are, with usual numbering,

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \qquad \text{starting } F_0 = 0, F_1 = 1 \\ &= 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \qquad \text{A000045} \end{aligned}$$

The Lucas numbers V_n are

$$\begin{aligned} V_n &= V_{n-1} + V_{n-2} \qquad \text{starting } V_0 = 2, V_1 = 1 \\ &= 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots \qquad \text{A000032} \end{aligned}$$

1 Tree Summary

A positive rational is represented as a pair of integers p/q

$$\begin{aligned} p \geq 1,\ q \geq 1 \\ \gcd(p, q) = 1 \end{aligned} \tag{1}$$

A matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is considered to act on such a pair by left multiplication,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix} \tag{2}$$

There are pairs of matrices which can be applied in a binary tree to enumerate all and only positive rationals p/q without duplication.

Table 1 summarises the combinations.

Descent	Matrices	Encoding	High to Low	Low to High
$\frac{p}{p+q}, \frac{p+q}{q}$	$\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$	0000 1111 runs	Stern-Brocot, Johnston	Calkin-Wilf
$\frac{q}{p+q}, \frac{p+q}{p}$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$	0101 1010 phase shift	Bird	Drib
$\frac{p+q}{q}, \frac{q}{p+q}$	$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$	1000 1000 1-bit markers	Hanna, Czyz-Self	Yu-Ting, Andreev, and fractions Kepler, Benson

Table 1: Trees of rationals using a fixed set of matrices

The matrices represent an encoding of a subtraction-only Euclidean greatest common divisor algorithm. Or equivalently an encoding of the quotients in the Euclidean GCD which are also the terms in the continued fraction representation of the rational.

"High to low" or "low to high" are whether the steps or quotients are encoded into the tree row position by taking binary bits from most to least significant bit or least to most.

The low-to-high forms are the simplest. A given point p,q descends by multiplication of the left or right matrix. The high-to-low forms are a recursive definition so that the left sub-tree is a point-wise $L.tree$, ie. the left matrix multiplied against each point of the tree. Similarly the right $R.tree$.

2 Rationals Preserving Matrices

For a matrix to be used in a tree of rationals it must "preserve" rationals in the sense that if p,q satisfies conditions (1) then the resulting p',q' (2) should satisfy them too.

Theorem 1. *Conditions* (3) *through* (7) *are necessary and sufficient for a matrix* $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *to preserve* p,q *positive rationals without duplication.*

$$\begin{aligned} &a, b, c, d \text{ integers} && && (3)\\ &a, b, c, d \geq 0 && \text{non-negative} && (4)\\ &\gcd(a, c) = 1 && && (5)\\ &\gcd(b, d) = 1 && && (6)\\ &ad - bc = \pm 1 && \text{determinant, unimodular} && (7) \end{aligned}$$

The GCDs are taken with $\gcd(n, 0) = |n|$ in the usual way.

The identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ preserves p,q pairs without duplication and satisfies the conditions. Likewise a swap $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proof of Theorem 1. Take first the necessity, that when a matrix sends all positive coprime p,q to positive coprime p',q' and never duplicates p',q', then its a,b,c,d are as described.

Consider p=1, q=1 and p=2, q=1,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} p'_1 \\ q'_1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a+b \\ 2c+d \end{pmatrix} = \begin{pmatrix} p'_2 \\ q'_2 \end{pmatrix}$$

These are solved for a,b,c,d in terms of p'_1,p'_2,q'_1,q'_2

$$\begin{aligned} a &= p'_2 - p'_1 = \text{integer} & b &= 2p'_1 - p'_2 = \text{integer} \\ c &= q'_2 - q'_1 = \text{integer} & d &= 2q'_1 - q'_2 = \text{integer} \end{aligned}$$

Since p'_1,p'_2,q'_1,q'_2 are all integers, so a,b,c,d are all integers (3).

Consider $p = k$, $q = 1$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k \\ 1 \end{pmatrix} = \begin{pmatrix} ka+b \\ kc+d \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}$$

Must have $a \geq 0$ otherwise big enough k gives $p' < 1$. Similarly $c \geq 0$ (4) otherwise $q' < 1$. If b,d have a common factor $g = \gcd(b, d) > 1$ then $k = g$ gives that common factor in p',q', so must have $\gcd(b, d) = 1$ (6).

Consider $p = 1$, $q = k$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ k \end{pmatrix} = \begin{pmatrix} a+bk \\ c+dk \end{pmatrix} = \begin{pmatrix} p' \\ q' \end{pmatrix}$$

Must have $b \geq 0$ otherwise big enough k gives $p' < 1$. Similarly $d \geq 0$ (4) otherwise $q' < 1$. If a,c have a common factor $g = \gcd(a,c) > 1$ then $k = g$ gives that common factor in p',q', so must have $\gcd(a,c) = 1$ (5).

If determinant $\Delta = ad - bc = 0$ due to $ad{=}bc{=}0$ then one of a,d is zero and one of b,c is zero. If $a{=}b{=}0$ then $p' = 0$ always which fails $p' \geq 1$. Similarly if $d{=}c{=}0$ then $q' = 0$ always. If $a{=}c{=}0$ then p',q' does not depend on the parent p, so p',q' pairs are duplicated. Similarly if $b{=}d{=}0$ then p',q' does not depend on the parent q, so p',q' pairs are duplicated. Thus cannot have $ad = bc = 0$.

If determinant $\Delta = ad - bc = 0$ due to $ad = bc \neq 0$ then with $\gcd(a,c) = 1$ and $\gcd(b,d) = 1$ can only have $d = c$ and $b = a$. In that case

$$\begin{pmatrix} a & a \\ c & c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a \\ 2c \end{pmatrix}$$

If $a = 0$ then $p' = 0$ fails $p' \geq 1$. If $c = 0$ then $q' = 0$ fails $q' \geq 1$. Otherwise p',q' have common factor 2 in p',q'. Thus

$$\Delta \neq 0 \tag{8}$$

Since $\gcd(a,c) = 1$ there exist integers x,y which satisfy

$$-xc + ya = 1 \qquad \text{since } a,c \text{ coprime} \tag{9}$$

Consider p,q pair

$$\begin{aligned} p &= xd - yb + \Delta k && \text{integer } k \\ q &= 1 \end{aligned}$$

Choose k big enough positive or negative to make $p \geq 1$. This is possible since $\Delta \neq 0$ (8). The resulting p,q gives

$$\begin{aligned} p' &= a(xd - yb + \Delta k) + b \\ &= xad - yab + a\Delta k + b \\ &= xad - (1 + xc)b + a\Delta k + b && \text{since } ya = 1 + xc \text{ (9)} \\ &= x(ad - bc) + a\Delta k \\ &= x\Delta + a\Delta k && \text{multiple of } \Delta \end{aligned}$$

$$\begin{aligned} q' &= c(xd - yb + \Delta k) + d \\ &= xcd - ybc + c\Delta k + d \\ &= (ya - 1)d - ybc + c\Delta k + d && \text{since } xc = ya - 1 \text{ (9)} \\ &= y(ad - bc) + c\Delta k \\ &= y\Delta + c\Delta k && \text{multiple of } \Delta \end{aligned}$$

Δ is a common factor in this p',q' so must have $\Delta = ad - bc = \pm 1$ per (7).

As a remark, this p,q pair arises from M and its adjoint (inverse times determinant), in a way similar to an answer given by Thomas Jager [13] for all integers (and on $n{\times}n$ matrices).

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\Delta \\ y\Delta \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \Delta k \\ 0 \end{pmatrix} \right) = \begin{pmatrix} x\Delta + a\Delta k \\ y\Delta + c\Delta k \end{pmatrix}$$

This is common factor Δ in p',q' on the right, provided the vector part on the left

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \Delta k \\ 0 \end{pmatrix}$$

is an acceptable p,q pair (1) or can be made so. One way to make it so is q=1 by x,y from $\gcd(a,c) = 1$ per (9) and then $p \geq 1$ using k. The gcd in fact gives a whole class of solutions to $q = -cx + ay = 1$

$$\begin{aligned} x &= x_0 + fa && \text{any integer } f \\ y &= y_0 + fc \end{aligned}$$

Taking a different f adds $f(ad - bc) = f\Delta$ to p and it could be chosen to ensure $p \geq 1$, rather than a separate k.

Turn now to the sufficiency, ie. that if the above conditions hold then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ sends all good p,q to good p',q' without duplication.

$$\begin{aligned} p' &= ap + bq \\ &\geq a + b \\ &\geq 1 && \text{as } ad - bc = \pm 1 \text{ (7) means not both } a{=}0, b{=}0 \\ q' &= cp + dq \\ &\geq c + d \\ &\geq 1 && \text{as } ad - bc = \pm 1 \text{ (7) means not both } c{=}0, d{=}0 \end{aligned}$$

For the GCD of p',q', since $\gcd(p,q) = 1$ there exist integers x,y satisfying

$$xp + yq = 1 \qquad \text{since } p,q \text{ coprime}$$

and substituting into that the inverses p,q in terms of p',q'

$$\begin{aligned} p &= (dp' - bq')/\Delta \\ q &= (-cp' + aq')/\Delta \end{aligned}$$

gives

$$\begin{aligned} &x(dp' - bq')/\Delta + y(-cp' + aq')/\Delta = 1 \\ &(xd - yc)p' + (-xb + ya)q' = \pm 1 && \text{since } \Delta = \pm 1 \text{ (7)} \end{aligned}$$

This is integer multiples of p',q' adding up to ± 1. So $\gcd(p', q')$ must be a divisor of ± 1, hence $\gcd(p', q') = 1$.

Finally any p',q' is reached from just one p,q since $\Delta = \pm 1$ means M is invertible so p,q is uniquely determined by p',q'. □

3 Three Possible Trees

Theorem 2. *There are three possible trees of rationals descending by a fixed set of matrices from a single root 1/1. These are the trees summarised in Table 1.*

Proof. A rational preserving matrix from theorem 1 always has $p' \geq p$ and $q' \geq q$ since $a, b, c, d \geq 0$ (4). This means the parent of point 1,2 can only be 1,1. To send 1,1 to 1,2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

There are six solutions to $a + b = 1$ and $c + d = 2$ in non-negative integers (4). Two of them satisfy the conditions of theorem 1.

a	b	c	d	
1	0	2	0	fails $ad - bc = \pm 1$ (7)
0	1	2	0	fails $ad - bc = \pm 1$ (7)
1	0	1	1	matrix $A1$
0	1	1	1	matrix $A2$
1	0	2	0	fails $ad - bc = \pm 1$ (7)
0	1	2	0	fails $ad - bc = \pm 1$ (7)

$$A1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad A2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Similarly the parent of 2,1 can only be 1,1 and it can be reached only by B1 or B2.

$$B1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A1^T \qquad B2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \text{Fibonacci Q matrix}$$

So to cover points 2,1 and 1,2 the tree must include one A and one B. There are four combinations of these, and they are the trees summarised in Table 1 above.

A1	B1	Stern-Brocot / Calkin-Wilf
A1	B2	HCS / AYT
A2	B1	HCS / AYT, swap p,q
A2	B2	Bird / Drib

The combination A1,B2 is the same as A2,B1, just swapping p,q. A swap of p,q can be performed by matrix $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and it is seen that

$$S.A1.S^{-1} = B1$$
$$S.B2.S^{-1} = A2$$

so pair B1,A2 acting on p,q is the same as pair A1,B2 acting on swapped q,p. □

B2 is the Fibonacci Q matrix (Brenner [5]) and in consequence gives Fibonacci pairs on the right of the Bird and Drib trees (see section 6).

If an A,B matrix pair is swapped so that B is the left and A is the right the result is to reverse the nodes left-to-right within a row. So B,A read left-to-right is the same as A,B read right-to-left. This is found in the B1,A2 of Yu-Ting and Andreev (section 9).

4 Stern-Brocot Tree

The Stern-Brocot tree is matrices A1,B1 taken high-to-low.

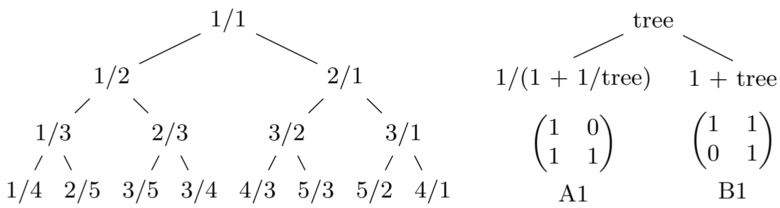


Figure 1: Stern-Brocot tree

$$\text{numerators row-wise} = 1,\ 1,2,\ 1,2,3,3,\ 1,2,3,3,4,5,5,4,\dots \qquad \text{A007305}$$
$$\text{denominators row-wise} = 1,\ 2,1,\ 3,3,2,1,\ 4,5,5,4,3,3,2,1,\dots \qquad \text{A047679}$$

Johnston [15] makes the same tree arrangement as a way to enumerate the rationals but as B1,A1 so read right to left in each row. Harrington [10] expresses this by row replications.

The rationals in each row are in ascending order. When plotted as Cartesian coordinates p,q this means clockwise around from the Y axis as shown in figure 2 below. The rows are symmetric in the sense that reading p/q left to right is the same as reading q/p right to left and hence the plot is symmetric across the leading diagonal $p = q$.

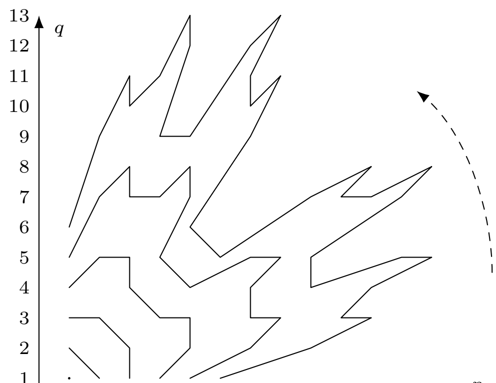


Figure 2: Stern-Brocot tree rows

The row lines do not intersect preceding rows and so give the shape of an expanding region of p,q coverage by the tree. This coverage is the same for all the tree forms since they are the same row points in different order.

It's convenient to number fractions in the tree row-wise starting from n=1 for the root and in general $2^d \le n < 2^{d+1}$ across the row at depth d down from the root.

In this numbering, rounding up to the next integer is given by the number of high 1-bits of n, since each right half of the tree, which is a further high 1-bit, is $1+$ tree by the action of matrix B1.

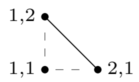
$$\begin{aligned}\left\lceil \frac{p_n}{q_n} \right\rceil &= \mathit{CountHighOnes}(n) \\ &= \text{number of 1s in the highest run of 1-bits} \\ &= 0, 1, 1, 2, 1, 1, 2, 3, 1, 1, 1, 1, 2, 2, 3, \ldots \qquad \text{A090996}\end{aligned}$$

4.1 Row Area

Theorem 3. *The area r_d between tree rows d and $d{+}1$, and the total area R_d up to row d, are*

$$\begin{aligned} r_d &= 5.2^{d-1} - 2 && \textit{area between rows } d \textit{ and } d{+}1 \qquad (10) \\ &= \tfrac{1}{2}, 3, 8, 18, 38, 78, 158, 318, 638, \ldots && d{\geq}1 \ \text{A051633} \\ R_d &= \sum_{i=0}^{d-1} r_i \ = \tfrac{5}{2}(2^d - 1) - 2d && \textit{total area to row } d \qquad (11) \\ &= 0, \tfrac{1}{2}, \tfrac{7}{2}, \tfrac{23}{2}, \tfrac{59}{2}, \tfrac{135}{2}, \tfrac{291}{2}, \tfrac{607}{2}, \ldots && {}^{-}\tfrac{1}{2}{+}\text{A097809},\ \tfrac{1}{2}{\times}\text{A126284} \end{aligned}$$

Proof. The area between row 0 and row 1 is the initial triangle 1,1–1,2–2,1 of area $\frac{1}{2}$ which is $r_0 = 5.2^{0-1} - 2 = \frac{1}{2}$.



area $r_0 = \frac{1}{2}$

For a subsequent row the area between rows d and $d{+}1$ is two copies of the preceding $d{-}1$ to d area transformed by multiplication on the left by matrices L and R as shown in figure 3. Those matrices are shears and so don't change the area.

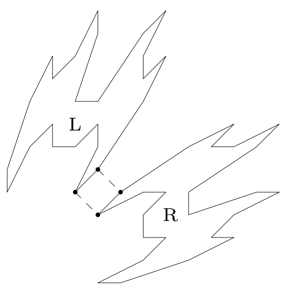


Figure 3: Stern-Brocot tree row area copies

Between the copies is a gap shown by dashed lines. The gap is a diamond shape as follows. The right edge of the L block is $L.R^n$. The left edge of the R block is $R.L^n$.

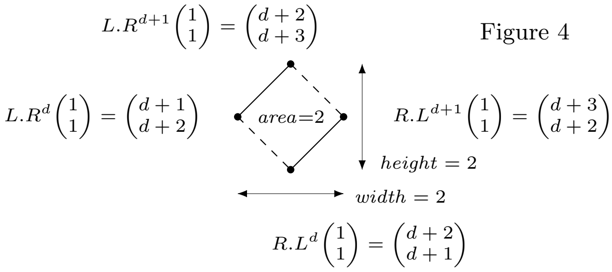


Figure 4

So the gap is area 2 and between rows is thus

$$\begin{aligned} r_d &= 2r_{d-1} + 2 && \text{recurrence } d \geq 1 \\ &= 5.2^{d-1} - 2 && \text{from start } r_0 = \tfrac{1}{2} \end{aligned} \qquad (12)$$

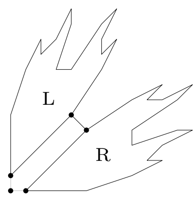
Summing for the total area,

$$R_d = \sum_{i=0}^{d-1} r_i \quad = \tfrac{5}{2}(2^d - 1) - 2d \qquad \square$$

The total area can be written as a recurrence too.

$$R_d = 2R_{d-1} + 2d - \tfrac{3}{2} \qquad \text{for } d \geq 1$$

This follows from the r_d recurrence, or by considering how the total area is sheared by the two matrices. R_d is 2 copies of R_{d-1} plus the middle diagonal area $2d - \frac{3}{2}$.



Stern-Brocot tree area copies

The copying means that each row comprises sheared copies of the initial triangle of area $\frac{1}{2}$ and the diamonds in between of area 2. This is a tiling of the first quadrant $p \geq 1$, $q \geq 1$ by those triangles and diamonds, though the repeated shears soon make them very elongated.

4.2 Area Centroid

Theorem 4. *The centroid (centre of gravity) of the area between rows d and d+1 of the Stern-Brocot tree is a point* (g_d, g_d) *which in terms of row area* r_d (10) *is*

$$\begin{aligned} g_d &= \frac{gtotal_d}{r_d} \\ &= \tfrac{4}{3}, 2, 3, \tfrac{40}{9}, \tfrac{125}{19}, \tfrac{127}{13}, \tfrac{1150}{79}, \tfrac{3458}{159}, \ldots \\ gtotal_d &= \tfrac{19}{6} 3^d - d - \tfrac{5}{2} \qquad (13) \\ &= \tfrac{2}{3}, 6, 24, 80, 250, 762, 2300, 6916, \ldots \end{aligned}$$

Proof. For $d = 0$ the row area is three points as follows. The centroid of a triangle is the mean of its three vertices.

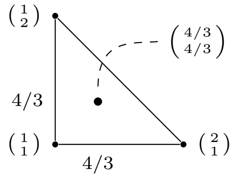
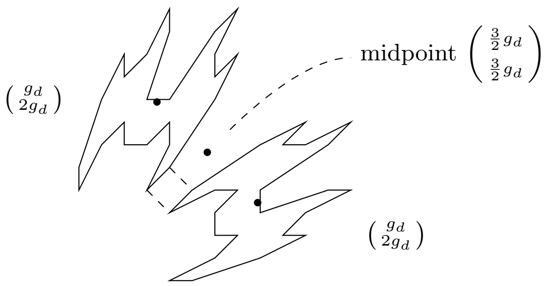


Figure 5: between rows d=0 and d=1
area $r_0 = 1/2$
centroid $g_0 = 4/3$
$gtotal_0 = 2/3$

The centroid of a shear is the shear of the centroid. So the centroid of the left and right sheared copies of the row area are at $\binom{2g_d}{g_d}$ and $\binom{g_d}{2g_d}$. Their midpoint is at $\frac{3}{2}g_d$.



The area between the two copies is the area 2 diamond centred at $(d{+}2, d{+}2)$ as from figure 4. So the weighted parts for $gtotal_{d+1}$ are

$$\begin{aligned} gtotal_{d+1} &= \tfrac{3}{2} g_d . 2r_d + 2(d{+}2) \\ &= 3 gtotal_d + 2d + 4 \end{aligned}$$

$gtotal_d$ is then of the form $W.3^d + Xd + Y$ in the usual way for linear recurrence and polynomial. The first three values starting $gtotal_0 = \frac{2}{3}$ give three equations in three unknowns leading to (13). □

The powers in r_d and $gtotal_d$ mean g_d grows roughly as $\frac{19}{15}(\frac{3}{2})^d$.

Theorem 5. *The centroid (centre of gravity) of the whole Stern-Brocot tree area to row d is a point* (G_d, G_d) *which in terms of the total area* R_d (11) *is*

$$G_d = \begin{cases} 1 & \text{for } d = 0 \\ \frac{Gtotal_d}{R_d} & \text{for } d \geq 1 \end{cases}$$

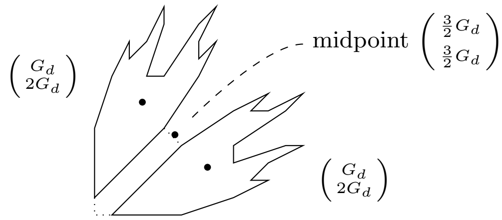
$$= 1, \tfrac{4}{3}, \tfrac{40}{21}, \tfrac{8}{3}, \tfrac{664}{177}, \tfrac{2164}{405}, \tfrac{6736}{873}, \tfrac{20536}{1821}, \tfrac{62032}{3729}, \ldots$$

$$Gtotal_d = \tfrac{1}{12}\left(19.3^d - 6d^2 - 24d - 19\right) \qquad \text{for } d \geq 1 \qquad (14)$$

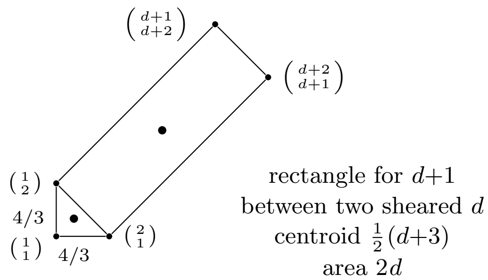
$$= \tfrac{2}{3}, \tfrac{20}{3}, \tfrac{92}{3}, \tfrac{332}{3}, \tfrac{1082}{3}, \tfrac{3368}{3}, \tfrac{10268}{3}, \tfrac{31016}{3}, \tfrac{93314}{3}, \ldots \qquad d \geq 1$$

Proof. For d=0 the tree is a single point 1,1 so centroid $G_0 = 1$. For d=1 the tree area is the area between d=0 and d=1 as in figure 5 so $G_1 = g_0$.

Again the centroid of a shear is the shear of the centroid. So the centroid of the left and right sheared copies of the tree area are at $\binom{2G_d}{G_d}$ and $\binom{G_d}{2G_d}$. Their midpoint is at $\frac{3}{2}G_d$.



The area between the two copies is a rectangle, plus initial triangle again from figure 5.



So the weighted parts for $Gtotal_{d+1}$ are

$$\begin{aligned} Gtotal_{d+1} &= \tfrac{3}{2}G_d\,.\,2R_d\ + \tfrac{1}{2}(d+3)\,.\,2d\ + \tfrac{4}{3}\cdot\tfrac{1}{2} \\ &= 3\,Gtotal_d + d^2 + 3d + \tfrac{2}{3} \end{aligned}$$

$Gtotal_d$ is then of the form $W.3^d + Xd^2 + Yd + Z$ for linear recurrence and polynomial. The first four values starting $Gtotal_1 = \frac{2}{3}$ give four equations in four unknowns leading to (14). □

The powers in R_d and $Gtotal_d$ mean G_d grows roughly as $\frac{19}{30}\,(\frac{3}{2})^d$.

The terms of $Gtotal$ shows $12\,Gtotal_d$ is a multiple of 8 so $Gtotal_d$ is an integer multiple of $\frac{2}{3}$, starting from $Gtotal_1 = \frac{2}{3}$.

4.3 Non-Coprime Points Between Rows

Lemma 1. *A line between consecutive points in a row of the Stern-Brocot tree does not pass through any other integer point.*

Proof. At depth d=0 there is a single point and the statement is true trivially.

Suppose the statement to be true within depth d. Then the points in depth d+1 are L and R sheared copies of row d. Those shears do not change the number of integer points because the transformed $\binom{p'}{q'} = L\binom{p}{q} = \binom{p}{p+q}$ is integers if and only if $\binom{p}{q}$ is integers. Similarly R.

The two middle points are d,d+1 and d+1,d. They are a unit diagonal and so do not pass through any integer point. □

Theorem 6. *The number of non-coprime points between row d and d+1 of the Stern-Brocot tree is*

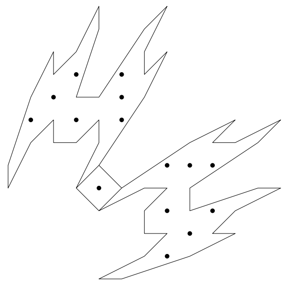
$$c_d = 2^d - 1$$

$$= 0, 1, 3, 7, 15, 31, 63, 127, 255, \ldots \qquad \text{A000225}$$

and the total non-coprime points up to row d is

$$C_d = \sum_{i=0}^{d-1} c_i \;= 2^d - d - 1$$
$$= 0, 0, 1, 4, 11, 26, 57, 120, 247, 502, \ldots \qquad \text{A000295}$$

Proof. The region between row d+1 and d+2 is formed by two sheared copies of d to d+1, similar to the row area in figure 3.



The shears preserve the coprime or non-coprime nature of the original p,q and by lemma 1 there are no points on the lines between row points.

The diamond between the two sheared copies contains a single point d+2, d+2 which is not coprime. So $c_{d+1} = 2c_d + 1$ and starting from initial triangle $c_0 = 0$ gives $c_d = 2^d - 1$. □

The non-coprime points between the rows are formed with the Stern-Brocot tree structure starting from point g, g so as to make points pg, qg with common factor g.

Second Proof of Theorem 6. Pick's theorem[18] for a polygon on a square lattice is

$$\mathit{Area} = \mathit{InsidePoints} + \mathit{BoundaryPoints}/2 - 1$$

In the area between row d and d+1, the points on the boundary are the p,q points of those two rows since by lemma 1 there are no integer points between those.

$$b_d = 2^d + 2^{d+1} \qquad \text{integer points on boundary}$$

The inside points are the non-coprime points and hence

$$r_d = c_d + b_d/2 - 1 \qquad \text{Pick's theorem}$$
$$c_d = 2^d - 1 \qquad \text{with } r_d \text{ from (12)}$$
□

Pick's theorem can also give the total non-coprime C_d from the total area R_d. In that case the inside points include the coprime points of preceding rows so they must be subtracted from the count.

4.4 Stern-Brocot Turn Sequence

Theorem 7. *The turn sequence of the Stern-Brocot tree as Cartesian points p,q is RL in row d=1 and then for each row $d \geq 2$ a repeating pattern LRRL except the first and last of each row is R.*

$$RRRL,\ LRRL,\ LRRL,\ \ldots,\ LRRL,\ LRRL,\ LRRR$$

Figure 6 below shows the turns for row d=4. The first turn in each row comes from the last point of the previous row. The last turn in each row goes towards the first point of the next row.

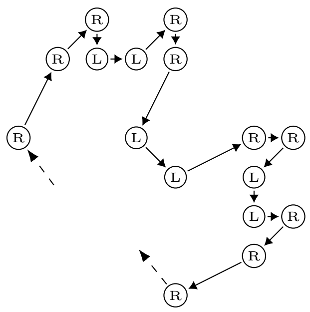
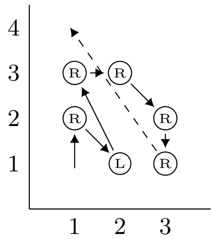


Figure 6: Stern-Brocot row d=4 turn sequence

RRRL
LRRL
LRRL
LRRR

Proof of Theorem 7. Row d=1 and d=2 are per the following diagram. d=1 is RL. Row d=2 is RRRR which is pattern LRRL except first and last are R.

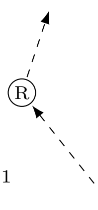


row d=1 RL
row d=2 RRRR

The L and R shears which copy row d to row d+1 do not change the turn directions. So the turns in row d+1 are two copies of the turns within row d, which means the turns except the first and last of row d. In row d+1 this is all turns except the first, last, and two middle points.

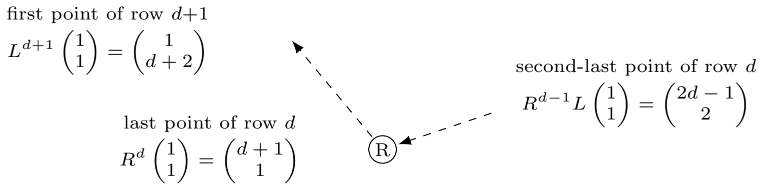
The first turn is always right since the second point $2, 2d-1$ is above and right of the diagonal $d-1, 1$ to $1, d$.

second point of row d $\quad L^{d-1}R\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2d-1\end{pmatrix}$

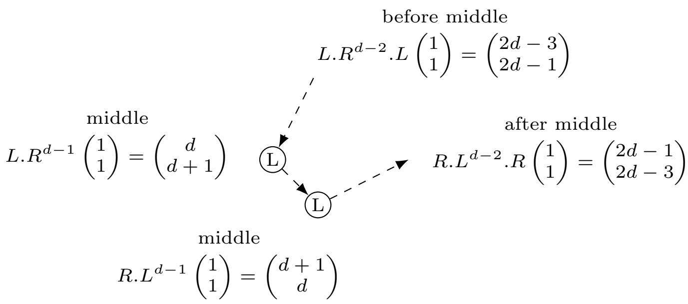
first point of row d  $\quad L^{d}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\d+1\end{pmatrix}$

last point of row $d-1$ $\quad R^{d-1}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}d\\1\end{pmatrix}$

The last turn is always right since the second-last point $2d-1, 2$ is above and right of the diagonal $d+1, 1$ to $1, d+2$.



The middle two turns are always left since they are a diagonal $d, d{+}1$ to $d{+}1, d$ and the second from middle is above that line.



□

5 Calkin-Wilf Tree

The Calkin-Wilf tree [6] is matrices A1,B1 taken low-to-high.

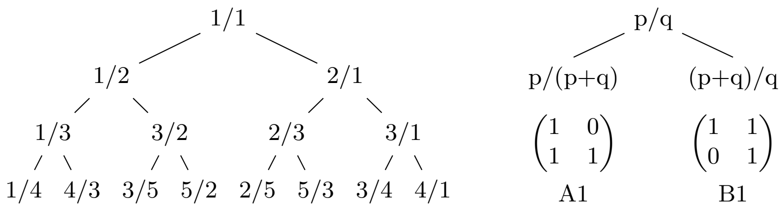


Figure 7: Calkin-Wilf tree

$$\text{numerators row-wise } = 1,\ 1,2,\ 1,3,2,3,\ 1,4,3,5,2,5,3,4,\ldots \qquad \text{A002487}$$
denominators row-wise, same but 1 later

Calkin and Wilf show the numerators row-wise are the Stern diatomic sequence (counts of hyperbinary representations), and the denominators the same sequence but 1 ahead. (This sequence is also “fusc” of Dijkstra [8].)

Stern established various properties of the sequence which are summarized by Lehmer [16].

1. Sum of numerators across a row is 3^d. This is the same in all the trees starting 1/1 since rows are permutations.
2. Adjacent terms have no common factor (hence rationals in reduced terms for the tree).
3. An adjacent pair a, b occurs only once (hence no duplication in the tree).
4. An adjacent pair a, b occurs in row one less than sum of quotients of continued fraction of a/b.

5. Numerators in a row read left to right are the same as read right to left, if the 1 of the first numerator of the next row is included. Or equivalently the same as denominators read right to left.

The tree can be iterated row-wise, and at the end of each row back to the start of the next, by

$$\begin{aligned} p_{next} &= q \\ q_{next} &= p + q - 2r \\ &\text{where } r \text{ remainder from } p/q, \text{ in range } 0 \le r < q \end{aligned} \tag{15}$$

(15) is given by Mike Stay in OEIS A002487. It follows from a problem posed by Knuth [17] to show that the following iteration x_n enumerates all rationals.

$$x_n = \frac{1}{1 + 2\,CountLowZeros(n) + x_{n-1}} \qquad \text{starting } x_0 = 0 \tag{16}$$

$$\begin{aligned} CountLowZeros(n) &= \text{number of low 0-bits of } n \text{ in binary} \\ &= 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, \ldots \quad n{\ge}1 \qquad \text{A007814} \end{aligned}$$

The answer by C. P. Rupert forms what is the Calkin-Wilf tree in these $x_n = p/q$ and shows how *CountLowZeros* is the number of the trailing R, so reversing those steps up, then going across, and then back down L steps gives next point row-wise, including wrapping around from the end of one row to the start of the next.

(16) uses $CountLowZeros(n)$ of the destination n. This is low 1-bits on the $n{-}1$ of the x_{n-1} since $CountLowOnes(n{-}1) = CountLowZeros(n)$ (increment turning low 1s into 0s).

A note in the answers by Moshe Newman is that the iteration can be written

$$\frac{p_{next}}{q_{next}} = \frac{1}{1 + 2\left\lfloor \frac{p}{q} \right\rfloor + \frac{p}{q}} \tag{17}$$

This is since an L step is $q/(p{+}q)$ so leaves $p{<}q$. An R is $(p{+}q)/q$ so repeated m times descends to $(p{+}mq)/q$. m is recovered by

$$m = \left\lfloor \frac{p}{q} \right\rfloor = CountLowOnes(n{-}1) = CountLowZeros(n)$$

A little rearrangement using $\frac{p}{q} = \left\lfloor \frac{p}{q} \right\rfloor + \frac{r}{q}$ where r is the remainder from division p/q turns (17) into (15).

The previous point row-wise follows in a similar way. The number of trailing 0-bits of n, which are L steps, is

$$m = \left\lceil \frac{q}{p} \right\rceil - 1 = CountLowZeros(n) \tag{18}$$

Going up by m, across, and back down, then becomes the following. Taking remainder r negative corresponds to the ceil in (18).

$$\begin{aligned} p_{prev} &= q - p - 2r \\ &\quad \text{where } r \text{ remainder from } q/p, \text{ in range } -p < r \le 0 \\ q_{prev} &= p \end{aligned}$$

The various trees can be taken as matrices instead of the final p,q points. Stange [20] for example takes the Calkin-Wilf tree this way. The iteration above can be adapted to iterate unimodular matrices across the tree.

Theorem 8. *For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the Calkin-Wilf tree, the next matrix row-wise, and at the end of a row back to the start of the next, is*

$$M_{next} = \begin{cases} \begin{pmatrix} 1 & 0 \\ b{+}1 & 1 \end{pmatrix} & \text{if } c = 0 \\ \begin{pmatrix} c & d \\ (2m{+}1)c - a & b + d - 2r \end{pmatrix} & \text{if } c \neq 0 \end{cases} \tag{19}$$

where division b/d so that $b = dm + r$ with $0 \le r < d$

Proof. The last matrix in a row is $R^m = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ which is $c{=}0$. Any other matrix has an L in its product which gives $c{\neq}0$. The start of the next row is $L^{m+1} = \left(\begin{smallmatrix} 1 & 0 \\ m+1 & 1 \end{smallmatrix}\right)$ hence (19).

Within a row a matrix and its row-wise next have some common ancestor $T = \left(\begin{smallmatrix} a_t & b_t \\ c_t & d_t \end{smallmatrix}\right)$. M is descent left and then some $m \ge 0$ many to the right,

$$M = R^m.L.\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = \begin{pmatrix} a_t + (a_t{+}c_t)m & b_t + (b_t{+}d_t)m \\ a_t{+}c_t & b_t{+}d_t \end{pmatrix} \tag{20}$$

Division b/d recovers the number of right steps m, and leaves remainder $r = b_t$, provided $d_t \ge 1$. All matrices in the tree start from the identity matrix where $d_t \ge 1$ and further L and R descents preserve that condition.

The step down on the other side from T to M_{next} is,

$$M_{next} = L^m.R.\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = \begin{pmatrix} a_t{+}c_t & b_t{+}d_t \\ c_t + (a_t{+}c_t)m & d_t + (b_t{+}d_t)m \end{pmatrix} \tag{21}$$

The top row is simply c, d from M at (20). c_{next} in (19) uses m to go from an a_t in M to a c_t in M_{next},

$$\begin{aligned} c_{next} &= (2m{+}1)c - a \\ &= (2m{+}1)(a_t{+}c_t) - \big(a_t + (a_t{+}c_t)m\big) \\ &= c_t + (a_t{+}c_t)m \qquad \text{per (21)} \end{aligned}$$

d_{next} can use the remainder r to go from a b_t in M to a d_t in M_{next},

$$\begin{aligned} d_{next} &= b + d - 2r \\ &= \big(b_t + (b_t{+}d_t)m\big) + (b_t{+}d_t) - 2b_t \\ &= d_t + (b_t{+}d_t)m \qquad \text{per (21)} \end{aligned}$$

□

Division a/c can be used too, though the remainder must be taken in the range $1 \le r \le d$. The remainder is $r = a_t$ and always have $a_t \ge 1$ (in the same manner as d_t above), so when $c_t{=}0$ want to leave a non-zero r. $c_t{=}0$ is when ancestor T is all R, per the $c{=}0$ case.

$$M_{next} = \begin{cases} \begin{pmatrix} 1 & 0 \\ b{+}1 & 1 \end{pmatrix} & \text{if } c = 0 \\ \begin{pmatrix} c & d \\ c+a-2r & (2m{+}1)\,b-d \end{pmatrix} & \text{if } c \neq 0 \end{cases}$$

where division a/c so that $a = cm + r$ with $1 \le r \le d$

The quotient m can be used for both $c_{next} = (2m{+}1)\,c - a$ and $d_{next} = (2m{+}1)\,b - d$ rather than remainder r in one of them if preferred.

6 Bird Tree

The Bird tree by Hinze [11] is matrices A2,B1 taken high-to-low.

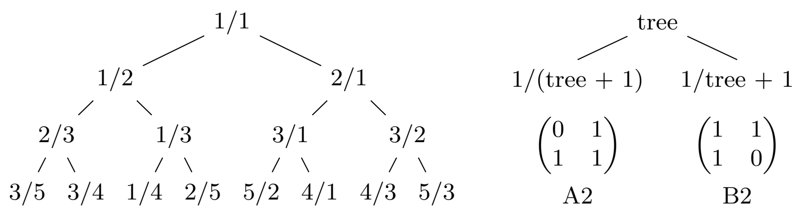


Figure 8: Bird tree

$$\begin{aligned} \text{numerators row-wise} &= 1,\ 1,2,\ 2,1,3,3,\ 3,3,1,2,5,4,4,5,\ldots && \text{A162909} \\ \text{denominators row-wise} &= 1,\ 2,1,\ 3,3,1,2,\ 5,4,4,5,2,1,3,3,\ldots && \text{A162910} \end{aligned}$$

Hinze expresses the tree by a recursive definition illustrating features of the Haskell programming language.

$$\begin{aligned} \text{left} \quad & \tfrac{1}{\text{bird}+1} = \tfrac{1}{p/q+1} = \tfrac{q}{p+q} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \\ \text{right} \quad & \tfrac{1}{\text{bird}} + 1 = \tfrac{q}{p} + 1 = \tfrac{p+q}{p} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \end{aligned}$$

6.1 Bird Turn Sequence

Theorem 9. *Let BirdTurn be the turn sequence of the Bird tree as Cartesian points p,q in the form* $+1$ *left,* -1 *right, and* 0 *straight ahead. For* $n = 2^d{+}r$ *with* $0 \le r < 2^d$*, so row d offset r,*

$$\mathit{BirdTurn}(n) = \begin{cases} -1,1,1,1,1,0,0 & \textit{if } n = 2 \text{ to } 8 \\ (-1)^{\,d-\max(2,\,\mathit{CountLows}(n))} & \textit{if } n \ge 9 \end{cases} \tag{22}$$

$$= -1,1,\ \ 1,1,1,0,\ \ 0,-1,-1,-1,\ -1,-1,-1,1,\ldots \quad n{\ge}2$$

$$\mathit{CountLows}(n) = \begin{cases} \textit{write } r \textit{ in binary using } d \textit{ many bits,} \\ \textit{length of lowest run of bits (0s or 1s)} \end{cases}$$

$$= 0,1,1,2,1,1,2,3,1,1,2,2,1,1,3,4,\ldots \quad n{\ge}1$$

n even A001511, n odd A091090

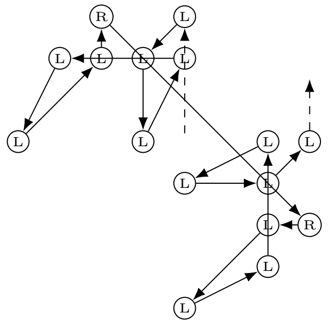


Figure 9:
Bird tree row d=4
turn sequence

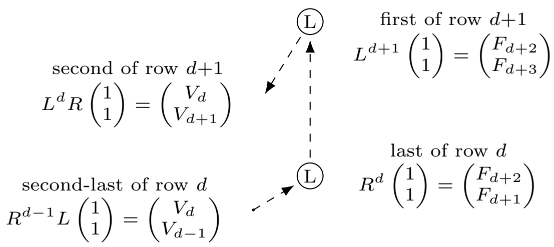
LLLL
LLLR
RLLL
LLLL

Proof. Turns in rows $d = 1$ to 4 inclusive can be verified explicitly.

Row d+1 is two copies of row d. The first copy is the left of the tree $q/(p+q)$ which is a shear and transpose. The shear leaves turns unchanged and the transpose swaps $L \leftrightarrow R$. The second copy $(p+q)/p$ too is a shear (the other way) and transpose.

This copy and flip applies to points which have their preceding and following points also copied, which means all except the first, last and middle two of a row.

For the last of row d and first of d+1, working through the matrix powers gives the following locations. Hinze notes the first and last points are Fibonacci pairs.



A cross-product determinant shows whether a point is on the left or right side of a preceding line,

$$\delta p_1 = p_2 - p_1,\ \ \delta p_2 = p_3 - p_2,\ \ \text{etc}$$

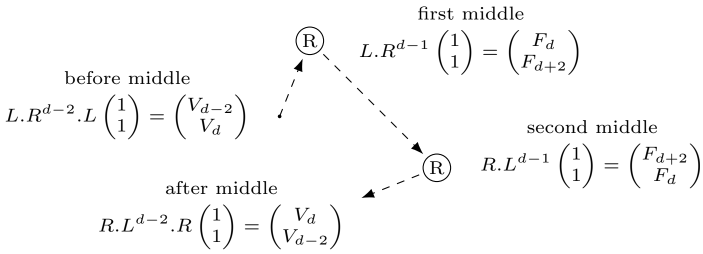
$$\mathit{cross} = \begin{pmatrix}\delta p_1\\ \delta q_1\end{pmatrix} \times \begin{pmatrix}\delta p_2\\ \delta q_2\end{pmatrix} = \delta p_1.\delta q_2 - \delta p_2.\delta q_1 \quad \begin{matrix} > 0 & \text{if turn left} \\ < 0 & \text{if turn right}\end{matrix} \tag{23}$$

For the last of row d,

$$\mathit{cross} = F_d^2 - (-1)^d \quad = 2, 0, 5, 8, 26, 63, 170, \ldots \quad d \ge 1 \qquad \text{A192883}$$
$$> 0 \ \text{ for } d \ge 3 \text{ so left turn}$$

The first of row d+1 has the same the cross-product so also a left turn.

The middle two points of row d are, again working through matrix products,



At the first middle point the cross-product is $cross = -F_{d-1}V_{d-1} - 3\,(-1)^d$ which is < 0 for $d \geq 3$ so a right turn. The second middle is the same, transposed and in reverse order, so also right turn.

These middle points are $L \leftrightarrow R$ flipped copies of the first and last of the preceding row. So row copying can begin at the first row with L for first and last, which is d=4.

$$\begin{aligned} row_{d+1} = \text{two copies of } \mathit{LRflip}(row_d), \qquad d \geq 4 \\ \text{except first and last points L} \end{aligned}$$

The first points with $\mathit{CountLows}(n) = l$ are the first and last of row l. The first point is row offset l many 0-bits 0000. When copied into the next row it is 10000 after the middle. The last point of row l is l many 1-bits 1111. When copied into the next row it is 01111. In both cases the copies are the same *CountLows*. Further copies into subsequent rows add further high bits and *CountLows* also unchanged. Each copy is a flip, so $d - \mathit{CountLows}(n)$ many flips. For $l \geq 4$ the row first and last points are L, so flips starting from that are $(-1)^{d-\mathit{CountLows}(n)}$.

For $l = 2, 3$, the points in row d=4 (as illustrated in figure 9) follow this formula too. But for l=1 they do not, they come out as R instead of L. Enforcing a minimum 2 in (22) is an extra flip when l=1, giving the desired L. □

The row copying and pattern of first and last same and middle two same means the turn sequence is left to right is the same as right to left across a row.

CountLows can also be conceived as the lowest location with different adjacent bits, up to maximum position d. That maximum only matters for the last point in row d which is $n = 2^{d+1}-1$ all 1s. Its low run would be d+1 many bits, but for *BirdTurn* just d is wanted. Another possible conception is by length of common prefix between n and $n \pm 1$, whichever is longer, but care is needed in cases where ± 1 goes to a different bit length.

7 Drib Tree

The Drib tree by Hinze [12] is matrices A2,B2 taken low-to-high

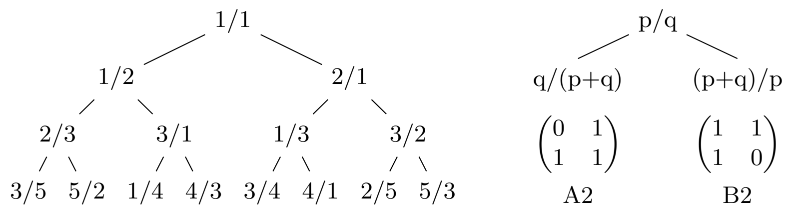


Figure 10: Drib tree

$$\begin{aligned} \text{numerators row-wise} &= 1,\ 1,2,\ 2,3,1,3,\ 3,5,1,4,3,4,2,5,\dots && \text{A162911} \\ \text{denominators row-wise} &= 1,\ 2,1,\ 3,1,3,2,\ 5,2,4,3,4,1,5,3,\dots && \text{A162912} \end{aligned}$$

8 HCS Tree

Matrices A1,B2 taken high-to-low give a tree which is an amalgam of ideas by Hanna [9] and Czyz and Self [7]. Call it "HCS".

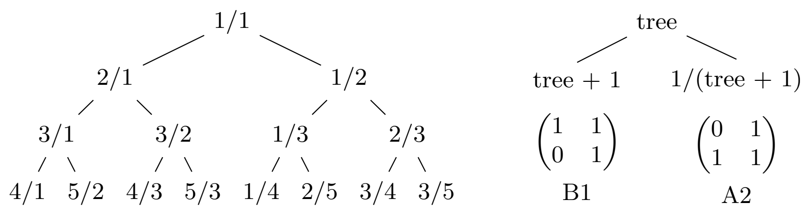


Figure 11: HCS tree

$$\begin{aligned} \text{numerators row-wise} &= 1,\ 2,1,\ 3,3,1,2,\ 4,5,4,5,1,2,3,3,\dots && \text{A229742} \\ \text{denominators row-wise} &= 1,\ 1,2,\ 1,2,3,3,\ 1,2,3,3,4,5,4,5,\dots && \text{A071766} \end{aligned}$$

Hanna makes the left half of the tree, being rationals $p/q > 1$, by encoding continued fraction quotients into integers using 1-bit markers. Taking $p/q - 1 = (p-q)/q$ gives the tree starting 1/1.

$$n = \underbrace{10\dots0}_{a_1}\ \underbrace{10\dots0}_{a_2}\ \underset{\dots}{\dots}\ \underbrace{10\dots0}_{a_n - 1} \qquad \text{binary}$$

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_n}}}$$

Czyz and Self encode the quotients in the same way by counting dots between digits. When p/q is an integer, they take the last run as a_n rather than $a_n - 1$.

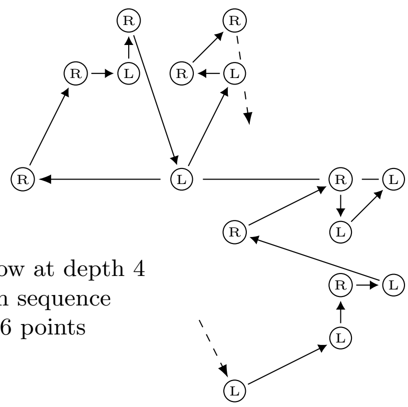
This has the effect of making the left side 0/1, 1/1, 2/1, 3/1, etc. This form includes 0 in the rationals, but the descent is then not by a fixed pair of matrices.

Shallit [19] makes a digit-based encoding of continued fraction terms using binary and ternary with digits 1,2,3 (rather than the usual 0,1,2). The same encoding can be done with radix 1 and binary, where radix 1 means an integer a is a run $111\ldots111$ of a many 1 digits. Such a variation gives the HCS tree with an offset -1 to the n numbering.

8.1 HCS Turn Sequence

Theorem 10. *The turn sequence of the HCS tree as Cartesian points p,q is the Thue-Morse sequence: count 1-bits mod 2 of* $n+1$.

The following diagram shows an example of the turns in row d=4.



HCS row at depth 4
turn sequence
16 points

n+1	1-bits	turn
10001	E	L
10010	E	L
10011	O	R
10100	E	L
10101	O	R
10110	O	R
10111	E	L
11000	E	L
11001	O	R
11010	O	R
11011	E	L
11100	O	R
11101	E	L
11110	E	L
11111	O	R
100000	O	R

E=even=left, O=odd=right

A turn is left or right according to whether the next point is on the left or right side of the preceding vector. Some turns are by a small angle. Some turns go nearly 180° back around.

The starting dashed line comes from the last point of the preceding row. The final dashed line goes towards the first of the next row. The theorem applies to all the points in the tree, including these row wrap-arounds.

Proof. Number the tree $n=1$ for the single point in row d=0, then $n=2,3$ the two points of row d=1, and in general $n=2^d$ to $2^d{-}1$ for row d. The claim is that the turn at point n is

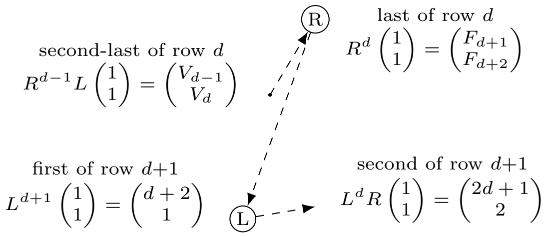
	count 1-bits in n+1
left	even
right	odd

Row d is two copies of row $d{-}1$. The first copy is the left of the tree and is a shear by matrix B1. The shear does not change the turn sequence within that copy. The n point numbers in that copy have a 0-bit introduced as a new second-highest bit. Take point $n=2^{d-1}+t$ in row $d{-}1$, where $1 \le t < 2^{d-1}{-}1$. This t is all points except the first and last. The first copy has $n_L = 2^d + t$. The number of 1-bits in n_L+1 is the same as in $n+1$.

The second copy is the right of the tree and is a transpose and shear by matrix A2. The shear does not change the turn sequence. The transpose changes the turns by flipping $L \leftrightarrow R$. The n point numbers in the second copy have a high 1-bit introduced as a new second-highest bit. Take point $n = 2^{d-1} + t$ in row $d{-}1$. The second copy is $n_R = 2^d + 2^{d-1} + t$. In $n_R + 1$ the new 1-bit 2^{d-1} is not affected by any carry from the $+1$ since $t < 2^{d-1}{-}1$. So there is one more 1-bit in $n_R + 1$ than in $n + 1$. This flips the 1-bit parity *odd* $\leftrightarrow$ even, corresponding to the turn flip $L \leftrightarrow R$.

Consider now the first, last and middle two points of each row.

The turns at the last point of row d and first point of $d{+}1$ are as follows



The second-last point in each row is a pair of Lucas numbers V_{d-1},V_d. The last point in each row is a pair of Fibonacci numbers F_{d+1},F_{d+2}. Those pairs both fall near a line from the origin 0,0 of slope golden ratio $\phi = \frac{\sqrt{5}+1}{2}$. Calculate a cross-product in the manner of (23) to be sure the first of row $d{+}1$ is on the right of this line.

$$\begin{aligned}
\delta p_1 = p_2 - p_1 &= F_{d+1} - V_{d-1} = F_{d-3}\\
\delta q_1 = q_2 - q_1 &= F_{d+2} - V_d \quad = F_{d-2}\\
\delta p_2 = p_3 - p_2 &= d{+}2 - F_{d+1}\\
\delta q_2 = q_3 - q_2 &= \quad 1 - F_{d+2}
\end{aligned}$$

$$\begin{aligned}
\mathit{cross} &= F_{d-3}(1 - F_{d+2}) - (d{+}2 - F_{d+1})F_{d-2}\\
&= F_{d-3} - (d{+}2)F_{d-2} \ + \ (F_{d+1}F_{d-2} - F_{d-3}F_{d+2})
\end{aligned}$$

then with d'Ocagne's identity

$$F_m F_{n+1} - F_n F_{m+1} = (-1)^n F_{m-n} \tag{24}$$

have

$$\begin{aligned}
\mathit{cross} &= -\Big((d{+}1)F_{d-2} + F_{d-4} + 3(-1)^d\Big)\\
&< 0 \qquad \text{so turn right}
\end{aligned}$$

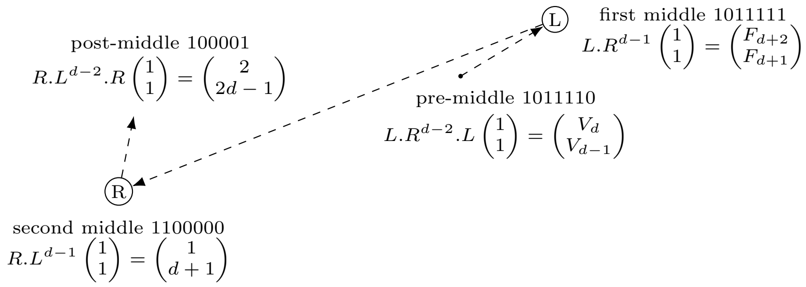
< 0 can be verified explicitly for $d{=}1$ through $d{=}3$. Then for $d \geq 4$ have $(d{+}1)F_{d-2} + F_{d-4} > 3$ exceeding the possibly-negative $3(-1)^d$. So the last turn is always right. This is point number $n = 2^d{-}1$ and $n{+}1 = 2^d$ has an odd number of bits (a single bit) which corresponds to right.

For the first point of row $d{+}1$

$$\begin{aligned}
\mathit{cross} &= \delta p_2.\delta q_3 - \delta p_3.\delta q_2\\
&= (d{+}2 - F_{d+1})(2 - 1) - (2d{+}1 - (d{+}2))(1 - F_{d+2})\\
&= (d{-}2)\,F_{d+2} + F_d + 3\\
&> 0 \quad \text{so turn left}
\end{aligned}$$

For d=1 have *cross* $= 2 > 0$. For $d \geq 2$ all terms are positive. This is point number $n = 2^{d+1}$. $n{+}1 = 2^{d+1} + 1$ has an even number of bits (two bits) which corresponds to left.

The middle two points are as follows



These points are almost a transpose of the previous diagram. The first middle and pre-middle are Fibonacci and Lucas pairs which are transposes of the last and second-last points of the row. The second middle and post-middle points are almost transposes of the first and second points except for -1 and -2 in q.

The same cross-product calculations as above can be made to see that the middle turns are left then right. This is for $d \geq 2$ since the middle points exist only for $d \geq 2$.

$$\begin{aligned}
\text{first middle} \quad \mathit{cross} &= d.F_{d-2} + F_{d-4} + 3(-1)^d \ > 0 \quad \text{turn left} \\
&= 2, 1, 7, 8, 22, 34, 70, \ldots \ d{\geq}2 \\
\text{second middle} \quad \mathit{cross} &= -\Big((d-3)\,F_{d+2} + F_d + 3\Big) \ < 0 \quad \text{turn right} \\
&= -1, -5, -14, -34, -74, -152, \ldots \ d{\geq}2 \qquad -\ \text{A094584}
\end{aligned}$$

The first middle is at $n = 2^d + 2^{d-1} - 1$ and $n{+}1 = 2^d + 2^{d-1}$ has an even number of bits (two bits) which corresponds to left.

The second middle is at $n = 2^d + 2^{d-1}$ and $n{+}1 = 2^d + 2^{d-1} + 1$ has an odd number of bits (three bits) which corresponds to right. □

This turn sequence result was found by searching for the values in Sloane's Online Encyclopedia of Integer Sequences. In retrospect, "first copy extra 0-bit turns unchanged" and "second copy extra 1-bit transpose" might have suggested a Thue-Morse parity of transposes. Suitable fixed directions at the middle points are necessary since they are copied into subsequent rows. The first and last turns are not copied and so could have been exceptions, but they follow the parity too.

8.2 HCS Rows Reversed

Rows of the tree can be read right to left instead. This is a form considered by Yosu Yurramendi in OEIS A245325. The matrices are A2, B1.

$$\begin{aligned}
\text{reverse numerators} &= 1,\ 1,2,\ 2,1,3,3,\ 3,3,2,1,5,4,5,4,\ldots \qquad \text{A245325} \\
\text{reverse denominators} &= 1,\ 2,1,\ 3,3,2,1,\ 5,4,5,4,3,3,2,1,\ldots \qquad \text{A245326}
\end{aligned}$$

9 Yu-Ting, Andreev Tree

Matrices B1,A2 taken low-to-high give a tree which is the enumerations of the rationals by Yu-Ting [21, 1980] and later independently David W. Wilson (OEIS A020650, 1996) and Andreev[1997] [1].

Each express the enumeration in terms of a recurrence

$$\begin{aligned} \gamma_{2n} &= \gamma_n + 1 \qquad\qquad \text{starting } \gamma_1 = 1/1 \\ \gamma_{2n+1} &= 1/(\gamma_n + 1) \end{aligned}$$

Arranging γ_{2n} and γ_{2n+1} as children of γ_n gives a tree of rationals.

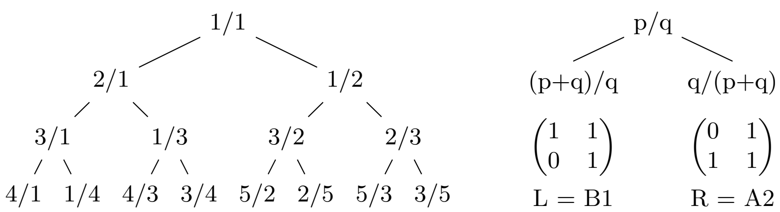


Figure 12: Yu-Ting, Wilson, Andreev tree (AYT)

$$\begin{aligned} \text{numerators row-wise } &= 1,\ 2,1,\ 3,1,3,2,\ 4,1,4,3,5,2,5,3,\dots && \text{A020650} \\ \text{denominators row-wise } &= 1,\ 1,2,\ 1,3,2,3,\ 1,4,3,4,2,5,3,5,\dots && \text{A020651} \end{aligned}$$

The previous point row-wise in the tree is

$$\begin{aligned} p_{prev} &= qF_{m-1} + rF_m \\ q_{prev} &= qF_m \quad + rF_{m+1} \end{aligned} = R^m \begin{pmatrix} q \\ r \end{pmatrix}$$

$$\text{where division } p/q \text{ gives } p = qm + r \text{ with } 0 \le r < q$$

This follows in a similar way to the Calkin-Wilf tree from section 5. Low 0-bits of n are L steps. They iterate $(p{+}q)/q$ so $(p{+}mq)/q$ and quotient m is how many trailing 0-bits. Reversing them up the tree goes to $\left(\begin{smallmatrix} r \\ q \end{smallmatrix}\right)$ and stepping across leftwards is $\left(\begin{smallmatrix} q \\ r \end{smallmatrix}\right)$, from which descend again by R^m.

Rows of the tree can be read right to left instead. This is a form considered by Yosu Yurramendi in OEIS A245327. The matrices are A2, B1 and each pair of children are still reciprocals, but the smaller one first.

$$\begin{aligned} \text{reverse numerators } &= 1,\ 1,2,\ 2,3,1,3,\ 3,5,2,5,3,4,1,4,\dots && \text{A245327} \\ \text{reverse denominators } &= 1,\ 2,1,\ 3,2,3,1,\ 5,3,5,2,4,3,4,1,\dots && \text{A245328} \end{aligned}$$

10 Kepler Fractions Tree

Matrices A1,A2 taken low-to-high starting from $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ is the descent by Kepler [14] and Benson [3]. Kepler extends the tree only as far as to make $p{+}q$ sums which are seven string harmonics.

This tree is fractions $0 < p/q < 1$.

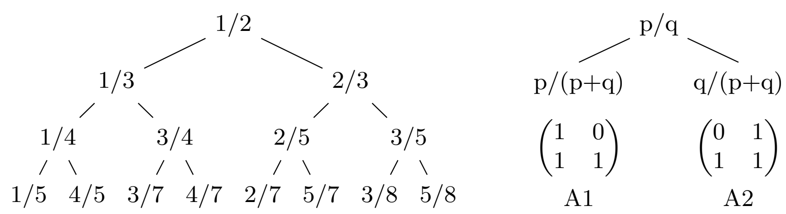


Figure 13: Kepler and Benson tree of fractions

$$\text{numerators row-wise} = 1,\ 1,2,\ 1,3,2,3,\ 1,4,3,4,2,5,3,5,\dots \qquad \text{A020651}$$
$$\text{denominators row-wise} = 2,\ 3,3,\ 4,4,5,5,\ 5,5,7,7,7,7,8,8,\dots \qquad \text{A086592}$$

This is equivalent to the AYT tree of Yu-Ting et al (section 9). If a given node of Kepler is p/q then the same node in AYT is $q/(p+q)$. Express this by a matrix C

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} p_{ayt} \\ q_{ayt} \end{pmatrix} = C \begin{pmatrix} p_{kepler} \\ q_{kepler} \end{pmatrix} \tag{25}$$

Then the Kepler and AYT trees are equivalent because

$$\begin{aligned} C^{-1}.A1.C &= B1 && \text{left matrices} \\ C^{-1}.A2.C &= A2 && \text{right matrices} \end{aligned} \tag{26}$$

At (25), the Kepler p,q is a product of Kepler matrices onto its root $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$. Using (26) and cancelling adjacent $C.C^{-1}$ leaves a single C^{-1} on that root,

$$\begin{aligned} C.\begin{pmatrix} p_{kepler} \\ q_{kepler} \end{pmatrix} &= C.(\text{Kepler LRs}) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= (\text{AYT LRs}).C^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} && C \text{ commute as inverse} \\ &= (\text{AYT LRs}). \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} p_{ayt} \\ q_{ayt} \end{pmatrix} && \text{is (25)} \end{aligned}$$

It happens that $C = A2$, but it is used here for a sum and swap.

Benson considers p+q of points in Kepler's tree and shows that p+q at position r across row d is equal to p+q of $bitrev_d(r)$ in that row, ie. reversing the d many bits of r. With each $x_i = A1 \text{ or } A2$, a point is $x_1 x_2 \dots x_d \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ and this p+q equality is

$$\left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) x_1 x_2 \dots x_d \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) x_d \dots x_2 x_1 \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \tag{27}$$

Using C to go from $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, rather than $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$, this is

$$\left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) x_1 x_2 \dots x_d\, C \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) x_d \dots x_2 x_1\, C \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$$

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, product $\left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right) M \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = a$+$b$+$c$+$d$, ie. sum of elements, so that Benson's result shows this sum of elements is equal on bit reversal. In fact reversal is the matrix transpose.

Theorem 11. *With C from* (25) *and each $x_i = A1 \text{ or } A2$,*

$$x_1 x_2 \ldots x_d\, C \;=\; (x_d \ldots x_2 x_1\, C\,)^T \tag{28}$$

Proof. Starting with the left of (28), a transpose reverses the matrix product,

$$x_1 x_2 \ldots x_d.C = (C^T.x_d^T \ldots x_2^T x_1^T)^T$$

On the right side of this, $A2^T = A2$ removes the transpose from those $x_i = A2$. $C = A2$ so likewise C^T becomes C and it commutes to the right across any $x_i = A2$. At an $x_i = A1$, have

$$C.A1^T = A1.C$$

so C commuting to the right across each $A1^T$ removes those transposes, leaving (28). □

Reversal (27) gives the right half of the HCS tree in section 8, ie. the part starting at 1/2. Bit reversal is the same set of p/q points in the row but in a different order. Benson's result is that $p{+}q$ across the row are the same.

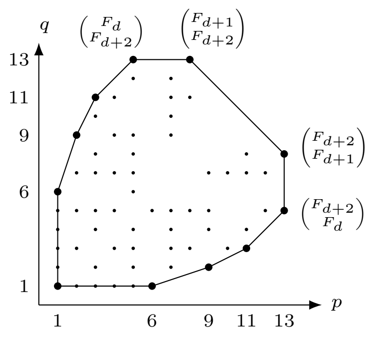
11 Convex Hull

A convex hull is the smallest convex polygon which can be drawn around a given set of points.

Theorem 12. *The vertices of the of the convex hull around the Stern-Brocot tree to depth d are the set of points*

$$HV_d = \begin{cases} \begin{pmatrix} F_{i+1} \\ F_{i+2} + (d-i)F_{i+1} \end{pmatrix} & \text{for } i{=}1 \text{ to } d \text{ inclusive} \\ \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) & \\ \begin{pmatrix} F_{i+2} + (d-i)F_{i+1} \\ F_{i+1} \end{pmatrix} & \text{for } i{=}1 \text{ to } d \text{ inclusive} \end{cases}$$

For $d{=}0$ the list $i = 1$ to $i = d$ is taken as empty so that HV_0 is the single point $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$.



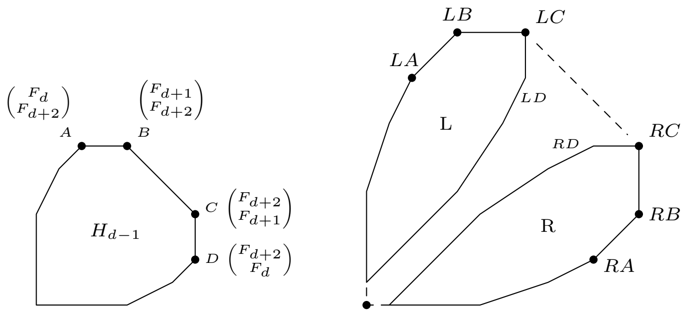
hull $d{=}5$

vertices

1,1	
1,6	6,1
2,9	9,2
3,11	11,3
5,13	13,5
8,13	13,8

Proof of Theorem 12. At $d = 0$ the tree has a single point $HV_0 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$. At $d = 1$ the hull gains the point $\left(\begin{smallmatrix}F_{1+1}\\F_{1+2}+(1-1)F_{1+1}\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)$ and its transpose $\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)$.

Proceed then by induction. Suppose HV_d is the hull vertices at depth d. The points at depth $d{+}1$ are left and right sheared copies of row d. The hull around $d{+}1$ is the hull around the two sheared d hulls, plus point $\binom{1}{1}$.



The left shear gives the upper side points as follows. Similarly p' the right shear gives the lower side points.

$$\begin{aligned} q' &= p + q \\ &= F_{i+1} \;+\; F_{i+2} \;+\; (d-i)F_{i+1} \\ &= F_{i+2} \;+\; ((d{+}1)-i)F_{i+1} && \text{is } HV_{d+1} \end{aligned}$$

These are all the points from 1,1 around to LB above and RB below. The point LC shown is a new vertex as follows which is the $i = d+1$ point in HV_{d+1}. Similarly RC.

$$\begin{aligned} L\begin{pmatrix} F_{d+2} \\ F_{d+1} \end{pmatrix} &= \begin{pmatrix} F_{d+2} \\ F_{d+2} + F_{d+1} \end{pmatrix} \\ &= \begin{pmatrix} F_{d+2} \\ F_{d+3} + ((d{+}1)-i)F_{d+2} \end{pmatrix} && \begin{array}{l} i = d{+}1 \\ \text{so } (d{+}1){-}i = 0 \end{array} \end{aligned}$$ □

Corollary 1. *The slopes of the boundary lines of the Stern-Brocot convex hull are* $d{-}2, d{-}3, \ldots, 3, 2, 1, 0$.

Proof. For $2 \le i \le d$ the slope from point $i{-}1$ to i is

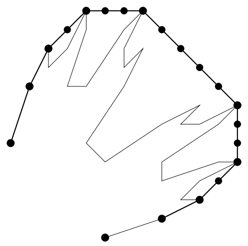
$$\begin{aligned} s &= \frac{q_i - q_{i-1}}{p_i - p_{i-1}} \\ &= \frac{\bigl(F_{i+2} \;+\; (d{-}i)F_{i+1}\bigr) - \bigl(F_{i+1} \;+\; (d-(i{-}1))F_i\bigr)}{F_{i+1} - F_i} \\ &= \frac{(d{-}i)(F_{i+1} - F_i)}{F_{i+1} - F_i} \qquad (29) \\ &= d - i \qquad \text{slopes } d{-}2 \text{ to } 0 \end{aligned}$$ □

Geometrically the segment LC to RC is new in each level and has slope −1. The shear in each subsequent level increases that slope by 1 so becomes successively slope 0, 1, 2, etc in each deeper row.

The maximum extent F_{d+2} is shown by Lehmer [16] in the context of dyads in the Stern diatomic sequence which is the Calkin-Wilf tree.

Theorem 13. *The number of integer points around the outside of the Stern-Brocot tree convex hull to row d is*

$$F_{d+3} - 1 \quad = 1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, \ldots \qquad \text{A000071}$$



row d=5 hull
outer integer points
$F_{5+3} - 1 = 20$

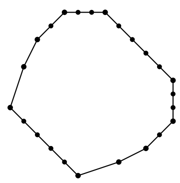
Proof. The centre diagonal $\binom{F_{d+1}}{F_{d+2}}$ to $\binom{F_{d+2}}{F_{d+1}}$ has $F_{d+2} - F_{d+1} = F_d$ many points, counting just one of its ends.

This centre diagonal is sheared in subsequent levels up and right to make the pairs of sides as described above. Those shears do not change the number of integer points on the lines. So the total integer points is

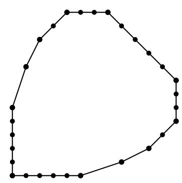
$$\begin{aligned} &2\left(F_1 + F_2 + F_3 + \cdots + F_{d-1}\right) + F_d + 1 \\ &\quad = 2\left(F_{d+1} - 1\right) + F_d + 1 \\ &\quad = F_{d+3} - 1 \end{aligned}$$

□

Taking the hull all the way around a single row adds a further $d-1$ points across the base (looking ahead to theorem 15 that there are no points below the diagonal). Taking the hull all the way around the whole tree adds $2d-1$ points across the base.



$$\begin{cases} 1 & \text{if } d = 0 \\ F_{d+3} + d - 2 & \text{if } d \geq 1 \end{cases}$$
$$= 1, 2, 5, 9, 15, 24, 38, 60, 95, \ldots$$



$$\begin{cases} 1 & \text{if } d = 0 \\ F_{d+3} + 2d - 2 & \text{if } d \geq 1 \end{cases}$$
$$= 1, 3, 7, 12, 19, 29, 44, 67, 103, \ldots$$

11.1 Convex Hull Area

Theorem 14. *The area of the convex hull around the tree points to depth d is*

$$\begin{aligned} H_d &= \tfrac{1}{2} F_{2d+3} - d - 1 \qquad (30) \\ &= 0, \tfrac{1}{2}, \tfrac{7}{2}, \tfrac{26}{2}, \tfrac{79}{2}, \tfrac{221}{2}, \tfrac{596}{2}, \tfrac{1581}{2}, \tfrac{4163}{2}, \ldots \qquad \tfrac{1}{2}\text{A027937} \end{aligned}$$

Proof. For d=0 the hull is a single point of no area $\frac{1}{2}F_{0+3} - 0 - 1 = 0$.

For d=1 the hull is a half unit triangle $\frac{1}{2}F_{2+3} - 1 - 1 = \frac{1}{2}$

For $d \geq 2$ figure 14 below shows the hull boundary within an F_{d+2} square and with side boundary lines extended to the p and q axes.

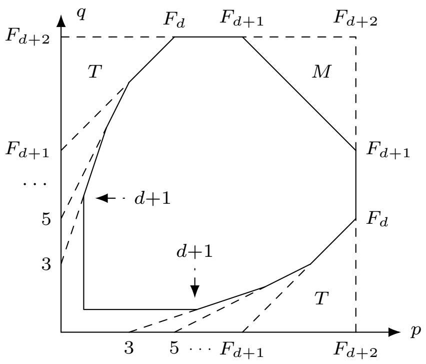


Figure 14:
convex hull
enclosing square
and side triangles

The top-right triangle M is area $\frac{1}{2}F_{d+1}^2$.
The side lines reach the q axis at Fibonacci numbers.

$$\begin{aligned} qaxis(i) &= q_i - p_i(d{-}i) && \text{slope } d{-}i \quad (29) \\ &= F_{i+2} \ + (d{-}i)F_{i+1} \ - (d{-}i)F_{i+1} \\ &= F_{i+2} && \text{on } q \text{ axis} \end{aligned}$$

The side triangles are sheared triangles of height $F_{i+2} - F_{i+1} = F_i$ and width F_i. The first triangle is the 3 to 5 which is 2×2. The top-most triangle T is $F_{d+1} \times F_{d+1}$ (and which happens to be the same as M).

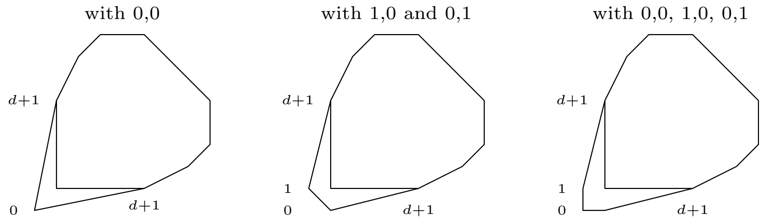
The hull is symmetric across the diagonal so the triangles along the p axis are the same as along the q axis. Hence the triangles become squares F_3^2 through F_d^2.

The net hull area for $d \geq 2$ is then as follows. The terms are simplified by usual Fibonacci identities for sum of squares and doubling.

$$\begin{aligned} H_d &= F_{d+2}^2 && \text{enclosing square} \quad (31) \\ &\quad - \tfrac{1}{2}F_d^2 && M \text{ triangle} \\ &\quad - (F_3^2 + \cdots + F_d^2) && \text{side triangles} \\ &\quad - (d+1-3) && 3 \text{ to } d{+}1 \\ &\quad - 5 && 0{,}0 \text{ to } 0{,}3 \text{ and } 3{,}0 \\ &= F_{d+2}^2 - \tfrac{1}{2}F_d^2 - F_dF_{d+1} - d - 1 \\ &= \tfrac{1}{2}F_{2d+3} - d - 1 \end{aligned}$$

□

The term $-d$ can be eliminated by adding point 0,0; or adding points 0,1 and 1,0; or adding all three of those points.



$\dfrac{F_{2d+3}-2}{2}$ $\quad$ $\dfrac{F_{2d+3}-1}{2}$ $\quad$ $\dfrac{F_{2d+3}}{2}$

$\frac{1}{2}$ A027941 $\quad$ $\frac{1}{2}$ A001519

In each case two $d{\times}1$ triangles are added on the sides, so area $+d$. For 0,0 these triangles are sheared down to 0,0 and nothing in between. For 1,0 and 0,1 the triangles are not sheared and in between there's a further triangle of area 1/2. For all three points there's a unit square in between. The latter means $+d{+}1$ which gives total area half odd Fibonaccis $F_{2d+3}/2$.

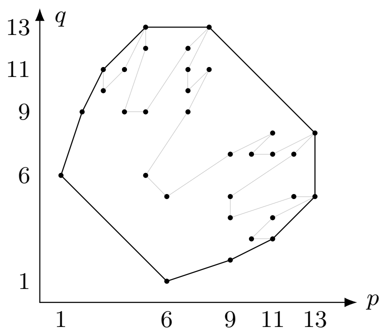
Points 0,1 and 1,0 are often included in the tree as parents of 1,1.

$$L^{-1}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\qquad R^{-1}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

Theorem 15. *The area of the convex hull around a single row d of the Stern-Brocot tree is*

$$\begin{aligned}h_d &= \frac{F_{2d+3}}{2}-\frac{d^2}{2}-d-1\\ &= 0,0,\tfrac{3}{2},\tfrac{17}{2},\tfrac{63}{2},\tfrac{196}{2},\tfrac{560}{2},\tfrac{1532}{2},\tfrac{4099}{2},\ldots\end{aligned}$$

$h_0=0$ since row d=0 is a single point (the same as H_0). $h_1=0$ since row d=1 is two points so a line segment.



Row d=5 hull
area $h_5=\frac{196}{2}$

Proof. In row d=0 the single point 1,1 has $p+q\geq d+2$. Suppose this to be true of all points in a row d. That row descends by the L matrix to

$$\begin{pmatrix}p'\\q'\end{pmatrix}=L\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}p+q\\q\end{pmatrix}$$

so

$$\begin{aligned} p' + q' &= p + q + q \\ &\geq d + 2 + q \\ &\geq d + 3 && \text{in row } d{+}1 \end{aligned}$$

Similarly the R matrix. So all points in a row are on or above the line from 1,d+1 to d+1,1. The first point $L^d\binom{1}{1} = \binom{1}{d+1}$ and last point $R^d\binom{1}{1} = \binom{d+1}{1}$ are on that line.

So the triangle below $1, d{+}1$ to $d{+}1, 1$ is removed from the full hull H_n,

$$h_n = H_n - (d{+}1 - 1)^2/2 \qquad \square$$

11.2 Convex Hull Centroid

Theorem 16. *The centroid (centre of gravity) of the convex hull of the Stern-Brocot tree is a point* (HG_d, HG_d) *which in terms of the hull area* H_d (30) *is*

$$\begin{aligned} HG_d &= \begin{cases} 1 & \text{for } d = 0 \\ \frac{HGtotal_d}{H_d} & \text{for } d \geq 1 \end{cases} \\ &= 1, \tfrac{4}{3}, \tfrac{40}{21}, \tfrac{109}{39}, \tfrac{1004}{237}, \tfrac{4372}{663}, \tfrac{1557}{149}, \tfrac{8818}{527}, \tfrac{112152}{4163}, \tfrac{237598}{5463}, \ldots \\ HGtotal_d &= \tfrac{1}{24}\left(F_{3d+7} - 4d^2 - 22d - 13\right) && \text{for } d \geq 1 \\ &= \frac{2}{3}, \frac{20}{3}, \frac{109}{3}, \frac{502}{3}, \frac{2186}{3}, 3114, 13227, 56076, 237598, \frac{3019648}{3}, \ldots && d{\geq}1 \end{aligned}$$

Proof. For $d \leq 1$ the hull is the same as the area so $HG_0{=}G_0$ and $HG_1{=}G_1$ as from theorem 5.

For $d \geq 2$ apply the hull parts (31) as weights positive and negative on the centroids of each of those parts. The centroid of a triangle is the mean of its vertices. The side triangles are in transposed pairs and the centroid of a transposed pair is the mean of the coordinates.

$$\begin{aligned} HGtotal_d = {} & F_{d+2}^2 \,.\, F_{d+2}/2 && \text{enclosing square} \\ & - F_d^2/2 \,.\, (F_{d+1}{+}F_{d+2}{+}F_{d+2})/3 && M \text{ triangle} \\ & - \sum_{i=3}^{d} F_i^2 \,.\, \begin{pmatrix} 0 + 0 + F_i \\ +F_{i+1} + F_{i+2} \\ +F_{i+1} + (d - i + 1)F_i \end{pmatrix} /6 && \text{side triangles} \\ & - (d{+}1{-}3) \,.\, (0{+}1{+}1 + 3{+}3{+}d{+}1)/6 && 3 \text{ to } d{+}1 \\ & - 5 \,.\, 11/10 && \text{0,0 to 0,3 and 3,0} \end{aligned}$$

For d=2 the sum i=3 to d is taken to be empty. The cubic products give terms in F_{3d} etc, as does the sum of cubes for the side triangles. These simplify to F_{3d+7} and the stated quadratic in d. $\square$

The numerator in $HGtotal_d$ is always a multiple of 8 so $HGtotal_d$ is a multiple of $\frac{1}{3}$. The numerator mod 3 goes in a 24-long pattern so $HGtotal_d$ is an integer or not in a 24-long pattern.

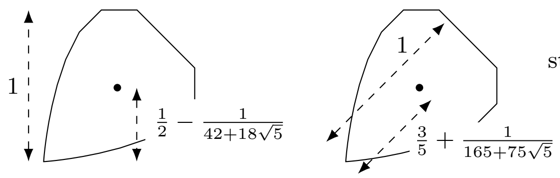
HG_d is close to the middle of the enclosing F_{d+2} square (and which in section 12 will be shown to be the minimum-area enclosing rectangle). With

$\phi = (1+\sqrt{5})/2$ the golden ratio,

$$\frac{HG_d - 1}{F_{d+2} - 1} \to \frac{\frac{1}{24}\,\phi^{3d+7}/\sqrt{5}}{\frac{1}{2}\phi^{2d+3}/\sqrt{5}\ .\ \phi^{d+2}/\sqrt{5}}$$
$$= \tfrac{1}{24}\left(5 + 3\sqrt{5}\right) \quad = 0.48784183\ldots$$
$$= \tfrac{1}{2} - \tfrac{1}{42+18\sqrt{5}} \quad = \tfrac{1}{2} - 0.012158169\ldots$$

The centroid is also close to $\frac{3}{5}$ along the diagonal hull extent, ie. without the top triangle M.

$$\frac{HG_d - 1}{F_{d+2} - \frac{1}{2}F_d - 1} \to \frac{\frac{1}{24}\,\phi^{3d+7}/\sqrt{5}}{\frac{1}{2}\phi^{2d+3}/\sqrt{5}.\ \left(\phi^{d+2}/\sqrt{5} - \frac{1}{2}\phi^{d}/\sqrt{5}\right)}$$
$$= \tfrac{1}{12}\left(5+\sqrt{5}\right) \quad = 0.60300566\ldots \qquad \text{A179641}$$
$$= \tfrac{3}{5} + \tfrac{1}{165+75\sqrt{5}} = \tfrac{3}{5}+0.00300566\ldots$$

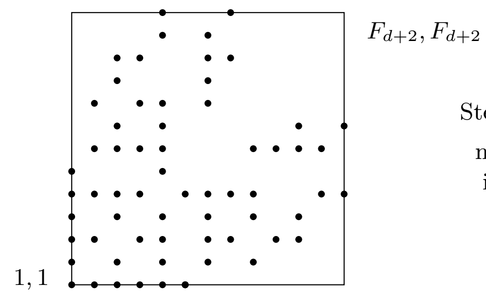


Hull centroid,
straight or diagonal,
as $k \to \infty$

12 Minimum Area Rectangle

Theorem 17. *The minimum-area rectangle around the Stern-Brocot tree points to depth d is a square aligned vertically and horizontally. For $d \geq 2$, this minimum rectangle is unique. For d=1, a second rectangle has equal minimum area. In all cases the area and boundary are*

$$AMR_d = (F_{d+2} - 1)^2 \qquad \textit{rectangle area}$$
$$= 0, 1, 4, 16, 49, 144, 400, 1089, 2916, 7744, 20449, \ldots \qquad \text{A188516}$$
$$BMR_d = 4\,(F_{d+2} - 1) \qquad \textit{boundary length}$$
$$= 0, 4, 8, 16, 28, 48, 80, 132, 216, 352, 572, \ldots \qquad d \geq 2 \ \ \text{A204644}$$

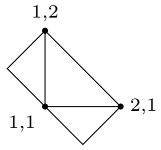


Stern-Brocot to depth d=5
minimum area rectangle
is square side $F_{d+2} - 1$

Proof. Any minimum area rectangle shares at least one side with the convex hull so it suffices to consider rectangles aligned to the sides of the hull from section 11.

For d=0, the tree is a single point enclosed by an empty square of side $F_{0+2} - 1 = 0$.

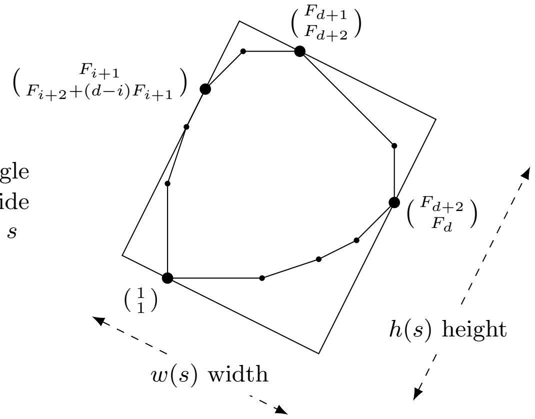
For d=1, the tree is three points and the square is $AMR_1 = 1$. A rectangle on the long side of the triangle is the only other alignment and it has equal minimum area 1 too.



$d = 1$

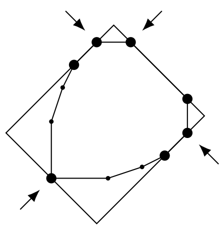
$area = 1$

For $d \geq 3$, consider a rectangle on one of the upper sloping sides as from theorem 12. For $2 \leq i \leq d-1$, the side $i-1$ to i has slope $s = d - i$ from corollary 1. The hull is symmetric across the leading diagonal so rectangles on the lower sloping sides are the same area as the upper.



The further three vertices of the hull shown are on the rectangle boundary since the line segments before and after those vertices have slopes either above and below s, or equal to s.

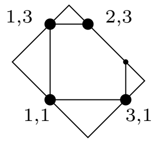
When s=1, the upper right diagonal of the convex hull is on the rectangle boundary, so that possibility is covered too.



$d \geq 3, \quad s = 1$

rectangle for upper right diagonal side

When d=2, the four vertex points above are a slope s=1 rectangle which is the only other alignment for this d. Point 1,3 is i=1 in the vertex numbering. There is no $i-1$ point as such but this i=1 suits the slope 1 upper right side. So the four vertices suit d=2 as well as $d \geq 3$.



d=2, s=1
points give
rectangle on upper right
diagonal side

Projecting the four vertex X and Y extents onto the rectangle slopes is factors $1/\sqrt{s^2+1}$ and $s/\sqrt{s^2+1}$. So with $i = d - s$, the width and height of the rectangle as a function of s (for given d) are

$$\begin{aligned} w(s) &= \big((F_{d+2} - F_{i+1})s \ + \ (F_{i+2} + sF_{i+1} - F_d)\big) \,/\sqrt{s^2+1} \\ &= \big(sF_{d+2} - F_d + F_{d-s+2}\big) \,/\sqrt{s^2+1} \\ h(s) &= \big((F_{d+2} - 1)s \ + \ (F_{d+1} - 1)\big) \,/\sqrt{s^2+1} \\ &= \big(sF_{d+2} + F_{d+1} - (s+1)\big) \,/\sqrt{s^2+1} \end{aligned}$$

Let $\mathit{diff}(s)$ be the area difference between the rectangle and AMR_d

$$\mathit{diff}(s) = w(s).h(s) - AMR_d$$

When s=1, the rectangle is bigger than the square since

$$\begin{aligned} \mathit{diff}(1) &= (F_{d+2} - F_d + F_{d+1})(F_{d+2} + F_{d+1} - 2)/2 \ - \ (F_{d+2} - 1)^2 \\ &= F_{d+1}F_{d+3} - F_{d+2}^2 + 2F_d - 1 && 2{\times}\text{A074331} \\ &= 2F_d - 1 + (-1)^d && \text{by Cassini's identity} \\ &> 0 && \text{when } d \geq 2 \end{aligned}$$

For d=2 and d=3, this $\mathit{diff}(1)$ is the only s. For $d \geq 4$, consider the area difference multiplied by $s^2 + 1$,

$$\mathit{dnum}(s) = (s^2+1)\ \mathit{diff}(s)$$

Calculate an increment $\mathit{dnum}(s{+}1) - \mathit{dnum}(s)$. This is simplified by some tedious cancellations and d'Ocagne's identity (24) to

$$\begin{aligned} &\mathit{dnum}(s{+}1) - \mathit{dnum}(s) \\ &\quad = \quad \big(F_s - (s{-}1)\big)\,F_{d-s}F_{d+2} && (32) \\ &\qquad + \ F_{s-1}F_{d-s-1}F_{d+2} \\ &\qquad + \ 2s\,(F_{d+2} - 1) - 1 && (33) \\ &\qquad + \ F_d + (-1)^{d+s}F_{s+2} && (34) \\ &\qquad + \ (s{-}1)F_{d-s} \\ &\qquad + \ F_{d-s-2} \\ &\qquad\quad > 0 \qquad\qquad \text{for } 1 \leq s \leq d{-}3 \text{ and } d \geq 4 \end{aligned}$$

Term (32) is ≥ 0 since $F_s \geq s{-}1$. Term (33) is > 0 since $F_{d+2} - 1 \geq 7$. Term (34) is > 0 since $s{+}2 \leq d{-}1$ means $F_d > F_{s+2}$.

So $\mathit{dnum}(s)$ is an increasing function of s. The sign of $\mathit{diff}(s)$ is the same as $\mathit{dnum}(s)$ so from initial $\mathit{diff}(1) > 0$ have all $\mathit{diff}(s) > 0$. Hence all the rectangles are bigger than the square AMR_d. □

As a remark, *diff*(s) does not in general increase the way *dnum*(s) with its factor $s^2{+}1$ does, but *diff* does remain positive.

A question by Michael Biro on Maths Overflow [4] answered by Andrew D. King notes that the ratio of minimum bounding rectangle area to convex polygon area is always between 1 (for a square) and 2 (for a triangle). For the minimum rectangle and convex hull around the Stern-Brocot tree this ratio is

$$\begin{aligned} AMR_d/H_d &= (F_{d+2}-1)^2/(\tfrac{1}{2}F_{2d+3}-d-1) \\ &\to \left(\phi^{2d+4}/5\right) \,/\, \left(\tfrac{1}{2}\phi^{2d+3}/\sqrt{5}\right) \\ &= 1+\tfrac{1}{\sqrt{5}} \;= 1.44721359\ldots \qquad \text{frac part A020762} \end{aligned}$$

13 Inertia

OEIS A052913 is a Lucas sequence. Yosu Yurramendi notes there that it is total of products pq across a tree row. Similar holds for sum of squares across a tree row.

Theorem 18. *Total squares p^2, or products $p.q$, across a tree row are respectively*

$$\begin{aligned} rpp_d &= \sum_{row\ d} p^2 \;= \sum_{row\ d} q^2 \\ &= 5\,rpp_{d-1} - 2\,rpp_{d-2} \qquad starting\ 1,5 \\ &= (\tfrac{1}{2}+\tfrac{5}{34}\sqrt{17})\,(\tfrac{5}{2}+\tfrac{1}{2}\sqrt{17})^d \;+(\tfrac{1}{2}-\tfrac{5}{34}\sqrt{17})\,(\tfrac{5}{2}-\tfrac{1}{2}\sqrt{17})^d \qquad (35) \\ &= 1,5,23,105,479,2185,9967,45465,\ldots \qquad \text{A107839} \end{aligned}$$

$$\begin{aligned} rpq_d &= \sum_{row\ d} pq \\ &= 5\,rpq_{d-1} - 2\,rpq_{d-2} \qquad starting\ 1,4 \\ &= (\tfrac{1}{2}+\tfrac{3}{34}\sqrt{17})\,(\tfrac{5}{2}+\tfrac{1}{2}\sqrt{17})^d \;+(\tfrac{1}{2}-\tfrac{3}{34}\sqrt{17})\,(\tfrac{5}{2}-\tfrac{1}{2}\sqrt{17})^d \qquad (36) \\ &= 1,4,18,82,374,1706,7782,35498,\ldots \qquad \text{A052913} \end{aligned}$$

$$grpp(x) = \frac{1}{1-5x+2x^2} \qquad grpq(x) = \frac{1-x}{1-5x+2x^2}$$

Proof. Each p/q in row d becomes $p/(p{+}q)$ and reciprocal $(p{+}q)/p$ in row $d{+}1$. Both are present so total squares p^2 is the same as total squares q^2. The total squares and products of the new fractions are

$$rpp_{d+1} = \sum_{\text{row } d} p^2 + (p+q)^2 \;= 3\,rpp_d + 2\,rpq_d \qquad (37)$$

$$rpq_{d+1} = \sum_{\text{row } d} 2p\,(p{+}q) \qquad = 2\,rpp_d + 2\,rpq_d \qquad (38)$$

Using (37) for *rpq* in terms of *rpp* and substituting that into (38) gives the recurrence for *rpp*. Converse substitution gives *rpq*.

These recurrences are Lucas sequences. The powers (35),(36) are the usual way to write a linear recurrence using powers of the roots of the characteristic polynomial, in this case $x^2 - 5x + 2$.

The generating functions follow from the recurrences and initial values. $\square$

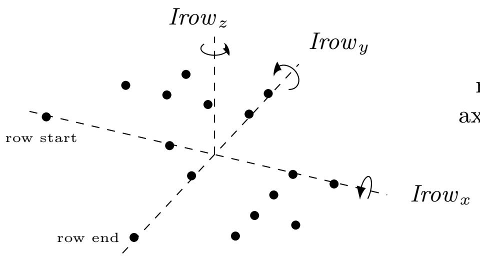
Difference (37) − (38) is $rpp_{d+1} - rpq_{d+1} = rpp_d$ so rpp increments by rpq, so cumulative

$$rpp_d = \sum_{j=0}^{d} rpq_j$$

This is also seen in the generating functions. Factor $1{-}x$ on a generating function is first differences, $grpq(x) = (1{-}x)\, grpp(x)$. Or conversely factor $\frac{1}{1-x}$ is cumulative.

rpp and rpq grow as $ppow$, the larger root in the powers forms (35),(36)

$$ppow = \tfrac{5}{2} + \tfrac{1}{2}\sqrt{17} = 4.561552\ldots \qquad \text{A082486}$$
$$psmall = \tfrac{5}{2} - \tfrac{1}{2}\sqrt{17} = 0.438447\ldots$$



d=4 row points
moment of inertia,
axes through centroid

Theorem 19. *Consider each point p, q in a tree row to have unit mass. With axes through the centroid, the mass moment of inertia tensor is*

$$\begin{pmatrix} Irow_x & -Irow_{xy} & 0 \\ -Irow_{xy} & Irow_y & 0 \\ 0 & 0 & Irow_z \end{pmatrix} \qquad \begin{matrix} I_x = \sum y^2 & I_{xy} = \sum xy \\ I_y = \sum x^2 & I_z = \sum x^2{+}y^2 \end{matrix}$$

where

$$\begin{aligned} Irow_x(d) &= rpp(d) - (\tfrac{9}{2})^d \\ &= 0, \tfrac{1}{2}, \tfrac{11}{4}, \tfrac{111}{8}, \tfrac{1103}{16}, \tfrac{10871}{32}, \ldots \\ Irow_y(d) &= Irow_x(d) \\ Irow_z(d) &= Irow_x(d) + Irow_y(d) = 2\, Irow_x(d) \\ Irow_{xy}(d) &= rpq(d) - (\tfrac{9}{2})^d \\ &= 0, -\tfrac{1}{2}, -\tfrac{9}{4}, -\tfrac{73}{8}, -\tfrac{577}{16}, -\tfrac{4457}{32}, \ldots \end{aligned}$$

Proof. The centroid of the points in a row is their mean. Per Stern (summarized by Lehmer [16]), the p or q total is 3^d for 2^d points, so centroid at $(\frac{3}{2})^d, (\frac{3}{2})^d$.

The sum rpp is squared x or y, and the sum rpq is product xy coordinates, both relative to the origin. By the parallel axis theorem, these are shifts from the centroid to there,

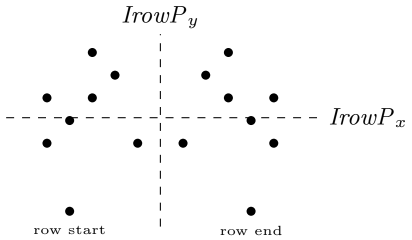
$$Irow_x(d) \;\; + 2^d \left((\tfrac{3}{2})^d \right)^2 = rpp_d$$

$$Irow_{xy}(d) + 2^d \left((\tfrac{3}{2})^d\right)^2 = rpq_d \qquad \square$$

The first few $Irow_{xy}$ are negative (and then negated again to positive in the tensor matrix). Roughly speaking, this is since more p, q points are in the 2nd and 4th quadrants relative to the centroid. But rpq has its $ppow > \frac{9}{2}$ so eventually $Irow_{xy}(d) > 0$. This happens for $d \geq 11$.

Rotated by 45° to principal axes, with x as an anti-diagonal $-45°$ and y as leading diagonal at $+45°$, the inertia becomes

$$\begin{aligned} IrowP_x(d) &= \tfrac{1}{2} Irow_x(d) + Irow_{xy}(d) + \tfrac{1}{2} Irow_y(d) && (39)\\ &= rpp(d) + rpq(d) - 2(\tfrac{9}{2})^d \\ &= \tfrac{1}{2} rpq(d{+}1) - 2(\tfrac{9}{2})^d \\ &= 0, 0, \tfrac{1}{2}, \tfrac{19}{4}, \tfrac{263}{8}, \tfrac{3207}{16}, \ldots \\ IrowP_y(d) &= \tfrac{1}{2} Irow_y(d) - Irow_{xy}(d) + \tfrac{1}{2} Irow_y(d) && (40)\\ &= rpp(d) - rpq(d) \\ &= rpp(d{-}1) && (41)\\ &= 0, 1, 5, 23, 105, 479, \ldots && d{\geq}1 \text{ A107839} \end{aligned}$$



d=4 row points moment of inertia, principal axes

For d=0 at (41), $rpp_{-1} = 0$ is by extending its recurrence backwards, or its powers form at -1.

Initially $Irow_x(d) < Irow_y(d)$ which is roughly speaking a distribution of points wider than high measured by squared distance, so more inertia when rotating about the vertical than horizontal. Difference (39) − (40) = $2\, Irow_{xy}$ shows $Irow_x$ bigger when $Irow_{xy}$ is positive which is $d \geq 11$ from above.

Ratio $IrowP_x$ over $IrowP_y$ has a limit from coefficients of the power forms of rpq and rpp

$$\begin{aligned} \frac{IrowP_x(d)}{IrowP_y(d)} &\to 4 + \sqrt{17} && (42)\\ &= 8.123105\ldots && \text{A176458} \end{aligned}$$

The inertia of the area r_d between rows d and d+1 from theorem 3 also goes as rpp and rpq.

Starting from the triangle between d=0 and d=1, the next depth row area from theorem 3 is the previous sheared up and across plus the square in between. A shear does not change the area, so the mass part of sheared inertia does not change. The coordinates do change, in the manner of (37),(38). Working through the resulting recurrences gives inertia about the origin

$$IOr_x = IOr_y = \tfrac{5}{3} rpp_d + \tfrac{37}{12} rpq_d - d^2 - 3d - \tfrac{23}{6} \qquad \text{of row area}$$

$$\begin{aligned} &= \tfrac{11}{12},\ \tfrac{77}{6},\ 80, 406,\ \tfrac{5759}{3},\ 8858,\ \tfrac{121645}{3}, \dots \\ IOr_{xy} &= \tfrac{37}{12} rpp_d + \tfrac{1}{8} rpq_d - 2d - \tfrac{7}{3} \\ &= \tfrac{7}{8},\ \tfrac{139}{12},\ \tfrac{401}{6},\ \tfrac{977}{3},\ \tfrac{4540}{3}, 6938, 31690, \dots \end{aligned}$$

The centroid is subtract $r_d\big(gtotal_d/r_d\big)^2$, which goes as $(\frac{9}{2})^2$ like the points centroid. Turned to principal axes there the ratio I_x/I_y for row area is the same as the points (42).

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